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Measure and Integration on non-Locally Compact Spaces and Representation Theory

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In memory of
Kenneth Burton
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ABSTRACT

The problem of studying representations of algebraic groups over higher dimensional local fields has already attracted a lot of attention. While many of the advances in this area involve very technical machinery and produce highly abstract results, in this thesis we approach this problem from a much more explicit angle, where we encounter phenomena which do not present themselves in the earlier approaches.

Along the way, we come to several developments which are interesting in their own right. These include a generalisation of the "higher measure" of Fesenko, and the study of topological spaces which are "not too far" from being (locally) compact.

By following Fesenko's explicit approach to defining a shift-invariant measure on a higher-dimensional local field F , we construct such a measure instead on the general linear group $\mathrm{GL}_2(F)$. The integral against this measure coincides with an integral previously defined by Morrow using lifts from the residue field, but its explicit nature makes the measure constructed here rather versatile.

Motivated by several patterns that appear while constructing the higher measures, we then study a new class of objects consisting of a group together with a topological framework called a level structure. These groups not only generalise the groups F and $\mathrm{GL}_2(F)$ associated to higher-dimensional local fields, but several standard topological concepts as well. In particular, the level structure gives rise to a very natural generalisation of compactness.

Following the more general digression, we then return to the concrete example of $\mathrm{GL}_2(F)$, using the new tools. We use the level structure to study its principal series representations, and unlike in previous, very technical incarnations of this theory, our methodology makes this much more simple.

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"We should always aim to climb Everest. Perhaps we may not get there, but hopefully we will at least manage to climb some smaller mountains along the way."

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INTRODUCTION

One of the first uses of mathematics one meets in school is the measurement of perceived quantities: the length of a line, the area of a shape, the volume of an object, and so on. The content of this thesis goes in a very similar direction to this, albeit in a much more abstract manner.

The word "measure" here may in fact be used almost interchangeably with the word "volume", and one may think of integration as being a process of calculating the volume enclosed by a particular shape (where the shape is the image of a function). The rest of the title, however, requires a little more explanation.

In terms of physical intuition, one may think of a compact object as being something that is bounded (i.e. doesn't disappear off to infinity) and closed (roughly speaking this means it contains all of its boundary). For example, if you own a field (the muddy kind, not the mathematical one!) and every point on its boundary, this would be a compact object. If you own the pastureland but not the boundary, it is not compact.

Being locally compact is slightly more general. As we noted above, if you have a field but not any point on its boundary, this is not a compact object. However, if you stand at any point in the field, it is possible to cut out a small circle around yourself that lies entirely inside the field, and since you *do* own the boundary of this small circle, it is compact. We thus say that the big field is locally compact, which in general means that every point has a small "neighbourhood" which is compact.

More general locally compact spaces are very important in classical mathematics. In particular, they are (roughly speaking) the most general situation in which the intuitive notion of volume makes sense.

This brings us quite nicely back to the title of this thesis. Here we are trying to understand volumes on spaces which are not locally compact. In general this would be a completely hopeless exercise, but we will consider simply a slight generalisation of what it means to be locally compact, and under these conditions it again becomes possible to compute volumes. However, in this more general situation, the volume of an object is not necessarily given by a number, but instead by a possibly infinite list of numbers (or, in more concrete terms, it is given by a power series).

1.1 ON LOCAL AND GLOBAL QUALITIES

One may consider the term "local field" to be a shortening of "locally compact (topological) field". However, in any dimension $n > 1$ (assuming, for now, we actually do have some idea what "dimension" is) this point of view begins to feel rather unsatisfactory, since an n -dimensional local field is never locally compact. Indeed, it is precisely due to this fact that most of the constructions in this thesis became "necessary".

Perhaps a better motivation for these terms comes from geometry. Suppose, as our starting object, we take a "surface" S (without being too precise about what that means at the moment). Such an object is surely two-dimensional, being the prototypical example of what our intuition says two-dimensional looks like. It also gives us some impression of the meanings of the terms "global" (related to the behaviour of the surface as a whole) and "local" (related to the behaviour only in some small enough section of the surface).

The first global object one usually attaches to a surface (or, indeed, to any geometric object) is its function field K , consisting of rational functions which are "defined on S ". One may then be tempted to say that an n -

dimensional global field should be the function field of some n -dimensional geometric object. However, here things begin to get a little messy.

For example, rather than a surface, let us think for a moment of a curve C . Classically, algebraic geometry would have us study C as a curve over the field $\mathbb{C}$ of complex numbers. In other words, we view the curve C as being the set of solutions in $\mathbb{C}$ to some polynomial equation $f(x, y) = 0$. Viewed in this way, the function field of C is (essentially) the field $\mathbb{C}(X)$, consisting of ratios of complex polynomials in one variable. However, it is not quite clear whether such a field is truly a "global" field, since from my point of view the field $\mathbb{C}$ is already a local field, and so $\mathbb{C}(X)$ should have both local and global aspects.

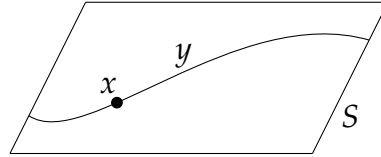
The mathematician's conundrum thus becomes "how can we adjust our point of view so that we get the expected outcome?". For me, the motivation comes from one of the most elementary results in the theory of fields, namely the existence of a prime subfield. (Slightly) more generally, let R be an integral domain. Then there exists a unique ring homomorphism $\mathbb{Z} \rightarrow R$, which sends 1 to the unit element 1_R . This map is either injective, i.e. R contains a copy of $\mathbb{Z}$, or its kernel is a prime ideal $p\mathbb{Z}$ for some prime p , and R contains a copy of the finite field $\mathbb{F}_p$.

Using a very convenient (anti-)equivalence of categories, this says roughly that a (nice enough) curve C either comes equipped with a surjective finite morphism $C \rightarrow \operatorname{Spec} \mathbb{Z}$ (and so is an "arithmetic curve"), or C is defined over a finite extension $\mathbb{F}_q$ of the field $\mathbb{F}_p$ (and so is an "algebraic curve"). In the first case, the function field of C will be a number field (i.e. a finite extension of $\mathbb{Q}$), and in the second case the function field will be a finite extension of $\mathbb{F}_q(X)$. These are precisely the fields which one would usually call "global fields".

One may thus, for instance, define a two-dimensional global field to be the function field of either an "algebraic surface" S defined over a finite field, or the function field of an "arithmetic surface", which is a scheme S equipped with a surjective morphism $S \rightarrow \operatorname{Spec} \mathbb{Z}$ such that the fibres are

one-dimensional (...plus some other technical conditions). While this definition will not be terribly important here, one should note that all surfaces in this text (unless otherwise stated) should be assumed to fall into one of these two cases.

To define the local objects, one has to apply a little more algebra and geometry. Starting with our surface S , we may look at some curve y which lies on the surface. For example, if S is an arithmetic surface, the curve can be classified as either vertical (i.e. it lies inside some fibre of the morphism $S \rightarrow \operatorname{Spec} \mathbb{Z}$, and so may be viewed as an algebraic curve over a finite field) or horizontal (i.e. y is an "arithmetic curve"). Finally, we then consider a particular point x on the surface which lies on the curve y :



I have seen some authors abbreviate the phrase "two-dimensional local field" to "two-local field". In addition to halving the number of syllables, this also has the advantage of giving the impression that such fields are somehow "twice local". This is indeed the case since two-local fields may be obtained from the above situation as follows.

First, one takes the (completed) local ring $\mathcal{O}_x$ of the surface S at the point x . Roughly, this consists of rational functions on S which are regular (i.e. don't have poles) at x . Assuming the surface S is nice enough, $\mathcal{O}_x$ is a regular local ring of Krull dimension two.

Next, we consider a local equation for the curve y at the point x . Informally (as the name suggests), this is an equation that defines the curve y in some neighbourhood of the point x , but is perhaps not valid everywhere on the surface S . Algebraically, this corresponds to some particular prime ideal $\mathfrak{p}_y$ of the ring $\mathcal{O}_x$.

Finally, we take the completion $\mathcal{O}_{x,y}$ of the localisation of $\mathcal{O}_x$ at $\mathfrak{p}_y$. (And so the ring $\mathcal{O}_{x,y}$ is the result of two localisations.) Assuming that the surface

S and the curve y are nice enough (in particular, x must be a regular point of the curve y), $\mathcal{O}_{x,y}$ is an integral domain, and its fraction field $K_{x,y}$ is a two-dimensional local field.

Unfortunately, this process does not quite account for *all* two-dimensional local fields - it only gives us the nonarchimedean ones. Such fields must necessarily be one of the following (see [FK00]):

- A finite extension of the Laurent series field $F((t))$;
- A finite extension of the "standard field" $F\{\{t\}\}$,

where F is a complete discrete valuation field with finite residue field. (The definition of $F\{\{t\}\}$ will be given in Chapter 2.)

On the other hand, not all (one-dimensional) local fields are discrete valuation fields - we also have the archimedean fields $\mathbb{R}$ and $\mathbb{C}$. From the above classification, this means we should certainly include the fields $\mathbb{R}((t))$ and $\mathbb{C}((t))$ in our definition of two-dimensional local fields, and by slight abuse of language we call these fields "two-dimensional archimedean".

At this point, the astute reader is (hopefully) now puzzling over the same question that has been burning in my mind for several years now: *are there archimedean analogues $\mathbb{R}\{\{t\}\}$ and $\mathbb{C}\{\{t\}\}$ of the standard fields?* To see why this is not a particularly simple problem to answer, the reader should note that the residue field of $F\{\{t\}\}$ is $\overline{F}((t))$, where $\overline{F}$ denotes the (finite) residue field of the local field F . This would suggest, for example, that the residue field of $\mathbb{R}\{\{t\}\}$ should be $\overline{\mathbb{R}}((t))$ - but the field of real numbers is not a discrete valuation field, and so it is not clear what we would mean by its "residue field".

These archimedean residue fields have puzzled me ever since I was introduced to two-dimensional arithmetic geometry, and it is reasonably clear that the answer will not be straightforward. Nevertheless, at this point I would like to mention a few speculations.

The residue field of a discrete valuation field F is defined as the quotient $\mathcal{O}_F/\mathcal{M}_F$ of the ring of integers $\mathcal{O}_F$ of F by its unique maximal ideal $\mathcal{M}_F$.

If we denote by $|\cdot|_F$ the norm induced by the discrete valuation on F , we have $\mathcal{O}_F = \{x \in F : |x|_F \leq 1\}$. By analogy, one may then think of $\mathcal{O}_{\mathbb{R}}$ as being the interval $[-1, 1]$ and $\mathcal{O}_{\mathbb{C}}$ as the closed unit disc. (Again, I would like to emphasise that this is speculation, and so this is not necessarily the correct way of looking at things.) However, these are no longer algebraic objects, and so using these to define a residue "field" in some way would not be straightforward.

I hear that there are some others who are thinking about these problems, using sophisticated techniques coming from analytic geometry. I have also heard whispers that there may be some connection to the mysterious "field of one element" $\mathbb{F}_1$. There is also the possibility that something related to the "arithmetic site" is involved, and I doubt very much that it is merely a coincidence that these three things have deep relations with each other.

Alternatively, the rings of integers of nonarchimedean local fields miraculously coincide with the maximal compact subgroups (of the additive group), and their unit groups coincide with the maximal compact subgroup of the multiplicative group.

In this text we will spend a lot of time dealing with groups of invertible 2×2 matrices, which represent a different kind of "higher dimensional" generalisation of the multiplicative group of a field. In the nonarchimedean case, the "maximal compact" subgroups are again intimately related to integral structures. In the archimedean case, one naturally comes across orthogonal and unitary groups, and so I would expect to see archimedean residue structures somehow reflected in such groups.

Taking this even further, the orthogonal group is also related to the complex upper half-plane, which is not only a sacred component of the study of modular forms, but forms a bridge between a great many different areas of mathematics. I would guess that there exists some "two dimensional" (in an appropriate sense) generalisation of the upper half-plane which plays a similar unifying role in "two dimensional" mathematics, though I am far from knowing where to look for such a creature. (I am aware that there are

already several known generalisations of the upper half-plane, but not of them seem to be *quite* what I am searching for.)

Of course, trying to identify archimedean residue fields using this kind of approach is rather like trying to study a tornado by looking at the trail of destruction left behind. (And perhaps, if the destruction is disorderly enough, not being sure that a tornado is the culprit at all.) Spectacularly, this also serves as a good analogy for representation theory, which also makes up a significant portion of this text!

With that lengthy digression out of the way, let us now return briefly to the issue that the geometric motivation does not account for the archimedean fields. We know that archimedean data is important - one only has to look at the product formula for number fields, for example, or consider the wealth of information given to us by analytic number theory. The exclusion of anything archimedean also has geometric significance: intersection theory does not work as expected on arithmetic surfaces, as one may seemingly make intersection points disappear by translating by a principal divisor. This problem, at least, was partially addressed by Arakelov, who "completed" an arithmetic surface to include also "fibres at infinity".

Unfortunately (or fortunately!), Arakelov geometry lies a little outside conventional scheme theory, and as such the "geometry" happening at infinity is still not so clear to me (as well as to many other mathematicians). On the other hand, a rather important open problem in two-dimensional adelic analysis involves finding an adelic description of the Arakelov intersection pairing, and it is possible that studying such connections may make various things more transparent.

On that note, once one has at hand all of the "local data" attached to a surface S , one may glue it together to form various other global objects. In particular, there are the geometric and analytic adeles $\mathbf{A}_S$ and $\mathbb{A}_S$, associated to S , which arise as restricted products of the two-local fields $K_{x,y}$ with respect to their integral structures. The function field K is contained in the

geometric adelic space, and in the arithmetic case another important open problem is to show that K is a discrete subspace of $\mathbf{A}_S$.

The global adelic objects mentioned above are involved in the study of several famous open problems in number theory - most notably the generalised Riemann Hypothesis and the Birch and Swinnerton-Dyer Conjecture (see [Fes10], [Fes17] for more details). While this thesis falls very far from making any breakthroughs related to such problems, at the very least we may hope to uncover new insights related to these important arithmetic objects.

1.2 ON THE AUTHOR'S HISTORY

As an undergraduate, I once heard number theory described as being "full of little tricks". Sadly, my initial exposure to the subject did nothing but support this sentiment, learning little theorems here and there and having each one appear seemingly independent from all of the others.

Soon after I would come across many fundamental objects, such as L -functions, Gaussian integers, and p -adic numbers. Yet all of these were presented in such a way as to seem completely unrelated, with no discussion of the deeper connections between them, and so number theory was left looking like a cluster of separatist movements with no appreciable goals for the future.

The first "saving grace" came in the form of Galois theory. Not only was it remarkably simple, but it finally gave a taste of how sometimes it is easier to study (for example) all number fields at the same time than it is to investigate one particular chosen field. Furthermore, there were concrete (and open!) problems to study, and these problems linked several other areas of mathematics together.

Alongside Galois theory, I also wound up studying algebraic number theory. During those few months, number theory went from being an overall underwhelming experience to becoming my principal area of study. I be-

gan to see that many of the things people saw as "tricks" actually came from much deeper ideas, and that all of those "distinct" topics were in fact intimately related.

My final year as an undergraduate had me studying class field theory. While already a fascinating subject in its own right, this also revealed many new mathematical paths to travel along. It led to adventures in higher dimensions and altercations with representation theory which eventually culminated in the writing of this text. It also leaves open several doors to future prospects - there should almost certainly be a modification of the representation-theoretic parts of this text which incorporate aspects of two-dimensional class field theory in a natural way.

Early on as a graduate student I came to realise that I was significantly far behind my peers (those "siblings" mentioned already in the acknowledgements) in terms of mathematical foundations. After several years it seems like that knowledge gap has only widened. Nonetheless, I continue to press forward. One gets nowhere standing still.

1.3 A MORE TECHNICAL OVERVIEW

When trying to establish analogues of classically known results in higher dimensional number theory, one comes across a very interesting dichotomy. Either the results generalise in an extremely straightforward way, or some essential obstruction presents itself which in some cases makes it impossible to "lift" results to higher dimensions.

In the former case there is usually not much to say. In the latter case, there are essentially two directions in which one may proceed. The first is to find a completely new approach to classical results which generalises more appropriately to higher dimensions. The alternative is to assess the nature of the "essential obstruction", and determine how much data may "pass through" this obstruction and how much must be left behind. This

second approach has the advantage of having a more definite starting point, but it only works as long as we don't lose too much in this transfer process.

As an example of this approach is the study of translation-invariant measure and integration on a higher dimensional local field F due to Fesenko ([Fes03], [Fes05]). Since such fields are not locally compact, there can be no real-valued Haar measure on them, and the problem of finding a suitable replacement remained open for several decades. Fesenko's remarkable observation was that, if one extends the space of values from $\mathbb{R}$ to $\mathbb{R}((X_2)) \cdots ((X_n))$, one obtains a theory of Harmonic analysis on an n -dimensional local field F which generalises the one-dimensional theory quite well.

By making the above extension, Fesenko defined a translation-invariant measure μ on the ring of subsets of F generated by translates of fraction ideals of the ring O_F of integers of F with respect to a discrete valuation of rank n . Interestingly, he noted that his generalised measure may fail to be countably additive in certain cases, although this can only happen if the infinite series which arise do not absolutely converge.

Several years later a similar idea was considered by Morrow in [Mor10]. Rather than appealing to a measure and writing down an explicit family of measurable sets, he defined an integral on F directly by lifting integrable functions from the residue field. In fact, his constructions work in the more general setting when F is any split valuation field (i.e. a field equipped with a valuation $v : F^\times \rightarrow \Gamma$ such that there is a map $t : \Gamma \rightarrow F^\times$ from the value group Γ into $F^\times$ such that $v \circ t$ is the identity on Γ) whose residue field is a local field (which may be archimedean). In the case that F is an n -dimensional local field, Fesenko's integral is recovered.

Morrow then went on to apply his construction to the group $GL_n(F)$ in [Moro8], and he considered in detail the effect of performing linear changes of variables. He then went on in [Mora] to notice that when one performs changes of variables which are not linear, Fubini's Theorem may fail. He

also noted that to define an integral on an arbitrary algebraic group would probably require a more general approach.

In the first part of this thesis, we will give an explicit construction of an invariant measure on $\mathrm{GL}_2(F)$ following Fesenko's approach, and show that the associated integral again coincides with the one defined by Morrow. As our 'generating sets' we choose analogues of the compact subgroups K_m of GL_2 of a local field, namely

$$K_{i,j} := I_2 + t_1^i t_2^j M_2(O_F),$$

where O_F is the rank two ring of integers of F , and t_1, t_2 are local parameters for F . (And all of these things will be appropriately defined later!)

From the point of view of this theory, both F and $\mathrm{GL}_2(F)$ behave to some extent as if they are locally compact groups, with maximal compact subgroups O_F and $\mathrm{GL}_2(O_F)$ respectively. It turns out that there are other situations where similar "locally compact" behaviour occurs, which naturally motivates us to look for an explanation. We explore one possible direction here by investigating the notion of level-compactness, which makes the above observations more precise. This will form the second part of the text, and while it draws inspiration heavily from the first part, this can mostly be read entirely independently if the reader wishes to focus on only the compactness properties.

In the third and final part, we develop the beginnings of a theory of admissible representations for $\mathrm{GL}_2(F)$ following (for example) Chapter 6 of [GH11]. Once again, if the reader is willing to take for granted a few results from the previous parts, this can be read independently of the rest.

The representation theory of $\mathrm{GL}_2(F)$ and related objects has also been considered previously by various authors such as Kapranov, Kazhdan and Gaitsgory ([KGo4]), Kim, and Lee ([KL04] [Lee10]). In particular, the work of Kazhdan and various collaborators uses very powerful category- and model-theoretic machinery to obtain extremely deep results (see also [HKo6] and [HKo8]), while Kim and Lee use combinatorial methods to study Hecke

Algebras in the absence of an invariant integral. The approach in this text seems to be of quite a different flavour, and relies heavily on the concept of "level", which (surprisingly!) does not appear to have been studied by anyone else.

1.4 THE BENEFITS OF HIGHER DIMENSIONAL NUMBER THEORY

Since this text will mostly be concerned with local objects, it makes sense for us to start our adventure there. For a full introduction to the theory of local fields one may consult [FVo2], and for the higher-dimensional case [FKoo] and [Morb], for example. We give only a basic overview to outline the most important aspects from the point of view of this text.

By "local field" we mean a locally compact topological field. Such fields may be split into the archimedean fields ($\mathbb{R}$ and $\mathbb{C}$) and the nonarchimedean ones. Many authors only consider the nonarchimedean case, but for global considerations it is important that we keep in mind the archimedean ones too.

The nonarchimedean fields have an important alternative characterisation. A discrete valuation field is a field F together with a surjective group homomorphism (called a discrete valuation) $v : F^\times \rightarrow \mathbb{Z}$ satisfying $v(a + b) \geq \min\{v(a), v(b)\}$ for any $a, b \in F^\times$. For convenience we let $v(0) = \infty$, which we take to be larger than every integer.

The set $\mathcal{O}_F := \{x \in F : v(x) \geq 0\}$ is a subring of F (this follows simply from the defining properties of v), which is called its ring of integers. This is a local ring with unique maximal ideal $\mathcal{M}_F := \{x \in F : v(x) > 0\}$ (since the units of $\mathcal{O}_F$ must have valuation equal to 0), and the field $\bar{F} := \mathcal{O}_F / \mathcal{M}_F$ is called the residue field of F .

It can be shown that the ring $\mathcal{O}_F$ is a principal ideal domain, and that the maximal ideal $\mathcal{M}_F$ can be generated by any element $t \in \mathcal{O}_F$ with $v(t) = 1$. Such an element t is called a prime or uniformising element. In general,

any local PID R is called a discrete valuation ring, and its field of fractions is always a discrete valuation field.

The valuation v induces a metric on F , given by $d_v(a, b) = 2^{-v(a-b)}$. The open (and also closed) balls with respect to this metric are exactly the fractional ideals $t^n \mathcal{O}_F$, where n is an integer and t is a prime element.

Definition 1.4.1. *A nonarchimedean local field is a discrete valuation field F which is complete with respect to the metric d_v and whose residue field $\bar{F}$ is finite.*

One is then naturally prompted to ask about the fields whose residue field is not finite. Here, one of the most simple cases to consider turns out to be when the residue field is a local field, and this process leads to the definition of higher dimensional fields.

Definition 1.4.2. *An n -dimensional local field F is defined inductively as follows. If $n = 1$ then we take F to be a local field as above. For $n > 1$ we then say that F is an n -dimensional local field if it is a complete discrete valuation field whose residue field $\bar{F}$ is an $(n - 1)$ -dimensional local field.*

In this text we will mostly concern ourselves with the case $n = 2$. It turns out that this is a very important boundary case. By this we mean that there are several important differences as one moves from one-dimensional to two-dimensional fields, and also when one moves from two- to three-dimensional fields. However, for $n \geq 3$ there are no fundamental consequences which arise simply from increasing the dimension. This then allows us to leave the reader with the first of several open questions.

Problem 1. *Extend the results of this text, and also those of (for example) [Fes10], [Fes15], to dimension three.*

We again write $\mathcal{O}_F$ for the ring of integers of a two-dimensional local field F with respect to its discrete valuation v , and $\mathcal{M}_F$ for its maximal ideal. We call v the rank one discrete valuation, and $\mathcal{O}_F$ the rank one ring of integers.

In addition to its discrete valuation, a two-dimensional local field F also comes equipped with a rank two discrete valuation, which is a homomor-

phism $\mathbf{v} : F^\times \rightarrow \mathbb{Z} \oplus \mathbb{Z}$ satisfying $\mathbf{v}(a + b) \geq \min\{\mathbf{v}(a), \mathbf{v}(b)\}$. Here, we order $\mathbb{Z} \oplus \mathbb{Z}$ lexicographically from the right. This means that $(a, b) < (x, y)$ if either $b < y$ or $b = y$ and $a < x$.

To define $\mathbf{v}$ we first must note that a two dimensional field also comes with two local parameters. The first local parameter can be taken to be any element $t_1 \in F$ such that t_1 is a unit of $\mathcal{O}_F$ which projects to a prime element of the residue field. The second local parameter is a prime element t_2 of F with respect to the rank one discrete valuation.

With respect to chosen local parameters $t_1, t_2 \in F$, for $x \in F$ we define $v_2(x) = v(x)$, and $v_1(x) = \bar{v}(t_2^{-v(x)}x)$, where $\bar{v}$ denotes the discrete valuation on the residue field $\bar{F}$ and $\bar{y}$ denotes the image of $y \in \mathcal{O}_F$ after projecting to the residue field. The rank two valuation is then given by $\mathbf{v}(x) = (v_1(x), v_2(x))$.

The rank two valuation may depend on the choice of local parameters. However, the following objects are independent of any choice of parameters.

Definition 1.4.3. *Let F be a two-dimensional local field and let $\mathbf{v}$ be a rank two discrete valuation. The set $\mathcal{O}_F := \{x \in F : \mathbf{v}(x) \geq 0\} = \{x \in \mathcal{O}_F : \bar{x} \in \mathcal{O}_{\bar{F}}\}$ is a subring of F , called the rank two ring of integers of F .*

The ring $\mathcal{O}_F$ is a local ring, with maximal ideal $M_F = (t_1, t_2)\mathcal{O}_F$ generated by the local parameters. (Note that $t_2 = t_1(t_1^{-1}t_2)$, and $t_1^{-1}t_2 \in \mathcal{O}_F$ since the valuation is weighted towards the second local parameter, and so in fact M_F is generated by t_1 alone). It is not a Noetherian ring. For example, it contains the strictly increasing chain of ideals $\cdots \subset t_1 t_2 \mathcal{O}_F \subset t_2 \mathcal{O}_F \subset t_1^{-1} t_2 \mathcal{O}_F \subset t_1^{-2} t_2 \mathcal{O}_F \subset \dots$. However, the fractional ideals of $\mathcal{O}_F$ are either principal of the form $t_1^i t_2^j \mathcal{O}_F$ or are powers of $\mathcal{M}_F$, which is not finitely generated.

A two-dimensional local field is also given a certain topology which takes into account the topology on the residue field. This topology is defined explicitly in [MZ95]; we do not reproduce it here since for all topological considerations in this text we will use a different topology which we will

introduce later. However, it is important to note that with respect to its two-dimensional topology a two-dimensional field is not locally compact. In particular, there is no Haar measure on F .

The Haar measure can be defined for any locally compact group, but we give it here only for local fields. For the general case see [RV99], for example, or for a fully detailed approach [Rud87].

For the archimedean fields, the Haar measure is just the usual Lebesgue measure. (On $\mathbb{R}$, for example, this says that the length $\mu([a, b])$ of the interval $[a, b]$ is $(b - a)$.) On a nonarchimedean local field, the volume $\mu(t^n \mathcal{O}_F)$ of the open (and closed) ball $t^n \mathcal{O}_F$ is defined to be q^{-n} , where q is the number of elements in the finite residue field $\bar{F}$.

The Haar measure on a locally compact group allows us to compute integrals. Furthermore, if the group is abelian (like a local field, for example), we can define a Fourier transform, which in the case of the field $\mathbb{R}$ coincides with the usual Fourier transform. Since local fields in particular arise as completions of number fields, this allows us to do harmonic analysis on number fields "locally".

In his PhD thesis, Tate established a remarkable connection between harmonic analysis on number fields and their zeta functions. The famous Euler-Riemann zeta function is given by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1 - p^{-s}}$$

where the product is taken over all primes p . The function $\zeta(s)$ converges for all complex numbers s with $\operatorname{Re}(s) > 1$. It extends to a function on the whole complex plane which is holomorphic everywhere except for at $s = 1$, where it has a simple pole.

The Riemann zeta function satisfies the functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s)$$

where

$$\Gamma(s) = \int_0^{\infty} x^{s-1} e^{-x} dx$$

is the Gamma function. The Gamma function also extends to the whole complex plane, having simple poles at nonpositive integers. The functional equation thus implies that $\zeta(s) = 0$ whenever s is a negative even integer.

It is known that any other zero of $\zeta(s)$ must satisfy $0 < \operatorname{Re}(s) < 1$. The Riemann Hypothesis states that all of these zeroes in fact satisfy $\operatorname{Re}(s) = \frac{1}{2}$, and is possibly the most famous unsolved problem in mathematics.

The connection with harmonic analysis is then as follows. For each prime p , we define the p -adic valuation $v_p(n)$ of an integer n to be the largest power of p that divides n . Since any integer may be decomposed uniquely into a product of primes, this is well-defined. We then extend to a map $v_p : \mathbb{Q}^\times \rightarrow \mathbb{Z}$ by setting $v_p\left(\frac{a}{b}\right) = v_p(a) - v_p(b)$. This is a discrete valuation on $\mathbb{Q}$, and the completion of $\mathbb{Q}$ with respect to the induced metric is a local field. This field is denoted $\mathbb{Q}_p$, and its elements are called p -adic numbers. The ring of integers of this field is denoted $\mathbb{Z}_p$.

Now we combine all of these local objects to form a global object, the ring of adeles. We define $\mathbb{A}_{\mathbb{Q}} \subset \mathbb{R} \times \prod_p \mathbb{Q}_p$ to be the set of elements $(a_\infty, a_p)_p$ in the product such that $a_p \in \mathbb{Z}_p$ for all but finitely many primes. Note that this includes any element $(a, a, a, a, \dots)$ with $a \in \mathbb{Q}$, since the denominator of a has only finitely many prime factors.

If we have a continuous function $f : \mathbb{A}_{\mathbb{Q}}^\times \rightarrow \mathbb{C}$ which we can factor as a product $f = f_\infty \cdot \prod_p f_p$, where $f_\infty : \mathbb{R}^\times \rightarrow \mathbb{C}$ and $f_p : \mathbb{Q}_p^\times \rightarrow \mathbb{C}$, we may define the integral

$$\int_{\mathbb{A}_{\mathbb{Q}}^\times} f(a) d^\times a = \left(\int_{\mathbb{R}^\times} f_\infty(a_\infty) d^\times a_\infty \right) \cdot \prod_p \int_{\mathbb{Q}_p^\times} f_p(a_p) d^\times a_p,$$

where $a = (a_\infty, a_p)_p \in \mathbb{A}_{\mathbb{Q}}^\times$.

One of the main results of Tate's thesis states that there exists an explicitly defined function $\Psi : \mathbb{A}_{\mathbb{Q}}^\times \rightarrow \mathbb{C}$ such that

$$\int_{\mathbb{A}_{\mathbb{Q}}^\times} \Psi(a) |a|_{\mathbb{A}_{\mathbb{Q}}}^s d^\times a = \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s),$$

where $|a|_{\mathbb{A}_{\mathbb{Q}}} = |a_\infty| \cdot \prod_p p^{-v_p(a_p)}$. Integrals of this form are usually called zeta integrals.

There is nothing particularly special about the field of rational numbers here. If we take any number field F (i.e. a field obtained by adjoining a root of some polynomial to $\mathbb{Q}$), we may similarly construct its ring of adeles $\mathbb{A}_F$. In this case, the zeta integral gives us not the Riemann zeta function, but a function $\zeta_F(s)$ related to F called the Dedekind zeta function. Formally, we may define it to be the Euler product

$$\zeta_F = \prod_{\mathfrak{p}} \frac{1}{1 - N(\mathfrak{p})^{-s}},$$

where the product runs over all prime ideals of the ring of integers of F and $N(\mathfrak{p})$ denoted the norm of an ideal. In the case $F = \mathbb{Q}$ this gives exactly the Riemann zeta function.

For the reader familiar with algebraic geometry, one can perform a similar construction given any scheme X of finite type over $\text{Spec } \mathbb{Z}$. Here the zeta function is given by

$$\zeta_X(s) = \prod_x \frac{1}{1 - |\kappa(x)|^{-s}},$$

where the product is over all closed points of X and $|\kappa(x)|$ is the number of elements in the residue field of the local ring at x . The product converges for $\text{Re}(s)$ large enough since X is of finite type. If $X = \text{Spec } O_F$, where O_F is the ring of integers of a number field F , then $\zeta_X(s)$ coincides with the Dedekind zeta function $\zeta_F(s)$.

If furthermore X is finite over $\text{Spec } \mathbb{Z}$, one may define the adelic object $\mathbb{A}_X$ associated to X as a restricted product over all closed points of x of certain fields K_x . (More precisely, if X is regular then K_x is the fraction field of the completion of the stalk of X at the point x .) In order to obtain the correct global theory, one also has to add extra factors which take into account archimedean data. In any case, we may obtain a zeta integral of the form

$$\int_{\mathbb{A}_X^\times} \Psi(a) |a|^s d^\times a = \delta(s) \zeta_X(s)$$

for an appropriate function Ψ and an appropriately defined valuation $|a|$ (which the experienced geometer will have no trouble identifying, and other

readers will simply not care about, so we don't write it here). The function δ is said to "complete" the zeta function; it consists of gamma factors arising from the archimedean points of X , with real places contributing $\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)$ and complex places contributing $2(2\pi)^{-s}\Gamma(s)$. These factors are often called the "local L -factors at infinity".

If X is not finite over $\text{Spec } \mathbb{Z}$ then the situation becomes much more complicated. Even in the simplest case, where the fibres of the morphism $X \rightarrow \text{Spec } \mathbb{Z}$ are curves, one has to do significantly more work to arrive at the correct adelic objects. In fact, this has only been done explicitly in the case where X is an arithmetic surface. (See [Fes10] for the constructions. From the work of Parshin and Beilinson it is possible to write down an adelic complex for more general schemes, as discussed in [Hub91]. However, it is only for arithmetic surfaces that we have an explicit definition of adeles as a restricted product.)

One issue in the case of a surface is that two-dimensional local fields begin to appear. As previously mentioned, there does not exist a Haar measure on such fields, and so the theory of harmonic analysis needed for zeta integrals is missing. However, at the turn of the century came Fesenko's fundamental construction of a more general invariant measure on a two-dimensional field F . In [Fes03], [Fes05], Fesenko defines an invariant measure on the class of subsets of F generated by the principal fractional ideals $t_1^i t_2^j O_F$ via $\mu(t_1^i t_2^j O_F) = q^{-i} X^j$, where q is the number of elements in the second residue field of F and X is an indeterminate. This measure is shift-invariant, but takes values in the larger field $\mathbb{R}((X))$.

With this measure in hand, Fesenko was able to define the Fourier transformation on two dimensional local objects. In his later work, he observes that one cannot define a reasonable shift-invariant measure on the adelic objects $\mathbf{A}_X$ considered by Parshin, Beilinson, and others which is compatible with the one-dimensional theory. This motivates the construction of a second "analytic" or "arithmetic" adelic object $\mathbb{A}_X$ in [Fes10], on which there does exist an appropriate theory of harmonic analysis. In the final part of

this paper, he constructs zeta integrals in the case where $X = \mathcal{E}$ is a proper regular model of an elliptic curve E over a number field K having at worst split multiplicative reduction.

The zeta integrals in dimension two constructed by Fesenko are significantly more involved than for the one-dimensional case discussed above. However, by looking at higher dimensions one may obtain further insights into the one-dimensional setting. For example, the zeta function of the surface $\mathcal{E}$ satisfies

$$\zeta_{\mathcal{E}}(s) = \frac{\zeta_K(s)\zeta_K(s-1)}{L_E(s)} n_{\mathcal{E}}(s),$$

where ζ_K is the Dedekind zeta function of the number field K , $n_{\mathcal{E}}$ is an auxiliary factor coming from nonsmooth points, and $L_E(s)$ is the L -function of the elliptic curve E .

The (generalised) Riemann Hypothesis is thus related to the study of the zeroes and poles of $\zeta_{\mathcal{E}}$. If we write $\zeta_{\mathcal{E},S'}(s)$ for Fesenko's zeta integral (which depends on a chosen set S' of curves on the surface $\mathcal{E}$), from [Fes17] we have the following.

Theorem 1.4.4. *There is an entire function ξ and a function H such that*

$$\zeta_{\mathcal{E},S'}(s) = \xi(s) + \xi(2-s) + \int_0^\infty H(t)e^{-(s-2)t} dt.$$

Moreover if $H'''(t)$ has no zeroes for all sufficiently large t , if the real part of GRH holds for $\zeta_{\mathcal{E}}(s)$, and if $\zeta_{\mathcal{E}}(s)$ has meromorphic continuation, then GRH holds for $\zeta_{\mathcal{E}}(s)$.

The real part of GRH means that any real pole of $\zeta_{\mathcal{E}}(s)$ with $0 < \operatorname{Re}(s) < 2$ is at $s = 1$, and this can be checked by computers. It is known that $H(t)$, $H'(t)$, $H''(t)$, and $H'''(t)$ all have no zeroes for sufficiently large t . In [Fes10], Fesenko also shows that meromorphic continuation of $\zeta_{\mathcal{E}}$ is equivalent to a certain family of functions having a "generalised Fourier series".

The two-dimensional zeta integral is also related to the conjecture of Birch and Swinnerton-Dyer, which concerns the behaviour of the L -function $L_E(s)$ as $s \rightarrow 1$. In particular, the conjecture states that the order of the zero

of $L_E(s)$ at the point $s = 1$ is equal to the rank of the Mordell-Weil group of the elliptic curve E , and this is directly related to the study of poles of $\zeta_E(s)$.

1.5 CLASS FIELD THEORY AND THE LANGLANDS PROGRAM

To an algebraic extension of fields $F \subset L$, one may associate the Galois group $G(L/F)$, which consists of symmetries of the field L which fix all elements of F . The extension is called Galois if these are the only elements fixed by all elements of $G(L/F)$. Writing F^s for the compositum of all finite separable extensions of F (i.e. those obtained by adjoining roots of a polynomial whose roots are all distinct), we define the absolute Galois group of F to be $G_F = G(F^s/F)$.

The main results of Galois theory give a functorial correspondence between Galois extensions of F inside F^s and subgroups of the Galois group G_F . For any extension L/F contained in F^s , the absolute Galois group G_L of L is a subgroup of G_F . The extension L/F is Galois if and only if G_L is a closed (in an appropriate topology) normal subgroup of G_F , in which case $G(L/F) = G_F/G_L$.

The absolute Galois group thus contains a great deal of information about the field F , since it governs (in particular) all finite algebraic extensions. In a few cases, the group is easy to calculate. For example, the only algebraic extension of $\mathbb{R}$ is $\mathbb{C}$ by the fundamental theorem of algebra, and the only automorphism σ of $\mathbb{C}$ fixing $\mathbb{R}$ is complex conjugation, and so we have $G_{\mathbb{R}} = \{1, \sigma\} \simeq \mathbb{Z}/2\mathbb{Z}$.

For a slightly more involved example, for a prime number p one may start with the finite field $\mathbb{F}_p$ containing p elements. It can be shown that for every positive integer r there is a unique extension of $\mathbb{F}_p$ of degree r , which is a field $\mathbb{F}_{p^r}$ consisting of p^r elements. The Galois group of this extension is $G(\mathbb{F}_{p^r}/\mathbb{F}_p) \simeq \mathbb{Z}/r\mathbb{Z}$, and we have $G_{\mathbb{F}_p} \simeq \varprojlim_n \mathbb{Z}/n\mathbb{Z} = \hat{\mathbb{Z}}$. (In general, the absolute Galois group of any field F is always a profinite group, equal

to the projective limit of the Galois groups of all finite Galois extensions of F .)

On the other hand, we cannot really say much in general. We do not even know, for example, the structure of the group $G_{\mathbb{Q}}$, despite many decades of devoted study. A popular method in this area is to study representations of $G_{\mathbb{Q}}$; one hopes that by studying how it acts on other objects, we may gain some insight into the structure of the group itself.

If we are a little less ambitious, however, we certainly can make progress. Suppose, instead of asking about the structure of *all* finite Galois extensions, we look only at those whose Galois group is abelian. In this case, the answer has been known to number theorists for almost 100 years thanks to class field theory.

For a local field F , class field theory gives a reciprocity isomorphism between the maximal abelian quotient of G_F and the profinite completion $\widehat{F^\times}$ of the multiplicative group. For any finite abelian extension L/F , this induces an isomorphism $G(L/F) \simeq F^\times / N_{L/F} L^\times$, where $N_{L/F}(\alpha)$ is the determinant of the linear operator $x \mapsto \alpha x$ on L .

If we now consider F a number field, we may glue together all of the local reciprocity maps for the completions at each prime ideal $\mathfrak{p}$ of O_F to obtain a global reciprocity map. Furthermore, this turns out to be trivial on $F^\times$, and so the global reciprocity map is defined on the idele class group $\mathbb{A}_F^\times / F^\times$. This generalises, for example, the quadratic reciprocity law of Gauss.

Another important aspect of class field theory is the existence theorem. This gives a correspondence between finite abelian extensions of F and closed subgroups of finite index in either the multiplicative group (in the local case) or the idele class group (in the global case). This is quite an astounding result, and shows just how much we gain by looking only at abelian extensions - in this case, we have a complete classification, while in the fully general case we know very little.

While there have been some attempts to consider reciprocity laws in the non-abelian setting (including some work by Fesenko), it is apparent that

the study of the full Galois group will require even more sophisticated techniques.

In particular, one such technique to study groups like G_F is to use representation theory, and to try and deduce the structure of G_F from its action on various spaces. This is one of the main paths of study in what is known as the Langlands program. As a starting point, one may reformulate the reciprocity law of class field theory as an association between one-dimensional representations of the Galois group and Hecke L -functions. In the Langlands program, one attempts to generalise this result by assigning L -functions to automorphic representations of algebraic groups.

In this sense, the Langlands program may be considered to be a generalisation of class field theory, and in fact uses a lot of similar terminology. However, there are several important details from class field theory which do not appear (in particular, there is no existence theorem), and so the extent to which it is a true generalisation remains up for debate.

In the later sections, we will consider very briefly the beginnings of the "2d local" Langlands for GL_2 , made possible by the study of this group from the point of view of defining an invariant measure.

1.6 SUMMARY OF MAIN RESULTS AND A SHORT ROADMAP

The main body of this text is split into three chapters.

In Chapter 2, we construct a translation-invariant, finitely additive, $\mathbb{R}((X))$ -valued measure on the group $GL_2(F)$ with F a two-dimensional local field with local parameters t_1, t_2 . After introducing all of the required tools, we prove that the measure exists and is well-defined in Theorem 2.6.3:

Theorem. *For $(i, j) > (0, 0) \in \mathbb{Z} \oplus \mathbb{Z}$ (with lexicographic ordering from the right), let $K_{i,j} = I_2 + t_1^i t_2^j M_2(O_F)$, and put $K = GL_2(O_F)$. Let $\mathcal{R}$ be the*

ring of sets generated by the distinguished sets $gK_{i,j}$ for $g \in G = \mathrm{GL}_2(F)$ and $(i, j) > (0, 0)$. The map $\mu : \mathcal{R} \rightarrow \mathbb{R}((X))$ given by

$$\mu(gK_{i,j}) = \frac{q^3}{(q^2 - 1)(q - 1)} q^{-4i} X^{4j}$$

on distinguished sets and extended to $\mathcal{R}$ by finite additivity is a well-defined, finitely additive, left invariant measure satisfying $\mu(K) = 1$.

Following this, we then define an integral against this measure, which is a linear functional on the $\mathbb{C}((X))$ -vector space spanned by indicator functions of elements of $\mathcal{R}$. Theorem 2.7.5 then gives a relationship between this integral and the integral against Fesenko's measure on F :

Theorem. Let $f = \sum_i \mathbb{I}_{U_i}$ for finitely many measurable sets U_i , and let $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}_2(F)$. Then

$$\int_{\mathrm{GL}_2(F)} f(g) d\mu(g) = \frac{q^3}{(q^2 - 1)(q - 1)} \int_F \int_F \int_F \int_F \frac{1}{|\alpha\delta - \beta\gamma|_F^2} f\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right) d\alpha d\beta d\gamma d\delta,$$

where the integrals on the right hand side are against the measure defined by Fesenko in [Fes03].

This Theorem then has several important applications.

Corollary. This integral, and hence also the measure, is both left and right shift-invariant.

Corollary. The measure on $\mathrm{GL}_2(F)$ is countably additive in the following refined sense. If $U = \bigcup_n U_n \in \mathcal{R}$ such that $\{U_n\}$ are countably many disjoint measurable sets and $\sum_n \mu(U_n)$ is an absolutely convergent series in $\mathbb{R}((X))$ then $\mu(\bigcup_n U_n) = \sum_n \mu(U_n)$.

In Chapter 3, we define and examine the properties of groups G with a level structure over a locally compact group X . In some sense, the "main results" of this chapter are the various important definitions - in particular

the definition of a level structure (3.1.1) and of level-compactness (3.2.5). After the many definitions, we come to an important class of groups G which are similar enough to $\mathrm{GL}_2(F)$ that we may restate the main results of Chapter 2 in more general terms.

Theorem. *Suppose G has a level structure of elevation e over a locally compact group X . If G is “nice enough”, and if the equality*

$$\mu_X(V) = |G_{V,\gamma} : G_{U,\gamma}| \mu_X(U)$$

holds for every $\gamma = (\gamma_1, \dots, \gamma_e) \in \mathbb{Z}^e$ and every pair $U \subset V$ of neighbourhoods of the identity in X , the map

$$\mu : g G_{U,\gamma} \mapsto \mu_X(U) Y_1^{\gamma_1} \cdots Y_e^{\gamma_e}$$

is a well-defined, finitely additive, left invariant measure taking values in the field $\mathbb{R}((Y_1)) \cdots ((Y_e))$, and is compatible with the Haar measure μ_X on X .

Here the groups $G_{U,\gamma}$ are the abstract analogues of the $K_{i,j}$. An important concrete application of this Theorem is to algebraic groups over higher-dimensional local fields.

Theorem. *Let G be an algebraic subgroup of $\mathrm{GL}_m(F)$ defined by the equations $f_1, \dots, f_k$, and let $\overline{G}$ be the subgroup of $\mathrm{GL}_m(F_1)$ defined by the reductions $\bar{f}_1, \dots, \bar{f}_k$, where F_1 is the $(n-1)^{\mathrm{st}}$ residue field of F . If G is “nice enough”, if $|K_{i,\gamma} \cap G : K_{j,\gamma} \cap G| = |K_i \cap \overline{G} : K_j \cap \overline{G}|$ for all $0 < i < j$ and all $\gamma > 0 \in \mathbb{Z}^{n-1}$ (lexicographically ordered from the right), and if the polynomials $\bar{f}_1, \dots, \bar{f}_k$ have no multiple roots, there is a left-invariant, finitely additive measure on G given by*

$$\mu(K_{i,\gamma} \cap G) = \bar{\mu}(K_i \cap \overline{G}) Y^\gamma,$$

where $\bar{\mu}$ is the Haar measure on $\overline{G}$.

Finally, in Chapter 4 we return to the case of $\mathrm{GL}_2(F)$ and define the principal series representations $(\pi, \mathcal{V}_{p;0})$ in level zero. We begin by first

with the generalisation of the full principal series to the two-dimensional local setting, i.e. the action by right-translation on the vector space

$$\mathcal{V}_p(\chi_1, \chi_2) = \left\{ f : \mathrm{GL}_2(F) \rightarrow \mathbb{C}((X)) \text{ locally constant such that} \right. \\ \left. f \left(\begin{pmatrix} y_1 & x \\ 0 & y_2 \end{pmatrix} g \right) = \chi_1(y_1) \chi_2(y_2) f(g) \right\}$$

associated to a pair of characters χ_1, χ_2 of the multiplicative group $F^\times$.

It appears that this vector space is far too large to work with directly - a problem that, for example, led Kazhdan and Gaitsgory to search for a different approach to representation theory in dimension 2 using iterated ind-pro categories (see [KGo4]).

However, due to insights gained in Chapters 2 and 3, we instead choose to attack the problem by considering instead the representation on the smaller spaces

$$\mathcal{V}_{p;j}(\chi_1, \chi_2) = \{ f \in \mathcal{V}_p(\chi_1, \chi_2) : \forall g \exists i \in \mathbb{Z} \text{ such that} \\ f(gk) = f(g) \forall k \in K_{i,j} \}.$$

consisting of the "level j locally" constant functions. We then focus on the easiest case, namely $j = 0$. If $\chi_i = |\cdot|_{\alpha_i, \beta_i} \omega_i$ for $i = 1, 2$ (where $\omega : F^\times \rightarrow \mathbb{C}^\times$ is a normalised unitary character and $|t_1^m t_2^n u|_{a,b} = q^{-am} q^{bn}$ for $a, b \in \mathbb{C}$), we use the notation $\mathcal{V}_{p;0}(\alpha, \beta, \omega)$ for the space $\mathcal{V}_{p;0}(\chi_1, \chi_2)$.

This chapter contains many computations, and culminates in the analytic continuation of the two-dimensional Jacquet integral for flat sections (which are particular elements of the space $\mathcal{V}_{p;0}(\alpha, \beta, \omega)$).

Theorem. *The integral*

$$(J_{\alpha, \beta, \omega}(f))(g) = \int_F f \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} g \right) \psi_F(-u) du$$

is a polynomial in $q^{\pm \alpha_i}$ and $q^{\pm \beta_i}$ for $i = 1, 2$ for any flat 0-section f and any fixed $g \in G$. $(J_{\alpha, \beta, \omega}(f))(g)$ thus has analytic continuation to all $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{C}$ for all $g \in G$ and $f \in \mathcal{V}_{p;0}(\alpha, \beta, \omega)$.

Of course, this is only the beginning of the study of such representation, and the techniques of this text can certainly be extended much further in this area.

For a thorough understanding of all of the material, it is of course best to read this text (and perhaps some of the reference material) in full from beginning to end. However, if the reader wishes to focus only on some particular aspect, it is possible to streamline the reading as follows.

For the reader interested in the various aspects of measure and integration, Chapter 2 should be read in full. The main goal will then be to study Section 3.5, and for this it is required to read Sections 3.1, 3.2, 3.3.1, and 3.4. The remaining sections of Chapter 3 may be also be helpful but are not strictly required.

The reader mainly interested in the topological aspects of level structures should briefly read Chapter 2 since this contains many of the motivating examples, and then read Chapter 3 in more detail. Sections 3.4 and 3.5 may be omitted if one doesn't care a great deal about the precise details regarding the invariant measure, but these sections still contain various open questions for study.

Finally, the reader interested in only the representation theoretic aspects should familiarise themselves with the basics of the higher dimensional local theory in Sections 2.2, 2.3, and 2.4, and then follow this up with reading 3.1 and 3.2 for the definition of level-compactness before moving on to Chapter 4.

MEASURE AND INTEGRATION ON GL_2 OVER A TWO-DIMENSIONAL LOCAL FIELD

In this chapter we discuss the theory of measure and integration on the group $GL_2(F)$, where F is a two-dimensional local field. This, along with the theory of harmonic analysis on F itself developed by Fesenko in [Fes03] and [Fes05], constitute the prototypical examples for the more general theory which will appear in the next chapter.

We begin in 2.1 with a review of the Haar measure on $GL_2(F)$ with F a local field. Since this group is locally compact, this theory is entirely classical, and we only include the parts that will be useful for the two-dimensional case.

In 2.2, we begin with a slightly more detailed introduction to higher dimensional local fields (again, with emphasis on what is to come in later chapters), and we then give an outline of Fesenko's construction of an invariant measure on a higher dimensional local field. This section contains far less classical material, and will be the main motivating example for both the remainder of this chapter and also most of Chapter 3, and so even a reader well-versed in this theory is asked to pay particular attention here. Fesenko's constructions continue into the shorter Section 2.3, where we pay close attention to the distinguished sets which we later generalise, and also recall some important integrals that will be used in Chapter 4.

We then move swiftly onto the new material in 2.4, where we combine Fesenko's construction in 2.3 with the classical theory in 2.1 to define an

appropriate class of measurable sets on the group $\mathrm{GL}_2(F)$ with F a two-dimensional local field.

In 2.5, with inspiration from Riemann integration we consider refinements of measurable sets; in particular, we show that if we write a measurable set in two different ways, there exists a common refinement of both presentations. This important observation is then applied in 2.6, where we define an invariant measure on $\mathrm{GL}_2(F)$ and show that it is well-defined when one passes to refinements, which is essentially enough to complete the proof that the measure is completely well-defined. In 2.7 we then define the integral against our measure and obtain a compatibility with Fesenko's integral on F as defined in 2.3.

We conclude the chapter with a few remarks in 2.8 about the similarities and differences between the approach here and the definition of the higher integral given previously by Morrow in [Moro8].

2.1 THE HAAR MEASURE ON LINEAR GROUPS OVER A LOCAL FIELD

We begin with the more classical Haar measure in the one-dimensional case. Throughout this section, F will denote a (nonarchimedean) local field, that is, F is a complete discrete valuation field with finite residue field $\overline{F} = \mathbb{F}_q$. Since the theory for locally compact groups is well known, we include in this section only the results that will be relevant for the two-dimensional study.

It is well known that F is a locally compact topological field. (See chapter IV of [FV02], for instance. A similar proof will also be given in the next chapter regarding the local level-compactness of a two-dimensional field.) It thus comes equipped with a unique (up to constant multiple) translation-invariant Borel measure, called a Haar measure.

Theorem 2.1.1. *Let F be a local field with ring of integers $\mathcal{O}$ and let $\pi \in \mathcal{O}$ be a generator of the maximal ideal. Then there is a unique Haar measure μ on F which satisfies $\mu(\alpha + \pi^i \mathcal{O}) = q^{-i}$ for all $\alpha \in F, i \in \mathbb{Z}$.*

Proof. The fact that there exists a Haar measure μ is a well-known fact in the theory of locally compact groups - see the first chapter of [RV99], for example. This measure is unique up to multiplication by a constant, and so we fix the constant such that $\mu(\mathcal{O}) = 1$. For $i > 0$ we have $|\mathcal{O} : \pi^i \mathcal{O}| = q^i$, and since μ is translation invariant we obtain $\mu(\mathcal{O}) = q^i \mu(\pi^i \mathcal{O})$ by writing $\mathcal{O}$ as a disjoint union of cosets. It thus follows that $\mu(\pi^i \mathcal{O}) = q^{-i}$. For $i < 0$ we instead note that $|\pi^i \mathcal{O} : \mathcal{O}| = q^{-i}$, from which we obtain $\mu(\pi^i \mathcal{O}) = q^{-i} \mu(\mathcal{O}) = q^{-i}$. The rest follows from translation invariance of the Haar measure. $\square$

As an open subset of the vector space $F \times F \times F \times F$, the group $\mathrm{GL}_2(F)$ is also locally compact. It thus comes with its own Haar measure.

Theorem 2.1.2. *Let $G = \mathrm{GL}_2(F)$, let $K = \mathrm{GL}_2(\mathcal{O})$, and for $n \geq 1$ let $K_n = I_2 + \pi^n M_2(\mathcal{O})$, where I_2 is the 2×2 identity matrix and $M_2(\mathcal{O})$ denotes the set of 2×2 matrices with integral coefficients. Then there is a unique Haar measure μ on G which satisfies $\mu(K) = 1$ and $\mu(g K_n) = |K : K_n|^{-1}$ for any $g \in G$.*

Proof. Similar to the case for F . $\square$

One may also define the integral against the Haar measure on G . For a locally constant function $f = \sum_i c_i \cdot \mathbb{I}_{U_i}$, where $c_i \in \mathbb{C}$ and $U_i \subset G$ are measurable, we have $\int_G f d\mu = \sum c_i \mu(U_i)$.

Theorem 2.1.3. *Suppose $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G$. Then*

$$\int_G f(g) d\mu(g) = \int_F \int_F \int_F \int_F \frac{1}{|\alpha\delta - \beta\gamma|_F^2} f\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right) d\alpha d\beta d\gamma d\delta.$$

Here, if μ_F denotes the measure on F , and $d\alpha = d\mu_F(\alpha)$, $d\beta = d\mu_F(\beta)$, $d\gamma = d\mu_F(\gamma)$, $d\delta = d\mu_F(\delta)$. The absolute value $|x|_F$ is induced by the discrete valuation on F , but is equal to $\frac{\mu_F(xU)}{\mu_F(U)}$ for any measurable subset $U \subset F$ of nonzero measure. The constant $c = \frac{q^3}{(q^2 - 1)(q - 1)}$ for the normalisation $\mu(K) = \mu_F(\mathcal{O}) = 1$.

Proof. See [GH11]. □

2.2 TWO-DIMENSIONAL LOCAL FIELDS AND THE FESENKO MEASURE

Since this section covers much less classical material, we present a little more background, and also repeat several definitions from the introduction. As before, we still only present the elements of the theory which are most relevant to us - for a fuller treatment of the theory of higher-dimensional local fields see [FKoo] and [Morb] (and also the older [Fes92a] and [Fes92b]), and for complete details on the higher measure see [Fes03] and [Fes05].

Although we will mostly work in the two-dimensional case, it is convenient to recall the definition of an n -dimensional local field for any $n \geq 0$.

Definition 2.2.1. *An n -dimensional local field F is defined inductively as follows. If $n = 1$ then we take F to be a local field (i.e. either complete discrete valuation field with finite residue field, or an archimedean field $F = \mathbb{R}$ or $F = \mathbb{C}$). For $n > 1$ we then say that F is an n -dimensional local field if it is a complete discrete valuation field whose residue field $\bar{F}$ is an $(n - 1)$ -dimensional local field. Finally, we define a 0-dimensional local field to be a finite field.*

Remark. In much of the literature on higher dimensional fields, the archimedean ones are typically given much less study, and in many cases none at all. However, due to (for example) adelic constructions on arithmetic surfaces, it is important that we include them. In particular, the most general constructions in the next chapter will equally apply to both archimedean and nonarchimedean fields.

Cohen structure theory gives a complete classification of local fields in dimension one. For a classification in higher dimensions one must first consider a "standard field" which does not appear in dimensions lower than two.

Let F be a complete discrete valuation field with valuation $w : F^\times \rightarrow \mathbb{Z}$. We define the "standard field"

$$F\{\{t\}\} = \left\{ \sum_{-\infty}^{\infty} a_i t^i : a_i \in F, \inf w(a_i) > -\infty, \lim_{i \rightarrow -\infty} w(a_i) = +\infty \right\}$$

consisting of doubly infinite series with coefficients in F such that all of the coefficients are bounded and tend to zero as the powers of t become large and negative.

One may define a discrete valuation on $F\{\{t\}\}$ by $v(\sum_{-\infty}^{\infty} a_i t^i) = \min w(a_i)$. Together with this valuation, $F\{\{t\}\}$ is a complete discrete valuation field. The ring of integers consists of those series with all coefficients integral in F , and is often denoted $\mathcal{O}_F\{\{t\}\}$. The maximal ideal of this ring consists of those series whose coefficients are in the maximal ideal $\mathcal{M}_F$ of $\mathcal{O}_F$. The condition $\lim_{i \rightarrow -\infty} w(a_i) = +\infty$ means in particular that for large enough negative i all of the coefficients are in $\mathcal{M}_F$, and so we have $\overline{F\{\{t\}\}} = \overline{F}((t))$.

Remark. By thinking of $F\{\{t\}\}$ as simply being series with bounded coefficients that tend to zero at large negative powers of t , one may view (for example) $\mathbb{Q}_p = \mathbb{F}_p\{\{t\}\}$, at least on the level of sets.

With the definition of a standard field in hand, one can give the full classification of local fields in any dimension. For this we will use the following inductive notation for the residue fields. If F is an n -dimensional local field, we write F_{n-1} for the first residue field $\overline{F}$, F_{n-2} for the second residue field $\overline{F_{n-1}}$, and so on, ending with the last residue field $F_0 = \overline{F_1} = \mathbb{F}_q$. Note that some authors use the reverse ordering; we choose this notation since the index denotes the dimension of each residue field.

Theorem 2.2.2 (Classification Theorem for n -dimensional Local Fields). *Let F be an n -dimensional local field. Then F is isomorphic to*

1. $\mathbb{F}_q((t_1)) \dots ((t_n))$ if $\text{char} F = p > 0$;
2. $K((t_2)) \dots ((t_n))$ if $\text{char} F_1 = 0$, where K is a local field;
3. a finite extension of a standard field

$$K\{\{t_1\}\} \dots \{\{t_m\}\}((t_{m+2})) \dots ((t_n))$$

if $\text{char} F_{m+1} = 0$, $\text{char} F_m = p > 0$. In this case, there is also a finite extension of F which is a standard field.

Idea of proof. The first two cases follow from the usual Cohen structure theory. For the third case, one constructs the required standard field using Witt vectors. To see that there are no further cases one also must note that $K((X))\{\{Y\}\}$ is isomorphic to $K((Y))((X))$. $\square$

Remark. In dimension two, the fields $F = K\{\{t\}\}$ with K a characteristic zero local field are perhaps the most arithmetically interesting. Not only do we have the characteristic shift as we pass to the residue field $\overline{F} = \overline{K}((t))$, but F also contains the local field K as a subfield. These mixed characteristic fields are also precisely the ones that appear as local data associated to points on vertical curves on an arithmetic surface.

Remark. In the above classification, the only place where archimedean fields currently appear is in the second case. However, as noted in the introduction, it would be interesting to attempt to also construct archimedean fields of the type $\mathbb{R}\{\{t\}\}$, for example. In terms of the underlying set, one could possibly attempt this by replacing discrete valuations with the appropriate archimedean absolute value $|\cdot|_\infty$. However, in order for this to fit into the above framework, we also expect (for example) that $\overline{\mathbb{R}\{\{t\}\}} = \overline{\mathbb{R}}((t))$, which then brings up the very interesting issue of archimedean residue fields.

Problem 2. *Study and develop the theory of archimedean local fields. In particular, identify what should be the residue field of the fields $\mathbb{R}$ and $\mathbb{C}$, and unify this with the construction of possible higher dimensional fields such as $\mathbb{R}\{\{t\}\}$.*

In addition to the usual valuation structure given by the discrete valuation, higher dimensional fields also contain more refined structures. We will concentrate our attention on the case of two-dimensional local fields since this will be the main focus of the text, but similar definitions exist also in higher dimensions.

Definition 2.2.3. *Let F be a two-dimensional local field with discrete valuation v , let $\mathcal{O}_F$ be the ring of integers with respect to this valuation, and let $\mathcal{O}_{\bar{F}}$ be the ring of integers of the residue field. We define the rank two ring of integers of F to be*

$$\mathcal{O}_F = \{x \in \mathcal{O}_F : \bar{x} \in \mathcal{O}_{\bar{F}}\},$$

where $\bar{x}$ denoted the image of $x \in \mathcal{O}_F$ in the residue field.

Examples. 1. If $F = \mathbb{F}_q((s))((t))$, we have $\mathcal{O}_F = \mathbb{F}_q((s))[[t]]$ and $\mathcal{O}_F = \mathbb{F}_q[[s]] + t\mathbb{F}_q((s))[[t]]$.

2. If $F = \mathbb{Q}_p((t))$ then $\mathcal{O}_F = \mathbb{Q}_p[[t]]$ and $\mathcal{O}_F = \mathbb{Z}_p + t\mathbb{Q}_p[[t]]$.

3. If $F = \mathbb{Q}_p\{\{t\}\}$ then $\mathcal{O}_F = \mathbb{Z}_p\{\{t\}\}$ and $\mathcal{O}_F$ consists of those series $\sum_{-\infty}^{\infty} a_i t^i \in \mathbb{Z}_p\{\{t\}\}$ with $a_i \in p\mathbb{Z}_p$ for all $i < 0$, or in other words $\mathcal{O}_F = \mathbb{Z}_p[[t]] + p\mathbb{Z}_p\{\{t\}\}$.

The rank two ring of integers may also be described in terms of a valuation. In order to define this, we first order the group $\mathbb{Z} \oplus \mathbb{Z}$ lexicographically from the right. In other words, $(a, b) \leq (x, y)$ if and only if $b < y$ or $b = y$ and $a \leq x$.

Definition 2.2.4. *Let F be a two-dimensional local field. Let v_2 be the discrete valuation on F , and let v_1 be the discrete valuation on $\bar{F}$. A pair of elements $t_1, t_2 \in F$ is called a system of local parameters if $v_2(t_2) = 1$ and $v_2(t_1) = 0$, $v_1(\bar{t}_1) = 1$. In other words, t_2 is a generator of the maximal ideal of $\mathcal{O}_F$, and t_1 is a unit in $\mathcal{O}_F$ whose image under the residue map generates the maximal ideal of $\mathcal{O}_{\bar{F}}$.*

Definition 2.2.5. *Let F be a two-dimensional local field. Let v_2 be the discrete valuation on F , let v_1 be the discrete valuation on $\bar{F}$, and let t_1, t_2 be local*

parameters. The rank two discrete valuation (associated to the parameters t_1 and t_2) is the map

$$v : F^\times \rightarrow \mathbb{Z} \oplus \mathbb{Z}$$

given by

$$v(x) = \left(v_1 \left(\overline{x t_2^{-v_2(x)}} \right), v_2(x) \right).$$

Remark. In general, the rank two valuation will depend on the choice of local parameters t_1, t_2 . For most of this text, we will generally assume that these local parameters are given, and so the reader should be aware that many constructions will possibly depend on the choice of parameters.

On the other hand, certain structures do not depend on such choices.

Lemma 2.2.6. *Let F be a two-dimensional local field with rank two discrete valuation v . Then*

$$O_F = \{x \in F : v(x) \geq (0, 0)\}.$$

Proof. This is immediate from the definitions of v and O_F . □

One also easily checks that the maximal ideal of O_F is given by

$$M_F = \{x \in F : v(x) > (0, 0)\}.$$

Given local parameters t_1, t_2 , we have a decomposition

$$F^\times \simeq t_1^{\mathbb{Z}} \times t_2^{\mathbb{Z}} \times O_F^\times,$$

and so every element of $F^\times$ may be written uniquely in the form $x = t_1^i t_2^j \varepsilon$ with $i, j \in \mathbb{Z}$ and $v(\varepsilon) = (0, 0)$.

With most of the relevant background on higher dimensional local fields now covered, we thus turn to the measure defined by Fesenko in [Fes03] and [Fes05]. For the remainder of this section we will let F be a two-dimensional (nonarchimedean) local field with local parameters t_1, t_2 .

A two-dimensional local field F admits many different topologies, depending on the situation. (For more details on the higher topology see [FK00] and [MZ95]; this is the most natural topology on F in the sense that

it carries information about both residue fields, though in later sections we will use a different topology on F .) However, it is not locally compact in any of these natural topologies.

Since F is not locally compact, it has no Haar measure. Instead, Fesenko found an explicit way of performing harmonic analysis on such fields by defining a more general measure. We start with the following.

Lemma 2.2.7. *Let $(i, j) \leq (k, \ell)$ in $\mathbb{Z} \oplus \mathbb{Z}$ (lexicographically ordered from the right as before). Then $t_1^k t_2^\ell O_F$ is a normal subgroup of $t_1^i t_2^j O_F$, and the index $|t_1^i t_2^j O_F : t_1^k t_2^\ell O_F|$ is finite if and only if $j = \ell$, in which case it is equal to q^{k-i} , where q is the cardinality of the last residue field F_0 of F .*

Proof. One sees that $t_1^k t_2^\ell O_F \subset t_1^i t_2^j O_F$ by observing that $t_1^\alpha t_2^\beta O_F = \{x \in F : v(x) \geq (\alpha, \beta)\}$, and since both groups are abelian it follows that it is a normal subgroup.

To compute the index, we first observe that

$$t_1^i t_2^j O_F / t_1^k t_2^\ell O_F \simeq O_F / t_1^{k-i} t_2^{\ell-j} O_F.$$

Now, the map $O_F \rightarrow F_0$ given by reduction mod $\mathcal{M}_F$ followed by reduction mod $\mathcal{M}_{\overline{F}}$ is surjective (since O_F maps surjectively onto $\mathcal{O}_{\overline{F}}$ via the first map), and the kernel consists of elements divisible by t_1 , namely $t_1 O_F$. We thus have an isomorphism $O_F / t_1 O_F \simeq F_0$, and so $|O_F : t_1^\alpha O_F| = |O_F : t_1 O_F|^\alpha = q^\alpha$. This proves the assertion in the case that $j = \ell$.

For the $j \neq \ell$ case, one simply has to note that $\mathcal{O}_{\overline{F}}$ maps injectively into $O_F / t_2 O_F$ via $y \mapsto \hat{y} + t_2 O_F$, where $\hat{y}$ is any element of O_F whose image in $\mathcal{O}_{\overline{F}}$ is y . Indeed, if $\hat{y} - \hat{z} \in t_2 O_F$ then $y - z = 0$, since t_2 generates the maximal ideal of $\mathcal{O}_F$ and so all elements divisible by it are in the kernel of the first residue map, and so the above map is indeed injective. This shows in particular that $O_F / t_2 O_F$ has cardinality at least that of $\mathcal{O}_{\overline{F}}$, which is already infinite. $\square$

Recall that even a Haar measure is only unique up to multiplication by a constant. For convenience, in the search for an invariant measure on F

we begin with a normalisation so that $\mu(O_F) = 1$. Lemma 2.2.7 then immediately tells us that any translation-invariant measure must satisfy $\mu(\alpha + t_1^i O_F) = q^{-i}$ for any $\alpha \in F, i \geq 0$, since O_F is the disjoint union of q^i cosets of this form. By similar index calculations, we see that in fact we must have $\mu(\alpha + t_1^i O_F) = q^{-i}$ for all integers i .

However, when we wish to consider translates of principal fractional ideals $\alpha + t_1^i t_2^j O_F$ with $j \neq 0$, problems immediately arise. For example, the set $t_2^{-1} O_F$ contains every $t_1^i O_F$ for all integers i , and since usual measures are monotone this in particular would imply that $\mu(t_2^{-1} O_F) \geq \mu(t_1^i O_F) = q^{-i}$ for every integer i . Similar computations would also force $\mu(t_2 O_F) \leq q^{-i}$ for every integer i .

For a real-valued measure this would indeed be an issue. However, by using techniques from non-standard analysis, Fesenko realised that one could instead define a measure on F which takes values in the two-dimensional archimedean field $\mathbb{R}((X))$ rather than in $\mathbb{R}$, by defining $\mu(\alpha + t_1^i t_2^j O_F) = q^{-i} X^j$ for $\alpha \in F, i, j \in \mathbb{Z}$. The indeterminate X may be thought of an infinitesimal positive element, and this interpretation then agrees with the monotonicity computations in the previous paragraph.

Before stating the fundamental Theorem concerning this measure, we require the following definition.

Definition 2.2.8. *We say that a series in $\mathbb{R}((X))$ (respectively a series in $\mathbb{C}((X))$) absolutely converges if it converges and furthermore the coefficient of each X^m is an absolutely convergent series in $\mathbb{R}$ (respectively in $\mathbb{C}$).*

Theorem 2.2.9. *Let F be a two-dimensional local field with local parameters t_1, t_2 . Then the map $\mu(\alpha + t_1^i t_2^j O_F) = q^{-i} X^j$ extends to a finitely additive, shift-invariant, $\mathbb{R}((X))$ -valued measure on the ring of subsets of F generated by the sets $\{\alpha + t_1^i t_2^j O_F : \alpha \in F, i, j \in \mathbb{Z}\}$. The measure is countably additive in the following refined sense: if $\{D_k\}$ are countably many disjoint measurable sets and $\sum_k \mu(D_k)$ converges absolutely in $\mathbb{R}((X))$ then we have*

$$\mu\left(\bigcup_k D_k\right) = \sum_k \mu(D_k).$$

Proof. See [Fes03] for the first construction of the measure, and [Fes05] for some additional details. $\square$

We end this section with an important definition.

Definition 2.2.10. For $\alpha \in F^\times$, let $|\alpha|_F = \frac{\mu(\alpha U)}{\mu(U)}$, where U is any measurable subset of F with nonzero measure.

Remark. This does not depend on the choice of U . We also put $|0|_F = 0$. This is an extension of the notion of absolute value $|\cdot|_L$ on a local field L , since the definition we have just given is equivalent to the usual definition in this case. However, since $|\cdot|_F$ takes values in $\mathbb{R}((X))$ rather than $\mathbb{R}$, we do not use the term ‘absolute value’ for this function. (In [Fes03], [Fes05] this is called the module.) If $\varepsilon \in O_F^\times$, we have $|t_1^i t_2^j \varepsilon|_F = q^{-i} X^j$.

2.3 DISTINGUISHED SETS - FIRST ENCOUNTER

The ring of subsets on which Fesenko’s measure is defined can in fact be given explicitly. We will repeat the construction here, since it is a very important prototype for what is to come later.

Definition 2.3.1. Let F be a two-dimensional local field with local parameters t_1, t_2 . A set of the form $\alpha + t_1^i t_2^j O_F$ for some $\alpha \in F, i, j \in \mathbb{Z}$ is called a distinguished set. We also say that the empty set is distinguished.

Lemma 2.3.2. If D_1 and D_2 are distinguished sets then $D_1 \cap D_2$ is either empty or equal to one of D_1, D_2 .

Proof. Let $D_1 = \alpha + t_1^i t_2^j O_F$ and $D_2 = \beta + t_1^m t_2^n O_F$, and without loss of generality suppose that $(i, j) \leq (m, n)$. Then we have $t_1^m t_2^n O_F \subset t_1^i t_2^j O_F$, hence $D_1 \cap D_2 = (\alpha + t_1^i t_2^j O_F) \cap (\beta + t_1^m t_2^n O_F) \subset (\alpha + t_1^i t_2^j O_F) \cap (\beta + t_1^i t_2^j O_F)$. The latter two sets are F -cosets of the same subgroup O_F , hence are disjoint or equal. If they are disjoint then D_1 and D_2 are also disjoint, and if they are equal then $D_1 \cap D_2 = (\alpha +$

$$t_1^i t_2^j O_F) \cap (\beta + t_1^m t_2^n O_F) = (\beta + t_1^i t_2^j O_F) \cap (\beta + t_1^m t_2^n O_F) = \beta + t_1^m t_2^n O_F = D_2. \quad \square$$

Definition 2.3.3. Let F be a two-dimensional local field. A *dd-set* is a set of the form $A = \bigcup_i A_i \setminus \bigcup_j B_j$ with pairwise disjoint distinguished sets A_i , pairwise disjoint distinguished sets B_j , and $\bigcup_j B_j \subset \bigcup_i A_i$. A *ddd-set* is a disjoint union of *dd-sets*.

Remark. By Lemma 2.3.2, we may arrange so that each B_j is contained inside some A_i .

Proposition 2.3.4. The class $\mathcal{R}$ of *ddd-sets* is closed under finite unions, intersections, and set difference. It coincides with the minimal ring of sets containing all of the distinguished sets.

Proof. First we show that the class of *dd-sets* is closed under intersection. Since intersection distributes over union, it is enough to check this for two *dd-sets* of the form $E_1 = A \setminus \bigcup_i B_i$ and $E_2 = C \setminus \bigcup_j D_j$ with A, B_i, C , and D_j all distinguished. We have

$$E_1 \cap E_2 = (A \cap C) \setminus \left(\bigcup_i B_i \cup \bigcup_j D_j \right),$$

and since A and C are distinguished their intersection is distinguished also by Lemma 2.3.2. $E_1 \cap E_2$ is thus a *dd-set* by definition.

Since the classes of distinguished sets and *dd-sets* are not closed under unions, if we show that the class $\mathcal{R}$ of *ddd-sets* is closed under union, intersection, and difference then by construction it is the minimal ring containing the distinguished sets, and by extension the ring of subsets of F on which Fesenko's measure is defined. However, $\mathcal{R}$ is closed under unions by definition, and is closed under intersections by the same argument in the previous paragraph, and so it remains to prove that it is closed under differences.

As before let $E_1 = A \setminus \bigcup_i B_i$ and $E_2 = C \setminus \bigcup_j D_j$ with A, B_i, C, D_j all distinguished. Using de Morgan's laws one obtains

$$E_1 \setminus E_2 = \left(A \setminus \left(C \cup \bigcup_i B_i \right) \right) \cup \left(\bigcup_j (A \cap D_j) \setminus \bigcup_i B_i \right).$$

Both components are ddd-sets by definition, hence the union is a ddd-set as required. $\square$

Given an invariant measure, it is then possible to define an integral against it. The most general integrable functions considered by Fesenko are the following.

Definition 2.3.5. Let $f(x) = \sum_i c_i \chi_i(x) f_i(x)$, where each $c_i \in \mathbb{C}$, $f_i(x) = \mathbb{I}_{U_i}(x)$ is the indicator function of a measurable $U_i \in \mathcal{R}$, and $\chi_i(x) : F \rightarrow \mathbb{C}^\times$ is a character of F . Then we define

$$\int_F f(x) d\mu(x) = \sum_i c_i \chi_i(U_i) \mu(U_i),$$

where $\chi_i(U_i)$ is equal to zero if χ_i takes more than one value on the set U_i and is equal to the constant value of χ_i on this set otherwise. We also define the integral of a function which is zero outside finitely many points to be zero.

It is often also useful to consider the integral over a subset of F .

Definition 2.3.6. Let $f(x) = \sum_i c_i \chi_i(x) f_i(x)$ be an integrable function as in Definition 2.3.5, and let E be a measurable subset of F . Then $f(x) \mathbb{I}_E(x)$ is also integrable, and we define

$$\int_E f(x) d\mu(x) = \int_F f(x) \mathbb{I}_E(x) d\mu(x).$$

One of the most important integrals, as computed in [Fes03], is the following.

Lemma 2.3.7. Let ψ_F be a character of F which is trivial on O_F but nontrivial on $t_1^\alpha t_2^\beta O_F$ for all $(\alpha, \beta) \leq (0, 0)$. Then

$$\int_{t_1^i t_2^j O_F^\times} \psi_F(-uy) du = \begin{cases} 0 & \text{if } y \notin t_1^{-i-1} t_2^{-j} O_F \\ -q^{-1-i} X^j & \text{if } y \in t_1^{-i-1} t_2^{-j} O_F^\times \\ (1 - q^{-1}) q^{-i} X^j & \text{if } y \in t_1^{-i} t_2^{-j} O_F. \end{cases}$$

The proof essentially follows from the fact that $\int_{t_1^i t_2^j O_F} \psi_F(-u y) du$ is equal to $q^{-i} X^j$ if $y \in t_1^{-i} t_2^{-j} O_F$ and 0 otherwise, and that

$$\int_{t_1^i t_2^j O_F^\times} \psi_F(-u y) du = \int_{t_1^i t_2^j O_F} \psi_F(-u y) du - \int_{t_1^{i+1} t_2^j O_F} \psi_F(-u y) du.$$

When we define the integral over $\mathrm{GL}_2(F)$ in later sections, we will not be concerned with integrals concerning characters. However, for the representation theory that will appear in later chapters, characters of F are quite important. We will also require the Fourier transform.

Definition 2.3.8. Let $f(x)$ be an integrable function as in Definition 2.3.5, and let ψ_F be a character of F which is trivial on O_F but nontrivial on $t_1^\alpha t_2^\beta O_F$ for all $(\alpha, \beta) \leq (0, 0)$. The Fourier transform of f is given by

$$\mathcal{F}(f)(x) = \int_F f(y) \psi_F(-x y) dx.$$

Remark. The Fourier transform of f may also be denoted by $\hat{f}$. In [Fes03] the Fourier transform is instead defined by $\int_F f(y) \psi_F(x y) dx$; whether or not one includes the minus sign is purely a matter of taste. (So long as one corrects for this at archimedean places when computing global integrals.)

Theorem 2.3.9. Let Q_F be the space of integrable functions f on F which are supported in O_F such that $f|_{O_F} = g \circ p$, where $p : O_F \rightarrow \overline{F}$ is the projection to the residue field and g is a Bruhat-Schwarz function on the local field $\overline{F}$. Then we have the double transform formula

$$\widehat{\widehat{f}}(x) = f(-x).$$

Proof. See section 9 of [Fes03]. We do not repeat the proof here since it does not contain any important ideas from the point of view of this text, but it essentially consists of reducing to the double transform on the residue field. $\square$

Remark. Kapranov suggested in [Kap] a completely different approach to analysis on higher dimensional local fields, working with distributions and functions rather than introducing a measure. See also Remark 1 in Section 13 of [Fes03].

2.4 DISTINGUISHED SETS FOR GL_2

In this section we finally move on to the new material. With the relevant background on two-dimensional local fields now covered, we will aim to emulate the one-dimensional results of Section 2.1 for GL_2 in the two-dimensional case, following the blueprints of Fesenko's measure on F via distinguished sets as in Section 2.3.

We begin with the two-dimensional analogue of the subgroups K_n , which will eventually become our distinguished sets. Recall that we order the group $\mathbb{Z} \oplus \mathbb{Z}$ lexicographically from the right.

Definition 2.4.1. For $(i, j) > (0, 0)$ in $\mathbb{Z} \oplus \mathbb{Z}$, put $K_{i,j} = I_2 + t_1^i t_2^j M_2(O_F)$, where $M_2(O_F)$ denotes the set of 2×2 matrices with entries in O_F . It will be convenient to also define $K = \mathrm{GL}_2(O_F)$.

Remark. One easily checks that $K_{i,j} \subset K_{m,n}$ if $(m, n) \leq (i, j)$. In previous iterations of this work, we took $K_{0,0}$ to be equal to K . However, while this slightly simplifies some statements in this particular case, it does not fit properly with the more general theory of level structures in the following chapter, and so we no longer make this definition.

Lemma 2.4.2. $K_{i,j}$ is a normal subgroup of K .

Proof. Let

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K, \quad k = \begin{pmatrix} 1 + t_1^i t_2^j \alpha & t_1^i t_2^j \beta \\ t_1^i t_2^j \gamma & 1 + t_1^i t_2^j \delta \end{pmatrix} \in K_{i,j}.$$

Then

$$gkg^{-1} = \begin{pmatrix} 1 + t_1^i t_2^j w & t_1^i t_2^j x \\ t_1^i t_2^j y & 1 + t_1^i t_2^j z \end{pmatrix} \in K_{i,j},$$

since $w = \frac{\alpha ad + \gamma bd - \beta ac - \delta bc}{ad - bc}$, $x = \frac{\beta a^2 + (\delta - \alpha)ab - \gamma b^2}{ad - bc}$, $y = \frac{\gamma d^2 + (\alpha - \delta)cd - \beta c^2}{ad - bc}$, $z = \frac{\beta ac + \delta ad - \alpha bc - \gamma bd}{ad - bc}$ are all elements of O_F (note that $ad - bc = \det g \in O_F^\times$ by assumption). $\square$

We now prove the important ordering property of these sets, similarly to the corresponding result for distinguished subsets of F .

Lemma 2.4.3. *Let $(i, j) \leq (m, n)$, and let $g, h \in GL_2(F)$. Then the intersection of $gK_{i,j}$ and $hK_{m,n}$ is either empty or equal to one of them.*

Proof. By assumption $K_{m,n} \subset K_{i,j}$, hence $gK_{i,j} \cap hK_{m,n} \subset gK_{i,j} \cap hK_{i,j}$. The latter two sets are $GL_2(F)$ -cosets of the same subgroup $K_{i,j}$, hence are disjoint or equal. If they are disjoint then $gK_{i,j}$ and $hK_{m,n}$ are also disjoint. If they are equal then $gK_{i,j} \cap hK_{m,n} = hK_{i,j} \cap hK_{m,n} = hK_{m,n}$. $\square$

Note that, by this intersection property, any union $gK_{i,j} \cup hK_{m,n}$ is either disjoint or equal to one of the components.

Remark. Although Lemma 2.4.3 is very simple, it will be used more frequently than any other result in this section, and so we would like to draw attention to its importance here. For archimedean fields, for example, this result does not hold, and this makes many things much trickier to work with.

Following the previous section, we then make the following definitions.

Definition 2.4.4. *A distinguished set is either empty or a set of the form $gK_{i,j}$ with $g \in GL_2(F)$ and $(i, j) > (0, 0)$. A dd-set is a set of the form $A = \bigcup_i A_i \setminus (\bigcup_j B_j)$, with pairwise disjoint distinguished sets A_i , pairwise disjoint distinguished sets B_j , and $\bigcup_j B_j \subset \bigcup_i A_i$. A ddd-set is a disjoint union of dd-sets.*

As before, for a dd-set we may again arrange that each B_j is contained in some A_i , and so we will make this assumption henceforth without any loss of generality. Indeed: $A_i \setminus B_j = A_i \setminus (A_i \cap B_j)$ and if $A_i \cap B_j$ is nonempty then it is a distinguished set by Lemma 2.4.3.

Lemma 2.4.5. *The class $\mathcal{R}$ of ddd-sets is closed under finite union, intersection, and difference. $\mathcal{R}$ is thus the minimal ring (of sets) which contains all of the distinguished sets.*

Proof. All that was required for the proof for ddd-sets of F were de Morgan's laws and Lemma 2.3.2. Substituting the latter with Lemma 2.4.3 thus makes the exact same proof (the proof of Proposition 2.3.4) work here. $\square$

The main goal of this chapter is to define a Fesenko-type measure on the collection $\mathcal{R}$ for $GL_2(F)$. In order to do this, we must consider the most important properties of ddd-sets.

Definition 2.4.6. For two distinguished sets $gK_{i,j} \supset hK_{m,n}$, by abuse of language we define the index $|gK_{i,j} : hK_{m,n}|$ of $hK_{m,n}$ in $gK_{i,j}$ to be the index $|K_{i,j} : K_{m,n}|$.

Remark. This definition makes sense since $g^{-1}hK_{m,n}$ is a coset of $K_{m,n}$ inside $K_{i,j}$.

Recall from Lemma 2.2.7 that $|O_F : t_1 O_F| = q$ and $|O_F : t_2 O_F|$ is infinite.

Lemma 2.4.7. If $j = n$, $|gK_{i,j} : hK_{m,n}| = q^{4(m-i)}$. Otherwise, the index is infinite.

Proof. The map $K_{i,j} \rightarrow (t_1^i t_2^j O_F / t_1^{i+1} t_2^j O_F)^4 \simeq (O_F / t_1 O_F)^4$ given by

$$\begin{pmatrix} 1 + t_1^i t_2^j \alpha & t_1^i t_2^j \beta \\ t_1^i t_2^j \gamma & 1 + t_1^i t_2^j \delta \end{pmatrix} \mapsto (\alpha, \beta, \gamma, \delta) \pmod{t_1 O_F}$$

induces an isomorphism $K_{i,j}/K_{i+1,j} \simeq (O_F/t_1 O_F)^4$. We thus have $|gK_{i,j} : hK_{m,j}| = |K_{i,j} : K_{m,j}| = \prod_{r=i}^{m-1} |K_{r,j} : K_{r+1,j}| = |O_F/t_1 O_F|^{4(m-i)} = q^{4(m-i)}$.

On the other hand, the map $K_{i,j} \rightarrow (t_1^i t_2^j O_F / t_1^i t_2^{j+1} O_F)^4 \simeq (O_F/t_2 O_F)^4$ given by

$$\begin{pmatrix} 1 + t_1^i t_2^j \alpha & t_1^i t_2^j \beta \\ t_1^i t_2^j \gamma & 1 + t_1^i t_2^j \delta \end{pmatrix} \mapsto (\alpha, \beta, \gamma, \delta) \pmod{t_2 O_F}$$

induces an isomorphism $K_{i,j}/K_{i,j+1} \simeq (O_F/t_2 O_F)^4$, and the latter group has infinite order. $\square$

While in the case of F the indices all took a similar form, in the case of $GL_2(F)$ the indices in the "maximal compact" subgroup K are slightly different.

Lemma 2.4.8. The index $|K : K_{i,0}| = q^{4i-3}(q^2-1)(q-1)$, and for $j > 0$ the index $|K : K_{i,j}|$ is infinite for any i .

Proof. The map $K \rightarrow GL_2(O_F/t_1O_F)$, $g \mapsto g \bmod t_1O_F$ induces an isomorphism $K/K_{1,0} \simeq GL_2(O_F/t_1O_F) \simeq GL_2(\mathbb{F}_q)$, and the latter is well known to have order $(q^2 - 1)(q^2 - q)$. We thus have $|K : K_{i,0}| = |K : K_{1,0}| \cdot |K_{1,0} : K_{i,0}| = q^{4(i-1)}(q^2 - 1)(q^2 - q) = q^{4i-3}(q^2 - 1)(q - 1)$.

Now suppose $j > 0$. For $i \geq 0$, $|K : K_{i,j}| \geq |K_{0,j} : K_{i,j}|$, and the latter is infinite by Lemma 2.4.7. On the other hand, if $i < 0$, the index $|K : K_{i,j}|$ differs from $|K_{0,j} : K_{i,j}|$ only by a finite constant, and so in either case $|K : K_{i,j}|$ is infinite. $\square$

It is important to see what happens for more general disjoint unions of other distinguished sets.

Proposition 2.4.9. *If D is a distinguished set such that $D = \bigcup_{r=1}^n D_r$ is a disjoint union of finitely many distinguished sets D_r then the indices $|D : D_r|$ are all finite.*

Proof. By making a translation if necessary, we may assume that $D = K_{i,j}$ for some (i, j) . The first step is to show that at least one D_r has finite index in D . If $D = \bigcup_{r=1}^n D_r = \bigcup_{r=1}^n g_r K_{i_r, j_r}$, let $(i_*, j_*) = \min_r \{(i_r, j_r)\}$. Then $\bigcup_r g_r K_{i_*, j_*} \supset \bigcup_r g_r K_{i_r, j_r} = D$; in other words, D is covered by finitely many copies of K_{i_*, j_*} , and so K_{i_*, j_*} has finite index in D .

By relabelling if necessary, we may thus assume that $|D : D_1|$ is finite, and so we may take a complete system of coset representatives $S = \{h_1 = I_2, h_2, \dots, h_m\}$. We thus have $D \setminus D_1 = \bigcup_{r=2}^n D_r = \bigcup_{s=2}^m h_s D_1$. Taking the intersection with any D_r , this gives $D_r = \bigcup_{s=2}^m (D_r \cap h_s D_1)$.

For each $r > 1$, let $S_r = \{h \in S : D_r \cap h D_1 \neq \emptyset\}$. Each S_r is nonempty, since $\bigcup_{h \in S_r} (D_r \cap h D_1) = D_r$. By Lemma 2.4.3, $D_r \cap h D_1$ is thus equal to either D_r or $h D_1$ for any $h \in S_r$.

If $D_r \cap h D_1 = D_r$ for any h then we must have $S_r = \{h\}$ and $D_r \subset h D_1$. But since $D \setminus D_1 = \bigcup_{r=2}^n D_r$, we have $h D_1 = \bigcup_{r \in R} D_r$, where R is a subset of $\{2, \dots, n\}$. In particular, we have a distinguished union of disjoint distinguished sets of shorter length, and so by induction each index $|h D_1 : D_r|$ is finite. The tower law for indices then implies that $|D : D_r| = |D : D_1| \cdot |D_1 : D_r|$ is finite.

On the other hand, if $D_r \cap hD_1 = hD_1$ for all $h \in S_r$ then we have $hD_1 \subset D_r$. By the tower law for indices we thus have $|D : D_1| = |D : hD_1| = |D : D_r| \cdot |D_r : hD_1|$, hence $|D : D_r| \leq |D : D_1|$ is finite. $\square$

Corollary 2.4.10. *If $D, D_1, \dots, D_n$ are finitely many distinguished sets such that $D_r \subset D$ and $|D : D_r|$ is infinite for each r , there do not exist finitely many distinguished sets $C_1, \dots, C_m$ with $D \setminus \bigcup_{r=1}^n D_r = \bigcup_{s=1}^m C_s$.*

Proof. We may assume that the D_r are all disjoint by deleting any which are contained in some larger one. Similarly, we may assume that the C_s are all disjoint. If there did exist finitely many such C_s , we would thus have $D = \bigcup_{r=1}^n D_r \cup \bigcup_{s=1}^m C_s$, a union of finitely many disjoint distinguished sets. By Proposition 2.4.9 this in particular implies that each $|D : D_r|$ is finite, which contradicts our assumption that these indices are infinite. $\square$

In other words, the above Corollary states that a distinguished set with finitely many "holes" of infinite index cannot be written as a finite union of distinguished sets.

2.5 REFINEMENTS

The results of the previous section will be enough to show that (up to scaling) there is at most one Fesenko-type measure on $\mathrm{GL}_2(F)$. In this section, we craft the tools which allow us to show that our measure is well-defined, or in other words that there is at least one such measure.

Definition 2.5.1. *Let $A = \bigcup_i B_i$ be a ddd-set, where the*

$$B_i = \bigcup_j C_{i,j} \setminus \bigcup_k D_{i,k}$$

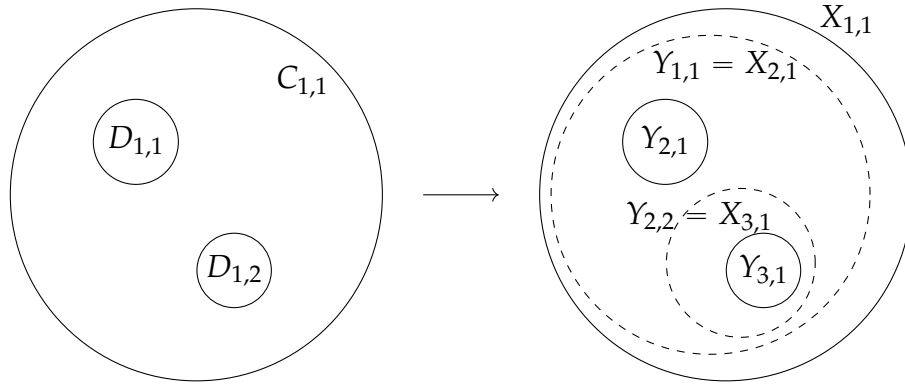
are disjoint dd-sets made from distinguished sets $C_{i,j}$ and $D_{i,k}$. A refinement of A is a ddd-set

$$\tilde{A} = \bigcup_p \left(\bigcup_q X_{p,q} \setminus \bigcup_r Y_{p,r} \right)$$

satisfying the following conditions:

1. $A = \tilde{A}$ as sets;
2. For every (i, j) there is some (p, q) such that $X_{p,q} = C_{i,j}$;
3. For every (i, k) there is some (p, r) such that $Y_{p,r} = D_{i,k}$.

The idea of a refinement can be best understood from the following picture.



Here, the ddd-set on the left is $A = C_{1,1} \setminus (D_{1,1} \cup D_{1,2})$. The set on the right $\tilde{A} = (X_{1,1} \setminus Y_{1,1}) \cup (X_{2,1} \setminus (Y_{2,1} \cup Y_{2,2})) \cup (X_{3,1} \setminus Y_{3,1})$ is a refinement of A since $C_{1,1} = X_{1,1}$, $D_{1,1} = Y_{2,1}$, and $D_{1,2} = Y_{3,1}$.

Remark. The idea of refinements of ddd-sets comes from the use of refinements of open intervals in the construction of the Riemann integral.

Before coming to the fundamental result regarding refinements, it is useful to introduce some terminology for ddd-sets.

Definition 2.5.2. Let

$$A = \bigcup_i \left(\bigcup_j C_{i,j} \setminus \bigcup_k D_{i,k} \right)$$

be a ddd-set. We call A reduced if it does not contain any dd-components of the form $B \setminus B$ for a distinguished set B .

Remark. The property of being reduced depends on the particular components of a ddd-set (or, more precisely, depends on the specific presentation

of a given ddd-set). For example, if B, C, D are disjoint nonempty distinguished sets such that $A = B \setminus (C \cup D)$, A is reduced even if $C \cup D = B$. However, $A' = (B \setminus (C \cup D)) \cup (E \setminus E)$ is not reduced, even though $A = A'$ at the level of sets.

Note that for any ddd-set A we may form a reduced ddd-set A_{red} by removing all of the superfluous components $B \setminus B$. Since the components we delete are empty, $A = A_{red}$ at the level of sets.

Definition 2.5.3. *Let*

$$A = \bigcup_i \left(\bigcup_j C_{i,j} \setminus \bigcup_k D_{i,k} \right)$$

be a ddd-set. The components $C_{i,j}$ are called the big shells, and the components $D_{i,k}$ are called the small shells.

Remark. By the above definition, we can reformulate the definition of a refinement as follows. A refinement of a ddd-set A is a ddd-set $\tilde{A}$ such that $A = \tilde{A}$ as sets, every big shell of A is a big shell of $\tilde{A}$, and every small shell of A is a small shell of $\tilde{A}$.

The most fundamental result regarding refinements is as follows.

Theorem 2.5.4. *Let A and A' be reduced ddd-sets with $A = A'$ as sets. Then there exists a reduced ddd-set $\tilde{A}$ which is a refinement of both A and A' .*

Proof. Suppose

$$A = \bigcup_i \left(\bigcup_j C_{i,j} \setminus \bigcup_k D_{i,k} \right),$$

$$A' = \bigcup_\ell \left(\bigcup_m C'_{\ell,m} \setminus \bigcup_n D'_{\ell,n} \right),$$

for distinguished sets $C_{i,j}, D_{i,k}, C'_{\ell,m}, D'_{\ell,n}$. Starting with $S = A$, the following algorithm will give the required $\tilde{A}$.

Step 1: The set S is a reduced ddd-set which is a refinement of A by construction. If S is a refinement of A' then we may take $\tilde{A} = S$ and we are done. If there is some $C'_{\ell,m}$ which is not a big shell of S , go to Step 2.

Otherwise, there is some $D'_{\ell,n}$ which is not a small cell of S , in which case go to step 4.

Step 2: If $C'_{\ell,m} \not\subset C$ for all big shells C of S then go to Step 3. Otherwise, the set of all big shells of S containing $C'_{\ell,m}$ is nonempty and totally ordered by inclusion by Lemma 2.4.3 (since their intersection in particular contains $C'_{\ell,m}$), and so there is a minimal one $C_{\min}$.

Let S' be the same as S but with $C_{\min} \setminus \bigcup_{D \subset C_{\min}} D$ replaced with

$$\left(C_{\min} \setminus \left(C'_{\ell,m} \cup \bigcup_{D \cap C'_{\ell,m} = \emptyset} D \right) \right) \cup \left(C'_{\ell,m} \setminus \bigcup_{D \subset C'_{\ell,m}} D \right).$$

(Note that none of these small shells can contain $C'_{\ell,m}$, since otherwise there would be a big shell smaller than $C_{\min}$ containing $C'_{\ell,m}$.) Since every big shell of S is still a big shell in S' , and every small shell of S is still a small shell in S' , S' is a refinement of S , and hence of A . Furthermore, S' contains one more big shell of A' than S . Thus if we put $S = S'$ and return to Step 1, after finitely many iterations there will be no big shells of A' which are not contained in at least one big shell of S .

Step 3: Since $C'_{\ell,m} \not\subset C$ for all big shells C of S , either $C'_{\ell,m} \cap C = \emptyset$ for all such C , or by Lemma 2.4.3 there is at least one C with $C \subset C'_{\ell,m}$.

In the first case, since $S = A'$ as sets, we must have $C'_{\ell,m} \setminus \bigcup_n D'_{\ell,n} = \emptyset$. We thus define S' to be the disjoint union $S \cup (C'_{\ell,m} \setminus \bigcup_n D'_{\ell,n})$. S' is a refinement of A , and again contains one more big shell of A' than S does, and so returning to Step 1 with $S = S'$ will eliminate this case after finitely many iterations.

In the second case, we can take a collection $\{C_x\}$ of maximal big shells of S contained in $C'_{\ell,m}$; in other words, $C_x \subset C'_{\ell,m}$ for all x , and for every big cell of S satisfying $C \subset C'_{\ell,m}$ there is exactly one x with $C \subset C_x$. (The fact that we may have such a collection, and that the collection will be nonempty, is guaranteed by Lemma 2.4.3.)

Suppose that $C'_{\ell,m} = \bigcup_x C_x$. In this case, we can let S' be the same as S but with the component $\bigcup_x (C_x \setminus \bigcup_{D \subset C_x} D)$ replaced by $(C'_{\ell,m} \setminus \bigcup_x C_x) \cup$

$\bigcup_x (C_x \setminus \bigcup_{D \subset C_x} D)$. This is again a refinement of A and contains one more big shell of A' .

On the other hand, suppose that $C'_{\ell,m} \neq \bigcup_x C_x$. Then there can't be another component of A to make up the difference, since this would give another big shell with nonempty intersection, so by assumption would have to be contained already in a C_x . This means there must be some small shells of A' which cut out the remaining part. In other words, $C'_{\ell,m} = \bigcup_x C_x \cup \bigcup_{D'_{\ell,n} \subset C'_{\ell,m}} D'_{\ell,n}$. Since this is a union of distinguished sets, pairwise intersections are either empty or equal to one of the components, in which case we can discard the superfluous components until we have a disjoint union $C'_{\ell,m} = \bigcup_x C_x \cup \bigcup_y D_y$.

We then let S' be the same as S but with the component $\bigcup_x (C_x \setminus \bigcup_{D \subset C_x} D)$ replaced by $\left(C'_{\ell,m} \setminus \left(\bigcup_x C_x \cup \bigcup_y D_y\right)\right) \cup \bigcup_x (C_x \setminus \bigcup_{D \subset C_x} D)$. This gives a refinement of A and contains one more big shell of A' , and since this also exhausts all possible big shell cases, returning to Step 1 with $S = S'$ will after finitely many iterations lead to all big shells of A' being contained in S .

Step 4: Since S contains all big shells of both A and A' , $D'_{\ell,n}$ must be contained in a big shell of S . Since by Lemma 2.4.3 the set of all big shells containing $D'_{\ell,n}$ is totally ordered by inclusion, there is a minimal one $C_{\min}$. We then let S' be the same as S but with $C_{\min} \setminus \bigcup_{D \subset C_{\min}} D$ replaced with

$$\left(C_{\min} \setminus \left(D'_{\ell,n} \cup \bigcup_{D \cap D'_{\ell,n} = \emptyset} D\right)\right) \cup \left(D'_{\ell,n} \setminus \bigcup_{D \subset D'_{\ell,n}} D\right).$$

(Note that as in Step 2 none of the small shells can contain $D'_{\ell,n}$ since otherwise there would be a big shell smaller than $C_{\min}$ containing $D'_{\ell,n}$.) Then S' is a refinement of S , and hence of A , and contains one more small shell of A' than S . Thus by returning to Step 1 with $S = S'$, after finitely many iterations all small shells of A' will be included in S . Furthermore, since this process does not remove any big shells, at this stage S will be a refinement of A' , and so we can set $\tilde{A} = S$. $\square$

Example. Let $D_1, \dots, D_n$ be finitely many disjoint distinguished sets such that $D = \bigcup_i D_i$ is distinguished. Taking $A = D$, $A' = \bigcup_i D_i$, the algorithm gives the chain of refinements $S_0 = D$, $S_1 = (D \setminus D_1) \cup D_1$, $S_2 = (D \setminus (D_1 \cup D_2)) \cup D_1 \cup D_2$, and so on, until finally after n iterations we obtain $\tilde{A} = (D \setminus \bigcup_i D_i) \cup \bigcup_i D_i$. On the other hand, if we instead run the algorithm with $A = \bigcup_i D_i$, $A' = D$, we obtain $\tilde{A} = (D \setminus \bigcup_i D_i) \cup \bigcup_i D_i$ after a single iteration.

Remark. One of the most important uses of the algorithm is when we take A' to be a refinement of A . In this case, the output $\tilde{A}$ of the algorithm will be A' , and so it gives a precise construction of A' from A .

2.6 MEASURE ON $\mathrm{GL}_2(F)$

We shall now utilise the results of the previous two sections to define an invariant measure μ on $\mathrm{GL}_2(F)$, and show that this measure is well-defined. We begin with the value of μ on distinguished sets.

To begin with, we would like the measure we define to be left-invariant, and so we insist that $\mu(gK_{i,j}) = \mu(K_{i,j})$ for all $g \in \mathrm{GL}_2(F)$ and all $(i, j) > (0, 0)$. Next, we recall that $K_{i,0}$ is of finite index in K for every $i \geq 0$ by Lemma 2.4.8. By writing K as a finite disjoint union of cosets, and using left-invariance and finite additivity, we obtain $\mu(K) = |K : K_{i,0}| \mu(K_{i,0})$. If we normalise so that $\mu(K) = 1$, this gives

$$\mu(K_{i,0}) = \frac{1}{|K : K_{i,0}|} = \frac{q^3}{(q^2 - 1)(q - 1)} q^{-4i}.$$

If $j > 0$ then the index is no longer finite. In this case, if also $i > 0$ we follow Fesenko's method and define

$$\mu(K_{i,j}) = \frac{X^{4j}}{|K : K_{i,0}|} = \frac{q^3}{(q^2 - 1)(q - 1)} q^{-4i} X^{4j}.$$

Remark. In terms of this particular definition, the use of the indeterminate X^4 in the above definition may appear quite arbitrary. Indeed, the definition will still work if we replace X^4 with any indeterminate Y . However, we will see in Theorem 2.7.5 that the particular choice $Y = X^4$ ensures compatibility

with Fesenko's measure on F - in other words, the X above is exactly the same X which appears in Fesenko's measure.

To see what should be the measure for nonpositive i , note that for any fixed $j > 0$ the index $|K_{i,j} : K_{\ell,j}| = q^{4(\ell-i)}$ for $i \leq \ell$ by Lemma 2.4.7. Finite additivity thus forces $\mu(K_{i,j}) = q^{4(\ell-i)}\mu(K_{\ell,j})$. If $i > 0$, this is in agreement with the definition given in the paragraph above. If $i < 0$, setting $\ell = -i$ gives

$$\mu(K_{i,j}) = q^{-8i}\mu(K_{-i,j}) = \frac{q^{-8i}X^{4j}}{|K : K_{-i,0}|} = \frac{q^3}{(q^2-1)(q-1)}q^{-4i}X^{4j}.$$

In particular, the formula is the same as for positive i . (Note that it would not be reasonable to write something like $|K : K_{i,0}|$ in this case, since $K_{i,0}$ is not even a group for $i < 0$.)

To determine what should be the value of $\mu(K_{0,j})$ for $j > 0$, we instead apply the argument of the previous paragraph with $i = 0, \ell = 1$ to obtain

$$\mu(K_{0,j}) = q^4\mu(K_{1,j}) = \frac{q^4X^{4j}}{|K : K_{1,0}|} = \frac{q^3}{(q^2-1)(q-1)}X^{4j}.$$

We have thus proved the following.

Proposition 2.6.1. *Any left-invariant, finitely additive measure μ on $\mathcal{R}$ which takes values in $\mathbb{R}((X^4))$ must satisfy $\mu(gK_{i,j}) = \lambda q^{-4i}X^{4j}$ for some $\lambda \in \mathbb{R}^\times$. In particular, if one normalises so that $\mu(K) = 1$ then we have*

$$\lambda = \frac{q^3}{(q^2-1)(q-1)}.$$

Remark. Note that we have not yet shown that the map $\mu(gK_{i,j}) = \lambda q^{-4i}X^{4j}$ extends to a measure on $\mathcal{R}$ - the above Proposition merely states that if there exists a left-invariant, finitely additive measure on $\mathcal{R}$ then it must be of this form when restricted to distinguished sets.

We thus want to extend μ to dd-sets via

$$\mu\left(\bigcup_i A_i \setminus \bigcup_j B_j\right) = \sum_i \mu(A_i) - \sum_j \mu(B_j)$$

and then to ddd-sets via $\mu(C \cup D) = \mu(C) + \mu(D)$ for disjoint dd-sets C and D . Under this extension, it is possible that μ depends heavily on the given presentation of a particular ddd-set, and so our task is now to show that if we have two different presentations A and A' of the same ddd-set then we in fact have $\mu(A) = \mu(A')$. We will do this using refinements.

Proposition 2.6.2. *Let D be a ddd-set, and let $\tilde{D}$ be a refinement of D . For any $\lambda \in \mathbb{R}^\times$, the map μ which satisfies $\mu(gK_{i,j}) = \lambda q^{-4i} X^{4j}$ on distinguished sets, when extended to $\mathcal{R}$ by additivity, satisfies $\mu(D) = \mu(\tilde{D})$.*

Proof. Using the algorithm of Theorem 2.5.4 with $A = D$ and $A' = \tilde{D}$, the resulting output will be $\tilde{A} = \tilde{D}$. In other words, the only operations that may occur to pass from a ddd-set to its refinement are exactly those in the proof of Theorem 2.5.4, and so we only need to check that the value of μ is preserved by these operations.

In Step 2, the single component $C_{\min} \setminus \bigcup_{D \subset C_{\min}} D$ is replaced with

$$\left(C_{\min} \setminus \left(C'_{\ell,m} \cup \bigcup_{D \cap C'_{\ell,m} = \emptyset} D \right) \right) \cup \left(C'_{\ell,m} \setminus \bigcup_{D \subset C'_{\ell,m}} D \right).$$

The former has measure

$$\mu \left(C_{\min} \setminus \bigcup_{D \subset C_{\min}} D \right) = \mu(C_{\min}) - \sum_{D \subset C_{\min}} \mu(D).$$

The measure of the latter is given by

$$\begin{aligned} & \left(\mu(C_{\min}) - \left(\mu(C'_{\ell,m}) + \sum_{D \cap C'_{\ell,m} = \emptyset} \mu(D) \right) \right) + \left(\mu(C'_{\ell,m}) - \sum_{D \subset C'_{\ell,m}} \mu(D) \right) \\ &= \mu(C_{\min}) - \left(\left(\sum_{D \cap C'_{\ell,m} = \emptyset} \mu(D) \right) + \left(\sum_{D \subset C'_{\ell,m}} \mu(D) \right) \right). \end{aligned}$$

This agrees with the former, since by the assumption of Step 2 each of the small shells D is either contained inside $C'_{\ell,m}$ or is disjoint from it, and so

$$\sum_{D \subset C_{\min}} \mu(D) = \left(\sum_{D \cap C'_{\ell,m} = \emptyset} \mu(D) \right) + \left(\sum_{D \subset C'_{\ell,m}} \mu(D) \right).$$

In the first case of Step 3, we add the single component $C'_{\ell,m} \setminus \bigcup_n D'_{\ell,n} = \emptyset$. In other words, we have to show that μ is well defined for a distinguished set D which is a disjoint union $D = \bigcup_i D_i$ of finitely many distinguished sets. However, in the Example following Theorem 2.5.4 we saw that we may start with D to obtain a chain of refinements $S_0 = D$, $S_1 = (D \setminus D_1) \cup D_1$, $\dots$, $S_n = (D \setminus \bigcup_i D_i) \cup \bigcup_i D_i$. Moreover, this chain of refinements is constructed entirely using Step 2 of the algorithm, for which we have already proved that the value of μ is preserved. We thus have $\mu(D) = \mu(S_0) = \mu(S_n) = \mu(D \setminus \bigcup_i D_i) + \sum_i \mu(D_i)$ as required.

In the second case of Step 3, we instead add one of the the single components

$$\left(C'_{\ell,m} \setminus \bigcup_x C_x \right) = \emptyset, \quad \left(C'_{\ell,m} \setminus \bigcup_x C_x \cup \bigcup_y D_y \right) = \emptyset,$$

which is again the case of a distinguished union of disjoint distinguished sets. The argument of the previous paragraph then applies.

Finally, in Step 4, $C_{\min} \setminus \bigcup_{D \subset C_{\min}} D$ is replaced with

$$\left(C_{\min} \setminus \left(D'_{\ell,n} \cup \bigcup_{D \cap D'_{\ell,n} = \emptyset} D \right) \right) \cup \left(D'_{\ell,n} \setminus \bigcup_{D \subset D'_{\ell,n}} D \right).$$

The former has measure $\mu(C_{\min}) - \sum_{D \subset C_{\min}} \mu(D)$. The latter has measure

$$\begin{aligned} & \left(\mu(C_{\min}) - \left(\mu(D'_{\ell,n}) + \sum_{D \cap D'_{\ell,n} = \emptyset} \mu(D) \right) \right) + \left(\mu(D'_{\ell,n}) - \sum_{D \subset D'_{\ell,n}} \mu(D) \right) \\ &= \mu(C_{\min}) - \left(\left(\sum_{D \cap D'_{\ell,n} = \emptyset} \mu(D) \right) + \left(\sum_{D \subset D'_{\ell,n}} \mu(D) \right) \right). \end{aligned}$$

This is equal to the former since if any D contains $D'_{\ell,n}$ this would contradict the minimality of $C_{\min}$, and so we have

$$\sum_{D \subset C_{\min}} \mu(D) = \left(\left(\sum_{D \cap D'_{\ell,n} = \emptyset} \mu(D) \right) + \left(\sum_{D \subset D'_{\ell,n}} \mu(D) \right) \right).$$

□

Theorem 2.6.3. *The map $\mu : \mathcal{R} \rightarrow \mathbb{R}((X))$ given by*

$$\mu(gK_{i,j}) = \frac{q^3}{(q^2 - 1)(q - 1)} q^{-4i} X^{4j}$$

on distinguished sets and extended to $\mathcal{R}$ by finite additivity is a well-defined, finitely additive, left invariant measure satisfying $\mu(K) = 1$.

Proof. Translation invariance on the left and finite additivity follow immediately from the definition. Proposition 2.6.2 gives that $\mu(A) = \mu(\tilde{A})$ for any refinement $\tilde{A}$ of A , and Theorem 2.5.4 says that for any pair $A, A' \in \mathcal{R}$ with $A = A'$ as sets there is some $\tilde{A}$ which is a refinement of both, hence we have $\mu(A) = \mu(\tilde{A}) = \mu(A')$ - in other words the extension of μ to $\mathcal{R}$ is well-defined. Finally, Proposition 2.6.1 gives the required volume of K . $\square$

Remark. Let $\bar{\mu}$ be the unique Haar measure on $\mathrm{GL}_2(\bar{F})$ satisfying $\bar{\mu}(\mathcal{O}_{\bar{F}}) = 1$, and let $p : \mathrm{GL}_2(\mathcal{O}_F) \rightarrow \mathrm{GL}_2(\bar{F})$ be the projection induced by the residue map. Let $K_n = I_2 + \bar{t}_1^n M_2(\mathcal{O}_{\bar{F}})$ as in Section 2.1. Then $\mu(I_2 + t_2^j p^{-1}(K_i - I_2)) = X^{4j} \bar{\mu}(K_i)$. In this way we may view μ as a lift of the measure $\bar{\mu}$ on $\mathrm{GL}_2(\bar{F})$.

Remark. In [KLo4] and [Lee10], Kim and Lee use combinatorial methods to study Hecke Algebras over groups closely related to $\mathrm{GL}_2(F)$ in the absence of an invariant measure. In an unreleased manuscript [KL], they construct a full σ -algebra of subsets of $\mathrm{GL}_2(F)$ on which they may define a measure, rather than just a ring of subsets. The measure μ_{KL} they construct may also be seen to take values in $\mathbb{R}((X))$, but in fact their measure is confined to the smaller space of monomials.

The measure μ_{KL} may be recovered by "taking the dominant term" of the measure μ we have defined in this chapter, and so we may in fact interpret the two as being "infinitesimally close". The "truncated" convolution product defined in [KLo4] and [Lee10] can be expressed in terms of an integral against μ_{KH} , and so it would be interesting to consider whether one can use the integral against μ defined in the following section to study "non-truncated" convolution product. See also the examples in the following section.

2.7 INTEGRATION ON $\mathrm{GL}_2(F)$

In classical analysis, the existence of an invariant measure on a locally compact space X is synonymous with the existence of an invariant integral, i.e. a linear functional on the space of continuous, compactly supported, complex-valued functions on X . (This is essentially the statement of the Riesz Representation Theorem, see [Rud87].)

In this section, we will consider a nice class of functions on $\mathrm{GL}_2(F)$ for which we can construct an integral against the measure defined in the previous section. These functions will be the analogue of the functions at the residue level which are locally constant and compactly supported. They are also analogous to the integrable functions against Fesenko's measure on F discussed in Section 2.3.

Definition 2.7.1. Let $f : \mathrm{GL}_2(F) \rightarrow \mathbb{C}((X))$ be a function of the form

$$f = \sum_{i=1}^n c_i \mathbb{I}_{U_i}$$

where each $c_i \in \mathbb{C}((X))$ and $\mathbb{I}_{U_i}$ is the indicator function of a dd -set U_i . Suppose that the U_i are pairwise disjoint, and let μ be the measure on $\mathrm{GL}_2(F)$ satisfying $\mu(K) = 1$ as in the previous section. We define the integral of f against μ to be

$$\int_{\mathrm{GL}_2(F)} f(g) d\mu(g) = \sum_{i=1}^n c_i \mu(U_i).$$

We also define the integral of a function which is zero outside finitely many points to be 0.

Remark. As in Section 2.3, if f is an integrable function on $\mathrm{GL}_2(F)$ and E is a measurable subset of $\mathrm{GL}_2(F)$, the function $f(g)\mathbb{I}_E(g)$ is also integrable, and we define

$$\int_E f(g) d\mu(g) = \int_{\mathrm{GL}_2(F)} f(g)\mathbb{I}_E(g) d\mu(g).$$

Proposition 2.7.2. Let R_G be the vector space generated by the simple functions $f = \sum_{i=1}^n c_i \mathbb{I}_{U_i}$ and the functions which are zero outside of a finite set. Then the

map $f(g) \mapsto \int_{GL_2(F)} f(g) d\mu(g)$ is a well-defined, left-invariant linear functional $R_G \rightarrow \mathbb{C}((X))$.

Proof. Suppose that

$$f = \sum_i c_i \mathbb{I}_{U_i} = \sum_j d_j \mathbb{I}_{V_j}$$

are two different ways of expressing f as a simple function. We need to show that $\sum_i c_i \mu(U_i) = \sum_j d_j \mu(V_j)$.

By finite additivity of the measure and the property that $\mathbb{I}_{A \cup B} = \mathbb{I}_A + \mathbb{I}_B$ for disjoint sets A and B , we may assume that each U_i and each V_j is a dd-set of the form $A \setminus \bigcup_r B_r$ with A and all B_r distinguished sets.

Furthermore, we may arrange that each U_i is in fact a distinguished set as follows. Suppose $U_i = A_i \setminus \bigcup_r B_{i,r}$. For a given r , let $W_{i,r} = (B_{i,r} \setminus \bigcup_k U_k)$ be the part of $B_{i,r}$ which is not contained in any U_k . Since $W_{i,r}$ is disjoint from all U_k and V_j , adding $c_i \mathbb{I}_{W_{i,r}}$ to both expressions for f changes both integrals by the same value. Doing this for all r and for all i then leaves the left hand expression for f (after simplification) in the form $\sum_i c_i \mathbb{I}_{U'_i}$ with U'_i a distinguished set.

For the next reduction, Lemma 2.4.3 implies that either U_1 is the union $\bigcup_{j \in J} V_j$ of some collection of the V_j , or the union $\bigcup_{i \in I} U_i$ of U_1 with some other of the U_i is equal to one of the V_j (which we may say is V_1 after relabelling). In the first case we have $f = c_1 \mathbb{I}_{U_1} + \sum_{i>1} c_i \mathbb{I}_{U_i} = \sum_{j \in J} d_j \mathbb{I}_{V_j} + \sum_{j \notin J} d_j \mathbb{I}_{V_j}$, and in the second case we have instead $f = \sum_{i \in I} c_i \mathbb{I}_{U_i} + \sum_{i \notin I} c_i \mathbb{I}_{U_i} = d_1 \mathbb{I}_{V_1} + \sum_{j>1} d_j \mathbb{I}_{V_j}$.

By induction on the lengths of the sums (since the result is clear when there is only one set on each side), it then suffices to prove the result for expressions of the form $f^* = c_1 \mathbb{I}_{U_1} = \sum_j d_j \mathbb{I}_{V_j}$ and $f^\dagger = \sum_i c_i \mathbb{I}_{U_i} = d_1 \mathbb{I}_{V_1}$. Moreover, since V_1 is of the form $A \setminus \bigcup_\ell B_\ell$, and each B_ℓ is necessarily disjoint from all of the U_i , adding $\sum_\ell d_1 \mathbb{I}_{B_\ell}$ to both sides of $f^\dagger$ leaves us in the f^* case.

After dividing by the constant, it remains to prove that if $\mathbb{I}_U = \sum d_j \mathbb{I}_{V_j}$ with U distinguished and V_j disjoint dd-sets with $U = \bigcup_j V_j$ then we have

$\mu(U) = \sum_j d_j \mu(V_j)$. To do so, we first note that since the V_j are disjoint and sum to U we must have $d_j = 1$ for all j , and the result then follows from finite additivity of the measure.

It remains to show that the integral is a left-invariant linear map. But if f and g can be written as sums of indicator functions then so can $f + g$, so additivity of the integral follows from the definition. Similarly, if $f = \sum c_i \mathbb{I}_{U_i}$ and $c \in \mathbb{C}((X))$ then $cf = \sum c \cdot c_i \mathbb{I}_{U_i}$ and $\int cf \, d\mu = \sum c \cdot c_i \mu(U_i) = c \sum c_i \mu(U_i) = c \int f \, d\mu$. Left-invariance follows immediately from the corresponding property of the measure.

□

Example. It is expected that the convolution product of two integrable functions will be important, and so as a first example we compute the convolution product of the characteristic function of K with itself.

Let $f(x) = \mathbb{I}_K(x)$. Then $(f * f)(x) = \int_G f(y)f(y^{-1}x) \, d\mu(y)$. The integrand is nonzero (and hence equal to 1) if and only if both $y \in K$ and $y^{-1}x \in K$. The second condition can be rewritten as $x \in yK$, and by the first condition we have $yK = K$, and so the integrand is nonzero if and only if both $y \in K$ and $x \in K$. Thus we have $(f * f)(x) = \int_G f(x)f(y) \, d\mu(y) = f(x)\mu(K) = f(x)$.

We will now work towards showing that the integral can be factored into a multiple integral over F , similarly to the one-dimensional result in Theorem 2.1.3. Since we already know significantly more about integration over F due to [Fes03], [Fes05], this allows us to deduce some important properties of the integral over $\mathrm{GL}_2(F)$.

We begin by showing that, for indicator functions of the $K_{i,j}$, we may take integrals over F in matrix coordinates.

Lemma 2.7.3. *Let $f = \mathbb{I}_{K_{i,j}}$, and let $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}_2(F)$. Then $f\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right)$ may be considered a function of 4 variables $\alpha, \beta, \gamma, \delta$ on F . This function is integrable on F with respect to any one of the variables (treating all of the others as constants). Moreover, the resulting functions remain integrable with respect to each of the three remaining variables.*

Proof. By the definitions of f and $K_{i,j}$, $f(g) = 1$ if and only if the following conditions are satisfied simultaneously: (1) $\alpha \in 1 + t_1^i t_2^j O_F$, (2) $\beta \in t_1^i t_2^j O_F$, (3) $\gamma \in t_1^i t_2^j O_F$, (4) $\delta \in 1 + t_1^i t_2^j O_F$; otherwise $f(g) = 0$. We can thus express $f(g)$ as a product of functions

$$f(g) = \mathbb{I}_{1+t_1^i t_2^j O_F}(\alpha) \mathbb{I}_{t_1^i t_2^j O_F}(\beta) \mathbb{I}_{t_1^i t_2^j O_F}(\gamma) \mathbb{I}_{1+t_1^i t_2^j O_F}(\delta)$$

with $\alpha, \beta, \gamma, \delta$ appearing independently of each other. If we treat any three of the variables as a constant, we obtain a multiple of an indicator function of a measurable subset of F (which is an integrable function on F) in the remaining variable. Integrating over this variable leaves a product of integrable functions of a similar form, but of shorter length, and so by induction one sees that we may successively integrate over each variable. $\square$

If we want to integrate successively over the entries, it is possible that the result may depend on the order that we integrate out the variables. Indeed, in [Mora] Morrow shows that Fubini's Theorem may fail in the higher dimensional case under a nonlinear change of variables. Since we will later want to make certain changes of variables which may permute the order of the differentials, it is thus important that we check that the multiple integrals in question do not depend on the order of integration.

Lemma 2.7.4. *Let $f = \mathbb{I}_{K_{i,j}}$, and let $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}_2(F)$. The multiple integral*

$$\int_F \int_F \int_F \int_F f\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right) d\alpha d\beta d\gamma d\delta,$$

is independent of the order of integration. Here $d\alpha = d\mu_F(\alpha)$, $d\beta = d\mu_F(\beta)$, and so on, where μ_F denotes the measure on F defined in Section 2.2.

Proof. It is enough to show that

$$\int_F \int_F \lambda \mathbb{I}_A(x) \mathbb{I}_B(y) dx dy$$

is independent of the order of integration for A, B measurable subsets of F and $\lambda \in \mathbb{C}$, since the desired result follows from this by induction. We have

$$\begin{aligned}
 \int_F \left(\int_F \lambda \mathbb{I}_A(x) \mathbb{I}_B(y) dx \right) dy &= \int_F \lambda \mathbb{I}_B(y) \left(\int_F \mathbb{I}_A(x) dx \right) dy \\
 &= \int_F \lambda \mu_F(A) \mathbb{I}_B(y) dy \\
 &= \lambda \mu_F(A) \mu_F(B) \\
 &= \int_F \lambda \mu_F(B) \mathbb{I}_A(x) dx \\
 &= \int_F \lambda \mathbb{I}_A(x) \left(\int_F \mathbb{I}_B(y) dy \right) dx \\
 &= \int_F \left(\int_F \lambda \mathbb{I}_A(x) \mathbb{I}_B(y) dy \right) dx.
 \end{aligned}$$

□

Remark. More generally, for a function $h(x, y)$ on $F \times F$ which is integrable with respect to each of x and y and can be written as a product $h(x, y) = h_1(x)h_2(y)$, the integral $\int_F \int_F h(x, y) dx dy$ does not depend on the order of integration. As noted above, in classical analysis one has Fubini's Theorem, which ensures that *any* integrable function $h(x, y)$ satisfies the same property, but this is not necessarily true in the higher dimensional case. For the remainder of this text, however, we will only be working with functions where one can "separate the variables", and so the above Lemma will be good enough here.

We now establish the main result of this section. Recall that for $\varepsilon \in O_F^\times$ we have $|t_1^i t_2^j \varepsilon|_F = q^{-i} X^j$.

Theorem 2.7.5. *Let $f \in R_G$, and let $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}_2(F)$. Then*

$$\begin{aligned}
 \int_{\mathrm{GL}_2(F)} f(g) d\mu(g) &= \\
 &= \frac{q^3}{(q^2 - 1)(q - 1)} \int_F \int_F \int_F \int_F \frac{1}{|\alpha\delta - \beta\gamma|_F^2} f\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right) d\alpha d\beta d\gamma d\delta.
 \end{aligned}$$

Proof. First we check that the integrals agree for $f = \mathbb{I}_{K_{i,j}}$. In this case, the left hand integral is equal to the volume of $K_{i,j}$. By our choice of normalisation

$\mu(K) = 1$, we have $\mu(K_{i,j}) = \frac{q^3}{(q^2-1)(q-1)} q^{-4i} X^{4j}$, and so we must check that this agrees with the right hand expression.

By the definition of $K_{i,j}$, we have

$$f(g) = \mathbb{I}_{1+t_1^i t_2^j O_F}(\alpha) \cdot \mathbb{I}_{t_1^i t_2^j O_F}(\beta) \cdot \mathbb{I}_{t_1^i t_2^j O_F}(\gamma) \cdot \mathbb{I}_{1+t_1^i t_2^j O_F}(\delta).$$

Furthermore, for $g \in K_{i,j}$, $\det g = \alpha\delta - \beta\gamma \in 1 + t_1^i t_2^j O_F$, hence $|\det g|_F^2 = 1$. We can now directly compute the integrals on the right hand side to obtain $\mu_F(1 + t_1^i t_2^j O_F)^2 \mu_F(t_1^i t_2^j O_F)^2 = q^{-4i} X^{4j}$. Multiplying by the factor $\frac{q^3}{(q^2-1)(q-1)}$ then yields the same value as the left hand side.

Since both integrals are linear, to complete the proof of the Theorem we must show that the integrals agree for $f = \mathbb{I}_{hK_{i,j}}$ with $h \in GL_2(F)$. Since the integral on the left is left-invariant, this is equivalent to proving that the integral on the right is left-invariant.

To show this, it is enough to consider translation by elementary matrices, since these generate $GL_2(F)$. First suppose $h = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$ with $x \in F^\times$. Then $hg = \begin{pmatrix} x\alpha & x\beta \\ \gamma & \delta \end{pmatrix}$, $|\det(hg)|_F^2 = |x|_F^2 |\det g|_F^2$, and under the change of variables $g \mapsto hg$ the differentials become $d\alpha \mapsto d(x\alpha) = |x|_F d\alpha$, $d\beta \mapsto d(x\beta) = |x|_F d\beta$, $d\gamma \mapsto d\gamma$, $d\delta \mapsto d\delta$. The factors of $|x|_F$ then cancel with those appearing in $|\det(hg)|_F^2$, leaving the integral invariant under this change of variables.

The case $h = \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}$ works almost identically, except the factors of $|x|_F$ come instead from $d\gamma$ and $d\delta$. For $h = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$ with $u \in F$, $\det h = 1$, and the change of variables $g \mapsto hg$ results in an additive shift inside the differentials. Since the measure on F is invariant under such transformations, the integral remains invariant. Exactly the same is true when we take h to be a lower-triangular unipotent matrix.

The final case we must check is $h = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, which has determinant -1 (and $|-1|_F = 1$) and simply permutes the differentials when we make the change $g \mapsto hg$, which leaves the integral invariant by Lemma 2.7.4. $\square$

While this Theorem is important on its own, since it gives a way of computing certain integrals much more easily by reducing to integrals over F , it also has several important Corollaries. First of all, we note in particular that the choice of X^4 as an indeterminate for the measure on $GL_2(F)$ is exactly the correct choice to ensure compatibility with Fesenko's original measure on F .

Corollary 2.7.6. *For $(i, j) > (0, 0)$, the unique $\mathbb{R}((X))$ -valued measure on $GL_2(F)$ which is compatible with the measure on F (in the sense of the above Theorem) satisfies*

$$\mu(K_{i,j}) = \frac{q^3}{(q^2 - 1)(q - 1)} q^{-4i} X^{4j}.$$

Next, while our definition of the measure on $GL_2(F)$ only ensures left-invariance, we note that the proof of Theorem 2.7.5 also implies invariance under translation on the right.

Corollary 2.7.7. *This integral, and hence the measure defined in the previous section, is both left and right translation-invariant.*

Proof. By Theorem 2.7.5, it suffices to check invariance of the integral on the right hand side. However, this can be done in exactly the same way that we checked left-invariance in the above proof. $\square$

Recall that a series absolutely converges in $\mathbb{R}((X))$ if it converges and in addition the coefficient of each X^j is an absolutely convergent series in $\mathbb{R}$.

Corollary 2.7.8. *The measure on $GL_2(F)$ is countably additive in the following refined sense. If $U = \bigcup_n U_n \in \mathcal{R}$ such that $\{U_n\}$ are countably many disjoint measurable sets and $\sum_n \mu(U_n)$ is an absolutely convergent series in $\mathbb{R}((X))$ then $\mu(\bigcup_n U_n) = \sum_n \mu(U_n)$.*

Proof. For the measure μ_F on F this is in [Fes05], and Theorem 2.7.5 allows us to reduce to this case as follows. First of all we may assume each $U_n = C_n \setminus \bigcup_m D_{m,n}$ with $C_n, D_{m,n}$ finitely many distinguished sets by finite additivity.

By the definition of the integral we have

$$\mu \left(C_n \setminus \bigcup_m D_{m,n} \right) = \int_{GL_2(F)} \mathbb{I}_{\bigcup_n (C_n \setminus \bigcup_m D_{m,n})}(g) dg,$$

and since the union is disjoint the integrand is equal to

$$\sum_n \mathbb{I}_{C_n \setminus \bigcup_m D_{m,n}}(g).$$

By applying Theorem 2.7.5 in the case $f(g) = \sum_n \mathbb{I}_{C_n \setminus \bigcup_m D_{m,n}}(g)$, we obtain the integral

$$\frac{q^3}{(q^2 - 1)(q - 1)} \int_F \int_F \int_F \int_F \frac{1}{|\alpha\delta - \beta\gamma|_F^2} \sum_n \mathbb{I}_{C_n \setminus \bigcup_m D_{m,n}} \left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \right) d\alpha d\beta d\gamma d\delta. \quad (1)$$

Since now we have a (multiple) integral over F , we can take the sum outside the integral to obtain

$$\sum_n \frac{q^3}{(q^2 - 1)(q - 1)} \int_F \int_F \int_F \int_F \frac{1}{|\alpha\delta - \beta\gamma|_F^2} \mathbb{I}_{C_n \setminus \bigcup_m D_{m,n}} \left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \right) d\alpha d\beta d\gamma d\delta, \quad (2)$$

which is equal to the previous integral if this sum absolutely converges.

However, we can apply Theorem 2.7.5 to each of the functions

$$\mathbb{I}_{C_n \setminus \bigcup_m D_{m,n}}(g)$$

to rewrite this as

$$\sum_n \int_{GL_2(F)} \mathbb{I}_{C_n \setminus \bigcup_m D_{m,n}}(g) dg,$$

which equals

$$\sum_n \mu \left(C_n \setminus \bigcup_m D_{m,n} \right)$$

by the definition of the integral, and this absolutely converges by assumption. This shows that the two integrals (1) and (2) are equal, and hence we have

$$\mu \left(\bigcup_n \left(C_n \setminus \bigcup_m D_{m,n} \right) \right) = \sum_n \mu \left(C_n \setminus \bigcup_m D_{m,n} \right).$$

□

Remark. We may in fact use the above Corollary to extend the class of measurable sets to all non-ddd-sets which are countable disjoint unions of dd-sets such that the relevant series converges absolutely.

We end this section with the computation of certain "reasonable" integrals (i.e. not simply characteristic functions of ddd-sets) which do not take values $\mathbb{C}$.

Example. Let $D = \{g \in M_2(O_F) : \det g \neq 0\}$ be the "punctured unit disc" in $\mathrm{GL}_2(F)$. For a positive integer s , we would like to compute the integral

$$I = \int_D |\det g|_F^s dg := \int_{\mathrm{GL}_2(F)} |\det g|_F^s \mathbb{I}_D(g) dg.$$

However, this integral does not converge.

Indeed, we may first write D as the disjoint union over all $(i, j) \geq (0, 0)$ of the "circles" $D_{i,j} = \{g \in M_2(O_F) : \det g \in t_1^i t_2^j O_F^\times\}$, on which the function $|\det g|_F^s$ takes the constant value $q^{-is} X^{js}$. We may realise the $D_{i,j}$ as cosets of $K = \mathrm{GL}_2(O_F)$ in D by observing that K is the kernel of the map $D \rightarrow \mathbb{C}((X))$ sending a matrix g to $|\det g|_F$, and so in particular $\mu(D_{i,j}) = 1$ for all i and j . We thus have

$$\int_D |\det g|_F^s dg = \sum_{(i,j) \geq (0,0)} q^{-is} X^{js} = \sum_{i=0}^{\infty} q^{-is} + \sum_{j=1}^{\infty} \left(\sum_{i=-\infty}^{\infty} q^{-is} \right) X^{js},$$

and while the first term converges for $s > 0$, the inner sum in the second term does not converge for any value of s .

However, we may try to approximate this integral by integrating over larger and larger "quarter planes". More precisely, for a pair of integers m and n , define $Q_{m,n}$ to be the disjoint union of all $D_{i,j}$ over $i \geq m$ and $j \geq n$ such that $(i, j) \geq (0, 0)$. Clearly we have $D = \bigcup_{m \in \mathbb{Z}} Q_{m,0}$, or in other words if we integrate over $Q_{m,0}$ for large negative m we should approach the integral I . Indeed, if $m < 0$ then

$$I_m = \int_{Q_{m,0}} |\det g|_F^s dg = \sum_{i=0}^{\infty} q^{-is} + \sum_{j=1}^{\infty} \sum_{i=m}^{\infty} q^{-is} X^{js},$$

and now each term converges for $s > 0$, giving

$$I_m = \frac{1}{1 - q^{-s}} + \sum_{j=1}^{\infty} \frac{q^{-ms}}{1 - q^{-s}} X^{js}.$$

Remark. From the point of view of the "identity" $\sum_{n \in \mathbb{Z}} q^n = 0$ discussed in [Fes03], the integral I above "agrees" with the classical determinant integral in the one-dimensional case, which has value $(1 - q^{-s})^{-1}$. Alternatively, if one considers X as an infinitesimal element, taking the limit as $m \rightarrow -\infty$ of the "dominant term" of I_m also gives $(1 - q^{-s})^{-1}$ (recall from the previous section that the measure of Kim and Lee in [KL] may also be seen as taking the "dominant term" of our measure). Note that in the one-dimensional case s may be any complex number with $\text{Re}(s) > 0$, whereas we must restrict to positive integers due to the appearance of the indeterminate X .

We conclude with an example of an integral which gives a power series with nonconstant coefficients.

Example. Let $D_{i,j}$ be as in the previous example, and let T be the disjoint union $T = \bigcup_{i \geq j \geq 0} D_{i,j}$. For an integer $s > 0$ we have

$$\int_T |\det g|_F^s = \sum_{j=0}^{\infty} \sum_{i=j}^{\infty} q^{-is} X^{js} = \sum_{j=0}^{\infty} \frac{q^{-js}}{1 - q^{-s}} X^{js}.$$

2.8 ASIDE: COMPARING WITH MORROW'S INTEGRAL

The formula in Theorem 2.7.5 agrees (up to a constant multiple) with the integral defined by Morrow in [Moro8]. In this short section we will briefly compare the two different approaches. First of all, we recall Morrow's method of integration by lifting.

Definition 2.8.1. Let $g : \bar{F} \rightarrow \mathbb{C}$ be a complex-valued function on the residue field of F , let $a \in F$, and let $n \in \mathbb{Z}$. The lift of g at (a, n) is the function

$$g_{a,n}(x) = \begin{cases} g\left(\overline{(x-a)t_2^{-n}}\right) & \text{if } x \in a + t_2^n \mathcal{O}_F \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.8.2. Let $\mathcal{L}(F)$ be the $\mathbb{C}((X))$ -vector space spanned by functions of the form $x \mapsto g_{a,n}(x)X^r$ for g a Haar-integrable function on $\bar{F}$, $a \in F$, and $n, r \in \mathbb{Z}$. Elements of $\mathcal{L}(F)$ are called integrable functions on F .

Theorem 2.8.3. *There exists a unique linear functional $\int_F(\cdot)dx$ on $\mathcal{L}(F)$ which is invariant under the action of F by additive translations and satisfies*

$$\int_F g_{a,n}(x)dx = X^n \int_{\overline{F}} g(u)du$$

for any Haar integrable function g on F , where the latter denotes the Haar integral on $\overline{F}$

Proof. See [Mor10]. □

This integral was then extended by Morrow to compute an integral over a finite dimensional F -vector space V . Since he had shown in [Mora] that Fubini's Theorem may fail for his integral, he considers the integral of Fubini functions. In other words, he considers only functions f on V for which the iterated integral

$$\int_F \dots \int_F f(x_1, \dots, x_n) dx_1 \dots dx_n$$

(where n is the F -dimension of V and we have for simplicity fixed an isomorphism $V \simeq F^n$) does not depend on the order of integration. Morrow then defines the value of

$$\int_V f(x)dx$$

to be the common value of these iterated integrals.

In particular, the space $M_n(F)$ of $n \times n$ matrices with entries in F may be realised in a natural way as an F -vector space of dimension n^2 , and so Morrow's integral naturally gives us a way of integrating on $M_n(F)$.

To define an integral on $\mathrm{GL}_n(F)$, Morrow considers functions ϕ on $\mathrm{GL}_n(F)$ such that $\tau \mapsto \phi(\tau) |\det \tau|_F^{-n}$ extends to a Fubini function on $M_n(F)$. He then defines the integral of ϕ via

$$\int_{\mathrm{GL}_n(F)} \phi(\tau) d\tau = \int_{M_n(F)} \phi(x) |\det x|^{-n} dx.$$

In the case $n = 2$, upon fixing an isomorphism $M_2(F) \simeq F^4$ the right hand integral can be written in the more familiar form

$$\int_F \int_F \int_F \int_F \phi(x) |\det x|^{-2} dx_1 dx_2 dx_3 dx_4,$$

which we now recognise from the right hand side of Theorem 2.7.5. In fact, this Theorem says exactly that our integral agrees (up to a constant multiple) with the one constructed by Morrow.

The advantage of Morrow's approach is that the integrable functions are related in a direct way to functions on the residue field. This is useful (especially if one considers higher dimensional local fields) if we want to make comparisons between residue levels. This theory is very powerful in the sense that one can deduce many things regarding integrals by appealing to corresponding properties at the residue level which are already known, and it also applies to more general objects than higher local fields. (See [Mor10] for examples of this, including applications to zeta integrals.)

On the other hand, it is too much to hope that *all* of the information about integrals on higher dimensional objects is already contained in the residue field. Indeed, if this were the case then it would not be of much interest to study higher dimensional fields at all. For this reason, it is also useful to have an approach like ours which works directly with the higher dimensional objects, but still satisfies the relevant compatibility conditions with residue structures. This allows us to draw information from the residue level when such information is available, but does not restrict us so severely when it is not available.

An example of where this is useful is that we do not need to write our functions in matrix coordinates in order to integrate them. This leaves our theory more open to modification to work for algebraic groups more general than $GL_n(F)$, and perhaps even more general spaces (as will be discussed in Chapter 3), whereas Morrow notes that his approach does not easily generalise to this situation.

Another of the main advantages our approach has over Morrow's is that the integrals on the right hand side of Theorem 2.7.5 are against the measure defined by Fesenko in [Fes03], [Fes05], and so we obtain Corollary 2.7.8 as an easy consequence of one of his results, while Morrow does not discuss such "countable additivity" results using his integral.

There is, of course, still significant overlap between Morrow's approach and the approach presented here. In particular, the functions $\mathbb{I}_{K_{i,j}}$ are integrable in both cases. This simple observation allows us to transfer some rather interesting consequences of Morrow's work into our setting.

For example, using Proposition 7.1 of [Mora] and induction, one may construct a function on $\mathrm{GL}_2(F)$ which is integrable in the sense of this text such that the multiple integral in matrix coordinates (see Lemma 2.7.4 above) does not satisfy the Fubini property. All we require to do this is a function of several variables which is integrable in both senses as a starting point, and for this we can just take $\mathbb{I}_{K_{i,j}}$ in matrix coordinates.

THE LEVEL TOPOLOGY

In this chapter we consider topological spaces whose topology may be reconstructed in a particular way from a locally compact group. This construction leads to a very natural generalisation of compactness, and in particular allows us to apply certain compactness arguments to groups related to higher dimensional local fields.

The main focus of this chapter is thus the definition and properties of groups with a level structure. The main motivation for the definition comes from the study of two-dimensional local fields as follows. Consider $F = \mathbb{Q}_p((t))$, and its rank two ring of integers $O_F = \mathbb{Z}_p + t\mathbb{Q}_p[[t]]$. One can define the level of a subset $S \subset F$ as the least integer j such that S contains a translate of a fractional ideal $p^i t^j O_F$ for some $i \in \mathbb{Z}$. The remarkable observation is then that, although O_F is not compact, if one looks only at its open covers (in a particular topology) by sets of the same level (level 0), all of them have finite subcovers. We thus have a weaker substitute which works "on the level".

Since this chapter is dedicated to the development of a new, more general theory of compactness, there will be a substantial number of definitions given in quick succession. We try to give as much motivation as possible to show that each definition is important and, where possible, we demonstrate the kind of pathological cases that may arise if one doesn't take care to make the required assumptions.

The reader is thus asked to persevere with the abnormal Definition-to-Theorem ratio, if only because of the possible wide-reaching applications. Indeed, unlike the other parts of this text, the notion of level-compactness is not at all limited to problems related to higher-dimensional number theory, and can almost certainly be studied in a variety of other contexts.

We begin our journey in 3.1 with the definition of a level structure on a group G . The remainder of the chapter will be devoted to the study of such groups, and so it is paramount that the reader familiarises themselves with this definition as thoroughly as possible. An important example to keep in mind is that of a higher dimensional local field as discussed in the previous chapter.

Section 3.1 then contains several other important definitions. In addition to the general notion of a ddd-set, we also define the notions of level and uniform level. We continue into 3.2 with the notion of rigidity, and after several elementary results concerning levels we arrive at the definition of level-compactness. This is again a definition which the reader should take due time to become familiar with, as it is not only the main focus of the following sections, but possibly the most far-reaching idea in the entire text.

Following this, we work through several properties of level-compactness in 3.3, including (though not limited to) many elementary topological properties which may be reformulated in this context. If nothing else, this section should allow the reader to become accustomed to working with all of the new definitions.

In Section 3.4 we then briefly narrow our scope, and consider a specific family of level structures which are analogous to two-dimensional nonarchimedean local fields. This continues into 3.5, where we generalise the constructions of Chapter 2 to define an invariant measure in the more general setting of groups of ordered type.

We then return to the more general case in 3.6 where we discuss how to pass a level structure from a group G to its subgroup, and also how to (trivially) inflate the level structure. We conclude in 3.7 with a few notes on

how to generalise some of the results in this chapter, and some special cases which may be worth investigating in more detail.

Due to the huge scope for development of the theory introduced in this chapter, many potentially interesting questions will arise. We endeavour to highlight these where possible.

Here, and henceforth, we write $\mathbb{Z}^n$ for the n -fold direct sum $\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$, and our convention will be that $\mathbb{Z}^0 = \{0\}$.

3.1 LEVEL STRUCTURES OVER LOCALLY COMPACT GROUPS

We begin with what is possibly the most important definition of the entire text.

Definition 3.1.1. *Let X be a locally compact topological group, and let $e \geq 0$ be an integer. A group G is levelled over X (with elevation e) if there is a collection $\mathcal{L}$ of subsets of G satisfying the following conditions:*

- (1) *Each element of $\mathcal{L}$ contains the identity element e_G of G .*
 - (2) *$\mathcal{L}$ indexed by $\mathcal{U}(1) \times \mathbb{Z}^e$, where $\mathcal{U}(1)$ is a basis of neighbourhoods of the identity in X and $\mathbb{Z}^e$ is lexicographically ordered from the right.*
 - (3) *For any $U, V \in \mathcal{U}(1)$ with $V \subset U$, if $G_{V,\gamma}, G_{U,\delta} \in \mathcal{L}$ with $\gamma \leq \delta$ then $G_{V,\gamma} \cap G_{U,\delta} = G_{V,\delta}$.*
 - (4) *For any fixed $\gamma \in \mathbb{Z}^e$, $G_{U,\gamma} \cup G_{V,\gamma} = G_{U \cup V,\gamma}$ and $G_{U,\gamma} \cap G_{V,\gamma} = G_{U \cap V,\gamma}$.*
- The collection $\mathcal{L}$ is called a level structure.*

Remark. If $U = V$, condition (3) simply says that $G_{U,\delta} \subset G_{U,\gamma}$ for $\gamma \leq \delta$. For further discussion on the generalities of this definition, see Section 3.7.

Examples. (1) Any locally compact group G is levelled over itself with elevation 0. In this case $\mathcal{L} = \mathcal{U}(1)$.

(2) An n -dimensional nonarchimedean local field F is levelled over the one-element group with elevation n , where $\mathcal{L}$ consists of all principal fractional ideals of the rank- n ring of integers O_F .

(3) Let F be an n -dimensional local field (which may now be archimedean) and F_1 is its $(n-1)^{st}$ residue field. Then F is levelled over F_1 with elevation $n-1$. If F and F_1 have the same characteristic, so that F is isomorphic to $F_1((t_2)) \dots ((t_n))$, $\mathcal{L}$ consists of sets of the form

$$t_2^{i_2} \dots t_n^{i_n} B(0, r) + \sum_{j=2}^n t_j^{i_j+1} t_{j+1}^{i_{j+1}} \dots t_n^{i_n} F_1((t_2)) \dots ((t_{j-1}))[[t_j]],$$

where $B(0, r)$ is the open ball of radius r in F_1 . In the mixed characteristic case, we associate to the pair $(t_1^{i_1} \mathcal{O}_{F_1}, (i_2, \dots, i_n)) \in \mathcal{U}(1) \times \mathbb{Z}^{n-1}$ the set $t_1^{i_1} \dots t_n^{i_n} \mathcal{O}_F \subset F$.

In the nonarchimedean case we may note that the neighbourhoods of the identity are themselves indexed by the totally ordered group $\mathbb{Z}$; doing so recovers the previous example for such fields.

(4) Since the form taken by elements of $\mathcal{L}$ in the previous example may look quite complicated, we give a concrete example in dimension 4 to illustrate the general phenomenon. Let $F = \mathbb{Q}_p((t_2))((t_3))((t_4))$, so that $F_1 = \mathbb{Q}_p$. The open balls in F_1 are then simply the fractional ideals $p^{i_1} \mathbb{Z}_p$ for $i_1 \in \mathbb{Z}$. Elements of $\mathcal{L}$ are thus of the form

$$p^{i_1} t_2^{i_2} t_3^{i_3} t_4^{i_4} \mathbb{Z}_p + t_2^{i_2+1} t_3^{i_3} t_4^{i_4} \mathbb{Q}_p[[t_2]] + t_3^{i_3+1} t_4^{i_4} \mathbb{Q}_p((t_2))[[t_3]] + t_4^{i_4+1} \mathbb{Q}_p((t_2))((t_3))[[t_4]].$$

Note that this is exactly the set $p^{i_1} t_2^{i_2} t_3^{i_3} t_4^{i_4} \mathcal{O}_F$.

(5) If F is an n -dimensional nonarchimedean local field and F_1 is its $(n-1)^{st}$ residue field as before, the association

$$t_1^{i_1} \mathcal{O}_{F_1} \mapsto K_{i_1, \dots, i_n} = I_2 + t_1^{i_1} \dots t_n^{i_n} M_m(\mathcal{O}_F)$$

for $(i_1, \dots, i_n) > 0$ defines a partial level structure for $\mathrm{GL}_m(F)$ over F_1 of elevation $n-1$ (in the sense that the indexing set is only a subset of $\mathcal{U}(1) \times \mathbb{Z}^{n-1}$ consisting of positive elements, but all other axioms are still satisfied). For $m = n = 2$, the elements of $\mathcal{L}$ are simply the subgroups $K_{i,j}$ of $\mathrm{GL}_2(F)$ studied in Chapter 2.

Remark. We will often refer to the level structures in the above examples as the "usual level structure" or "canonical level structure" when discussing any of these particular cases.

Lemma 3.1.2. *Let G be levelled over X with elevation e . The set $\mathcal{L}$ is closed under finite intersection.*

Proof. Let $G_{U,\gamma}, G_{V,\delta} \in \mathcal{L}$, and assume without loss of generality that $\gamma \leq \delta$. In this case, $G_{V,\gamma} \cap G_{V,\delta} = G_{V,\delta}$ by condition (3), hence $G_{U,\gamma} \cap G_{V,\delta} = G_{U,\gamma} \cap G_{V,\gamma} \cap G_{V,\delta} = G_{U \cap V, \gamma} \cap G_{V,\delta}$ by condition (4). But $U \cap V \in \mathcal{U}(1)$ and is a subset of V , hence by (3) we have $G_{U \cap V, \gamma} \cap G_{V,\delta} = G_{U \cap V, \delta} \in \mathcal{L}$. $\square$

By the above Lemma, the collection $\mathcal{L}$ is a basis of neighbourhoods of the identity for a topology on G .

Definition 3.1.3. *We equip G with the level topology as follows. We take $\mathcal{L}$ as a basis of neighbourhoods of the identity, and then extend to other points of G by insisting that multiplication by any fixed element be continuous.*

If G , as in the first example, is a locally compact group viewed as being levelled over itself with $e = 0$, this is just the original topology on G . On the other hand, if G is (the additive group of) a two-dimensional local field as in the third example, the level topology on G is not the usual two-dimensional topology as defined (for example) in [MZ95] - in this topology elements of $\mathcal{L}$ are closed but not open, for example.

Definition 3.1.4. *An element of $G\mathcal{L} = \{gH : g \in G, H \in \mathcal{L}\}$ is called a distinguished set. We also allow the empty set to be distinguished.*

Example. The distinguished subsets of a two-dimensional local field F as defined in Section 2.3 are exactly the distinguished sets of the elevation 1 level structure of F over its residue field. Similarly, the distinguished subsets of $\mathrm{GL}_2(F)$ as defined in 2.4 are exactly those appearing in the partial elevation 1 level structure of $\mathrm{GL}_2(F)$ over the residue field.

The following result shows that the level topology is equivalent to the topology generated by the distinguished sets.

Proposition 3.1.5. *$G\mathcal{L} \cup \{\emptyset\}$ is closed under finite intersection.*

Proof. We want to consider the intersection of $gG_{U,\gamma}$ and $hG_{V,\delta}$ with $g, h \in G$ and $G_{U,\gamma}, G_{V,\delta} \in \mathcal{L}$. If the intersection is empty we are done, so assume otherwise, so that there is an element $\alpha^{-1} \in G$ which is contained in the intersection. Translating by α then implies that e_G is contained in both $\alpha gG_{U,\gamma}$ and $\alpha hG_{V,\delta}$. By continuity, both of these are basic open neighbourhoods of e_G in the level topology, and hence by the above Lemma so is their intersection: $\alpha gG_{U,\gamma} \cap \alpha hG_{V,\delta} = G_{W,\beta}$. Translating back then gives $gG_{U,\gamma} \cap hG_{V,\delta} = \alpha^{-1}G_{W,\beta}$. $\square$

The following definition won't be of immediate interest to us, but will be useful to have in hand when we attempt to construct an invariant measure on a certain class of groups with level structure.

Definition 3.1.6. *The ring of ddd-sets of G with respect to the level structure $\mathcal{L}$ is the minimal ring of sets containing $G\mathcal{L}$.*

We now arrive at the second most important definition of this text.

Definition 3.1.7. *For $G_{U,\gamma} \in \mathcal{L}$, its level*

$$\text{lv}(G_{U,\gamma}) = \max\{\delta \in \mathbb{Z}^e : G_{U,\gamma} \subset G_{U,\delta}\}.$$

We then put $\text{lv}(gG_{U,\gamma}) = \text{lv}(G_{U,\gamma})$ for any $g \in G$. For a general subset $S \subset G$, the level of S (if it exists) is the minimal level of any subset of S of the form $gG_{U,\gamma}$ for $g \in G$, $U \in \mathcal{U}(1)$, $\gamma \in \mathbb{Z}^e$. We write $\text{lv}(S)$ for the level of S .

Remark. Note that we in fact have

$$\text{lv}(G_{U,\gamma}) = \max\{\delta \in \mathbb{Z}^e : G_{U,\gamma} \subset G_{U,\delta}\} = \max\{\delta \in \mathbb{Z}^e : G_{U,\gamma} = G_{U,\delta}\}.$$

One would like to simply define the level of $gG_{U,\gamma}$ to be γ , but since we have not ruled out the possibility that, say, $G_{U,\gamma} = G_{U,\delta}$ for $\gamma \neq \delta$, this would not be a consistent definition. The advantage of allowing such "degeneracy" is that one may define the induced level structure on a subgroup (which we will do in Section 3.6).

Clearly we have the equality

$$\text{lv}(S) = \min\{\text{lv}(S') : S' \subset S \text{ has a level}\}.$$

This leads to the following first observation concerning levels.

Lemma 3.1.8. *If $\text{lv}(A) < \text{lv}(B)$ for two subsets A and B of G then it cannot be the case that $A \subset B$. Moreover, there can be no $g \in G$ such that $A \subset gB$.*

Proof. If $\text{lv}(A) < \text{lv}(B)$ then there is $\gamma \in \mathbb{Z}^e$ such that A contains a distinguished set $gG_{U,\gamma}$ but there is no $\delta \leq \gamma$ with $hG_{V,\delta} \subset B$ for any choice of h and V . In particular, there is at least one element of $gG_{U,\gamma}$ which is contained in A but not in B . The second statement follows from the fact that level is invariant under the action of G by translation. $\square$

Remark. We may interpret level as being related to the size of a subset, with a higher level indicating a smaller size. Lemma 3.1.8 is in agreement with this interpretation.

Example. Let F be a two-dimensional nonarchimedean local field with local parameters t_1, t_2 , and let $\mathcal{O}_F$ and O_F be the rank-one and rank-two rings of integers of F . If $S \subset F$ is a finite set then S does not have a level. For $i, j \in \mathbb{Z}$ the set $t_1^i t_2^j \mathcal{O}_F$ is a distinguished set of level j , while the set $t_2^j \mathcal{O}_F$ is a non-distinguished set of level j .

One of the main purposes of this chapter is to introduce the notion of "level-compactness", for which the definition of level above will be very important. Using the intuitive interpretation of the level of a subset of G as indicating its relative size, level-compactness will be equivalent to saying that any open cover by "large enough" sets has a finite subcover (this will be made precise later, see Definition 3.2.5). However, our current definition of level is not quite strong enough for this, as the next example illustrates.

Example. Let F be a two-dimensional nonarchimedean local field with t_1, t_2 , O_F as in the previous example. For all $\alpha, \beta \in F$ the set $A_{\alpha,\beta} = (\alpha + t_2 O_F) \cup (\beta + O_F)$ has level 0. We have $O_F \subset \bigcup_{\alpha} A_{\alpha,\beta}$, where the union is over a

complete (and infinite) set of representatives of O_F/t_2O_F in O_F . Taking any $\beta \notin O_F$, we obtain in this way an open cover of O_F by sets of level 0 with no finite subcover.

The problem in the above example is that, although the open cover we construct is essentially a cover of sets of level 1 (since the $(\beta + O_F)$ components contribute nothing, being disjoint from O_F), the presence of this extra component formally lowers the level even though it does not contribute. This shows that in order to make reasonable progress towards any kind of "compactness", we must consider only those open covers by sets which are "uniformly large", which motivates the following definition.

Definition 3.1.9. *A subset $S \subset G$ has uniform level γ if $\text{lv}(S) = \gamma$ and for every point $s \in S$ there is a distinguished set D_s of level γ with $s \in D_s$ and $D_s \subset S$.*

It is immediate from the definition that any subset with uniform level is open. The sets $A_{\alpha,\beta} = (\alpha + t_2O_F) \cup (\beta + O_F)$ in the previous example are not uniformly of level 0, since α (for example) is not contained in any distinguished set of level 0 lying inside $A_{\alpha,\beta}$. This additional condition is enough to eliminate such pathological examples; we will see later (Proposition 3.2.6) that O_F is in fact compact with respect to open covers of uniform level 0.

3.2 RIGIDITY AND LEVEL-COMPACTNESS

In the lead up to the third and final "most important definition", we first examine a few more properties of the level of a subset. In particular, we would like this notion to be well behaved with respect to certain set theoretic operations. In order to achieve this, it is convenient to include the following rigidity assumption, which will also be very important in the next section when we look at properties of level-compactness.

Definition 3.2.1. *A group G levelled over X with elevation e is rigid if it satisfies the following condition: for any $\gamma \in \mathbb{Z}^e$, if G contains at least one subset of level γ then $\text{lv}(G_{U,\gamma}) = \gamma$ for all $U \in \mathcal{U}(1)$.*

Remark. Note that we always have $\text{lv}(G_{U,\gamma}) \geq \gamma$ from the definition of level. The level of a subset was defined in a way as to allow for a consistent definition, for instance, in the case when there are no subsets having level $\leq \delta$ for some δ by setting the level of all "large enough" distinguished sets to be the maximum possible. In theory, this definition could also allow pathological cases where the level of a distinguished set no longer matches the intuitive notion of its "size"; the notion of rigidity is intended to exclude these possible cases.

Example. If F is an n -dimensional local field with its canonical level structure, the subset $t_1^{i_1} \cdots t_n^{i_n} O_F$ has level $(i_1, \dots, i_n)$, and so F is rigid. Similarly, the subgroup $K_{i_1, \dots, i_n}$ of $\text{GL}_m(F)$ has level $(i_1, \dots, i_n)$, and so $\text{GL}_m(F)$ is rigid with the canonical partial level structure for all $m \geq 1$.

Problem 3. Give an example of a non-rigid group with level structure, or prove that no such spaces exist. If examples do exist, can they be classified in some reasonable way.

We now begin our investigation of the interaction between levels and set operations with the following preliminary Lemma.

Lemma 3.2.2. Let G be rigid, let $S \subset G$ with $\text{lv}(S) = \gamma$, and for any $\delta \geq \gamma$ suppose that G contains at least one subset of level δ . Then for every $\delta \geq \gamma$ there is a subset $S' \subset S$ with $\text{lv}(S') = \delta$. If, moreover, S has uniform level γ , for each $s \in S$ there is a distinguished set $D_{s,\delta}$ of level δ with $s \in D_{s,\delta} \subset S$.

Proof. By rigidity and the definition of level, S contains some distinguished set $gG_{U,\gamma}$ of level γ . By property (3) in the definition of level structure we have $gG_{U,\delta} \subset gG_{U,\gamma} \subset S$, and by rigidity we have $\text{lv}(gG_{U,\delta}) = \delta$. This proves the first assertion.

Now suppose that S has uniform level γ , and take any $s \in S$. Then there is some $g \in G$ such that $s \in gG_{U,\gamma} \subset S$. On the other hand, $s \in sG_{U,\delta}$, and from the proof of Proposition 3.1.5 we see that, if $\delta \geq \gamma$, $gG_{U,\gamma} \cap sG_{U,\delta}$ is a distinguished set $D_{s,\delta}$ which has level δ , and $s \in D_{s,\delta} \subset S$ by construction. $\square$

Proposition 3.2.3. *If A and B are both open subsets of G which have a level and $A \cap B \neq \emptyset$ has a level, $\text{lv}(A \cap B) \geq \max\{\text{lv}(A), \text{lv}(B)\}$. Furthermore, if A and B both have uniform level then so does $A \cap B$, and the inequality is in fact an equality.*

Proof. If A and B are both distinguished sets then it follows from the proofs of Lemma 3.1.2 and Proposition 3.1.5 that $\text{lv}(A \cap B) = \max\{\text{lv}(A), \text{lv}(B)\}$. In the general case, let $x \in A \cap B$. Let D_A be a distinguished open neighbourhood of x in A , and let D_B be a distinguished open neighbourhood of x in B . By Proposition 3.1.5 $D = D_A \cap D_B \subset A \cap B$ is a distinguished set, and by the first line it has level $\max\{\text{lv}(D_A), \text{lv}(D_B)\} \geq \max\{\text{lv}(A), \text{lv}(B)\}$.

It thus remains to show that $A \cap B$ cannot contain a distinguished set of any smaller level. Indeed, if it did contain such a set D' of level less than $\max\{\text{lv}(A), \text{lv}(B)\}$, then both A and B would also contain D' , which is impossible since at least one of them has strictly larger level.

For the final assertion, note that if A and B have uniform level we may choose D_A and D_B to have levels $\text{lv}(D_A) = \text{lv}(A)$ and $\text{lv}(D_B) = \text{lv}(B)$, and then from what we have already proved it follows that

$$\text{lv}(D) = \max\{\text{lv}(D_A), \text{lv}(D_B)\} = \max\{\text{lv}(A), \text{lv}(B)\} \leq \text{lv}(A \cap B).$$

But since $D \subset A \cap B$ we have $\text{lv}(D) \geq \text{lv}(A \cap B)$ by definition, hence we have equality. $\square$

Remark. Note that this Proposition *not* prove that $\text{lv}(A \cap B)$ exists in the non-uniform case, since $A \cap B$ may still contain distinguished sets of level γ with $\max\{\text{lv}(A), \text{lv}(B)\} < \gamma < \max\{\text{lv}(D_A), \text{lv}(D_B)\}$. In the case that the elevation $e = 1$, this is indeed enough to prove that the level exists, since there must be a minimal such γ , but for $e > 1$ this is not necessarily the case, since bounded sequences in $\mathbb{Z}^e$ do not necessarily have extrema.

Corollary 3.2.4. *Let A and B are subsets of G which have a level such that $A \cap B$ has a level. If $\text{int}A \cap \text{int}B \neq \emptyset$ then $\text{lv}(A \cap B) \geq \max\{\text{lv}(A), \text{lv}(B)\}$. (Here $\text{int}S$ denotes the interior of a subset $S \subset G$.)*

Proof. We can apply Proposition 3.2.3 to $\text{int}A$ and $\text{int}B$ to find a distinguished set D inside $A \cap B$ of level $\text{lv}(D) \geq \max\{\text{lv}(\text{int}A), \text{lv}(\text{int}B)\} \geq \max\{\text{lv}(A), \text{lv}(B)\}$. The same argument in the second paragraph of the proof of Proposition 3.2.3 then shows that $A \cap B$ cannot contain a distinguished set whose level is lower than the latter. $\square$

To deal with unions in the general case is a lot harder. With some extra restrictions we can still cover some ground, however, and we will do this in section 3.4.

Before this, we come to the fundamental notion of level-compactness.

Definition 3.2.5. *Let G be a group with level structure, and let $\gamma \in \mathbb{Z}^e$. A subset $S \subset G$ is called γ -compact if every open cover (in the level topology) of S by sets of uniform level γ has a finite subcover. We will call S level-compact if there is some $\gamma \in \mathbb{Z}^e$ such that S is γ -compact.*

Remark. As was hinted previously, it is important that each set in the cover has uniform level γ . Note that although we refer to open covers in the definition (so that the reader may immediately see the connection with compactness), we may in fact omit the word "open" since we saw earlier that any set of uniform level is necessarily open.

Possibly the most important example to keep in mind is the following, which is the main motivating example for the definition of level-compactness (and hence for this entire chapter).

Proposition 3.2.6. *Let F be a d -dimensional nonarchimedean local field with parameters $t_1, \dots, t_d$. If F is given the level structure of elevation $d - 1$ over its 1-dimensional residue field then the subset $t_1^{i_1} \cdots t_d^{i_d} O_F$ is γ -compact with $\gamma = (i_2, \dots, i_d)$.*

Proof. Assume otherwise, i.e. there is a γ -cover $(V_m)_{m \in M}$ which admits no finite subcover. Since $t_1^{i_1} \cdots t_d^{i_d} O_F / t_1^{i_1+1} \cdots t_d^{i_d} O_F \simeq O_F / t_1 O_F$ is finite, there is $\theta_0 \in t_1^{i_1} \cdots t_d^{i_d} O_F$ with $\theta_0 + t_1^{i_1+1} \cdots t_d^{i_d} O_F$ not contained in a finite union of the V_m (since otherwise we would have a finite subcover).

Similarly, there are $\theta_1, \dots, \theta_n \in t_1^{i_1} \dots t_d^{i_d} O_F$ such that $\alpha_n + t_1^{i_1+n+1} \dots t_d^{i_d} O_F = \theta_0 + \theta_1 t_1 + \dots + \theta_n t_1^n + t_1^{i_1+n+1} \dots t_d^{i_d} O_F$ is not covered by a finite union of V_m . But since F is complete, $\alpha = \lim_{n \rightarrow \infty} \alpha_n$ belongs to some V_ℓ .

Now, since V_ℓ has uniform level γ , there is a distinguished set D with $\text{lv}(D) = \gamma$ and $\alpha \in D \subset V_\ell$. On the other hand, we also have $\alpha \in A_n = \alpha_n + t_1^{i_1+n+1} \dots t_d^{i_d} O_F$ for $n \geq 0$. (Note that A_n has uniform level γ .) By Lemma 2.3.2, the intersection of two distinguished sets is either empty or equal to one of them, and so we have $D \cap A_n = D$ or A_n . If $D \cap A_n = A_n$ for every n , we have $D \subset \bigcap A_n = t_2^{i_2+1} \dots t_d^{i_d} O_F^{(d-1)}$, where $O_F^{(d-1)}$ is the rank $(d-1)$ ring of integers of F . But then we have $\gamma = \text{lv}(D) \geq \text{lv}(\bigcap A_n) > \gamma$, which is a contradiction, hence we have $A_n \subset D \subset V_\ell$ for n large enough. But we have previously concluded that no A_n can be covered by a finite number of the V_m , and so we have reached the desired contradiction. $\square$

Remark. It is essential that the elements of the cover all have the same level as $t_1^{i_1} \dots t_d^{i_d} O_F$ in this Proposition. Indeed, $O_F = \bigcup \alpha + t_2 O_F$ where α runs through an (infinite!) set of representatives for $O_F/t_2 O_F$, and since the union is disjoint there can be no finite subcover. It is equally important that the elements of the cover have uniform level, as we saw in an earlier example.

The proof of Proposition 3.2.6 uses the fact that F is complete in an essential way. As some of the consequences of completeness will be crucial later, it is worthwhile to ask if the completeness property (or perhaps a weaker alternative which still works for the above proof) can be restated purely in terms of the level structure.

Problem 4. *Investigate possible candidates for an appropriate notion of "level-completeness", and give a proof of Proposition 3.2.6 purely in terms of the level structure on F .*

Definition 3.2.7. *A group G levelled over a locally compact group X is called locally level-compact if for every $g \in G$ there is some $\gamma \in \mathbb{Z}^e$ such that g has a γ -compact neighbourhood.*

Example. An n -dimensional local field F is locally level-compact over its local residue field.

3.3 PROPERTIES OF LEVEL-COMPACTNESS

In this section we will study various elementary topological properties of level-compactness. Since we may ask the question "can we replace compactness by level-compactness" in almost every definition and theorem concerning compactness, we will of course not cover all possibilities here. Instead we focus as much as possible on results that are useful from the point of view of the remainder of this text (or in some cases when there are possible important applications to areas of higher dimensional number theory and arithmetic geometry not covered here).

3.3.1 Some elementary properties

Proposition 3.3.1. *Let G be levelled over X with elevation e with the level topology, suppose that G is rigid, and suppose that G contains at least one set of level γ . Then the following properties hold.*

- (1) *If $S \subset G$ is γ -compact then S is also δ -compact for all $\delta \leq \gamma$.*
- (2) *If $S \subset G$ is γ -compact and C is a closed subset of S such that $S \setminus C$ is uniformly of level $\delta \leq \gamma$ then C is δ -compact.*

Proof. If G has no subsets of level δ then the result trivially holds. Otherwise, let $S = \bigcup_m U_m$ be a uniform open δ -cover of S . We may assume without loss of generality that each U_m is a basic open set of level δ . By rigidity, it follows from Lemma 3.2.2 that we may write $U_m = \bigcup_{\alpha \in U_m} V_\alpha$, where V_α is a distinguished set of level γ containing α . Then $S = \bigcup_{m, \alpha \in U_m} V_\alpha$ is a uniform open γ -cover of S , and by γ -compactness it has a finite subcover. It thus follows that for each V_α in this finite subcover we may take some U_m containing it, and doing so gives a finite δ -subcover of S .

The proof of (2) follows the same reasoning as the proof that closed subsets of compact spaces are compact. Indeed, first note that we know from (1) that S is also δ -compact. If we take any uniform open δ -cover $C = \bigcup_m U_m$ of C , then $G = (G \setminus C) \cup \bigcup_m U_m$ is a uniform open δ -cover of S , hence it has a finite subcover, and this gives us also a finite subcover of C . $\square$

Remark. Property (2) may be thought of as saying that sufficiently small closed subsets of γ -compact sets are level-compact.

We can actually improve property (1) of Proposition 3.3.1 quite substantially. In order to do this, we first note that we may classify subsets that have no level into three distinct categories.

Definition 3.3.2. A subset $A \subset G$ is of type S if there exists no distinguished subset D with $D \subset A$.

Remark. The subsets of type S should be thought of as those which are "too small" to have a level. In all examples we have given so far, finite sets have always been of type S .

Definition 3.3.3. A subset $A \subset G$ is of type L if for every $\gamma \in \mathbb{Z}^e$ there is a distinguished set D_γ of level $\delta \leq \gamma$ with $D_\gamma \subset A$.

Remark. Subsets of type L are the opposite extreme to those of type S ; they are the subsets which are "too large" to have a level. If the map $G\mathcal{L} \rightarrow \mathbb{Z}^e$ which sends every distinguished set to its level is surjective, the whole group G is always of type L , although there may be more subsets of this type.

Definition 3.3.4. Let A be a subset of G with no level. We say that A is of type E if it is not of type S or of type L .

Remark. The subsets of type E are "exceptional" subsets. If G is of elevation $e \leq 1$ and the level map is surjective then there are no subsets of type E . For $e = 0$ this is trivial, and for $e = 1$ it is an immediate consequence of the fact that any sequence in $\mathbb{Z}$ which is bounded below has a minimum. When the level map is not surjective, many more sets of type E may appear.

Problem 5. Give an example of a group G with level structure and an explicit subset $A \subset G$ of type E , or prove that no such examples exist.

We noted earlier that level-compactness should be thought of as being compactness with respect to open covers by "sufficiently large" sets. The intuition from the previous definition suggests that we should also attempt to allow sets of type L in our covers. Of course, due to the same issues which appeared previously, we are guided towards the following subcollection of sets of type L .

Definition 3.3.5. Let $A \subset G$ be a subset of type L . For $\gamma \in \mathbb{Z}^e$ we say that A is γ -uniform if A has an open covering $A = \bigcup U_i$ with each $U_i \subset A$ uniformly of level γ .

The refinement of Proposition 3.3.1 is the following.

Proposition 3.3.6. Let G be levelled over X with elevation e , and suppose that G is rigid. Let $A \subset G$ be γ -compact for some $\gamma \in \mathbb{Z}^e$ such that G contains a set of level γ , and let $A \subset \bigcup U_i$ be an open cover of A . Suppose that for each i we have either (i) U_i is uniformly of level $\gamma_i \leq \gamma$, or (ii) U_i is of type L and is γ -uniform. Then $\bigcup U_i$ has a finite subcover.

Proof. If U_i is of the form (i), we saw in the proof of Proposition 3.3.1 that we may cover U_i by distinguished sets $\{V_\alpha^{(i)}\}$ of uniform level γ . If U_i is of the form (ii) then from the definition of γ -uniformity we also have an open covering of U_i by sets $\{V_\alpha^{(i)}\}$ of uniform level γ . This gives us a uniform open γ -cover $A \subset \bigcup_{i,\alpha} V_\alpha^{(i)}$, which has a finite subcover by γ -compactness. For each $V_\alpha^{(i)}$ in this subcover, choosing one of the $U_i \supset V_\alpha^{(i)}$ gives the required subcover of $\bigcup U_i$. $\square$

Remark. One may similarly define the notion of γ -uniformity for sets of type E (note that for sets of type S the condition can never be satisfied), and further refine Proposition 3.3.6 to include these sets as well. These exceptional sets of type E seem quite mysterious, and it may be interesting

to study their properties. There seems to be some link between the presence of exceptional sets and how badly G can behave.

We end this section with one final result on unions.

Proposition 3.3.7. *Let G be rigid. If $K_1 \subset G$ is γ -compact and $K_2 \subset G$ is δ -compact with $\gamma \geq \text{lv}(K_1)$, $\delta \geq \text{lv}(K_2)$ then $K = K_1 \cup K_2$ is η -compact for some $\eta \geq \text{lv}(K)$, if this level exists.*

Proof. Let $\eta = \min\{\gamma, \delta\}$. Then any uniform open η -cover of K is an open cover of each of the η -compact sets K_1 and K_2 , which both have finite subcovers. Taking the union of these subcovers then gives a finite subcover of K , hence K is η -compact.

Since K contains both K_1 and K_2 ,

$$\text{lv}(K) \leq \min\{\text{lv}(K_1), \text{lv}(K_2)\} \leq \min\{\gamma, \delta\} = \eta,$$

if the level of K exists. □

3.3.2 Product spaces

When speaking about compactness one also expects to consider products of spaces. If G is levelled over X and H is levelled over Y with the same elevation e , then $G \times H$ is naturally levelled over the product space $X \times Y$ with elevation e via $(G \times H)_{(U \times V), \gamma} = G_{U, \gamma} \times H_{V, \gamma}$. The level topology on the product coincides with the product topology.

Since we will want to apply the earlier results of this chapter, the following easy Lemma is important.

Lemma 3.3.8. *Let G be levelled over X and H be levelled over Y , both with elevation e . If G and H are rigid then $G \times H$ is rigid over $X \times Y$.*

Proof. We need to show that $\text{lv}((G \times H)_{(U \times V), \gamma}) = \gamma$. However, this follows from the definitions, since if $(G \times H)_{(U \times V), \gamma}$ is contained in some other $(G \times H)_{(U \times V), \delta}$ with $\delta > \gamma$ then (for example) $G_{U, \gamma} \subset G_{U, \delta}$, and so $\text{lv}(G_{U, \gamma}) \geq \delta > \gamma$, which contradicts rigidity of G . □

Remark. Since the direct product of two spaces is commutative, the proof of Lemma 3.3.8 actually shows that $G \times H$ is rigid as long as at least one of G and H is rigid.

Proposition 3.3.9. *If G is γ_1 -compact and rigid over X , and if H is γ_2 -compact and rigid over Y , the product $G \times H$ is γ -compact over $X \times Y$, where*

$$\gamma = \min\{\gamma_1, \gamma_2\}.$$

Proof. We know from Proposition 3.3.1 that G and H are both γ -compact. Thus if we take any uniform open γ -cover $G \times H = \bigcup_m (U_m \times V_m)$, we know that $G = \bigcup_m U_m$ has a finite subcover $G = \bigcup_{m \in M_1} U_m$ and $H = \bigcup_m V_m$ has a finite subcover $H = \bigcup_{m \in M_2} V_m$. It then follows that $\bigcup_{m \in M_1 \cup M_2} U_m \times V_m$ is a finite subcover of $G \times H$. $\square$

Remark. This, along with the following results, is also true for the appropriate level-compact subsets of G and H . However, for the sake of brevity we will formulate the statements only in terms of the full group G , and so on.

The definitions also work for infinite products, but for the analogue of Tychonoff's theorem we will need to do a little more work.

Lemma 3.3.10. *Let $G = \prod_{i \in I} G_i$ with each G_i γ -compact over a space X_i . Then any open cover of G by sets of the form $\pi_j^{-1}(U)$ with $U \subset G_j$ open of uniform level γ has a finite subcover. (Here $\pi_j : G \rightarrow G_j$ is the projection map.)*

Proof. Let $\mathcal{U}$ be such a cover, and let $\mathcal{U}_j$ be the collection of $U \subset G_j$ such that $\pi_j^{-1}(U) \in \mathcal{U}$. Suppose that there is no j such that $\mathcal{U}_j$ covers G_j , so that for each j we find $g_j \in G_j$ with g_j not in the union of all elements of $\mathcal{U}_j$. But then $(g_i)_{i \in I}$ is not contained in any element of $\mathcal{U}$, which is not possible since this is a cover of G .

We can thus find some j such that $\mathcal{U}_j$ is a cover of G_j , and by γ -compactness we can find a finite subcover $G_j \subset \bigcup_{k=1}^n U_k$, hence we have a finite subcover $G = \bigcup_{k=1}^n \pi_j^{-1}(U_k)$ of $\mathcal{U}$. $\square$

Now we may prove a modification of the Alexander Subbase Theorem. For this, the set theory enthusiasts will note that we must assume the axiom of choice, since the proof requires the use of Zorn's Lemma.

Lemma 3.3.11. *Let G be levelled over X , and let $\mathcal{V}$ be a subbase for the level topology on G . If every collection of sets of uniform level γ from $\mathcal{V}$ which covers G has a finite subcover then G is γ -compact.*

Proof. Suppose that every such cover has a finite subcover, but G is not γ -compact. Then the collection $\mathcal{F}$ of all open γ -covers of G with no finite subcover is nonempty and is partially ordered by inclusion. Let $\{E_\alpha\}$ be any totally ordered subset of $\mathcal{F}$, and put $E = \bigcup_\alpha E_\alpha$.

By definition, E is a uniform open γ -cover of G . For any finite collection $U_1, \dots, U_n$ of elements of the cover, we have $U_j \in E_{\alpha_j}$ for some α_j , and since the E_α are totally ordered there is some E_{α_0} containing all of them. It follows that E has no finite subcover, and so E is an upper bound for $\{E_\alpha\}$, thus by Zorn's Lemma there is a maximal element $\mathcal{M}$ of $\mathcal{F}$.

Let $S = \mathcal{V} \cap \mathcal{M}$, and suppose there is $g \in G$ that is not inside any element of S . Since $\mathcal{M}$ is a cover of G there is some $U \in \mathcal{M}$ with $g \in U$, and since $\mathcal{V}$ is a subbase for the topology there are $V_1, \dots, V_n \in \mathcal{V}$ with $g \in \bigcap_{i=1}^n V_i \subset U$. By assumption none of the V_i are in $\mathcal{M}$, so by maximality $\mathcal{M} \cup \{V_i\}$ contains a finite subcover $G = V_i \cup U_i$, where U_i is a finite union of sets in $\mathcal{M}$. Then $U \cup \bigcup_{i=1}^n U_i \supset (\bigcap_{i=1}^n V_i) \cup (\bigcup_{i=1}^n U_i) \supset \bigcap_{i=1}^n (V_i \cup U_i) \supset G$. But this is a finite cover of G by elements of $\mathcal{M}$, which contradicts $\mathcal{M}$ having no finite subcover.

It thus follows that S is a cover of G , and since $S \subset \mathcal{V}$ it has a finite subcover by assumption. But S is also contained in $\mathcal{M}$, and so it cannot have a finite subcover, hence the collection $\mathcal{F}$ must be empty, i.e. G is γ -compact. $\square$

This allows us to deduce the extension of Tychonoff's Theorem.

Theorem 3.3.12. *If $G = \prod_{i \in I} G_i$ with each G_i γ -compact over X_i then G is γ -compact over $X = \prod_{i \in I} X_i$.*

Proof. The collection $\{\pi_j^{-1}(U_j)\}$ is a subbase for the product topology. By Lemma 3.3.10 any subcollection of this set which covers G has a finite subcover, and then the Alexander Subbase Theorem shows that G is γ -compact. $\square$

Remark. If all of the G_i are rigid, we don't need that all of the G_i are γ -compact with the same γ . As long as there is a minimum $\gamma_{\min}$ across all G_i , this $\gamma_{\min}$ will be the (maximal) γ that works for the product.

For a collection of spaces $\{A_i\}$ with subspaces $B_i \subset A_i$, recall that the restricted product $\prod' A_i$ of the A_i with respect to B_i consists of sequences $(a_i)_i \in \prod A_i$ such that $a_i \in B_i$ for all but finitely many i . It follows from Theorem 3.3.12 that the restricted product of a collection of rigid locally level-compact groups $\{A_i\}$ with respect to a system of γ -compact subgroups $\{B_i\}$ is again locally level-compact.

Corollary 3.3.13. *Let k be either a number field or the function field of a proper, smooth, connected curve over a finite field, and let $\mathcal{S}$ be an arithmetic surface which is a proper regular model of a smooth, projective, geometrically irreducible curve X/k . Then groups $A_{\mathcal{S}}$ of geometric adeles and $\mathbb{A}_{\mathcal{S}}$, of analytic adeles, associated to $\mathcal{S}$ and a given set $\mathcal{S}$ of all fibres and finitely many horizontal curves on $\mathcal{S}$, are locally level-compact.*

Proof. For a two-dimensional nonarchimedean local field, we saw in Proposition 3.2.6 that $t_1^i t_2^j \mathcal{O}_F$ is j -compact, and since $t_2^j \mathcal{O}_F$ is contained in all of the distinguished sets $t_1^i t_2^{j-1} \mathcal{O}_F$ and disjoint from any of their F -translates it is trivially $(j-1)$ -compact. For the archimedean components, the j -compactness of the subsets $\alpha + Ct^j + t^{j+1}F[[t]]$ for $C \subset F$ compact follows immediately from the compactness of C , and so the archimedean fields $F((t))$ are also locally level compact. The adelic spaces are then restricted products of locally level-compact spaces with respect to level-compact subgroups (see [Fes10]). $\square$

Remark. The space $A_{\mathcal{S}}$ was first considered by Parshin, Beilinson and others, while $\mathbb{A}_{\mathcal{S}}$ was first defined by Fesenko (see [Fes10] for the definitions,

which are too lengthy to reproduce here). In particular when $\mathcal{S} = \mathcal{E}$ is the minimal regular model of an elliptic curve over a number field, the adelic spaces $\mathbf{A}_{\mathcal{E}}$ and $\mathbb{A}_{\mathcal{E}}$, are related to several important open problems in number theory and arithmetic geometry, including the Riemann Hypothesis and the Birch and Swinnerton-Dyer Conjecture. For details see [Fes10], [Fes17].

3.3.3 Discreteness

We now consider connections with discreteness. It follows immediately from the definitions that a topological space which is both discrete and compact must be finite. Here we discuss what happens when we replace "compact" with " γ -compact".

Remark. If every element of $\mathcal{L}$ is equal to $\{e_G\}$, discreteness essentially forces the elevation e to be equal to 0, since all distinguished sets will have the same level. However, if there are non-singleton distinguished sets, this may not be the case, as the following example illustrates.

Example. Let $X = \{x\}$ be the single element group. We may endow $G = \mathbb{Z}$ with an elevation $e = 1$ level structure over X via $G_{\{x\}, \gamma} = \{0\}$ for $\gamma \geq 0$ and $G_{\{x\}, \gamma} = \{0, 1, \dots, n-1\}$ for $\gamma = -n$ with n a positive integer. The former sets all have level -1 , while the latter have level $-n$.

The level topology coincides with the discrete topology since $\{0\} \in \mathcal{L}$, and in this case $-\text{lv}(U)$ gives the length of the longest chain of consecutive integers in a subset U of G . In this example, we easily see that G has no subsets of type S , and that every finite set has a level. Since G has elevation $e = 1$, there are also no subsets of type E . We see that G is of type L , but we may also construct infinitely many examples of proper subsets of type L . One such example is the set

$$\{0, 2, 3, 5, 6, 7, 9, 10, 11, 12, 14, \dots\}$$

obtained by adding consecutive chains of longer lengths each time *ad infinitum*.

Remark. Note that not all infinite subsets of G in the above example are of type L : the set $2\mathbb{Z}$ of even integers, for example, has (uniform) level -1 , while the set of all prime numbers has (non-uniform) level -2 .

Remark. Since, as we have noted now on several occasions, the level of a set is intuitively related to a notion of "size", introducing various level structures on $\mathbb{Z}$ may be of some interest in analytic number theory, as it gives an alternative way of defining the "volume" of a set of primes. For example, one could state the twin prime conjecture in terms of the level structure $G'_{\{x\},\gamma} = 2G_{\{x\},\gamma}$, with $G_{\{x\},\gamma}$ the level structure in the previous example.

Suppose G is levelled over X , and that $S \subset G$ is a γ -compact, discrete subset. By the results of the previous section, distinguished sets are the smallest open subsets of G , and so for every $s \in S$ there is a distinguished subset D_s with $D_s \cap S = \{s\}$.

Now, in general the distinguished sets D_s may change wildly as s varies, or simply all be of too high level for γ -compactness to come into play. However, if we have some control over these factors then we can indeed say something.

Proposition 3.3.14. *Let G be levelled over X and let $S \subset G$ be discrete and γ -compact. If there exists a $\delta \leq \gamma$ such that for every $s \in S$ there is a distinguished $D_s \subset G$ with $\text{lv}(D_s) = \delta$ and $D_s \cap S = \{s\}$ then S is finite.*

Proof. We have a uniform δ open cover $S \subset \bigcup_{s \in S} D_s$, and since $\delta \leq \gamma$ S is δ -compact and so we have a finite subcover. However, since each $s \in S$ appears in exactly one element of the cover (namely D_s), this implies that S must be finite. $\square$

Corollary 3.3.15. *If S is discrete and γ -compact over X for all $\gamma \in \mathbb{Z}^e$, and furthermore S can be completely disconnected by open subsets of some level δ , then S is finite.*

3.3.4 Compactness at all levels

Clearly if G is compact then it is γ -compact for all γ . However, it is not clear when (if at all) the converse holds. Obtaining results in this direction is almost equivalent to controlling the subsets of type L and type E .

Proposition 3.3.16. *Suppose G is rigid and γ -compact for every $\gamma \in \mathbb{Z}^e$, and that the following conditions are satisfied:*

- (i) G contains no open subsets of type E ;
- (ii) For all $\gamma \in \mathbb{Z}^e$ and for every open subset $A \subset G$ of type L , A is γ -uniform.

Then any open cover $G = \bigcup U_i$ such that the set $\{\text{lv}(U_i)\}_i$ has an upper bound in $\mathbb{Z}^e$ has a finite subcover. (Here we allow the possibility that U_i has no level.)

Proof. Essentially all that we have to do is describe the possible structures of open subsets of G . If $U \subset G$ is open and does not have a level, it must be of type L or of type E , since the only other possibility is that U is of type S and contains no nonempty open set (which is clearly false). Since by assumption no subset of type E is open, the open subsets of G either have a level or are of type L .

Now, let $G = \bigcup U_i$ be an open cover such that the levels of the U_i are bounded above by $\gamma \in \mathbb{Z}^e$. If U_i is of type L , by (ii) we can cover U_i by open sets $\{V_{i,m}\}$ which are uniformly of level γ , and if U_i is distinguished then we can do the same by Lemma 3.2.2. This gives a uniform open γ -cover $G = \bigcup V_{i,m}$, from which we can take a finite subcover, and then take one U_i containing each element of this subcover to obtain a finite subcover of $\{U_i\}$. $\square$

3.3.5 The finite intersection property

An alternative characterisation of compactness can be formulated in terms of the finite intersection property. Recall that a family $\{S_i\}$ of subsets of a topological space has the finite intersection property if any intersection of

finitely many elements of the family is nonempty. It is well known that a topological space is compact if and only if every family of closed subsets satisfying the finite intersection property has nonempty intersection. Similarly, we have the following.

Lemma 3.3.17. *G is γ -compact if and only if any collection $\{E_i\}$ of closed sets with $G \setminus E_i$ uniformly of level γ satisfying the finite intersection property has nonempty intersection.*

Proof. First let G be γ -compact, and suppose $\bigcap E_i = \emptyset$. Then $G = G \setminus \emptyset = G \setminus \bigcap_i E_i = \bigcup_i (G \setminus E_i)$ is a uniform open γ -cover of G , and so there is a finite subcover $G = \bigcup_{i=1}^n (G \setminus E_i) = G \setminus \bigcap_{i=1}^n E_i$. But the latter implies that $\bigcap_{i=1}^n E_i$ is empty, which is a contradiction.

Now suppose that any collection $\{E_i\}$ of closed sets with $G \setminus E_i$ uniformly of level γ satisfying the finite intersection property has nonempty intersection. Let $G = \bigcup_i U_i$ be a uniform open γ -cover, and suppose there is no finite subcover. Then for any finite subcollection $U_1, \dots, U_n$ we have $\bigcap_{i=1}^n (G \setminus U_i) = G \setminus (\bigcup_{i=1}^n U_i) \neq \emptyset$, and so the collection $\{G \setminus U_i\}$ satisfies the finite intersection property. We thus have $\emptyset \neq \bigcap_i (G \setminus U_i) = G \setminus (\bigcup_i U_i) = G \setminus G = \emptyset$, which is clearly impossible. $\square$

3.4 GROUPS OF ORDERED TYPE

In this section we restrict attention to a particular kind of level structure which exhibits the same desirable properties as higher dimensional nonarchimedean fields. While it is unfortunate that this class of spaces does not include the archimedean case, this restriction makes so many things simpler that it is well worth studying as a special case.

We begin with the definition, which the reader may recognise from the statement of Lemma 2.4.3. Given how powerful this result was in Chapter 2, the reader should not be surprised that we want to consider how it can be applied in the more general setting.

Definition 3.4.1. A group G levelled over X is of ordered type if $D_1 \cap D_2 \in \{D_1, D_2, \emptyset\}$ for any two distinguished subsets D_1, D_2 of G .

Remark. The terminology comes from the following observation: G is of ordered type if and only if every family of pairwise nondisjoint distinguished subsets of G is totally ordered with respect to inclusion.

Example. Nonarchimedean higher local fields F are of ordered type (see [Fes03]), as are the groups $\mathrm{GL}_2(F)$ by Lemma 2.4.3. Archimedean fields are not: $(0, 2) + t\mathbb{R}[[t]]$ and $(1, 3) + t\mathbb{R}[[t]]$ are distinguished subsets of $\mathbb{R}((t))$ with nonempty intersection but neither is contained inside the other.

For groups of ordered type, the level is much more controllable.

Proposition 3.4.2. Let G levelled over X be of ordered type, and let A, B be a distinguished subsets of G with levels γ, δ respectively. If $A \cup B$ is not a distinguished set then it has level $\min\{\gamma, \delta\}$.

Proof. Let $\beta = \min\{\gamma, \delta\}$. Then certainly $A \cup B$ contains a distinguished set of level β , and so if it has a level then the level must be at most β . Suppose that $A \cup B$ contains a distinguished set D of level $< \beta$.

Now, since they are distinguished, $D \cap A$ is either empty or equal to A (since A has the higher level), and similarly $D \cap B$ is either empty or equal to B . Thus $D = D \cap (A \cup B) = (D \cap A) \cup (D \cap B) \in \{\emptyset, A, B, A \cup B\}$. But since D is distinguished it cannot be empty, and since D has lower level than both A and B it cannot be equal to either of them. Thus the only possibility is that $D = A \cup B$ is distinguished, which is not true by assumption.

We have thus shown that $A \cup B$ contains no distinguished set of level lower than β , but since it contains both A and B it must contain a distinguished set of level β , and so it follows that $A \cup B$ has level β . $\square$

Remark. If we remove the assumption that $A \cup B$ not be distinguished then the level may decrease, and it is easy to construct examples to show that this decrease can be arbitrarily large. However, $A \cup B$ still has a level in this case, since it is distinguished, and so we see that any union of two distinguished

sets has a level. By imposing certain extra conditions one may obtain a bound on the possible decrease in level.

Corollary 3.4.3. *If G is of ordered type and $A_1, \dots, A_r$ are finitely many distinguished subsets of G such that no union of any collection of the A_i is a distinguished set, then $A = \bigcup_{i=1}^r A_i$ has level $\min\{\text{lv}(A_i)\}$.*

Proof. Let $\beta = \min\{\text{lv}(A_i)\}$ and suppose that A contains a distinguished set D of level $< \beta$. Then as in the proof of the Proposition we must have $D \cap A_i \in \{\emptyset, A_i\}$ for every i , hence $D = D \cap A$ is either empty or equal to a union of some number ≥ 2 of A_i , which is a contradiction since neither of these are distinguished. We then conclude as before that $\text{lv } A = \beta$. $\square$

Corollary 3.4.4. *If G is of ordered type and $A_1, \dots, A_r$ are finitely many distinguished sets then $A = \bigcup_{i=1}^r A_i$ has a level.*

Proof. If no subunion of $A = \bigcup_{i=1}^r A_i$ is equal to a distinguished set then we use the previous Corollary. If there is a subunion $\bigcup_j A_{i_j}$ which is equal to a distinguished set B , we replace the subcollection $\{A_{i_j}\}$ with the set B . Since there are only finitely many A_i to begin with, after at most finitely many replacements we will be in the first case. $\square$

Problem 6. *Investigate other possible conditions other than G being of ordered type for which the above Proposition and Corollary hold.*

Unfortunately, the notion of having uniform level is particularly badly behaved with respect to unions, even when G is of ordered type. (Indeed, the trouble that unions can cause were the motivation for the definition!) In general it is not even true that the union of two sets of the same uniform level γ has uniform level.

Example. Let $F = \mathbb{Q}_p((t))$, and $O_F = \mathbb{Z}_p + t\mathbb{Q}_p[[t]]$ be the rank two ring of integers of F . Put $A = tO_F \cup (t^{-1} + tO_F)$ and $B = O_F \setminus tO_F$. Both A and B have uniform level 1, but $A \cup B = O_F \cup (t^{-1} + tO_F)$ does not have uniform level. (It has level 0 since it contains O_F , but the point t^{-1} has no distinguished neighbourhood of level 0.)

Let us now consider what happens with differences.

Proposition 3.4.5. *Let G be Hausdorff and of ordered type. Then any distinguished subset of G is closed, and if A and B are distinguished subsets of G with $B \subset A$ such that $\text{lv}(B) > \text{lv}(A)$ and $A \setminus B$ has a level then $\text{lv}(A \setminus B) \geq \text{lv}(A)$.*

Proof. Clearly $A \setminus B$ cannot contain any distinguished set of level $< \text{lv}(A)$, and so it is enough to show that it contains a distinguished set of some level.

Let $b \in B$, $x \in A \setminus B$. Then $B_x = xb^{-1}B$ is a distinguished subset of G , and since it contains $x = xb^{-1}b$ we have $B_x \cap A \neq \emptyset$. We thus have $B_x \cap A \in \{B_x, A\}$, and since $\text{lv}(B_x) = \text{lv}(B) > \text{lv}(A)$, from Proposition 3.2.3 we have $\text{lv}(B_x \cap A) \geq \text{lv}(B_x) > \text{lv}(A)$, hence we must have $B_x \cap A = B_x$, i.e. $B_x \subset A$.

Now, since G is Hausdorff we can find disjoint open subsets $U_x, U_b \subset G$ with $x \in U_x$ and $b \in U_b$. Since B_x and B are both open sets we may assume that $U_x \subset B_x$ and $U_b \subset B$. Furthermore, since open sets are unions of distinguished sets, we may assume that U_x and U_b are in fact distinguished sets. But $x \in U_x \setminus B$ and $b \in B \setminus U_x$ implies $U_x \cap B = \emptyset$, hence U_x is a distinguished subset of $A \setminus B$.

It remains to show that distinguished sets are closed. Let D be any distinguished subset of G . By the same argument as in the previous paragraph, for any $x \in G \setminus D$ we can find a distinguished set U_x which contains x and is disjoint from D . But then $G \setminus D = \bigcup_{x \in G \setminus D} U_x$ is an open set, hence D is closed. $\square$

The following Proposition is not important for the remainder of the current text, but we present it here as an example of the kind of conditions one can impose in order to get a bound on the level. In particular, this result holds for higher dimensional local fields (although in that case one can easily compute the level directly).

Proposition 3.4.6. *Let G be Hausdorff and of ordered type, and suppose that each element of $\mathcal{L}$ is a subgroup of G . Let A be a distinguished subset of G of level γ*

and let $B \subset A$ be a distinguished subset of G of level $\delta > \gamma$. If $A \setminus B$ has a level then $\gamma \leq \text{lv}(A \setminus B) \leq \delta$.

Proof. The first inequality was shown in the previous Proposition, and so it remains to show that $A \setminus B$ contains a distinguished set of level δ . Write $A = gG_{U,\gamma}$, $B = hG_{V,\delta}$ with $g, h \in G$ and $G_{U,\gamma}, G_{V,\delta} \in \mathcal{L}$. Then $g^{-1}hG_{V,\delta}$ is a coset of $G_{V,\delta}$ in $G_{U,\gamma}$. Since we assumed that B has strictly larger level than A , $A \setminus B$ is nonempty, hence there exists at least one other coset $kG_{V,\delta}$. Thus $gkG_{V,\delta} \subset A$ and $B \cap gkG_{V,\delta} = g(g^{-1}hG_{V,\delta} \cap kG_{V,\delta}) = \emptyset$, hence $A \setminus B$ contains the distinguished set $gkG_{V,\delta}$ of level δ . $\square$

Problem 7. Show that the result of Proposition 3.4.6 still holds without the assumption that elements of $\mathcal{L}$ be subgroups of G , or give a counterexample to show that the result is false.

We would now like to combine some of our results in order to get something for ddd-sets.

Proposition 3.4.7. Let G be Hausdorff and of ordered type, and for $1 \leq i \leq r$ let A_i, B_j be distinguished (or empty) subsets of G with each $B_j \subset A_i$ for some i and the sets $A_i \setminus \bigcup_j B_j$ pairwise disjoint. If no union of any collection of $A_i \setminus \bigcup_j B_j$ is a distinguished set and $C = \bigcup_i A_i \setminus \bigcup_j B_j$ is not of type E then C has level at least $\min\{\text{lv}(A_i)\}$.

Proof. Let $\gamma = \min\{\text{lv}(A_i)\}$, $\delta = \min\{\text{lv}(A_i \setminus \bigcup_j B_j)\}$. By construction C contains a distinguished set of level δ , and so all we must show is that it contains no distinguished subset of level lower than γ .

Suppose C contains a distinguished set D of level $< \gamma$. First of all this means that $\bigcup_i A_i$ contains D . Since they are distinguished we have $A_i \cap D \in \{A_i, D, \emptyset\}$, and since D has lower level than all of the A_i we in fact have $A_i \cap D \in \{A_i, \emptyset\}$, and so $(A_i \setminus \bigcup_j B_j) \cap D \in \{(A_i \setminus \bigcup_j B_j), \emptyset\}$. But $D = D \cap C$ cannot be empty or equal to a single $A_i \setminus \bigcup_j B_j$, hence it is a union of two or more of the $A_i \setminus \bigcup_j B_j$, which violates our assumption that no subunion inside C be distinguished. $\square$

Corollary 3.4.8. *If G is Hausdorff and of ordered type then every ddd-set of G either has a level or is of type E .*

Proof. By definition any ddd-set can be written as a difference of finite disjoint unions $C = \bigcup_i A_i \setminus \bigcup_j B_j$ with each B_j contained in some A_i . If any subunion $\bigcup_k A_{i_k} \setminus \bigcup_j B_j$ inside C is equal to a distinguished set D , replace the collection $\bigcup_k A_{i_k} \setminus \bigcup_j B_j$ with the set D (which has a level since it is distinguished). Since we begin with only finitely many sets, after finitely many such replacements we will be in the situation of the previous Proposition. $\square$

The above result may still hold even if G is not of ordered type. (It may be proven directly for an archimedean higher local field, for example.)

Problem 8. *Give an example of a group G and a ddd-set $D \subset G$ which does not have a level, or show that such examples do not exist.*

Problem 9. *Investigate the possibilities for subsets of type S , L , and E for a group G of ordered type, and determine whether or not it is possible to classify them further in this case.*

3.5 AN INVARIANT MEASURE ON GROUPS OF ORDERED TYPE

We saw in Chapter 2 that one of the most crucial results that facilitated the construction of an invariant measure on F and $\mathrm{GL}_2(F)$ for a two-dimensional local field F were the intersection properties in Lemma 2.3.2 and Lemma 2.4.3. Since this intersection property is the very definition of a group G being of ordered type, we are naturally inclined to search for an invariant measure on such spaces in general.

In this section we will thus reformulate the relevant results from Chapter 2 in the more general setting of groups of ordered type. In particular, the groups F and $\mathrm{GL}_2(F)$ are good examples to keep in mind for all of the constructions in this section.

First of all, we need to extend the notion of index of a subgroup to distinguished sets. The following definitions make sense for all groups with level structure, not just those of ordered type.

Definition 3.5.1. Let D_1 and D_2 be distinguished subsets of G such that $D_1 \subset D_2$. We say that D_1 has finite index in D_2 if there are finitely many $g_i \in G$ such that $\bigcup_i g_i D_1$ is an open cover of D_2 .

Definition 3.5.2. Let D_1 and D_2 be distinguished subsets of G and suppose that D_1 has finite index in D_2 . Then the index $|D_2 : D_1|$ of D_1 in D_2 is defined to be the minimum cardinality

$$\min \left\{ \#I : \#I < \infty, D_2 \subset \bigcup_{i \in I} g_i D_1 \right\}$$

of all finite open covers of D_2 by translates of D_1 .

Remark. The set $\{\#I : \#I < \infty, D_2 \subset \bigcup_{i \in I} g_i D_1\}$ is a nonempty subset of $\mathbb{Z}$ and is bounded below by 1, and so this minimum always exists. If D_1 is a subgroup of D_2 , this coincides with the usual definition of index, and it also coincides with the use of this terminology in the previous chapter for the group $G = \mathrm{GL}_2(F)$ with F a two-dimensional local field.

By abuse of notation we will write $|D_2 : D_1| < \infty$ if D_1 has finite index in D_2 .

Lemma 3.5.3. Let D_1, D_2 , and D_3 be distinguished subsets of G with $D_1 \subset D_2 \subset D_3$. Then $|D_3 : D_1|$ is finite if and only if both $|D_3 : D_2|$ and $|D_2 : D_1|$ are finite, in which case $|D_3 : D_1| \leq |D_3 : D_2| \cdot |D_2 : D_1|$.

Proof. First suppose $|D_3 : D_2|$ and $|D_2 : D_1|$ are finite, so that $D_3 \subset \bigcup_{i=1}^m g_i D_2$ and $D_2 \subset \bigcup_{j=1}^n h_j D_1$ for some $g_i, h_j \in G$. This gives $D_3 \subset \bigcup_{i=1}^m \bigcup_{j=1}^n g_i h_j D_1$, and so D_1 has finite index in D_3 , and taking m and n to be minimal gives the required inequality of indices.

Conversely, suppose that $|D_3 : D_1|$ is finite, so that $D_3 \subset \bigcup_{k=1}^\ell f_k D_1$. Since $D_2 \subset D_3$, $\bigcup_{k=1}^\ell f_k D_1$ is already a finite cover of D_2 , and so it follows immediately that $|D_2 : D_1|$ is finite. On the other hand, since $D_1 \subset D_2$ we have $\bigcup_{k=1}^\ell f_k D_2 \supset \bigcup_{k=1}^\ell f_k D_1 \supset D_3$, and so $|D_3 : D_2|$ is also finite. $\square$

Problem 10. Give an example of D_1, D_2, D_3 such that the inequality is strict, or prove that no such examples exist (i.e. that it is in fact an equality). If it is possible for the inequality to be strict, find reasonable additional conditions on G which force it to be an equality.

One such "reasonable condition" which is of immediate interest to us is the following.

Corollary 3.5.4. Let G be of ordered type and let $D_1 \subset D_2 \subset D_3$ be distinguished subsets of G . If $|D_3 : D_1|$ is finite then so are both $|D_3 : D_2|$ and $|D_2 : D_1|$, and $|D_3 : D_1| = |D_3 : D_2| \cdot |D_2 : D_1|$

Proof. Since G is of ordered type we may in fact arrange that $D_3 = \bigcup_{i=1}^m g_i D_2$ is a disjoint union of translates of D_2 and $D_2 = \bigcup_{j=1}^n h_j D_1$ is a disjoint union of translates of D_1 . This gives a disjoint union $D_3 = \bigcup_{i=1}^m \bigcup_{j=1}^n g_i h_j D_1$, which shows that at least mn translates of D_1 are required to cover D_3 . $\square$

Remark. If G is of ordered type, any two translates of the same distinguished set D are either disjoint or equal, and so in this case distinguished subsets behave exactly like cosets. In particular, if we have two distinguished subsets $D_1 \subset D_2$, we can take a system of representatives $\{g_i\} \subset G$ such that we have a disjoint union $D_2 = \bigcup_i g_i D_1$.

If F is a two-dimensional local field with rank two ring of integers O_F , the index $|t_1^{i_1} t_2^{j_1} O_F : t_1^{i_2} t_2^{j_2} O_F|$ is finite if and only if $j_1 = j_2$. Furthermore, this fact plays a significant part in the definition of the measure on F , and so if we wish to extend such constructions to a more general setting it makes sense to restrict to examples where similar results hold.

Definition 3.5.5. Let G be levelled over X . We say that G is properly suspended if it satisfies the following property: for any two distinguished subsets D_1 and D_2 of G with $D_1 \subset D_2$, D_1 has finite index in D_2 if and only if $\text{lv}(D_1) = \text{lv}(D_2)$.

Unfortunately, being properly suspended and of ordered type isn't quite strong enough to imply strong statements about levels. However, with additional assumptions, we do obtain such statements, such as the following.

Proposition 3.5.6. *Let G be properly suspended and of ordered type, and let $D \subset G$ be distinguished. Then every open cover $D = \bigcup_i D_i$ of D by distinguished sets of level $\text{lv}(D)$ which has a minimal element, in the sense that either $D \subset D_i$ for all i or there is some j with $|D : D_j| \geq |D : D_i|$ for all $i \neq j$, has a finite subcover.*

Proof. Let D be a distinguished set of level γ , and take any open cover $D \subset \bigcup_i D_i$ by distinguished sets D_i of uniform level γ . We may assume that $D_i \cap D \neq \emptyset$ for all i , since any set not meeting D can be removed from the cover. If any $D_i \supset D$ then D_i is already a finite subcover of D . If no $D_i \supset D$, then all $D_i \subset D$ since G is of ordered type.

Let D_j be a minimal element of the open cover. Since we assume that G is properly suspended, there are finitely many $g_k \in G$ such that $D \subset \bigcup_k g_k D_j$. By taking a minimal such cover, we may assume that each $g_k D_j$ contains an element of D which is not contained in any of the others, and hence the $g_k D_j$ are all disjoint since G is of ordered type.

As each $g_k D_j$ is also a distinguished set, for all $i \neq j$ we have $g_k D_j \cap D_i \in \{g_k D_j, D_i, \emptyset\}$. Since the $g_k D_j$ cover D , for each i there is at least one k such that this intersection is nonempty. Furthermore, we can never have D_i properly contained inside $g_k D_j$, since otherwise we would have $|D : D_i| > |D : D_j|$ which contradicts our assumption.

We thus have the following: for each $i \neq j$ there exists some k such that $g_k D_j \subset D_i$. By Corollary 3.5.4, $|D_i : D_j|$ is finite, and so each D_i can be covered by finitely many translates of D_j : $D_i = \bigcup_k g_{i,k} D_j$. We thus have $D = \bigcup_{i,k} g_{i,k} D_j$. Since D_j has finite index in D , this cover has a finite subcover by $\{g_{i,k} D_j : i \in I\}$ with I a finite set. In particular, this implies that D is covered by $\{D_i : i \in I\} \cup \{D_j\}$, which is a finite subcover of our original cover $D = \bigcup_i D_i$. $\square$

Problem 11. *Investigate other conditions which, along with ordered type and properly suspended, force a group G to be locally level-compact.*

For the remainder of this section, we work with a group G levelled over X that is properly suspended and of ordered type.

We now come to the generalisation of Proposition 2.4.9.

Proposition 3.5.7. *If $D \subset G$ is a distinguished set such that $D = \bigcup_{r=1}^n D_r$ is a disjoint union of finitely many distinguished sets D_r then at least one of the indices $|D : D_r|$ is finite.*

Proof. First of all, note that we may order the D_r as follows. If there is some $g \in G$ such that $gD_x \subset D_y$ then we say $D_x \leq D_y$. Since G is of ordered type, this is a total ordering on the set of all D_r , and by relabelling if necessary we may assume that $D_1 \leq D_2 \leq \dots \leq D_n$. For $1 \leq i < n$, taking $g_i \in G$ such that $D_i \subset g_i D_n$ and letting $g_n = 1$ gives $\bigcup_{r=1}^n g_r D_n \supset \bigcup_{r=1}^n D_r = D$, which shows that $|D : D_n|$ is finite. $\square$

Proposition 3.5.8. *If G is of ordered type and $D \subset G$ is a distinguished set such that $D = \bigcup_{r=1}^n D_r$ is a disjoint union of finitely many distinguished sets D_r then the indices $|D : D_r|$ are all finite.*

Proof. By Proposition 3.5.7, at least one of the indices is finite, and so by relabelling if necessary we may assume that $|D : D_1|$ is finite. Since G is of ordered type, we may take a complete system of representatives $S = \{h_1 = I_2, h_2, \dots, h_m\}$, so that $D = \bigcup_i h_i D_1$. We thus have

$$D \setminus D_1 = \bigcup_{r=2}^n D_r = \bigcup_{s=2}^m h_s D_1.$$

Taking the intersection with any D_r , this gives

$$D_r = \bigcup_{s=2}^m (D_r \cap h_s D_1).$$

For each $r > 1$, let $S_r = \{h \in S : D_r \cap hD_1 \neq \emptyset\}$. Each S_r is nonempty, since $\bigcup_{h \in S_r} (D_r \cap hD_1) = D_r$. Since G is of ordered type, $D_r \cap hD_1$ is thus equal to either D_r or hD_1 for any $h \in S_r$.

If $D_r \cap hD_1 = D_r$ for any h then we must have $S_r = \{h\}$ and $D_r \subset hD_1$. But since $D \setminus D_1 = \bigcup_{r=2}^n D_r$, we have $hD_1 = \bigcup_{r \in R} D_r$, where R is a subset of $\{2, \dots, n\}$. In particular, we have a distinguished union of disjoint distinguished sets of shorter length, and so by induction each index $|hD_1 : D_r|$ is

finite. The tower law for indices then implies that $|D : D_r| = |D : D_1| \cdot |D_1 : D_r|$ is finite.

On the other hand, if $D_r \cap hD_1 = hD_1$ for all $h \in S_r$ then we have $hD_1 \subset D_r$. By Corollary 3.5.4 we thus have $|D : D_1| = |D : hD_1| = |D : D_r| \cdot |D_r : hD_1|$, hence $|D : D_r| \leq |D : D_1|$ is finite. $\square$

Corollary 3.5.9. *If G is properly suspended and of ordered type and $D \subset G$ is a distinguished set such that $D = \bigcup_{r=1}^n D_r$ is a disjoint union of finitely many distinguished sets D_r then $\text{lv}(D) = \text{lv}(D_r)$ for all r .*

Proof. Immediate from the above Proposition and the definition of properly suspended. $\square$

We now generalise the idea of refinements to our current setting. As in the case of $\text{GL}_2(F)$, the following definition is convenient.

Definition 3.5.10. *Let*

$$A = \bigcup_i \left(\bigcup_j C_{i,j} \setminus \bigcup_k D_{i,k} \right)$$

be a ddd-set. The components $C_{i,j}$ are called the big shells, and the components $D_{i,k}$ are called the small shells.

We now define refinements in the more compact formulation using this terminology.

Definition 3.5.11. *Let $A \subset G$ be a ddd-set. A refinement of A is a ddd-set $\tilde{A}$ such that $A = \tilde{A}$ as sets, every big shell of A is a big shell of $\tilde{A}$, and every small shell of A is a small shell of $\tilde{A}$.*

We also recall the definition of a ddd-set to be reduced.

Definition 3.5.12. *Let*

$$A = \bigcup_i \left(\bigcup_j C_{i,j} \setminus \bigcup_k D_{i,k} \right)$$

be a ddd-set. We call A reduced if it does not contain any dd-components of the form $B \setminus B$ for a distinguished set B .

As in the particular case of $\mathrm{GL}_2(F)$, the property of being reduced depends the specific presentation of a given ddd-set, and given a non-reduced ddd-set A we may remove all of the superfluous components to obtain a reduced ddd-set A_{red} .

Again, we have the following important result concerning refinements.

Theorem 3.5.13. *Let A and A' be reduced ddd-sets with $A = A'$ as sets. Then there exists a reduced ddd-set $\tilde{A}$ which is a refinement of both A and A' .*

The proof is exactly the same as that of Theorem 2.5.4, but with Lemma 2.4.3 replaced by G being of ordered type, and so we do not reproduce it again here. In particular, we have exactly the same algorithm for constructing a refinement of a ddd-set D .

Problem 12. *Extend the construction of refinements to groups such as $\mathbb{R}((t))$ and $\mathrm{GL}_2(\mathbb{R}((t)))$ which are not of ordered type. Determine how far this construction may be extended, and to what extent it facilitates the definition of an invariant measure.*

We now turn to the definition of an invariant measure on a general properly suspended group G of ordered type. Since G is levelled over a locally compact topological group X , in particular X is equipped with a Haar measure μ_X , which is unique up to multiplication by a real constant.

For what follows we will require the following convenient notation. For $\gamma = (\gamma_1, \dots, \gamma_e) \in \mathbb{Z}^e$, we will write Y^γ to denote the element $Y_1^{\gamma_1} \dots Y_e^{\gamma_e} \in \mathbb{R}((Y_1)) \dots ((Y_e))$. We will also denote the latter group simply by $\mathbb{R}((Y))$ for convenience.

As for the cases of F and $\mathrm{GL}_2(F)$, we wish to define a finitely additive, $\mathbb{R}((Y))$ -valued left-invariant measure on the family of ddd-sets of G . Since the framework of refinements already transfers conveniently to our current situation, this essentially amounts to defining such a measure on distinguished sets.

Let $G_{U,\gamma} \subset G$ be a distinguished set. As far as possible, we would like the measure μ we will define on G to be compatible with the Haar measure

μ_X on X . Since μ_X is unique only up to a real constant, our μ will (at best!) also be unique only up to the same scaling. The measure we will define will also be intimately tied to the indexing of particular distinguished sets, and so we will further assume that G is rigid and that the level map $G\mathcal{L} \rightarrow \mathbb{Z}^e$ is surjective.

To begin with, since we want μ to be left-invariant, we impose the condition $\mu(gG_{U,\gamma}) = \mu(G_{U,\gamma})$ for all $g \in G$. For compatibility with μ_X , we then set $\mu(G_{U,\gamma}) = \mu_X(U)Y^\gamma$. However, we must check that this definition (which essentially comes from the level of the base X) agrees with the computation of indices at the level of G .

Since G is properly suspended, $G_{U,\gamma} \subset G_{V,\delta}$ will have finite index if and only if $\gamma = \delta$, in which case $U \subset V$ also. In particular, our definition of μ does not cause any issues between different levels, and so we only have to consider one particular level γ .

Since G is of ordered type, we have seen previously that for $G_{U,\gamma} \subset G_{V,\gamma}$, the index $|G_{V,\gamma} : G_{U,\gamma}|$ is finite, and that we may in fact write $G_{V,\gamma}$ as a disjoint union of finitely many (i.e. equal to the index) G -translates of $G_{U,\gamma}$. In order for μ to be finitely additive, we thus require that

$$\mu(G_{V,\gamma}) = |G_{V,\gamma} : G_{U,\gamma}| \mu(G_{U,\gamma}).$$

In terms of our tentative definition of μ , this requires the following definition to hold.

Definition 3.5.14. *Let G levelled over X be properly suspended, rigid, and of ordered type. Then G is called compatible if, for every $U \subset V$, $|G_{V,\gamma} : G_{U,\gamma}|$ is constant as γ varies throughout $\mathbb{Z}^e$.*

Assuming that G is compatible, we may write the above equality of measures in terms of μ_X :

$$\mu_X(V) = |G_{V,\gamma} : G_{U,\gamma}| \mu_X(U).$$

This proves the following.

Proposition 3.5.15. *Suppose G levelled over X is compatible, and that the equality*

$$\mu_X(V) = |G_{V,\gamma} : G_{U,\gamma}| \mu_X(U)$$

holds for every $U \subset V \in \mathcal{U}(1)$. Then any left-invariant, finitely additive measure on G must take the form

$$\mu : gG_{U,\gamma} \mapsto \mu_X(U)Y^\gamma$$

on distinguished sets.

Remark. As in the case of $\mathrm{GL}_2(F)$ in the previous chapter, we have not yet shown that this map is a measure; the above Proposition merely states that any possible measure must be of this form.

Theorem 3.5.16. *Suppose G levelled over X is compatible, and that the equality*

$$\mu_X(V) = |G_{V,\gamma} : G_{U,\gamma}| \mu_X(U)$$

holds for every $U \subset V \in \mathcal{U}(1)$. Let D be a ddd-set, and let $\tilde{D}$ be a refinement of D . The map

$$\mu : gG_{U,\gamma} \mapsto \mu_X(U)Y^\gamma,$$

when extended to $\mathcal{R}$ by additivity, satisfies $\mu(D) = \mu(\tilde{D})$, and hence is a well-defined, finitely additive, left-invariant measure compatible with the Haar measure μ_X on X .

Proof. The proof that μ is well-defined on refinements is again exactly the same as the proof of Proposition 2.6.2, since this uses only the definition of ddd-sets and the algorithm of Theorem 2.5.4 (which we have in our setting via Theorem 3.5.13). The argument of Theorem 2.6.3, combined with the earlier computations in this section, then completes the proof. $\square$

Remark. All of the constructions in this section (and similarly, the constructions in the previous chapter) could have equally been used to define instead a right-invariant measure by replacing $gG_{U,\gamma}$ everywhere with $G_{U,\gamma}g$. Corollary 2.7.7 states that the group $\mathrm{GL}_2(F)$ is unimodular, i.e. that the left- and right-invariant measures coincide, but this is not necessarily true for more general G (it is not even true in general for the Haar measure, for instance).

Example. Let F be an n -dimensional nonarchimedean local field, and let F_1 be the $(n-1)^{\text{st}}$ residue field, which is a local field. The level structure $F_{t_1^{\gamma_1} \mathcal{O}_{F_1}(\gamma_2, \dots, \gamma_n)} = t_1^{\gamma_1} \dots t_n^{\gamma_n} \mathcal{O}_F$, where $\mathcal{O}_{F_1}$ is the ring of integers of F_1 and $\mathcal{O}_F$ is the rank n ring of integers of F , makes F a compatible group. Furthermore, the Haar measure μ_{F_1} on F_1 such that $\mu_{F_1}(\mathcal{O}_{F_1}) = 1$ satisfies the required formula $\mu_{F_1}(V) = |G_{V,\gamma} : G_{U,\gamma}| \mu_{F_1}(U)$, and so the map

$$\mu(\alpha + t_1^{\gamma_1} \dots t_n^{\gamma_n} \mathcal{O}_F) = q^{-\gamma_1} Y_2^{\gamma_2} \dots Y_n^{\gamma_n}$$

extends to Fesenko's measure on ddd-sets.

Remark. We may also apply the results of this section to only a partial level structure. For example, we may consider the subgroups $K_{i,j}$ of $\text{GL}_2(F)$ (as in Chapter 2) with $i, j > 0$ as forming part of a level structure over $\text{GL}_2(\bar{F})$, with $K_{i,j}$ corresponding to the pair (K_i, j) . This does not define a level structure on $\text{GL}_2(F)$ as per our definition since, for example, there is no distinguished set corresponding to $(K_1, -1)$. However as we saw in Chapter 2 one may define a measure on the ddd-sets generated by the $K_{i,j}$, and this coincides with the measure above wherever both are defined.

Problem 13. Extend $\{K_{i,j} : i, j > 0\}$ to a full level structure for $\text{GL}_2(F)$ over $\text{GL}_2(\bar{F})$.

We note in particular the following example, which directly generalises the content of Chapter 2

Theorem 3.5.17. Let F be an n -dimensional nonarchimedean local field, and let $G = \text{GL}_m(F)$. For $\gamma_1 \in \mathbb{Z}$, $\gamma = (\gamma_2, \dots, \gamma_n) \in \mathbb{Z}^{n-1}$, such that $(\gamma_1, \dots, \gamma_n) > (0, \dots, 0)$, we may consider the subgroups

$$K_{\gamma_1, \dots, \gamma_n} = I_m + t_1^{\gamma_1} \dots t_n^{\gamma_n} M_m(\mathcal{O}_F),$$

where $\mathcal{O}_F$ is the rank n ring of integers of F . The association

$$(K_{\gamma_1}, \gamma) \mapsto K_{\gamma_1, \dots, \gamma_n},$$

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equips G with a partial level structure over $\mathrm{GL}_m(F_1)$ of elevation $(n-1)$, where F_1 is the $(n-1)^{\mathrm{st}}$ residue field of F . The map

$$\mu(K_{\gamma_1, \dots, \gamma_n}) = \frac{q^{\frac{1}{2}n(n+1)}}{(q^n - 1)(q^{n-1} - 1) \dots (q - 1)} q^{-n^2 \gamma_1} Y_2^{\gamma_2} \dots Y_n^{\gamma_n} = \mu_{\mathrm{GL}_m(F_1)}(K_m) Y^\gamma$$

is a left-invariant, finitely additive measure on the ddd-sets generated by the $K_{\gamma_1, \dots, \gamma_n}$ which satisfies $\mu(\mathrm{GL}_m(O_F)) = 1$.

Remark. Setting $Y_k = X_k^{n^2}$, one obtains in a similar fashion to Theorem 2.7.5 a nice compatibility between the measure on G and the measure on F . More precisely, for $g = (g_{r,s}) \in G$ we have the equality of differentials

$$dg = |\det g|_F^{-n} \prod dg_{r,s}.$$

Problem 14. Following the constructions in Chapter 2, construct an invariant measure on groups G which are not necessarily compatible. Show that this will agree with the definition in Theorem 3.5.16 in the compatible case, but in general will not be compatible with the Haar measure on X .

Problem 15. Prove or disprove the following conjecture: any finite union of subsets of type S of a (compatible) group G is also of type S . If this conjecture is indeed true, it also makes sense to extend the measure by defining all subsets of type S to have zero volume.

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We now return to the general situation and consider how we can transfer a level structure from one group to another. Let G be levelled over X , and let H be a subgroup of G . One can define a level structure on H over X in a canonical way.

Definition 3.6.1. If G has a level structure $\mathcal{L}$ over X , and H is a subgroup of G , then the collection $\mathcal{L}_H = \{H_{U,\gamma} = H \cap G_{U,\gamma} : G_{U,\gamma} \in \mathcal{L}\}$ of subsets of H is called the induced level structure.

Lemma 3.6.2. *The collection $\mathcal{L}_H$ is a level structure for H over X .*

Proof. Property (1) is satisfied since H is a subgroup of G , and property (2) holds by construction. Properties (3) and (4) follow from the associativity and distributivity properties of union and intersection. $\square$

It is clear from the definition that the level topology on H given by $\mathcal{L}_H$ coincides with the topology induced on H as a subspace of G .

Up until this point, it would have been possible to restrict attention only to spaces where the level map $G\mathcal{L} \rightarrow \mathbb{Z}^e$ is surjective (and in fact in the previous section we did so at various points). However, if we want induces level structures to make sense in general it is important that we do not make this assumption. For example, if H is a subgroup of G of level δ , the induced level structure on H contains no sets of level smaller than δ .

Proposition 3.6.3. *If G is rigid and (locally) level-compact and H is a closed subgroup of G with the induced level structure such that $G \setminus H$ has a level in G then H is (locally) level-compact.*

Proof. This follows immediately from Proposition 3.3.1. $\square$

Corollary 3.6.4. *Any algebraic group G over a higher dimensional local field F which is closed in $\mathrm{GL}_n(F)$ such that $\mathrm{GL}_n(F) \setminus G$ has a level is locally level-compact.*

In the case of algebraic subgroups, one may in fact consider an induced (partial) level structure over a more appropriate base than the original one.

Proposition 3.6.5. *Suppose $G = \mathrm{GL}_m(F)$ for an n -dimensional nonarchimedean field F , with the partial level structure over $\mathrm{GL}_m(F_1)$ given by the distinguished subgroups $K_{\gamma_1, \dots, \gamma_n}$. Let H be a subgroup of G defined by finitely many polynomial equations $f_1, \dots, f_k \in O_F[[X_1, \dots, X_{m^2}]]$, i.e.*

$$H = \{(g_{r,s}) \in G : f_i((g_{r,s})) = 0, 1 \leq i \leq k\},$$

and let $\bar{H}$ be the subgroup of $\mathrm{GL}_m(F_1)$ defined by the reductions $\bar{f}_1, \dots, \bar{f}_k \in F_1[[X_1, \dots, X_{m^2}]]$. If all of the polynomials $\bar{f}_i$ are separable (i.e. all the roots are

simple), the association $(U \cap \overline{H}, \gamma) \mapsto H \cap G_{U, \gamma}$ defines a (partial) level structure for H over $\overline{H}$ whose distinguished sets coincide with those of the induced level structure.

Proof. We must first check that the association is well-defined. In other words, if $\overline{H} \cap K_\alpha = \overline{H} \cap K_\beta$ for $\alpha, \beta \in \mathbb{Z}$ we must make sure that $H \cap K_{\alpha, \gamma} = H \cap K_{\beta, \gamma}$ for all $\gamma \in \mathbb{Z}^{n-1}$. Without loss of generality, we may assume that $\beta \leq \alpha$, so that we have $H \cap K_{\alpha, \gamma} \subset H \cap K_{\beta, \gamma}$.

Let $g = (g_{r,s}) \in H \cap K_{\beta, 0}$. Then the reduction $\bar{g} = (\bar{g}_{r,s}) \in K_\beta$ is a root of all of the polynomials $\bar{f}_i$, and so we have $\bar{g} \in \overline{H} \cap K_\beta = \overline{H} \cap K_\alpha$. By Hensel's Lemma, there exists a unique $g' \in K_{\beta, 0}$ with $\bar{g}' = \bar{g}$ and $f_i(g') = 0$, and there exists a unique $g'' \in K_{\alpha, 0}$ with $\bar{g}'' = \bar{g}$ and $f_i(g'') = 0$. Since $f_i(g) = 0$, uniqueness of g' forces $g = g'$. Furthermore, since $K_{\alpha, 0} \subset K_{\beta, 0}$, we have $g'' \in K_{\beta, 0}$, and so uniqueness of g' forces also $g' = g''$. We thus have $g = g'' \in H \cap K_{\alpha, 0}$ and so we have the required equality for $\gamma = 0$. The result for general γ then follows from the fact that the map $I_m + t_1^{\gamma_1} \dots t_n^{\gamma_n} M \mapsto I_m + t_1^{\gamma_1} M$ is an isomorphism $K_{\gamma_1, \gamma} \rightarrow K_{\gamma_1, 0}$.

The fact that the distinguished sets coincide with those of the induced level structure is then immediate from the definition, and this implies that conditions (1) and (2) in the definition of a level structure hold. It remains to check (3) and (4).

Let $U, V \in \mathcal{U}(1)$ and $\gamma \leq \delta \in \mathbb{Z}^{n-1}$. We have

$$G_{V \cap \overline{H}, \gamma} \cap G_{U \cap \overline{H}, \delta} = (H \cap G_{V, \gamma}) \cap (H \cap G_{U, \delta}) = H \cap G_{V, \delta} = G_{V \cap \overline{H}, \delta},$$

and so (3) is satisfied.

Similarly,

$$\begin{aligned} G_{U \cap \overline{H}, \gamma} \cup G_{V \cap \overline{H}, \gamma} &= (H \cap G_{U, \gamma}) \cup (H \cap G_{V, \gamma}) \\ &= H \cap (G_{U, \gamma} \cup G_{V, \gamma}) \\ &= H \cap G_{U \cup V, \gamma} = G_{(U \cup V) \cap \overline{H}, \gamma} \end{aligned}$$

and

$$\begin{aligned}
 G_{U \cap \bar{H}, \gamma} \cap G_{V \cap \bar{H}, \gamma} &= (H \cap G_{U, \gamma}) \cap (H \cap G_{V, \gamma}) \\
 &= H \cap (G_{U, \gamma} \cap G_{V, \gamma}) \\
 &= H \cap G_{U \cap V, \gamma} \\
 &= G_{(U \cap V) \cap \bar{H}, \gamma},
 \end{aligned}$$

and so (4) is satisfied. $\square$

In particular, we have the following.

Theorem 3.6.6. *Let G be an algebraic subgroup of $\mathrm{GL}_m(F)$ defined by the equations $f_1, \dots, f_k$, and let $\bar{G}$ be the subgroup of $\mathrm{GL}_m(F_1)$ defined by the reductions $\bar{f}_1, \dots, \bar{f}_k$. If G is properly suspended and rigid over $\bar{G}$ with respect to the induced (partial) level structure, if $|K_{i, \gamma} \cap G : K_{j, \gamma} \cap G| = |K_i \cap \bar{G} : K_j \cap \bar{G}|$ for all $0 < i \leq j$ and all $\gamma > 0 \in \mathbb{Z}^{n-1}$, and if the polynomials $\bar{f}_1, \dots, \bar{f}_k$ have no multiple roots, there is a left-invariant, finitely additive measure on G given by*

$$\mu(K_{i, \gamma} \cap G) = \bar{\mu}(K_i \cap \bar{G}) Y^\gamma,$$

where $\bar{\mu}$ is the Haar measure on $\bar{G}$.

Proof. The equality $|K_{i, \gamma} \cap G : K_{j, \gamma} \cap G| = |K_i \cap \bar{G} : K_j \cap \bar{G}|$ is exactly the one that is required for the measure to exist in the previous section under the conditions of being properly suspended, rigid (which are both true for G by assumption), and of ordered type (which is true of G as a subgroup of a group of ordered type). $\square$

Once we already know examples where this Theorem holds, one may then use the following Lemma to easily produce further examples.

Lemma 3.6.7. *Let*

$$1 \rightarrow N \rightarrow G \rightarrow Q \rightarrow 1$$

be a short exact sequence of algebraic subgroups of $\mathrm{GL}_m(F)$. If any two of N, G, Q satisfy the conditions of Theorem 3.6.6, then so does the third.

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Proof. Write $N_{i,\gamma} = K_{i,\gamma} \cap N$, $G_{i,\gamma} = K_{i,\gamma} \cap G$, $Q_{i,\gamma} = K_{i,\gamma} \cap Q$, and similarly write $N_i = K_i \cap N$, $G_i = K_i \cap G$, $Q_i = K_i \cap Q$. For $i < \ell$, applying the snake lemma to the commutative diagrams

$$\begin{array}{ccccccc} 1 & \longrightarrow & N_\ell & \longrightarrow & G_\ell & \longrightarrow & Q_\ell \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & N_i & \longrightarrow & G_i & \longrightarrow & Q_i \longrightarrow 1 \end{array}$$

and

$$\begin{array}{ccccccc} 1 & \longrightarrow & N_{\ell,\gamma} & \longrightarrow & G_{\ell,\gamma} & \longrightarrow & Q_{\ell,\gamma} \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & N_{i,\gamma} & \longrightarrow & G_{i,\gamma} & \longrightarrow & Q_{i,\gamma} \longrightarrow 1 \end{array}$$

gives the short exact sequences of cokernels

$$1 \rightarrow N_i/N_\ell \rightarrow G_i/G_\ell \rightarrow Q_i/Q_\ell \rightarrow 1,$$

$$1 \rightarrow N_{i,\gamma}/N_{\ell,\gamma} \rightarrow G_{i,\gamma}/G_{\ell,\gamma} \rightarrow Q_{i,\gamma}/Q_{\ell,\gamma} \rightarrow 1.$$

In particular, we have the equalities of indices

$$|G_i : G_\ell| = |N_i : N_\ell| \cdot |Q_i : Q_\ell|$$

$$|G_{i,\gamma} : G_{\ell,\gamma}| = |N_{i,\gamma} : N_{\ell,\gamma}| \cdot |Q_{i,\gamma} : Q_{\ell,\gamma}|,$$

and so if any two of N, G, Q satisfy the required index equalities of Theorem 3.6.6 then so must the third. $\square$

As a first example, we obtain an invariant measure on the group $SL_m(F)$.

Theorem 3.6.8. *Let F be a two-dimensional local field and let $G = SL_m(F)$. For $(i, j) > (0, 0)$ let $\widetilde{K}_{i,j} = K_{i,j} \cap G$. The map*

$$\mu(g\widetilde{K}_{i,j}) = \lambda q^{-(m^2-1)i} X^{(m^2-1)j}$$

for $\lambda \in \mathbb{R}((X))^\times$ is a left-invariant, finitely additive measure on G .

Proof. Here the required polynomial is simply $\det -1$, and we may apply Lemma 3.6.7 to the short exact sequence

$$1 \rightarrow F^\times \rightarrow GL_m(F) \rightarrow SL_m(F) \rightarrow 1$$

to deduce the existence of an invariant measure μ_{SL_m} on $SL_m(F)$. To see that it may be written in the form $\mu_{SL_m}(g\widetilde{K}_{i,j}) = \lambda q^{-(m^2-1)i} X^{(m^2-1)j}$, note that the proof of Lemma 3.6.7 implies that we may organise so that

$$\mu_Q(D) = \frac{\mu_G(D)}{\mu_N(D)}$$

for any distinguished set D . □

Example. Recall that the groups $\mathbf{A}_S$ and $\mathbb{A}_S$, of geometric and analytic adeles associated to a surface S arise as restricted products of two-dimensional local fields. Since the functor $\mathrm{GL}_2(\cdot)$ behaves well with respect to (restricted) products, we may consider the groups $\mathrm{GL}_2(\mathbf{A}_S)$ and $\mathrm{GL}_2(\mathbb{A}_S)$, along with their algebraic subgroups with the induced level structure. Since the formula

$$\mu_X(V) = |G_{V,\gamma} : G_{U,\gamma}| \mu_X(U)$$

holds at each local component, and since all but finitely many will be equal to 1, it in fact holds globally. There thus exists a left-invariant measure on any properly suspended, rigid adelic algebraic group G defined by polynomials with separable reduction.

Remark. In [Fes10], Fesenko shows that it is not possible to define a measure on the geometric adeles $\mathbf{A}_S$ which is both compatible with the one-dimensional theory (for example, in the sense of the above example) and satisfies the important property $\mu(\mathcal{O}\mathbf{A}_S) = 1$ for an appropriate integral structure $\mathcal{O}\mathbf{A}_S$. Indeed, this is one of the major motivations for the introduction of the analytic adelic structures.

Problem 16. *For an algebraic group G over a two dimensional field F , or for a two-dimensional adelic algebraic group, determine conditions which give "compatibility with the measure on F " (respectively the measure on $\mathbf{A}_S$ or $\mathbb{A}_S$), in the sense of Theorem 2.7.5. The dimension of G as an affine variety is expected to appear (for example, for GL_2 the required indeterminate takes the form X^4 , and for SL_2 it takes the form X^3).*

Remark. We have seen that the existence of an invariant measure on a group G which is compatible with the measure on X heavily depends on the choice of the base space X (as one would expect!). However, once defined, the measure can be seen to be intrinsic to G ; if we afterwards consider G with the same level structure but over a different base X' , the measure on G still exists, it just may not have anything to do with the space X' . (Consider, for example, $\mathrm{GL}_2(F)$ levelled over $\mathrm{GL}_2(\bar{F})$ versus $\mathrm{GL}_2(F)$ levelled over F .)

This seems to be a very common theme concerning groups with level structure. While the topological and analytic properties of G should be essentially self-contained (since the information is bound to the level structure, which is in principle just a collection of subsets of G), by choosing the correct base X for the level structure we may see various analogies between G and X which makes certain properties of G appear more clearly.

One may, for example, utilise the induced level structures to study the above remark. If we consider G with the same level structure over two different bases X and X' , we may take the product to obtain a level structure for $G \times G$ over $X \times X'$. The induced level structure on the diagonal image of G inside $G \times G$ will then be related to both X and X' .

To end this section, we return briefly to the problem of degeneracy for induced level structures. Consider the following example.

Example. Consider $G = \mathbb{Q}_p((t))$ with its usual level structure of elevation $e = 1$ over $\mathbb{Q}_p$. We consider the induced level structure on $H = \mathbb{Q}_p$. For $G_{i,j} = p^i t^j \mathbb{Z}_p + t^{j+1} \mathbb{Q}_p[[t]] \in \mathcal{L}$ we have $H_{i,j} = 0$ for $j > 0$, $H_{i,0} = p^i \mathbb{Z}_p$, and $H_{i,j} = \mathbb{Q}_p$ for $j < 0$. One sees at once that the induced level topology coincides with the p -adic topology, and that the level structure has essentially fallen away completely. In this example it is indeed true that j -compactness for all $j \in \mathbb{Z}$ implies compactness, since one only needs to check the $j = 0$ case.

In the above example we see that (in a certain sense) the elevation may drop when inducing a level structure on a subgroup. To make this more precise, we first consider how to inflate the elevation.

Proposition 3.6.9. *Let G be levelled over X with elevation e , and let $\mathcal{L}$ be the level structure. Then for any $w \in \mathbb{Z}$ the collection $\mathcal{L}'$ of subsets of G indexed by $\mathcal{U}(1) \times \mathbb{Z}^{e+1} \simeq \mathcal{U}(1) \times \mathbb{Z}^e \times \mathbb{Z}$ given by $G_{U,\gamma,j} = G$ for $j < w$, $G_{U,\gamma,w} = G_{U,\gamma}$, and $G_{U,\gamma,j} = \{e_G\}$ for $j > w$ defines a level structure for G over X of elevation $e + 1$.*

Proof. It follows immediately from the definitions and the fact that we order lexicographically from the right. $\square$

By iterating the above procedure we may inflate the elevation from e to $e + n$ for any positive integer n . While inflation may not produce anything particularly interesting, it does allow us to be precise about saying that the level structure on $\mathbb{Q}_p$ induced from $\mathbb{Q}_p((t))$ is essentially of elevation 0. It also allows us to take products of groups with different elevations, since we may inflate everywhere to the maximal elevation among the components.

3.7 ASIDE: A FEW PARTICULARS AND GENERALITIES

In this section we will briefly discuss a few ideas that aren't necessarily important for the overall theme of the text, but may nonetheless be interesting to think about.

First, the reader will surely have noted that, aside from in Section 3.5, we do not use anywhere the fact that the base space X of a group with level structure is locally compact. Moreover, the property we use in Section 3.5 is not the local compactness per se, but the existence of a (not necessarily real-valued!) Haar measure on X .

One may thus relax the requirements on the base X to the following much less stringent conditions: X must be a pointed set, and the set must come equipped with a collection $\mathcal{U}(1)$ of subsets which contain the distinguished point of X and satisfies the defining condition to be a "basis of neighbourhoods" at this point. (The set X itself need not be a topological space, it just needs to look like one around the distinguished point.)

One possibly important example is to take X to be a group levelled over a second space X' .

Lemma 3.7.1. *Let X have level structure $\mathcal{L}_X$ over X' with elevation e_1 , and let G be levelled over X with elevation e_2 and level structure $\mathcal{L}_G$ indexed by $\mathcal{L}_X \times \mathbb{Z}^{e_2}$. Then G is levelled over X' with elevation $e_1 + e_2$ and level structure $\mathcal{L}_G$.*

Proof. To an open neighbourhood U of 1 in X' and to $(\gamma, \delta) \in \mathbb{Z}^{e_1+e_2}$ with $\gamma \in \mathbb{Z}^{e_1}$ and $\delta \in \mathbb{Z}^{e_2}$ we associate the distinguished set $G_{U,(\gamma,\delta)} = G_{X_{U,\gamma},\delta}$. The fact that this defines a level structure immediately follows from the fact that $\mathcal{L}_X$ and $\mathcal{L}_G$ are both level structures. $\square$

Problem 17. *Verify that the results of Section 3.5 hold for a group G levelled over X such that X has an invariant, finitely additive (not necessarily real-valued) measure. In particular, the measure μ on a higher-dimensional local field F should be compatible with the measure on each successive residue field.*

Next, we would like to consider further the influence of the base space. (Or, more precisely, how much information about G we can extract from the base.) We noted earlier that, while various properties (such as the existence of an invariant measure) are intrinsic to G , a convenient choice of base can make these properties easier to study.

One possibility in this direction (for which the author thanks Tom Oliver for the suggestion!) is the following. Consider the two-dimensional local field $F = \mathbb{Q}_p((t))$. We obtain the natural level structure of F over the residue field $\bar{F} = \mathbb{Q}_p$ via the association $(p^i \mathbb{Z}_p, j) \mapsto p^i t^j O_F$. Alternatively, we obtain the same level structure on F over the field $\mathbb{F}_p((t))$ via the association $(t^j \mathbb{F}_p[[t]], i) \mapsto p^i t^j O_F$. Depending on how much information may be "shared" between a group and its base, it may thus be possible to obtain new analogies between the two different local fields $\mathbb{Q}_p$ and $\mathbb{F}_p((t))$. Alternatively, by going up one more level to a three-dimensional field, it may be interesting to investigate possible analogies between two-dimensional fields that arise in a similar way.

Finally, we conclude this chapter with a short remark on the special case where the elevation $e = 1$. In this case, the indexing group is simply $\mathbb{Z}$ (or, if we only take a partial level structure, a subset of this). In particular, we have the following elementary Lemma which may have some interesting consequences.

Lemma 3.7.2. *Any sequence in $\mathbb{Z}$ which is bounded below has a minimum element.*

We have already noted this previously in passing, when we remarked that no subsets of type E can exist in the case $e = 1$ due to this property when the level map is surjective. Clearly the Lemma is not true for higher powers of $\mathbb{Z}$: the sequence $(-n, 0)_{n \geq 0}$ is bounded below by $(0, -1)$ since we order from the right, but it is strictly decreasing.

Problem 18. *Investigate how far one may refine the results of Chapter 3 in the elevation $e = 1$ case using elementary properties of $\mathbb{Z}$ such as Lemma 3.7.2.*

The above problem may have important consequences regarding the difference between arithmetic geometry in dimension 2 ($e = 1$ case) and in dimensions ≥ 3 ($e \geq 2$ case).

THE LEVEL 0 PRINCIPAL SERIES IN DIMENSION TWO

In this chapter we study the level 0 part of the principal series representations of $GL_2(F)$ with F a two-dimensional local field, following the classical one-dimensional approach. We study the space of flat 0-sections, and compute their image under Jacquet's map when the ramification is relatively nice, and for all types of ramification in most cases.

The principal series arise as representations which are induced on the group $GL_2(F)$ from a representation of the Borel subgroup of upper triangular matrices given by a pair of characters on the multiplicative group F . In the case where F is a local field, the principal series representations are irreducible outside particular special cases, and the irreducible admissible (infinite dimensional) representations of $GL_2(F)$ are given by the irreducible principal series, a closely related representation in the aforementioned special cases, and the supercuspidal representations.

For F a two-dimensional local field, the principal series representations are not so easy to study as in dimension one. Indeed, if one attempts to work with the full principal series, one almost surely arrives at all manner of roadblocks, such as doubly infinite series and integrals which do not converge.

To get around this issue, one may study instead various subrepresentations which are more amenable to study using classical techniques. In the current text, we split up the two-dimensional principal series by their "level", and focus most of our attention to the level 0 case where, for the most part,

we may proceed similarly to the one-dimensional situation. Nevertheless, we attempt to make our constructions in such a way that they should easily carry over to the higher level situation.

Until the turn of the century, another obstruction to the two-dimensional theory was the lack of harmonic analysis for two-dimensional local fields. Thankfully, Fesenko's constructions (as discussed in Chapter 2) suffice for the current text, and the measure on $GL_2(F)$ (as also discussed in Chapter 2) will also be important for the study of supercuspidal representations in higher dimensions.

Beginning in 4.1 by recalling a few important notions from earlier chapters, we start off slowly with a few elementary matrix decomposition theorems in 4.2, and a brief discussion of the theory of characters on the multiplicative group $F^\times$ in 4.3.

The main part of the chapter then begins in Section 4.4, where we define the principal series representations of $GL_2(F)$ as representations induced by a pair of multiplicative characters acting on the Borel subgroup of upper triangular matrices. Since the full principal series representation is difficult to access, we also define the principal series of level j for $j \geq 0$. For the remainder of the text we mostly focus on the easiest case $j = 0$. In Section 4.5 we then study the flat sections, in particular those of level zero, for which we can exhibit bases for various spaces similarly to the one-dimensional case.

Finally, in Section 4.6 we compute the image of the level zero principal series under the two-dimensional version of Jacquet's map, which is given by the integral

$$(J_{\alpha,\beta,\omega}(f))(g) = \int_F f \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} g \right) \psi_F(-u) du.$$

In contrast to the theory-heavy Chapters 2 and 3, in this chapter we will mostly proceed by making various computations. While there will still be several definitions (especially in the preliminary sections), many of these are simply generalisations of classical ones.

On the other hand, the content of this chapter barely scratches the surface of what can be done in the way of representation theory using the structures studied in the earlier parts of the text. It is expected that the relationship between harmonic analysis and representation theory in higher dimensions should have many similarities to the one-dimensional case.

4.1 PREREQUISITES

Let F be a two-dimensional local field with rank two ring of integers O_F and local parameters t_1, t_2 . We will use the following notation throughout: $G(\cdot)$ is the group $\mathrm{GL}_2(\cdot)$, $B(\cdot)$ the Borel subgroup of upper triangular matrices. For the case of F itself we will simply write $G = \mathrm{GL}_2(F)$ and $B = B(F)$. Write $K = G(O_F)$, and $K_{i,j}$ for the subgroup of K consisting of matrices which reduce to the identity mod $t_1^i t_2^j$ as in the previous chapters.

We consider G with the following topology: a subset $U \subset G$ is open if for every $x \in U$ there is some $g \in G$ and $(i, j) \geq (0, 0)$ with $x \in gK_{i,j} \subset U$. One easily verifies that this coincides with the level topology on G as defined in Chapter 3 (see Definition 3.1.1).

Recall (Definition 3.1.7) that the level $\mathrm{lv}(S)$ of a subset $S \subset G$ is the minimum (if it exists) $\mathrm{lv}(S) = \min\{j \in \mathbb{Z} : gK_{i,j} \subset S \text{ for some } g \in G, i \in \mathbb{Z}\}$. In particular, $\mathrm{lv}(gK_{i,j}) = j$.

Recall also (Definition 3.1.9) that S is of uniform level j if $\mathrm{lv}(S) = j$ and for every $s \in S$ there is an open set U with $s \in U \subset S$ and $\mathrm{lv}(U) = j$. A subset $C \subset G$ is j -compact if every open covering of C by sets of uniform level j has a finite subcover.

Lemma 4.1.1. *The set $gK_{i,j}$ is j -compact.*

Proof. It follows from the fact that $t_1^i t_2^j O_F$ is j -compact, which we proved in Proposition 3.2.6. □

As suggested by the title of this chapter, we will mostly be interested in the level 0 situation. There are several reasons for this restriction, for

instance the 0-compactness of the subgroup K , and the finiteness of certain indices (such as $|K : K_{i,0}|$, as seen in Chapter 2).

4.2 DECOMPOSITION THEOREMS

It will be important to have some knowledge of the structure of G in terms of various subgroups. First of all we have the two-dimensional Iwasawa decomposition.

Lemma 4.2.1. *For every $g \in G$ there is $h \in K$ such that gh is upper triangular.*

Proof. Write

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

If $c = 0$ the assertion is obvious, and if $d = 0$ then we can take

$$h = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

so assume c and d are nonzero. If $cd^{-1} \in O_F$ then

$$g \begin{pmatrix} 1 & 0 \\ -cd^{-1} & 1 \end{pmatrix} = \begin{pmatrix} d^{-1} \det g & b \\ 0 & a \end{pmatrix}.$$

Otherwise $dc^{-1} \in O_F$ since O_F is a valuation ring, and

$$g \begin{pmatrix} -dc^{-1} & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -c^{-1} \det g & a \\ 0 & c \end{pmatrix}.$$

□

Proposition 4.2.2 (2d Iwasawa decomposition). *Every $g \in G$ may be expressed in the form $g = udk$ with u a unipotent upper triangular matrix, d a diagonal matrix whose entries are products of powers of the local parameters t_1, t_2 , and $k \in K$. In particular, g may be written in the form $g = bk$ with $b \in B$ and $k \in K$.*

Proof. By Lemma 4.2.1 $g = bc^{-1}$ with b upper triangular and $c^{-1} \in K$. We may further decompose

$$b = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} = \begin{pmatrix} 1 & yz^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & z \end{pmatrix} = u \begin{pmatrix} x & 0 \\ 0 & z \end{pmatrix}.$$

We may express $x, z \in F$ as uniquely determined products $x = t_1^{i_1} t_2^{j_1} \varepsilon_1$, $z = t_1^{i_2} t_2^{j_2} \varepsilon_2$ with $i_1, i_2, j_1, j_2 \in \mathbb{Z}$ and $\varepsilon_1, \varepsilon_2 \in O_F^\times$. Thus $g = u d_1 d_2 k$, where

$$d_1 = \begin{pmatrix} t_1^{i_1} & 0 \\ 0 & t_1^{i_2} \end{pmatrix}, d_2 = \begin{pmatrix} t_2^{j_1} & 0 \\ 0 & t_2^{j_2} \end{pmatrix}, k = \begin{pmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{pmatrix} P^{-1}.$$

Now suppose

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t_1^{i_1} & 0 \\ 0 & t_1^{i_2} \end{pmatrix} \begin{pmatrix} t_2^{j_1} & 0 \\ 0 & t_2^{j_2} \end{pmatrix} k = \begin{pmatrix} 1 & a' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t_1^{e_1} & 0 \\ 0 & t_1^{e_2} \end{pmatrix} \begin{pmatrix} t_2^{f_1} & 0 \\ 0 & t_2^{f_2} \end{pmatrix} k'.$$

Then

$$K \ni k' k^{-1} = \begin{pmatrix} t_1^{i_1-e_1} t_2^{j_1-f_1} & t_1^{i_2-e_1} t_2^{j_2-f_1} (a-a') \\ 0 & t_1^{i_2-e_2} t_2^{j_2-f_2} \end{pmatrix},$$

thus $t_1^{i_*-e_*} t_2^{j_*-f_*} \in O_F^\times$ for $*$ = 1, 2. Comparing the valuation $v(t_1^{i_*-e_*} t_2^{j_*-f_*}) = (i_* - e_*, j_* - f_*) = (0, 0)$ then implies $i_1 = e_1$, $i_2 = e_2$, $j_1 = f_1$, $j_2 = f_2$, i.e. the diagonal matrices in the decomposition are uniquely determined. $\square$

Remark. This proof is almost identical to the proof given for $F = \mathbb{Q}_p$ in 4.2 of [GH11]. In the case of $\mathbb{Q}_p$, one can impose convenient conditions on u to make the full Iwasawa decomposition unique (and not just the diagonal part). In the two dimensional case this appears to be a much more difficult task, since one has to manipulate doubly infinite series of the form $\sum_{(i,j) \in \Omega} \theta_{i,j} t_1^i t_2^j$ for some appropriate $\Omega \subset \mathbb{Z} \oplus \mathbb{Z}$.

We will also need to consider at various points certain spaces of double cosets. For this, we fix once and for all a set R_i of representatives in O_F of the (finite) quotient $O_F/t_1^i O_F$ for each $i > 0$, taken so that $R_m \subset R_n$ for $m \leq n$.

Proposition 4.2.3. *The elements $\begin{pmatrix} 0 & 1 \\ 1 & d \end{pmatrix}$ for $d \in R_i$ and $\begin{pmatrix} 1 & 0 \\ t_1^i c & 1 \end{pmatrix}$ for $c \in R_{i-1}$ give a complete system representatives of the double coset space $B(O_F) \backslash K / K_{i,0}$.*

Proof. We have $K/K_{i,0} \simeq G(O_F/t_1^i O_F)$, and the image of $B(O_F)$ in the quotient is $B(O_F/t_1^i O_F)$. Since $K_{i,0}$ is a normal subgroup of K , we may identify $B(O_F) \backslash K / K_{i,0}$ with the coset space $G(O_F/t_1^i O_F) / B(O_F/t_1^i O_F)$.

Let E be the residue field of F , so that E is a local field. In this case, it is known that a complete system of representatives for $B(\mathcal{O}_E) \backslash G(\mathcal{O}_E) / K_i$ may

be given as in the statement of the Proposition. Since $O_F/t_1^i O_F = \mathcal{O}_E/\bar{t}_1^i \mathcal{O}_E$ one obtains that the number of double coset representatives should then be the same in the two dimensional case, and by considering the reduction of the above matrices modulo t_2 we see that the above system of representatives works. $\square$

Remark. Here it is really important that we are in level 0: for $j > 0$ the space $B(O_F) \backslash K/K_{i,j}$ is uncountable, and we certainly cannot use the proof as above. However, we get a similar explicit description of all representatives by replacing R_i with a set $R_{i,j}$ of representatives of $O_F/t_1^i t_2^j O_F$, and such a description would be required in order to do the computations in later sections for higher levels.

Write $\mathcal{C}_i$ for the set of double coset representatives given in Proposition 4.2.3.

Corollary 4.2.4. *Every $g \in G$ may be written in the form $g = b\zeta k_i$ with $b \in B(F)$, $\zeta \in \mathcal{C}_i$ and $k_i \in K_{i,0}$.*

Proof. It is immediate from Proposition 4.2.3 and the two-dimensional Iwasawa decomposition. $\square$

4.3 CHARACTERS OF THE MULTIPLICATIVE GROUP

Since the principal series representations are induced from representations of the Borel subgroup given by (quasi)characters of the multiplicative group of the underlying field, in order to proceed in a similar manner in two dimensions we make a short discussion of multiplicative characters (which for the remainder we simply refer to as characters).

Definition 4.3.1. *If $\chi(u) = 1$ for all $u \in O_F^\times$ then we say that χ is an unramified character. If there is some $u \in O_F^\times$ with $\chi(u) \neq 1$ then χ is ramified.*

If χ is unramified, for $x = t_1^i t_2^j u \in F^\times$ we have $\chi(x) = \chi(t_1)^i \chi(t_2)^j$. In other words, an unramified character is completely determined by its values on local parameters.

If χ is a ramified character, we would like to define the conductor of χ . In dimension one this is straightforward, but in dimension two there are two options.

Let $V_{i,j} = 1 + t_1^i t_2^j \mathcal{O}_F$. If there is some $V_{i,j}$ such that χ is trivial on $V_{i,j}$ but nontrivial on $V_{k,\ell}$ for all $(k,\ell) < (i,j)$, we define $V_{i,j}$ to be the conductor of χ . Since $\mathcal{O}_F^\times / V_{i,j} \simeq (\mathcal{O}_F / t_1^i t_2^j \mathcal{O}_F)^\times$, a ramified character with conductor $V_{i,j}$ is determined by its values on local parameters, along with a character of the group $(\mathcal{O}_F / t_1^i t_2^j \mathcal{O}_F)^\times$. Note that in the level 0 case this group is finite.

If no such $V_{i,j}$ exists, there exists some j such that χ is trivial on $V_j = 1 + t_2^j \mathcal{O}_F$ but nontrivial on $V_\ell = 1 + t_2^\ell \mathcal{O}_F$ for all $\ell < j$, where $\mathcal{O}_F$ is the rank one ring of integers of F . In this case, we define V_j to be the conductor of χ .

Definition 4.3.2. *A normalised unitary character is a continuous homomorphism $\omega : F^\times \rightarrow S^1$ satisfying $\omega(t_1) = \omega(t_2) = 1$.*

Note that, in the case where F is a (one-dimensional) local field, every character χ , may be written as a product $\chi = |\cdot| \omega$, where $\omega : F^\times \rightarrow S^1$ is a normalised unitary character and $|\cdot| : F^\times \rightarrow \mathbb{R}_+$ is a power of the absolute value induced by the discrete valuation.

In dimension two, our "usual" choice for the absolute value $|\cdot|_F$ takes values instead in $\mathbb{R}_+((X))$, and so we do not get a character from this. However, there do exist alternatives. For example, we may define $|\cdot|_{a,b} : F^\times \rightarrow \mathbb{R}_+$ by $|t_1^i t_2^j u|_{a,b} = q^{bj - ai}$.

It may also be useful to consider the special case where the second local parameter t_2 is not involved as part of the level 0 situation. We thus make the following final definitions.

Definition 4.3.3. *A character $\chi : F^\times \rightarrow \mathbb{C}^\times$ is called strongly unramified if $\chi(v) = 1$ for all $v \in \mathcal{O}_F$. A character is called finitely ramified if it is ramified*

4.4 THE PRINCIPAL SERIES FOR $\mathrm{GL}_2(F)$

with conductor $V_{i,0}$ for some i . A character is called *deeply ramified* if it is ramified with conductor V_j for some j

Remark. Since $\mathcal{O}_F \subset \mathcal{O}_F$, a strongly unramified character is indeed unramified. The term finitely ramified refers to the fact that the ramification is constrained to a character of a finite group. In the one dimensional case, the definitions of unramified and strongly unramified characters coincide, as do the definitions of ramified, finitely ramified, and deeply ramified characters. On the other hand, if we move to higher dimensions the spectrum of ramification becomes larger, since we obtain higher rank rings of integers.

Example. Let $F = \mathbb{Q}_p((t))$, and let ω be a unitary character of $\mathbb{Q}_p^\times$. Consider the character given by $\chi(p^i t^j u) = p^{-ai} \omega(\bar{u})$ for some positive integer a . If ω is unramified then χ is strongly unramified. If ω is ramified with conductor $V_i = 1 + p^i \mathbb{Z}_p$ then χ is finitely ramified with conductor $V_{i,0} = 1 + p^i \mathbb{Z}_p + t\mathbb{Q}_p[[t]]$.

We will not discuss deeply ramified characters in any detail here. In fact, in most cases we will only consider characters which are at worst finitely ramified.

4.4 THE PRINCIPAL SERIES FOR $\mathrm{GL}_2(F)$

Given a pair of characters χ_1, χ_2 of the multiplicative group $F^\times$, we wish to construct the representation induced by the representation of the Borel subgroup given by $\begin{pmatrix} y_1 & x \\ 0 & y_2 \end{pmatrix} \cdot z = \chi_1(y_1)\chi_2(y_2)z$. This leads us to consider the $\mathbb{C}((X))$ -vector space

$$\left\{ f : \mathrm{GL}_2(F) \rightarrow \mathbb{C}((X)) \text{ locally constant such that } f\left(\begin{pmatrix} y_1 & x \\ 0 & y_2 \end{pmatrix} g\right) = \chi_1(y_1)\chi_2(y_2)f(g) \right\},$$

which we denote by $\mathcal{V}_p(\chi_1, \chi_2)$. (Here we are using the level topology on $\mathrm{GL}_2(F)$ defined in Chapter 3, which is generated by all of the sets $gK_{i,j}$.

In particular, the locally constant functions arise as sums of integrable functions - though these sums may be infinite.)

Definition 4.4.1. *The representation π of G on $\mathcal{V}_p(\chi_1, \chi_2)$ given by*

$$(\pi(h) \cdot f)(g) = f(gh)$$

is the principal series representation of G associated to the characters χ_1, χ_2 . If both χ_1 and χ_2 are (strongly) unramified, we call the corresponding representation (strongly) unramified. We call the representation finitely ramified if one of χ_1, χ_2 is finitely ramified and the other is either unramified or finitely ramified.

Remark. In the one dimensional case, the unramified representations are those corresponding to $\chi_i(x) = |x|_F^{s_i}$ for some $s_i \in \mathbb{C}$. It would be interesting to try and obtain such a characterisation in dimension two.

Remark. Note that some authors use the term "principal series" only for the irreducible representations of the above type. We do not require *a priori* that the representations be irreducible.

The full space $\mathcal{V}_p(\chi_1, \chi_2)$ seems very difficult to try and access directly. Indeed, when working with similar two-dimensional representations, Kazhdan and Gaitsgory ([KGo4]) were forced to use a very different approach involving ind-pro categories. However, in our case we have the additional guidance of the topological framework given to us by the level structure. Rather than attempting to study $\mathcal{V}_p(\chi_1, \chi_2)$ as a whole, we thus attempt to attack the problem on a smaller scale by studying the following level subspaces.

Definition 4.4.2. *For $j \geq 0$ define the $\mathbb{C}((X))$ -vector space*

$$\mathcal{V}_{p;j}(\chi_1, \chi_2) = \{f \in \mathcal{V}_p(\chi_1, \chi_2) : \forall g \exists i \in \mathbb{Z} \text{ such that } f(gk) = f(g) \forall k \in K_{i,j}\}.$$

$\mathcal{V}_{p;j}(\chi_1, \chi_2)$ *thus consists of the "level j locally" constant functions.*

In this text we will mostly be concerned with the space $\mathcal{V}_{p;0}(\chi_1, \chi_2)$ of "level zero locally" constant functions.

Proposition 4.4.3. *The principal series representation of G of level 0, given by $(\pi(h) \cdot f)(g) = f(gh)$ for $f \in \mathcal{V}_{p;0}(\chi_1, \chi_2)$, is smooth.*

Proof. For each $k \in K$ there is some $i(k) \geq 1$ with $f(kk') = f(k)$ for all $k' \in K_{i(k),0}$, and so it remains to show that we may take some i which works for all k . However, the cosets $kK_{i(k),0}$ form a uniform open 0-cover of K , and so by 0-compactness we may take a finite subcover. If we take i to be the maximum of all $i(k)$ from this subcover then $f(kk') = f(k)$ for all $k \in K, k' \in K_{i,0}$. The result then follows for $g \in G$ by Iwasawa decomposition and the definition of the space $\mathcal{V}_{p;0}(\chi_1, \chi_2)$. $\square$

4.5 FLAT SECTIONS

In this section (and all that follows), we will restrict attention to multiplicative characters of the form $\chi = |\cdot|_{a,b}\omega$, where $\omega : F^\times \rightarrow \mathbb{C}^\times$ is a normalised unitary character and $|t_1^i t_2^j u|_{a,b} = q^{-ai} q^{bj}$ for $a, b \in \mathbb{C}$ as in Section 4.3.

If $\chi_i = |\cdot|_{\alpha_i, \beta_i} \omega_i$ for $i = 1, 2$, we will use the notation $\mathcal{V}_{p;0}(\alpha, \beta, \omega)$ for the space $\mathcal{V}_{p;0}(\chi_1, \chi_2)$, where we write $\alpha = (\alpha_1, \alpha_2)$, $\beta = (\beta_1, \beta_2)$, $\omega = (\omega_1, \omega_2)$.

Definition 4.5.1. *Fix a pair $\omega = (\omega_1, \omega_2)$ of normalised unitary characters. For an integer $j \geq 0$, a flat j -section is a function $f : \mathrm{GL}_2(F) \times \mathbb{C}^4 \rightarrow \mathbb{C}((X))$ satisfying the following two properties:*

- (1) *For fixed $(\alpha, \beta) \in \mathbb{C}^4$, the function $f(*; \alpha, \beta) : \mathrm{GL}_2(F) \rightarrow \mathbb{C}((X))$ is in the space $\mathcal{V}_{p;j}(\alpha, \beta, \omega)$.*
- (2) *For fixed $k \in K$, the function $f(k; \star) : \mathbb{C}^4 \rightarrow \mathbb{C}((X))$ is constant.*

The $\mathbb{C}((X))$ -vector space of all flat j -sections will be denoted $\mathrm{Flat}_j(\omega)$.

Remark. In the one-dimensional case, the principal series (and hence the flat sections) depend only on a single pair of complex numbers $s = (s_1, s_2)$. Compare this with the above definition in which we need two pairs α and β (one for each local parameter), or in other words a 4-tuple of complex numbers. However, if we are in the strongly unramified situation then we

have $\beta = (0, 0)$, and so in this case we may simply consider a single pair of complex numbers as in the one-dimensional case.

Lemma 4.5.2. *If f is a flat j -section and there is some particular choice of α and β such that $f(*; \alpha, \beta)$ is constant on left cosets of $K_{i,j}$ then $f(*; \alpha, \beta)$ is constant on left cosets of $K_{i,j}$ for all α, β .*

Proof. Let $g \in G$ and write $g = udk$ by Iwasawa decomposition. For $k_{i,j} \in K_{i,j}$ we have $f(gk_{i,j}; \alpha, \beta) = |d_{11}|_{\alpha_1, \beta_1} |d_{22}|_{\alpha_2, \beta_2} f(kk_{i,j}; \alpha, \beta)$ since $f(*; \alpha, \beta) \in \mathcal{V}_{p,j}(\alpha, \beta)$ by the first property of flat sections. But by the second property of flat sections $f(kk_{i,j}; \alpha, \beta) = f(kk_{i,j}; \alpha', \beta')$ for any $\alpha', \beta' \in \mathbb{C}^2$, and so we obtain $f(gk_{i,j}; \alpha, \beta) = f(gk_{i,j}; \alpha', \beta')$, and the result immediately follows from this equality. $\square$

Remark. If f is a flat j -section then by definition we may find some i such that f is constant on left cosets of $K_{i,j}$.

Recall that we have an explicit description of a set $\mathcal{C}_i$ of double coset representatives for $B(O_F) \backslash K / K_{i,0}$ from Section 4.2, which is given in terms of sets of representatives R_i and R_{i-1} in O_F for $O_F / t_1^i O_F$ and $O_F / t_1^{i-1} O_F$ respectively.

Definition 4.5.3. *For $i > 0$ and $d \in R_i$, define flat 0-sections*

$$f_{1,d}^i(g; \alpha, \beta) = \begin{cases} |y_1|_{\alpha_1, \beta_1} \omega_1(y_1) \cdot |y_2|_{\alpha_2, \beta_2} \omega_2(y_2) & \text{if } g = \begin{pmatrix} y_1 & x \\ 0 & y_2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & d \end{pmatrix} k_i \\ 0 & \text{otherwise,} \end{cases}$$

where $y_1, y_2 \in F^\times$, $x \in F$, $k_i \in K_{i,0}$.

Definition 4.5.4. *For $i > 0$ and $c \in R_{i-1}$, define flat 0-sections*

$$f_{t_1 c, 1}^i(g; \alpha, \beta) = \begin{cases} |y_1|_{\alpha_1, \beta_1} \omega_1(y_1) \cdot |y_2|_{\alpha_2, \beta_2} \omega_2(y_2) & \text{if } g = \begin{pmatrix} y_1 & x \\ 0 & y_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t_1 c & 1 \end{pmatrix} k_i \\ 0 & \text{otherwise,} \end{cases}$$

where $y_1, y_2 \in F^\times$, $x \in F$, $k_i \in K_{i,0}$.

Problem 19. Give a full proof of Proposition 4.2.3 in the case $j > 0$, and use this to define the analogous flat j -sections to those above. Note that in the case $j > 0$ their number will be (uncountably) infinite.

Proposition 4.5.5. The set $\mathcal{B}_{\text{Flat}}^i(\omega) = \{f_{1,d}^i : d \in R_i\} \cup \{f_{t_1c,1}^i : c \in R_{i-1}\}$ is a basis for the $\mathbb{C}((X))$ -vector space $\text{Flat}_0^i(\omega)$ of flat 0-sections which are constant on left cosets of $K_{i,0}$.

Proof. That elements of $\mathcal{B}_{\text{Flat}}^i(\omega)$ are linearly independent follows from the fact that $\mathcal{C}_i$ is a set of double coset representatives. To see that it spans, note that every element $\varphi \in \text{Flat}_0^i(\omega)$ is completely determined by its values on left K -cosets of $K_{i,0}$, since $G = BK$ by Iwasawa decomposition and the value of φ on elements of B given in the definition. Specifically, we have

$$\varphi = \sum_d \varphi \left(\begin{pmatrix} 0 & 1 \\ 1 & d \end{pmatrix} \right) f_{1,d}^i + \sum_c \varphi \left(\begin{pmatrix} 1 & 0 \\ t_1c & 1 \end{pmatrix} \right) f_{t_1c,1}^i.$$

□

Remark. If we establish an explicit system of double coset representatives $\mathcal{C}_{i,j}$ for $B(O_F) \backslash K / K_{i,j}$ then the above constructions should be easily extendable to flat j -sections with $j > 0$. Again, in the case $j > 0$ the above basis will no longer be finite.

4.6 THE JACQUET INTEGRAL ON FLAT 0-SECTIONS

Consider the map $J_{\alpha,\beta,\omega} : \mathcal{V}_p(\alpha, \beta, \omega) \rightarrow \text{Maps}(G, \mathbb{C}((X)))$ given by

$$(J_{\alpha,\beta,\omega}(f))(g) = \int_F f \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} g \right) \psi_F(-u) du$$

whenever this integral converges, where ψ_F is an additive character on F with conductor O_F as in [Fes03].

In this section we will determine the image of flat 0-sections under $J_{\alpha,\beta,\omega}$. We first make a few simple reductions.

Lemma 4.6.1. *For $f \in \mathcal{B}_{\text{Flat}}^i(\omega)$ and $g = bk$ with $b \in B$ and $k \in K$, there exists some $\tilde{f} \in \mathcal{B}_{\text{Flat}}^i(\omega)$ with $f(g) = \tilde{f}(b)$.*

Proof. The same as the proof of Lemma 6.6.6 in [GH11]. $\square$

By Lemma 4.6.1, to find the image of $\text{Flat}_0(\omega)$ under $J_{\alpha,\beta,\omega}$ it is enough to consider the values on upper triangular matrices. By extracting a scalar matrix (i.e. a multiple of the identity), we may moreover assume that the lower right entry is equal to 1. Finally, may we reduce to the case of diagonal matrices via the following Lemma.

Lemma 4.6.2.

$$J_{\alpha,\beta,\omega}(f) \left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \right) = \psi_F(x) \cdot J_{\alpha,\beta,\omega}(f) \left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \right)$$

for any $f \in \mathcal{V}_p(\alpha, \beta, \omega)$.

Proof.

$$\begin{aligned} J_{\alpha,\beta,\omega}(f) \left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \right) &= \int_F f \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \right) \psi_F(-u) du \\ &= \int_F f \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & u+x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \right) \psi_F(-u) du \\ &= \int_F f \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \right) \psi_F(x-u) du \\ &= \psi_F(x) \cdot J_{\alpha,\beta,\omega}(f) \left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \right). \end{aligned}$$

$\square$

It thus remains to compute the latter integral for $f \in \mathcal{B}_{\text{Flat}}^i$. Recall that $|y|_F = q^{-m}X^n$ for $y = t_1^m t_2^n u$ with $u \in O_F^\times$.

Proposition 4.6.3.

$$J_{\alpha,\beta,\omega}(f_{1,d}^i) \left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \right) = \begin{cases} \psi_F(x - dy) |y|_{\alpha_2, \beta_2} \omega_2(y) q^{-i} |y|_F & \text{if } y \in t_1^{-i} O_F \\ 0 & \text{otherwise.} \end{cases}$$

Proof.

$$\begin{aligned}
 J_{\alpha,\beta,\omega}(f_{1,d}^i) \left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \right) &= \int_F f_{i,d}^i \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}; \alpha, \beta \right) \psi_F(-u) du \\
 &= \int_F f_{1,d}^i \left(\begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & y^{-1}u \\ 0 & 1 \end{pmatrix}; \alpha, \beta \right) \psi_F(-u) du \\
 &= |y|_{\alpha_2, \beta_2} \omega_2(y) \int_F f_{i,d}^i \left(\begin{pmatrix} 0 & 1 \\ 1 & y^{-1}u \end{pmatrix}; \alpha, \beta \right) \psi_F(-u) du \\
 &= |y|_{\alpha_2, \beta_2} \omega_2(y) |y|_F \int_F f_{i,d}^i \left(\begin{pmatrix} 0 & 1 \\ 1 & u \end{pmatrix}; \alpha, \beta \right) \psi_F(-yu) du.
 \end{aligned}$$

To compute the latter integral we first consider the Iwasawa decomposition

$$\begin{pmatrix} 0 & 1 \\ 1 & u \end{pmatrix} = \begin{cases} \begin{pmatrix} 0 & 1 \\ 1 & u \end{pmatrix} & \text{if } u \in O_F \\ \begin{pmatrix} u^{-1} & 1 \\ 0 & u \end{pmatrix} \begin{pmatrix} -1 & 0 \\ u^{-1} & 1 \end{pmatrix} & \text{otherwise.} \end{cases}$$

In the latter case, $\begin{pmatrix} 0 & 1 \\ 1 & u \end{pmatrix}$ is never in a Borel translate of the double coset represented by $\begin{pmatrix} 0 & 1 \\ 1 & d \end{pmatrix}$, since $\begin{pmatrix} -1 & 0 \\ u^{-1} & 1 \end{pmatrix}$ clearly belongs to a coset with a lower triangular representative. Thus for $u \notin O_F$ the value of

$$f_{i,d}^i \left(\begin{pmatrix} 0 & 1 \\ 1 & u \end{pmatrix}; \alpha, \beta \right) = 0,$$

and hence so does the integral.

On the other hand, if $u \in O_F$ then by comparing coefficients we see that $\begin{pmatrix} 0 & 1 \\ 1 & u \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & d \end{pmatrix} \begin{pmatrix} 1+t_1^i \kappa & t_2^i \lambda \\ t_2^i \mu & 1+t_2^i \nu \end{pmatrix}$ if and only if $\kappa = \mu = \nu = 0$ and $u = d + t_1^i \lambda$. In other words,

$$f_{i,d}^i \left(\begin{pmatrix} 0 & 1 \\ 1 & u \end{pmatrix}; \alpha, \beta \right) = \begin{cases} 1 & \text{if } u - d \in t_1^i O_F \\ 0 & \text{otherwise.} \end{cases}$$

We thus have

$$J_{\alpha,\beta,\omega}(f_{1,d}^i) \left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \right) = |y|_{\alpha_2, \beta_2} \omega_2(y) |y|_F \int_{d+t_1^i O_F} \psi_F(-yu) du,$$

and from [Fes03] the latter integral is equal to $q^{-i}\psi(-dy)$ if $y \in t_1^{-i}O_F$ and is zero otherwise. The Proposition then follows immediately from Lemma 4.6.2. $\square$

Before doing the computations for $f_{t_1c,1}^i$, we evaluate the following integral of characters.

Lemma 4.6.4. *Let ω_0 be a nontrivial normalised unitary character of $F^\times$ with conductor $V_{i,0} = 1 + t_1^i O_F$, let $y_0 \in O_F^\times$, and let $Y \in \mathbb{Z}$. If R^* is a set of representatives in $O_F^\times$ for $O_F^\times/V_{i,0}$ then*

$$\int_{O_F^\times} \omega_0(u) \psi_F(uy_0 t_1^{-Y}) du = \begin{cases} \frac{\omega_0^{-1}(y_0)}{q^Y} \sum_{r \in R^*} \omega_0(r) \psi_F(t_1^{-Y} r) & \text{if } i=Y \\ 0 & \text{otherwise.} \end{cases}$$

Proof. This is essentially the same as Lemma 6.6.9 in [GH11] $\square$

Remark. To do this computation for $j > 0$ would be tricky, since the cardinality of R^* is uncountable in this case. Unfortunately, this means that at present we will have to restrict to the finitely ramified situation whenever we want to use this Lemma.

Proposition 4.6.5. *Suppose that either $\omega_1 = \omega_2$ or that $\omega_1^{-1}\omega_2$ is finitely ramified. For $c \in R_i \setminus \{0\}$*

$$J_{\alpha,\beta,\omega}(f_{t_1c,1}^i) \left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \right) = \psi_F(x) \omega_2(y) |y|_{\alpha_2, \beta_2} |y|_F \cdot \phi(y)$$

for some locally constant function $\phi : F \rightarrow \mathbb{C}((X))$ depending on i, c , and ω . Moreover there is $N \in \mathbb{Z}^2$ such that $\phi(y) = 0$ whenever $v(y) < N$, where v is the rank two discrete valuation on F .

Proof. As in Proposition 4.6.3, we have

$$J_{\alpha,\beta,\omega}(f_{t_1c,1}^i) \left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \right) = |y|_{\alpha_2, \beta_2} \omega_2(y) |y|_F \int_F f_{t_1c,1}^i \left(\begin{pmatrix} 0 & 1 \\ 1 & u \end{pmatrix}; \alpha, \beta \right) \psi_F(-yu) du,$$

and again using the Iwasawa decomposition and comparing coefficients we have

$$\begin{aligned} f_{t_1c,1}^i \left(\begin{pmatrix} 0 & 1 \\ 1 & u \end{pmatrix}; \alpha, \beta \right) &= |u^{-1}|_{\alpha_1, \beta_1} \omega_1(u^{-1}) |u|_{\alpha_2, \beta_2} \omega_2(u) \\ &= |u|_{\alpha_2 - \alpha_1, \beta_2 - \beta_1} \omega_1^{-1}(u) \omega_2(u) \end{aligned}$$

if $u^{-1} - t_1c \in t_1^i O_F$, and is zero otherwise. Now, we have $|t_1c|_F = q^{-k}$ for some positive integer k , and $u^{-1} - t_1c \in t_1^i O_F$ if and only if $u - (t_1c)^{-1} \in t_1^{i-2k} O_F$. Indeed, if $u^{-1} = t_1c + t_1^i \xi$ then $u = (t_1c)^{-1} - (t_1c)^{-2} t_1^i \xi + \dots$. We thus have to compute

$$\int_{(t_1c)^{-1} + t_1^{i-2k} O_F} |u|_{\alpha_2 - \alpha_1, \beta_2 - \beta_1} \omega_1^{-1}(u) \omega_2(u) \psi(-uy) du.$$

Since $|u|_{\alpha_2 - \alpha_1, \beta_2 - \beta_1}$ has a constant value $|t_1c|_{\alpha_2 - \alpha_1, \beta_2 - \beta_1}^{-1} = q^{k(\alpha_2 - \alpha_1)}$ on $(t_1c)^{-1} + t_1^{i-2k} O_F$, if $\omega_1 = \omega_2$ the integral equals $\psi_F(-(t_1c)^{-1}y) q^{2k-i} q^{k(\alpha_2 - \alpha_1)}$ if $y \in t_1^{i-2k} O_F$ and is zero otherwise.

Suppose now that $\omega_1 \neq \omega_2$, and suppose $\omega_1^{-1} \omega_2$ has conductor $V_{\ell,0} = 1 + t_1^\ell O_F$. In this case the integral is equal to

$$\omega_1(t_1c) \omega_2^{-1}(t_1c) |t_1c|_{\alpha_1 - \alpha_2 - 1, \beta_2 - \beta_1} \int_{1 + t_1^{i-k} O_F} \omega_1^{-1}(u) \omega_2(u) \psi_F \left(-u \frac{y}{t_1c} \right) du.$$

If $\ell \leq i - k$ then $\omega_1^{-1}(u) \omega_2(u) = 1$ on the domain of integration, and so we simply obtain $\omega_1(t_1c) \omega_2^{-1}(t_1c)$ times the value in the case where the characters were equal. If $\ell > i - k$ then from the proof of Lemma 4.6.4 we see that the integral vanishes unless $\frac{y}{t_1c} = y_0 t_1^{-\ell}$ for some $y_0 \in O_F$, in which case the value of the integral is given by Lemma 4.6.4. $\square$

For the $c = 0$ case, we first make some preliminary computations in order to prevent the proof from becoming too cluttered.

Lemma 4.6.6. *For $y = y_0 t_1^m t_2^n$ with $y_0 \in O_F^\times$, and for any two complex numbers s_1, s_2 , the value of*

$$S = \sum_{k=i}^{\infty} \sum_{\ell=0}^{\infty} q^{ks_1} q^{-\ell s_2} \int_{O_F^\times} \psi_F \left(-u \frac{y}{t_1^k t_2^\ell} \right) du$$

is given by

$$\left\{ \begin{array}{ll} (n+1)(m-i+1)(1-q^{-1}) + n(1-q^{-1}) - q^{-1} & \text{if } m \geq i-1, n \geq 0, \\ & \text{and } q^{s_1} = 1 = q^{s_2} \\ \left(n(1-q^{-1}) - q^{-1} - \frac{(n+1)(1-q^{-1})}{1-q^{s_1}} \right) q^{(m+1)s_1} & \text{if } m \geq i-1, n \geq 0, \\ + \frac{(n+1)(1-q^{-1})}{1-q^{s_1}} q^{is_1} & \text{and } q^{s_1} \neq 1 = q^{s_2} \\ \left((m-i+1)(1-q^{-1}) - q^{-1} - \frac{(m-i+2)(1-q^{-1})}{1-q^{-s_2}} \right) q^{-ns_2} & \text{if } m \geq i-1, n \geq 0, \\ + \frac{(m-i+2)(1-q^{-1})}{1-q^{-s_2}} & \text{and } q^{s_1} = 1 \neq q^{s_2} \\ (1-q^{-1})q^{(m+1)s_1} \left(\frac{1-q^{-ns_2}}{1-q^{-s_2}} \right) - q^{-1}q^{(m+1)s_1}q^{-ns_2} & \text{if } m \geq i-1, n \geq 0, \\ + (1-q^{-1}) \left(\frac{q^{is_1} - q^{(m+1)s_1}}{1-q^{s_1}} \right) \left(\frac{1-q^{-(n+1)s_2}}{1-q^{-s_2}} \right) & \text{and } q^{s_1} \neq 1 \neq q^{s_2} \end{array} \right.$$

and is equal to 0 in all other cases.

Proof. From Lemma 2.3.7 we have

$$\int_{O_F^\times} \psi_F \left(-u \frac{y}{t_1^k t_2^\ell} \right) du = \begin{cases} 0 & \text{if } y \notin t_1^{k-1} t_2^\ell O_F \\ -q^{-1} & \text{if } y \in t_1^{k-1} t_2^\ell O_F^\times \\ 1 - q^{-1} & \text{if } y \in t_1^k t_2^\ell O_F. \end{cases}$$

In particular, when k and ℓ are large enough (more precisely, when $k-1 > m$ or $\ell > n$), the integral vanishes, and so both of the sums are in fact finite.

Furthermore, if $m < i - 1$ or $n < 0$ then the sum is empty, and so is again equal to zero. It thus remains to compute

$$S = \sum_{k=i}^{m+1} \sum_{\ell=0}^n q^{ks_1} q^{-\ell s_2} \int_{O_F^\times} \psi_F \left(-u \frac{y}{t_1^k t_2^\ell} \right) du \quad (3)$$

$$= -q^{-1} q^{(m+1)s_1} q^{-ns_2} + (1 - q^{-1}) \sum_{k=i}^m \sum_{\ell=0}^n q^{ks_1} q^{-\ell s_2} + (1 - q^{-1}) q^{(m+1)s_1} \sum_{\ell=0}^{n-1} q^{-\ell s_2} \quad (4)$$

$$= -q^{-1} q^{(m+1)s_1} q^{-ns_2} + (1 - q^{-1}) \sum_{\ell=0}^{n-1} \sum_{k=i}^{m+1} q^{ks_1} q^{-\ell s_2} + (1 - q^{-1}) q^{-ns_2} \sum_{k=i}^m q^{ks_1}. \quad (5)$$

If $q^{s_1} = q^{s_2} = 1$ then $S = -q^{-1} + (n+1)(m-i+1)(1-q^{-1}) + n(1-q^{-1})$.

If $q^{s_2} = 1 \neq q^{s_1}$ then from (4) we have

$$\begin{aligned} S &= -q^{-1} q^{(m+1)s_1} + (n+1)(1-q^{-1}) \sum_{k=i}^m q^{ks_1} + n(1-q^{-1}) q^{(m+1)s_1} \\ &= -q^{-1} q^{(m+1)s_1} + (n+1)(1-q^{-1}) \frac{q^{is_1}(1-q^{(m-i+1)s_1})}{1-q^{s_1}} + n(1-q^{-1}) q^{(m+1)s_1} \\ &= \frac{(n+1)(1-q^{-1})}{1-q^{s_1}} q^{is_1} + \left(n(1-q^{-1}) - q^{-1} - \frac{(n+1)(1-q^{-1})}{1-q^{s_1}} \right) q^{(m+1)s_1}. \end{aligned}$$

Similarly, if $q^{s_1} = 1 \neq q^{s_2}$ then from (5) we have

$$\begin{aligned} S &= -q^{-1} q^{-ns_2} + (m-i+2)(1-q^{-1}) \sum_{\ell=0}^{n-1} q^{-\ell s_2} + (m-i+1)(1-q^{-1}) q^{-ns_2} \\ &= -q^{-1} q^{-ns_2} + (m-i+2)(1-q^{-1}) \frac{1-q^{-ns_2}}{1-q^{-s_2}} + (m-i+1)(1-q^{-1}) q^{-ns_2} \\ &= \frac{(m-i+2)(1-q^{-1})}{1-q^{-s_2}} \\ &\quad + \left((m-i+1)(1-q^{-1}) - q^{-1} - \frac{(m-i+2)(1-q^{-1})}{1-q^{-s_2}} \right) q^{-ns_2}. \end{aligned}$$

All that remains is the case $q^{s_1} \neq 1 \neq q^{s_2}$. Since neither expression for S is particularly convenient in this case we simply evaluate (4) to obtain

$$\begin{aligned} &-q^{-1} q^{(m+1)s_1} q^{-ns_2} + (1-q^{-1}) \left(\frac{q^{is_1} - q^{(m+1)s_1}}{1-q^{s_1}} \right) \left(\frac{1-q^{-(n+1)s_2}}{1-q^{-s_2}} \right) \\ &\quad + (1-q^{-1}) q^{(m+1)s_1} \left(\frac{1-q^{-ns_2}}{1-q^{-s_2}} \right). \end{aligned}$$

□

Proposition 4.6.7. *The value of*

$$J_{\alpha,\beta,\omega}(f_{0,1}^i) \left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \right)$$

is given by $|y|_{\alpha_2,\beta_2}\omega_2(y)|y|_F S(y)$ if $\omega_1 = \omega_2$, with $S(y)$ denoting the sum evaluated in Lemma 4.6.6 with $s_1 = \alpha_2 - \alpha_1 + 1$ and $s_2 = \beta_2 - \beta_1 + 1$.

If $\omega_1 \neq \omega_2$ and $\omega_0 = \omega_1^{-1}\omega_2$ is finitely ramified with conductor $V_{a,0}$ then the value is instead

$$|y|_{\alpha_2,\beta_2}\omega_2(y)|y|_F q^{(a+m)(\alpha_2-\alpha_1+1)} q^{-n(\beta_2-\beta_1+1)} \frac{\omega_0^{-1}(y_0)}{q^a} \sum_{r \in R^*} \omega_0(r) \psi_F(t_1^{-a}r)$$

if $y = y_0 t_1^m t_2^n$ with $y_0 \in O_F^\times$, $a + m \geq i$, $n \geq 0$, and is zero otherwise.

Proof. As for previous computations we have

$$J_{\alpha,\beta,\omega}(f_{0,1}^i) \left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \right) = |y|_{\alpha_2,\beta_2}\omega_2(y)|y|_F \int_F f_{0,1}^i \left(\begin{pmatrix} 0 & 1 \\ 1 & u \end{pmatrix}; \alpha, \beta \right) \psi_F(-yu) du,$$

and in the case under consideration we have

$$f_{0,1}^i \left(\begin{pmatrix} 0 & 1 \\ 1 & u \end{pmatrix}; \alpha, \beta \right) = \begin{cases} \omega_1^{-1}(u)\omega_2(u)|u|_{\alpha_2-\alpha_1,\beta_2-\beta_1} & \text{if } v_1(u) \leq -i \text{ and } v_2(u) \leq 0 \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathbf{v} = (v_1, v_2)$ is the rank two discrete valuation on F . The integral

$$I = \int_F f_{0,1}^i \left(\begin{pmatrix} 0 & 1 \\ 1 & u \end{pmatrix}; \alpha, \beta \right) \psi_F(-yu) du$$

is thus equal to

$$\begin{aligned} I &= \sum_{k=i}^{\infty} \sum_{\ell=0}^{\infty} \int_{t_1^{-k} t_2^{-\ell} O_F^\times} \omega_1^{-1}(u)\omega_2(u)|u|_{\alpha_2-\alpha_1,\beta_2-\beta_1} \psi_F(-uy) du \\ &= \sum_{k=i}^{\infty} \sum_{\ell=0}^{\infty} q^{k(\alpha_2-\alpha_1+1)} q^{-\ell(\beta_2-\beta_1+1)} \int_{O_F^\times} \omega_1^{-1}(u)\omega_2(u) \psi_F \left(-u \frac{y}{t_1^k t_2^\ell} \right) du. \end{aligned}$$

If $\omega_1 = \omega_2$ the result then follows from Lemma 4.6.6, and if $\omega \neq \omega_2$ and $\omega_1^{-1}\omega_2$ has conductor $V_{a,0}$ the result follows from Lemma 4.6.4. $\square$

Remark. For the infinitely ramified characters we would need to see what we can do with Lemma 4.6.4 in higher levels.

Theorem 4.6.8. *The integral*

$$(J_{\alpha,\beta,\omega}(f))(g) = \int_F f \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} g \right) \psi_F(-u) du$$

is a polynomial in $q^{\pm\alpha_i}$ and $q^{\pm\beta_i}$ for $i = 1, 2$ for any flat 0-section f and any fixed $g \in G$. $(J_{\alpha,\beta,\omega}(f))(g)$ thus has analytic continuation to all $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{C}$ for all $g \in G$ and $f \in \mathcal{V}_{p,0}(\alpha, \beta, \omega)$.

Proof. The fact that $(J_{\alpha,\beta,\omega}(f))(g)$ is a polynomial follows from all of the previous results in this section, and from this it follows that there is analytic continuation for all flat sections. To extend to $f \in \mathcal{V}_{p,0}(\alpha, \beta, \omega)$ one simply notes that any such f is completely determined by its values on the double cosets $B(O_F) \backslash K/K_{i,0}$, and so is in fact a linear combination of flat 0-sections.

□



LIST OF PROBLEMS

Problem 1. *Extend the results of this text, and also those of (for example) [Fes10], [Fes15], to dimension three (and higher).*

Problem 2. *Study and develop the theory of archimedean local fields. In particular, identify what should be the residue field of the fields $\mathbb{R}$ and $\mathbb{C}$, and unify this with the construction of possible higher dimensional fields such as $\mathbb{R}\{\{t\}\}$.*

Problem 3. *Give an example of a non-rigid group with level structure, or prove that no such spaces exist. If examples do exist, can they be classified in some reasonable way.*

Problem 4. *Investigate possible candidates for an appropriate notion of "level-completeness", and give a proof of Proposition 3.2.6 purely in terms of the level structure on F .*

Problem 5. *Give an example of a group G with level structure and an explicit subset $A \subset G$ of type E , or prove that no such examples exist.*

Problem 6. *Investigate other possible conditions other than G being of ordered type for which Proposition 3.4.2 and Corollary 3.4.4 hold.*

Problem 7. *Show that the result of Proposition 3.4.6 still holds without the assumption that elements of $\mathcal{L}$ be subgroups of G , or give a counterexample to show that the result is false.*

Problem 8. *Give an example of a group G and a ddd-set $D \subset G$ which does not have a level, or show that such examples do not exist.*

Problem 9. Investigate the possibilities for subsets of type S , L , and E for a group G of ordered type, and determine whether or not it is possible to classify them further in this case.

Problem 10. Give an example of D_1, D_2, D_3 such that the inequality in Lemma 3.5.3 is strict, or prove that no such examples exist (i.e. that it is in fact an equality). If it is possible for the inequality to be strict, find reasonable additional conditions on G which force it to be an equality.

Problem 11. Investigate other conditions which, along with ordered type and properly suspended, force a group G to be locally level-compact.

Problem 12. Extend the construction of refinements to groups such as $\mathbb{R}((t))$ and $\mathrm{GL}_2(\mathbb{R}((t)))$ which are not of ordered type. Determine how far this construction may be extended, and to what extent it facilitates the definition of an invariant measure.

Problem 13. Extend $\{K_{i,j} : i, j > 0\}$ to a full level structure for $\mathrm{GL}_2(F)$ over $\mathrm{GL}_2(\bar{F})$.

Problem 14. Following the constructions in Chapter 2, construct an invariant measure on groups G which are not necessarily compatible. Show that this will agree with the definition in Theorem 3.5.16 in the compatible case, but in general will not be compatible with the Haar measure on X .

Problem 15. Prove or disprove the following conjecture: any finite union of subsets of type S of a (compatible) group G is also of type S . If this conjecture is indeed true, it also makes sense to extend the measure by defining all subsets of type S to have zero volume.

Problem 16. For an algebraic group G over a two dimensional field F , or for a two-dimensional adelic algebraic group, determine conditions which give "compatibility with the measure on F " (respectively the measure on A_S or $\mathbb{A}_S$) in the sense of Theorem 2.7.5. The dimension of G as an affine variety is expected to appear (for example, for GL_2 the required indeterminate takes the form X^4 , and for SL_2 it takes the form X^3).

Problem 17. *Verify that the results of Section 3.5 hold for a group G levelled over X such that X has an invariant, finitely additive (not necessarily real-valued) measure. In particular, the measure μ on a higher-dimensional local field F should be compatible with the measure on each successive residue field.*

Problem 18. *Investigate how far one may refine the results of Chapter 3 in the elevation $e = 1$ case using elementary properties of $\mathbb{Z}$ such as Lemma 3.7.2.*

Problem 19. *Give a full proof of Proposition 4.2.3 in the case $j > 0$, and use this to define the analogous flat j -sections to those above. Note that in the case $j > 0$ their number will be (uncountably) infinite.*

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