

Regularity and extensions of Banach function algebras

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*Thesis submitted to the University of Nottingham
for the degree of Doctor of Philosophy*

December 2017

Abstract

In this thesis we investigate the properties of various Banach function algebras and uniform algebras. We are particularly interested in regularity of Banach function algebras and extensions of uniform algebras.

The first chapter contains the background in normed algebras, Banach function algebras, and uniform algebras which will be required throughout the thesis.

In the second chapter we investigate the classicalisation of certain compact subsets of the complex plane obtained by deleting a sequence of open disks from a closed disk. Sets obtained in this manner are called Swiss cheese sets. We give a new topological proof of the Feinstein–Heath classicalisation theorem along with similar results. We conclude the chapter with an application of the classicalisation results. The results in this chapter are joint with H. Yang.

In the third chapter we study Banach function algebras of functions satisfying a generalised notion of differentiability. These algebras were first investigated by Bland and Feinstein as a way to describe the completion of certain normed algebras of complex-differentiable functions. We prove a new version of chain rule in this setting, generalising a result of Chaobankoh, and use this chain rule to give a new proof of the quotient rule. We also investigate naturality and homomorphisms between these algebras.

In the fourth chapter we continue the study of the notion of differentiability from the third chapter. We investigate a new notion of quasianalyticity in this setting and prove an analogue of the classical Denjoy–Carleman theorem. We describe those functions which satisfy a notion of analyticity, and give an application of these results.

In the fifth chapter we investigate various methods for constructing extensions of uniform algebras. We study the structure of Cole extensions, introduced by Cole and later investigated by Dawson, relative to certain projections. We also discuss a larger class of extensions, which we call generalised Cole extensions, originally introduced by Cole and Feinstein.

In the final chapter we investigate extensions of derivations from uniform algebras. We prove that there exists a non-trivial uniform algebra such that every derivation extends with the same norm to every generalised Cole extension of that algebra. A non-trivial, weakly amenable uniform algebra satisfies this property. We also investigate a sequence of extensions of a derivation from the disk algebra.

Acknowledgements

I would like to thank my supervisor, Dr. Joel Feinstein, for his guidance, encouragement, and many helpful discussions during the writing of this thesis.

I would like to thank the Engineering and Physical Sciences Research Council for providing financial support.

I would like to thank my parents for their encouragement and guidance. Without their support, none of this would have been possible.

I would like to thank the organisers of the various conferences, workshops, and meetings I have attended. These have been thoroughly enjoyable.

Finally, I would like to thank my friends from the School of Mathematical Sciences and beyond.

Contents

Abstract	ii
Acknowledgements	iv
Contents	v
1 Introduction and basic results	1
1.1 Outline and major results	1
1.2 Notation and background	9
1.3 Tensor products	18
1.4 Banach algebras	19
1.5 Banach function algebras	24
1.6 Uniform algebras	25
2 Abstract Swiss cheeses	28
2.1 Abstract Swiss cheese space	30
2.2 Classicalisation of abstract Swiss cheeses	35
2.3 Controlled classicalisation	42
2.4 Annular classicalisation	49
2.5 Regularity for $R(X)$	54
2.6 Classicalisation of an example of O’Farrell	56
2.7 Open problems and discussion	61
3 Algebras of differentiable functions	62
3.1 Paths in the plane	64
3.2 Algebras of differentiable functions	68
3.3 Bland–Feinstein algebras	70
3.4 Composition of $\mathcal{F}$ -differentiable functions	81
3.5 Bland–Feinstein algebra homomorphisms	84
3.6 Naturality of Bland–Feinstein algebras	89

3.7	Open problems and discussion	97
4	Quasianalyticity and $\mathcal{F}$-quasianalyticity	99
4.1	Quasianalytic Banach function algebras	100
4.2	$\mathcal{F}$ -quasianalyticity	101
4.3	$\mathcal{F}$ -analyticity	112
4.4	Sets of quasianalyticity	118
4.5	Trivial Jensen measures without regularity	123
4.6	Open problems and discussion	129
5	Extensions of uniform algebras	131
5.1	Direct limits and inverse limits	132
5.2	Some results for Banach algebras	135
5.3	Systems of Cole extensions	140
5.4	The Shilov boundary for Cole extensions	144
5.5	Properties of Simple Cole extensions	146
5.6	Generalised Cole extensions	151
5.7	Open problems and discussion	158
6	Extensions of derivations	160
6.1	Extensions of derivations	161
6.2	An interesting dichotomy	167
6.3	An example: the disk algebra	171
6.4	Open problems and discussion	175
A	Operator spaces and operator algebras	177
A.1	C^* -algebras	177
A.2	Operator spaces and operator algebras	178
	Bibliography	181
	Index of Symbols	189
	Index	191

Mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true.

Bertrand Russell *Mysticism and Logic* (1918)

CHAPTER 1

Introduction and basic results

This chapter contains an outline of the thesis and the major results, along with the basic definitions, standard results, notations, and conventions that will be used throughout this thesis. All the definitions and results from this chapter are standard and, where no explicit reference is given, we refer the reader to standard texts on Banach algebras [13, 27] for details.

We assume that the reader is familiar with standard results from the following subjects (references given in brackets): complex analysis ([3, 26, 93]), general topology ([74]), functional analysis ([5, 25, 94]), measure theory ([60, 93]), rings and algebras ([14, 53]), and linear algebra.

In this thesis, we label prominent results as either *theorems* or *propositions*. We reserve the former for new results, and the latter for results which are not original and where a reference is given. Where possible, we provide references for those lemmas and corollaries which are not original.

Each chapter begins with a short introduction containing historical context and motivation for the content of that chapter. In addition, we conclude each chapter with a section containing open problems (and accompanying discussion) related to that chapter.

1.1 Outline and major results

In this section we give an outline of the thesis and state the major results. These results are displayed in order of appearance, and are sectioned according to chapter. We direct the reader to the respective chapter for the terminology and symbols used in the statement of each theorem.

1.1.1 Chapter 2

Chapter two contains a study of abstract Swiss cheese space. We use the topology on this space to prove various classicalisation theorems for Swiss cheese sets, including the Feinstein–Heath theorem.

In Section 2.1 we introduce abstract Swiss cheese space, and prove some elementary facts about this space. We also introduce some functions and terminology that will be used throughout the chapter.

In Section 2.2 we give a topological proof of the Feinstein–Heath theorem. This is achieved by constructing compact subsets $\mathcal{S}_2 \subseteq \mathcal{S}_1$, related to a given abstract Swiss cheese, of abstract Swiss cheese space. We first prove that all abstract Swiss cheeses in $\mathcal{S}_1$ are semiclassical, thus providing a ‘semiclassicalisation’ theorem.

Theorem 2.2.7. *All abstract Swiss cheeses in $\mathcal{S}_1$ are semiclassical.*

We then show that $\mathcal{S}_2$ contains only classical abstract Swiss cheeses, from which we obtain the Feinstein–Heath theorem.

Theorem 2.2.9. *All abstract Swiss cheeses in $\mathcal{S}_2$ are classical.*

Finally we show that, under a mild assumption, we need not consider $\mathcal{S}_2$ to find classical abstract Swiss cheeses.

Theorem 2.2.10. *Suppose that $\text{int}(X_A) = \emptyset$. Then each abstract Swiss cheese in $\mathcal{S}_1$ is classical.*

In Section 2.3 we prove a new classicalisation theorem, which we call the ‘controlled classicalisation’ theorem. Here we place restrictions on the changes that can be made, thus providing finer control on the resulting classical abstract Swiss cheese. This is achieved by considering controlling collections.

Theorem 2.3.7. *Let $I \subseteq \mathbb{N}$ be non-empty. Let $(K_n)_{n \in I}$ be a collection of compact plane sets and let $(M_n)_{n \in I}$ be a collection of positive real numbers. For each $n \in I$, let $U_n := U(K_n, M_n)$. Suppose that $U_k \subseteq B(a_0, r_0)$ and $\rho_{U_k}(A) < M_k/2$ for all $k \in I$ and suppose that $U_k \cap U_\ell = \emptyset$ for all distinct $k, \ell \in I$. Let $\mathcal{C}$ be the controlling collection $((K_n, U_n))_{n \in I}$ and suppose $E(A) \subseteq F(\mathcal{C})$. Then there exists $B = ((b_n, s_n)) \in \mathcal{L}_A^*(\mathcal{C})$ such that $X_B \setminus V(\mathcal{C}) = X_A \setminus V(\mathcal{C})$ and B is classical.*

In Section 2.4 we prove an analogue of the Feinstein–Heath theorem for annular abstract Swiss cheeses using techniques similar to those in Section 2.2.

Theorem 2.4.5. *Let $\mathcal{A}$ be as in Lemma 2.4.3. Then there exists a classical, annular Swiss cheese $B = ((b_n, s_n)) \in \mathcal{A}$ with $\delta_{\text{ann}}(B) \geq \delta_{\text{ann}}(A)$ and $X_B \subseteq X_A$. Moreover, $r_0 - 2\rho_{\text{ann}}(A) \leq s_0 \leq r_0$ and $r_1 \leq s_1 \leq r_1 + 2\rho_{\text{ann}}(A)$.*

In Section 2.5 we investigate the regularity of $R(X)$ where X is obtained from the union of a family of compact plane sets.

Theorem 2.5.4. *Let X be a compact plane set and let E be a countable subset of X . Let (X_α) be a family of compact plane sets such that $R(X_\alpha)$ is regular for all α and $\bigcup_\alpha \text{int}_X(X \cap X_\alpha) \supseteq X \setminus E$. Then $R(X)$ is regular.*

Section 2.6 contains the construction of a classical Swiss cheese to replace an example due to O’Farrell, and uses the techniques and results developed in the chapter.

Theorem 2.6.6. *Let $\varepsilon > 0$. Then there exists a classical abstract Swiss cheese $B = ((b_n, s_n))$ with $0 \in X_B$ and $\rho(B) < \varepsilon$ and such that $R(X_B)$ is regular and admits a non-degenerate bounded point derivation of infinite order at 0.*

1.1.2 Chapter 3

In Chapter 3 we study algebras of functions that satisfy a generalised notion of (complex) differentiability, which we call $\mathcal{F}$ -differentiation. The main aim in this chapter is to prove a general version of the chain rule in this setting, and investigate the properties of these algebras.

In Section 3.1 we recall some definitions and results related to paths in the complex plane, and in Section 3.2 we recall some definitions related to algebras of (continuously) complex-differentiable functions on certain subsets of the complex plane.

In Section 3.3 we provide a short introduction to $\mathcal{F}$ -differentiation and Bland–Feinstein algebras. We then investigate maximal collections of paths; this is the content of Section 3.3.2. These maximal collections of paths are used to introduce a notion of compatibility between collections of paths. We prove that these maximal collections of paths are determined by dense sets of functions, in an appropriate space. This has some important corollaries. In particular, it allows us to connect our general version of the chain rule (below) with previous results.

Theorem 3.3.22. *Let $S \subseteq C(X) \oplus_1 C(X)$. Then $\mathfrak{p}(\overline{S}) = \mathfrak{p}(S)$.*

In Section 3.4 we give the proof of the general chain rule for $\mathcal{F}$ -differentiation, and list several consequences. For this version of the chain rule we utilise a notion of compatibility between collections of paths.

Theorem 3.4.2. *Let X, Y be semi-rectifiable compact plane sets, let $\mathcal{F}$ be an effective collection of paths in X , and let $\mathcal{G}$ be an effective collection of paths in Y . Let $\varphi \in D_{\mathcal{F}}^{(1)}(X)$ with $\varphi(X) \subseteq Y$. Suppose that φ is $\mathcal{F}$ - $\mathcal{G}$ -compatible. Then, for all $f \in D_{\mathcal{G}}^{(1)}(Y)$, $f \circ \varphi$ is $\mathcal{F}$ -differentiable and $(f \circ \varphi)^{[1]} = (f^{[1]} \circ \varphi)^{[1]}$.*

In section 3.5 we give two further applications of the new chain rule to find homomorphisms between algebras of $\mathcal{F}$ -differentiable functions. The first is for algebras of n -times $\mathcal{F}$ -differentiable functions, and is an immediate consequence of the Faà di Bruno formula for $\mathcal{F}$ -derivatives.

Theorem 3.5.1. *Let X, Y be semi-rectifiable compact plane sets and let $n \in \mathbb{N}$. Let $\mathcal{F}$ be an effective collection of paths in X , let $\mathcal{G}$ be an effective collection of paths in Y , and let $\varphi \in D_{\mathcal{F}}^{(n)}(X)$. Suppose that $\varphi(X) \subseteq Y$ and that φ is $\mathcal{F}$ - $\mathcal{G}$ -compatible. Then φ^* is a unital algebra homomorphism from $D_{\mathcal{F}}^{(n)}(Y)$ into $D_{\mathcal{F}}^{(n)}(X)$.*

The second is for the $\mathcal{F}$ -differentiation analogue of Dales–Davie algebras.

Theorem 3.5.3. *Let X, Y be semi-rectifiable compact plane sets, let $n \in \mathbb{N}$, let $\mathcal{F}$ be an effective collection of paths in X , let $\mathcal{G}$ be an effective collection of paths in Y , let $M = (M_n)_{n=0}^{\infty}$ be a non-analytic algebra sequence, and let $\varphi \in D_{\mathcal{F}}^{(\infty)}(X)$. Suppose that φ is $\mathcal{F}$ -analytic, that $\varphi(X) \subseteq Y$, and that φ is $\mathcal{F}$ - $\mathcal{G}$ -compatible.*

- (a) *If φ satisfies and $|\varphi^{[1]}|_X < 1$ then φ induces a homomorphism from $D_{\mathcal{G}}(Y, M)$ into $D_{\mathcal{F}}(X, M)$.*
- (b) *If the sequence $(n^2 M_n / M_{n+1})$ is bounded and $|\varphi^{[1]}|_X \leq 1$ then φ induces a homomorphism from $D_{\mathcal{G}}(Y, M)$ into $D_{\mathcal{F}}(X, M)$.*

In Section 3.6 we investigate sufficient conditions for algebras of $\mathcal{F}$ -differentiable functions to be natural Banach function algebras. We characterise those collections of paths $\mathcal{F}$ such that the complex conjugate function is $\mathcal{F}$ -differentiable and has constant $\mathcal{F}$ -derivative. This gives some sufficient conditions for these algebras to be natural.

Theorem 3.6.11. *Let X be a semi-rectifiable compact plane set and let $\mathcal{F}$ be an effective collection of paths in X . Then $\overline{Z} \in D_{\mathcal{F}}^{(1)}(X)$ and $\overline{Z}^{[1]}$ is constant on X if and only if there exists $\theta \in (-\pi, \pi]$ such that, for each $\gamma \in \mathcal{F}$, there exists $x_{\gamma} \in X$ with $\gamma^* \subseteq L(x_{\gamma}, \theta) \cap X$.*

1.1.3 Chapter 4

In Chapter 4 we continue our study of $\mathcal{F}$ -differentiable functions. In particular, we study new notions of quasianalyticity and analyticity for $\mathcal{F}$ -differentiation, which we call $\mathcal{F}$ -quasianalyticity and $\mathcal{F}$ -analyticity, respectively.

Section 4.1 contains a short survey of quasianalyticity in the theory of Banach function algebras.

In Section 4.2 we introduce a new notion of quasianalyticity for $\mathcal{F}$ -differentiation, and prove a new version of the classical Denjoy–Carleman theorem in this setting.

Theorem 4.2.7. *Let $\Gamma : [a, b] \rightarrow \mathbb{C}$ be an admissible path and let $\mathcal{F}$ denote the collection of all subpaths and reverses of subpaths of Γ . Let $(M_n)_{n=0}^\infty$ be a sequence of positive real numbers such that $\sum_{n=1}^\infty M_n^{-1/n} = \infty$. Suppose that $f \in D_{\mathcal{F}}^{(\infty)}(\Gamma^*)$ and $z_0 \in \Gamma^*$ satisfy*

$$|f^{[k]}|_{\Gamma^*} \leq M_k \quad \text{and} \quad f^{[k]}(z_0) = 0 \quad (1.1)$$

for all $k \in \mathbb{N}_0$. Then $f(z) = 0$ for all $z \in \Gamma^*$.

In Section 4.3 we study the differences between various notions of analyticity. The next theorem gives an example of a compact set in $\mathbb{C}$ and a collection of paths where these notions of analyticity differ.

Theorem 4.3.4. *There exist a simple compact plane set X , an effective collection of paths $\mathcal{F}$ in X , and $g \in D_{\mathcal{F}}^{(\infty)}(X)$ such that g is $\mathcal{F}$ -analytic but not analytic.*

We then give a characterisation of $\mathcal{F}$ -analytic functions in terms of their behaviour on each of the paths under consideration.

Theorem 4.3.5. *Let X be a semi-rectifiable compact plane set, let $\mathcal{G}$ be an effective collection of paths in X , and let $g \in D_{\mathcal{G}}^{(\infty)}(X)$ be $\mathcal{G}$ -analytic. Then, for each $\gamma \in \mathcal{G}$ there exist an open neighbourhood U of γ^* and an analytic function $f : U \rightarrow \mathbb{C}$ such that $f|_{\gamma^*} = g|_{\gamma^*}$.*

In Section 4.4 we study sets of quasianalyticity to determine the relationship $\mathcal{F}$ -quasianalyticity and the collection of paths $\mathcal{F}$. The next theorem asserts that, in some cases, sets of quasianalyticity are always connected.

Theorem 4.4.5. *Let $x \in X$. Suppose that A is a natural Banach function algebra on X . Then $Q_{A, \mathcal{F}}(x)$ is a connected subset of X .*

We show that sets of quasianalyticity can be used to determine whether an algebra A is quasianalytic or $\mathcal{F}$ -quasianalytic.

Theorem 4.4.6. *The algebra A is quasianalytic if and only if, for each $x \in X$, $Q_{A,\mathcal{F}}(x)$ contains the component of X containing x , and A is $\mathcal{F}$ -quasianalytic if and only if, for each $x \in X$ and $\gamma \in \mathcal{F}$ with $x \in \gamma^*$, we have $\gamma^* \subseteq Q_{A,\mathcal{F}}(x)$.*

We then establish the connection between sets of quasianalyticity relative to a given collection of paths and to the associated maximal collection of paths. The next theorem provides the main tool for proving this connection.

Theorem 4.4.10. *Let $\mathcal{G}$ be an effective collection of paths in X , let $E \subseteq X$, and let $z_0 \in X$. Suppose that $A \subseteq D_{\mathcal{G}}^{(\infty)}(X)$ and that $\iota_{\mathcal{G}}(f) = \iota_{\mathcal{F}}(f)$ for all $f \in A$. Then E is set of A -quasianalyticity for z_0 with respect to $\mathcal{F}$ if and only if E is a set of A -quasianalyticity for z_0 with respect to $\mathcal{G}$.*

Section 4.5 contains the construction of a classical Swiss cheese set to replace an example due to Feinstein.

Theorem 4.5.4. *There exists a classical abstract Swiss cheese $A = ((a_n, r_n))$ such that $R(X_A)$ is not regular and does not admit any non-trivial Jensen measures.*

1.1.4 Chapter 5

In Chapter 5 we investigate certain methods for extending uniform algebras. We begin by exploring the properties of Cole extensions. We then explore a more general class of extensions which preserve non-triviality of the original algebra.

Section 5.1 contains a brief introduction to direct limits and inverse limits that will be used throughout the chapter.

In Section 5.2 we prove some facts for Banach algebras and uniform algebras that we use throughout the chapter. In particular, we investigate the Shilov boundary for certain uniform algebra extensions.

Theorem 5.2.7. *Let A be a uniform algebra on Φ_A , let B be a uniform algebra on Φ_B , and let $\theta : A \rightarrow B$ be an isometric, unital algebra homomorphism. Set $\tau := \theta^*|_{\Phi_B} : \Phi_B \rightarrow \Phi_A$. Suppose that $\tau(\Gamma(B)) \subseteq \Gamma(A)$ and that, for each $\psi \in \Gamma(A)$, $B|_{\tau^{-1}(\{\psi\})}$ is uniformly dense in $C(\tau^{-1}(\{\psi\}))$. Then $\Gamma(B) = \tau^{-1}(\Gamma(A))$.*

Section 5.3 contains a short introduction to (systems of) Cole extensions, and in Section 5.4 we provide new proofs for some facts about the Shilov boundary for Cole extensions.

In Section 5.5 we study simple Cole extensions of a uniform algebra. In particular, we study the structure of Cole extensions relative to certain projections.

We first show that every Cole extension admits a projection with some desirable properties.

Theorem 5.5.2. *Let $q(t) = f_0 + f_1t + \cdots + f_{n-1}t^{n-1} + t^n \in A[t]$. Then there exists a contractive, unital projection $P : A^q \rightarrow A^q$ such that $P(A^q) = \pi^*(A)$ and $P(p_q) = 0$. Moreover, P is a $\pi^*(A)$ -module homomorphism.*

We then state and prove a structure theorem for Cole extensions relative to certain projections. We prove that, in some cases, we are not required to take uniform closures when constructing a simple Cole extension.

Theorem 5.5.6. *Let X be a compact space, let A be a uniform algebra on X , and let $q(t) \in A[t]$ be a monic polynomial of degree $n \in \mathbb{N}$.*

- (a) *Let $P \in \mathcal{B}(A^q)$ be a continuous, unital projection such $P(A^q) = \pi^*(A)$ and such that P is a $\pi^*(A)$ -module homomorphism. Set $Q := \text{id}_{A^q} - P$. Then*

$$A^q = \pi^*(A) + \overline{\sum_{j=1}^{n-1} Q(p_q^j) \cdot \pi^*(A)}.$$

- (b) *Suppose that $q(0)$ is not a zero divisor in A . The following are equivalent.*

- (i) *The function $q(0)$ has no zeros on the Shilov boundary for A .*
 - (ii) *The map $L_{p_q} \in \mathcal{B}(C(X^q))$ given by $L_{p_q}(f) = p_q f$ is bounded below on A^q .*
 - (iii) *The subspace $p_q^j \cdot \pi^*(A)$ is closed in A^q for $j = 0, \dots, n-1$.*
- (c) *Suppose that $q(0)$ has no zeros on $\Gamma(A)$ and that there exists a continuous, projection P on A^q with $P(A^q) = \pi^*(A)$ such that P is an $\pi_q^*(A)$ -module homomorphism and $(\text{id}_{A^q} - P)(p_q^j) = p_q^j$ for $j = 1, \dots, n-1$. Then*

$$A^q = \pi^*(A) + \sum_{j=1}^{n-1} p_q^j \cdot \pi^*(A).$$

In Section 5.6 we discuss a new class of uniform algebra extensions that are a generalisation of Cole extensions. In particular, these extensions always preserve non-triviality of the original algebra. However, in some cases, these new extensions are Cole extensions.

Theorem 5.6.8. *Let X be a compact space, let A be a uniform algebra on X , and let $P \in \mathcal{B}(A)$ be a contractive, unital projection. Suppose that $\text{id}_A - P$ is also*

contractive. Set $B := P(A)$ and set $E := \ker P$. Suppose that B is a uniform algebra on a compact space Y , that E is a Banach B -bimodule, and that there exists $h_0 \in E$ such that $E = \overline{h_0 \cdot B}$. Then there exists $h \in B$ and an isometric, unital algebra isomorphism $\theta : B^q \rightarrow A$, where $q(t) = t^2 - h$, such that $\theta \circ \pi_q^* = \text{id}_B$ and $\theta(p_q) = h_0$.

In Section 5.6.2, we explore the existence and properties of maps from a generalised Cole extension onto the original algebra. In particular we seek sufficient conditions for the existence of algebra homomorphisms from the extension onto the original algebra.

Theorem 5.6.16. *Let X, Y be compact spaces, let π_X and π_Y be the coordinate projections from $X \times Y$ onto X and Y respectively, let Z be a closed subset of $X \times Y$, let A be a uniform algebra on X , let $S \subseteq C(Z)$ such that $\pi_X^*(A) \subseteq S$, and let $\varphi : X \rightarrow Y$ be a continuous function. Set $\tilde{\varphi}(x) = (x, \varphi(x))$ ($x \in X$). Suppose that, for all $x \in X$, we have $\tilde{\varphi}(x) \in Z$ and that, for all $f \in S$, we have $f \circ \tilde{\varphi} \in A$. Then there exists a contractive, unital algebra homomorphism $\theta : C(Z) \rightarrow C(X)$ such that $\theta \circ \pi_X^*(X) = \text{id}_{C(X)}$ and $\theta(f) = f \circ \tilde{\varphi}$ for all $f \in C(Z)$. Moreover, we have $\theta(\overline{\text{alg}_{C(Z)} S}) = A$.*

1.1.5 Chapter 6

In Chapter 6 we study methods for extending continuous derivations from a given Banach algebra to a Banach algebra extension. In particular, we study the extension of derivations via certain projections.

In Section 6.1 we give some general results for extending continuous derivations. We prove that a continuous homomorphism is not always necessary to find derivation extensions.

Theorem 6.1.9. *Let $\mathfrak{B} \subseteq \mathfrak{A}$ be Banach algebras. Let P be a continuous, unital projection on $\mathfrak{A}$ such that $P(\mathfrak{A}) = \mathfrak{B}$ and set $Q := \text{id}_{\mathfrak{A}} - P$. Suppose that P is a $\mathfrak{B}$ -module homomorphism, $E := Q(\mathfrak{A})$ is a Banach $\mathfrak{B}$ -bimodule, and $Q(E^2) = \{0\}$. Suppose further that there exist bounded $\mathfrak{B}$ -module homomorphisms $\delta, \mu : E \rightarrow \mathfrak{B}$ such that*

$$P(xy) = xy = \delta(x)\mu(y) = \mu(x)\delta(y) \quad (1.2)$$

for all $x, y \in E$. Let $D_1 \in \mathcal{D}(\mathfrak{B})$ and set

$$D_2 = (\mu^* \circ D_1 \circ \delta + \delta^* \circ D_1 \circ \mu)/2.$$

Then $D := P^* \circ D_1 \circ P + Q^* \circ D_2 \circ Q$ is a continuous derivation from $\mathfrak{A}$ into $\mathfrak{A}^*$. Moreover, for all $a \in \mathfrak{A}$, we have

$$\|D(a)\| \leq \|D_1\|(\|P\|\|P(a)\| + \|\delta\|\|\mu\|\|Q\|\|Q(a)\|).$$

In Section 6.2 we show that there is a non-trivial uniform algebra such that whenever a generalised Cole extension of this algebra is considered, any continuous derivation that extends must do so without increasing the operator norm. The proof is established by contradiction, which is obtained by constructing a non-trivial, weakly amenable uniform algebra. Thus if one can show that for any non-trivial uniform algebra there exists a generalised Cole extension and a non-zero, continuous derivation which extends with a strictly larger norm, then this theorem gives a non-trivial, weakly amenable uniform algebra.

Theorem 6.2.4. *There exists a non-trivial uniform algebra A on a compact space X with the following property: for every $D \in \mathcal{D}_0(A)$ and every uniform algebra B on a compact space Y , such that B is a generalised Cole extension of A with associated maps $\pi : Y \rightarrow X$ and $T : C(Y) \rightarrow C(X)$, if there exists $\tilde{D} \in \mathcal{D}_0(B)$ such that $\tilde{D}(\pi^*(f))(\pi^*(g)) = D(f)(g)$ for all $f, g \in A$ then $\|\tilde{D}\| = \|D\|$.*

In Section 6.3, we consider the construction discussed in the previous section for the specific case of the disk algebra. In particular, we study a sequence of continuous derivations—from the disk algebra into its dual—that correspond to derivations from certain Cole extensions of the disk algebra. In some cases, there is an algebra homomorphism from the Cole extension onto the original algebra, which is the content of the following theorem.

Theorem 6.3.6. *Let $\alpha \in \mathbb{C} \setminus \mathbb{D}$, let $h = Z - \alpha$, and let $\rho(t) = t^2 - h \in A[t]$. Then there exists a contractive, surjective, unital algebra homomorphism $\theta : A^\rho \rightarrow A$ with $\theta \circ \pi_\rho^* = \text{id}_A$.*

1.2 Notation and background

We now fix our notation and recall some elementary facts which will be required throughout this thesis. Throughout this thesis, all linear spaces and algebras are over $\mathbb{C}$. We also adopt the convention that algebras which are not necessarily commutative are denoted by the letters $\mathfrak{A}, \mathfrak{B}$, for example, and we reserve the letters A, B for commutative algebras. We shall recall the definition of an algebra in Section 1.2.2 for the convenience of the reader.

We write $\mathbb{N}$ for the set of positive integers, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ for the set of non-negative integers. A subset I of $\mathbb{N}$ (or $\mathbb{N}_0$) is an *initial segment* if, for each $n \in I$, whenever $k \in \mathbb{N}$ (or $\mathbb{N}_0$) such that $k \leq n$, then $k \in I$. The *open unit disk* in the complex plane is denoted $\mathbb{D}$ (that is, the set of all complex numbers of modulus strictly less than one), and the closure, in the standard topology on $\mathbb{C}$, of $\mathbb{D}$ is the *closed unit disk* denoted by $\overline{\mathbb{D}}$. The *unit circle* in $\mathbb{C}$ is denoted $\mathbb{T}$.

When using transfinite induction, we sometimes index by ordinals rather than positive integers. We refer the reader to [68, Chapter 1] for a formal introduction to ordinals. Let α, β be ordinals. We write $\alpha < \beta$ if $\alpha \in \beta$ (or, equivalently, if $\alpha \subset \beta$), and we write $\alpha \leq \beta$ if we also allow for equality. It is standard that the class of all ordinals is well-ordered with this ordering. The first infinite ordinal will usually be denoted by ω , and the first uncountable ordinal will be denoted by ω_1 . Let κ be a fixed ordinal. We write $\{\alpha : \alpha \leq \kappa\}$ for the set of ordinals α such that $\alpha \leq \kappa$, and $(O_\alpha)_{\alpha \leq \kappa}$ is short-hand for the family of O_α where $\alpha \in \{\beta : \beta \leq \kappa\}$. The *successor* of an ordinal α is denoted $\alpha + 1$.

Let X, Y be sets. We denote the set of all functions from X into Y by Y^X . Let $f \in Y^X$. We write $f|E$ for the restriction of f to a subset E of X , and, if $S \subseteq Y^X$, we write $S|E$ for the set $\{f|E : f \in S\}$. The *preimage* of a subset F of Y under f , denoted $f^{-1}(F)$, is the set $\{x \in X : f(x) \in F\}$. If Z is another set and $\varphi : X \rightarrow Z$ is a function, then $\varphi^* : Y^Z \rightarrow Y^X$ defined by $\varphi^*(f) = f \circ \varphi$ ($f \in Y^Z$) is induced by φ . By abuse of notation, we shall often use φ^* to mean the restriction to some subset of Y^Z . The identity map on X is denoted by id_X .

In the special case of $Y = \mathbb{C}$, $\mathbb{C}^X$ is the space of complex-valued functions on X . Given pointwise operations, $\mathbb{C}^X$ is a commutative algebra. Let $f \in \mathbb{C}^X$. We write $\overline{f}$ for the function given by $\overline{f}(x) = \overline{f(x)}$ ($x \in X$). If X is a subset of $\mathbb{C}$, we shall often write Z for the identity function on X which we call the *coordinate functional*.

1.2.1 Topological preliminaries

Throughout this thesis, we say *compact space* to mean a non-empty, compact, Hausdorff topological space. We say *compact plane set* to mean a non-empty, compact subset of the complex plane $\mathbb{C}$.

Let X be a topological space and let $E \subseteq X$. We write $\text{int}_X(E)$ for the interior of E relative to X ; that is, $\text{int}_X(E)$ is the largest open set of X contained in E . If $X = \mathbb{C}$, we write $\text{int}(E)$ for $\text{int}_{\mathbb{C}}(E)$. We denote the closure of E in X by $\overline{E}$. (We shall specify the topology, or space, in which the closure should be taken if there is

a risk of ambiguity.) We write $\partial_X E$ for the *boundary of E in X* . Again, if $E \subseteq \mathbb{C}$, we write ∂E for $\partial_{\mathbb{C}} E$.

We denote the set of continuous, complex-valued functions on X by $C(X)$. Let $E \subseteq X$ and let $f \in C(X)$. We write

$$|f|_E := \sup\{|f(x)| : x \in E\}. \quad (1.3)$$

We give $C(X)$ pointwise operations of addition, scalar multiplication, and multiplication to make $C(X)$ into a commutative algebra. If X is a compact space, then $C(X)$ is a Banach space with the *uniform norm* $|\cdot|_X$ given by (1.3), and $|fg|_X \leq |f|_X |g|_X$ for all $f, g \in C(X)$. We write $C_0(X)$ for the subset of $C(X)$ consisting of all functions which *vanish at infinity*: $f \in C_0(X)$ if and only if $f \in C(X)$ and, for each $\varepsilon > 0$, the set $\{x \in X : |f(x)| \geq \varepsilon\}$ is compact in X . If X is compact, then $C_0(X) = C(X)$. Let $f \in \mathbb{C}^X$. We write $\text{supp } f$ for the *closed support of f* , which is the closure in X of the set $\{x \in X : f(x) \neq 0\}$.

Let Y be a second topological space, and let $\varphi : Y \rightarrow X$ be a continuous function. Then φ^* maps $C(X)$ into $C(Y)$. Note that φ^* is linear and satisfies $\varphi^*(fg) = \varphi^*(f)\varphi^*(g)$ for all $f, g \in C(X)$. If X, Y are compact spaces and φ is surjective, then $|\varphi^*(f)|_Y = |f|_X$ for all $f \in C(X)$.

Let $f : X \rightarrow \mathbb{R}$ be a function. We say that f is *upper-semicontinuous* if, for each $t \in \mathbb{R}$, the set $\{x \in X : f(x) < t\}$ is open in X . We say that f is *lower-semicontinuous* if, for each $t \in \mathbb{R}$, the set $\{x \in X : f(x) > t\}$ is open in X . Clearly f is upper-semicontinuous if and only if $-f$ is lower-semicontinuous, and f is continuous if and only if f is both upper- and lower-semicontinuous. It is standard that if K is a non-empty compact subset of X , and f is upper-semicontinuous, then $s := \sup\{|f(x)| : x \in K\}$ is finite and there exists $x_0 \in K$ such that $|f(x_0)| = s$. An analogous statement holds if f is lower-semicontinuous and the supremum is replaced by the infimum.

Let (X, d) be a metric space, let $x \in X$, and let $r > 0$. We write $B_X(x, r)$ for the open ball, centred at x of radius r in X and we write $\bar{B}_X(x, r)$ for the corresponding closed ball. That is, $B_X(x, r) := \{y \in X : d(x, y) < r\}$ and $\bar{B}_X(x, r) := \{y \in X : d(x, y) \leq r\}$. In the special case where $X = \mathbb{C}$ and d is the usual metric on $\mathbb{C}$, we write $B(x, r)$ for $B_{\mathbb{C}}(x, r)$ and $\bar{B}(x, r)$ for $\bar{B}_{\mathbb{C}}(x, r)$. We also set $B(x, 0) := \emptyset$ and $\bar{B}(x, 0) := \{x\}$ for each $x \in \mathbb{C}$.

Let $E \subseteq X$ be non-empty and let $x \in X$. We write $\text{dist}_X(x, E)$ for the *distance from x to E* given by

$$\text{dist}_X(x, E) = \inf\{d(x, y) : y \in E\}.$$

Note that $\text{dist}_X(x, E) \geq 0$, and $\text{dist}_X(x, E) = 0$ if and only if $x \in \overline{E}$. We write $\text{dist}(z, E)$ for $\text{dist}_{\mathbb{C}}(z, E)$ for $z \in \mathbb{C}$ and $E \subseteq \mathbb{C}$.

Let $S \subseteq \mathbb{C}$. We say that S is *convex* if, for each $z, w \in S$ and $t \in [0, 1]$, we have $tz + (1 - t)w \in S$. By convention, the empty set is convex. We write $\text{conv } S$ for the smallest convex subset of $\mathbb{C}$ which contains S . We write $\overline{\text{conv}} S$ for the smallest closed convex subset of $\mathbb{C}$ which contains S . We call $\text{conv } S$ the *convex hull* of S , and $\overline{\text{conv}} S$ is the *closed convex hull* of S .

1.2.2 Algebraic preliminaries

Recall that a (complex) *algebra* is a (complex) vector space $\mathfrak{A}$ with a multiplication $\mathfrak{A} \times \mathfrak{A} \rightarrow \mathfrak{A} : (a, b) \mapsto ab$ such that $(ab)c = a(bc)$, $(a + b)c = ac + bc$, $a(b + c) = ab + ac$ for all $a, b, c \in \mathfrak{A}$, and $\alpha(ab) = (\alpha a)b = a(\alpha b)$ for all $a, b \in \mathfrak{A}$, $\alpha \in \mathbb{C}$. We say that $\mathfrak{A}$ is *commutative* if $ab = ba$ for all $a, b \in \mathfrak{A}$; we say that $\mathfrak{A}$ is *unital* if there exists an element $1_{\mathfrak{A}} \neq 0$ such that $a1_{\mathfrak{A}} = 1_{\mathfrak{A}}a = a$ for all $a \in \mathfrak{A}$. An element $1_{\mathfrak{A}}$ with these properties is necessarily unique, and is called the *identity* of $\mathfrak{A}$. If $\mathfrak{A}$ is unital, we shall often identify the complex numbers with the subalgebra $\mathbb{C} \cdot 1_{\mathfrak{A}} := \{z1_{\mathfrak{A}} : z \in \mathbb{C}\}$, and write 1 rather than $1_{\mathfrak{A}}$ for the identity element of $\mathfrak{A}$.

Let $\mathfrak{A}$ be an algebra. A *subalgebra* of $\mathfrak{A}$ is a linear subspace $\mathfrak{B}$ of $\mathfrak{A}$ such that $bb' \in \mathfrak{B}$ for all $b, b' \in \mathfrak{B}$. If $\mathfrak{A}$ is a unital algebra, we say that a subalgebra $\mathfrak{B}$ (of $\mathfrak{A}$) is *unital* if $1_{\mathfrak{A}} \in \mathfrak{B}$. A *left* (respectively, *right*) *ideal* in $\mathfrak{A}$ is a linear subspace I of $\mathfrak{A}$ such that $ab \in I$ (respectively, $ba \in I$) for all $a \in \mathfrak{A}$ and $b \in I$. An *ideal* in $\mathfrak{A}$ is a linear subspace I which is both a left ideal and a right ideal in $\mathfrak{A}$.

Let $a \in \mathfrak{A}$. We say that a is a *zero divisor* if there exists $b \in \mathfrak{A}$ such that $ab = ba = 0$. We say that a is an *idempotent* if $a^2 = a$.

Let $\mathfrak{X}$ be a vector space and let $S \subseteq \mathfrak{X}$. We write $\text{lin}_{\mathfrak{X}} S$ for the smallest linear subspace of $\mathfrak{X}$ containing S ; where if $S = \emptyset$, we set $\text{lin}_{\mathfrak{X}} S = \{0\}$. Let $T \subseteq \mathfrak{X}$. We write $S + T$ for the subset $\{x + y : x \in S, y \in T\}$ of $\mathfrak{X}$. If S, T are linear subspaces of $\mathfrak{X}$ then $S + T$ is a linear subspace of $\mathfrak{X}$. The (algebraic) *direct sum* of $\mathfrak{X}$ and the vector space $\mathfrak{Y}$ is denoted $\mathfrak{X} \oplus \mathfrak{Y}$.

Let $\mathfrak{A}$ be an algebra and let $S \subseteq \mathfrak{A}$. We write $\text{alg}_{\mathfrak{A}} S$ for the smallest subalgebra of $\mathfrak{A}$ containing S . Let $T \subseteq \mathfrak{A}$. We write $S \cdot T$ for the set $\{st : s \in S, t \in T\}$. In the special case where $S = \{a\}$ for some $a \in \mathfrak{A}$, we write $a \cdot T$ for $S \cdot T$.

Definition 1.2.1. Let $\mathfrak{A}$ be a unital algebra and let $a \in \mathfrak{A}$. We say that a is *invertible* if there exists a (necessarily unique) element $b \in \mathfrak{A}$ such that $ab = ba = 1$. In this case, the element b is called the *inverse* of a and we write $a^{-1} = b$. We denote the set of all invertible elements in $\mathfrak{A}$ by $\text{inv } \mathfrak{A}$.

Let $\mathfrak{A}$ be a unital algebra and let $a \in \mathfrak{A}$. The *resolvent* for a (with respect to $\mathfrak{A}$) is the set

$$\rho_{\mathfrak{A}}(a) := \{\zeta \in \mathbb{C} : (a - \zeta 1) \in \text{inv } \mathfrak{A}\},$$

and the *spectrum* of a (with respect to $\mathfrak{A}$) is the set $\sigma_{\mathfrak{A}}(a) := \mathbb{C} \setminus \rho_{\mathfrak{A}}(a)$. The *spectral radius* of a is $\nu_{\mathfrak{A}}(a) := \sup\{|\zeta| : \zeta \in \sigma_{\mathfrak{A}}(a)\}$. It is possible to define the spectrum for non-unital algebras. However, this is beyond the scope of this thesis.

Definition 1.2.2. Let $\mathfrak{A}, \mathfrak{B}$ be algebras and let $\theta : \mathfrak{A} \rightarrow \mathfrak{B}$ be a function. We say that θ is an *algebra homomorphism* if θ is linear and $\theta(ab) = \theta(a)\theta(b)$ for all $a, b \in \mathfrak{A}$. If both $\mathfrak{A}$ and $\mathfrak{B}$ are unital then we say that θ is a *unital map* if $\theta(1_{\mathfrak{A}}) = 1_{\mathfrak{B}}$.

Let $\mathfrak{A}$ be an algebra. A *character* on $\mathfrak{A}$ is a non-zero, algebra homomorphism from $\mathfrak{A}$ onto $\mathbb{C}$. The set of all characters on $\mathfrak{A}$ is denoted $\Phi_{\mathfrak{A}}$, which we call the *character space* of $\mathfrak{A}$.

If $\mathfrak{A}$ is unital and $\varphi \in \Phi_{\mathfrak{A}}$ then we have $\varphi(1_{\mathfrak{A}}) = 1$ and $\varphi(a) \neq 0$ for each $a \in \text{inv } \mathfrak{A}$. Moreover, if $\varphi, \psi \in \Phi_{\mathfrak{A}}$ such that $\varphi \neq \psi$, then there exists $a \in \mathfrak{A}$ such that $\varphi(a) = 0$ and $\psi(a) = 1$.

The following proposition will be useful throughout this thesis.

Proposition 1.2.3 ([27, Proposition 1.5.28]). *Let $\mathfrak{A}, \mathfrak{B}$ be unital algebras and let $\theta : \mathfrak{A} \rightarrow \mathfrak{B}$ be a unital algebra homomorphism. Then, for each $a \in \mathfrak{A}$, we have $\sigma_{\mathfrak{B}}(\theta(a)) \subseteq \sigma_{\mathfrak{A}}(a)$ and $\nu_{\mathfrak{B}}(\theta(a)) \leq \nu_{\mathfrak{A}}(a)$.*

We next define modules over algebras.

Definition 1.2.4. Let $\mathfrak{A}$ be an algebra. A *left $\mathfrak{A}$ -module*, respectively *right $\mathfrak{A}$ -module*, is a linear space $\mathfrak{X}$ with an operation $\mathfrak{A} \times \mathfrak{X} \rightarrow \mathfrak{X}$, $(a, x) \mapsto a \cdot x$, respectively $(a, x) \mapsto x \cdot a$, satisfying $ab \cdot x = a \cdot (b \cdot x)$, respectively $x \cdot ab = (x \cdot a) \cdot b$, for all $a, b \in \mathfrak{A}$ and for all $x \in \mathfrak{X}$. An *$\mathfrak{A}$ -bimodule* is a linear space $\mathfrak{X}$ which is both a left $\mathfrak{A}$ -module and a right $\mathfrak{A}$ -module, and such that $a \cdot (x \cdot b) = (a \cdot x) \cdot b$ for all $a, b \in \mathfrak{A}$ and $x \in \mathfrak{X}$.

The following definitions are stated only for left $\mathfrak{A}$ -modules but, with suitable modifications, are also valid for right $\mathfrak{A}$ -modules and $\mathfrak{A}$ -bimodules. Let $\mathfrak{A}$ be an unital algebra and let $\mathfrak{X}$ be a left $\mathfrak{A}$ -module. We say that $\mathfrak{X}$ is *unital* if $1_{\mathfrak{A}} \cdot x = x$ for all $x \in \mathfrak{X}$.

A *left $\mathfrak{A}$ -submodule* of $\mathfrak{X}$ is a linear subspace $\mathfrak{Y}$ of $\mathfrak{X}$ which is also a left $\mathfrak{A}$ -module with respect to the module action inherited from $\mathfrak{X}$. Right $\mathfrak{A}$ -submodules and $\mathfrak{A}$ -sub-bimodules similarly.

Let $S \subseteq \mathfrak{A}$ and $T \subseteq \mathfrak{X}$. We write $S \cdot T$ for the set $\{a \cdot x : a \in S, x \in T\}$. In the special case where $S = \{a\}$ for some $a \in \mathfrak{A}$, we write $a \cdot S$ for $\{a\} \cdot S$. Similarly, if $T = \{x\}$ for some $x \in \mathfrak{X}$, we write $S \cdot x$ for $S \cdot \{x\}$. Let $\mathfrak{Y}$ be a left $\mathfrak{A}$ -submodule of $\mathfrak{X}$. The intersection of all left $\mathfrak{A}$ -submodules of $\mathfrak{X}$ containing T is called the submodule *generated by T* .

Definition 1.2.5. Let $\mathfrak{A}$ be an algebra and let $\mathfrak{X}$ be an $\mathfrak{A}$ -bimodule. We say that $\mathfrak{X}$ is *commutative* if $a \cdot x = x \cdot a$ for all $a \in \mathfrak{A}$ and all $x \in \mathfrak{X}$.

If $\mathfrak{A}$ is a commutative algebra, then we say $\mathfrak{A}$ -module to mean a commutative $\mathfrak{A}$ -bimodule.

Definition 1.2.6. Let $\mathfrak{A}$ be an algebra, let $\mathfrak{X}, \mathfrak{Y}$ be left $\mathfrak{A}$ -modules, and let $T : \mathfrak{X} \rightarrow \mathfrak{Y}$ be a map. We say that T is a *left $\mathfrak{A}$ -module homomorphism* if T is linear and $T(a \cdot x) = a \cdot T(x)$ for all $a \in \mathfrak{A}$ and $x \in \mathfrak{X}$.

We define right $\mathfrak{A}$ -module and $\mathfrak{A}$ -bimodule homomorphisms similarly. We shall usually say $\mathfrak{A}$ -module homomorphism in place of $\mathfrak{A}$ -bimodule homomorphism.

Definition 1.2.7. Let $\mathfrak{A}$ be an algebra, and let $\mathfrak{X}$ be a $\mathfrak{A}$ -bimodule. A *derivation* from $\mathfrak{A}$ into $\mathfrak{X}$ is a linear map $D : \mathfrak{A} \rightarrow \mathfrak{X}$ such that, for each $a, b \in \mathfrak{A}$, we have

$$D(ab) = a \cdot D(b) + D(a) \cdot b.$$

We say that a derivation $D : \mathfrak{A} \rightarrow \mathfrak{X}$ is *inner* if there exists $x_0 \in \mathfrak{X}$ such that $D(a) = a \cdot x_0 - x_0 \cdot a$.

Let $\mathfrak{A}$ be an algebra, and let φ be a character on $\mathfrak{A}$. A *point derivation (on $\mathfrak{A}$) at φ* is a linear functional $d : \mathfrak{A} \rightarrow \mathbb{C}$ such that, for each $a, b \in \mathfrak{A}$, we have

$$d(ab) = \varphi(a)d(b) + d(a)\varphi(b).$$

We are also interested in higher order point derivations. (It is easy to see that the following definition generalises the notion of a point derivation.)

Definition 1.2.8. Let $\mathfrak{A}$ be an algebra and let $\varphi \in \Phi_{\mathfrak{A}}$. A *point derivation of order $n \in \mathbb{N}$ (respectively, ∞) at φ* is a sequence $d_0, d_1, \dots, d_n$ (respectively, $d_0, d_1, \dots$) of linear functionals with $d_0 = \varphi$, satisfying

$$d_j(ab) = \sum_{k=0}^j d_k(a)d_{j-k}(b) \quad (a, b \in \mathfrak{A}),$$

for $j = 1, 2, \dots, n$ (respectively, $j = 1, 2, \dots$).

Let $\mathbf{d} = (d_j)_{j=0}^n$ be a point derivation of order n at x (where we include the possibility that $n = \infty$ when (d_j) is a point derivation of infinite order). We say that $\mathbf{d}$ is *non-degenerate* if $d_1 \neq 0$.

We refer the reader to [30] (especially Lemma 2.1 and p. 170) for further details, related results and comments concerning non-degenerate higher point derivations.

The following construction is useful when dealing with derivations.

Proposition 1.2.9. *Let $\mathfrak{A}$ be an algebra, and let $\mathfrak{X}$ be an $\mathfrak{A}$ -bimodule. Let $\mathfrak{B} := \mathfrak{A} \oplus \mathfrak{X}$. Define the product*

$$(a_1, x_1) \cdot (a_2, x_2) = (a_1 a_2, a_1 \cdot x_2 + x_1 \cdot a_2) \quad (a_1, a_2 \in \mathfrak{A}, x_1, x_2 \in \mathfrak{X}) \quad (1.4)$$

on $\mathfrak{B}$. Then $\mathfrak{B}$ is an algebra with this product. Let $D : \mathfrak{A} \rightarrow \mathfrak{X}$ be a derivation, and let $\theta : \mathfrak{A} \rightarrow \mathfrak{B}$ be given by $\theta(a) = (a, D(a))$ ($a \in \mathfrak{A}$). Then θ is an algebra homomorphism. If $\mathfrak{A}$ and $\mathfrak{X}$ are unital, then both $\mathfrak{B}$ and θ are unital. If both $\mathfrak{A}$ and $\mathfrak{X}$ are commutative, then $\mathfrak{B}$ is commutative.

1.2.3 Banach spaces and bounded linear maps

We now fix our notation and terminology regarding normed spaces, Banach spaces, and bounded linear maps.

Let $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$, $(\mathfrak{Y}, \|\cdot\|_{\mathfrak{Y}})$ be normed spaces. We write $\mathfrak{X} \oplus_1 \mathfrak{Y}$ for the algebraic direct sum $\mathfrak{X} \oplus \mathfrak{Y}$ equipped with the norm $\|(x, y)\|_1 := \|x\|_{\mathfrak{X}} + \|y\|_{\mathfrak{Y}}$. If both $\mathfrak{X}$ and $\mathfrak{Y}$ are Banach spaces, then so is $\mathfrak{X} \oplus_1 \mathfrak{Y}$. Let S be a subset of $\mathfrak{X}$. We write $\overline{\text{lin}}_{\mathfrak{X}} S$ for the closure in $\mathfrak{X}$ of $\text{lin}_{\mathfrak{X}} S$.

We denote the space of all bounded (continuous) linear maps from $\mathfrak{X}$ into $\mathfrak{Y}$ be $\mathcal{B}(\mathfrak{X}; \mathfrak{Y})$. We give $\mathcal{B}(\mathfrak{X}; \mathfrak{Y})$ the *operator norm*, which is denoted $\|\cdot\|$ to make $\mathcal{B}(\mathfrak{X}; \mathfrak{Y})$ a normed space. (It should be clear from context when this is the norm that we are using.) If $\mathfrak{Y}$ is a Banach space, then $\mathcal{B}(\mathfrak{X}; \mathfrak{Y})$ is a Banach space.

We write $\mathcal{B}(\mathfrak{X})$ for $\mathcal{B}(\mathfrak{X}; \mathfrak{X})$, and we write $\mathfrak{X}^*$ for the dual space of $\mathfrak{X}$, given by $\mathcal{B}(\mathfrak{X}; \mathbb{C})$. It is standard that $\mathcal{B}(\mathfrak{X})$ is a (non-commutative) unital algebra with multiplication given by composition, and $\|S \circ T\| \leq \|S\| \|T\|$ for all $S, T \in \mathcal{B}(\mathfrak{X})$. (If $\mathfrak{X}$ is a Hilbert space, then $\mathcal{B}(\mathfrak{X})$ is a C^* -algebra see Appendix A.1.)

The *kernel* of $T \in \mathcal{B}(\mathfrak{X}; \mathfrak{Y})$ is denoted by $\ker T$ and the *image* of T is denoted $T(\mathfrak{X})$. It is standard that $\ker T$ is a closed linear subspace of $\mathfrak{X}$, and $T(\mathfrak{X})$ is a linear subspace of $\mathfrak{Y}$.

Definition 1.2.10. Let $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}}), (\mathfrak{Y}, \|\cdot\|_{\mathfrak{Y}})$ be normed spaces and $T \in \mathcal{B}(\mathfrak{X}; \mathfrak{Y})$. We say that T is *contractive* if $\|T\| \leq 1$, and we say that T is *isometric* if $\|T(x)\|_{\mathfrak{Y}} = \|x\|_{\mathfrak{X}}$ for all $x \in \mathfrak{X}$.

Let $\mathfrak{X}$ be a normed space and let $T \in \mathcal{B}(\mathfrak{X})$. We say that T is a *projection* (on $\mathfrak{X}$) if $T \circ T = T$. We have $\ker T \cap T(\mathfrak{X}) = \{0\}$ and $\mathfrak{X} = T(\mathfrak{X}) + \ker T$. Let $S = \text{id}_{\mathfrak{X}} - T$. Then S is also a projection on $\mathfrak{X}$, $S(\mathfrak{X}) = \ker T$, $\ker S = T(\mathfrak{X})$. Note that $\|T\| \geq 1$ and $\|S\| \leq 1 + \|T\|$.

Let E be a closed linear subspace of $\mathfrak{X}$. We say that E is *complemented in $\mathfrak{X}$* if there exists a closed linear subspace F of $\mathfrak{X}$ such that $E \cap F = \{0\}$ and $E + F = \mathfrak{X}$. Equivalently, there is a projection $P \in \mathcal{B}(\mathfrak{X})$ such that $P(\mathfrak{X}) = E$.

Definition 1.2.11. Let $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}}), (\mathfrak{Y}, \|\cdot\|_{\mathfrak{Y}})$ be normed spaces and $T \in \mathcal{B}(\mathfrak{X}; \mathfrak{Y})$. We say that T is an *open map* if there exists $C > 0$ such that, for each $y \in T(\mathfrak{X})$, there exists $x \in \mathfrak{X}$ such that $T(x) = y$ and $\|x\|_{\mathfrak{X}} \leq C\|y\|_{\mathfrak{Y}}$. We say that T is a *linear homeomorphism* if T is a bijection and T is an open map. (Equivalently, T is an invertible bounded linear map and T^{-1} is also bounded.)

If $\mathfrak{X}$ and $\mathfrak{Y}$ are Banach spaces then: every surjective $T \in \mathcal{B}(\mathfrak{X}; \mathfrak{Y})$ is an open map (open mapping theorem); every bijective $T \in \mathcal{B}(\mathfrak{X}; \mathfrak{Y})$ is a linear homeomorphism (Banach isomorphism theorem); if $T : \mathfrak{X} \rightarrow \mathfrak{Y}$ is a linear map then T is bounded if and only if $\{(x, T(x)) : x \in \mathfrak{X}\}$ is closed in $\mathfrak{X} \oplus_1 \mathfrak{Y}$ (closed graph theorem).

1.2.4 Measures

In this thesis we say *measure* to mean a regular, complex Borel measure. We denote the *total variation* of a measure μ by $|\mu|$, and we write $\|\mu\|$ for the *total variation norm* of μ . A *probability measure* is a positive measure with norm one.

Let X be a locally compact topological space and let μ be a measure on X . We write $\text{supp } \mu$ for the largest closed set E of X such that, for each $x \in E$ and each neighbourhood U of x , we have $|\mu|(U) > 0$.

Let $x \in X$. We write δ_x for the *point-mass measure* at x ; that is, δ_x is the measure on X such that $\delta_x(E) = 1$ if $x \in E$ and $\delta_x(E) = 0$ otherwise, where E is a Borel set in X .

We refer the reader to [93] for details on measure theory and standard results, and to [13, 16] and [54] for information on the use of measures in functional analysis.

1.2.5 Polynomials

We now recall the definitions, notation, and standard results about polynomials over a commutative algebra. Throughout this section, let A be a commutative algebra.

Definition 1.2.12. A *polynomial with coefficients in A* is a formal sum

$$q(t) := \sum_{j=0}^{\infty} a_j t^j, \quad (1.5)$$

where t is an *indeterminate over A* , $a_0 t^0 = a_0$, and $a_j \in A$ for each $j \in \mathbb{N}_0$, and all but finitely many of the a_j are zero. We denote the set of all polynomials with coefficients in A (in the indeterminate t) by $A[t]$.

We shall often write q for $q(t)$ when we do not refer to the indeterminate explicitly. Let q be a polynomial with coefficients in A . The largest integer n such that $a_n \neq 0$ is called the *degree* of q , and we denote this n by $\deg q$. If A is unital, we say that q is *monic* if $a_{\deg q} = 1$. We shall usually write $q(t) = a_0 + a_1 t + \cdots + a_n t^n$, where $n = \deg q(t)$, and omit the following terms (which are all zero).

We give $A[t]$ the operations

$$\begin{aligned} \sum_{j=0}^{\infty} a_j t^j + \sum_{j=0}^{\infty} b_j t^j &:= \sum_{j=0}^{\infty} (a_j + b_j) t^j, \\ \left(\sum_{j=0}^{\infty} a_j t^j \right) \cdot \left(\sum_{j=0}^{\infty} b_j t^j \right) &:= \sum_{j=0}^{\infty} \left(\sum_{k=0}^j a_k b_{j-k} \right) t^j, \\ \alpha \left(\sum_{j=0}^{\infty} a_j t^j \right) &:= \sum_{j=0}^{\infty} \alpha a_j t^j, \end{aligned}$$

to make $A[t]$ into a commutative algebra which contains A as a subalgebra. If A is unital, then so is $A[t]$.

Let B be a second commutative algebra with $A \subseteq B$, and let $q(t) \in A[t]$. For each $b \in B$, we write $q(b)$ to mean the element of B obtained by replacing t with b in $q(t)$. The map $q(t) \mapsto q(b) : A[t] \rightarrow B$ is called the *evaluation homomorphism*. Indeed this map is an algebra homomorphism. Let $q(t) \in A[t]$. We say that $b \in B$ is a *root* of q if $q(b) = 0$.

The following proposition is elementary, and will be used later.

Proposition 1.2.13. *Let $q(t) \in A[t]$, let $a, b \in A$, and let $\rho(t) := q(t - a)$. Then b is a root of ρ if and only if $b + a$ is a root of q .*

Let B be a commutative algebra and let $\theta : A \rightarrow B$ be an algebra homomorphism. If $q(t) \in A[t]$ and is given by (1.5), then

$$\rho_q(t) := \sum_{j=0}^{\infty} \theta(a_j) t^j$$

is an element of $B[t]$. The map $q(t) \mapsto \rho_q(t) : A[t] \rightarrow B[t]$ is an algebra homomorphism. By abuse of notation, we write $\theta(q)(t) := \rho_q(t)$ to mean the homomorphism $A[t] \rightarrow B[t]$.

More formal introductions to polynomials over rings (algebras) can be found in numerous texts on abstract algebra. We refer the reader to [27, Section 1.6] for further details.

1.3 Tensor products

Let $\mathfrak{X}_1, \dots, \mathfrak{X}_n, \mathfrak{X}$ be vector spaces. An n -linear map T from $\prod_{j=1}^n \mathfrak{X}_j$ into $\mathfrak{X}$, is a map such that, for each $k \in \{1, \dots, n\}$ and $x_j \in \mathfrak{X}_j$ for each $j \in \{1, \dots, n\} \setminus \{k\}$, $x \mapsto T(x_1, \dots, x_{k-1}, x, x_{k+1}, \dots, x_n)$ ($x \in \mathfrak{X}_k$) is a linear map from $\mathfrak{X}_k$ into $\mathfrak{X}$.

Definition 1.3.1. Let $\mathfrak{X}_1, \dots, \mathfrak{X}_n$ be vector spaces. Then the *tensor product* of $\mathfrak{X}_1, \dots, \mathfrak{X}_n$ is a pair $(\iota, \mathfrak{X})$, where $\mathfrak{X}$ is a vector space and ι is an n -linear map from $\prod_{j=1}^n \mathfrak{X}_j$ into $\mathfrak{X}$, which satisfies the following universal property: for each vector space $\mathfrak{Y}$ and each n -linear map $T : \prod_{j=1}^n \mathfrak{X}_j \rightarrow \mathfrak{Y}$ there exists a unique linear map $S : \mathfrak{X} \rightarrow \mathfrak{Y}$ such that $T = S \circ \iota$. For $(x_1, \dots, x_n) \in \prod_{j=1}^n \mathfrak{X}_j$ we write $x_1 \otimes \dots \otimes x_n$ for $\iota(x_1, \dots, x_n) \in \mathfrak{X}$.

Tensor products are unique up to linear isomorphism and we write $\bigotimes_{j=1}^n \mathfrak{X}_j$ or $\mathfrak{X}_1 \otimes \dots \otimes \mathfrak{X}_n$ for the tensor product of the vector spaces $\mathfrak{X}_1, \dots, \mathfrak{X}_n$. We refer the reader to [27, pp. 26–33] and [13, Chapter VI] for a more detailed discussion of tensor products.

Let $\mathfrak{X}_1, \dots, \mathfrak{X}_n$ be vector spaces and let $z \in \bigotimes_{j=1}^n \mathfrak{X}_j$. Then there exist $m \in \mathbb{N}$, and $x_{i,j} \in \mathfrak{X}_j$ for $i = 1, \dots, m$ and $j = 1, \dots, n$ such that

$$z = \sum_{i=1}^m x_{i,1} \otimes x_{i,2} \otimes \dots \otimes x_{i,n}.$$

However, such representations need not be unique.

Proposition 1.3.2 ([27, Proposition 1.3.11]). *Let $\mathfrak{A}_1, \dots, \mathfrak{A}_n$ be algebras and let $\mathfrak{A}$ be the vector space $\bigotimes_{j=1}^n \mathfrak{A}_j$. Then there exists a unique product on $\mathfrak{A}$ so that $\mathfrak{A}$*

is an algebra with

$$(a_1 \otimes \cdots \otimes a_n)(b_1 \otimes \cdots \otimes b_n) = a_1 b_1 \otimes \cdots \otimes a_n b_n \quad (a_i, b_i \in \mathfrak{A}_i, i = 1, \dots, n).$$

If each $\mathfrak{A}_i$ is commutative then $\mathfrak{A}$ is commutative; if each $\mathfrak{A}_i$ is unital then $\mathfrak{A}$ is unital with identity $1_{\mathfrak{A}_1} \otimes \cdots \otimes 1_{\mathfrak{A}_n}$.

The tensor product of algebras will always be given the above product and considered as an algebra unless otherwise stated.

1.4 Banach algebras

Let $\mathfrak{A}$ be an algebra, and let $\|\cdot\|$ be a norm on the underlying vector space of $\mathfrak{A}$. We say that $\|\cdot\|$ is an *algebra norm* on $\mathfrak{A}$ if $\|ab\| \leq \|a\|\|b\|$ for all $a, b \in \mathfrak{A}$. If $\|\cdot\|$ is an algebra norm on $\mathfrak{A}$, then we say that $\mathfrak{A}$ is a *normed algebra*. A *Banach algebra* is a normed algebra which is also a Banach space. If $\mathfrak{A}$ is a unital algebra, we usually insist that $\|1_{\mathfrak{A}}\| = 1$. However, by moving to an equivalent norm, this can always be achieved.

Let X be a locally compact, Hausdorff topological space. Then $C_0(X)$ is a commutative Banach algebra, with pointwise operations and the uniform norm. If X is a compact space then $C_0(X) = C(X)$ and so $C(X)$ is a commutative, unital Banach algebra with the uniform norm.

We say that an algebra homomorphism $\theta : \mathfrak{A} \rightarrow \mathfrak{B}$, where $\mathfrak{A}$ and $\mathfrak{B}$ are normed algebras, is *continuous* if θ is a bounded linear map. We write $\|\theta\|$ for the operator norm of θ .

Let $\mathfrak{A}$ be a normed algebra, and let $S \subseteq \mathfrak{A}$. We write $\overline{\text{alg}}_{\mathfrak{A}} S$ for the closure in $\mathfrak{A}$ of $\text{alg}_{\mathfrak{A}} S$. In other words, $\overline{\text{alg}}_{\mathfrak{A}} S$ is the smallest closed subalgebra of $\mathfrak{A}$ containing S .

1.4.1 Gelfand theory

Let $\mathfrak{A}$ be a Banach algebra. If φ is a character on $\mathfrak{A}$, then φ is continuous and $\|\varphi\| \leq 1$. Moreover, if $\mathfrak{A}$ is unital then $\|\varphi\| = 1$.

We give $\Phi_{\mathfrak{A}}$ the relative weak-* topology (from $\mathfrak{A}^*$), which we call the *Gelfand topology*, and this makes $\Phi_{\mathfrak{A}}$ a locally compact, Hausdorff, topological space (which may be empty). If $\mathfrak{A}$ is unital, then $\Phi_{\mathfrak{A}}$ is compact in the Gelfand topology.

Suppose that $\Phi_{\mathfrak{A}}$ is non-empty. For each $a \in \mathfrak{A}$, we define the complex-valued function $\hat{a}$ on $\Phi_{\mathfrak{A}}$ by $\hat{a}(\varphi) := \varphi(a)$ ($\varphi \in \Phi_{\mathfrak{A}}$). Clearly $\hat{a}$ is continuous on $\Phi_{\mathfrak{A}}$.

Definition 1.4.1. Let $\mathfrak{A}$ be a Banach algebra with $\Phi_{\mathfrak{A}} \neq \emptyset$. Then the function $\mathcal{G} : \mathfrak{A} \rightarrow C_0(\Phi_{\mathfrak{A}}) : a \mapsto \hat{a}$ is called the *Gelfand transform*. We write $\hat{\mathfrak{A}}$ for the image of the Gelfand transform; i.e. $\hat{\mathfrak{A}} = \{\hat{a} : a \in \mathfrak{A}\}$.

Let $\mathfrak{A}$ be a Banach algebra with $\Phi_{\mathfrak{A}} \neq \emptyset$. It is standard that $\mathcal{G}$ is a continuous algebra homomorphism with $\|\mathcal{G}\| \leq 1$, and if $\mathfrak{A}$ is unital then so is $\mathcal{G}$. Moreover, $\hat{\mathfrak{A}}$ is a subalgebra of $C_0(\Phi_{\mathfrak{A}})$ (but need not be uniformly closed). We say that $\mathfrak{A}$ is *semisimple* if $\mathcal{G}$ is injective. The kernel of $\mathcal{G}$ is $\bigcap \{\ker \varphi : \varphi \in \Phi_{\mathfrak{A}}\}$, which is called the (Jacobson) *radical* of $\mathfrak{A}$.

Note that if A is a commutative, unital Banach algebra then Φ_A is a non-empty and compact. If, in addition, A is semisimple then ν_A is an algebra norm on A , and $\nu_A(a) = |\hat{a}|_{\Phi_A}$ for each $a \in A$.

1.4.2 Properties of Banach algebras

We now recall some standard results regarding Banach algebras. The first is (mostly) part of [27, Theorem 2.3.8]. Statement (a) is sometimes called the *fundamental theorem of Banach algebras*, and statement (b) is the spectral radius formula.

Proposition 1.4.2. *Let $\mathfrak{A}$ be a Banach algebra, and let $a \in \mathfrak{A}$.*

- (a) *The spectrum $\sigma_{\mathfrak{A}}(a)$ is a compact plane set.*
- (b) *We have $\nu_{\mathfrak{A}}(a) = \lim_{n \rightarrow \infty} \|a^n\|^{1/n} = \inf_{n \in \mathbb{N}} \|a^n\|^{1/n}$.*
- (c) *We have $\nu_{\mathfrak{A}}(a) \leq \|a\|$.*

The second is the (single-variable) holomorphic functional calculus ([27, Theorem 2.4.4]). This result is stated for commutative unital Banach algebras, but holds in more generality.

Proposition 1.4.3 (Holomorphic functional calculus). *Let A be a commutative, unital Banach algebra, and let $a \in A$. Let U be an open set in $\mathbb{C}$ such that $\sigma_A(a) \subseteq U$, and let $f : U \rightarrow \mathbb{C}$ be an analytic function. Then there exists a unique element $f(a)$ in A such that $\mathcal{G}(f(a)) = f \circ \hat{a}$. Moreover $\sigma_A(f(a)) = f(\sigma_A(a))$, and $Z(a) = a$.*

In fact, the holomorphic functional calculus states that there is a unital algebra homomorphism from $\mathcal{O}_{\sigma_A(a)}$ into A , where $\mathcal{O}_{\sigma_A(a)}$ denotes the family of germs of analytic functions on $\sigma_A(a)$. (See [27, Appendix A.2] for the relevant

definitions.) We will not require the full strength of the holomorphic functional calculus in this thesis.

The next result is a consequence of the multi-variable holomorphic functional calculus, called the *Shilov idempotent theorem* ([27, Theorem 2.4.33]).

Proposition 1.4.4. *Let A be a commutative, unital Banach algebra. Suppose that K is a non-empty compact and open subset of Φ_A . Then there exists a unique idempotent $p \in A$ such that $\hat{p} = \chi_K$, where χ_K is the characteristic function of K .*

Our final result is an automatic continuity result that will be useful to us. The first statement is [27, Theorem 2.3.3], the second follows from Proposition 1.2.3.

Proposition 1.4.5. *Let A be a Banach algebra, let B be a commutative, semisimple Banach algebra, and let $\theta : A \rightarrow B$ be an algebra homomorphism. Then θ is continuous. If $\|b\|_B = \nu_B(b)$ for all $b \in B$, then $\|\theta\| \leq 1$.*

1.4.3 Modules and derivations

We now discuss the theory of Banach modules and derivations. Throughout this section, let $\mathfrak{A}$ be a Banach algebra.

Definition 1.4.6. A *Banach left $\mathfrak{A}$ -module* is a Banach space $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ which is also a left $\mathfrak{A}$ -module such that $\|a \cdot x\|_{\mathfrak{X}} \leq \|a\| \|x\|_{\mathfrak{X}}$. We define *Banach right $\mathfrak{A}$ -modules* and *Banach $\mathfrak{A}$ -bimodules* similarly.

Again, we usually say *Banach A -module* in place of *commutative Banach A -bimodule* when A is a commutative Banach algebra.

Before we continue, we note some useful examples of Banach modules.

- (a) Suppose that φ is a character on $\mathfrak{A}$. Let $\mathfrak{X} := \mathbb{C}$ with the module operation $a \cdot \zeta = \varphi(a)\zeta$ ($a \in \mathfrak{A}$, $\zeta \in \mathbb{C}$). Then $\mathfrak{X}$ is a Banach left $\mathfrak{A}$ -module (or a commutative $\mathfrak{A}$ -bimodule). The module $\mathfrak{X}$ is usually denoted by $\mathbb{C}_{\varphi}$. If $\mathfrak{A}$ is unital, then $\mathbb{C}_{\varphi}$ is unital.
- (b) Let $\mathfrak{Y}$ be a Banach $\mathfrak{A}$ -bimodule and set $\mathfrak{X} := \mathfrak{Y}^*$ with the operations given by

$$(a \cdot \lambda)(y) = \lambda(y \cdot a) \quad (\lambda \cdot a)(y) = \lambda(a \cdot y) \quad (a \in \mathfrak{A}, y \in \mathfrak{Y}, \lambda \in \mathfrak{X}). \quad (1.6)$$

Then $\mathfrak{X}$ is a Banach $\mathfrak{A}$ -bimodule. If $\mathfrak{Y}$ is unital $\mathfrak{A}$ -bimodule, then $\mathfrak{X}$ is unital. The module $\mathfrak{X}$ is the *dual module* of $\mathfrak{Y}$, and we shall usually denote this

simply by $\mathfrak{Y}^*$. In particular, we can apply this in the case where $\mathfrak{Y} = \mathfrak{A}$. In this case, $\mathfrak{X} = \mathfrak{A}^*$ is commutative if and only if $\mathfrak{A}$ is commutative.

- (c) Let $\mathfrak{B}$ be a second Banach algebra, let $\mathfrak{X}$ be a Banach left $\mathfrak{B}$ -module, and let $\theta : \mathfrak{A} \rightarrow \mathfrak{B}$ be a contractive algebra homomorphism. Then $\mathfrak{X}$ is a Banach left $\mathfrak{A}$ -module with the operation $a \cdot x = \theta(a) \cdot x$ ($a \in \mathfrak{A}$, $x \in \mathfrak{X}$). This also applies if $\mathfrak{X}$ is a Banach right $\mathfrak{B}$ -module or a Banach $\mathfrak{B}$ -bimodule by a suitable modification to the action. If $\mathfrak{A}$ and $\mathfrak{B}$ are unital algebras, $\mathfrak{X}$ is a unital left $\mathfrak{B}$ -module, and θ is a unital algebra homomorphism, then $\mathfrak{X}$ is a unital left $\mathfrak{A}$ -module with this action.

We are mostly concerned with continuous derivations from $\mathfrak{A}$ into some Banach $\mathfrak{A}$ -bimodule $\mathfrak{X}$. Recall that derivation from $\mathfrak{A}$ into $\mathfrak{X}$ is continuous if it is a bounded linear map.

Definition 1.4.7. We say that $\mathfrak{A}$ is *amenable* if, for each Banach $\mathfrak{A}$ -bimodule $\mathfrak{X}$, every continuous derivation from $\mathfrak{A}$ into $\mathfrak{X}^*$ is inner. We say that $\mathfrak{A}$ is *weakly amenable* if every continuous derivation from $\mathfrak{A}$ into $\mathfrak{A}^*$ is inner.

In both of the above definitions, the module action is the one given by (1.6).

Note that, if $\mathfrak{A}$ is commutative then $\mathfrak{A}$ is weakly amenable if and only if every continuous derivation from $\mathfrak{A}$ into $\mathfrak{A}^*$ is necessarily 0.

We denote the set of all continuous derivations from $\mathfrak{A}$ into $\mathfrak{A}^*$ by $\mathcal{D}(\mathfrak{A})$. By the above, if $\mathfrak{A}$ is commutative then $\mathfrak{A}$ is weakly amenable if and only if $\mathcal{D}(\mathfrak{A}) = \{0\}$.

Definition 1.4.8. Let $\mathbf{d} = (d_j)_{j=0}^n$ be a higher order point derivation of order n (where we include the possibility that $n = \infty$ when $(d_j)_{j=0}^\infty$ is a point derivation of infinite order). We say that $\mathbf{d}$ is *continuous* if d_j is a bounded linear functional for each j with $j \leq n$ (or $j < \infty$ if $n = \infty$).

The following proposition is a Banach algebra version of Proposition 1.2.9.

Proposition 1.4.9. Let $\mathfrak{X}$ be a Banach $\mathfrak{A}$ -bimodule. Set $\mathfrak{B} := \mathfrak{A} \oplus_1 \mathfrak{X}$ with the product given by (1.4). Then $\mathfrak{B}$ is a Banach algebra. Let $D : \mathfrak{A} \rightarrow \mathfrak{X}$ be a continuous derivation, and let $\theta : \mathfrak{A} \rightarrow \mathfrak{B}$ be the map given by $\theta(a) = (a, D(a))$ ($a \in \mathfrak{A}$). Then θ is a continuous algebra homomorphism with $\|\theta\| = 1 + \|D\|$. If $\mathfrak{A}$ and $\mathfrak{X}$ are unital, then both $\mathfrak{B}$ and θ are unital. If both $\mathfrak{A}$ and $\mathfrak{X}$ are commutative, then $\mathfrak{B}$ is commutative.

1.4.4 Tensor products

We now introduce some norms on tensor products of normed spaces.

Definition 1.4.10. Let $\mathfrak{X}_1, \dots, \mathfrak{X}_n$ be normed spaces. The *injective tensor norm*, denoted $\|\cdot\|_\varepsilon$, on $\bigotimes_{j=1}^n \mathfrak{X}_j$ is the norm given by

$$\|z\|_\varepsilon = \sup \left\{ \left| \sum_{i=1}^m \lambda_i(x_{i,1}) \cdots \lambda_i(x_{i,n}) \right| \right\} \quad \left(z \in \bigotimes_{j=1}^n \mathfrak{X}_j \right)$$

where the supremum is taken over all representations $\sum_{i=1}^m x_{i,1} \otimes x_{i,2} \otimes \cdots \otimes x_{i,n}$ of z and all $\lambda_j \in \mathfrak{X}_j^*$ with $\|\lambda_j\| \leq 1$ for $j = 1, \dots, n$. The *projective tensor norm*, denoted $\|\cdot\|_\pi$, on $\bigotimes_{j=1}^n \mathfrak{X}_j$ is the norm given by

$$\|z\|_\pi = \inf \left\{ \sum_{i=1}^m \|x_{i,1}\| \cdots \|x_{i,n}\| \right\} \quad \left(z \in \bigotimes_{j=1}^n \mathfrak{X}_j \right)$$

where the infimum is taken over all representations $z = \sum_{i=1}^m x_{i,1} \otimes x_{i,2} \otimes \cdots \otimes x_{i,n}$.

Let $\mathfrak{X}_1, \dots, \mathfrak{X}_n$ be normed spaces. It is standard that $\|\cdot\|_\varepsilon$ is the norm inherited from the natural embedding of $\bigotimes_{j=1}^n \mathfrak{X}_j$ in the Banach space $\mathcal{B}(\mathfrak{X}_1, \dots, \mathfrak{X}_n; \mathbb{C})$ of all bounded n -linear maps from $\prod_{j=1}^n \mathfrak{X}_j$ into $\mathbb{C}$. The *injective tensor product* $\check{\bigotimes}_{j=1}^n \mathfrak{X}_j$ of $\mathfrak{X}_1, \dots, \mathfrak{X}_n$ is the completion of $(\bigotimes_{j=1}^n \mathfrak{X}_j, \|\cdot\|_\varepsilon)$. The *projective tensor product* $\widehat{\bigotimes}_{j=1}^n \mathfrak{X}_j$ of $\mathfrak{X}_1, \dots, \mathfrak{X}_n$ is the completion of $(\bigotimes_{j=1}^n \mathfrak{X}_j, \|\cdot\|_\pi)$. We shall usually write $\mathfrak{X}_1 \check{\otimes} \cdots \check{\otimes} \mathfrak{X}_n$ for the injective tensor product of $\mathfrak{X}_1, \dots, \mathfrak{X}_n$, and write $\mathfrak{X}_1 \widehat{\otimes} \cdots \widehat{\otimes} \mathfrak{X}_n$ for the projective tensor product of $\mathfrak{X}_1, \dots, \mathfrak{X}_n$.

Both the injective tensor norm and the projective tensor norm are *cross norms*; i.e. if $\|\cdot\|$ denotes either the injective or projective tensor norm on $\bigotimes_{j=1}^n \mathfrak{X}_j$ then

$$\|x_1 \otimes \cdots \otimes x_n\| = \|x_1\| \cdots \|x_n\| \quad \left((x_1, \dots, x_n) \in \prod_{j=1}^n \mathfrak{X}_j \right).$$

Let $\mathfrak{A}_1, \dots, \mathfrak{A}_n$ be Banach algebras. Then $\bigotimes_{j=1}^n \mathfrak{A}_j$ is a normed algebra when given either the injective tensor norm or the projective tensor norm. Moreover, both $\check{\bigotimes}_{j=1}^n \mathfrak{A}_j$ and $\widehat{\bigotimes}_{j=1}^n \mathfrak{A}_j$ are Banach algebras.

We now give two elementary properties of the tensor products of Banach algebras.

Proposition 1.4.11 ([13, Proposition VI.19]). *Let A, B be commutative Banach algebras. Then $\Phi_{A \widehat{\otimes} B}$ is homeomorphic to $\Phi_A \times \Phi_B$.*

The following is effectively [90, Proposition 1.10.21].

Proposition 1.4.12. *Let X, Y be compact spaces. Then there exists an isometric algebra isomorphism from $C(X) \check{\otimes} C(Y)$ onto $C(X \times Y)$.*

1.5 Banach function algebras

We now give the definitions and basic theory of Banach function algebras, which is the main topic of this thesis. Throughout this section, let X be a compact space.

Let S be a subset of $C(X)$. We say that S *separates the points of X* if, for each $x, y \in X$ with $x \neq y$, there exists $f \in S$ such that $f(x) \neq f(y)$.

Definition 1.5.1. Let A be a subalgebra of $C(X)$, and let $\|\cdot\|$ be an algebra norm on A . We say that A is a *normed function algebra (on X)* if A contains all constant functions, A separates the points of X , and $\|f\| \geq |f|_X$ for all $f \in A$. We say that A is a *Banach function algebra (on X)* if A is a normed function algebra (on X) and the norm $\|\cdot\|$ is complete. We say that A is a *uniform algebra (on X)* if A is a Banach function algebra (on X) and $\|\cdot\| = |\cdot|_X$.

Note that every commutative, unital, semisimple Banach algebra A can be identified with a Banach function algebra on Φ_A via the Gelfand transform. (Since A is semisimple, it follows that $\hat{A}$ separates the points of Φ_A .) Also note that, if A is a uniform algebra on X , then the norm on A is ν_A , and the Gelfand transform is isometric. Thus, by Proposition 1.4.2(b), any commutative, unital, semisimple Banach algebra A such that $\|a^2\| = \|a\|^2$ ($a \in A$) is isometrically isomorphic (via the Gelfand transform) to a uniform algebra on Φ_A .

Let A be a normed function algebra on X . For each $x \in X$, let $\varepsilon_x : A \rightarrow \mathbb{C}$ be the map given by $\varepsilon_x(f) = f(x)$. Then ε_x is a character on $C(X)$. The map $x \mapsto \varepsilon_x : X \rightarrow \Phi_A$ identifies X with a subset of Φ_A . The topology on X agrees with relative Gelfand topology from Φ_A . (That is, the map $x \mapsto \varepsilon_x$ is continuous.) Each of the characters ε_x ($x \in X$) are continuous. We usually identify ε_x with x . For example, we usually write d is a *point derivation on A at x* to mean d is a *point derivation on A at ε_x* .

Definition 1.5.2. Let A be a Banach function algebra on X . We say that X is *natural (on X)* if, for each $\varphi \in \Phi_A$, there exists $x \in X$ such that $\varphi = \varepsilon_x$.

It is standard that $C(X)$ is a natural uniform algebra on X .

1.5.1 Regularity

We now discuss regularity for Banach function algebras. Recall that X is a compact space throughout. For each $x \in X$, we define

$$J_x(A) := \{f \in A : x \in \text{int } f^{-1}(\{0\})\} \quad M_x(A) := \{f \in A : f(x) = 0\} = \ker \varepsilon_x.$$

Then both $J_x(A)$ and $M_x(A)$ are ideals in A , and $J_x(A) \subseteq M_x(A)$. If there is no risk of ambiguity, we shall write J_x for $J_x(A)$ and M_x for $M_x(A)$.

Definition 1.5.3. Let A be a Banach function algebra on X , and let $x \in X$. We say that x is an *R-point* (for A) if $J_x \not\subseteq M_y$ for all $y \in X \setminus \{x\}$. We say that x is a *point of continuity* (for A) if $J_y \not\subseteq M_x$ for all $y \in X \setminus \{x\}$.

Note that, in [50], points of continuity and *R*-points were referred to as *points of regularity of types I and II*, respectively.

Definition 1.5.4. Let A be a Banach function algebra on X . We say that A is *regular on X* if, for each closed set E of X , and each $x \in X \setminus E$, there exists $f \in A$ such that $f(x) = 1$ and $f(E) \subseteq \{0\}$. We say that A is *normal on X* if, for each pair of closed subsets E, F of X with $E \cap F = \emptyset$, there exists $f \in A$ such that $f(E) \subseteq \{0\}$ and $f(F) \subseteq \{1\}$. We say that A is *regular* if $\hat{A}$ is regular on Φ_A , and we say that A is *normal* if $\hat{A}$ is normal on Φ_A .

Let A be a Banach function algebra on X . It is easy to see that, if K is a compact subset of X such that each point of K is an *R*-point, and if $y \in X \setminus K$ then there exists $f \in A$ such that $f^{-1}(\{0\})$ is a neighbourhood of K in X and $f(y) = 1$. In addition, it is standard that A is regular on X if and only if each point $x \in X$ is a point of continuity for A , and this occurs if and only if every point $x \in X$ is an *R*-point for A . It is also standard that, if A is natural on X , then A is regular on X if and only if A is normal on X ([27, Proposition 4.1.18]).

Proposition 1.5.5 ([27, Corollary 4.3.10]). *Let A be a uniform algebra on X . If A is normal on X then A is natural on X .*

We refer the reader to [50–52] for details on regularity, *R*-points, and points of continuity for Banach function algebras.

1.6 Uniform algebras

We now discuss uniform algebras in more detail. We refer the reader to [16, 54, 97] for further details on the theory of uniform algebras. The following are standard examples of uniform algebras.

Definition 1.6.1. Let X be a compact plane set. Let $P_0(X)$ denote the algebra of restrictions to X of polynomial functions and let $R_0(X)$ denote the algebra of restrictions to X of all rational functions with no poles on X . We define $P(X)$ and

$R(X)$ to be the uniform closure in $C(X)$ of $P_0(X)$ and $R_0(X)$, respectively. We denote by $A(X)$ the algebra of all continuous functions on X which are analytic on $\text{int } X$.

Let X be a compact plane set. It is easy to see that $P(X)$, $R(X)$, and $A(X)$ are uniform algebras on X , and that following inclusions hold.

$$P(X) \subseteq R(X) \subseteq A(X) \subseteq C(X).$$

It is standard that $R(X)$ and $A(X)$ are natural uniform algebras on X , and $P(X)$ is natural on X if and only if $\mathbb{C} \setminus X$ is connected; in this case, we have $P(X) = R(X)$. It is trivial that $A(X) = C(X)$ if and only if $\text{int } X = \emptyset$.

Definition 1.6.2. Let X be a compact space, let A be a uniform algebra on X , and let $\varphi \in \Phi_A$. A *representing measure for φ (on X , with respect to A)* is a probability measure μ on X such that

$$\varphi(f) = \int_X f \, d\mu \quad (f \in A).$$

For $x \in X$, we say that a representing measure μ for ε_x is *trivial* if $\mu = \delta_x$.

We use the phrase *μ is a representing measure for x* to mean *μ is a representing measure for ε_x* .

Definition 1.6.3. Let X be a compact space, let A be a uniform algebra on X , let $E \subseteq X$, and let $x \in X$. We say that a function $f \in A$ *peaks on E* if $f(E) \subseteq \{1\}$ and $|f(y)| < 1$ for all $y \in X \setminus E$. We say that E is a *peak set (for A)* if there exists $f \in A$ such that f peaks on E . We say that x is a *peak point (for A)* if $\{x\}$ is a peak set. We say that E is a *peak set in the weak sense (for A)* if there is a family (E_α) of peak sets for A such that $E = \bigcap_\alpha E_\alpha$. We say that x is a *strong boundary point (for A)* (or *peak point in the weak sense*) if $\{x\}$ is a peak set in the weak sense for A .

The set of all strong boundary points for A is denoted $\Gamma_0(A)$, and is called the *Choquet boundary* for A . It is standard that if X is metrisable then every strong boundary point is a peak point.

We summarise some properties of strong boundary points in the following (see [16, Theorem 2.3.4], [27, Theorem 4.3.5] and [97, Theorem 7.18]).

Proposition 1.6.4. *Let X be a compact space, let A be a uniform algebra on X , and let $x \in X$. Then the following are equivalent:*

- (a) $x \in \Gamma_0(A)$;
- (b) the unique representing measure for x on X is δ_x ;
- (c) there exists $\alpha, \beta \in (0, 1)$ with $\alpha < \beta$ such that, for each open neighbourhood U of x in X , there exists $f \in A$ such that $|f|_X \leq 1$, $f(x) > \beta$, and $|f|_{X \setminus U} < \alpha$;
- (d) for each open neighbourhood U of x in X , there exists $f \in A$ such that $|f|_X \leq 1$, $|f(x)| > 3/4$, and $|f|_{X \setminus U} < 1/4$.

Let A be a uniform algebra on a compact space X . It is standard that if $x \in X$ is a strong boundary point for A , then there is no non-zero point derivation on A at x ([27, Corollary 4.3.6]). (In fact, more is true: if x is a strong boundary point then M_x has a bounded approximate identity.)

Definition 1.6.5. Let X be a compact space, let A be a uniform algebra on X , and let $E \subseteq X$. We say that E is a *boundary* for A if, for each $f \in A$, there exists $x \in E$ such that $|f(x)| = |f|_X$. We say that E is a *closed boundary* for A if E is closed and E is a boundary for A . The *Shilov boundary* for A is the intersection of all closed boundaries for A , and is denoted by $\Gamma(A)$.

Let A be a uniform algebra on a compact space X . It is standard that the Shilov boundary for A is indeed a closed boundary for A , and that $\overline{\Gamma_0(A)} = \Gamma(A)$.

Definition 1.6.6. Let A be a uniform algebra on compact space X . The *essential set* for A is the smallest closed set E of X such that the conditions $f \in C(X)$ and $f(E) \subseteq \{0\}$ imply that $f \in A$. We say that A is *essential* if X is the essential set for A .

A uniform algebra A on compact space X with essential set E is trivial if and only if $E = \emptyset$. In addition, $A|E$ is uniformly closed in $C(E)$, and

$$A = \{f \in C(X) : f|E \in A|E\}.$$

For further information regarding peak sets, peak points, and boundaries we refer the reader to [16, 54, 97]. For further information about the essential set, see [16, Section 2.8].

CHAPTER 2

Swiss Cheese Sets, Abstract Swiss Cheeses, and Classicalisation

Swiss cheese sets are compact subsets of the complex plane obtained by deleting a sequence of open disks from a closed disk. By carefully choosing the centre and radius of each open disk, we can construct a Swiss cheese set X such that $R(X)$ has various combinations of properties. If no restrictions are placed on the open disks, then any compact subset of $\mathbb{C}$ can be obtained in this way. We say that a Swiss cheese set is classical if the sum of the radii of the open disks is finite and if the open disks and the complement of the closed disk have pairwise disjoint closures. (This will be made precise in Section 2.1.)

Swiss cheese sets were introduced by Roth [92] (see also [86]), where she gave the first example of a compact plane set X with $\text{int } X = \emptyset$ and such that $R(X) \neq A(X)$. The Swiss cheese set constructed by Roth is classical in the sense described above. Swiss cheese sets have also been used extensively in the study of the uniform algebra $R(X)$. Note that, in much of the literature Swiss cheese sets are referred to simply as ‘Swiss cheeses’.

There are several examples of Swiss cheese sets which are important in the theory of uniform algebras: in [85], McKissick constructs a Swiss cheese set X such that $R(X)$ is regular and non-trivial; in [99], Wermer constructs a Swiss cheese set X such that $R(X)$ does not admit any bounded point derivations but is non-trivial; in [88], O’Farrell constructs a Swiss cheese set X such that $R(X)$ is regular and admits a bounded point derivation of infinite order. Of these examples, only Wermer’s construction yields a classical Swiss cheese set. Many other examples of the use of Swiss cheese sets exist in the literature. A short survey of these constructions is contained in [48].

Classical Swiss cheese sets have many desirable properties. A classical Swiss cheese set is path connected and locally path connected, and hence has no isolated points. In fact, every classical Swiss cheese set is uniformly regular with constant π (see Section 3.1 for the definition) [29]. In [40], Feinstein and Heath note that every classical Swiss cheese set with empty interior is homeomorphic to the Sierpiński carpet, which follows from a theorem of Whyburn [101]. Thus any two classical Swiss cheese sets which have empty interior are homeomorphic. If X is a classical Swiss cheese set then the uniform algebra $R(X)$ is essential ([16, p. 163–167], see also [40, Theorem 1.8]) and X has positive area—a consequence of the Hartogs–Rosenthal theorem. A direct proof that classical Swiss cheese set has positive area is also given in [16]. This fact can also be deduced from a theorem of Wesler [100].

It is easy to see that non-classical Swiss cheese sets need not be connected, and may have many isolated points. For those examples of Swiss cheese sets in the literature which are not classical, we seek to construct classical Swiss cheese sets which solve the same problem. (For example, we seek to construct a classical Swiss cheese set X such that $R(X)$ is regular and non-trivial.) The process of creating such classical Swiss cheeses is what we call classicalisation.

The first classicalisation result is called the Feinstein–Heath classicalisation theorem (Corollary 2.2.11), proved by Feinstein and Heath in [40]. Their method used certain maps between underlying Swiss cheese objects and Zorn’s lemma to find classical Swiss cheese subsets of certain Swiss cheese sets. Mason later gave an inductive proof of the Feinstein–Heath theorem [84].

In this chapter we give a topological proof of the Feinstein–Heath classicalisation theorem, and discuss new techniques for classicalisation of Swiss cheese sets. To do this, we construct certain compact subsets of what we call abstract Swiss cheese space (see Definition 2.1.1), and then use certain functions to show that the desired classical Swiss cheese set exists. We also prove a ‘controlled classicalisation’ theorem where, under some technical assumptions, we only allow open disks which lie in a certain region to be changed. We then prove an analogue of the Feinstein–Heath theorem for annular Swiss cheese sets, which are obtained by deleting a sequence of open disks from a closed annulus. We conclude the chapter with several results about the regularity of the uniform algebra $R(X)$, and the construction of a classical Swiss cheese set X such that $R(X)$ is regular and admits a bounded point derivation of infinite order. That is, we classicalise the example of O’Farrell mentioned above.

All of the work in this chapter is joint with H. Yang and J. Feinstein, and the new results are contained in [49]. See also [48], which contains further applications

of the results from this chapter, and a comparison of the methods used in [40, 84] and our topological method.

2.1 Abstract Swiss cheese space

In this section, we discuss abstract Swiss cheeses, abstract Swiss cheese space and various subspaces thereof which were introduced in [49] (see also [48]).

Throughout the chapter, we will work in *abstract Swiss cheese space* $\mathcal{F}$, where $\mathcal{F} = (\mathbb{C} \times \mathbb{R}^+)^{\mathbb{N}_0}$ is equipped with the product topology, which makes $\mathcal{F}$ into a metrisable topological space.

Definition 2.1.1. An *abstract Swiss cheese* is an element $A = ((a_n, r_n))_{n \geq 0}$ in $\mathcal{F}$. For an abstract Swiss cheese A as above we define the following.

- (a) The *significant index set* of A is $S_A := \{n \in \mathbb{N} : r_n > 0\}$. We say that A is *finite* if S_A is a finite set, otherwise it is *infinite*.
- (b) The *associated Swiss cheese set* X_A is given by

$$X_A := \bar{B}(a_0, r_0) \setminus \left(\bigcup_{n=1}^{\infty} B(a_n, r_n) \right) = \bar{B}(a_0, r_0) \setminus \left(\bigcup_{n \in S_A} B(a_n, r_n) \right). \quad (2.1)$$

- (c) We say that A is *semiclassical* if $\sum_{n=1}^{\infty} r_n < \infty$, $r_0 > 0$ and for all $k \in S_A$ the following hold:
 - (i) $B(a_k, r_k) \subseteq B(a_0, r_0)$;
 - (ii) whenever $\ell \in S_A$ has $\ell \neq k$, we have $B(a_k, r_k) \cap B(a_\ell, r_\ell) = \emptyset$.
- (d) We say that A is *classical* if $\sum_{n=1}^{\infty} r_n < \infty$, $r_0 > 0$ and for all $k \in S_A$ the following hold:
 - (i) $\bar{B}(a_k, r_k) \subseteq B(a_0, r_0)$;
 - (ii) whenever $\ell \in S_A$ with $\ell \neq k$, we have $\bar{B}(a_k, r_k) \cap \bar{B}(a_\ell, r_\ell) = \emptyset$.
- (e) We say that A is *annular* if $a_0 = a_1$ and $r_0 > r_1 > 0$.

We usually write $A = ((a_n, r_n))$ for an abstract Swiss cheese and it should be understood from context that the sequence is indexed by the non-negative integers. Let X be a Swiss cheese set. We say that X is *semiclassical* if there exists a semiclassical abstract Swiss cheese A such that $X_A = X$, and we say that X is *classical* if there exists a classical abstract Swiss cheese A such that $X_A = X$.

We define the following functions on abstract Swiss cheese space.

Definition 2.1.2. For each $\alpha \geq 1$, we define the *discrepancy function of order α* , $\delta_\alpha : \mathcal{F} \rightarrow [-\infty, \infty)$, by

$$\delta_\alpha(A) := r_0^\alpha - \sum_{n=1}^{\infty} r_n^\alpha \quad (A = ((a_n, r_n)) \in \mathcal{F}). \quad (2.2)$$

The *annular discrepancy function* $\delta_{\text{ann}} : \mathcal{F} \rightarrow [-\infty, \infty)$ is given by

$$\delta_{\text{ann}}(A) := r_0 - r_1 - 2 \sum_{n=2}^{\infty} r_n \quad (A = ((a_n, r_n)) \in \mathcal{F}).$$

The *radius sum function* is the map $\rho : \mathcal{F} \rightarrow [0, \infty]$ given by

$$\rho(A) := \sum_{n=1}^{\infty} r_n, \quad (A = ((a_n, r_n)) \in \mathcal{F}).$$

The *centre bound function* is the map $\mu : \mathcal{F} \rightarrow [0, \infty]$ given by

$$\mu(A) := \sup_{n \in \mathbb{N}} |a_n|, \quad (A = ((a_n, r_n)) \in \mathcal{F}).$$

Let $E \subseteq \mathbb{C}$. For an abstract Swiss Cheese $A = ((a_n, r_n))$ we define $H_A(E)$ to be the set of $n \in S_A$ such that $\bar{B}(a_n, r_n) \cap E \neq \emptyset$. The *local radius sum function on E* is the function $\rho_E : \mathcal{F} \rightarrow [0, \infty]$ given by

$$\rho_E(A) := \sum_{n \in H_A(E)} r_n, \quad (A = ((a_n, r_n)) \in \mathcal{F}).$$

The *annular radius sum function* is the map $\rho_{\text{ann}} : \mathcal{F} \rightarrow [0, \infty]$ given by

$$\rho_{\text{ann}}(A) := \sum_{n=2}^{\infty} r_n \quad (A = ((a_n, r_n)) \in \mathcal{F})$$

Let $\mathcal{N}$ denote the set of all abstract Swiss cheeses $A = ((a_n, r_n)) \in \mathcal{F}$ such that $\rho(A) < \infty$ the sequence $(r_n)_{n=1}^{\infty}$ is non-increasing. In addition, for each $M > 0$ and $R > 0$, we define the set $\mathcal{N}(M, R)$ to be the set of all $A = ((a_n, r_n)) \in \mathcal{N}$ such that $\mu(A) \leq M$, and $\rho(A) \leq R$. Throughout this chapter, for each $n \in \mathbb{N}_0$, $\pi_n : \mathcal{F} \rightarrow (\mathbb{C} \times \mathbb{R}^+)$ denotes the projection onto the n th coordinate.

Lemma 2.1.3. *Let $M, R > 0$ and let $\alpha \geq 1$.*

- (a) *The set $\mathcal{N}(M, R)$ is closed in $\mathcal{F}$. Moreover, $S \subseteq \mathcal{N}(M, R)$ is compact if and only if S is closed and $\pi_0(S) \subseteq (\mathbb{C} \times \mathbb{R}^+)$ is compact.*
- (b) *For all $A = ((a_n, r_n)) \in \mathcal{N}(M, R)$ we have $r_n \leq R/n$ for all $n \in \mathbb{N}$.*
- (c) *The function $\delta_\alpha : \mathcal{F} \rightarrow [-\infty, \infty)$ is upper-semicontinuous.*

(d) If $\alpha > 1$ then the function $\delta_\alpha|_{\mathcal{N}(M, R)} : \mathcal{N}(M, R) \rightarrow \mathbb{R}$ is continuous.

Proof. (a) We first show that $\mathcal{N}(M, R)$ is closed in $\mathcal{F}$. For each $m \in \mathbb{N}$, let $A^{(m)} = ((a_n^{(m)}, r_n^{(m)}))$ be an abstract Swiss cheese in $\mathcal{N}(M, R)$. Suppose that $(A^{(m)})$ converges to $A = ((a_n, r_n)) \in \mathcal{F}$ as $m \rightarrow \infty$. We show $A \in \mathcal{N}(M, R)$. Since convergence is pointwise, for each $n \in \mathbb{N}_0$, we have $a_n = \lim_{m \rightarrow \infty} a_n^{(m)} \in \bar{B}(0, M)$ and $r_n = \lim_{m \rightarrow \infty} r_n^{(m)} \in [0, R]$. By Fatou's lemma for series, we have

$$\rho(A) = \sum_{n=1}^{\infty} r_n \leq \liminf_{m \rightarrow \infty} \sum_{n=1}^{\infty} r_n^{(m)} = \liminf_{m \rightarrow \infty} \rho(A^{(m)}) \leq R.$$

It follows that $A \in \mathcal{N}(M, R)$ and hence $\mathcal{N}(M, R)$ is closed.

Let $S \subseteq \mathcal{N}(M, R)$. Set $T := \pi_0(S) \times (\bar{B}(0, M) \times [0, R])^{\mathbb{N}}$. Then we have $S \subseteq T$. It is easy to see, by Tychonoff's theorem, that T is compact if and only if $\pi_0(S)$ is compact in $\mathbb{C} \times \mathbb{R}^+$. Thus S is compact if and only if $\pi_0(S)$ is compact in $\mathbb{C} \times \mathbb{R}^+$ and S is closed in T .

(b) This is clear.

(c) For each $m \in \mathbb{N}$, let $A^{(m)} = ((a_n^{(m)}, r_n^{(m)})) \in \mathcal{F}$ and suppose that $(A^{(m)})$ converges to $A = ((a_n, r_n)) \in \mathcal{F}$ as $m \rightarrow \infty$. By Fatou's lemma for series, we have $\sum_{n=1}^{\infty} r_n^\alpha \leq \liminf_{m \rightarrow \infty} \sum_{n=1}^{\infty} (r_n^{(m)})^\alpha$ so that

$$\delta_\alpha(A) = r_0^\alpha - \sum_{n=1}^{\infty} r_n^\alpha \geq \limsup_{m \rightarrow \infty} \left((r_0^{(m)})^\alpha - \sum_{n=1}^{\infty} (r_n^{(m)})^\alpha \right) = \limsup_{m \rightarrow \infty} \delta_\alpha(A^{(m)}).$$

It follows that δ_α is upper-semicontinuous on $\mathcal{F}$.

(d) For each $m \in \mathbb{N}$, let $A^{(m)} = ((a_n^{(m)}, r_n^{(m)})) \in \mathcal{N}(M, R)$ and suppose that $(A^{(m)})$ converges and has limit $A = ((a_n, r_n)) \in \mathcal{N}(M, R)$. By (b), for each $m, n \in \mathbb{N}$, we have $r_n^{(m)} \leq R/n$, and, with $\alpha > 1$, the series $\sum_{n=1}^{\infty} (R/n)^\alpha$ converges. Thus, by the dominated convergence theorem, we have

$$\delta_\alpha(A) = r_0^\alpha - \sum_{n=1}^{\infty} r_n^\alpha = \lim_{m \rightarrow \infty} \left((r_0^{(m)})^\alpha - \sum_{n=1}^{\infty} (r_n^{(m)})^\alpha \right) = \lim_{m \rightarrow \infty} \delta_\alpha(A^{(m)}).$$

It follows that $|\delta_\alpha(A)| < \infty$ and δ_α is continuous from $\mathcal{N}(M, R)$ to $\mathbb{R}$. $\square$

Note that, if A is a semi-classical or classical abstract Swiss cheese, then $\pi\delta_2(A)$ is the area of the Swiss cheese set X_A . Also note that, in the proof of (d), we do not use the fact that $\mu(A) \leq M$, so we could relax the conditions to only restrict the radius sum. The upper-semicontinuity of δ_α , for $\alpha \geq 1$, implies that the function $\rho : \mathcal{F} \rightarrow [0, \infty]$ is lower-semicontinuous; this is also a trivial consequence of Fatou's lemma for series.

The following example demonstrates that the discrepancy function of order 1 need not be lower-semicontinuous on $\mathcal{N}(M, R)$.

Example 2.1.4. For each $m \in \mathbb{N}$, let $A^{(m)} = ((a_n^{(m)}, r_n^{(m)}))$ be the abstract Swiss cheese such that $a_n^{(m)} = 0$ for all $n \in \mathbb{N}_0$, $r_0^{(m)} = 1$ and $r_n^{(m)} = 1/2m$ for each $n = 1, 2, \dots, m$ and $r_n^{(m)} = 0$ for all $n > m$. Then the sequence $(A^{(m)})_{m=1}^\infty$ converges to the abstract Swiss cheese $A = ((a_n, r_n))$ where $a_n = 0$ for all $n \in \mathbb{N}_0$, $r_0 = 1$, and $r_n = 0$ for all $n \in \mathbb{N}$ so that $\delta_1(A) = 1$. But, for all $m \in \mathbb{N}$, we have

$$\delta_1(A^{(m)}) = 1 - \sum_{n=1}^m \frac{1}{2m} = \frac{1}{2}.$$

Note that $A^{(m)} \in \mathcal{N}(1, 1)$ for all $m \in \mathbb{N}$, so δ_1 fails to be continuous on $\mathcal{N}(M, R)$ for any $M, R > 0$. ■

We shall often work with abstract Swiss cheeses which do not have any ‘redundant’ disks. We therefore introduce the following terminology.

Definition 2.1.5. Let $A = ((a_n, r_n))$ be an abstract Swiss cheese. We say that A is *redundancy-free* if, for all $k \in S_A$, we have $B(a_k, r_k) \cap \bar{B}(a_0, r_0) \neq \emptyset$ and, for all $\ell \in S_A$ with $\ell \neq k$, we have $B(a_k, r_k) \not\subseteq B(a_\ell, r_\ell)$.

The following lemma shows that we may restrict our attention to redundancy-free abstract Swiss cheeses in many cases.

Lemma 2.1.6. *Let $A = ((a_n, r_n)) \in \mathcal{F}$ with $\rho(A) < \infty$. Then there exists a redundancy-free abstract Swiss cheese $B = ((b_n, s_n)) \in \mathcal{N}$ such that $X_B = X_A$, $\mu(A) < \infty$, $\bar{B}(b_0, s_0) = \bar{B}(a_0, r_0)$. Moreover, for each subset E of $\mathbb{C}$, we have $\rho_E(B) \leq \rho_E(A)$. In particular, $\rho(B) \leq \rho(A)$.*

Proof. Since the sum of the radii of A is finite we have $r_n \rightarrow 0$ as $n \rightarrow \infty$, so there exists an initial segment P of $\mathbb{N}$ (possibly equal) such that S_A can be enumerated as $(n_j)_{j \in P}$ (without repeats) arranged so that $(r_{n_j})_{j \in P}$ is non-increasing. For each $j \in P$, define the set

$$I_j := \{n_k : k \in P, k > j, B(a_{n_k}, r_{n_k}) \subseteq B(a_{n_j}, r_{n_j})\}.$$

Let $I = \bigcup_{j \in P} I_j$. Let J denote the set of all $k \in S_A$ with $B(a_k, r_k) \cap \bar{B}(a_0, r_0) = \emptyset$.

Let $S = S_A \setminus (I \cup J)$, let $b_0 = a_0$ and $s_0 = r_0$. There exists an initial segment Q of $\mathbb{N}$ (possibly equal) and a bijective function $\sigma : Q \rightarrow S$ such that $(r_{\sigma(n)})_{n \in Q}$ is a non-increasing (possibly finite) sequence. For each $n \in \mathbb{N}$, let $b_n = a_{\sigma(n)}$, $s_n = r_{\sigma(n)}$ if $n \in Q$ and $b_n = a_0$, $s_n = 0$ if $n \notin Q$. Let $B = ((b_n, s_n))$ be the resulting abstract Swiss cheese.

We now *claim* that B has the desired properties. Clearly $X_A \subseteq X_B$. Let $z \in \mathbb{C} \setminus X_A$, we show $z \notin X_B$. If $z \in \mathbb{C} \setminus \bar{B}(a_0, r_0)$ then we have $z \in \mathbb{C} \setminus \bar{B}(b_0, s_0)$,

so we can find a minimum $m \in \mathbb{N}_0$ such that $z \in B(a_{n_m}, r_{n_m})$. Then we have $B(a_{n_m}, r_{n_m}) \not\subseteq B(a_{n_j}, r_{n_j})$ so that $m \notin I_j$ for all $j \in \mathbb{N}_0$. It follows that there exists $\ell \in S_B$ such that $z \in B(b_\ell, s_\ell)$. Thus $z \notin X_B$ and $X_A = X_B$. By the construction above, it is not hard to see that B is redundancy-free. It is also clear that $\rho_E(B) \leq \rho_E(A)$ for any non-empty set $E \subseteq \mathbb{C}$ so the result follows. $\square$

Note that the conditions $\bar{B}(b_0, s_0) = \bar{B}(a_0, r_0)$ and $\rho(B) \leq \rho(A)$ together imply that $\delta_1(B) \geq \delta_1(A)$.

We now introduce some terminology for constructing new Swiss cheeses by deleting, inserting, and replacing disks from a given abstract Swiss cheese.

Definition 2.1.7. Let $A = ((a_n, r_n))$ be an abstract Swiss cheese.

- (a) Let $a \in \mathbb{C}$ and $r > 0$ and let $m \in \mathbb{N}_0$. We say that an abstract Swiss cheese $B = ((b_n, s_n))$ is obtained from A by *inserting* a disk $B(a, r)$ at index m if, for each $n \in \mathbb{N}_0$ with $0 \leq n < m$, we have $b_n = a_n$, $s_n = r_n$; for each $n \in \mathbb{N}_0$ with $n > m$ we have $b_n = a_{n-1}$, $s_n = r_{n-1}$, and $b_m = a$, $s_m = r$.
- (b) Let $m \in \mathbb{N}_0$. We say that an abstract Swiss cheese $B = ((b_n, s_n))$ is obtained from A by *deleting* the disk at index m if, for each $n \in \mathbb{N}_0$ with $0 \leq n < m$, we have $b_n = a_n$, $s_n = r_n$ and for each $n \in \mathbb{N}_0$ with $n \geq m$ we have $b_n = a_{n+1}$, $s_n = r_{n+1}$.
- (c) Suppose that $A \in \mathcal{N}$. Let $a \in \mathbb{C}$ and $r > 0$ and $k, \ell \in \mathbb{N}$ with $k \neq \ell$. We say that an abstract Swiss cheese $B = ((b_n, s_n))$ is obtained from A by *replacing* the disks $B(a_k, r_k), B(a_\ell, r_\ell)$ by $B(a, r)$ if B is obtained by deleting the disks at indices k, ℓ and inserting the disk $B(a, r)$ at the first index such that the sequence (s_n) is non-increasing.

Note that if B is an abstract Swiss cheese obtained by deleting a disk or replacing a pair of disks from an abstract Swiss cheese $A \in \mathcal{N}$, then we also have $B \in \mathcal{N}$.

2.1.1 Geometric Lemmas

The following elementary lemmas will be required throughout the remainder of the chapter. The first is probably well known, but we have not been able to find a reference. The proof is omitted.

Lemma 2.1.8. Let $z, w \in \mathbb{C}$ and $r, s \in \mathbb{R}^+$, then $\bar{B}(z, r) \subseteq \bar{B}(w, s)$ if and only if $|z - w| \leq s - r$. If $r > 0$, then $B(z, r) \subseteq \mathbb{C} \setminus \bar{B}(w, s)$ if and only if $|w - z| \geq s + r$.

The next two lemmas are essentially those used in [40, 84], but including some additional information distilled from the original proofs. These lemmas are summarised in Figure 2.1. In the first lemma, we allow for the line segment to be degenerate.

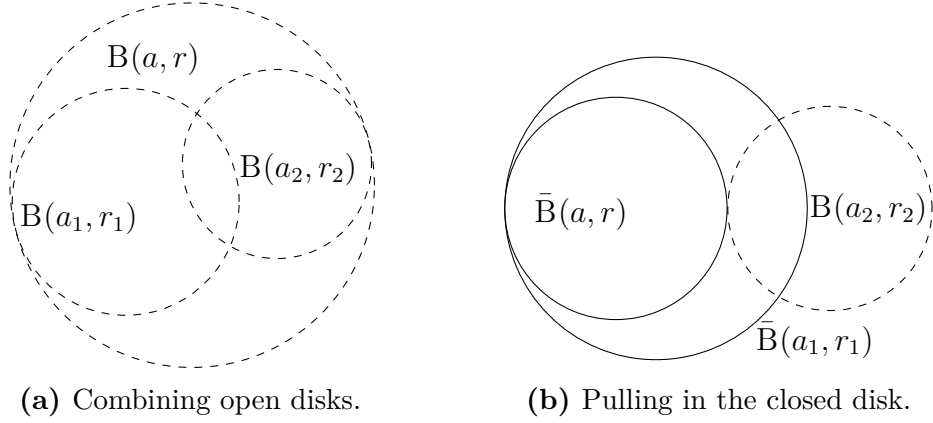


Figure 2.1: Elementary lemmas for combining and pulling in disks.

The first lemma describes a process that we call *combining* disks.

Lemma 2.1.9. *Let $a_1, a_2 \in \mathbb{C}$ and $r_1, r_2 > 0$. Then there exists a unique pair $(a, r) \in \mathbb{C} \times \mathbb{R}^+$ with $B(a_1, r_1) \cup B(a_2, r_2) \subseteq B(a, r)$ such that r is minimal. Moreover, the point a lies on the line segment joining a_1 and a_2 . Suppose further that $\bar{B}(a_1, r_1) \cap \bar{B}(a_2, r_2) \neq \emptyset$. Then we have $r \leq r_1 + r_2$, and equality holds if and only if $B(a_1, r_1) \cap B(a_2, r_2) = \emptyset$.*

The second lemma describes a process that we call *pulling in*.

Lemma 2.1.10. *Let $a_1, a_2 \in \mathbb{C}$ and $r_1 > r_2 > 0$ with $\bar{B}(a_2, r_2) \not\subseteq B(a_1, r_1)$. Then there exists a unique pair $(a, r) \in \mathbb{C} \times \mathbb{R}^+$ with $\bar{B}(a, r) \subseteq \bar{B}(a_1, r_1)$ and $B(a_2, r_2) \cap \bar{B}(a, r) = \emptyset$ such that r is maximal. Moreover, we have $r \geq r_1 - r_2$ and equality holds if and only if $B(a_2, r_2) \subseteq B(a_1, r_1)$.*

The cases in which equality holds in Lemmas 2.1.9 and 2.1.10 are illustrated in Figure 2.2.

2.2 Classicalisation of abstract Swiss cheeses

In this section we give a new, topological proof of the Feinstein–Heath classicalisation theorem, and several related results. Our methods involve constructing a non-empty compact subset of abstract Swiss cheese space, on which we can

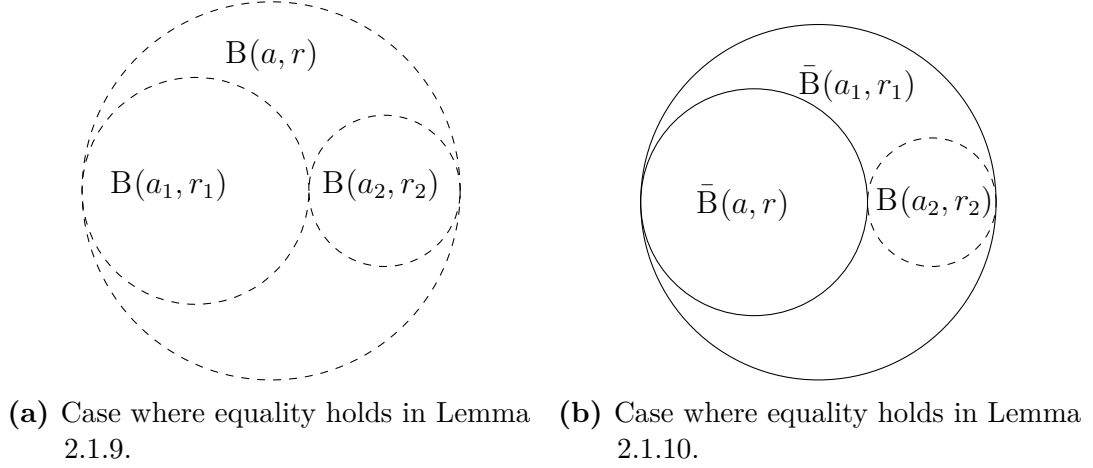


Figure 2.2: Extreme cases in the combining and pulling in lemmas.

maximise the discrepancy function of order 1. We use the resulting set of abstract Swiss cheeses to prove the Feinstein–Heath classicalisation theorem.

We now define a relation on abstract Swiss cheese space.

Definition 2.2.1. Let $A = ((a_n, r_n))$ and $B = ((b_n, s_n))$ be abstract Swiss cheeses. We say that B is *partially above* A if $\bar{B}(b_0, s_0) \subseteq \bar{B}(a_0, r_0)$, and, for each $n \in \mathbb{N}$, either $B(a_n, r_n) \subseteq \mathbb{C} \setminus \bar{B}(b_0, s_0)$, or there exists $m \in \mathbb{N}$ such that $B(a_n, r_n) \subseteq B(b_m, s_m)$, or both.

Note that A is partially above itself and that if B is partially above A then $X_B \subseteq X_A$. The following example shows that the partially above relation is not a partial order on abstract Swiss cheese space.

Example 2.2.2. Choose sequences (α_n) and (β_n) of positive real numbers such that $\alpha_n, \beta_n \rightarrow 1$ as $n \rightarrow \infty$ and, for each $n \in \mathbb{N}$, $0 < \alpha_n < \beta_n < \alpha_{n+1} < 1$. Let $A = ((a_n, r_n))$ be an abstract Swiss cheese with $a_n = 0$ for all $n \in \mathbb{N}_0$, $r_0 = 1$ and $r_n = \alpha_n$ for all $n \in \mathbb{N}$. Let $B = ((b_n, s_n))$ be an abstract Swiss cheese $b_n = 0$ for all $n \in \mathbb{N}_0$, $s_0 = 1$ and $s_n = \beta_n$ for all $n \in \mathbb{N}$. Clearly A and B are not the same abstract Swiss cheese. We *claim* that A is partially above B and B is partially above A .

We first note that $\bar{B}(a_0, r_0) = \bar{B}(b_0, s_0)$. So we need to show that, for all $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that $B(a_n, r_n) \subseteq B(b_m, s_m)$ and, similarly, for all $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that $B(b_n, s_n) \subseteq B(a_m, r_m)$. For each $n \in \mathbb{N}$, we have

$$B(a_n, r_n) \subseteq B(b_n, s_n) \subseteq B(a_{n+1}, r_{n+1}).$$

It follows that A is partially above B and B is partially above A . ■

Remark 2.2.3. In the above, neither A nor B is in $\mathcal{N}$, and $\rho(A) = \rho(B) = \infty$.

For the remainder of this section, let $A = ((a_n, r_n)) \in \mathcal{N}$ be a redundancy-free abstract Swiss cheese with $\mu(A) < \infty$ and $\delta_1(A) > 0$. Let $M := \mu(A)$ and let $R := \rho(A)$. Let $\mathcal{S}(A)$ denote the set of all abstract Swiss cheeses $B \in \mathcal{N}(M, R)$ such that B is partially above A . We seek to prove that $\mathcal{S}(A)$ is compact in $\mathcal{F}$. This is achieved by means of the following lemmas.

Lemma 2.2.4. *For each $m \in \mathbb{N}$, let $A^{(m)} = ((a_n^{(m)}, r_n^{(m)})) \in \mathcal{S}(A)$. There exists a subsequence $(A^{(m_i)})_{i=1}^\infty$ of $(A^{(m)})$ such that, for each $k \in \mathbb{N}$, one of the following holds:*

- (a) $r_k = 0$;
- (b) $|a_k - a_0^{(m_i)}| \geq r_0^{(m_i)} + r_k$ for all $i \in \mathbb{N}$ with $i \geq k$;
- (c) there exists $p_k \in \mathbb{N}$ such that $|a_k - a_{p_k}^{(m_i)}| \leq r_{p_k}^{(m_i)} - r_k$ for all $i \in \mathbb{N}$ with $i \geq k$.

Proof. We use a diagonalisation argument to construct inductively a sequence of nested subsequences of $(A^{(m)})$ and a sequence (p_k) of positive integers with the properties described above. Let $k = 1$. If $r_1 = 0$ then let $(A^{(m,1)})_{m=1}^\infty$ be the sequence $(A^{(m)})$ and let $p_1 = 1$. Otherwise we have $r_1 > 0$. If there are infinitely many $m \in \mathbb{N}$ such that $|a_1 - a_0^{(m)}| \geq r_1 + r_0^{(m)}$, then let $(A^{(m,1)})_m$ be a subsequence of $(A^{(m)})$ such that $|a_1 - a_0^{(m,1)}| \geq r_1 + r_0^{(m,1)}$ holds for all $m \in \mathbb{N}$, and let $p_1 = 1$. Otherwise, for all but finitely many $m \in \mathbb{N}$, there exists $\ell_m \in \mathbb{N}$ such that $|a_1 - a_{\ell_m}^{(m)}| \leq r_{\ell_m}^{(m)} - r_1$. Now since $A^{(m)} \in \mathcal{N}(M, R)$ for each $m \in \mathbb{N}$, where such an integer ℓ_m exists, it must satisfy $1 \leq \ell_m \leq R/r_1$. It follows that there is at least one integer p such that $\ell_m = p$ for infinitely many m . Set $p_1 = p$ and let $(A^{(m,1)})$ be a subsequence of $(A^{(m)})$ such that $|a_1 - a_{p_1}^{(m,1)}| \leq r_{p_1}^{(m,1)} - r_1$ for all $m \in \mathbb{N}$.

Let $k \in \mathbb{N}$ and suppose that we have constructed the nested subsequences $(A^{(m,j)})_m$ for $j = 1, 2, \dots, k-1$ and positive integers $p_1, \dots, p_{k-1}$. We construct the subsequence $(A^{(m,k)})_m$ and p_k as follows. If $r_k = 0$ then let $(A^{(m,k)})_m$ be the sequence $(A^{(m,k-1)})_m$, and let $p_k = 1$. If there are infinitely many $m \in \mathbb{N}$ such that $|a_k - a_0^{(m)}| \geq r_k + r_0^{(m)}$, then let $(A^{(m,k)})_m$ be a subsequence of $(A^{(m,k-1)})_m$ such that $|a_k - a_0^{(m,k)}| \geq r_k + r_0^{(m,k)}$ holds for all $m \in \mathbb{N}$, and let $p_k = 1$. Otherwise, for all but finitely many $m \in \mathbb{N}$, there exists $\ell_m \in \mathbb{N}$ such that $|a_k - a_{\ell_m}^{(m)}| \leq r_{\ell_m}^{(m)} - r_k$. Now since $A^{(m,k)} \in \mathcal{N}(M, R)$ for each $m \in \mathbb{N}$, where such an integer ℓ_m exists, it must satisfy $1 \leq \ell_m \leq R/r_k$. It follows that there is at least one integer p such

that $\ell_m = p$ for infinitely many m . Set $p_k = p$ and let $(A^{(m,k)})$ be a subsequence of $(A^{(m)})$ such that $|a_k - a_{p_k}^{(m,k)}| \leq r_{p_k}^{(m,k)} - r_k$ for all $m \in \mathbb{N}$.

The construction may now proceed. For each $i \in \mathbb{N}$, let $m_i \in \mathbb{N}$ such that $A^{(m_i)} = A^{(i,i)}$. We must verify that $(A^{(m_i)})_i$ has the desired properties. Fix $k \in \mathbb{N}$. If $r_k = 0$ then we are done, so suppose not. By our construction, the sequence $(A^{(m_i)})_{i=k}^\infty$ is a subsequence of $(A^{(m,k)})_{m=1}^\infty$ so that either $|a_k - a_0^{(m_i)}| \geq r_k + r_0^{(m_i)}$ or $|a_k - a_{p_k}^{(m_i)}| \leq r_{p_k}^{(m_i)} - r_k$ for all $i \in \mathbb{N}$ with $i \geq k$. Thus we have constructed a subsequence $(A^{(m_i)})_i$ with the desired properties. $\square$

Lemma 2.2.5. *The set $\mathcal{S}(A)$ is a compact subset of $\mathcal{F}$.*

Proof. It is not hard to see that $\pi_0(\mathcal{S}(A))$ is compact in $\mathbb{C} \times \mathbb{R}^+$ so, by Lemma 2.1.3, it remains only to show that $\mathcal{S}(A)$ is closed in $\mathcal{N}(M, R)$. For each $m \in \mathbb{N}$, let $A^{(m)} = ((a_n^{(m)}, r_n^{(m)}))$ be an abstract Swiss cheese in $\mathcal{S}(A)$, and suppose the sequence $(A^{(m)})$ converges and has limit $B = ((b_n, s_n)) \in \mathcal{N}(M, R)$. We need to show that $B \in \mathcal{S}(A)$.

Since $A^{(m)} \in \mathcal{S}(A)$ for each $m \in \mathbb{N}$, we have $|a_0^{(m)} - a_0| \leq r_0^{(m)} - r_0$ by Lemma 2.1.8. Letting $m \rightarrow \infty$, we have $|b_0 - a_0| \leq s_0 - r_0$, and hence $\bar{B}(b_0, s_0) \subseteq \bar{B}(a_0, r_0)$. By Lemma 2.2.4, there exists a subsequence $(A^{(m_i)})_{i=1}^\infty$ of $(A^{(m)})$ such that, for each $k \in \mathbb{N}$, one of the following holds:

- (a) $r_k = 0$;
- (b) $|a_k - a_0^{(m_i)}| \geq r_0^{(m_i)} + r_k$ for all $i \in \mathbb{N}$ with $i \geq k$;
- (c) there exists $p_k \in \mathbb{N}$ such that $|a_k - a_{p_k}^{(m_i)}| \leq r_{p_k}^{(m_i)} - r_k$ for all $i \in \mathbb{N}$ with $i \geq k$.

Now since $A^{(m)} \rightarrow B$ as $m \rightarrow \infty$, it follows that $A^{(m_i)} \rightarrow B$ as $i \rightarrow \infty$. Let $k \in \mathbb{N}$. If $r_k = 0$ then $B(a_k, r_k) = \emptyset$ so there is nothing to prove. If $|a_k - a_0^{(m_i)}| \geq r_0^{(m_i)} + r_k$ for all $i \in \mathbb{N}$ with $i \geq k$ then, taking $i \rightarrow \infty$, we have $|a_k - b_0| \geq s_0 + r_k$ so that, by Lemma 2.1.8, we have $B(a_k, r_k) \subseteq \mathbb{C} \setminus B(b_0, s_0)$. Otherwise, there exists $p_k \in \mathbb{N}$ such that $|a_k - a_{p_k}^{(m_i)}| \leq r_{p_k}^{(m_i)} - r_k$ for all $i \in \mathbb{N}$ with $i \geq k$. In this case, taking $i \rightarrow \infty$, we obtain $|a_k - b_{p_k}| \leq s_{p_k} - r_k$ so that, by Lemma 2.1.8, we have $B(a_k, r_k) \subseteq B(b_{p_k}, s_{p_k})$. This concludes the proof. $\square$

The function δ_1 is upper-semicontinuous on the non-empty, compact subset $\mathcal{S}(A)$ of $\mathcal{F}$, so δ_1 attains its maximum value on $\mathcal{S}(A)$. Let $\mathcal{S}_1$ denote the set of all abstract Swiss cheeses $B \in \mathcal{S}(A)$ such that $\delta_1(B) = \max \{\delta_1(C) : C \in \mathcal{S}(A)\}$. Then the set $\mathcal{S}_1$ is non-empty and compact in $\mathcal{S}(A)$. Note that, for all $B \in \mathcal{S}_1$, we have $\delta_1(B) \geq \delta_1(A) > 0$. The function δ_2 is continuous on $\mathcal{S}_1$, and so δ_2 attains

both minimum and maximum values on $\mathcal{S}_1$. Let $\mathcal{S}_2$ denote the set of all $B \in \mathcal{S}_1$ such that $\delta_2(B) = \min \{\delta_2(C) : C \in \mathcal{S}_1\}$. Then $\mathcal{S}_2$ is a non-empty, compact subset of $\mathcal{S}_1$, and hence is a non-empty, compact subset of $\mathcal{S}(A)$. We seek to prove that any abstract Swiss cheese $B \in \mathcal{S}_2$ is a classical abstract Swiss cheese.

We require the following ‘safe replacement’ lemma.

Lemma 2.2.6. *Let $B = ((b_n, s_n)) \in \mathcal{S}(A)$.*

- (a) *Suppose that $k, \ell \in S_B$ with $k \neq \ell$ such that $\bar{B}(b_k, s_k) \cap \bar{B}(b_\ell, s_\ell) \neq \emptyset$. If we have $B(b_k, s_k) \cap B(b_\ell, s_\ell) \neq \emptyset$ then there exists $B' \in \mathcal{S}(A)$ such that $\delta_1(B') > \delta_1(B)$. Otherwise, there exists $B' \in \mathcal{S}(A)$ such that $\delta_1(B') = \delta_1(B)$ and $\delta_2(B') < \delta_2(B)$.*
- (b) *Suppose that $k \in S_B$ with $s_k < s_0$ such that $\bar{B}(b_k, s_k) \not\subseteq B(b_0, s_0)$. If we have $B(b_k, s_k) \not\subseteq B(b_0, s_0)$ then there exists $B' \in \mathcal{S}(A)$ such that $\delta_1(B') > \delta_1(B)$. Otherwise, there exists $B' \in \mathcal{S}(A)$ with $\delta_1(B') = \delta_1(B)$ and $\delta_2(B') < \delta_2(B)$.*

Proof. (a) Let $B(b, s)$ be the open disk obtained by applying Lemma 2.1.9 to the disks $B(b_k, s_k)$ and $B(b_\ell, s_\ell)$. Let $B' = ((b'_n, s'_n))$ be obtained by replacing the disks $B(b_k, s_k)$ and $B(b_\ell, s_\ell)$ by $B(b, s)$.

If $B(b_k, s_k) \cap B(b_\ell, s_\ell) \neq \emptyset$ then we have $s < s_k + s_\ell$ and so $\delta_1(B') > \delta_1(B)$. Otherwise, we have $s = s_k + s_\ell$ and hence $s^2 > s_k^2 + s_\ell^2$. In this case, $\delta_1(B') = \delta_1(B)$ and $\delta_2(B') < \delta_2(B)$.

We now show that $B' \in \mathcal{S}(A)$. Clearly $B' \in \mathcal{N}$ by our definition of replacing disks in an abstract Swiss cheese. Since b lies on the line segment connecting b_k and b_ℓ , it follows that $\mu(B') \leq \mu(B)$ and since $s \leq s_k + s_\ell$ we have $\rho(B') \leq \rho(B)$. Thus $B' \in \mathcal{N}(M, R)$. It remains to show that B' is partially above A .

We have $\bar{B}(b'_0, s'_0) = \bar{B}(b_0, s_0)$ so that $\bar{B}(b'_0, s'_0) \subseteq B(a_0, r_0)$. Fix $p \in \mathbb{N}$. Since B is partially above A , we have either $B(a_p, r_p) \subseteq B(b_m, s_m)$, for some $m \in S_B$, or $B(a_p, r_p) \subseteq \mathbb{C} \setminus \bar{B}(b_0, s_0)$ (or both). If $B(a_p, r_p) \subseteq \mathbb{C} \setminus \bar{B}(b_0, s_0)$ then we also have $B(a_p, r_p) \subseteq \mathbb{C} \setminus \bar{B}(b'_0, s'_0)$. Otherwise, let $m \in S_B$ with $B(a_p, r_p) \subseteq B(b_m, s_m)$. If $m = k$ or $m = \ell$, then, with q as the index where $B(b, s)$ was inserted, we have $B(a_p, r_p) \subseteq B(b'_q, s'_q)$. If $m \neq k, \ell$, then there exists $q \in S_{B'}$ such that $B(b'_q, s'_q) = B(b_m, s_m)$. Thus $B(a_p, r_p) \subseteq B(b'_q, s'_q)$. Hence B' is partially above A , and so $B' \in \mathcal{S}(A)$ as required.

(b) Let $\bar{B}(b, s)$ be the closed disk obtained by applying Lemma 2.1.10 to the disks $B(b_0, s_0)$ and $B(b_k, s_k)$. Let $B' = ((b'_n, s'_n))$ be the abstract Swiss cheese obtained by deleting the disks at indices 0 and k and inserting the disk $\bar{B}(b, s)$ at index 0. It is easy to see that $B' \in \mathcal{N}(M, R)$ and we have $B(b, s) \subseteq B(b_0, s_0)$ by construction.

If $B(b_k, s_k) \not\subseteq B(b_0, s_0)$ then we have $s > s_0 - s_k$ so that $\delta_1(B') > \delta_1(B)$. Otherwise, we have $s_0 = s + s_k$ and $s_0^2 > s^2 + s_k^2$ so that $\delta_1(B') = \delta_1(B)$ and $\delta_2(B') < \delta_2(B)$.

Fix $p \in \mathbb{N}$. Again, we show that $B(a_p, r_p) \subseteq \mathbb{C} \setminus \bar{B}(b, s)$ or that there exists $q \in S_{B'}$ such that $B(a_p, r_p) \subseteq B(b'_q, s'_q)$. If $r_p = 0$ then the result is trivial, as above, so suppose that $p \in S_A$. Since $B \in \mathcal{S}(A)$, we have $B(a_p, r_p) \subseteq \mathbb{C} \setminus \bar{B}(b_0, s_0)$ or we have $B(a_p, r_p) \subseteq B(b_m, s_m)$ for some $m \in S_B$. If $B(a_p, r_p) \subseteq \mathbb{C} \setminus \bar{B}(b_0, s_0)$ then we also have $B(a_p, r_p) \subseteq \mathbb{C} \setminus \bar{B}(b, s)$. Let $m \in S_B$ and suppose that we have $B(a_p, r_p) \subseteq B(b_m, s_m)$. If $m \neq k$ then there exists $q \in S_{B'}$ such that $B(b'_q, s'_q) = B(b_m, s_m)$ and hence $B(a_p, r_p) \subseteq B(b'_q, s'_q)$. If $m = k$ then we now have $B(a_p, r_p) \subseteq \mathbb{C} \setminus \bar{B}(b, s)$. It follows that B' is partially above A , and so $B' \in \mathcal{S}(A)$ as required. This concludes the proof. $\square$

The following theorem about abstract Swiss cheeses in $\mathcal{S}_1$ yields a ‘semiclassicalisation’ theorem. See also [48, Theorem 7.3], which was originally proved in the MSc dissertation of H. Yang.

Theorem 2.2.7. *All abstract Swiss cheeses in $\mathcal{S}_1$ are semiclassical.*

Proof. Let $B = ((b_n, s_n)) \in \mathcal{S}_1$. Suppose for contradiction that B is not a semiclassical abstract Swiss cheese. Consider first the case where there are distinct $k, \ell \in S_B$ with $B(b_k, s_k) \cap B(b_\ell, s_\ell) \neq \emptyset$. By Lemma 2.2.6(a) there exists $B' \in \mathcal{S}(A)$ with $\delta_1(B') > \delta_1(B)$, which is a contradiction.

The remaining case is where there is a $k \in S_B$ with $B(b_k, s_k) \not\subseteq B(b_0, s_0)$. We have $\delta_1(B) \geq \delta_1(A) > 0$ so that $s_k < s_0$. By Lemma 2.2.6(b) there exists $B' \in \mathcal{S}(A)$ with $\delta_1(B') > \delta_1(B)$, which is a contradiction. $\square$

Remark 2.2.8. By the above theorem, all abstract Swiss cheeses in $\mathcal{S}_1$, and hence in $\mathcal{S}_2$, are semiclassical. Thus the function $\pi\delta_2 : \mathcal{S}_1 \rightarrow \mathbb{R}^+$ is the function which maps an abstract Swiss cheese B to the area of the associated Swiss cheese set X_B . By this observation, the abstract Swiss cheeses in $\mathcal{S}_2$ are obtained by minimising the value of the area of the associated Swiss cheese sets on $\mathcal{S}_1$.

We are now ready to prove the main theorem of this section, which provides the means for finding classical abstract Swiss cheeses.

Theorem 2.2.9. *All abstract Swiss cheeses in $\mathcal{S}_2$ are classical.*

Proof. Let $B = ((b_n, s_n)) \in \mathcal{S}_2$. Suppose for contradiction that B is not classical. If there are distinct $k, \ell \in S_B$ with $\bar{B}(b_k, s_k) \cap \bar{B}(b_\ell, s_\ell) \neq \emptyset$ then, by Lemma 2.2.6(a),

there exists $B' \in \mathcal{S}(A)$ such that either $\delta_1(B') > \delta_1(B)$ or $\delta_1(B') = \delta_1(B)$ and $\delta_2(B') < \delta_2(B)$. In either case we obtain a contradiction since $B \in \mathcal{S}_2$.

Otherwise there exists $k \in S_B$ with $\bar{B}(b_k, s_k) \not\subseteq B(b_0, s_0)$. Note that $s_k < s_0$ since $\delta_1(B) > 0$. By Lemma 2.2.6(b) there exists $B' \in \mathcal{S}(A)$ such that either $\delta_1(B') > \delta_1(B)$ or $\delta_1(B') = \delta_1(B)$ and $\delta_2(B') < \delta_2(B)$. In either case we obtain a contradiction since $B \in \mathcal{S}_2$. $\square$

In the next theorem, we show that if X_A has empty interior then we do not have to minimise δ_2 on $\mathcal{S}_1$ to find classical abstract Swiss cheeses.

Theorem 2.2.10. *Suppose that $\text{int}(X_A) = \emptyset$. Then each abstract Swiss cheese in $\mathcal{S}_1$ is classical.*

Proof. Let $B = ((b_n, s_n)) \in \mathcal{S}_1$. Then, by Theorem 2.2.7, B is semiclassical. Suppose, for contradiction, that B is not classical. Then there are two cases summarised in Figure 2.3. First suppose that there exist distinct $k, \ell \in S_B$ with $\bar{B}(b_k, s_k) \cap \bar{B}(b_\ell, s_\ell) \neq \emptyset$. Then by Lemma 2.1.9, since $B(b_k, s_k) \cap B(b_\ell, s_\ell) = \emptyset$, there exists an open disk $B(a, r) \supseteq B(b_k, s_k) \cup B(b_\ell, s_\ell)$ with $r = s_k + s_\ell$. By replacing the disks $B(b_k, s_k)$ and $B(b_\ell, s_\ell)$ with $B(a, r)$ we obtain a new abstract Swiss cheese $B' = ((b'_n, s'_n))$ such that $B' \in \mathcal{S}_1$ (following the proof of Lemma 2.2.6). Let p be the index at which the disk $B(a, r)$ was inserted. Since X_B has empty interior, there exists $m \in S_B$ with $m \neq p$ such that $B(a, r) \cap B(b_m, s_m) \neq \emptyset$. Let $q \in S_{B'}$ be such that $B(b'_q, s'_q) = B(b_m, s_m)$. Note that $p \neq q$. Applying Lemma 2.2.6(a) to $p, q \in S_{B'}$ and B' , we obtain an abstract Swiss cheese $B'' \in \mathcal{S}(A)$ which has $\delta_1(B'') > \delta_1(B')$. But this is a contradiction.

Now suppose there exists $k \in S_B$ with $\bar{B}(b_k, s_k) \not\subseteq B(b_0, s_0)$. Then, by applying Lemma 2.1.10, there exists a closed disk $\bar{B}(b, s)$ contained in $\bar{B}(b_0, s_0) \setminus B(b_k, s_k)$. By deleting the disks at indices 0 and k and inserting $B(b, s)$ at index 0, we obtain a new abstract Swiss cheese $B' = ((b'_n, s'_n)) \in \mathcal{S}_1$ such that $\delta_1(B') = \delta_1(B)$ (again following the proof of Lemma 2.2.6). Since X_B has empty interior, there exists $q \in S_{B'}$ such that $B(b_q, s_q) \not\subseteq \bar{B}(b, s)$. Applying Lemma 2.2.6(b) to m and B' , we obtain an abstract Swiss cheese $B'' \in \mathcal{S}(A)$ which has $\delta_1(B'') > \delta_1(B')$. But this is a contradiction. $\square$

We now obtain the Feinstein–Heath classicalisation theorem as a corollary of Theorem 2.2.9, stated below in the language of abstract Swiss cheeses.

Corollary 2.2.11 (Feinstein–Heath theorem). *Let $A_0 = ((a_n^{(0)}, r_n^{(0)}))$ be an abstract Swiss cheese with $\delta_1(A_0) > 0$. Then there exists a classical abstract Swiss cheese $B \in \mathcal{F}$ such that $X_B \subseteq X_{A_0}$ and $\delta_1(B) \geq \delta_1(A_0)$.*

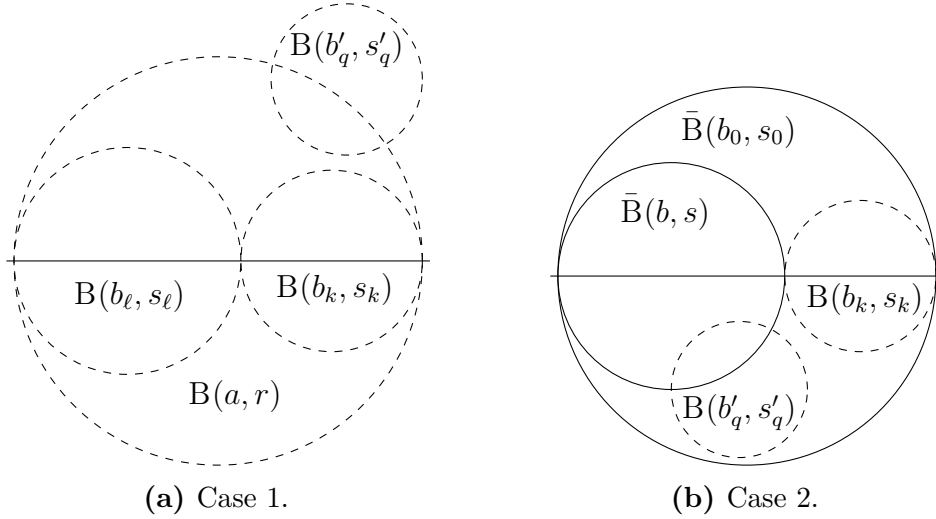


Figure 2.3: The two cases in the proof of Theorem 2.2.10

Proof. Since $\delta_1(A_0) > 0$, we have $\rho(A_0) < \infty$ so, by Lemma 2.1.6, there exists a redundancy-free abstract Swiss cheese $A \in \mathcal{N}$ such that $X_A \subseteq X_{A_0}$, $\mu(A) < \infty$, $\bar{B}(a_0, r_0) = \bar{B}(a_0^{(0)}, r_0^{(0)})$ and such that $\rho_E(A) \leq \rho_E(A_0)$ for each subset $E \subseteq \mathbb{C}$. In particular, we have $\rho(A) \leq \rho(A_0)$ (taking, for example, $E = \mathbb{C}$) and hence $\delta_1(A) \geq \delta_1(A_0)$. Let $\mathcal{S}(A)$ be the compact subset of $\mathcal{N}(\mu(A), \rho(A))$, constructed as above, and let $\mathcal{S}_1$ and $\mathcal{S}_2$ be the subsets obtained by maximising the value of δ_1 on $\mathcal{S}(A)$ and minimising the value of δ_2 on $\mathcal{S}_1$, as described above. Let $B \in \mathcal{S}_2$. By Theorem 2.2.9, the abstract Swiss cheese B is classical, and we have $\delta_1(B) \geq \delta_1(A) \geq \delta_1(A_0)$. Moreover, B is partially above A , so $X_B \subseteq X_A = X_{A_0}$. Thus B is the desired abstract Swiss cheese. $\square$

As an immediate corollary of the Feinstein–Heath classicalisation theorem and the result in [85], after translation and scaling, we obtain the following result from [40] which will be used later.

Corollary 2.2.12. *Let $\varepsilon > 0$, let $\lambda > 0$, and let $a \in \mathbb{C}$. There exists a classical abstract Swiss cheese $A = ((a_n, r_n))$ with $a_0 = a$, $r_0 = \lambda$, and $\rho(A) < \varepsilon$ such that $R(X_A)$ is non-trivial and regular.*

2.3 Controlled classicalisation

In this section we discuss some situations in which it is possible to make a Swiss cheese classical without changing certain disks. This process we call ‘controlled classicalisation’.

Recall that, for $E \subseteq \mathbb{C}$ and an abstract Swiss cheese $A = ((a_n, r_n))$, the set $H_A(E)$ is the set of all $n \in S_A$ such that $\bar{B}(a_n, r_n) \cap E \neq \emptyset$.

Lemma 2.3.1. *Let $U \subseteq \mathbb{C}$ be a non-empty open set. For each $m \in \mathbb{N}_0$, let $A^{(m)} = ((a_n^{(m)}, r_n^{(m)})) \in \mathcal{F}$ and suppose that $A^{(m)} \rightarrow A = ((a_n, r_n)) \in \mathcal{F}$ as $m \rightarrow \infty$. Then $\rho_U(A) \leq \liminf_{m \rightarrow \infty} \rho_U(A^{(m)})$.*

Proof. Since U is open and $A^{(m)} \rightarrow A$ as $m \rightarrow \infty$, for each $k \in H_A(U)$ there exists $m_0 \in \mathbb{N}_0$ such that, for all $m \geq m_0$, we have $k \in S_{A^{(m)}}$ and $\bar{B}(a_k^{(m)}, r_k^{(m)}) \cap U \neq \emptyset$. Let χ_m denote the characteristic function of $H_{A^{(m)}}(U) \cap H_A(U)$. Then χ_m converges pointwise to $\chi := \chi_{H_A(U)}$ as $m \rightarrow \infty$. Since $r_k^{(m)} \rightarrow r_k$ as $m \rightarrow \infty$ for each k , by Fatou's lemma for series, we have

$$\rho_U(A) = \sum_{n=1}^{\infty} \chi(n) r_n \leq \liminf_{m \rightarrow \infty} \sum_{n=1}^{\infty} \chi_m(n) r_n^{(m)} \leq \liminf_{m \rightarrow \infty} \rho_U(A^{(m)}),$$

as required. $\square$

For the rest of this section $A = ((a_n, r_n)) \in \mathcal{N}$ will be a fixed redundancy-free abstract Swiss cheese. Note that both $\rho(A)$ and $\mu(A)$ are finite and $r_n \leq \rho(A)/n$ for all $n \in \mathbb{N}$. We define the (classical) *error set* of A to be

$$E(A) := \bigcup_{\substack{m, n \in S_A \\ m \neq n}} (\bar{B}(a_m, r_m) \cap \bar{B}(a_n, r_n)) \cup \bigcup_{n \in S_A} ((\mathbb{C} \setminus B(a_0, r_0)) \cap \bar{B}(a_n, r_n)).$$

Note that if $E(A) \subseteq B(a_0, r_0)$ then $\bar{B}(a_n, r_n) \subseteq B(a_0, r_0)$ for all $n \in S_A$. We aim to prove that, under suitable conditions, we can classicalise A while leaving many of the open disks unchanged.

As in Section 2.2, we seek to construct a compact subset of $\mathcal{F}$ on which the function δ_1 can be maximised and then the function δ_2 minimised to give a suitable classical abstract Swiss cheese.

In the rest of this section, we will frequently need to consider indexed collections of pairs of sets of the following form. Let $I \subseteq \mathbb{N}$ be non-empty. Let $\mathcal{C} = ((K_n, U_n))_{n \in I}$, where each K_n is a compact plane set and each U_n is an open set with $K_n \subseteq U_n$. We call such an indexed collection a *controlling collection of pairs*. In the special case where I has only one member, we say that $\mathcal{C}$ is a *controlling pair* and write $\mathcal{C} = (K, U)$.

Definition 2.3.2. Let $\mathcal{C} = ((K_n, U_n))_{n \in I}$ be a controlling collection of pairs. Define

$$V(\mathcal{C}) := \bigcup_{n \in I} U_n, \quad F(\mathcal{C}) := \bigcup_{n \in I} K_n.$$

Let $\mathcal{L}_A(\mathcal{C})$ denote the set of all $B = ((b_n, s_n)) \in \mathcal{N}(\mu(A), \rho(A))$ such that:

- (a) for each $(K, U) \in \mathcal{C}$ we have $\rho_U(B) \leq \rho_U(A)$;
- (b) $\bar{B}(b_0, s_0) = \bar{B}(a_0, r_0)$;
- (c) for all $k \in S_A$ with $\bar{B}(a_k, r_k) \cap V(\mathcal{C}) = \emptyset$ there exists $\ell \in S_B$ such that $B(b_\ell, s_\ell) = B(a_k, r_k)$;
- (d) for each $n \in I$ and for all $k \in S_A$ such that $\bar{B}(a_k, r_k) \cap U_n \neq \emptyset$:
 - (i) there exists $\ell \in S_B$ with $B(b_\ell, s_\ell) = B(a_k, r_k)$; or
 - (ii) there exists $\ell \in H_B(K_n)$ with $B(a_k, r_k) \subseteq B(b_\ell, s_\ell)$.

Note that $A \in \mathcal{L}_A(\mathcal{C})$, and if $B \in \mathcal{L}_A(\mathcal{C})$ then B is partially above A . Thus if $B \in \mathcal{L}_A(\mathcal{C})$ then $X_B \subseteq X_A$. The properties (a)–(d) reflect the properties we desire for the final abstract Swiss cheese. We will use the open sets U to bound the error set $E(A)$. Under some technical assumptions, conditions (c) and (d) ensure that abstract Swiss cheeses maximising δ_1 in $\mathcal{L}_A(\mathcal{C})$ have the property that any open disk which lies outside $V(\mathcal{C})$ is the same as an open disk from A .

We first require some preliminary lemmas. The following lemma is probably well known and can be proved using a Hausdorff metric argument, but we include an elementary proof for the convenience of the reader.

Lemma 2.3.3. *Let K be a compact plane set. Let (z_n) be a sequence in $\mathbb{C}$ and (t_n) be a sequence in $\mathbb{R}^+$. Suppose that $\bar{B}(z_n, t_n) \cap K \neq \emptyset$ for all n and $z_n \rightarrow z$ and $t_n \rightarrow t$ as $n \rightarrow \infty$. Then $\bar{B}(z, t) \cap K \neq \emptyset$.*

Proof. For each $n \in \mathbb{N}_0$ there exists a point $w_n \in \bar{B}(z_n, t_n) \cap K$. Now since (w_n) is a sequence in K there is a convergent subsequence (w_{n_k}) converging to a point $w \in K$. For each $k \in \mathbb{N}_0$, we have $w_{n_k} \in \bar{B}(z_{n_k}, t_{n_k})$ so that $|w_{n_k} - z_{n_k}| \leq t_{n_k}$. Hence, taking the limit as $k \rightarrow \infty$, we have $|w - z| \leq t$ so that $w \in \bar{B}(z, t) \cap K$ as required. $\square$

We now prove that the space $\mathcal{L}_A(\mathcal{C})$ is a compact subspace of $\mathcal{F}$ for an arbitrary controlling collection of pairs $\mathcal{C}$.

Lemma 2.3.4. *Let $\mathcal{C} := ((K_n, U_n))_{n \in I}$ be a controlling collection of pairs. Then the set $\mathcal{L}_A(\mathcal{C}) \subseteq \mathcal{F}$ is compact.*

Proof. It is clear that $\pi_0(\mathcal{L}_A(\mathcal{C})) = \{(a_0, r_0)\}$ is compact in $\mathbb{C} \times \mathbb{R}^+$. By Lemma 2.1.3, we need to show that $\mathcal{L}_A(\mathcal{C})$ is closed in $\mathcal{N}(\mu(A), \rho(A))$. For each $m \in \mathbb{N}_0$, let $A^{(m)} = ((a_n^{(m)}, r_n^{(m)}))_{n=0}^\infty \in \mathcal{L}_A(\mathcal{C})$. Let $B = ((b_n, s_n))$ and suppose that $(A^{(m)})$ converges to $B \in \mathcal{N}(\mu(A), \rho(A))$ as $m \rightarrow \infty$. We need to show that $B \in \mathcal{L}_A(\mathcal{C})$.

By Lemma 2.3.1 we see that B also satisfies (a), and it is immediate that (b) is also satisfied.

It remains to prove (c) and (d) hold for B . Fix $k \in S_A$. First suppose that $\bar{B}(a_k, r_k) \cap V(\mathcal{C}) = \emptyset$. Since, for each $m \in \mathbb{N}_0$, we have $A^{(m)} \in \mathcal{L}_A(\mathcal{C})$ it follows that for each m there exists an integer ℓ_m such that $B(a_k, r_k) = B(a_{\ell_m}^{(m)}, r_{\ell_m}^{(m)})$. Now since $r_k^{(m)} = r_k$ for each m we have $1 \leq \ell_m \leq \rho(A)/r_k$ for all m . But then there must exist an integer $1 \leq p \leq \rho(A)/r_k$ such that $\ell_m = p$ infinitely often so we can find a subsequence $(A^{(m_j)})_j$ such that $\ell_{m_j} = p$ for all j . Since $B(a_k, r_k) = B(a_k^{(m_j)}, r_k^{(m_j)})$ for all j and $A^{(m_j)} \rightarrow B$, it follows that $B(a_k, r_k) = B(b_p, s_p)$. This proves that (c) holds for B .

Now suppose that $\bar{B}(a_k, r_k) \cap U \neq \emptyset$ for some $(K, U) \in \mathcal{C}$. For each $m \in \mathbb{N}_0$ there exists an integer ℓ_m such that $B(a_k, r_k) \subseteq B(a_{\ell_m}^{(m)}, r_{\ell_m}^{(m)})$ and $r_{\ell_m}^{(m)} \geq r_k$. We choose ℓ_m as follows: if in $A^{(m)}$ there is an open disk $B(a, r) = B(a_k, r_k)$ then we pick ℓ_m to be the index of that open disk, otherwise we choose ℓ_m to be the index of an open disk $B(a, r)$ that properly contains $B(a_k, r_k)$ and $\bar{B}(a, r) \cap F(\mathcal{C}) \neq \emptyset$. Hence we have $1 \leq \ell_m \leq \rho(A)/r_k$ for all m and so there exists an integer $1 \leq p \leq \rho(A)/r_k$ such that $\ell_m = p$ infinitely often. By considering a subsequence we can assume that $\ell_m = p$ for all m . If $B(a_p^{(m)}, r_p^{(m)}) = B(a_k, r_k)$ holds for infinitely many m then there is a subsequence $(A^{(m_j)})_j$ such that $B(a_k, r_k) = B(a_p^{(m_j)}, r_p^{(m_j)})$ for all j . Since $A^{(m_j)} \rightarrow B$ as $j \rightarrow \infty$ it follows that $B(a_k, r_k) = B(b_p, s_p)$. If $B(a_k, r_k) = B(a_p^{(m)}, r_p^{(m)})$ for finitely many m then, for infinitely many m , we must have

$$B(a_k, r_k) \subseteq B(a_p^{(m)}, r_p^{(m)}) \quad \text{and} \quad \bar{B}(a_p^{(m)}, r_p^{(m)}) \cap K \neq \emptyset.$$

Then there exists a subsequence $(A^{(m_j)})_j$ such that $B(a_k, r_k) \subseteq B(a_p^{(m_j)}, r_p^{(m_j)})$ and $\bar{B}(a_p^{(m_j)}, r_p^{(m_j)}) \cap K \neq \emptyset$ for all j . But then $B(a_k, r_k) \subseteq B(b_p, s_p)$ and, by Lemma 2.3.3, we have $\bar{B}(b_p, s_p) \cap K \neq \emptyset$. This proves that (d) holds for B .

Thus we have proved that $B \in \mathcal{L}_A(\mathcal{C})$ and hence $\mathcal{L}_A(\mathcal{C})$ is compact. $\square$

We are interested in those abstract Swiss cheeses B in a space $\mathcal{L}_A(\mathcal{C})$ on which the discrepancy function δ_1 is maximised. These abstract Swiss cheeses have some desirable properties. Let $\mathcal{L}_A^*(\mathcal{C})$ denote the subset of $\mathcal{L}_A(\mathcal{C})$ of all abstract Swiss cheeses where δ_1 achieves its maximum. Since $\mathcal{L}_A(\mathcal{C})$ is non-empty and compact, $\mathcal{L}_A^*(\mathcal{C})$ is non-empty and compact. Recall that $A \in \mathcal{N}$ is assumed to be redundancy-free.

Lemma 2.3.5. *Let $\mathcal{C} := ((K_n, U_n))_{n \in I}$ be a controlling collection of pairs. Let $B = ((b_n, s_n)) \in \mathcal{L}_A^*(\mathcal{C})$. Then B has the following properties.*

- (a) If $k, \ell \in S_B$ with $k \neq \ell$ then $B(b_k, s_k) \neq B(b_\ell, s_\ell)$.
- (b) Let $k \in S_B$. Then there exists $\ell \in S_A$ with $B(a_\ell, r_\ell) \subseteq B(b_k, s_k)$. Moreover, if $\bar{B}(b_k, s_k) \cap F(\mathcal{C}) = \emptyset$ then this $\ell \in S_A$ is unique, and $B(b_k, s_k) = B(a_\ell, r_\ell)$.
- (c) Fix $E \subseteq \mathbb{C}$. Let $H_1 := H_B(E) \setminus H_B(V(\mathcal{C}))$ and let $H_2 := H_A(E) \setminus H_A(V(\mathcal{C}))$. There exists a bijection $\sigma : H_1 \rightarrow H_2$ satisfying the following condition: for each $k \in H_1$ and $\ell \in H_2$, we have $\sigma(k) = \ell$ if and only if $B(b_k, s_k) = B(a_\ell, r_\ell)$. In particular,

$$\sum_{n \in H_1} s_n = \sum_{n \in H_2} r_n.$$

Proof. (a) If $k, \ell \in S_B$ with $k \neq \ell$ such that $B(b_k, s_k) = B(b_\ell, s_\ell)$ then we can obtain an abstract Swiss cheese B' by deleting the disk at index ℓ which has $\delta_1(B') > \delta_1(B)$. It is easy to see that $B' \in \mathcal{L}_A(\mathcal{C})$, which is a contradiction.

(b) Let $k \in S_B$. Assume, for contradiction, there does not exist $\ell \in S_A$ such that $B(a_\ell, r_\ell) \subseteq B(b_k, s_k)$. Then we can delete the disk at index k from B to obtain an abstract Swiss cheese B' which has $\delta_1(B') > \delta_1(B)$. It is clear that $B' \in \mathcal{L}_A(\mathcal{C})$, which contradicts the maximality of $\delta_1(B)$. Thus there exists $\ell \in S_A$ such that $B(a_\ell, r_\ell) \subseteq B(b_k, s_k)$.

Now suppose, in addition, that $\bar{B}(b_k, s_k) \cap F(\mathcal{C}) = \emptyset$. We show that the $\ell \in S_A$ found above with $B(a_\ell, r_\ell) \subseteq B(b_k, s_k)$ is unique and satisfies $B(a_\ell, r_\ell) = B(b_k, s_k)$. Assume, for contradiction, that $B(a_\ell, r_\ell) \neq B(b_k, s_k)$. Then, since A is redundancy-free, we must have $B(a_m, r_m) \neq B(b_k, s_k)$ for all $m \in S_A$. We *claim* that the abstract Swiss cheese B' obtained by deleting the disk at index k from B has $B' \in \mathcal{L}_A(\mathcal{C})$; this will lead to a contradiction.

Clearly $B' \in \mathcal{N}(\mu(A), \rho(A))$ and it is also clear that B' satisfies conditions (a) and (b) of Definition 2.3.2. Since $B(a_m, r_m) \neq B(b_k, s_k)$ for all $m \in S_A$, it follows that 2.3.2(c) remains true for B' . Similarly, since $\bar{B}(b_k, s_k) \cap F(\mathcal{C}) = \emptyset$, 2.3.2(d) remains true for B' . This proves our claim.

But now $\delta_1(B') > \delta_1(B)$, which contradicts the maximality of $\delta_1(B)$. Thus we must have $B(a_\ell, r_\ell) = B(b_k, s_k)$. The uniqueness of ℓ follows from the fact that A is redundancy-free.

(c) Note that if, for some $k \in S_B$ and $\ell \in S_A$, $B(b_k, s_k) = B(a_\ell, r_\ell)$ then $k \in H_1$ if and only if $\ell \in H_2$. Combining this with (b), for each $k \in H_1$ there exists a unique $\ell \in H_2$ such that $B(b_k, s_k) = B(a_\ell, r_\ell)$. Thus we may define $\sigma(k) = \ell$ for such k, ℓ . We must show that σ is a bijection. By (a), σ is injective. Let $\ell \in H_2$. By 2.3.2(c), there exists $k \in S_B$ with $B(b_k, s_k) = B(a_\ell, r_\ell)$. By the remark above,

$k \in H_1$, and so $\sigma(k) = \ell$. This proves that σ is surjective. It is now immediate that $\sum_{n \in H_1} s_n = \sum_{n \in H_2} r_n$. This completes the proof. $\square$

In order to obtain a controlled classicalisation theorem, we need to impose some technical conditions on $\mathcal{C}$. Let K be a compact plane set and let $M > 0$. We define $U(K, M) := \{z \in \mathbb{C} : \text{dist}(z, K) < M\}$. Clearly $U(K, M)$ is non-empty and open.

Our next lemma is a ‘safe replacement’ result for controlled classicalisation.

Lemma 2.3.6. *Let $I \subseteq \mathbb{N}$ be non-empty. Let $(K_n)_{n \in I}$ be a collection of compact plane sets and let $(M_n)_{n \in I}$ be a collection of positive real numbers. For each $n \in I$, let $U_n := U(K_n, M_n)$. Suppose that $\rho_{U_k}(A) < M_k/2$ and $U_k \subseteq B(a_0, r_0)$ for all $k \in I$ and suppose that $U_k \cap U_\ell = \emptyset$ for all distinct $k, \ell \in I$. Let $\mathcal{C}$ be the controlling collection $((K_n, U_n))_{n \in I}$. Let $B = ((b_n, s_n)) \in \mathcal{L}_A(\mathcal{C})$ and fix $m \in I$. Suppose there exists $k, \ell \in S_B$ with $k \neq \ell$ such that $\bar{B}(b_k, s_k) \cap K_m \neq \emptyset$ and $\bar{B}(b_k, s_k) \cap \bar{B}(b_\ell, s_\ell) \neq \emptyset$. Then there exists $B' \in \mathcal{L}_A(\mathcal{C})$ such that either $\delta_1(B') > \delta_1(B)$ or $\delta_1(B') = \delta_1(B)$ and $\delta_2(B') < \delta_2(B)$.*

Proof. Let $B(b, s)$ be the disk obtained by the application of Lemma 2.1.9 to the disks $B(b_k, s_k)$ and $B(b_\ell, s_\ell)$. Let $B' = ((b'_n, s'_n))$ be an abstract Swiss cheese obtained from B by replacing the disks at indices k, ℓ with the disk $B(b, s)$. Since $B \in \mathcal{L}_A(\mathcal{C})$ we have $\rho_{U_m}(B) \leq \rho_{U_m}(A) < M_m/2$, so that $s \leq s_k + s_\ell < M_m/2$. Since $\bar{B}(b_k, s_k) \cap K_m \neq \emptyset$, we must have $\bar{B}(b, s) \subseteq U_m$ and hence $\bar{B}(b, s) \cap U_n = \emptyset$ for all $n \in I$ with $n \neq m$.

It is clear now that either $\delta_1(B') > \delta_1(B)$, when $s < s_k + s_\ell$, or we have $\delta_1(B') = \delta_1(B)$ and $\delta_2(B') < \delta_2(B)$, when $s = s_k + s_\ell$, so it remains to show that $B' \in \mathcal{L}_A(\mathcal{C})$. By construction, and since $\bar{B}(b, s) \subseteq U_m$ and $\bar{B}(b, s) \cap U_n = \emptyset$ for $n \in I$ with $n \neq m$, we have $B' \in \mathcal{N}(\mu(A), \rho(A))$ and satisfies (a) and (b) in Definition 2.3.2.

Fix $j \in S_A$. If $\bar{B}(a_j, r_j) \cap V(\mathcal{C}) = \emptyset$, then there exists $p \in S_B$ with $p \neq k, \ell$ and $B(b_p, s_p) = B(a_j, r_j)$. Hence there exists $p' \in S_{B'}$ such that $B(b'_{p'}, s'_{p'}) = B(a_j, r_j)$ and B' satisfies (c) in Definition 2.3.2.

Suppose that $\bar{B}(a_j, r_j) \cap V(\mathcal{C}) \neq \emptyset$. Let $n \in I$ such that $\bar{B}(a_j, r_j) \cap U_n \neq \emptyset$. Since $B \in \mathcal{L}_A(\mathcal{C})$, there exists $p \in S_B$ such that $B(a_j, r_j) \subseteq B(b_p, s_p)$, where equality holds unless $\bar{B}(b_p, s_p) \cap K_n \neq \emptyset$. If $p \neq k, \ell$, then there exists $q \in S_{B'}$ such that $B(b'_q, s'_q) = B(b_p, s_p)$. Thus $B(a_j, r_j) \subseteq B(b'_q, s'_q)$ and equality holds if $\bar{B}(b'_q, s'_q) \cap K_n = \emptyset$. If $n \neq m$ then we cannot have $p = k$ or $p = \ell$ since $\bar{B}(b, s) \subseteq U_m$ and $U_n \cap U_m = \emptyset$. If $n = m$ and either $p = k$ or $p = \ell$, then there exists $q \in S_{B'}$

such that $B(b'_q, s'_q) = B(b, s)$, so that $B(a_j, r_j) \subseteq B(b'_q, s'_q)$ and $\bar{B}(b'_q, s'_q) \cap K_n \neq \emptyset$. Moreover, $\bar{B}(b'_q, s'_q) \cap U_i = \emptyset$ for all $i \in I$ with $i \neq m$. It follows that B' satisfies 2.3.2(d) and hence $B' \in \mathcal{L}_A(\mathcal{C})$. This completes the proof. $\square$

Similar geometric reasoning and induction shows that, under the conditions of the lemma, given $n_1, \dots, n_p \in S_A$ and $m \in I$ such that

$$\bar{B}(a_{n_1}, r_{n_1}) \cap K_m \neq \emptyset \quad \text{and} \quad \bar{B}(a_{n_{j-1}}, r_{n_{j-1}}) \cap \bar{B}(a_{n_j}, r_{n_j}) \neq \emptyset$$

for $j = 2, \dots, p$ we have $\bar{B}(b_{n_j}, r_{n_j}) \subseteq U_m$ for each $j = 1, \dots, p$.

We are now ready to prove the controlled classicalisation theorem.

Theorem 2.3.7. *Let $I \subseteq \mathbb{N}$ be non-empty. Let $(K_n)_{n \in I}$ be a collection of compact plane sets and let $(M_n)_{n \in I}$ be a collection of positive real numbers. For each $n \in I$, let $U_n := U(K_n, M_n)$. Suppose that $U_k \subseteq B(a_0, r_0)$ and $\rho_{U_k}(A) < M_k/2$ for all $k \in I$ and suppose that $U_k \cap U_\ell = \emptyset$ for all distinct $k, \ell \in I$. Let $\mathcal{C}$ be the controlling collection $((K_n, U_n))_{n \in I}$ and suppose $E(A) \subseteq F(\mathcal{C})$. Then there exists $B = ((b_n, s_n)) \in \mathcal{L}_A^*(\mathcal{C})$ such that $X_B \setminus V(\mathcal{C}) = X_A \setminus V(\mathcal{C})$ and B is classical.*

Proof. We know that $\mathcal{L}_A^*(\mathcal{C})$ is non-empty and compact so δ_2 obtains its minimum on $\mathcal{L}_A^*(\mathcal{C})$. Let $B \in \mathcal{L}_A^*(\mathcal{C})$ such that δ_2 is minimised on $\mathcal{L}_A^*(\mathcal{C})$ at B . We first show that $\bar{B}(b_k, s_k) \subseteq B(b_0, s_0)$ for all $k \in S_B$. Let C be the complement of the disk $B(a_0, r_0) = B(b_0, s_0)$. Let $k \in S_B$ and assume, for contradiction, that $C \cap \bar{B}(b_k, s_k) \neq \emptyset$. If there exists $u \in S_A$ such that $B(a_u, r_u) = B(b_k, s_k)$ then

$$\emptyset \neq \bar{B}(b_k, s_k) \cap C = \bar{B}(a_u, r_u) \cap C \subseteq C \cap E(A) = \emptyset,$$

which is impossible. Otherwise, by Lemma 2.3.5, there exists $u \in S_A$ such that $B(a_u, r_u) \subseteq B(b_k, s_k)$. Since $B \in \mathcal{L}_A^*(\mathcal{C})$ there exists $m \in I$ such that

$$\bar{B}(b_k, s_k) \cap K_m \neq \emptyset,$$

and it follows that $\bar{B}(b_m, s_m) \subseteq U_m \subseteq B(a_0, r_0)$, which is a contradiction as $B(a_0, r_0) = B(b_0, s_0)$.

We must now show that there do not exist distinct $k, \ell \in S_B$ such that we have $\bar{B}(b_k, s_k) \cap \bar{B}(b_\ell, s_\ell) \neq \emptyset$. Suppose, for contradiction, that such a pair exists. If $\bar{B}(b_k, s_k) \cap F(\mathcal{C}) = \emptyset$ and $\bar{B}(b_\ell, s_\ell) \cap F(\mathcal{C}) = \emptyset$ then there exists $u, v \in S_A$ with $B(a_u, r_u) = B(b_k, s_k)$ and $B(a_v, r_v) = B(b_\ell, s_\ell)$, which is a contradiction since $E(A) \subseteq F(\mathcal{C})$. Thus at least one of these disks has non-empty intersection with at least one compact set K_m .

We may assume, without loss of generality, that $B(b_k, s_k) \cap K_m \neq \emptyset$ for some $m \in I$. It follows that $s_k, s_\ell < M_m/2$ and $\bar{B}(b_k, s_k) \subseteq U_m$ and $\bar{B}(b_\ell, s_\ell) \cap U_m \neq \emptyset$. Let $B(b, s)$ be the open disk obtained by an application of Lemma 2.1.9 to the disks $B(b_k, s_k)$ and $B(b_\ell, s_\ell)$. Then, by Lemma 2.3.6, the abstract Swiss cheese $B' \in \mathcal{L}_A(\mathcal{C})$ obtained by replacing the disks $B(b_k, s_k)$ and $B(b_\ell, s_\ell)$ with $B(b, s)$ has either $\delta_1(B') > \delta_1(B)$ or $\delta_1(B') = \delta_1(B)$ and $\delta_2(B') < \delta_2(B)$. Both of these cases are impossible since we assumed that δ_1 was maximised on B and δ_2 was minimised on B . It follows that no such pair k, ℓ can exist and hence B is classical.

It remains to show that $X_B \setminus V(\mathcal{C}) = X_A \setminus V(\mathcal{C})$. Note that $B \in \mathcal{L}_A(\mathcal{C})$ so $X_B \subseteq X_A$, thus $X_B \setminus V(\mathcal{C}) \subseteq X_A \setminus V(\mathcal{C})$. Let $U_A := (\mathbb{C} \setminus X_A) \cup V(\mathcal{C})$ and $U_B := (\mathbb{C} \setminus X_B) \cup V(\mathcal{C})$. Let $z \in U_B$, we show that $z \in U_A$. If z is outside of $\bar{B}(b_0, s_0)$ then it is also outside of $\bar{B}(a_0, r_0)$ since the closed balls are the same. If $z \in \bar{B}(b_0, s_0)$, then there exists $k \in S_B$ such that $z \in B(b_k, s_k)$. Note that $\bar{B}(b_k, s_k) \cap F(\mathcal{C}) = \emptyset$, otherwise $\bar{B}(b_k, s_k) \subseteq V(\mathcal{C})$. By Lemma 2.3.5, there exists $\ell \in S_A$ such that $B(a_\ell, r_\ell) = B(b_k, s_k)$. Thus $z \in B(a_\ell, r_\ell)$ and $U_B \subseteq U_A$. It follows that $\mathbb{C} \setminus U_B \supseteq \mathbb{C} \setminus U_A$ and hence $X_B \setminus V(\mathcal{C}) = X_A \setminus V(\mathcal{C})$. $\square$

Note here that the classical, abstract Swiss cheese B obtained from this theorem is an element of $\mathcal{L}_A^*(\mathcal{C})$ and therefore satisfies properties (a)–(d) of Definition 2.3.2, and the conclusion of Lemma 2.3.5 holds for B . Note also that, in contrast to the Feinstein–Heath classicalisation theorem, $\delta_1(B)$ may be negative here. We can obtain similar results using transfinite induction.

Taking I to have just one element in Theorem 2.3.7, we obtain the following corollary.

Corollary 2.3.8. *Let K be a compact plane set and let M be a positive real number. Let $U = U(K, M)$ and let $\mathcal{C}$ be the controlling pair (K, U) . Suppose that $\rho_U(A) < M$ and $E(A) \subseteq K$. Then there exists $B = ((b_n, s_n)) \in \mathcal{L}_A^*(\mathcal{C})$ such that $X_B \setminus U = X_A \setminus U$ and B is classical.*

2.4 Annular classicalisation

In this section we give some results about Swiss cheese like sets obtained by deleting open disks from a closed annulus, rather than a closed disk. If K is a closed annulus in the plane, we can write $K = \bar{B}(a_0, r_0) \setminus B(a_1, r_1)$ for some $a_0 = a_1 \in \mathbb{C}$ and $r_0 > r_1 > 0$ real. Recall that an abstract Swiss cheese $A = ((a_n, r_n))$ is *annular* if $a_0 = a_1$ and $0 < r_1 < r_0$, and let K_A denote the annulus $\bar{B}(a_0, r_0) \setminus B(a_1, r_1)$. We

shall usually omit ‘abstract’ from the statement *A is an annular abstract Swiss cheese*.

The following lemma is an annular analogue of Lemma 2.1.10. The statement is presumably well known, but we include the details for the convenience of the reader.

Lemma 2.4.1. *Let $a \in \mathbb{C}$ and $r_0 > r_1 > 0$ and let $K := \bar{B}(a, r_0) \setminus B(a, r_1)$. Let $b \in \mathbb{C}$ and $0 < s < (r_0 - r_1)/2$ such that $\bar{B}(b, s) \cap \overline{\mathbb{C} \setminus K} \neq \emptyset$. Then there exists $r'_0, r'_1 > 0$ such that $K' := \bar{B}(a, r'_0) \setminus B(a, r'_1) \subseteq K$ with $K' \cap B(b, s) = \emptyset$ and $r'_0 - r'_1 \geq r_0 - r_1 - 2s$.*

Proof. Set $D = B(b, s)$. If $D \subseteq \mathbb{C} \setminus K$ then there is nothing to prove so suppose not. Since $s < (r_0 - r_1)/2$ there are only two possible cases. We must have either $\bar{D} \cap \bar{B}(a, r_1) \neq \emptyset$ or $D \cap \mathbb{C} \setminus B(a, r_0) \neq \emptyset$.

In the first case, where $\bar{D} \cap \bar{B}(a, r_1) \neq \emptyset$, let $r'_0 = r_0$ and $r'_1 = |b - a| + s$. We have $|b - a| > r_1 - s$ and $|b - a| \leq r_1 + s$. Hence $r'_1 > r_1 - s + s = r_1$ and $r'_1 \leq r_1 + 2s < r_1 + r_0 - r_1 = r_0$ and

$$r'_0 - r'_1 = r_0 - (|b - a| + s) \geq r_0 - s - r_1 - s = r_0 - r_1 - 2s.$$

Since for each $z \in D$ we have $|b - a| - s < |z - a| < |b - a| + s$ it follows immediately that $D \subseteq \mathbb{C} \setminus K$.

In the second case, where $D \cap \mathbb{C} \setminus B(a, r_0) \neq \emptyset$, let $r'_0 = |b - a| - s$ and $r'_1 = r_1$. We have $|b - a| < r_0 + s$ and $|b - a| \geq r_0 - s$. Hence $r'_0 < r_0 + s - s = r_0$ and

$$r'_0 > r_0 - s - s > r_0 - (r_0 - r_1) = r_1$$

and so

$$r'_0 - r'_1 = |b - a| - s - r_1 \geq r_0 - r_1 - 2s.$$

Similarly, for all $z \in D$ we have $|b - a| - s < |z - a| < |b - a| + s$ and so $D \subseteq \mathbb{C} \setminus K$. This completes the proof. $\square$

Let $A = ((a_n, r_n))$ be an abstract Swiss cheese. Recall that

$$\delta_{\text{ann}}(A) = r_0 - r_1 - 2\rho_{\text{ann}}(A),$$

as defined in Definition 2.1.3. In particular, if $\delta_{\text{ann}}(A) > 0$ then we have $r_0 > r_1$.

We aim to prove an analogue of the Feinstein–Heath classicalisation theorem (Corollary 2.2.11) for annular Swiss cheeses by constructing a suitable compact subset of $\mathcal{F}$.

The following is an analogue of Lemma 2.1.6 for annular Swiss cheeses.

Lemma 2.4.2. *Let A be an annular Swiss cheese with $\rho_{\text{ann}}(A) < \infty$. Then there exists an annular Swiss cheese $B = ((b_n, s_n))$ with the following properties:*

- (a) $\rho_{\text{ann}}(B) \leq \rho_{\text{ann}}(A)$ and $\mu(B) < \infty$;
- (b) $X_B = X_A$ and $K_B = K_A$;
- (c) $(s_n)_{n \geq 2}$ is non-increasing;
- (d) for each $j \in S_B \setminus \{1\}$ we have $B(b_j, s_j) \cap K_B \neq \emptyset$;
- (e) for each $j, k \in S_B \setminus \{1\}$ with $j \neq k$, we have $B(b_j, s_j) \not\subseteq B(b_k, s_k)$;
- (f) for each $E \subseteq \mathbb{C}$ we have $\rho_E(B) \leq \rho_E(A)$.

Proof. Since the sum of the radii of A is finite we have $r_n \rightarrow 0$ as $n \rightarrow \infty$, so there exists an initial segment P of $\mathbb{N}$ (possibly equal) such that S_A can be enumerated as $(n_j)_{j \in P}$ (without repeats) arranged so that $(a_{n_1}, r_{n_1}) = (a_1, r_1)$ and $(r_{n_j})_{j \in P \setminus \{1\}}$ is non-increasing. For each $j \in P$, define the set

$$I_j := \{n_k : k \in P, k > j, B(a_{n_k}, r_{n_k}) \subseteq B(a_{n_j}, r_{n_j})\}.$$

Let $I = \bigcup_{j \in P} I_j$. Let J denote the set of all $k \in S_A$ with $B(a_k, r_k) \cap \bar{B}(a_0, r_0) = \emptyset$.

Let $S = S_A \setminus (I \cup J)$, let $b_0 = a_0$ and $s_0 = r_0$. Note that $1 \in S$. There exists an initial segment Q of $\mathbb{N}$ (possibly equal) and a bijective function $\sigma : Q \rightarrow S$ such that $\sigma(1) = 1$ and $(r_{\sigma(n)})_{n \in Q \setminus \{1\}}$ is a non-increasing (possibly finite) sequence. For each $n \in \mathbb{N}$ let $b_n = a_{\sigma(n)}$, $s_n = r_{\sigma(n)}$ if $n \in Q$ and $b_n = a_0$, $s_n = 0$ if $n \notin Q$. Let $B = ((b_n, s_n))$ be the resulting abstract Swiss cheese.

We now *claim* that B has the desired properties. Clearly $X_A \subseteq X_B$. In addition, B is annular since A is annular and $(b_1, s_1) = (a_1, r_1)$. Let $z \in \mathbb{C} \setminus X_A$, we show $z \notin X_B$. If $z \in \mathbb{C} \setminus \bar{B}(a_0, r_0)$ then we have $z \in \mathbb{C} \setminus \bar{B}(b_0, s_0)$, so we can find a minimum $m \in \mathbb{N}_0$ such that $z \in B(a_{n_m}, r_{n_m})$. Then we have $B(a_{n_m}, r_{n_m}) \not\subseteq B(a_{n_j}, r_{n_j})$ so that $m \notin I_j$ for all $j \in \mathbb{N}_0$. It follows that there exists $\ell \in S_B$ such that $z \in B(b_\ell, s_\ell)$. Thus $z \notin X_B$ and $X_A = X_B$ and $K_B = K_A$. It is clear that $\rho_E(B) \leq \rho_E(A)$ for any non-empty set $E \subseteq \mathbb{C}$. It is also clear that we must have $\rho_{\text{ann}}(B) \leq \rho_{\text{ann}}(A)$. It is not hard to see that B satisfies the other properties by the construction above. The result follows. $\square$

Note that, in the previous lemma, $K_B = K_A$ and $\rho_{\text{ann}}(B) \leq \rho_{\text{ann}}(A)$ together imply that $\delta_{\text{ann}}(B) \geq \delta_{\text{ann}}(A)$.

For the rest of this section, let $A = ((a_n, r_n))$ be an annular Swiss cheese with $\delta_{\text{ann}}(A) > 0$, such that $\mu(A) < \infty$ and $(r_n)_{n=2}^\infty$ is non-increasing.

Lemma 2.4.3. *Let $\mathcal{A}$ be the family of all $B = ((b_n, s_n)) \in \mathcal{F}$ such that*

- (a) *the sequence $(s_n)_{n \geq 2}$ is non-increasing,*
- (b) *$\rho_{\text{ann}}(B) \leq \rho_{\text{ann}}(A)$,*
- (c) *$\mu(B) \leq \mu(A)$,*
- (d) *B is partially above A , and*
- (e) *$b_0 = b_1 = a_0$, and $r_0 \geq s_0 \geq s_1 \geq r_1$.*

Then $\mathcal{A}$ is compact in $\mathcal{F}$, each abstract Swiss cheese $B \in \mathcal{A}$ with $\delta_{\text{ann}}(B) > 0$ is annular. Moreover, the function $\delta_{\text{ann}}|_{\mathcal{A}} : \mathcal{A} \rightarrow \mathbb{R}$ is upper-semicontinuous and the function $\delta_2|_{\mathcal{A}} : \mathcal{A} \rightarrow \mathbb{R}$ is continuous.

Proof. It is easy to see that the family $\mathcal{A}$ is pointwise bounded by properties (b), (c), and (e) so it remains only to prove that $\mathcal{A}$ is closed. For each $m \in \mathbb{N}_0$, let $A^{(m)} = ((a_n^{(m)}, r_n^{(m)}))_{n=0}^\infty \in \mathcal{A}$ and suppose that $A^{(m)} \rightarrow B \in \mathcal{F}$ as $m \rightarrow \infty$. It is clear that B satisfies (a)–(d) (as in the proof of Lemma 2.2.5). Since convergence is pointwise, we have $b_0 = a_0$ and $b_1 = a_1$. Since A was annular, it follows that $b_0 = b_1$.

Since each $A^{(m)} \in \mathcal{A}$ we have $r_0 \geq r_0^{(m)} \geq r_1^{(m)} \geq r_1$, by taking $m \rightarrow \infty$ we have

$$r_0 \geq s_0 \geq s_1 \geq r_1.$$

Hence $\mathcal{A}$ is closed and pointwise bounded and is therefore compact by Tychonoff's theorem.

Let $B = ((b_n, s_n)) \in \mathcal{A}$ with $\delta_{\text{ann}}(B) > 0$. Then we have $b_0 = b_1$ and $\delta_{\text{ann}}(B) > 0$ and this implies that $s_0 > s_1$ and it follows that B is annular.

The proof that δ_{ann} is upper-semicontinuous is an immediate consequence of Fatou's lemma for series, similar to the upper-semicontinuity of δ_1 .

To prove that the restriction of δ_2 to $\mathcal{A}$ is continuous note that, for $n \in \mathbb{N}$ with $n \geq 2$, we have $s_n^2 \leq \rho_{\text{ann}}(B)^2/n^2$ for each $B = ((b_n, s_n)) \in \mathcal{A}$. The result then follows from the dominated convergence theorem as in the proof of Lemma 2.1.3(d). $\square$

It is clear that $A \in \mathcal{A}$ and so $\mathcal{A}$ is non-empty. For all $B \in \mathcal{A}$ we also have $X_B \subseteq X_A$. We require one additional lemma before we prove the main theorem. The following lemma is the ‘safe replacement’ lemma for annular Swiss cheeses.

Lemma 2.4.4. *Let $\mathcal{A}$ be as in Lemma 2.4.3. Let $B = ((b_n, s_n)) \in \mathcal{A}$ be an annular Swiss cheese with $\delta_{\text{ann}}(B) \geq \delta_{\text{ann}}(A)$. Suppose there exists $k \in S_B \setminus \{1\}$ such that $\bar{B}(b_k, s_k) \cap \overline{\mathbb{C} \setminus K_B} \neq \emptyset$. Then there exists $B' = ((b'_n, s'_n)) \in \mathcal{A}$ with $\delta_{\text{ann}}(B') \geq \delta_{\text{ann}}(B)$. Moreover, if $\delta_{\text{ann}}(B') = \delta_{\text{ann}}(B)$ then $\delta_2(B') < \delta_2(B)$.*

Proof. Let $b'_0 = b'_1 = b_0$. As in Lemma 2.4.1, we can find $s'_0 > s'_1 > 0$ such that $K_{B'} := \bar{B}(b'_0, s'_0) \setminus B(b'_1, s'_1) \subseteq K_B$, $K_{B'} \cap B(b_k, s_k) = \emptyset$ and

$$s'_0 - s'_1 \geq s_0 - s_1 - 2s_k.$$

Let $b'_\ell = b_\ell$ and $s'_\ell = s_\ell$ if $2 \leq \ell < k$, $b'_\ell = b_{\ell+1}$ and $s'_\ell = s_{\ell+1}$ if $k < \ell$, we obtain an abstract Swiss cheese $B' = ((b'_n, s'_n))$.

From the construction we see that B' satisfies properties (a), (c), and (e) in Lemma 2.4.3. We have

$$\delta_{\text{ann}}(B') = s'_0 - s'_1 - 2 \sum_{n=2}^{\infty} s'_n \geq s_0 - s_1 - 2s_k - 2 \sum_{n=2}^{\infty} s_n + 2s_k = \delta_{\text{ann}}(B).$$

Since $s'_0 \leq s_0$ and $s'_1 \geq s_1$ we must have $\rho_{\text{ann}}(B') \leq \rho_{\text{ann}}(B) \leq \rho_{\text{ann}}(A)$, so (b) is satisfied.

We now show that B' is partially above A . Fix $j \in S_A$. If $B(a_j, s_j)$ lies in the complement of $B(b_0, s_0)$, then it lies in the complement of $B(b'_0, s'_0)$ and if $B(a_j, s_j) \subseteq B(b_1, s_1)$ then $B(a_j, s_j) \subseteq B(b'_1, s'_1)$. Suppose there exists $m \in S_B$ such that $B(a_j, s_j) \subseteq B(b_m, s_m)$. If we have $m \neq k$ then there exists $\ell \in S_{B'}$ such that $B(b'_\ell, s'_\ell) = B(b_m, s_m)$, and so $B(a_j, s_j) \subseteq B(b'_\ell, s'_\ell)$. If $m = k$ then either $B(a_j, s_j) \subseteq B(b'_1, s'_1)$ or $B(a_j, s_j)$ lies in the complement of $B(b'_0, s'_0)$. It follows that B' is partially above A , and satisfies 4 and hence $B' \in \mathcal{A}$. Since $\delta_{\text{ann}}(B') \geq \delta_{\text{ann}}(A) > 0$, it follows that B' is annular.

It remains to show that if $\delta_{\text{ann}}(B') = \delta_{\text{ann}}(B)$ then $\delta_2(B') < \delta_2(B)$. Assume that $\delta_{\text{ann}}(B') = \delta_{\text{ann}}(B)$. Then we must have either $s_0 = s'_0 + 2s_k$ or $s'_1 = s_1 + 2s_k$. In the first case we have $s_0^2 > (s'_0)^2 + s_k^2$, and in the second case we have $(s'_1)^2 > s_1^2 + s_k^2$. In either case, we have $\delta_2(B') < \delta_2(B)$. This completes the proof. $\square$

Note that, as for arbitrary abstract Swiss cheeses, if B is a semiclassical, annular Swiss cheese then $\pi\delta_2(B)$ is the area of X_B .

Theorem 2.4.5. *Let $\mathcal{A}$ be as in Lemma 2.4.3. Then there exists a classical, annular Swiss cheese $B = ((b_n, s_n)) \in \mathcal{A}$ with $\delta_{\text{ann}}(B) \geq \delta_{\text{ann}}(A)$ and $X_B \subseteq X_A$. Moreover, $r_0 - 2\rho_{\text{ann}}(A) \leq s_0 \leq r_0$ and $r_1 \leq s_1 \leq r_1 + 2\rho_{\text{ann}}(A)$.*

Proof. Since δ_{ann} is upper semicontinuous on $\mathcal{A}$ and $\mathcal{A}$ is compact and non-empty, it follows that δ_{ann} achieves its maximum on $\mathcal{A}$. Let $\mathcal{A}_1$ denote the non-empty, compact subset of $\mathcal{A}$ on which δ_{ann} is maximised. Then δ_2 , which is continuous on $\mathcal{A}_1$, achieves its minimum. Let $\mathcal{A}_2$ denote the non-empty, compact subset of $\mathcal{A}_1$ on which δ_2 is minimised and let $B = ((b_n, s_n)) \in \mathcal{A}_2$.

Since $\delta_{\text{ann}}(B) \geq \delta_{\text{ann}}(A) > 0$ it follows that B is annular and $X_B \subseteq X_A$. Suppose, for contradiction, that B is non-classical. There are two possible cases.

First suppose there exist $k, \ell \in S_B \setminus \{1\}$ with $k > \ell$ and such that $k, \ell \in S_B$ and $\bar{B}(b_k, s_k) \cap \bar{B}(b_\ell, s_\ell) \neq \emptyset$. Then, by Lemma 2.1.9 there exists $b \in \mathbb{C}$ and $s > 0$ such that

$$B(b_k, s_k) \cup B(b_\ell, s_\ell) \subseteq B(b, s)$$

and $s \leq s_k + s_\ell$. Let $B' = ((b'_n, s'_n))$ be the abstract Swiss cheese obtained by deleting the disks at indices k, ℓ from B and inserting the disk $B(b, s)$ at the first index in $\mathbb{N} \setminus \{1\}$ such that $(s'_n)_{n=2}^\infty$ is non-increasing. It is easy to see that $B' \in \mathcal{A}$ and

$$\rho_{\text{ann}}(B) \geq \rho_{\text{ann}}(B') - s_k - s_\ell + s = \rho_{\text{ann}}(B'), \quad (2.3)$$

so that $\delta_{\text{ann}}(B') \geq \delta_{\text{ann}}(B)$. By the maximality of $\delta_{\text{ann}}(B)$, equality must hold here and in (2.3). Thus $s = s_k + s_\ell$ and $s^2 = (s_k + s_\ell)^2 > s_k^2 + s_\ell^2$ so that $\delta_2(B') < \delta_2(B)$. This contradicts the minimality of $\delta_2(B)$. It follows that no such k, ℓ exist.

Now suppose there exists $k \in S_B \setminus \{1\}$ such that $\bar{B}(b_k, s_k) \cap \overline{\mathbb{C} \setminus K_B} \neq \emptyset$ and $s_k > 0$. By Lemma 2.4.4 there exists an annular Swiss cheese $B' \in \mathcal{A}$ with $\delta_{\text{ann}}(B') \geq \delta_{\text{ann}}(B)$ such that, if $\delta_{\text{ann}}(B') = \delta_{\text{ann}}(B)$ then $\delta_2(B') < \delta_2(B)$. This is a contradiction, so no such k can exist. It follows that B is classical.

Since $B \in \mathcal{A}$, we have $r_0 \geq s_0 \geq s_1 \geq r_1$. We also have

$$s_0 - s_1 \geq \delta_{\text{ann}}(B) \geq \delta_{\text{ann}}(A) = r_0 - r_1 - 2\rho_{\text{ann}}(A)$$

so that

$$s_0 \geq r_0 - 2\rho_{\text{ann}}(A) - (r_1 - s_1) \geq r_0 - 2\rho_{\text{ann}}(A)$$

and $s_1 \leq r_1 + 2\rho_{\text{ann}}(A) - (r_0 - s_0) \leq r_1 + 2\rho_{\text{ann}}(A)$. This completes the proof. $\square$

2.5 Regularity for $R(X)$

We now discuss some results on the regularity of $R(X)$ for various compact plane sets X . The following proposition is a special case of [51, Corollary 4.7].

Proposition 2.5.1. *Let X be a compact plane set such that $R(X)$ is not regular. Let E denote the set of non- R -points for $R(X)$. Then E contains a non-empty perfect subset. In particular, E is uncountable.*

The following proposition, stated in [40], lists some properties of $R(X)$ which are inherited when a subset of X is considered.

Proposition 2.5.2. *Let X and Y be compact plane sets with $Y \subseteq X$.*

- (a) *If $R(X) = C(X)$ then $R(Y) = C(Y)$.*
- (b) *If $R(X)$ does not have any non-zero bounded point derivations then neither does $R(Y)$.*
- (c) *If $R(X)$ is regular then so is $R(Y)$.*

In this section, we prove some results about regularity of $R(X)$ which we shall require for the construction in the next section. The following proposition is essentially [54, Corollary II.10.3].

Proposition 2.5.3. *Let X be a compact plane set and let $f \in C(X)$. Suppose that for each $x \in X$ there is a closed neighbourhood N_x of x in X such that $f|_{N_x} \in R(N_x)$. Then $f \in R(X)$.*

Our main theorem for this section is the following.

Theorem 2.5.4. *Let X be a compact plane set and let E be a countable subset of X . Let (X_α) be a family of compact plane sets such that $R(X_\alpha)$ is regular for all α and $\bigcup_\alpha \text{int}_X(X \cap X_\alpha) \supseteq X \setminus E$. Then $R(X)$ is regular.*

Proof. We first show that every point in $X \setminus E$ is an R -point for $R(X)$. Let $x \in X \setminus E$ and $y \in X$ with $x \neq y$. Then there exists α and $r > 0$ such that $\bar{B}(x, r) \cap X \subseteq X_\alpha$. Let $\delta < r/3$ such that $|x - y| > 2\delta$. Let F denote the complement of $X \cap \bar{B}(x, 2\delta)$ in $\bar{B}(x, r) \cap X$. Since $X \cap \bar{B}(x, r) \subseteq X_\alpha$, $R(X \cap \bar{B}(x, r))$ is regular by Proposition 2.5.2 (and hence normal). Thus there exists a function $g \in R(X \cap \bar{B}(x, r))$ with $g(z) = 0$ for all $z \in X \cap \bar{B}(x, \delta)$ and $g(z) = 1$ for all $z \in F$. Extend g to a function $f \in C(X)$ by setting $f(z) = g(z)$ for all $z \in (\bar{B}(x, r) \cap X)$ and $f(z) = 1$ for all $z \in X \setminus \bar{B}(x, r)$. Clearly f satisfies the conditions of Proposition 2.5.3, so $f \in R(X)$. By our choice of δ , we have $x \in U$ and $y \in F$ so f vanishes on a neighbourhood of x and $f(y) = 1$, so x is an R -point for $R(X)$.

It follows that $R(X)$ has at most countably many non- R -points. So, by Proposition 2.5.1, $R(X)$ is regular. $\square$

Note that we do not assume that $X_\alpha \subseteq X$. However, replacing X_α by $X \cap X_\alpha$ does not alter the result. We obtain the following corollaries.

Corollary 2.5.5. *Let X be a compact plane set and $x_0 \in X$. Let (X_α) be a family of compact plane sets such that $\bigcup_\alpha \text{int}_X(X \cap X_\alpha) = X \setminus \{x_0\}$ and $R(X_\alpha)$ is regular for all α . Then $R(X)$ is regular.*

Corollary 2.5.6. *Let X_1, X_2 be compact plane sets such that $X_1 \cap X_2$ is countable. If $R(X_1)$ and $R(X_2)$ are regular then $R(X_1 \cup X_2)$ is regular.*

2.6 Classicalisation of an example of O’Farrell

In this section we see an application of the results of this chapter. In [88], O’Farrell modified the construction of McKissick [85] to construct a Swiss cheese set X such that $R(X)$ is regular and admits a non-degenerate bounded point derivation of infinite order. However, this Swiss cheese set is not necessarily classical.

Following our general scheme of classicalisation, we construct a classical Swiss cheese set X such that $R(X)$ is regular and admits a bounded point derivation of infinite order at one of the points of X .

The first proposition is easily deduced from a theorem of Hallstrom [59], and some elementary facts about analytic capacity. (See [88].)

Proposition 2.6.1. *Let $A = ((a_n, r_n))$ be an abstract Swiss cheese, and let $x \in X_A$. For each $n \in \mathbb{N}$, let $K_n(x) := \{z \in \mathbb{C} : 2^{-1-n} \leq |z - x| \leq 2^{-n}\}$. Suppose that*

$$\sum_{n=0}^{\infty} n^n \rho_{K_n(x)}(A) < \infty.$$

Then $R(X_A)$ admits bounded point derivation of infinite order at x .

The next proposition is an immediate corollary of Proposition 2.5.2 and Corollary 2.2.12 (see also [85, 88] and [40, Proposition 1.10]).

Proposition 2.6.2. *Let $b_0 = b_1 \in \mathbb{C}$, let $s_0 > s_1 > 0$, and let $\varepsilon > 0$. Then there exists an annular Swiss cheese $A = ((a_n, r_n))$ with $\rho_{\text{ann}}(A) < \varepsilon$, $a_j = b_j$ and $r_j = s_j$ for $j = 0, 1$, and such that $R(X_A)$ is regular.*

We now use a sequence of lemmas to show that we can construct a classical annular Swiss cheese with the same properties as those in Proposition 2.6.2.

Lemma 2.6.3. *Let $\lambda_0 > \lambda_1 > 0$ and $\varepsilon, \eta > 0$ be given and let $a \in \mathbb{C}$. There exists a classical, annular Swiss cheese $B = ((b_n, s_n))$ with $b_0 = b_1 = a$ such that $\rho_{\text{ann}}(B) < \varepsilon$, $\lambda_0 \leq s_0 \leq \lambda_0 + \eta$ and $\lambda_1 - \eta \leq s_1 \leq \lambda_1$ such that $R(X_B)$ is regular.*

Proof. We may assume that $\eta < \lambda_1$ and $\varepsilon \leq \eta/2$. Let $A = ((a_n, r_n))$ be an abstract Swiss cheese obtained from Proposition 2.6.2 with $a_0 = a_1 = a$, $r_0 = \lambda_0 + \eta$, $r_1 = \lambda_1 - \eta$ and $\rho_{\text{ann}}(A) < \varepsilon$ and such that $R(X_A)$ is regular. By Lemma 2.4.2, we may assume that the sequence $(r_n)_{n=2}^\infty$ is non-increasing. Apply Theorem 2.4.5 to the abstract Swiss cheese A to obtain a classical, annular Swiss cheese $B = ((b_n, s_n))$ with

$$b_0 = b_1 = a_0 = a_1 = a, \quad r_0 - 2\varepsilon \leq s_0 \leq r_0, \quad \text{and} \quad r_1 \leq s_1 \leq r_1 + 2\varepsilon,$$

such that $\rho_{\text{ann}}(B) \leq \rho_{\text{ann}}(A)$ and $X_B \subseteq X_A$. By Lemma 2.5.2, $R(X_B)$ is regular. Since $2\varepsilon \leq \eta$, we have $\lambda_0 \leq s_0 \leq \lambda_0 + \eta$ and $\lambda_1 - \eta \leq s_0 \leq \lambda_1$. This completes the proof. $\square$

By instead taking $r_0 = \lambda_0$ and $r_1 = \lambda_1$ in the proof of the previous lemma we see that we could also approximate the desired annulus with a smaller annulus, rather than a larger annulus as in Lemma 2.6.3.

In the next lemma, we see how to obtain a classical, annular Swiss cheese A such that $R(X_B)$ is regular with one of the first two radii specified exactly while prescribing tight bounds on the other.

Lemma 2.6.4. *Let $\lambda_0 > \lambda_1 > 0$ and $\varepsilon, \eta > 0$ be given and let $a \in \mathbb{C}$.*

- (a) *There exists a classical, annular Swiss cheese $B^{(1)} = ((b_n^{(1)}, s_n^{(1)}))$ such that $s_0^{(1)} = \lambda_0$, $\lambda_1 - \eta \leq s_1^{(1)} \leq \lambda_1$, $b_0^{(1)} = b_1^{(1)} = a$, $\rho_{\text{ann}}(B^{(1)}) < \varepsilon$, and such that $R(X_{B^{(1)}})$ is regular.*
- (b) *There exists a classical, annular Swiss cheese $B^{(2)} = ((b_n^{(2)}, s_n^{(2)}))$ such that $b_0^{(2)} = b_1^{(2)} = a$, $\rho_{\text{ann}}(B^{(2)}) < \varepsilon$, $\lambda_0 \leq s_0^{(2)} \leq \lambda_0 + \eta$, $s_1^{(2)} = \lambda_1$, and such that $R(X_{B^{(2)}})$ is regular.*

Proof. We prove (b); the proof of (a) is similar but easier. We may assume, without loss of generality, that $a = 0$. Let $\gamma \in (0, \lambda_1)$ to be chosen later. Apply Lemma 2.6.3 to obtain a classical, annular Swiss cheese $A = ((a_n, r_n))$ such that $\rho_{\text{ann}}(A) < \gamma/2$, $\lambda_0 \leq r_0 \leq \lambda_0 + \gamma$ and $\lambda_1 - \gamma \leq r_1 \leq \lambda_1$ and such that $R(X_A)$ is regular.

For each $n \geq 0$ let $b_n^{(2)} := \lambda_1 a_n / r_1$ and $s_n^{(2)} := \lambda_1 r_n / r_1$. Then

$$\sum_{n=2}^{\infty} s_n^{(2)} \leq \frac{\lambda_1}{\lambda_1 - \gamma} \sum_{n=2}^{\infty} r_n < \frac{\lambda_1}{\lambda_1 - \gamma} \frac{\gamma}{2}.$$

Set $M_\gamma := \lambda_1 / (\lambda_1 - \gamma) > 1$. We have $s_1^{(2)} = \lambda_1$ and $s_0^{(2)} \geq \lambda_0$ so it remains to show that $s_0^{(2)} \leq \lambda_0 + \eta$ provided that γ is sufficiently small. We have

$$\lambda_0 \leq s_0^{(2)} = \frac{\lambda_1}{r_1} r_0 \leq M_\gamma r_0 \leq M_\gamma (\lambda_0 + \gamma).$$

Since $M_\gamma \rightarrow 1$ as $\gamma \rightarrow 0$, if γ is small enough then we have $\lambda_0 \leq s_0^{(1)} \leq \lambda_0 + \eta$ and $M_\gamma \gamma / 2 < \varepsilon$. Clearly $R(X_{B^{(2)}})$ is regular since $R(X_A)$ is regular. This completes the proof of the (b). $\square$

Our final lemma shows that we can obtain this type of annular Swiss cheese with s_0 and s_1 precisely prescribed.

Lemma 2.6.5. *Let $\lambda_0 > \lambda_1 > 0$ and $\varepsilon > 0$ be given and let $a \in \mathbb{C}$. There exists a classical, annular Swiss cheese $B = ((b_n, s_n))$ with $b_0 = b_1 = a$, $s_0 = \lambda_0$, $s_1 = \lambda_1$, $\rho_{\text{ann}}(B) < \varepsilon$ and such that $R(X_B)$ is regular.*

Proof. Let $\kappa = (\lambda_0 + \lambda_1)/2$ and let $\eta > 0$. By Lemma 2.6.4(a) there exists a classical, annular Swiss cheese $A^{(1)} = ((a_n^{(1)}, r_n^{(1)}))$ such that $a_0^{(1)} = a_1^{(1)} = a$, $\rho_{\text{ann}}(A^{(1)}) < \eta/16$, $r_0^{(1)} = \lambda_0$, $\kappa - \eta/4 \leq r_1^{(1)} \leq \kappa - \eta/8$, and $R(X_{A^{(1)}})$ is regular. By Lemma 2.6.4(b) there exists a classical, annular Swiss cheese $A^{(2)} = ((a_n^{(2)}, r_n^{(2)}))$ with $a_0^{(2)} = a_1^{(2)} = a$, $\rho_{\text{ann}}(A^{(2)}) < \eta/16$, $r_1^{(2)} = \lambda_1$, and $\kappa \leq r_0^{(2)} \leq \kappa + \eta/4$ such that $R(X_{A^{(2)}})$ is regular.

Let $(a_n^{(3)})_{n \geq 2}$ be a sequence containing all elements from the sequences $(a_n^{(1)})_{n \geq 2}$, and $(a_n^{(2)})_{n \geq 2}$ exactly once and let $(r_n^{(3)})_{n \geq 2}$ be the corresponding sequence containing all elements from the sequences $(r_n^{(1)})_{n \geq 2}$, and $(r_n^{(2)})_{n \geq 2}$ exactly once. Let $a_0^{(3)} := a$, $a_1^{(3)} := a$ and $r_0^{(3)} := \lambda_0$, $r_1^{(3)} := \lambda_1$, and let $A^{(3)} = ((a_n^{(3)}, r_n^{(3)}))$ be the corresponding annular Swiss cheese. Then

$$\rho_{\text{ann}}(A^{(3)}) = \sum_{n=2}^{\infty} r_n^{(3)} < \frac{\eta}{16} + \frac{\eta}{16} = \frac{\eta}{8}.$$

Let $X := X_{A^{(3)}}$, then we can easily check that $X = \text{int}_X X_{A^{(1)}} \cup \text{int}_X X_{A^{(2)}}$ so, by Corollary 2.5.6, $R(X)$ is regular.

Choose η small enough so that $\eta < (\lambda_0 - \lambda_1)/4$, $\eta/2 < \varepsilon$, and $\eta < \lambda_1$. Let

$$K := \{z \in \mathbb{C} : \kappa - \eta/4 \leq |z| \leq \kappa + \eta/4\},$$

and let $M := \eta/4$. Let $A = ((a_n, r_n)) \in \mathcal{N}$ be obtained by applying Lemma 2.4.2. Then, for each open $U \subseteq \mathbb{C}$, $\rho_U(A) \leq \rho_U(A^{(3)})$. It is now easy to see that A, K and M satisfy the conditions of Corollary 2.3.8. Note that $U := U(K, M)$ has $U \cap \bar{B}(a_1, r_1) = \emptyset$. Note that $X_A = X = X_{A^{(3)}}$.

Let $B = ((b_n, s_n))$ be the classical abstract Swiss cheese obtained by applying Corollary 2.3.8 to A, K and M . Then $B \in \mathcal{L}_A^*(\mathcal{C})$, where $\mathcal{C}$ is the controlling pair (K, U) . Thus, by Lemma 2.3.5, there exists $\ell \in S_B$ such that $B(b_\ell, s_\ell) = B(a_1, r_1)$. Since the sequence $(s_n)_{n \geq 1}$ is non-increasing and $\rho_{\text{ann}}(A) < r_1$, it follows that $\ell = 1$. It follows that B is annular and has $s_0 = \lambda_0$, $s_1 = \lambda_1$ and $\rho_{\text{ann}}(B) < \varepsilon$. Since $R(X_A)$ is regular, by Proposition 2.5.2, $R(X_B)$ is regular. This completes the proof. $\square$

We are now ready to construct a classical Swiss cheese set X such that $R(X)$ is regular and admits a bounded point derivation of infinite order.

Theorem 2.6.6. *Let $\varepsilon > 0$. Then there exists a classical abstract Swiss cheese $B = ((b_n, s_n))$ with $0 \in X_B$ and $\rho(B) < \varepsilon$ and such that $R(X_B)$ is regular and admits a non-degenerate bounded point derivation of infinite order at 0.*

Proof. We may assume $\varepsilon < 2^{-5}$. For each $n \in \mathbb{N}$, let $\gamma_n = (2n)^{-n}\varepsilon$. Note that $\sum_{n=1}^{\infty} \gamma_n < \varepsilon$. Let $A^{(1)} = ((a_n^{(1)}, r_n^{(1)}))$ be a classical, annular Swiss cheese, given by Lemma 2.6.5, such that $a_0^{(1)} = 0$, $\rho_{\text{ann}}(A^{(1)}) < \gamma_1 2^{-3}$, $r_0^{(1)} = 1$, $r_1^{(1)} = 2^{-1}$, and $R(X_1)$ is regular, where $X_1 := X_{A^{(1)}}$. For each $m \geq 2$ let $A^{(m)} = ((a_n^{(m)}, r_n^{(m)}))$ be a classical, annular Swiss cheese, given by Lemma 2.6.5, such that $a_0^{(m)} = 0$, $\rho_{\text{ann}}(A^{(m)}) < \gamma_m 2^{-m-2}$, $r_0^{(m)} = (33/32)2^{1-m}$, $r_1^{(m)} = 2^{-m}$, and $R(X_m)$ is regular, where $X_m := X_{A^{(m)}}$. Note that, for each $m \in \mathbb{N}$, since $A^{(m)}$ is annular we have $a_1^{(m)} = 0$.

For each $m \in \mathbb{N}$ we have

$$\rho_{\text{ann}}(A^{(m)}) + \rho_{\text{ann}}(A^{(m+1)}) \leq \frac{\gamma_m}{2^{m+2}} + \frac{\gamma_{m+1}}{2^{m+3}} < \frac{3}{2} \frac{\gamma_m}{2^{m+2}}. \quad (2.4)$$

The set $\{(a_n^{(m)}, r_n^{(m)}) : m, n \in \mathbb{N}, n \geq 2\}$ is countably infinite so we may enumerate it as a sequence of pairs (a_n, r_n) , so that each pair occurs exactly once. Let $a_0 := 0$ and $r_0 := 1$ and let $A = ((a_n, r_n))$ be the resulting abstract Swiss cheese. It is clear that $0 \in X_A$. It is easy to check that

$$\bigcup_{m=1}^{\infty} \text{int}_{X_A} X_m = X_A \setminus \{0\}$$

so that $R(X_A)$ is regular by Corollary 2.5.5.

Using the notation for closed annuli from Section 2.4, for each $m \in \mathbb{N}$, let $W_m = K_{A^{(m)}} \cup K_{A^{(m+1)}}$. Then, by (2.4), we see that

$$\rho_{W_m}(A) < (3/2)\gamma_m 2^{-m-2}. \quad (2.5)$$

We also have

$$\rho(A) < \sum_{m=1}^{\infty} 2^{-m-2}\gamma_m < \varepsilon. \quad (2.6)$$

By an application of Lemma 2.1.6, we may assume that A is redundancy-free and $A \in \mathcal{N}$ while preserving X_A , the regularity of $R(X_A)$ and the inequalities (2.5) and (2.6).

For each $m \in \mathbb{N}$, let $K_m := \{z \in \mathbb{C} : (15/16)2^{-m} \leq |z| \leq (17/16)2^{-m}\}$ and $M_m = 3\gamma_m/2^{m+2}$ so that, since $\gamma_m < 1$, we have

$$2^{-m-4} + M_m < 2^{-m-4}(1 + 3 \cdot 2^{-m}) \leq (5/32)2^{-m}. \quad (2.7)$$

For each $m \in \mathbb{N}$, define $U_m := U(K_m, M_m)$ as in the statement of Theorem 2.3.7, and let $\mathcal{C}$ be the controlling collection $((K_n, U_n))_{n \in \mathbb{N}}$. By (2.7)

$$U_m \subseteq \left\{ z \in \mathbb{C} : \frac{27}{32}2^{-m} \leq |z| \leq \frac{37}{32}2^{-m} \right\}$$

and hence $U_m \cap U_n = \emptyset$ for all $m, n \in \mathbb{N}$ with $m \neq n$. It is clear that $E(A) \subseteq F(\mathcal{C})$. Since $U_m \subseteq W_m$, for each $m \in \mathbb{N}$, we see that

$$\rho_{U_m}(A) \leq \rho_{W_m}(A) < \frac{3}{2} \frac{\gamma_m}{2^{m+2}}$$

This shows that the sequences $(K_n)_{n \geq 1}$ and $(M_n)_{n \geq 1}$ satisfy the conditions of Theorem 2.3.7.

Applying Theorem 2.3.7 to A , $(K_n)_{n \geq 1}$ and $(M_n)_{n \geq 1}$, there exists an abstract Swiss cheese $B = ((b_n, s_n)) \in \mathcal{L}_A^*(\mathcal{C})$ with $b_0 = 0$, $s_0 = 1$, $X_B \subseteq X_A$ and $\rho(B) < \varepsilon$ such that B is classical and $X_B \setminus V(\mathcal{C}) = X_A \setminus V(\mathcal{C})$; in particular $0 \in X_B$. Also, we have $\rho_{U_m}(B) \leq \rho_{U_m}(A)$ for each $m \in \mathbb{N}$. By Proposition 2.5.2, $R(X_B)$ is regular.

For each $m \in \mathbb{N}_0$, let $E_m := \{z \in \mathbb{C} : (3/2)2^{-m-1} \leq |z| \leq (3/2)2^{-m}\}$, then $U_m \subseteq E_m$ and $U_j \cap E_m = \emptyset$ for all $j \neq m$. For each $m \in \mathbb{N}$, let

$$I_A(m) := H_A(E_m) \setminus H_A(V(\mathcal{C})), \quad I_B(m) := H_B(E_m) \setminus H_B(V(\mathcal{C})).$$

Note that, $\rho_{E_m}(A) \leq \rho_{W_m}(A) < (3/2)\gamma_m 2^{-m-2}$. Since $\varepsilon < 2^{-5}$, it follows that, if $k \in H_{U_m}(A)$, we have

$$\bar{B}(a_k, r_k) \subseteq \left\{ z \in \mathbb{C} : \frac{3}{4}2^{-m} < |z| < \frac{5}{4}2^{-m} \right\} \subseteq \text{int } E_m$$

Hence $H_A(E_0) = I_A(0)$ and $H_B(E_0) = I_B(0)$ and, for each $m \geq 1$, we have $I_A(m) = H_A(E_m) \setminus H_A(U_m)$ and $I_B(m) = H_B(E_m) \setminus H_B(U_m)$. Since $B \in \mathcal{L}_A^*(\mathcal{C})$ we see that $\sum_{n \in I_B(m)} s_n = \sum_{n \in I_A(m)} r_n$, for all $m \in \mathbb{N}$, by Lemma 2.3.5. Hence $\rho_{E_0}(B) = \rho_{E_0}(A)$ and for $m \geq 1$ we have

$$\rho_{E_m}(B) = \sum_{n \in I_B(m)} s_n + \rho_{U_m}(B) \leq \sum_{n \in I_A(m)} r_n + \rho_{U_m}(A) = \rho_{E_m}(A).$$

Since $\rho_{E_m}(A) \leq \rho_{W_m}(A) < \gamma_m$, for each $m \in \mathbb{N}$, it now follows that

$$\sum_{m=1}^{\infty} m^m \rho_{E_m}(B) \leq \sum_{m=1}^{\infty} m^m \frac{\varepsilon}{2^m m^m} = \varepsilon < \infty.$$

Since each disk meets at most two of the E_m , $R(X_B)$ admits a non-degenerate bounded point derivation of infinite order at 0 by Proposition 2.6.1. This completes the proof. $\square$

2.7 Open problems and discussion

We conclude this chapter with a discussion of some open problems. In this section let X be a compact plane set.

Our first question relates to the algebra $R(X)$.

Question 2.7.1. Suppose that $R(X)$ has no non-zero bounded point derivations. Must $R(X)$ be regular?

Indeed we do not have enough information even when X is a classical Swiss cheese set, as constructed by Wermer [99].

CHAPTER 3

Algebras of differentiable functions

In this chapter, we discuss Banach function algebras consisting of functions which satisfy a generalised notion of differentiability, which we call $\mathcal{F}$ -differentiability. This notion of differentiability was introduced by Bland and Feinstein in [10] (see also [9]), and were further investigated by Dales and Feinstein [29]. We investigate the properties of algebras of $\mathcal{F}$ -differentiable functions, and the homomorphisms between such algebras.

Let X be a perfect compact plane set, and let $D^{(1)}(X)$ denote the algebra of all continuously complex-differentiable, complex-valued functions on X . Then $D^{(1)}(X)$ is a normed function algebra on X , which is not complete in general. (We give the formal definition of these algebras in Section 3.2.) In [28], Dales and Davie showed that if X is uniformly regular (see Section 3.1) then $D^{(1)}(X)$ is a Banach function algebra on X . In [10], Bland and Feinstein showed that if X has infinitely many components, then $D^{(1)}(X)$ is never complete. In [29], Dales and Feinstein discussed various conditions for $D^{(1)}(X)$ to be complete. They also gave many examples of sets X such that $D^{(1)}(X)$ is not complete. Also in [29], Dales and Feinstein showed that all characters on $D^{(1)}(X)$ correspond to point evaluation at points of X . This is valid even if $D^{(1)}(X)$ is not complete, so the characters on $D^{(1)}(X)$ are always continuous.

Bland and Feinstein introduced $\mathcal{F}$ -differentiation to study the completion of $D^{(1)}(X)$, which need not be well-behaved. Indeed, Bishop has given an example of a perfect compact plane set J for which the completion of $D^{(1)}(J)$ is not semisimple (see [29, Example 6.2]). However, in many cases, the completion of $D^{(1)}(X)$ can be realised as a closed subalgebra of an algebra of $\mathcal{F}$ -differentiable functions ([10]). In these cases, the completion of $D^{(1)}(X)$ is a Banach function algebra. A different

approach to describing the completion of $D^{(1)}(X)$ is given in [65] using the theory of unbounded linear operators.

In [10] (see also [29]) Bland and Feinstein showed that the product rule is valid for $\mathcal{F}$ -differentiation, and that $\mathcal{F}$ -differentiation generalises (continuous) complex-differentiation. In his thesis [18], Chaobankoh showed that the quotient rule and, under some technical assumptions, the chain rule are also valid for $\mathcal{F}$ -differentiation. However, the assumptions required to prove the chain rule are quite restrictive. Using a result of Honary [66], Chaobankoh showed that the algebra $D_{\mathcal{F}}^{(1)}(X)$, of all $\mathcal{F}$ -differentiable functions on X , is natural if the uniform closure in $C(X)$ of $D_{\mathcal{F}}^{(1)}(X)$ is a natural uniform algebra.

Dales–Davie algebras are normed function algebras consisting of infinitely continuously (complex) differentiable, complex-valued functions on a perfect compact subset of the complex plane. These algebras were introduced in [28] by Dales and Davie, where they investigated quasianalyticity for these algebras (see Chapter 4). Dales–Davie algebras are not complete or natural function algebras, in general. Dales and Davie gave some conditions Dales–Davie algebras to be complete and natural. Their conditions for naturality were slightly generalised in [2] by Abtahi and Honary.

The algebras $D_{\mathcal{F}}(X, M)$ were introduced by Bland and Feinstein ([10]), and are an $\mathcal{F}$ -differentiability analogue of the Dales–Davie algebras. Of course the algebras $D_{\mathcal{F}}(X, M)$ are always Banach function algebras, though they need not be natural. Sufficient conditions for $D_{\mathcal{F}}(X, M)$ to be natural on X are given in [18, Section 4.3]. This generalises the results of Abtahi and Honary for Dales–Davie algebras mentioned above.

Chaobankoh [18] discussed some sufficient conditions for a self map φ of X to induce an algebra endomorphism on the algebra $D_{\mathcal{F}}(X, M)$. (The formal definition of $D_{\mathcal{F}}(X, M)$ is given in Section 3.3.1.) These generalise results of Feinstein and Kamowitz for endomorphisms of Dales–Davie algebras from [42, 43, 70]. Feinstein and Kamowitz also gave related results for homomorphisms between Dales–Davie algebras in [44].

This chapter is divided into seven sections. In the first section, we discuss paths in the complex plane and path integrals. In the second section we give a formal introduction to the algebras $D^{(1)}(X)$ and Dales–Davie algebras. Next we discuss Bland–Feinstein algebras $D_{\mathcal{F}}^{(1)}(X)$ (and $D_{\mathcal{F}}(X, M)$) and their properties. We introduce maximal collections and compatibility. In the fourth section, we prove a new, general version of the chain rule for $\mathcal{F}$ -differentiation using our notion of compatibility. From this, we give a new proof of the quotient rule, and

deduce that the Faà di Bruno formula is also valid for $\mathcal{F}$ -differentiation. In the fifth section, we give some sufficient conditions for a map to induce an algebra homomorphism between algebras of the form $D_{\mathcal{F}}(X, M)$. In the sixth section, we give some sufficient conditions for the algebras $D_{\mathcal{F}}^{(1)}(X)$ and $D_{\mathcal{F}}(X, M)$ to be natural Banach function algebras on X . We also give several related results. We conclude the chapter with several open questions and a short discussion.

The first five sections of this chapter include the new results from [19].

3.1 Paths in the plane

We begin with a discussion of collections of paths in the complex plane, and some properties of path integrals. Many of the definitions and results in this section are standard.

Definition 3.1.1. A *path* in $\mathbb{C}$ is a continuous function $\gamma : [a, b] \rightarrow \mathbb{C}$, where $a < b$ are real numbers. Let $\gamma : [a, b] \rightarrow \mathbb{C}$ be a path. The *parameter interval* of γ is the interval $[a, b]$. The *endpoints* of γ are the points $\gamma(a)$ and $\gamma(b)$, which we denote by γ^- and γ^+ , respectively. We denote by γ^* the *image* $\gamma([a, b])$ of γ . A *subpath* of γ is a path obtained by restricting γ to a non-degenerate, closed subinterval of $[a, b]$. If X is a subset of $\mathbb{C}$ then we say that γ is a *path in X* if $\gamma^* \subseteq X$. We say that $x, y \in X$ are *connected by γ* if $\gamma^- = x$ and $\gamma^+ = y$.

We shall say that γ is a *path* when the parameter interval $[a, b]$ is not explicitly required. We say that γ is a *Jordan path* to mean that γ is an injective function.

Let $[a, b]$ be a non-degenerate interval. A *partition* $\mathcal{P}$ of $[a, b]$ is a finite set $\{x_0, \dots, x_n\} \subseteq [a, b]$ with $x_0 = a$, $x_n = b$ and such that $x_j < x_{j+1}$ for each $j \in \{0, 1, \dots, n-1\}$. If $\mathcal{P}$ and $\mathcal{P}'$ are partitions of $[a, b]$ then we say that $\mathcal{P}'$ is *finer* than $\mathcal{P}$ if $\mathcal{P} \subseteq \mathcal{P}'$.

Definition 3.1.2. Let $\gamma : [a, b] \rightarrow \mathbb{C}$ be a path and let $c, d \in [a, b]$ with $c < d$. The *total variation* of γ over $[c, d]$ is

$$V_c^d(\gamma) := \sup_{\mathcal{P}} \left\{ \sum_{j=0}^{n-1} |\gamma(x_{j+1}) - \gamma(x_j)| : \mathcal{P} = \{x_0, \dots, x_n\} \right\}$$

where the supremum is taken over all partitions $\mathcal{P}$ of $[c, d]$. We say that γ is *rectifiable* if $V_a^b(\gamma) < \infty$ and we write $\Lambda(\gamma) := V_a^b(\gamma)$; otherwise it is *non-rectifiable*. The *length* of a rectifiable path is $\Lambda(\gamma)$.

For a detailed discussion of total variation and path length, see [6].

We say that a path γ is *admissible* if γ is rectifiable and contains no constant subpaths. The following proposition summarises the discussion from [35, pp. 109–110] (see also [17, Lemma 7.9]).

Proposition 3.1.3. *Let $\gamma : [a, b] \rightarrow \mathbb{C}$ be a non-constant rectifiable path. Then there exists a unique path $\sigma : [0, \Lambda(\gamma)] \rightarrow \mathbb{C}$ satisfying $\sigma(V_a^t(\gamma)) = \gamma(t)$ ($t \in [a, b]$).*

Let $\gamma : [a, b] \rightarrow \mathbb{C}$ be a non-constant rectifiable path. We define the *path length parametrisation* $\gamma^{\text{pl}} : [0, \Lambda(\gamma)] \rightarrow \mathbb{C}$ of γ to be the unique path satisfying $\gamma^{\text{pl}}(V_a^t(\gamma)) = \gamma(t)$ ($t \in [a, b]$) given by the above proposition. We define the *normalised path length parametrisation* $\gamma^{\text{no}} : [0, 1] \rightarrow \mathbb{C}$ of γ to be the path such that $\gamma^{\text{no}}(t) = \gamma^{\text{pl}}(t\Lambda(\gamma))$ for each $t \in [0, 1]$. It is clear that γ^{pl} and γ^{no} are necessarily admissible paths and $(\gamma^{\text{pl}})^* = (\gamma^{\text{no}})^* = \gamma^*$.

We also recall the following definitions from [10, 28] and [29].

Definition 3.1.4. Let X be a compact plane set. We say that X is *regular at* $x \in X$ if there exists a constant $C_x > 0$ such that, for each $y \in X$, there exists a path γ in X with $\gamma^- = x$ and $\gamma^+ = y$ such that $\Lambda(\gamma) \leq C_x|x - y|$. We say that X is *pointwise regular* if X is regular at each $x \in X$. We say that X is *uniformly regular* if there exists a constant $C > 0$ such that for all $x, y \in X$ there exists a rectifiable path γ in X with $\gamma^- = x$ and $\gamma^+ = y$ such that $\Lambda(\gamma) \leq C|x - y|$. We say that X is *semi-rectifiable* if the union of the images of all rectifiable, Jordan paths in X is dense in X .

Let X be a compact plane set. It is easy to see that if X is pointwise regular then X is semi-rectifiable, and if X is uniformly regular then X is pointwise regular. Moreover, if X is pointwise regular and, for each $x \in X$, $C_x > 0$ is the constant as in Definition 3.1.4 and $\sup_{x \in X} C_x < \infty$, then X is uniformly regular. Each of these conditions imply that X is a perfect compact plane set.

Definition 3.1.5. Let X be a compact plane set. Suppose that $x, y \in X$ are connected by a rectifiable path γ in X . We define the *geodesic distance* $\mathbf{g}(x, y)$ between x and y to be $\inf \Lambda(\gamma)$, where the infimum is taken over all rectifiable paths γ in X connecting x and y .

Let X be a compact plane set. It is standard that, if $x, y \in X$ are connected by a rectifiable path, then there is a rectifiable path γ connecting x and y such that $\Lambda(\gamma) = \mathbf{g}(x, y)$ (see, for example, [17]). Such a path is called a *geodesic* and it is clear that any geodesic is a Jordan path. We say that X is *rectifiably connected*

if for each $x, y \in X$ there exists a rectifiable path in X connecting x and y . In this case, $(X, \mathfrak{g})$ is a metric space. We say that X is *geodesically bounded* if X is rectifiably connected and there exists $M > 0$ such that, for all $x, y \in X$, we have $\mathfrak{g}(x, y) \leq M$.

Definition 3.1.6. Let $\gamma : [a, b] \rightarrow \mathbb{C}$ be rectifiable path and let $f : [a, b] \rightarrow \mathbb{C}$ be a continuous function. The *Riemann–Stieltjes integral* of f with respect to γ is the complex number I satisfying the following property: for each $\varepsilon > 0$, there exists a partition $\mathcal{P}_0 = \{t_0, \dots, t_n\}$ of $[a, b]$ such that, for every partition $\mathcal{P} = \{t'_0, \dots, t'_m\}$ of $[a, b]$ which is finer than $\mathcal{P}_0$, we have

$$\left| I - \sum_{j=0}^{m-1} f(\tau_j) [\gamma(t'_{j+1}) - \gamma(t'_j)] \right| < \varepsilon$$

for each choice of $\tau_j \in [t'_j, t'_{j+1}]$ for $j = 0, 1, \dots, m-1$.

We refer the reader to [6] and [26] for definitions and standard results about Riemann–Stieltjes integrals and path integrals of complex functions. In this thesis, the Riemann–Stieltjes integrals we consider will always be with respect to a rectifiable path in $\mathbb{C}$, and we write

$$\int_{\gamma} f(z) dz$$

for the Riemann–Stieltjes integral of $f \circ \gamma$, for $f \in C(\gamma^*)$, with respect to a rectifiable path γ .

Let $[a, b]$ be a non-degenerate, compact interval in $\mathbb{R}$ and let $\gamma : [a, b] \rightarrow \mathbb{C}$ be a rectifiable path. Then the Riemann–Stieltjes integral with respect to γ defines a linear functional on $C([a, b])$ and so, by the Riesz representation theorem, is represented by integration with respect to a measure μ on $[a, b]$. The measure μ satisfies $\mu([c, d]) = \gamma(d) - \gamma(c)$ for each $[c, d] \subseteq [a, b]$; see, for example, [77, Section 2.3].

Let X, Y be compact spaces, and let $\varphi : X \rightarrow Y$ be a continuous function. If μ is a measure on X , then the formula $\varphi_*(\mu)(E) = \mu(\varphi^{-1}(E))$ (E is a Borel set of Y) defines a measure on Y . This measure is called the *pushforward measure*. See, for example, [35, 2.1.2, 2.2.17]. The following proposition is a special case of [35, Theorem 2.4.18]. (See also [63, 12.45–12.46].)

Proposition 3.1.7. *Let X, Y be compact spaces, let μ be a finite, positive measure on X , and let $\varphi : X \rightarrow Y$ be a continuous surjection. Let $f : Y \rightarrow \mathbb{R}$ be a Borel measurable function. Then $f \circ \varphi$ is Borel measurable and*

$$\int_X (f \circ \varphi)(x) d\mu(x) = \int_Y f(y) d(\varphi_*(\mu))(y).$$

The following lemma will be used implicitly throughout this chapter; the result is probably well known but we have been unable to find an explicit reference so we include the details of the proof for the convenience of the reader.

Lemma 3.1.8. *Let $\gamma : [a, b] \rightarrow \mathbb{C}$ be a non-constant rectifiable path and let $f \in C(\gamma^*)$. Then*

$$\int_{\gamma} f(z) dz = \int_{\gamma^{\text{pl}}} f(z) dz = \int_{\gamma^{\text{no}}} f(z) dz.$$

Proof. First suppose that

$$\int_{\gamma} f(z) dz = \int_{\gamma^{\text{pl}}} f(z) dz. \quad (3.1)$$

Let $\varphi : [0, 1] \rightarrow [0, \Lambda(\gamma)]$ be the map $\varphi(t) = t\Lambda(\gamma)$. Then φ is a strictly monotonic increasing bijection from $[0, 1]$ onto $[0, \Lambda(\gamma)]$ so that

$$\int_{\gamma^{\text{pl}}} f(z) dz = \int_{\gamma^{\text{no}}} f(z) dz$$

by [6, Theorem 7.7]. So it remains to show that (3.1) holds.

Let μ be the measure on $[a, b]$ such that $\mu([c, d]) = \gamma(d) - \gamma(c)$ for each non-degenerate, closed subinterval $[c, d]$ of $[a, b]$. Then, as described above, integrating with respect to μ yields the Riemann-Stieltjes integral with respect to γ . Similarly, let ν be the measure on $[0, \Lambda(\gamma)]$ such that $\nu([c, d]) = \gamma^{\text{pl}}(d) - \gamma^{\text{pl}}(c)$ for each non-degenerate, closed subinterval $[c, d]$ of $[0, \Lambda(\gamma)]$. Let $\varphi : [a, b] \rightarrow [0, \Lambda(\gamma)]$ be the map given by $\varphi(t) := V_a^t(\gamma)$. Then φ is a monotone increasing, continuous surjection from $[a, b]$ onto $[0, \Lambda(\gamma)]$. Let $\varphi_*(\mu)$ be the pushforward measure, given by

$$\varphi_*(\mu)(E) = \mu(\varphi^{-1}(E)) \quad (E \subseteq [0, \Lambda(\gamma)]).$$

We *claim* that $\varphi_*(\mu) = \nu$.

To prove the claim, first let $[c, d]$ be a non-degenerate, closed interval in $[0, \Lambda(\gamma)]$. Then $\varphi^{-1}([c, d])$ is a non-degenerate, closed interval in $[a, b]$, say $[x, y]$. Clearly we have

$$x = \inf\{t \in [a, b] : \varphi(t) = c\}, \quad y = \sup\{t \in [a, b] : \varphi(t) = d\}.$$

But now, since $\gamma^{\text{pl}} \circ \varphi = \gamma$, we have

$$\nu([c, d]) = \gamma^{\text{pl}}(d) - \gamma^{\text{pl}}(c) = \gamma^{\text{pl}}(\varphi(y)) - \gamma^{\text{pl}}(\varphi(x)) = \gamma(y) - \gamma(x),$$

and

$$\varphi_*(\mu)([c, d]) = \mu(\varphi^{-1}([c, d])) = \mu([x, y]) = \gamma(y) - \gamma(x).$$

Thus $\varphi_*(\mu)([c, d]) = \nu([c, d])$. It now follows from the regularity of μ and ν that $\varphi_*(\mu) = \nu$. This proves the claim.

By writing $f = f_1 - f_2 + i(f_3 - f_4)$, with f_j a positive valued function on γ^* ($j = 1, \dots, 4$), and considering the polar decomposition of μ , the result now follows from Proposition 3.1.7. $\square$

Remark 3.1.9. The author is grateful to Dr. Matthias Kurzke for several useful conversations on path integration and geometric measure theory.

3.2 Algebras of differentiable functions

We begin with some elementary definitions, terminology, and results. Throughout this section X will be a perfect compact plane set. Let $f : X \rightarrow \mathbb{C}$ be a function and let $z_0 \in X$. We say f is *complex differentiable* (or just *differentiable*) at z_0 if the limit

$$f'(z_0) := \lim_{\substack{z \rightarrow z_0 \\ z \in X \setminus \{z_0\}}} \frac{f(z) - f(z_0)}{z - z_0} \quad (3.2)$$

exists and is finite. We say that f is (*complex*) *differentiable on X* (or simply *differentiable on X*) if f is differentiable at each point $z_0 \in X$ and we say that f' (defined by (3.2)) is the *derivative of f* . We denote the set of all continuous functions $f : X \rightarrow \mathbb{C}$ which admit a continuous derivative on X by $D^{(1)}(X)$. For each $n \in \mathbb{N}$ with $n \geq 2$, we define the spaces $D^{(n)}(X)$ inductively by

$$D^{(n)}(X) = \{f \in D^{(1)}(X) : f' \in D^{(n-1)}(X)\}. \quad (3.3)$$

Let $n \in \mathbb{N}$. We denote the n th derivative of $f \in D^{(n)}(X)$ by $f^{(n)}$. We adopt the convention that $f^{(0)} = f$. We equip the set $D^{(n)}(X)$ with pointwise operations of addition, scalar multiplication and multiplication to make $D^{(n)}(X)$ into a commutative, unital algebra. Furthermore, we give $D^{(n)}(X)$ the norm

$$\|f\|_{D^{(n)}(X)} := \sum_{j=0}^n \frac{|f^{(j)}|_X}{j!} \quad (f \in D^{(n)}(X)), \quad (3.4)$$

to make $D^{(n)}(X)$ a normed function algebra on X .

Notation. We denote the algebra of all continuous functions on X with continuous derivatives on X of all orders by $D^{(\infty)}(X) = \bigcap_{n=1}^{\infty} D^{(n)}(X)$.

The following proposition from [10] shows that it is sufficient to investigate the completeness of the algebra $D^{(1)}(X)$.

Proposition 3.2.1. *Let $n \in \mathbb{N}$ and suppose that $D^{(n)}(X)$ is complete. Then $D^{(k)}(X)$ is complete for each $k \in \mathbb{N}$ with $k \geq n$.*

Let X be a perfect compact plane set. In [28], Dales and Davie proved that $D^{(1)}(X)$ is complete whenever X is a uniformly regular. However, as noted in [67] (see also [10]), the proof in [28] is valid if X is only assumed to be pointwise regular. Bland and Feinstein also proved that the same is true if X is the finite union of pointwise regular sets, and that if X has infinitely many components then $D^{(n)}(X)$ is incomplete for each $n \in \mathbb{N}$.

3.2.1 Dales–Davie algebras

We now discuss Dales–Davie algebras of infinitely differentiable functions, as introduced in [28]. We first describe some terminology associated with sequences of positive real numbers.

Definition 3.2.2. Let $M = (M_n)_{n=0}^\infty$ be a sequence of positive real numbers. We say that M is an *algebra sequence* if $M_0 = 1$ and, for each $k, n \in \mathbb{N}_0$ with $k \leq n$, we have

$$\frac{M_n}{M_{n-k}M_k} \geq \binom{n}{k}.$$

We say that M is *logarithmically convex* (*log-convex*) if, for each $k \in \mathbb{N}$, we have

$$M_k^2 \leq M_{k-1}M_{k+1}.$$

Let $d(M) = \limsup_{k \rightarrow \infty} (k!/M_k)^{1/k}$. We say that M is *non-analytic* if $d(M) = 0$ and we say that M is *analytic* if $d(M) > 0$.

The following class of algebras were introduced in [28], we call these algebras *Dales–Davie algebras*.

Definition 3.2.3. Let X be a perfect compact plane set and let $M = (M_n)_{n=0}^\infty$ be an algebra sequence. We define

$$D(X, M) := \left\{ f \in D^{(\infty)}(X) : \sum_{k=0}^{\infty} \frac{|f^{(k)}|_X}{M_k} < \infty \right\}$$

with pointwise operations. We give $D(X, M)$ the norm

$$\|f\|_{D(X, M)} := \sum_{k=0}^{\infty} \frac{|f^{(k)}|_X}{M_k} \quad (f \in D(X, M)),$$

to make $D(X, M)$ into a normed algebra of functions.

Note that, in general, $D(X, M)$ is not a Banach function algebra; however, if $D^{(1)}(X)$ is complete then so is $D(X, M)$.

3.3 Bland–Feinstein algebras

In this section, we discuss a class of algebras introduced by Bland and Feinstein [10] (see also [29]). In certain cases, these algebras can be viewed as the completion of the corresponding algebra $D^{(1)}(X)$.

In the remainder of this chapter we shall make use of various collections of rectifiable paths in certain compact plane sets. We now give some terminology associated with such collections.

The following definition is from [29] and generalises the definition of a ‘useful collection’ as introduced in [10].

Definition 3.3.1. Let X be a compact plane set and let $\mathcal{F}$ be a collection of paths in X . Set $\mathcal{F}^* := \{\gamma^* : \gamma \in \mathcal{F}\}$. We say that $\mathcal{F}$ is *effective* if $\bigcup \mathcal{F}^* = X$, each path in $\mathcal{F}$ is admissible, and every subpath of a path in $\mathcal{F}$ belongs to $\mathcal{F}$. We denote by $\mathcal{F}^{\text{no}}$ the collection $\{\gamma^{\text{no}} : \gamma \in \mathcal{F}, \gamma \text{ non-constant}\}$.

Remark 3.3.2. The notation $\bigcup \mathcal{F}^*$ used in this thesis differs from the notation $\mathcal{F}(X)$ used in [10] and [18].

It is clear that if X is a compact plane set and $\mathcal{F}$ is a collection of non-constant rectifiable paths in X then $(\mathcal{F}^{\text{no}})^* = \mathcal{F}^*$.

We also require the following definition from [29].

Definition 3.3.3. Let X be a compact plane set and let $\mathcal{F}$ be an effective collection of paths in X . We say that X is *$\mathcal{F}$ -regular at $x \in X$* if there exists a constant $C_x > 0$ such that for each $y \in X$ there exists $\gamma \in \mathcal{F}$ with $\gamma^- = x$, $\gamma^+ = y$ and $\Lambda(\gamma) \leq C_x |x - y|$. We say that X is *$\mathcal{F}$ -regular* if X is $\mathcal{F}$ -regular at each point $x \in X$.

Let X be a semi-rectifiable compact plane set, let $\mathcal{F}$ be an effective collection of paths in X , and let $x \in X$. It is easy to see that if X is $\mathcal{F}$ -regular at x , then X is regular at x . Consequently, if X is $\mathcal{F}$ -regular then X is pointwise regular. Let $\mathcal{A}$ denote the set of all admissible paths in X . Then X is *$\mathcal{A}$ -regular at x* if and only if X is regular at x , and X is *$\mathcal{A}$ -regular* if and only if X is pointwise regular.

3.3.1 $\mathcal{F}$ -differentiable functions

We now define $\mathcal{F}$ -derivatives and $\mathcal{F}$ -differentiable functions, as introduced by Bland and Feinstein ([10]) and discussed by Dales and Feinstein ([29]).

Our motivation for the definition of an $\mathcal{F}$ -derivative is the *fundamental theorem of calculus for rectifiable paths*. A proof of Proposition 3.3.4 by repeated bisection, due to Allan, is given in [9, Appendix A].

Proposition 3.3.4 (Fundamental theorem of calculus for rectifiable paths). *Let γ be a rectifiable path in $\mathbb{C}$ and let $f \in D^{(1)}(\gamma^*)$. Then*

$$\int_{\gamma} f'(z) dz = f(\gamma^+) - f(\gamma^-).$$

We now define $\mathcal{F}$ -derivatives and $\mathcal{F}$ -differentiable functions on certain compact plane sets.

Definition 3.3.5. Let X be a perfect compact plane set, let $\mathcal{F}$ be a collection of rectifiable paths in X , and let $f \in C(X)$. A function $g \in C(X)$ is an $\mathcal{F}$ -derivative for f if, for each $\gamma \in \mathcal{F}$, we have

$$\int_{\gamma} g(z) dz = f(\gamma^+) - f(\gamma^-).$$

If f has an $\mathcal{F}$ -derivative on X then we say that f is $\mathcal{F}$ -differentiable on X . We denote the set of $\mathcal{F}$ -differentiable functions on X by $D_{\mathcal{F}}^{(1)}(X)$.

Note that $\mathcal{F}$ -derivatives need not be unique in general. The following proposition summarises various properties of $\mathcal{F}$ -derivatives and $\mathcal{F}$ -differentiable functions from [10] and [29].

Proposition 3.3.6. *Let X be a semi-rectifiable compact plane set and let $\mathcal{F}$ be an effective collection of paths in X .*

- (a) *If $f, g, h \in C(X)$ such that g, h are $\mathcal{F}$ -derivatives for f , then $g = h$.*
- (b) *If $f \in D_{\mathcal{F}}^{(1)}(X)$ then f' is an $\mathcal{F}$ -derivative for f .*
- (c) *If $f_1, f_2, g_1, g_2 \in C(X)$ such that g_1 is an $\mathcal{F}$ -derivative for f_1 and g_2 is an $\mathcal{F}$ -derivative for f_2 then $f_1 g_2 + g_1 f_2$ is an $\mathcal{F}$ -derivative for $f_1 f_2$.*
- (d) *If $f_1, f_2, g_1, g_2 \in C(X)$ and $\alpha, \beta \in \mathbb{C}$ such that g_1 is an $\mathcal{F}$ -derivative for f_1 and g_2 is an $\mathcal{F}$ -derivative for f_2 then $\alpha g_1 + \beta g_2$ is an $\mathcal{F}$ -derivative for $\alpha f_1 + \beta f_2$.*

Given $X, \mathcal{F}$ as above and a compact plane set Y such that $X \subseteq Y$, then we see that $D_{\mathcal{F}}^{(1)}(Y)$ consists of all extensions $\tilde{f}$ (given by the Tietze extension theorem) of functions $f \in D_{\mathcal{F}}^{(1)}(X)$. It is therefore possible to relax the condition that $\bigcup \mathcal{F}^*$

be dense in X and consider the more general case where $\mathcal{F}$ is effective on $\overline{\bigcup \mathcal{F}^*}$. This is similar to the situation dealt with in [10]; however, we do not insist that the collection $\mathcal{F}$ contains only Jordan curves. Note that properties (b), (c), and (d) of Proposition 3.3.6 hold even if $\mathcal{F}$ is only effective on $\overline{\bigcup \mathcal{F}^*} \subset X$.

The properties (c) and (d) above show that $D_{\mathcal{F}}^{(1)}(X)$ is an algebra (given pointwise operations), and (b) shows that $D^{(1)}(X) \subseteq D_{\mathcal{F}}^{(1)}(X)$.

Let X be a perfect compact plane set and let $\mathcal{F}$ be a collection of admissible paths in X . Set $Y := \overline{\bigcup \mathcal{F}^*}$ and suppose that $\mathcal{F}$ is an effective collection of paths in Y . We define the norm

$$\|f\|_{\mathcal{F},1} := |f|_X + |g|_Y \quad (3.5)$$

where $f \in D_{\mathcal{F}}^{(1)}(X)$ and $g \in C(X)$ is an $\mathcal{F}$ -derivative for f . Note that, by Proposition 3.3.6(a), (3.5) is well defined.

The following proposition is an immediate consequence of [10, Theorem 4.15] and [29, Theorem 5.6].

Proposition 3.3.7. *Let X be a perfect compact plane set and let $\mathcal{F}$ be a collection of admissible paths in X . Set $Y := \overline{\bigcup \mathcal{F}^*}$ and suppose that $\mathcal{F}$ is an effective collection of paths in Y . Then $(D_{\mathcal{F}}^{(1)}(X), \|\cdot\|_{\mathcal{F},1})$ is a Banach function algebra on X , and $D_{\mathcal{F}}^{(1)}(X)$ contains $D^{(1)}(X)$ as a subalgebra.*

Let X be a perfect compact plane set and let $\mathcal{F}$ be an effective collection of paths in X . In this setting $\mathcal{F}$ -derivatives are unique and we shall usually write $f^{[1]}$ for the $\mathcal{F}$ -derivative of $f \in D_{\mathcal{F}}^{(1)}(X)$. We adopt the convention that $f^{[0]} = f$.

As in [10], we define the sets $D_{\mathcal{F}}^{(n)}(X)$ (for each $n \in \mathbb{N}$ with $n \geq 2$) of n -times $\mathcal{F}$ -differentiable functions on X inductively by

$$D_{\mathcal{F}}^{(n)}(X) = \{f \in D_{\mathcal{F}}^{(1)}(X) : f^{[1]} \in D_{\mathcal{F}}^{(n-1)}(X)\},$$

and we write $f^{[n]}$ for the n th $\mathcal{F}$ -derivative of $f \in D_{\mathcal{F}}^{(n)}(X)$. It follows from the product rule for $\mathcal{F}$ -derivatives (Proposition 3.3.6(c)) that the Leibniz law for n th $\mathcal{F}$ -derivatives of products holds so that $D_{\mathcal{F}}^{(n)}(X)$ is an algebra for each $n \in \mathbb{N}$. We give $D_{\mathcal{F}}^{(n)}(X)$ the norm

$$\|f\|_{\mathcal{F},n} := |f|_X + \sum_{j=1}^n |f^{[j]}|_{\overline{\bigcup \mathcal{F}^*}} \quad (f \in D_{\mathcal{F}}^{(n)}(X)). \quad (3.6)$$

The normed algebra $(D_{\mathcal{F}}^{(n)}(X), \|\cdot\|_{\mathcal{F},n})$ is also a Banach function algebra on X , as noted in [10]. We set $D_{\mathcal{F}}^{(\infty)}(X) := \bigcap_{j=1}^{\infty} D_{\mathcal{F}}^{(j)}(X)$.

Definition 3.3.8. Let X be a semi-rectifiable compact plane set and let $\mathcal{F}$ be an effective collection of paths in X . Let $M = (M_k)_{k=0}^\infty$ be an algebra sequence. We define the algebra

$$D_{\mathcal{F}}(X, M) := \left\{ f \in D_{\mathcal{F}}^{(\infty)}(X) : \|f\|_{\mathcal{F}, M} := \sum_{k=0}^{\infty} \frac{|f^{[k]}|_X}{M_k} < \infty \right\}.$$

It is easy to see that, with X , $\mathcal{F}$ and M as above, $(D_{\mathcal{F}}(X, M), \|\cdot\|_{\mathcal{F}, M})$ is a Banach function algebra on X and $D(X, M) \subseteq D_{\mathcal{F}}(X, M)$.

We shall refer to the algebras $D_{\mathcal{F}}^{(n)}(X)$ and $D_{\mathcal{F}}(X, M)$ as *Bland–Feinstein algebras*.

It is not known, in general, when the algebras $D_{\mathcal{F}}^{(n)}(X)$ and $D_{\mathcal{F}}(X, M)$ are natural Banach function algebras, although some sufficient conditions are given in [18]. We shall revisit this problem in Section 3.6.

3.3.2 Maximal collections

In this section we discuss collections of paths further and investigate when it is possible to add paths to the collection $\mathcal{F}$ without altering the associated algebra $D_{\mathcal{F}}^{(1)}(X)$. We discuss maximal collections of paths and the operations under which such a collection is closed. Finally, we discuss the compatibility of functions for given collections of paths, with an aim to prove a new chain rule for $\mathcal{F}$ -differentiation. Unless otherwise stated, all results of this section are new.

Throughout this section we fix a semi-rectifiable compact plane set X and we denote the collection of all admissible paths in X by $\mathcal{A}$. Let $\mathcal{F}$ be an effective collection of paths in X . We shall often identify the algebra $D_{\mathcal{F}}^{(1)}(X)$ with the subspace $S_{\mathcal{F}}$ of $C(X) \oplus_1 C(X)$ containing all pairs (f, g) such that $f \in D_{\mathcal{F}}^{(1)}(X)$ and g is an $\mathcal{F}$ -derivative for f . We begin with a definition.

Definition 3.3.9. Let γ be an admissible path in X and let $f, g \in C(X)$. We say that g is the γ -derivative for f if, for each subpath σ of γ , we have

$$\int_{\sigma} g(z) dz = f(\sigma^+) - f(\sigma^-).$$

Note that, taking $\mathcal{F}$ to be the collection of all subpaths of γ , then g is the γ -derivative for f if and only if g is an $\mathcal{F}$ -derivative for f . The collection $\mathcal{F}$ is effective, so the restriction of $\mathcal{F}$ -derivatives to γ^* are unique so the definition above is justified.

Definition 3.3.10. Let $S \subseteq C(X) \oplus_1 C(X)$. We define

$$\mathfrak{p}(S) := \{\gamma \in \mathcal{A} : \text{for all } (f, g) \in S, g \text{ is the } \gamma\text{-derivative of } f\}.$$

If $\mathcal{F}$ is an effective collection of paths in X then we write $\mathfrak{m}(\mathcal{F})$ for $\mathfrak{p}(S_{\mathcal{F}})$. We call $\mathfrak{m}(\mathcal{F})$ the *maximal collection* associated with $\mathcal{F}$.

The following lemma is immediate from the definition of $\mathfrak{m}(\mathcal{F})$. Note that $\mathcal{F} \subseteq \mathfrak{m}(\mathcal{F})$.

Lemma 3.3.11. Let $S, T \subseteq C(X) \oplus_1 C(X)$. If $S \subseteq T$ then $\mathfrak{p}(T) \subseteq \mathfrak{p}(S)$. Let $\mathcal{F}, \mathcal{G}$ be effective collections of paths in X . Then $D_{\mathfrak{m}(\mathcal{F})}^{(1)}(X) = D_{\mathcal{F}}^{(1)}(X)$. Moreover, $S_{\mathcal{F}} \subseteq S_{\mathcal{G}}$ if and only if $\mathfrak{m}(\mathcal{G}) \subseteq \mathfrak{m}(\mathcal{F})$.

Proof. Clearly, if $S \subseteq T$ then we have $\mathfrak{p}(T) \subseteq \mathfrak{p}(S)$.

We have $D_{\mathfrak{m}(\mathcal{F})}^{(1)}(X) \subseteq D_{\mathcal{F}}^{(1)}(X)$ since if $f \in D_{\mathfrak{m}(\mathcal{F})}^{(1)}(X)$ then, for each $\gamma \in \mathfrak{m}(\mathcal{F})$, $f^{[\gamma]}$ is a γ -derivative of f . In particular this holds whenever $\gamma \in \mathcal{F}$, and it follows that $f \in D_{\mathcal{F}}^{(1)}(X)$. On the other hand, it is clear that $D_{\mathcal{F}}^{(1)}(X) \subseteq D_{\mathfrak{m}(\mathcal{F})}^{(1)}(X)$. Thus equality holds.

For the final assertion, if $S_{\mathcal{F}} \subseteq S_{\mathcal{G}}$ then $\mathfrak{m}(\mathcal{G}) \subseteq \mathfrak{m}(\mathcal{F})$ by the first assertion. Now suppose that $\mathfrak{m}(\mathcal{G}) \subseteq \mathfrak{m}(\mathcal{F})$. Let $(f, g) \in S_{\mathcal{F}}$. Then, for each $\gamma \in \mathfrak{m}(\mathcal{F})$, g is the γ -derivative for f . Since $\mathfrak{m}(\mathcal{G}) \subseteq \mathfrak{m}(\mathcal{F})$, this holds for all $\gamma \in \mathfrak{m}(\mathcal{G})$. Thus $(f, g) \in S_{\mathfrak{m}(\mathcal{G})} = S_{\mathcal{G}}$ and so $S_{\mathcal{F}} \subseteq S_{\mathcal{G}}$. This completes the proof. $\square$

Note that the condition $S_{\mathcal{F}} \subseteq S_{\mathcal{G}}$ implies that $D_{\mathcal{F}}^{(1)}(X) \subseteq D_{\mathcal{G}}^{(1)}(X)$. There reverse inclusion need not hold in general; see Example 3.3.14.

We now consider some operations on paths in $\mathbb{C}$. Let $\gamma_1 : [a, b] \rightarrow \mathbb{C}$ and $\gamma_2 : [c, d] \rightarrow \mathbb{C}$ be paths such that $\gamma_1^+ = \gamma_2^-$. We write $\gamma_1 \dot{+} \gamma_2$ for the path given by

$$\gamma_1 \dot{+} \gamma_2 = \begin{cases} \gamma_1(a + 2t(b - a)) & t \in [0, 1/2), \\ \gamma_2(c + (2t - 1)(d - c)) & t \in [1/2, 1]. \end{cases}$$

We call the path $\gamma_1 \dot{+} \gamma_2$ the *join* of γ_1 and γ_2 . Note that, if γ_1 and γ_2 are admissible, then $\gamma_1 \dot{+} \gamma_2$ is admissible. The *reverse* of γ_1 , denoted by $-\gamma_1$, is given by $-\gamma_1(t) = \gamma_1(-t)$ ($t \in [-b, -a]$). Our notation for joining and reversing paths is not entirely standard and there are numerous ways to parametrise these paths.

Lemma 3.3.12. Let $\mathcal{F}$ be an effective collection of paths in X . Then $\mathfrak{m}(\mathcal{F})$ has the following properties:

- (a) if $\gamma \in \mathfrak{m}(\mathcal{F})$ then $-\gamma \in \mathfrak{m}(\mathcal{F})$;

- (b) if $\gamma \in \mathbf{m}(\mathcal{F})$ then $\gamma^{\text{pl}}, \gamma^{\text{no}} \in \mathbf{m}(\mathcal{F})$;
- (c) if $\gamma_1, \gamma_2 \in \mathbf{m}(\mathcal{F})$ such that $\gamma_1^+ = \gamma_2^-$ then $\gamma_1 \dot{+} \gamma_2 \in \mathbf{m}(\mathcal{F})$.

Proof. (a) This is clear from the definitions.

(b) This is Lemma 3.1.8.

(c) This is effectively [10, Theorem 4.5], and follows from the definitions. $\square$

By induction using part (c) of the above, we can join finitely many paths $\gamma_1, \dots, \gamma_n$ from $\mathcal{F}$, with $\gamma_j^+ = \gamma_{j+1}^-$ for $j = 1, \dots, n-1$, to obtain the path $\gamma_1 \dot{+} \dots \dot{+} \gamma_n$ which is in $\mathbf{m}(\mathcal{F})$.

The following lemma summarises several additional properties of maximal collections of paths.

Lemma 3.3.13. *Let $\mathcal{F}, \mathcal{G}$ be effective collections of paths in X . Set $A := D_{\mathcal{F}}^{(1)}(X)$ and $B := D_{\mathcal{G}}^{(1)}(X)$.*

- (a) We have $\mathbf{m}(\mathbf{m}(\mathcal{F})) = \mathbf{m}(\mathcal{F})$.
- (b) We have $\mathbf{p}(C(X) \oplus_1 C(X)) = \emptyset$ and, for each $S \subseteq \{(f, f^{(1)}) : f \in D^{(1)}(X)\}$, we have $\mathbf{p}(S) = \mathcal{A}$.
- (c) We have $D_{\mathcal{F} \cup \mathcal{G}}^{(1)}(X) \subseteq D_{\mathcal{F}}^{(1)}(X) \cap D_{\mathcal{G}}^{(1)}(X)$.
- (d) We have $\mathbf{m}(\mathcal{F}) \cup \mathbf{m}(\mathcal{G}) \subseteq \mathbf{m}(\mathcal{F} \cup \mathcal{G})$.

Proof. (a) This is immediate by Lemma 3.3.11.

(b) The first assertion is clear. The second assertion follows immediately from the fundamental theorem of calculus for rectifiable paths (Proposition 3.3.4).

(c) Let $\mathcal{H} := \mathcal{F} \cup \mathcal{G}$. Clearly $A \cap B$ consists of all $f \in C(X)$ such that there exists $g_1, g_2 \in C(X)$ with $(f, g_1) \in S_{\mathcal{F}}$ and $(f, g_2) \in S_{\mathcal{G}}$, and $S_{\mathcal{H}} = S_{\mathcal{F}} \cap S_{\mathcal{G}}$. Let $S \subseteq C(X) \oplus_1 C(X)$ be the subset of all $(f, g) \in S_{\mathcal{F}} \cup S_{\mathcal{G}}$ with $f \in A \cap B$. Then $S_{\mathcal{H}} \subseteq S$. It follows that $D_{\mathcal{H}}^{(1)}(X) \subseteq D_{\mathcal{F}}^{(1)}(X) \cap D_{\mathcal{G}}^{(1)}(X)$.

(d) Let $\mathcal{H} := \mathcal{F} \cup \mathcal{G}$, and let $C := D_{\mathcal{H}}^{(1)}(X)$. Then, by (c), we have $C \subseteq A \cap B$. It follows, from Lemma 3.3.11, that we have $\mathbf{m}(\mathcal{F}) \subseteq \mathbf{m}(\mathcal{H})$ and $\mathbf{m}(\mathcal{G}) \subseteq \mathbf{m}(\mathcal{H})$. Thus $\mathbf{m}(\mathcal{F}) \cup \mathbf{m}(\mathcal{G}) \subseteq \mathbf{m}(\mathcal{H})$. $\square$

In general, if $(f, g_1), (f, g_2) \in S$ with $g_1 \neq g_2$, then we have $\mathbf{p}(S) = \emptyset$. Note that the inclusions in (c) and (d) in the above can be strict, as the following example shows.

Example 3.3.14. Let $X := \{x + iy \in \mathbb{C} : x, y \in [0, 1]\}$. Let $\mathcal{F}$ be the collection of all line segment paths in X parallel to the real axis, and let $\mathcal{G}$ be the collection of all line segment paths in X parallel to the imaginary axis. It is easy to see that $\bar{Z}$ is both $\mathcal{F}$ - and $\mathcal{G}$ -differentiable, with $\mathcal{F}$ -derivative 1 and $\mathcal{G}$ -derivative -1 . Thus $\bar{Z} \in D_{\mathcal{F}}^{(1)}(X) \cap D_{\mathcal{G}}^{(1)}(X)$. However, $\mathcal{F} \cup \mathcal{G}$ consists of all vertical and horizontal line segments in X , and it is easy to see that X is $\mathbf{m}(\mathcal{F} \cup \mathcal{G})$ -regular. It follows that $D_{\mathcal{F} \cup \mathcal{G}}^{(1)}(X) = D^{(1)}(X)$ and $\mathbf{m}(\mathcal{F} \cup \mathcal{G})$ is the set of all admissible paths in X by Lemma 3.3.13(b). Thus we have a strict inequality in (c) and (d) of Lemma 3.3.13. In fact, it is not hard to see that if S is the subset of $C(X) \oplus_1 C(X)$ corresponding to $D_{\mathcal{F}}^{(1)}(X) \cap D_{\mathcal{G}}^{(1)}(X)$, then we have $\mathbf{p}(S) = \emptyset$. ■

Let $\text{BVC}([0, 1])$ denote the set of all continuous, complex-valued functions of bounded variation on $[0, 1]$. (See, for example, [27, Appendix A.4].) Then $\text{BVC}([0, 1])$ is a linear subspace of $C([0, 1])$. We give $\text{BVC}([0, 1])$ the norm

$$\|f\|_{\text{BVC}([0, 1])} := |f|_{[0, 1]} + V_0^1(f) \quad (f \in \text{BVC}([0, 1])).$$

Then, $(\text{BVC}([0, 1]), \|\cdot\|_{\text{BVC}([0, 1])})$ is a Banach space with this norm (see, for example, [27, Proposition A.4.14(i)]). Let $\mathcal{F}$ be an effective collection of paths in X . Given a non-constant rectifiable path γ , we have $\gamma^{\text{no}} \in \text{BVC}([0, 1])$ so we may consider sequences of such paths as sequences in $\text{BVC}([0, 1])$. However, the limit of a convergent sequence in $\text{BVC}([0, 1])$ may be constant on a non-degenerate subinterval of $[0, 1]$, so we cannot simply take limits when attempting to find admissible paths. The following lemma gives us the desired result.

Lemma 3.3.15. *Let $\mathcal{F}$ be an effective collection of paths in X and let (γ_n) be a sequence of paths in $\mathbf{m}(\mathcal{F})$. Suppose that (γ_n^{no}) is a convergent sequence in $\text{BVC}([0, 1])$ with limit $\gamma \in \text{BVC}([0, 1])$. If γ is non-constant then $\gamma^{\text{pl}} \in \mathbf{m}(\mathcal{F})$.*

Proof. Suppose that γ is non-constant. Fix $f \in D_{\mathcal{F}}^{(1)}(X)$ and set

$$I := \int_{\gamma^{\text{pl}}} f^{[1]}(z) dz = \int_{\gamma} f^{[1]}(z) dz, \quad I_n := \int_{\gamma_n^{\text{no}}} f^{[1]}(z) dz \quad (n \in \mathbb{N}).$$

The proof is completed by showing that $I = f(\gamma^+) - f(\gamma^-)$. For each $n \in \mathbb{N}$, we have $I_n = f(\gamma_n^+) - f(\gamma_n^-)$ and thus $\lim_{n \rightarrow \infty} I_n = f(\gamma^+) - f(\gamma^-)$. Note that $(\gamma_n^{\text{no}})^+ = \gamma_n^+$ and $(\gamma_n^{\text{no}})^- = \gamma_n^-$. So it remains to prove that $I = \lim_{n \rightarrow \infty} I_n$.

For each $n \in \mathbb{N}$, let $h_n := f^{[1]} \circ \gamma_n^{\text{no}}$ and let $h := f^{[1]} \circ \gamma$. Then $h_n \rightarrow h$ uniformly as $n \rightarrow \infty$ and hence $R := \sup_{n \in \mathbb{N}} |h_n|_{[0, 1]} < \infty$.

Let $\varepsilon > 0$ and choose $n_0 \in \mathbb{N}$ large enough so that $|h_n - h|_{[0, 1]} < \varepsilon/6R$ for all $n \geq n_0$. Since $V_0^1(\gamma_n^{\text{no}} - \gamma) \rightarrow 0$ as $n \rightarrow \infty$, there exists $n_1 \in \mathbb{N}$ such that

$V_0^1(\gamma_n^{\text{no}} - \gamma) < \varepsilon/6R$ for all $n \geq n_1$. Fix $n \in \mathbb{N}$ larger than both n_0 and n_1 . Let $\mathcal{P}_n$ be a partition of $[0, 1]$ such that, for every partition $\mathcal{P} = \{x_0, \dots, x_m\}$ of $[0, 1]$ finer than $\mathcal{P}_n$ and for any choice of $t_j \in [x_j, x_{j+1}]$ for $j = 0, 1, \dots, m-1$, we have

$$\left| \sum_{j=0}^{m-1} h_n(t_j)(\gamma_n(x_{j+1}) - \gamma_n(x_j)) - \int_{\gamma_n} f^{[1]}(z) dz \right| < \frac{\varepsilon}{3}.$$

Now choose a partition $\mathcal{P}_0$ of $[0, 1]$ such that, for every partition $\mathcal{P} = \{x_0, \dots, x_m\}$ of $[0, 1]$ finer than $\mathcal{P}_0$ and for any choice of $t_j \in [x_j, x_{j+1}]$ for $j = 0, 1, \dots, m-1$, we have

$$\left| \sum_{j=0}^{m-1} h(t_j)(\gamma(x_{j+1}) - \gamma(x_j)) - \int_{\gamma} f^{[1]}(z) dz \right| < \frac{\varepsilon}{3}.$$

Fix a partition $\mathcal{P} = \{x_0, \dots, x_m\}$ of $[0, 1]$ such that $\mathcal{P}$ is finer than both $\mathcal{P}_0$ and $\mathcal{P}_n$ and fix $t_j \in [x_j, x_{j+1}]$ for $j = 0, 1, \dots, m-1$. Let

$$S(h_n, \gamma_n) := \sum_{j=0}^{m-1} h_n(t_j)(\gamma_n(x_{j+1}) - \gamma_n(x_j))$$

and let

$$S(h, \gamma) := \sum_{j=0}^{m-1} h(t_j)(\gamma(x_{j+1}) - \gamma(x_j)).$$

We have

$$\begin{aligned} |S(h_n, \gamma_n) - S(h, \gamma)| &\leq \left| \sum_{j=0}^{m-1} h_n(t_j) [(\gamma_n(x_{j+1}) - \gamma_n(x_j)) - (\gamma(x_{j+1}) - \gamma(x_j))] \right| \\ &\quad + \left| \sum_{j=0}^{m-1} (h_n(t_j) - h(t_j))(\gamma(x_{j+1}) - \gamma(x_j)) \right| \\ &\leq |h_n|_{[0,1]} \sum_{j=0}^{m-1} |(\gamma_n(x_{j+1}) - \gamma_n(x_j)) - (\gamma(x_{j+1}) - \gamma(x_j))| \\ &\quad + |h_n - h|_{[0,1]} \sum_{j=0}^{m-1} |\gamma(x_{j+1}) - \gamma(x_j)| \\ &\leq |h_n|_{[0,1]} V_0^1(\gamma_n - \gamma) + |h_n - h|_{[0,1]} V_0^1(\gamma). \end{aligned}$$

It now follows that $|S(h_n, \gamma_n) - S(h, \gamma)| < \varepsilon/6 + \varepsilon/6 = \varepsilon/3$. Thus

$$|I_n - I| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

so that $I_n \rightarrow I$ as $n \rightarrow \infty$. This completes the proof. $\square$

We now use maximal collections to introduce a notion of compatibility of functions for given collections of paths.

We will often compose paths with $\mathcal{F}$ -differentiable functions. We now prove that such a composition yields a rectifiable path. The proof is elementary in the case where φ is continuously differentiable and we give the details in the more general setting.

Lemma 3.3.16. *Let $\mathcal{F}$ be an effective collection of paths in X . Let $\varphi \in D_{\mathcal{F}}^{(1)}(X)$ and let $\gamma : [a, b] \rightarrow X$ be a path in $\mathcal{F}$. If $\varphi \circ \gamma$ is non-constant then $\varphi \circ \gamma$ is rectifiable.*

Proof. If $\varphi \circ \gamma$ is constant then there is nothing to prove, so suppose not. Set $M := |\varphi^{[1]}|_X$ and let $\mathcal{P} = \{x_0, \dots, x_n\}$ be a partition of $[a, b]$. Set $\sigma := \varphi \circ \gamma$ and let γ_j be the subpath of γ obtained by restricting γ to $[x_j, x_{j+1}]$ for each $j \in \{0, \dots, n-1\}$. We have

$$V(\sigma, \mathcal{P}) := \sum_{j=0}^{n-1} |\sigma(x_{j+1}) - \sigma(x_j)| = \sum_{j=0}^{n-1} \left| \int_{\gamma_j} \varphi^{[1]}(z) dz \right| \leq M \Lambda(\gamma).$$

It follows that $\Lambda(\sigma) = \sup V(\sigma, \mathcal{P}) \leq M \Lambda(\gamma)$, where the supremum is taken over all partitions $\mathcal{P}$ of $[a, b]$. This completes the proof. $\square$

We now introduce our notion of compatibility.

Definition 3.3.17. Let Y be a semi-rectifiable compact plane set, let $\mathcal{F}$ be an effective collection of paths in X , and let $\mathcal{G}$ be an effective collection of paths in Y . Let $\varphi \in D_{\mathcal{F}}^{(1)}(X)$ such that $\varphi(X) \subseteq Y$. We say that φ is $\mathcal{F}$ - $\mathcal{G}$ -compatible if, for each $\gamma \in \mathcal{F}$, either $\varphi \circ \gamma$ is constant or we have $(\varphi \circ \gamma)^{\text{pl}} \in \mathbf{m}(\mathcal{G})$.

Let X, Y be semi-rectifiable compact plane sets and let $\mathcal{F}$ be the collection of all admissible paths in X , and let $\mathcal{G}$ be the collection of all admissible paths in Y . Then, for any $\varphi \in D_{\mathcal{F}}^{(1)}(X)$ with $\varphi(X) = Y$, φ is $\mathcal{F}$ - $\mathcal{G}$ -compatible. However, given effective collections $\mathcal{F}$ and $\mathcal{G}$ on X and Y respectively, a suitable map $\varphi : X \rightarrow Y$ need not be $\mathcal{F}$ - $\mathcal{G}$ -compatible, even if φ is continuously differentiable. The following example gives two collections of paths $\mathcal{F}$ and $\mathcal{G}$ and a suitable map which is not $\mathcal{F}$ - $\mathcal{G}$ -compatible. This example is effectively due to Chaobankoh [18].

Example 3.3.18. Let $X := \{x + iy \in \mathbb{C} : x, y \in [0, 1]\}$. Let $\mathcal{F}$ be the collection of all line segment paths in X parallel to the real axis, and let $\mathcal{G}$ be the collection of all line segment paths in X parallel to the imaginary axis. Set $\varphi(z) := z$ ($z \in X$). Then $\varphi : X \rightarrow X$ is continuously differentiable on X and $\varphi(X) = X$. Clearly $\varphi \circ \gamma \in \mathcal{F}$ for all $\gamma \in \mathcal{F}$. We *claim* that if $\gamma \in \mathbf{m}(\mathcal{F})$ then $\gamma^* \in \mathcal{F}^*$, and if $\gamma \in \mathbf{m}(\mathcal{G})$, then $\gamma^* \in \mathcal{G}^*$.

Suppose, towards a contradiction, that there exists $\gamma \in \mathbf{m}(\mathcal{F})$ with $\gamma^* \notin \mathcal{F}^*$. Then there exists a subpath σ of γ such that $\text{Im } \sigma^+ \neq \text{Im } \sigma^-$. Let $f(x+iy) = 1+iy$ so that $f \in D_{\mathcal{F}}^{(1)}(X)$ and $f^{[1]} = 0$. But then we must have

$$0 = \int_{\sigma} f^{[1]}(z) dz = f(\sigma^+) - f(\sigma^-) = \text{Im } \sigma^+ - \text{Im } \sigma^- \neq 0,$$

which is a contradiction. Thus such a γ cannot exist and so $\gamma^* \in \mathcal{F}^*$ for all $\gamma \in \mathbf{m}(\mathcal{F})$.

Similarly, $g(x+iy) = x+i$ is $\mathcal{G}$ -differentiable with $\mathcal{G}$ -derivative $g^{[1]} = 0$. We arrive at a similar contradiction so that $\gamma^* \in \mathcal{G}^*$ for all $\gamma \in \mathbf{m}(\mathcal{G})$. This proves the claim.

Thus $\mathbf{m}(\mathcal{F}) \cap \mathbf{m}(\mathcal{G}) = \emptyset$ and it follows that φ is not $\mathcal{F}$ - $\mathcal{G}$ -compatible. ■

We shall often make use of the following collection of paths.

Definition 3.3.19. Let Y be a semi-rectifiable compact plane set, let $\mathcal{F}$ be an effective collection of paths in X , and let $\varphi \in D_{\mathcal{F}}^{(1)}(X)$ such that $\varphi(X) \subseteq Y$. We define $\mathcal{F}_{\varphi}$ to be the set $\{(\varphi \circ \gamma)^{\text{pl}} : \gamma \in \mathcal{F}, \varphi \circ \gamma \text{ is non-constant}\}$

We summarise the properties of the collection the collections $\mathcal{F}_{\varphi}$ in the following lemma.

Lemma 3.3.20. *Let Y be a semi-rectifiable compact plane set, let $\mathcal{F}$ be an effective collection of paths in X , and let $\varphi \in D_{\mathcal{F}}^{(1)}(X)$ such that φ is non-constant and $\varphi(X) \subseteq Y$. Let $\mathcal{G} := \mathcal{F}_{\varphi}$. Then*

- (a) *if $\varphi(\cup \mathcal{F}^*) = \cup \mathcal{G}^*$ then $\cup \mathcal{G}^*$ is dense in $\varphi(X)$,*
- (b) *φ is $\mathcal{F}$ - $\mathcal{G}$ -compatible,*
- (c) *if $\mathcal{H}$ is an effective collection of paths in Y such that $\mathcal{G} \subseteq \mathcal{H}$ then φ is $\mathcal{F}$ - $\mathcal{H}$ -compatible.*

Proof. (a) Note that φ is a continuous surjection of X onto $\varphi(X)$ and, since $\cup \mathcal{F}^*$ is dense in X , it follows that $\varphi(\cup \mathcal{F}^*)$ is dense in $\varphi(X)$. Thus $\cup \mathcal{G}^*$ is dense in $\varphi(X)$.

(b) This is clear.

(c) This follows from (b). □

Note that if $\mathcal{F}$ is an effective collection of paths in X such that X is $\mathcal{F}$ -regular, then X is also $\mathbf{m}(\mathcal{F})$ -regular, though the converse need not hold. Our next result is a minor modification of [29, Theorem 7.2].

Lemma 3.3.21. *Let X be a semi-rectifiable compact plane set and let $\mathcal{F}$ be an effective collection of paths in X . Suppose that X is $\mathbf{m}(\mathcal{F})$ -regular. Then*

$$D_{\mathcal{F}}^{(1)}(X) = D^{(1)}(X)$$

and $\mathbf{m}(\mathcal{F}) = \mathcal{A}$. In particular, if X is $\mathcal{F}$ -regular then $\mathbf{m}(\mathcal{F}) = \mathcal{A}$.

We conclude this section with some results for the collections of paths generated by the closure of a subset of $C(X) \oplus_1 C(X)$. Our key result is the following theorem.

Theorem 3.3.22. *Let $S \subseteq C(X) \oplus_1 C(X)$. Then $\mathbf{p}(\overline{S}) = \mathbf{p}(S)$.*

Proof. By Lemma 3.3.11, we have $\mathbf{p}(\overline{S}) \subseteq \mathbf{p}(S)$. Let (f_n, g_n) be a sequence in S such that $(f_n, g_n) \rightarrow (f, g) \in \overline{S}$ as $n \rightarrow \infty$. Let $\gamma \in \mathbf{p}(S)$. Then we have

$$f_n(\gamma^+) - f_n(\gamma^-) = \int_{\gamma} g_n(z) \, dz,$$

for all $n \in \mathbb{N}$. But now $f_n \rightarrow f$ and $g_n \rightarrow g$ uniformly as $n \rightarrow \infty$, so

$$f(\gamma^+) - f(\gamma^-) = \lim_{n \rightarrow \infty} f_n(\gamma^+) - f_n(\gamma^-) = \lim_{n \rightarrow \infty} \int_{\gamma} g_n(z) \, dz = \int_{\gamma} g(z) \, dz.$$

Thus $\gamma \in \mathbf{p}(\overline{S})$. It follows that $\mathbf{p}(S) \subseteq \mathbf{p}(\overline{S})$ and this completes the proof. $\square$

The following is an interesting corollary follows immediately from Theorem 3.3.22 and Lemma 3.3.13.

Corollary 3.3.23. *Let $\mathcal{F}$ be an effective collection of paths in X . Suppose that $D^{(1)}(X)$ is dense in $D_{\mathcal{F}}^{(1)}(X)$. Then $\mathbf{m}(\mathcal{F}) = \mathcal{A}$.*

Combining Corollary 3.3.23 with Lemma 3.3.20 we obtain the following corollary.

Corollary 3.3.24. *Let Y be a semi-rectifiable compact plane set, let $\mathcal{F}$ be an effective collection of paths in X , and let $\mathcal{G}$ be an effective collection of paths in Y . Suppose that $D^{(1)}(Y)$ is dense in $D_{\mathcal{G}}^{(1)}(Y)$. Let $f \in D_{\mathcal{F}}^{(1)}(X)$ such that $f(X) \subseteq Y$. Then f is $\mathcal{F}$ - $\mathcal{G}$ -compatible.*

In particular, in the above, if $R_0(Y)$ is dense in $D_{\mathcal{G}}^{(1)}(Y)$, then we have $\mathbf{m}(\mathcal{F}) = \mathcal{A}$ and f , as in the above, is $\mathcal{F}$ - $\mathcal{G}$ -compatible.

3.4 Composition of $\mathcal{F}$ -differentiable functions

In this section we discuss the chain rule for $\mathcal{F}$ -differentiation. Unless otherwise stated, the results of this section are all new.

In his thesis, Chaobankoh proved a version of the chain rule ([18, Theorem 2.3.18]) for $\mathcal{F}$ -differentiable functions. (See Corollary 3.4.6 and the following comments in the present section.) We use a different approach, based on our notion of compatibility, to obtain a more general result. A similar technique was used by Chaobankoh to prove that a chain rule for $\mathcal{F}$ -differentiation holds when considering functions of the form $f(z) = az + b$ (where $a, b \in \mathbb{C}$), see [18, Theorem 2.3.26]. Note that the definition effective given in [18] differs from that given in Definition 3.3.1; however, since we consider $\mathfrak{m}(\mathcal{F})$, we avoid any problems of consistency.

Our main tool for this section is the following lemma. This is an easy modification of standard results. We include the details for the convenience of the reader.

Lemma 3.4.1. *Let X be a semi-rectifiable compact plane set and let $\mathcal{F}$ be an effective collection of paths in X . Let $\varphi \in D_{\mathcal{F}}^{(1)}(X)$ and let $\gamma \in \mathcal{F}$. Then, for each $f \in C(\varphi(\gamma^*))$, we have*

$$\int_{\gamma} (f \circ \varphi) \varphi^{[1]}(z) dz = \int_{\varphi \circ \gamma} f(z) dz. \quad (3.7)$$

Proof. By Lemma 3.3.16, $\sigma := \varphi \circ \gamma$ is a rectifiable path so that the integral on the right hand side of (3.7) exists. Fix $f \in C(\sigma^*)$ and let $\varepsilon > 0$. Set $M := |\varphi^{[1]}|_X \Lambda(\gamma)$ and let $h := f \circ \sigma : [a, b] \rightarrow \mathbb{C}$. Since h is uniformly continuous, there exists $\delta > 0$ such that, for each $s, t \in [a, b]$ with $|s - t| < \delta$, we have $|h(t) - h(s)| < \varepsilon/(2M)$. Choose a partition $\mathcal{P}_0 = \{t_0^{(0)}, \dots, t_m^{(0)}\}$ of $[a, b]$ such that $\max_{0 \leq j \leq m-1} |t_{j+1} - t_j| < \delta$. For any partition $\mathcal{P} = \{t_0, \dots, t_n\}$ finer than $\mathcal{P}_0$ and, for each $j \in \{0, \dots, n-1\}$, let $\gamma_j^{(\mathcal{P})}$ be the restriction of γ to $[t_j, t_{j+1}]$. We have

$$T(\mathcal{P}) := \left| \sum_{j=0}^{n-1} \int_{\gamma_j^{(\mathcal{P})}} (f(\varphi(z)) - h(s_j)) \varphi^{[1]}(z) dz \right| < \frac{\varepsilon}{2M} |\varphi^{[1]}|_X \Lambda(\gamma) = \frac{\varepsilon}{2}, \quad (3.8)$$

for any $s_j \in [t_j, t_{j+1}]$ ($j = 0, 1, \dots, n-1$).

Now fix a partition $\mathcal{P} = \{t_0, \dots, t_n\}$ of $[a, b]$ finer than $\mathcal{P}_0$ such that, viewing the integral in the right hand side of (3.7) as a Riemann-Stieltjes integral on $[a, b]$, we have

$$\left| \sum_{j=0}^{n-1} h(s_j) (\sigma(t_{j+1}) - \sigma(t_j)) - \int_{\sigma} f(z) dz \right| < \frac{\varepsilon}{2}, \quad (3.9)$$

for any choice of $s_j \in [t_j, t_{j+1}]$ ($j = 0, 1, \dots, n-1$).

We now *claim* that, for this partition $\mathcal{P}$, we have

$$\left| \int_{\gamma} f(\varphi(z)) \varphi^{[1]}(z) dz - \sum_{j=0}^{n-1} h(s_j) (\sigma(t_{j+1}) - \sigma(t_j)) \right| < \frac{\varepsilon}{2} \quad (3.10)$$

for any choice of $s_j \in [t_j, t_{j+1}]$ ($j = 0, 1, \dots, n-1$).

For the remainder of the proof, for each $j \in \{0, \dots, n-1\}$, fix $s_j \in [t_j, t_{j+1}]$ and let $S := \sum_{j=0}^{n-1} h(s_j) (\sigma(t_{j+1}) - \sigma(t_j))$.

By the definition of $\varphi^{[1]}$, we have

$$\sum_{j=0}^{n-1} h(s_j) \int_{\gamma_j^{(\mathcal{P})}} \varphi^{[1]}(z) dz = S.$$

We also have

$$\left| \int_{\gamma} f(\varphi(z)) \varphi^{[1]}(z) dz - \sum_{j=0}^{n-1} h(s_j) \int_{\gamma_j^{(\mathcal{P})}} \varphi^{[1]}(z) dz \right| = T(\mathcal{P}),$$

where $T(\mathcal{P}) < \varepsilon/2$ as in (3.8). This proves the claim and so (3.10) holds.

But now, by (3.8) and (3.10), we have

$$\left| \int_{\gamma} f(\varphi(z)) \varphi^{[1]}(z) dz - \int_{\sigma} f(w) dw \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This holds for all $\varepsilon > 0$ and any choice of the s_j , so the result follows. $\square$

We can now state and prove a new version of the chain rule for $\mathcal{F}$ -differentiable functions.

Theorem 3.4.2. *Let X, Y be a semi-rectifiable compact plane sets, let $\mathcal{F}$ be an effective collection of paths in X , and let $\mathcal{G}$ be an effective collection of paths in Y . Let $\varphi \in D_{\mathcal{F}}^{(1)}(X)$ with $\varphi(X) \subseteq Y$. Suppose that φ is $\mathcal{F}$ - $\mathcal{G}$ -compatible. Then, for all $f \in D_{\mathcal{G}}^{(1)}(Y)$, $f \circ \varphi$ is $\mathcal{F}$ -differentiable and $(f \circ \varphi)^{[1]} = (f^{[1]} \circ \varphi) \varphi^{[1]}$.*

Proof. Fix $f \in D_{\mathcal{G}}^{(1)}(Y)$ and $\gamma \in \mathcal{F}$. Then, by Lemma 3.4.1, we have

$$\int_{\gamma} (f^{[1]} \circ \varphi)(z) \varphi^{[1]}(z) dz = \int_{\varphi \circ \gamma} f^{[1]}(z) dz.$$

Since φ is $\mathcal{F}$ - $\mathcal{G}$ -compatible, we have $\varphi \circ \gamma \in \mathbf{m}(\mathcal{G})$ and so

$$\int_{\varphi \circ \gamma} f^{[1]}(z) dz = f((\varphi \circ \gamma)^+) - f((\varphi \circ \gamma)^-).$$

But $(\varphi \circ \gamma)^+ = \varphi(\gamma^+)$ and $(\varphi \circ \gamma)^- = \varphi(\gamma^-)$. Thus $f \circ \varphi$ is $\mathcal{F}$ -differentiable and has $\mathcal{F}$ -derivative $(f^{[1]} \circ \varphi) \varphi^{[1]}$. This completes the proof. $\square$

By applying Theorem 3.4.2 inductively, we obtain the Faà di Bruno formula for the composition of n -times $\mathcal{F}$ -differentiable functions.

Corollary 3.4.3 (Faà di Bruno formula). *Let X, Y be a semi-rectifiable compact plane sets, let $\mathcal{F}$ be an effective collection of paths in X , and let $\mathcal{G}$ be an effective collection of paths in Y . Fix $n \in \mathbb{N}$. Let $\varphi \in D_{\mathcal{F}}^{(n)}(X)$ with $\varphi(X) \subseteq Y$. Suppose that φ is $\mathcal{F}$ - $\mathcal{G}$ -compatible. Then, for all $f \in D_{\mathcal{G}}^{(n)}(Y)$, $f \circ \varphi \in D_{\mathcal{F}}^{(n)}(X)$ and, for each $k \in \{1, 2, \dots, n\}$, we have*

$$(f \circ \varphi)^{[k]} = \sum_{i=0}^k (f^{[i]} \circ \varphi) \sum \frac{k!}{a_1! \cdots a_k!} \prod_{j=1}^k \left(\frac{\varphi^{[j]}}{j!} \right)^{a_j},$$

where the inner sum is over all $a_1, \dots, a_k \in \mathbb{N}_0$ such that $a_1 + \cdots + a_k = i$ and $a_1 + 2a_2 + \cdots + ka_k = k$.

We also obtain the quotient rule for $\mathcal{F}$ -differentiable functions as a corollary of Theorem 3.4.2. This was originally proved by Chaobankoh ([18, Lemma 2.3.12, Theorem 2.3.13]) using a method of repeated bisection.

Corollary 3.4.4. *Let X be a semi-rectifiable compact plane set, let $\mathcal{F}$ be an effective collection of paths in X , and let $f, g \in D_{\mathcal{F}}^{(1)}(X)$. Suppose that $0 \notin g(X)$. Then $f/g \in D_{\mathcal{F}}^{(1)}(X)$ and*

$$\left(\frac{f}{g} \right)^{[1]} = \frac{gf^{[1]} - fg^{[1]}}{g^2}.$$

Proof. We first show that $h := 1/g \in D_{\mathcal{F}}^{(1)}(X)$ and that $h^{[1]} = -g^{[1]}/g^2$. Since $0 \notin g(X)$, the function $\varphi(z) := 1/z$ ($z \in g(X)$) is continuous and complex-differentiable on $g(X)$, i.e. $\varphi \in D^{(1)}(g(X))$. Let $\mathcal{G}$ be the collection of all admissible paths in $g(X)$. Then we have $\varphi \in D^{(1)}(g(X)) \subseteq D_{\mathcal{G}}^{(1)}(g(X))$ and g is $\mathcal{F}$ - $\mathcal{G}$ -compatible by Lemma 3.3.20(c). By Theorem 3.4.2, we have $\varphi \circ g \in D_{\mathcal{F}}^{(1)}(X)$ with $(\varphi \circ g)^{[1]} = (\varphi^{[1]} \circ g)g^{[1]}$. But $\varphi^{[1]}$ is just the ordinary complex derivative of φ so that $\varphi^{[1]} \circ g = -1/g^2$. Thus $h^{[1]} = -g^{[1]}/g^2$. The result now follows from the product rule for $\mathcal{F}$ -derivatives, Proposition 3.3.6(c). $\square$

In addition, combining Theorem 3.4.2 with Corollary 3.3.24, we obtain the following corollary

Corollary 3.4.5. *Let X be a semi-rectifiable compact plane set and let $\mathcal{F}$ be an effective collection of paths in X . Let $f, g \in D_{\mathcal{F}}^{(1)}(X)$ such that $g(X) \subseteq X$. Suppose $D^{(1)}(X)$ is dense in $D_{\mathcal{F}}^{(1)}(X)$. Then $f \circ g \in D_{\mathcal{F}}^{(1)}(X)$ and $(f^{[1]} \circ g)g^{[1]}$ is the $\mathcal{F}$ -derivative of $f \circ g$.*

In particular, we obtain the chain rule as given by Chaobankoh ([18, Theorem 2.3.18], stated below) when we impose the apparently stronger criteria that $R_0(X)$ be dense in $D_{\mathcal{F}}^{(1)}(X)$. This was originally proved using the quotient rule.

Corollary 3.4.6. *Let X be a semi-rectifiable compact plane set and let $\mathcal{F}$ be an effective collection of paths in X . Let $f, g \in D_{\mathcal{F}}^{(1)}(X)$ such that $g(X) \subseteq X$. Suppose $R_0(X)$ is dense in $D_{\mathcal{F}}^{(1)}(X)$. Then $f \circ g \in D_{\mathcal{F}}^{(1)}(X)$ and $(f^{[1]} \circ g)g^{[1]}$ is the $\mathcal{F}$ -derivative of $f \circ g$.*

3.5 Bland–Feinstein algebra homomorphisms

We now describe sufficient conditions under which a function can induce an algebra homomorphism Bland–Feinstein algebras, including those Bland–Feinstein algebras which are analogous to Dales–Davie algebras. All of the results of this section are new. Note that all algebra homomorphisms between Banach function algebras are automatically continuous, by Proposition 1.4.5, so we do not mention the continuity explicitly here.

The first theorem is an immediate consequence of the chain rule and Faà di Bruno formula for $\mathcal{F}$ -derivatives (Corollary 3.4.3).

Theorem 3.5.1. *Let X, Y be semi-rectifiable compact plane sets and let $n \in \mathbb{N}$. Let $\mathcal{F}$ be an effective collection of paths in X , let $\mathcal{G}$ be an effective collection of paths in Y , and let $\varphi \in D_{\mathcal{F}}^{(n)}(X)$. Suppose that $\varphi(X) \subseteq Y$ and that φ is $\mathcal{F}$ - $\mathcal{G}$ -compatible. Then φ^* is a unital algebra homomorphism from $D_{\mathcal{F}}^{(n)}(Y)$ into $D_{\mathcal{F}}^{(n)}(X)$.*

In the remainder of the section, we aim to prove a result similar to [44, Theorem 4] for the algebras $D_{\mathcal{F}}(X, M)$.

We first define the term $\mathcal{F}$ -analytic for infinitely $\mathcal{F}$ -differentiable functions. (See [18, Section 5.5] for a similar definition.) We shall discuss $\mathcal{F}$ -analyticity further in Section 4.3.

Definition 3.5.2. Let X be a semi-rectifiable compact plane set, and let $\mathcal{F}$ be an effective collection of paths in X . Let $f \in D_{\mathcal{F}}^{(\infty)}(X)$. We say that f is $\mathcal{F}$ -analytic if

$$\limsup_{k \rightarrow \infty} \left(\frac{|f^{[k]}|_X}{k!} \right)^{1/k} < \infty$$

We are now ready to prove the main result of this section. The calculations in the proof are identical to those in [43] (see also, [18, 42, 70]), but we include the

details for the convenience of the reader. Note that we do not assume that either algebra is natural.

Theorem 3.5.3. *Let X, Y be semi-rectifiable compact plane sets, let $n \in \mathbb{N}$, let $\mathcal{F}$ be an effective collection of paths in X , let $\mathcal{G}$ be an effective collection of paths in Y , let $M = (M_n)_{n=0}^\infty$ be a non-analytic algebra sequence, and let $\varphi \in D_{\mathcal{F}}^{(\infty)}(X)$. Suppose that φ is $\mathcal{F}$ -analytic, that $\varphi(X) \subseteq Y$, and that φ is $\mathcal{F}$ - $\mathcal{G}$ -compatible.*

- (a) *If φ satisfies and $|\varphi^{[1]}|_X < 1$ then φ induces a homomorphism from $D_{\mathcal{G}}(Y, M)$ into $D_{\mathcal{F}}(X, M)$.*
- (b) *If the sequence $(n^2 M_n / M_{n+1})$ is bounded and $|\varphi^{[1]}|_X \leq 1$ then φ induces a homomorphism from $D_{\mathcal{G}}(Y, M)$ into $D_{\mathcal{F}}(X, M)$.*

Proof. It is clear that, for each $f \in D_{\mathcal{G}}(Y, M)$ we have $f \circ \varphi \in D_{\mathcal{F}}^{(\infty)}(X)$ by Corollary 3.4.3. So it remains to show that $f \circ \varphi$ satisfies $\|f \circ \varphi\| < \infty$ in each case.

(a) Suppose that $|\varphi^{[1]}|_X < 1$ and fix $f \in D_{\mathcal{G}}(Y, M)$ and let $F := f \circ \varphi$. Then, for each $n \in \mathbb{N}$, we have

$$F^{[n]} = \sum_{k=0}^n (f^{[k]} \circ \varphi) \sum \frac{n!}{a_1! \cdots a_n!} \prod_{j=1}^n \left(\frac{\varphi^{[j]}}{j!} \right)^{a_j}$$

where the inner sum is taken over all non-negative integers $a_1, \dots, a_n$ such that $a_1 + a_2 + \cdots + a_n = k$ and $a_1 + 2a_2 + \cdots + na_n = n$.

In the remainder of the proof the inner sum is always be over all non-negative integers $a_1, \dots, a_n$ such that

$$a_1 + \cdots + a_n = k \quad \text{and} \quad a_1 + 2a_2 + \cdots + na_n = n.$$

We now see that, for each $n \in \mathbb{N}$, we have

$$|F^{[n]}|_X \leq \sum_{k=0}^n |f^{[k]}|_Y \sum \frac{n!}{a_1! \cdots a_n!} \prod_{j=1}^n \left(\frac{|\varphi^{[j]}|_X}{j!} \right)^{a_j}.$$

It follows that

$$\sum_{n=0}^{\infty} \frac{|F^{[n]}|_X}{M_n} \leq \sum_{n=0}^{\infty} \frac{1}{M_n} \sum_{k=0}^n |f^{[k]}|_Y \sum \frac{n!}{a_1! \cdots a_n!} \prod_{j=1}^n \left(\frac{|\varphi^{[j]}|_X}{j!} \right)^{a_j}.$$

Now, by interchanging the order of summation, we have

$$\sum_{n=0}^{\infty} \frac{|F^{[n]}|_X}{M_n} \leq \sum_{k=0}^{\infty} |f^{[k]}|_Y \sum_{n=k}^{\infty} M_n \sum \frac{n!}{a_1! \cdots a_n!} \prod_{j=1}^n \left(\frac{|\varphi^{[j]}|_Y}{j!} \right)^{a_j}. \quad (3.11)$$

Hence

$$\sum_{n=0}^{\infty} \frac{|F^{[n]}|_X}{M_n} \leq \sum_{k=0}^{\infty} \frac{|f^{[k]}|_Y}{M_k} k! \sum_{n=k}^{\infty} \frac{1}{n!} \sum \frac{M_k}{k!} \frac{n!}{M_n} \frac{n!}{a_1! \cdots a_n!} \prod_{j=1}^n \left(\frac{|\varphi^{[j]}|_Y}{j!} \right)^{a_j}. \quad (3.12)$$

Note that the function

$$z \mapsto \sum_{j=2}^{\infty} \frac{|\varphi^{[j]}|_X z^{j-1}}{|\varphi^{[1]}|_X j!}$$

takes the value 0 at 0 and is analytic on a neighbourhood of 0. Choose $\varepsilon \in (0, 1)$ such that $\delta := |\varphi^{[1]}|_X (1 + \sum_{j=2}^{\infty} (|\varphi^{[j]}|_X \varepsilon^{j-1}) / (|\varphi^{[1]}|_X j!)) < 1$. Since $d(M) = 0$, it follows that there exists $C > 0$ such that $m!/M_m < C\varepsilon^m$ for each $m \in \mathbb{N}_0$.

Moreover, since

$$\frac{M_p}{p!} \frac{q!}{M_q} \leq \frac{(q-p)!}{M_{q-p}} \quad (p, q \in \mathbb{N}_0, p \leq q),$$

we have $(M_p/p!)(q!/M_q) \leq C\varepsilon^{p-q}$ for all $p, q \in \mathbb{N}_0$ with $q \leq p$. Thus, by (3.12) and since $a_1 + \cdots + a_n = k$, we have

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{|F^{[n]}|_X}{M_n} &\leq \sum_{k=0}^{\infty} \frac{|f^{[k]}|_Y}{M_k} k! \sum_{n=k}^{\infty} \frac{1}{n!} \sum C\varepsilon^{n-k} \frac{n!}{a_1! \cdots a_n!} \prod_{j=1}^n \left(\frac{|\varphi^{[j]}|_Y}{j!} \right)^{a_j} \\ &= C \sum_{k=0}^{\infty} \frac{|f^{[k]}|_Y}{M_k} k! \sum_{n=k}^{\infty} \frac{\varepsilon^n}{n!} \sum \frac{n!}{a_1! \cdots a_n!} \prod_{j=1}^n \left(\frac{|\varphi^{[j]}|_Y}{\varepsilon j!} \right)^{a_j}. \end{aligned}$$

Now, from formula B3 (p. 823) of [1], we have

$$\begin{aligned} k! \sum_{n=k}^{\infty} \frac{\varepsilon^n}{n!} \sum \frac{n!}{a_1! \cdots a_n!} \prod_{j=1}^n \left(\frac{|\varphi^{[j]}|_Y}{\varepsilon j!} \right)^{a_j} &= \left(\sum_{m=1}^{\infty} \frac{|\varphi^{(m)}|_X}{\varepsilon m!} \varepsilon^m \right)^k \\ &= |\varphi^{[1]}|_X^k \left(1 + \sum_{m=2}^{\infty} \frac{|\varphi^{(m)}|_X}{|\varphi^{[1]}|_X m!} \varepsilon^{m-1} \right)^k \\ &= \delta^k \end{aligned}$$

Since $0 < \delta < 1$ and since $f \in D_G(Y, M)$ it follows that

$$\sum_{n=0}^{\infty} \frac{|F^{[n]}|_X}{M_n} \leq C \sum_{k=0}^{\infty} \frac{|f^{[k]}|_Y}{M_k} \delta^k < \infty.$$

Thus $F \in D_{\mathcal{F}}(X, M)$ as required.

(b) Suppose now that $|\varphi^{[1]}|_X \leq 1$ and that $(n^2 M_{n-1}/M_n)$ is a bounded sequence. Let $f \in D_G(Y, M)$ and set $F := f \circ \varphi$. We note that (3.11) still holds. Since $d(M) = 0$ and $(n^2 M_{n-1}/M_n)$ is bounded, there exists constants $B, C > 0$ such that

$$\frac{M_p}{p!} \frac{q!}{M_q} \leq B \left(\frac{C}{p} \right)^{q-p} \quad (p, q \in \mathbb{N}_0, p \leq q).$$

(This is observed in [43].) Also, since φ satisfies $\sup_{j \in \mathbb{N}_0} (|\varphi^{[j]}|_X / j!)^{1/j} < \infty$, there exists $D > 0$ such that $|\varphi^{[j]}|_X / j! \leq D^j$ for each $j \in \mathbb{N}_0$. Let N be the greatest integer with $N \leq CD$.

Then, by (3.11), we have

$$\sum_{n=0}^{\infty} \frac{|F^{[n]}|_X}{M_n} \leq S_1 + S_2,$$

where

$$S_1 = \sum_{k=0}^N |f^{[k]}|_Y \sum_{n=k}^{\infty} \frac{1}{M_n} \sum \frac{n!}{a_1! \cdots a_n!} \prod_{j=1}^n \left(\frac{|\varphi^{[j]}|_Y}{j!} \right)^{a_j},$$

and

$$S_2 = \sum_{k=N+1}^{\infty} |f^{[k]}|_Y \sum_{n=k}^{\infty} \frac{1}{M_n} \sum \frac{n!}{a_1! \cdots a_n!} \prod_{j=1}^n \left(\frac{|\varphi^{[j]}|_Y}{j!} \right)^{a_j}.$$

We first show S_1 is finite. First note that, for the $a_1, \dots, a_n$ under consideration, we have

$$\prod_{j=1}^n \left(\frac{M_j}{j!} \right)^{a_j} \leq \frac{M_n}{n!},$$

thus, for each $k \in \mathbb{N}$, we have

$$\begin{aligned} & \sum_{n=k}^{\infty} \frac{1}{M_n} \sum \frac{n!}{a_1! \cdots a_n!} \prod_{j=1}^n \left(\frac{|\varphi^{[j]}|_Y}{j!} \right)^{a_j} \\ &= \sum_{n=k}^{\infty} \frac{1}{n!} \sum \frac{n!}{a_1! \cdots a_n!} \frac{n!}{M_n} \prod_{j=1}^n \left(\frac{M_j}{j!} \frac{|\varphi^{[j]}|_Y}{M_j} \right)^{a_j} \\ &\leq \sum_{n=k}^{\infty} \frac{1}{n!} \sum \frac{n!}{a_1! \cdots a_n!} \prod_{j=1}^n \left(\frac{|\varphi^{[j]}|_Y}{M_j} \right)^{a_j}. \end{aligned}$$

Now, again using formula B3 (p. 823) in [1], we see that

$$\sum_{n=k}^{\infty} \frac{1}{n!} \sum \frac{n!}{a_1! \cdots a_n!} \prod_{j=1}^n \left(\frac{|\varphi^{[j]}|_Y}{M_j} \right)^{a_j} = \frac{1}{k!} \left(\sum_{m=1}^{\infty} \frac{|\varphi^{(m)}|_X}{M_m} \right)^k,$$

which is equal to $(\|\varphi\| - |\varphi|_X)^k / k!$. It now follows that

$$S_1 \leq \sum_{k=0}^N \frac{|f^{[k]}|_Y}{k!} (\|\varphi\| - |\varphi|_X)^k,$$

which is finite.

It remains to show that S_2 is finite. Indeed

$$\begin{aligned}
S_2 &= \sum_{k=N+1}^{\infty} |f^{[k]}|_Y \sum_{n=k}^{\infty} \frac{1}{M_n} \sum \frac{n!}{a_1! \cdots a_n!} \prod_{j=1}^n \left(\frac{|\varphi^{[j]}|_Y}{j!} \right)^{a_j} \\
&= \sum_{k=N+1}^{\infty} \frac{|f^{[k]}|_Y}{M_k} k! \sum_{n=k}^{\infty} \frac{1}{n!} \sum \frac{n!}{M_n} \frac{M_k}{k!} \frac{n!}{a_1! \cdots a_n!} \prod_{j=1}^n \left(\frac{|\varphi^{[j]}|_Y}{j!} \right)^{a_j} \\
&\leq B \sum_{k=N+1}^{\infty} \frac{|f^{[k]}|_Y}{M_k} k! \sum_{n=k}^{\infty} \frac{1}{n!} \sum \frac{n!}{a_1! \cdots a_n!} \prod_{j=1}^n \left(\frac{|\varphi^{[j]}|_Y C^{j-1}}{j! k^{j-1}} \right)^{a_j}.
\end{aligned}$$

Again using formula B3 (p. 823) from [1], we have

$$S_2 \leq B \sum_{k=N+1}^{\infty} \frac{|f^{[k]}|_Y}{M_k} \left(\sum_{j=1}^{\infty} \frac{|\varphi^{[j]}|_X C^{j-1}}{j! k^{j-1}} \right)^k.$$

Since $|\varphi^{[j]}|/j! < D^j$ for each j , we have

$$S_2 \leq B \sum_{k=N+1}^{\infty} \frac{|f^{[k]}|_Y}{M_k} |\varphi^{[1]}|_X^k \left(1 + \frac{D}{|\varphi^{[1]}|_X} \sum_{j=2}^{\infty} \frac{D^{j-1} C^{j-1}}{k^{j-1}} \right)^k,$$

and

$$\left(1 + \frac{D}{|\varphi^{[1]}|_X} \sum_{j=2}^{\infty} \frac{D^{j-1} C^{j-1}}{k^{j-1}} \right)^k = \left(1 + \frac{CD^2}{|\varphi^{[1]}|_X} \frac{1}{k - CD} \right)^k,$$

which can be shown to be bounded in m . It follows that

$$S_2 \leq B \sum_{k=N+1}^{\infty} \frac{|f^{[k]}|_Y}{M_k} |\varphi^{[1]}|_X^k \left(1 + \frac{CD^2}{|\varphi^{[1]}|_X} \frac{1}{k - CD} \right)^k,$$

and, since $|\varphi^{[1]}|_X \leq 1$, it follows that S_2 is finite. This completes the proof. $\square$

Note that, in the above, $D_{\mathcal{F}}(X, M)$ and $D_{\mathcal{G}}(Y, M)$ are always Banach function algebras, so we do not need to make any additional completeness assumptions.

Remark 3.5.4. We do not know whether the assumption that φ be $\mathcal{F}$ -analytic in Theorem 3.5.3 is a necessary condition; see the open problems for this chapter (Section 3.7).

In the case where $(n^2 M_{n-1}/M_n)$ is unbounded, the condition that $|f^{[1]}|_X \leq 1$ is no longer sufficient for φ^* to be a homomorphism from $D_{\mathcal{G}}(Y, M)$ into $D_{\mathcal{F}}(X, M)$. The following example is from [44].

Example 3.5.5. Let $I = [0, 1]$, let $\mathcal{F}$ be the collection of all admissible paths in I , and, for each $n \in \mathbb{N}$, let $M_n = (n!)^{3/2}$. Then $M = (M_n)_{n=0}^{\infty}$ is a non-analytic algebra sequence such that $n^2 M_{n-1}/M_n \rightarrow \infty$ as $n \rightarrow \infty$. Moreover, since I is

uniformly regular, we have $D_{\mathcal{F}}(I, M) = D(I, M)$, and $D_{\mathcal{F}}(I, M)$ is natural on I by [27, Theorem 4.4.16(ii)]. Let $\varphi(t) := (1 + t^2)/2$ ($t \in I$). Then φ is an $\mathcal{F}$ -analytic map from I into I , $|\varphi'|_X \leq 1$, and φ is $\mathcal{F}$ - $\mathcal{F}$ -compatible. However, by [44, Theorem 3.2], φ^* is not a homomorphism from $D_{\mathcal{F}}(I, M)$ into $D_{\mathcal{F}}(I, M)$. ■

3.6 Naturality of Bland–Feinstein algebras

In this section we investigate the naturality of Bland–Feinstein algebras as discussed earlier in this chapter. Unless otherwise stated, all results in this section are new (and do not appear in [19]).

The naturality of the algebras $D^{(1)}(X)$, where X is a perfect compact plane set, was investigated in [29], where they proved the following.

Proposition 3.6.1 ([29, Theorem 3.1]). *Let X be a perfect compact plane set. Then each character φ on $D^{(1)}(X)$ corresponds to evaluation at a point of X .*

Remark 3.6.2. The algebra $D^{(1)}(X)$ in the above proposition need not be complete, so another consequence of this result is that all characters on $D^{(1)}(X)$ are continuous.

Chaobankoh extended this result to the algebras $D^{(n)}(X)$, for each $n \in \mathbb{N}$ and where X a perfect compact plane set; see [18, Theorem 4.2.5]. Chaobankoh also investigated the naturality of the algebras $D_{\mathcal{F}}^{(1)}(X)$. Note that if we have $D_{\mathcal{F}}^{(1)}(X) = D^{(1)}(X)$, then Proposition 3.6.1 shows that $D_{\mathcal{F}}^{(1)}(X)$ is also natural. This is the case, for example, if X is $\mathcal{F}$ -regular. We wish to obtain results similar to the above for other Bland–Feinstein algebras. We focus on the algebras $D_{\mathcal{F}}^{(k)}(X)$ at present; the algebras $D_{\mathcal{F}}(X, M)$ will be considered separately.

Our main tool in this section is the following result due to Chaobankoh ([18, Theorem 4.3.2]); see also [2] and [66].

Proposition 3.6.3. *Let X be a semi-rectifiable compact plane set, let $\mathcal{F}$ be an effective collection of paths in X , let $n \in \mathbb{N}$, and let A be the uniform closure of $D_{\mathcal{F}}^{(n)}(X)$ in $C(X)$. If A is natural on X then $D_{\mathcal{F}}^{(n)}(X)$ is natural on X .*

In the remainder of this section, we give various sufficient conditions for the uniform closure of a Bland–Feinstein algebra to be natural.

Let $\mathcal{F}$ be a collection of paths in $\mathbb{C}$ and let $E \subseteq \mathbb{C}$. We denote by $\mathcal{F}|E$ the set of all $\gamma \in \mathcal{F}$ such that $\gamma^* \subseteq E$. Note that $\mathcal{F}|\emptyset = \emptyset$.

Lemma 3.6.4. *Let X, Y be semi-rectifiable compact plane sets with $Y \subseteq X$, let $\mathcal{F}$ be an effective collection of paths in X , and let $\mathcal{G} := \mathcal{F}|Y$. Suppose that $\mathcal{G}$ is an effective collection of paths in Y . Then $\mathbf{m}(\mathcal{G}) \subseteq \mathbf{m}(\mathcal{F})|Y$.*

Proof. Let S denote the set of all $(f, g) \in C(X) \oplus_1 C(X)$ such that $g|Y$ is the $\mathcal{G}$ -derivative of $f|Y$. Then we have $S_{\mathcal{F}} \subseteq S$ and so, by Lemma 3.3.11, we have $\mathbf{p}(S) \subseteq \mathbf{p}(S_{\mathcal{F}}) = \mathbf{m}(\mathcal{F})$. But now $\mathbf{p}(S) = \mathbf{m}(\mathcal{G})$ when S is considered as a subset of $C(Y) \oplus_1 C(Y)$. The result follows. $\square$

Note that the condition that $\mathcal{G}$ be effective in Y is necessary given the definition of $\mathbf{m}(\mathcal{G})$.

Let X be a compact plane set and let U be a non-empty open set in $\mathbb{C}$ such that $U \subseteq X$. We write $A(X, U)$ for the algebra of all continuous functions f on X such $f|U$ is analytic. It is standard that $A(X, U)$ is a natural uniform algebra on X (see [54, Chapter 2]). The proof of the following is an easy modification of the proof of [10, Lemma 5.3]

Lemma 3.6.5. *Let X be a semi-rectifiable compact plane set, let U be a non-empty open set in $\mathbb{C}$ such that $U \subseteq X$, and let $\mathcal{F}$ be an effective collection of paths in X . Then $D_{\mathcal{F}}^{(1)}(X) \subseteq A(X, U)$ if and only if $\mathbf{m}(\mathcal{F})|U$ is the collection of all admissible paths in U .*

Proof. First suppose that $D_{\mathcal{F}}^{(1)}(X) \subseteq A(X, U)$. Then, for each $f \in D_{\mathcal{F}}^{(1)}(X)$, $f|U$ is analytic. Let $z \in U$. There exists $r > 0$ such that $Y := \bar{B}(z, r) \subseteq U$ and let $\mathcal{G} := \mathcal{F}|Y$. Now each $f \in D_{\mathcal{F}}^{(1)}(X)$ is analytic on a neighbourhood of Y , so $f|Y \in D^{(1)}(Y)$. It follows that $D_{\mathcal{G}}^{(1)}(Y) = D^{(1)}(Y)$. By Lemma 3.3.13(b), $\mathbf{m}(\mathcal{G})$ is the set of all admissible paths in Y . By Lemma 3.6.4, we have $\mathbf{m}(\mathcal{G}) \subseteq \mathbf{m}(\mathcal{F})|Y$. This occurs for all $z \in U$, so $\mathbf{m}(\mathcal{F})$ must contain all admissible paths γ in $\mathbb{C}$ such that $\gamma^* \subseteq U$.

Now suppose that $\mathbf{m}(\mathcal{F})|U$ is the collection of all admissible path in U . Let $z \in U$. Then there exists $r > 0$ such that $Y := \bar{B}(z, r) \subseteq U$ and let $\mathcal{G} := \mathcal{F}|Y$. By assumption, $\mathbf{m}(\mathcal{G})$ is the collection of all admissible paths in Y . But Y is uniformly regular (which implies that Y is $\mathcal{G}$ -regular) so we have $D_{\mathcal{G}}^{(1)}(Y) = D^{(1)}(Y)$. Thus, for each $f \in D_{\mathcal{F}}^{(1)}(X)$, we have $f \in D_{\mathcal{G}}^{(1)}(Y) = D^{(1)}(Y)$ so that f is analytic at z . Thus f is analytic on U . It follows that $D_{\mathcal{F}}^{(1)}(X) \subseteq A(X, U)$. This completes the proof. $\square$

From the above we obtain the following immediate corollary.

Corollary 3.6.6. *Let X be a semi-rectifiable compact plane set, let $U := \text{int } X$, and let $\mathcal{F}$ be an effective collection of paths in X . Suppose that $U \neq \emptyset$. Then $D_{\mathcal{F}}^{(1)}(X) \subseteq A(X)$ if and only if $\mathbf{m}(\mathcal{F})|U$ is the set of all admissible paths in U .*

Remark 3.6.7. The ‘if’ implication of the above corollary is [10, Lemma 5.3].

Let X be a semi-rectifiable compact plane set such that $R(X) = A(X)$. Let $\mathcal{F}$ be the collection of all admissible paths in X . Then the uniform closure of $D_{\mathcal{F}}^{(1)}(X)$ in $C(X)$ is equal to $A(X)$, and so $D_{\mathcal{F}}^{(1)}(X)$ is a natural Banach function algebra.

The following also gives sufficient conditions for $D_{\mathcal{F}}^{(1)}(X)$ to be natural on X .

Lemma 3.6.8. *Let X be a semi-rectifiable compact plane set, let $\mathcal{F}$ be an effective collection of paths in X , and let B be the uniform closure of $D_{\mathcal{F}}^{(1)}(X)$ in $C(X)$. Suppose that $\overline{Z} \in D_{\mathcal{F}}^{(1)}(X)$. Then $B = C(X)$. In particular, $D_{\mathcal{F}}^{(1)}(X)$ is a natural Banach function algebra on X .*

Proof. Let $A := \text{alg}\{1, Z, \overline{Z}\}$. Then $A \subseteq D_{\mathcal{F}}^{(1)}(X)$ and A is uniformly dense in $C(X)$ by the Stone–Weierstrass theorem. Thus $B = C(X)$ also. The remainder of the result now follows from Proposition 3.6.3. $\square$

The following examples show that it is indeed possible to have $\overline{Z} \in D_{\mathcal{F}}^{(1)}(X)$. In these examples, all intervals $[a, b]$ are assumed to be non-degenerate.

Example 3.6.9. Let $X := \{x + iy : x, y \in [0, 1]\}$ and let

$$\mathcal{F} := \{x_0 + it \ (t \in [a, b]) : x_0 \in X, [a, b] \subseteq [0, 1]\}.$$

Then, for each $\gamma \in \mathcal{F}$, there exists a non-degenerate, closed interval $[a, b] \subseteq [0, 1]$ such that $\gamma(t) = x_0 + it$, for some $x_0 \in [0, 1]$, for each $t \in [a, b]$. We have

$$\int_{\gamma} -1 \, dz = \int_a^b -i \, dt = -ib + ia = (x_0 - ib) - (x_0 - ia).$$

It follows that $\overline{Z} \in D_{\mathcal{F}}^{(1)}(X)$ and $\overline{Z}^{[1]} = -1$. $\blacksquare$

Example 3.6.10. Let $R \in (0, 1)$ and let $X := \{z \in \mathbb{C} : R \leq |z| \leq 1\}$. Let

$$\mathcal{F} := \{r \exp it \ (t \in [a, b]) : r \in [R, 1], [a, b] \subseteq [0, 2\pi]\}.$$

Fix $r \in (0, 1]$, and let $[a, b] \subseteq [0, 2\pi]$. Then $\overline{Z} = r^2/Z$ on the path $\gamma(t) := r \exp it$ ($t \in [a, b]$). Thus $\overline{Z}$ is complex-differentiable on γ^* . It follows that $\overline{Z} \in D_{\mathcal{F}}^{(1)}(X)$. Note that $\overline{Z} \notin D^{(1)}(X)$. $\blacksquare$

Note that, in the above examples, for each $\gamma \in \mathbf{m}(\mathcal{F})$ we have $\gamma^* \in \mathcal{F}^*$. We do not know whether this is a necessary condition for the complex conjugate function to be $\mathcal{F}$ -differentiable. Also note that, in the first example the $\mathcal{F}$ -derivative of $\overline{Z}$ is a constant, while in the second example the $\mathcal{F}$ -derivative of $\overline{Z}$ is a non-constant function.

Let $\theta \in (-\pi, \pi]$ and $z_0 \in \mathbb{C}$. We denote the set containing all points on the line through z_0 which makes the angle θ with the real axis by $L(z_0, \theta)$. That is,

$$L(z_0, \theta) = \{z_0 + r \exp(i\theta) : r \in \mathbb{R}\}$$

We can generalise the case in Example 3.6.9 in the following theorem.

Theorem 3.6.11. *Let X be a semi-rectifiable compact plane set and let $\mathcal{F}$ be an effective collection of paths in X . Then $\overline{Z} \in D_{\mathcal{F}}^{(1)}(X)$ and $\overline{Z}^{[1]}$ is constant on X if and only if there exists $\theta \in (-\pi, \pi]$ such that, for each $\gamma \in \mathcal{F}$, there exists $x_\gamma \in X$ with $\gamma^* \subseteq L(x_\gamma, \theta) \cap X$.*

Proof. First suppose that $\overline{Z}$ is $\mathcal{F}$ -differentiable and $\overline{Z}^{[1]} = \alpha \in \mathbb{C}$. Let $\gamma \in \mathcal{F}$ such that $\gamma^+ \neq \gamma^-$. Then

$$\overline{\gamma^+} - \overline{\gamma^-} = \int_{\gamma} \alpha dz = \alpha(\gamma^+ - \gamma^-).$$

It follows that

$$\frac{\overline{\gamma^+} - \overline{\gamma^-}}{\gamma^+ - \gamma^-} = \alpha,$$

so we see immediately that $|\alpha| = 1$. Moreover,

$$\text{Arg}(\overline{\gamma^+} - \overline{\gamma^-}) - \text{Arg}(\gamma^+ - \gamma^-) = \text{Arg}\left(\frac{\overline{\gamma^+} - \overline{\gamma^-}}{\gamma^+ - \gamma^-}\right) = \text{Arg}(\alpha).$$

Set $\theta = -\text{Arg}(\alpha)/2$ and $x_\gamma := \gamma^-$. Then $\gamma^+ - \gamma^- = |\gamma^+ - \gamma^-| \exp(i\theta)$ so that $\gamma^* \subseteq L(x_\gamma, \theta)$. This holds for all $\gamma \in \mathcal{F}$ such that $\gamma^+ \neq \gamma^-$.

Let $\gamma : [a, b] \rightarrow X$ be a path in $\mathcal{F}$ and suppose that $\gamma^+ = \gamma^-$. Since γ is admissible, there is a sequence of subpaths (σ_n) of γ with $\sigma_n^+ \rightarrow \gamma^+$ as $n \rightarrow \infty$, and such that $\sigma_n^+ \neq \sigma_n^-$ and $\sigma_n^- = \gamma^-$ for each $n \in \mathbb{N}$. By the above, we have $\sigma_n^* \subseteq L(\gamma^-, \theta)$. It follows that $\gamma^* \subseteq L(\gamma^-, \theta)$.

Now suppose that there exists $\theta \in (-\pi, \pi]$ such that, for each $\gamma \in \mathcal{F}$, there exists $x_\gamma \in X$ with $\gamma^* \subseteq L(x_\gamma, \theta) \cap X$. Let $\mathcal{G}$ be the collection of all paths $\gamma : [a, b] \rightarrow X$ such that $\gamma(t) = x_\gamma + t \exp(i\theta)$ ($t \in [a, b]$) for some $x_\gamma \in X$. Clearly every path γ in X such that there exists $x_\gamma \in X$ with $\gamma^* \subseteq L(x_\gamma, \theta)$ has $\gamma \in \mathbf{m}(\mathcal{G})$. Clearly

$\mathcal{F} \subseteq \mathbf{m}(\mathcal{G})$ and so $\mathbf{m}(\mathcal{F}) \subseteq \mathbf{m}(\mathcal{G})$. By Lemma 3.3.11, we have $D_{\mathcal{G}}^{(1)}(X) \subseteq D_{\mathcal{F}}^{(1)}(X)$. So it is sufficient to show that $\overline{Z} \in D_{\mathcal{G}}^{(1)}(X)$ and that $\overline{Z}^{[1]}$ is constant on X .

Let $g(z) = \exp(-2i\theta)$ ($z \in X$). We show that g is an $\mathcal{G}$ -derivative of $\overline{Z}$. Let $\gamma : [a, b] \rightarrow X$ be a path in $\mathcal{G}$ given by $\gamma(t) = x_\gamma + \exp(i\theta)t$ ($t \in [a, b]$) for some $x_\gamma \in X$. Then we have

$$\int_\gamma g(z) dz = \int_a^b e^{-i\theta} dt = e^{-i\theta}(b - a) = \overline{(x_\gamma + e^{i\theta}b)} - \overline{(x_\gamma + e^{i\theta}a)}.$$

This holds for each path $\gamma \in \mathcal{G}$ and so g is the $\mathcal{G}$ -derivative of $\overline{Z}$. This completes the proof. $\square$

We obtain the following useful corollary of the above theorem and Lemma 3.6.8.

Corollary 3.6.12. *Let X be a semi-rectifiable compact plane set and let $\mathcal{F}$ be an effective collection of paths in X . Suppose that there exists $\theta \in (-\pi, \pi]$ such that, for each $\gamma \in \mathcal{F}$, there exists $x_\gamma \in X$ with $\gamma^* \subseteq L(x_\gamma, \theta) \cap X$. Let B denote the uniform closure of $D_{\mathcal{F}}^{(1)}(X)$ in $C(X)$. Then $B = C(X)$ and $D_{\mathcal{F}}^{(1)}(X)$ is a natural Banach function algebra on X .*

We can also attempt to characterise those semi-rectifiable compact plane sets X and effective collections of paths $\mathcal{F}$ in X such that $D_{\mathcal{F}}^{(1)}(X)$ is natural by means of the following observation.

Lemma 3.6.13. *Let X be a semi-rectifiable compact plane set, let $k \in \mathbb{N}$, let $\mathcal{F}$ be an effective collection of paths in X , and let B denote the uniform closure of $D_{\mathcal{F}}^{(k)}(X)$ in $C(X)$. Suppose that B is normal on X . Then $D_{\mathcal{F}}^{(k)}(X)$ is a natural Banach function algebra on X .*

Proof. Since B is normal on X , it follows from Proposition 1.5.5 that B is natural on X . The result follows from Proposition 3.6.3. $\square$

In particular, if $D_{\mathcal{F}}^{(1)}(X)$ is a normal Banach function algebra on X then B is also normal, and so $D_{\mathcal{F}}^{(1)}(X)$ is natural by the above.

Certainly it is not true that $D_{\mathcal{F}}^{(1)}(X)$ is always normal on X . For instance, if $D_{\mathcal{F}}^{(1)}(X) \subseteq A(X)$, and X has non-empty interior, then $D_{\mathcal{F}}^{(1)}(X)$ is not regular. Moreover, in this case, $D_{\mathcal{F}}^{(1)}(X)$ need not be uniformly dense in $A(X)$ and so it may not be possible to show that $D_{\mathcal{F}}^{(1)}(X)$ is natural on X .

The following further corollary of Lemma 3.6.13 is of more use to us.

Corollary 3.6.14. *Let X be a semi-rectifiable compact plane set and let $\mathcal{F}$ be an effective collection of paths in X . Suppose that $R(X)$ is normal on X . Then, for each $k \in \mathbb{N}$, $D_{\mathcal{F}}^{(k)}(X)$ is natural on X .*

Proof. Fix $k \in \mathbb{N}$. Let B denote the uniform closure of $D_{\mathcal{F}}^{(k)}(X)$ in $C(X)$. Then $R(X) \subseteq B$, and it follows that B is normal on X . The result now follows Lemma 3.6.13. $\square$

Remark 3.6.15. Note that the naturality of $D_{\mathcal{F}}^{(1)}(X)$ in the above is dependent only on X , and therefore is true for every effective collection of paths in X .

Of course, if $R(X) = C(X)$ then $R(X)$ is regular, and hence normal, so the above applies. However, in this case, naturality is given by Proposition 3.6.3. The following example illustrates an interesting case in the above corollary where we have $R(X) \neq C(X)$.

Example 3.6.16. Let $A = ((a_n, r_n))$ be a classical abstract Swiss cheese such that $R(X_A)$ is non-trivial and regular; for example, the Swiss cheese set obtained by Corollary 2.2.12. Set $X := X_A$. Then $R(X)$ is also normal on X . Let $\mathcal{F}$ be an effective collection of paths in X . Then $D_{\mathcal{F}}^{(1)}(X)$ is natural on X by Corollary 3.6.14. $\blacksquare$

The Swiss cheese set X in the above example is uniformly regular (see, for example, [29]) so if $\mathbf{m}(\mathcal{F})$ is the set of all admissible paths in X , then we have $D_{\mathcal{F}}^{(1)}(X) = D^{(1)}(X)$ and so is natural by Proposition 3.6.1. In fact, in this case, we have $D_{\mathcal{F}}^{(1)}(X) \subseteq R(X)$ by [28, Lemma 1.5(iv)]. Similarly, taking the collection $\mathcal{F}$ to be all line segments in X which are parallel to a given line—for example, the real axis or the imaginary axis—then we see that $D_{\mathcal{F}}^{(1)}(X)$ is uniformly dense in $C(X)$ by Corollary 3.6.12, and so is natural on X . (Note that the existence of an abundance of such line segments is a consequence of an argument of Allard which shows that a classical Swiss cheese set has positive area, see [16, pp. 163-164].)

In the following example (due to Feinstein), we use some techniques developed for uniform algebras in [41] by Feinstein and Izzo, to give a classical Swiss cheese set X and an effective collection of paths in X such that $\mathbf{m}(\mathcal{F})$ is not the collection of all admissible paths in X but $D_{\mathcal{F}}^{(1)}(X)$ is natural on X .

Example 3.6.17. Let $A = ((a_n, r_n))$ be the classical abstract Swiss cheese, as in Example 3.6.16, such that $R(X_A)$ is regular on X_A , and such that $\text{int } X_A = \emptyset$. Set $X := X_A$. For each $n \in \mathbb{N}$, let $\gamma_n(t) = a_n + r_n \exp(it)$ ($t \in [0, 2\pi]$), and let $\mathcal{F}$ be the collection of rectifiable paths consisting of the γ_n and all subpaths thereof. It is easy to see that $\bigcup \mathcal{F}^*$ is dense in X so that $\mathcal{F}$ is an effective collection of paths. By Corollary 3.6.14, $D_{\mathcal{F}}^{(1)}(X)$ is natural on X . We now *claim* that there exists $f \in D_{\mathcal{F}}^{(1)}(X)$ such that f is non-constant but $f^{[1]}$ is identically zero on X .

Let $\sim$ be the equivalence relation on X such that $x \sim y$ if and only if either $x, y \in X \setminus \bigcup \mathcal{F}^*$ and $x = y$, or there exists $\gamma \in \mathcal{F}$ such that $x, y \in \gamma^*$. Let Y be the quotient space $X/\sim$; that is, Y is the topological space obtained from X by contracting the sets γ_n^* ($n \in \mathbb{N}$) to a point. By [41, Lemma 2.5], Y is a compact metric space. Let $q : X \rightarrow Y$ be the quotient map. Then the map $f \mapsto f \circ q$ defines a isometric, unital algebra homomorphism from $C(Y)$ into $C(X)$ and, for each $f \in C(Y)$, $f \circ q$ is constant on γ_n^* for each $n \in \mathbb{N}$. Let $f \in C(Y)$ such that f is non-constant and let $g := f \circ q \in C(X)$. Then, as above, g is constant on γ_n^* for each $n \in \mathbb{N}$, and so $g \in D_{\mathcal{F}}^{(1)}(X)$ with $g^{[1]} = 0$. But f is non-constant and so g is non-constant on $X \setminus \bigcup \mathcal{F}^*$. It follows that $D_{\mathcal{F}}^{(1)}(X) \neq D^{(1)}(X)$. ■

Remark 3.6.18. We do not know whether $D_{\mathcal{F}}^{(1)}(X)$ is uniformly dense in $C(X)$ in this case.

The techniques from [41] can be applied in a slightly more general setting. Let X be a semi-rectifiable compact plane set, and let $\mathcal{F}$ be an effective collection of paths in X . Suppose that there exists a sequence (γ_n) of paths in $\mathcal{F}$ with the following properties: $\gamma_j^* \cap \gamma_k^* = \emptyset$ for all $j, k \in \mathbb{N}$ with $j \neq k$; for each $\sigma \in \mathcal{F}$ there exists $k \in \mathbb{N}$ with $\sigma^* \subseteq \gamma_k^*$; $\Lambda(\gamma_n) \rightarrow 0$ as $n \rightarrow \infty$. Then we can use the construction above to find non-constant functions $f \in D_{\mathcal{F}}^{(1)}(X)$ with $f^{[1]}$ identically zero on X .

3.6.1 Naturality of the $D_{\mathcal{F}}(X, M)$

We now consider the naturality of the algebras $D_{\mathcal{F}}(X, M)$. Let X be a perfect compact plane set and let $M = (M_n)_{n=0}^{\infty}$ be an algebra sequence. Then we have $R_0(X) \subseteq D(X, M)$ if and only if M is a non-analytic algebra sequence ([28]), so the condition that M is non-analytic is necessary for the algebras $D(X, M)$ and $D_{\mathcal{F}}(X, M)$, where $\mathcal{F}$ is an effective collection of paths in X .

We require the following definition from [2] (see also [18]).

Definition 3.6.19. Let $M = (M_n)_{n=0}^{\infty}$ be a sequence of positive real numbers. For each $k \in \mathbb{N}$, let $P_k = M_k/k!$ and let

$$A_k = \sup \left\{ \frac{1}{P_k} \prod_{j=1}^n P_j^{a_j} \right\},$$

where the supremum is taken over all $n \in \mathbb{N}$ with $n \geq k$ and non-negative integers $a_1, \dots, a_n$ such that $a_1 + \dots + a_n = k$ and $a_1 + 2a_2 + \dots + na_n = n$.

Let $M = (M_n)_{n=0}^\infty$ be an algebra sequence. It is easy to see that, for each $k \in \mathbb{N}$, we have $A_k \leq 1$ (the details are noted in [18, p. 52]). As in [18], we say that M is *naturalisable* if $\lim_{k \rightarrow \infty} A_k^{1/k} = 0$. It was shown in [2] that if M is naturalisable then M is non-analytic (see also [18]). Our main tool for this section will be the following proposition, due to Chaobankoh [18, Theorem 4.3.6] which is the $D_{\mathcal{F}}(X, M)$ analogue of Proposition 3.6.3.

Proposition 3.6.20. *Let X be a semi-rectifiable compact plane set, let $\mathcal{F}$ be an effective collection of paths in X , let $M = (M_n)_{n=0}^\infty$ be a naturalisable algebra sequence, and let B denote the uniform closure in $C(X)$ of $D_{\mathcal{F}}(X, M)$. Then $D_{\mathcal{F}}(X, M)$ is natural on X if and only if B is natural on X .*

Some of the results for the algebras $D_{\mathcal{F}}^{(1)}(X)$ discussed above are valid for the algebras $D_{\mathcal{F}}(X, M)$ too. We summarise these modified results in the following.

Lemma 3.6.21. *Let X be a semi-rectifiable compact plane set, let $\mathcal{F}$ be a effective collection of paths in X , let $M = (M_n)_{n=0}^\infty$ be an algebra sequence, and let B denote the uniform closure of $D_{\mathcal{F}}(X, M)$ in $C(X)$.*

- (a) *For each non-empty open set $U \subseteq \mathbb{C}$ with $U \subseteq X$, if $\mathbf{m}(\mathcal{F})|U$ is the collection of all admissible paths in U , then we have $D_{\mathcal{F}}(X, M) \subseteq A(X, U)$.*
- (b) *We have $\overline{Z} \in D_{\mathcal{F}}(X, M)$ and $\overline{Z}^{[1]}$ is constant on X if and only if there exists $\theta \in (-\pi, \pi]$ such that, for each $\gamma \in \mathcal{F}$, there exists $x_\gamma \in X$ with $\gamma^* \subseteq L(x_\gamma, \theta) \cap X$.*

Suppose further that M is naturalisable.

- (c) *If $\overline{Z} \in D_{\mathcal{F}}(X, M)$ then $B = C(X)$ and $D_{\mathcal{F}}(X, M)$ is natural on X .*
- (d) *If B is normal on X then $D_{\mathcal{F}}(X, M)$ is natural on X .*

Proof. (a) Let $U \subseteq \mathbb{C}$ be a non-empty open set such that $U \subseteq X$. First note that $D_{\mathcal{F}}(X, M) \subseteq D_{\mathcal{F}}^{(1)}(X)$, so it follows immediately that if $D_{\mathcal{F}}^{(1)}(X) \subseteq A(X, U)$ then $D_{\mathcal{F}}(X, M) \subseteq A(X, U)$. Thus, if $\mathbf{m}(\mathcal{F})|U$ is the collection of all admissible paths in U then, by Lemma 3.6.5, we have $D_{\mathcal{F}}(X, M) \subseteq A(X, U)$.

(b) Note that, if $\overline{Z} \in D_{\mathcal{F}}^{(1)}(X)$ such that $\overline{Z}^{[1]}$ is constant, then $\overline{Z} \in D_{\mathcal{F}}^{(\infty)}(X)$ and $\overline{Z}^{[k]} = 0$ for all $k \in \mathbb{N}$ with $k \geq 2$. Thus $\overline{Z} \in D_{\mathcal{F}}(X, M)$. But since $D_{\mathcal{F}}(X, M) \subseteq D_{\mathcal{F}}^{(1)}(X)$, it follows that $\overline{Z} \in D_{\mathcal{F}}(X, M)$ such that $\overline{Z}^{[1]}$ is constant if and only if the same is true of $D_{\mathcal{F}}^{(1)}(X)$. The result now follows from Theorem 3.6.11.

(c) Suppose that $\overline{Z} \in D_{\mathcal{F}}(X, M)$. Set $A := \text{alg}\{1, Z, \overline{Z}\} \subseteq D_{\mathcal{F}}(X, M)$. Let $\overline{A}$ be the uniform closure of A in $C(X)$. Then $\overline{A} \subseteq B \subseteq C(X)$. But, by the Stone–Weierstrass theorem, $\overline{A} = C(X)$. Thus $B = C(X)$ and, since M is naturalisable, the result follows from Proposition 3.6.20.

(d) Suppose that B is normal on X . As in Lemma 3.6.13, B is natural on X . Since M is naturalisable, the result follows from Proposition 3.6.20. $\square$

Let X be a semi-rectifiable compact plane set, let $\mathcal{F}$ be an effective collection of paths in X , and let $M = (M_n)_{n=0}^{\infty}$ be a naturalisable algebra sequence. Combining (b) and (c) from the above lemma, gives a result analogous to Corollary 3.6.12 for the algebras $D_{\mathcal{F}}(X, M)$.

We have the following analogue of Corollary 3.6.14 for the algebras $D_{\mathcal{F}}(X, M)$.

Corollary 3.6.22. *Let X be a semi-rectifiable compact plane set, let $\mathcal{F}$ be a effective collection of paths in X , and let $M = (M_n)_{n=0}^{\infty}$ be a naturalisable algebra sequence. If $R(X)$ is normal on X then $D_{\mathcal{F}}(X, M)$ is a natural Banach function algebra on X .*

Proof. Let B denote the uniform closure of $D_{\mathcal{F}}(X, M)$ in $C(X)$. Since M is non-analytic, we have $R_0(X) \subseteq D_{\mathcal{F}}(X, M)$ and so $R(X) \subseteq B \subseteq C(X)$. The proof is completed by applying Lemma 3.6.21(d). $\square$

Note that the same holds if $D_{\mathcal{F}}(X, M)$ is assumed to be normal.

3.7 Open problems and discussion

The following problems are related to the content of Chapter 3. Throughout this section, let X be a semi-rectifiable compact plane set, let $\mathcal{A}$ denote the collection of all admissible paths in X , and let $\mathcal{F}$ be an effective collection of paths in X .

The first two problems relate to approximation of paths in $\mathbb{C}$.

Question 3.7.1. Let $\mathcal{G}$ denote the set of all rectifiable Jordan paths in X . Do we have $\mathfrak{m}(\mathcal{G}) = \mathcal{A}$?

First note that $\mathcal{G}$ is effective. This answer is certainly affirmative if X is convex. In general, one can attempt to approximate a rectifiable path by the finite joining of geodesics in X .

Question 3.7.2. Let Γ be an admissible path in $\mathbb{C}$ and let $\mathcal{G}$ denote the collection of all subpaths of Γ . Does $\mathfrak{m}(\mathcal{G})$ include all geodesics in Γ^* ?

This certainly appears to be true in several simple cases. For example, it is certainly true if Γ is a Jordan path. Similarly, if Γ has only finitely many self-intersections then the above appears to be true; note that, in this case, the path Γ is a finite join of rectifiable Jordan paths.

The next question concerns the existence of homomorphisms between Bland–Feinstein algebras of the form $D_{\mathcal{F}}(X, M)$.

Question 3.7.3. Let $M = (M_n)_{n=0}^{\infty}$ be an algebra sequence, let Y be a semi-rectifiable compact plane set, and let $\mathcal{G}$ be an effective collection of paths in Y . Let $\varphi \in D_{\mathcal{F}}^{(\infty)}(X)$ with $\varphi(X) \subseteq Y$ such that φ is $\mathcal{F}$ - $\mathcal{G}$ -compatible. Can we remove the hypothesis that φ be $\mathcal{F}$ -analytic in Theorem 3.5.3?

Our next questions concern isomorphisms of Bland–Feinstein algebras.

Question 3.7.4. Let Y be a semi-rectifiable compact plane set and let $\mathcal{G}$ be an effective collection of paths in Y . Suppose that $\varphi \in D_{\mathcal{F}}^{(1)}(X)$ with $\varphi(X) \subseteq Y$ such that φ is $\mathcal{F}$ - $\mathcal{G}$ -compatible and φ is a bijection. Under what conditions on φ is it automatically true that $\varphi^{-1} \in D_{\mathcal{G}}^{(1)}(Y)$?

Question 3.7.5. Let $M = (M_n)_{n=0}^{\infty}$ be an algebra sequence, let Y be a semi-rectifiable compact plane set, and let $\mathcal{G}$ be an effective collection of paths in Y . Let $\varphi \in D_{\mathcal{F}}^{(\infty)}(X)$ with $\varphi(X) \subseteq Y$ such that φ is $\mathcal{F}$ - $\mathcal{G}$ -compatible. Suppose also that φ is bijective and induces a homomorphism φ^* from $D_{\mathcal{G}}(Y, M)$ into $D_{\mathcal{F}}(X, M)$. Under what conditions on φ is φ^* an isomorphism?

CHAPTER 4

Quasianalyticity and $\mathcal{F}$ -quasianalyticity

Traditionally, we say that a collection A of infinitely continuously differentiable functions on $\mathbb{R}$ is quasianalytic if every function f in A satisfying $f^{(k)}(x) = 0$ ($k \in \mathbb{N}_0$), for some $x \in \mathbb{R}$, is f constantly zero. The Denjoy–Carleman theorem (see [83] and [93, Section 19.6]) gives necessary and sufficient conditions for certain classes of functions to be quasianalytic. In [28], Dales and Davie gave a new definition of quasianalyticity for collections of infinitely continuously complex-differentiable functions defined on a compact plane set. They also gave sufficient conditions for certain collections of functions to be quasianalytic in this new sense. However, their results require that the compact plane set has certain properties.

Dales and Davie used the properties of quasianalytic Banach function algebras to show that there exists a natural Banach function algebra with only countably many peak points [28, Theorem 2.3]. They also constructed a quasianalytic Banach function algebra A such that there exists a discontinuous function F which operates on A [28, Theorem 3.8]. We refer the reader to [28] for the relevant definitions.

In [37], Feinstein constructed a Swiss cheese set X such that $R(X)$ is not regular but does not admit any non-trivial Jensen measures. This is achieved by constructing X so that the interval $I := [-1/2, 1/2]$ is contained in X and the algebra $R(X)|_I$ is quasianalytic but each point $x \in X \setminus I$ is a point of continuity for $R(X)$. It follows from this that $R(X)$ cannot admit any non-trivial Jensen measures but is not regular. (See Section 4.5 for the relevant definitions.) Feinstein and Yang [52] have improved this example by replacing the interval I with a set of positive area.

Quasianalyticity has also been investigated in [4] and [73]. In the former, Albrecht and Athar consider Banach function algebras similar to the algebras

$D(X, M)$ but, instead a sequence of positive real numbers, they allow M to be a sequence of bounded functions from X into $[1, \infty)$.

In this chapter, we introduce a notion of $\mathcal{F}$ -quasianalyticity for collections of infinitely $\mathcal{F}$ -differentiable functions on certain compact plane sets, and give new sufficient conditions for such collections to be $\mathcal{F}$ -quasianalytic. We prove an $\mathcal{F}$ -differentiation version of the Denjoy–Carleman theorem, which generalises the results of Dales and Davie. We then discuss several examples to illustrate the behaviour of $\mathcal{F}$ -quasianalytic algebras. Next, we discuss the connections between several different notions of analyticity, including $\mathcal{F}$ -analyticity. In Section 4.4, we introduce sets of quasianalyticity as a means of investigating the properties of $\mathcal{F}$ -quasianalyticity. We then give the construction of a classical Swiss cheese set X such that $R(X)$ admits no non-trivial Jensen measures on X but $R(X)$ is not regular. This improves the example of Feinstein (as above). We conclude the chapter with several open problems and a discussion.

Parts of this chapter are contained in [47].

4.1 Quasianalytic Banach function algebras

We begin with a discussion of the quasianalytic algebras of functions as defined by Dales and Davie [28]. The following is our definition of quasianalyticity that we consider here. We have adapted the definition given in [28], which could allow some undesirable cases.

Definition 4.1.1. Let X be a perfect compact plane set and let $A \subseteq D^{(\infty)}(X)$. Then A is *quasianalytic* if, for each $z_0 \in X$, the conditions

$$f \in A \quad \text{and} \quad f^{(k)}(z_0) = 0 \quad (k \in \mathbb{N}_0)$$

together imply that $f(z) = 0$ for all z in the component of X containing z_0 .

The definition of quasianalyticity given in [28] only requires that the function f , satisfying the conditions above, vanishes on a proper superset of $\{z_0\}$.

The following is a sufficient condition for quasianalyticity from [28].

Proposition 4.1.2. Let γ be a path in $\mathbb{C}$ such that γ^* is uniformly regular. Let $f \in D^{(\infty)}(X)$ and suppose that there exists a log-convex sequence $(M_k)_{k=0}^\infty$ such that $\sum_{k=0}^\infty M_k/M_{k+1} = \infty$. If there exists $z_0 \in \gamma^*$ such that $f^{(k)}(z_0) = 0$ for all $k \in \mathbb{N}_0$ then $f(z) = 0$ for all $z \in \gamma^*$.

Remark 4.1.3. A similar theorem, due to Davie, with the weaker condition that γ be rectifiable appears in [15].

The following are examples of sequences $M = (M_n)_{n=0}^\infty$ that we may wish to consider. The sequence given by $M_0 = 1$ and $M_n = n! \log 2 \log 3 \cdots \log(n+1)$ ($n = 1, 2, \dots$) is non-analytic (i.e. $d(M) = 0$) and satisfies the conditions in Proposition 4.1.2. The sequence given by $M_n = n!$ ($n \in \mathbb{N}_0$) also satisfies the condition in Proposition 4.1.2 but is analytic (i.e. $d(M) > 0$). For each $\alpha > 1$, the sequence $(n!^\alpha)_{n=0}^\infty$ does not satisfy the condition in Proposition 4.1.2, but is non-analytic.

4.2 $\mathcal{F}$ -quasianalyticity

In this section we aim to prove an analogue of Proposition 4.1.2 which can be applied to the algebras $D_{\mathcal{F}}(X, M)$ of $\mathcal{F}$ -differentiable functions introduced in Section 3.3. Let X be a semi-rectifiable compact plane set and let $\mathcal{F}$ be an effective collection of paths in X . When considering $\mathcal{F}$ -differentiable functions on X , we must restrict our attention to those points which lie on paths in $\mathcal{F}$ as we have little information about the behaviour of $\mathcal{F}$ -derivatives at the other points of X . We introduce the following notion of $\mathcal{F}$ -quasianalyticity.

Definition 4.2.1. Let X be a semi-rectifiable compact plane set, let $\mathcal{F}$ be an effective collection of paths in X , and let $A \subseteq D_{\mathcal{F}}^{(\infty)}(X)$. Then A is $\mathcal{F}$ -quasianalytic if, for each $\gamma \in \mathcal{F}$ and $z_0 \in \gamma^*$, the conditions

$$f \in A \quad \text{and} \quad f^{[k]}(z_0) = 0 \quad (k = 0, 1, \dots),$$

together imply that $f(z) = 0$ for all $z \in \gamma^*$.

In Section 4.4, we examine those subsets of a compact plane set X which satisfy a similar property to the above relative to a given algebra of functions $A \subseteq D_{\mathcal{F}}^{(\infty)}(X)$.

We shall discuss the relationship between $\mathcal{F}$ -quasianalyticity and quasianalyticity in more depth in Section 4.2.1, but first we prove some sufficient conditions for $\mathcal{F}$ -quasianalyticity.

For the remainder of the section we fix an admissible path Γ , and let $\mathcal{F}$ denote the collection of all paths in Γ^* which are subpaths of Γ , or the reverses of subpaths of Γ . In fact, since $D_{\mathcal{F}}^{(\infty)}(X) = D_{\mathfrak{m}(\mathcal{F})}^{(\infty)}(X)$, we could consider $\mathcal{F}$ to be the maximal collection generated by all subpaths of Γ . This is not necessary for our proofs.

We are interested in those sequences $M = (M_n)_{n=0}^\infty$ of positive real numbers satisfying

$$\sum_{n=1}^{\infty} M_n^{-1/n} = \infty. \quad (4.1)$$

We write ds for integrals with respect to path length measure. Set $D := D_{\mathcal{F}}^{(\infty)}(\Gamma^*)$ and, for each $n \in \mathbb{N}$, set $D_n := D_{\mathcal{F}}^{(n)}(\Gamma^*)$.

We aim to prove that, for each $f \in D$ and $z_0 \in \Gamma^*$, if $|f^{[k]}|_{\Gamma^*} \leq M_k$ and $f^{[k]}(z_0) = 0$, for all $k \in \mathbb{N}_0$, then f is identically zero on Γ^* . Our method will follow the proof in [22], of the traditional Denjoy–Carleman theorem, adapted for $\mathcal{F}$ -differentiation.

Note that, unlike Proposition 4.1.2, we do not require that the sequence M be log-convex. This is achieved by instead working with the greatest log-convex minorant of M . Moreover, we do not require that Γ^* be uniformly regular.

Before we can prove the theorem, we first require several lemmas. The first lemma is a summary of the properties of the *log-convex minorant* of a sequence of positive real numbers. We refer the reader to [75, Chapter IV] and [83, Chapter 1] for details and properties of the log-convex minorant. The properties listed below are from [22].

Lemma 4.2.2. *Let $(M_n)_{n=0}^\infty$ be a sequence of positive real numbers such that $\liminf_{n \rightarrow \infty} M_n^{1/n} = +\infty$. Then there exists a log-convex sequence $(M_n^c)_{n=0}^\infty$ and a strictly increasing sequence $(n_j)_{j=0}^\infty$ of integers with $n_0 = 0$ such that:*

- (a) $M_k^c \leq M_k$ for all $k \in \mathbb{N}_0$;
- (b) $M_{n_k}^c = M_{n_k}$ for all $k \in \mathbb{N}_0$;
- (c) for each $k \in \mathbb{N}_0$, we have $M_j^c/M_{j+1}^c = M_{n_k}^c/M_{n_k+1}^c$ for all $j \in \mathbb{N}_0$ with $n_k \leq j < n_{k+1}$.

It is easy to see that if a sequence $M = (M_n)_{n=0}^\infty$ of positive real numbers satisfies (4.1) then the log-convex minorant $M^c = (M_n^c)_{n=0}^\infty$, as given by Lemma 4.2.2, satisfies (4.1). Moreover, if M^c satisfies (4.1) then M^c satisfies $\sum_{n=0}^\infty M_n^c/M_{n+1}^c = \infty$; see [75, Chapter IV] for details.

We will require lemmas to prove the main result. The first lemma is standard; see, for example, [26, Proposition 1.17].

Lemma 4.2.3. *Let $\gamma \in \mathcal{F}$ and let $f \in D_1$. Then*

$$|f(\gamma^+)| \leq |f(\gamma^-)| + \Lambda(\gamma)|f^{[1]}|_{\gamma^*}.$$

Our next lemma is an $\mathcal{F}$ -differentiability analogue of [22, Lemma 2].

Lemma 4.2.4. *Let $\gamma \in \mathcal{F}$, let $M > 0$, let $m \in \mathbb{N}$, and let $\sigma = \gamma^{\text{pl}}$. Let $s_0, \dots, s_{m-1} \in [0, \Lambda(\sigma)]$ such that $0 \leq s_0 < s_1 < \dots < s_{m-1} < \Lambda(\sigma)$. Let $g \in D_m$ and suppose that $|g^{[m]}(\sigma(t))| \leq M$ for all $t \in [s_0, \Lambda(\gamma)]$. Then, for all $s \in (s_{m-1}, \Lambda(\gamma)]$, we have*

$$|g(\sigma(s))| \leq \sum_{p=0}^{m-1} \frac{|g^{[p]}(\sigma(s_{m-p-1}))|(s - s_{m-p-1})^p}{p!} + \frac{M}{m!}(s - s_0)^m.$$

Proof. Note that, for each $a, b \in [0, \Lambda(\gamma)]$ with $a < b$ we have $\Lambda(\sigma|[a, b]) = b - a$. We prove the result by induction on m . If $m = 1$ then, by Lemma 4.2.3 applied to $\sigma|[s_0, s]$, we have

$$|g(\sigma(s))| \leq |g(\sigma(s_0))| + M\Lambda(\sigma|[s_0, s]) = |g(\sigma(s_0))| + M(s - s_0).$$

Let $m > 1$ be an integer and assume that the result holds for $m - 1$. Since $g^{[1]} \in D_{m-1}$, it follows that, for each $s \in [s_{m-2}, \Lambda(\gamma)]$, we have

$$|g^{[1]}(\sigma(s))| \leq \sum_{p=0}^{m-2} \frac{|g^{[p+1]}(\sigma(s_{m-p-2}))|(s - s_{m-p-2})^p}{p!} + \frac{M}{(m-1)!}(s - s_0)^{m-1}.$$

Fix $t \in (s_{m-1}, \Lambda(\gamma)]$. For each $j \in \{1, \dots, m-1\}$ let $\sigma_j := \sigma|[s_j, t]$. Then

$$\begin{aligned} \int_{\sigma_{m-1}} |g^{[1]}(\sigma(s))| \, ds &\leq \int_{\sigma_{m-1}} \sum_{p=0}^{m-2} \frac{|g^{[p+1]}(\sigma(s_{m-p-2}))|(s - s_{m-p-2})^p}{p!} \, ds \\ &\quad + \int_{\sigma_{m-1}} \frac{M}{(m-1)!}(s - s_0)^{m-1} \, ds. \end{aligned} \quad (4.2)$$

For each $j \in \{1, \dots, m-2\}$, we have

$$\int_{\sigma_{m-1}} (s - s_{m-j-1})^{j-1} \, ds \leq \int_{\sigma_{m-j-1}} (s - s_{m-j-1})^{j-1} \, ds = \frac{(t - s_{m-j-1})^j}{j}, \quad (4.3)$$

and, by combining (4.3) and (4.2), we obtain

$$\int_{\sigma_{m-1}} |g^{[1]}(\sigma(s))| \, ds \leq \sum_{p=1}^{m-1} \frac{|g^{[p]}(\sigma(s_{m-p-1}))|(t - s_{m-p-1})^p}{p!} + \frac{M}{m!}(t - s_0)^m. \quad (4.4)$$

But now

$$|g(\sigma(t))| \leq |g(\sigma(s_{m-1}))| + \int_{\sigma_{m-1}} |g^{[1]}(\sigma(s))| \, ds,$$

and combining this with (4.4), we obtain the desired result. $\square$

We now check an easy special case of our result.

Lemma 4.2.5. *Let $M = (M_n)_{n=0}^\infty$ be a sequence of positive real numbers such that $\liminf_{n \rightarrow \infty} M_n^{1/n} < \infty$. Let $z_0 \in \Gamma^*$, and let $f \in D$ with $|f^{[k]}|_{\Gamma^*} \leq M_k$ and $f^{[k]}(z_0) = 0$ for all $k \in \mathbb{N}_0$. Then $f(z) = 0$ for all $z \in \Gamma^*$.*

Proof. Fix $z \in \Gamma^*$. Let $\gamma \in \mathcal{F}$ such that $\gamma^- = z_0$, $\gamma^+ = z$, and $\gamma = \gamma^{\text{pl}}$. We first claim that, for each $k, n \in \mathbb{N}_0$ with $k \leq n$, we have

$$|f^{[k]}(z)| \leq M_n \frac{s^{n-k}}{(n-k)!}, \quad (4.5)$$

where $s = \Lambda(\gamma)$.

Fix $n \in \mathbb{N}_0$. We prove our claim by backwards induction on $k \leq n$. The case where $k = n$ is trivial. Now assume the claim holds for some $k \in \mathbb{N}$ with $k \leq n$. Let $z \in \gamma^*$. We have

$$|f^{[k-1]}(z)| \leq |f^{[k-1]}(\gamma^+)| + \int_\gamma |f^{[k]}|_{\gamma^*} ds = \int_\gamma |f^{[k]}|_{\gamma^*} ds,$$

and applying the claim to $|f^{[k]}(\zeta)|$, we have

$$|f^{[k-1]}(z)| \leq \int_\gamma \frac{M_n s^{n-k}}{(n-k)!} ds = \frac{M_n s^{n-k+1}}{(n-k+1)!}.$$

This proves the claim.

We now see that $|f(z)| \leq M_n s^n / n!$ for all $n \in \mathbb{N}_0$. Since $\liminf_{n \rightarrow \infty} M_n^{1/n} < \infty$, there exists $R > 0$ such that $M_n < R^n$ for infinitely many $n \in \mathbb{N}$. Let $(n_k)_{k=1}^\infty$ be a strictly increasing sequence in $\mathbb{N}$ such that $M_{n_k} < R^{n_k}$ for all $k \in \mathbb{N}$. Then we have

$$|f(z)| \leq \frac{M_{n_k} s^{n_k}}{n_k!} < \frac{(Rs)^{n_k}}{n_k!} \rightarrow 0$$

as $k \rightarrow \infty$. This holds for all $z \in \Gamma^*$, so the result follows. $\square$

Let $0 < \alpha < 1$. As in [22], we define $B_{j,k}^{(\alpha)}$ for $j, k \in \mathbb{N}_0$ with $k \geq j \geq 0$ as follows. For each $j \in \mathbb{N}_0$, let $B_{0,j}^{(\alpha)} = 0$ and, for each $j \in \mathbb{N}$, let $B_{j,j}^{(\alpha)} = 1$. For each $j, k \in \mathbb{N}_0$ with $k > j$, define $B_{j+1,k+1}^{(\alpha)}$ inductively by setting $B_{j+1,k+1}^{(\alpha)} = B_{j,k+1}^{(\alpha)} + \alpha B_{j+1,k}^{(\alpha)}$.

Our main tool in the proof of the main theorem is the following lemma. Part of the proof we provide is derived from the proof of [22, Lemma 1]. We include the details for the convenience of the reader.

Lemma 4.2.6. *Let $\alpha \in (0, 1/2e^2)$. Then there exists a constant $K > 0$ such that $B_{j,k}^{(\alpha)} \leq K\alpha < 1/2$ for all $j, k \in \mathbb{N}$ with $j < k$. Moreover, for each $n \in \mathbb{N}$, we have*

$$\sum_{j=0}^{n-1} \frac{((j+1)\alpha)^j}{j!} + 2 \frac{(n\alpha)^n}{n!} < 2.$$

Proof. Fix $C = e^2$. Define $\phi_0(x) := 1$ ($x \in \mathbb{R}$). For each $j \in \mathbb{N}$, define $\phi_j(x) := (Cx/j)^j$ ($x \in \mathbb{R}$). First note that, for each $j, k \in \mathbb{N}$ with $k \geq j > 1$, we have (since $k^j \geq (k-1)^j + j(k-1)^{j-1}$)

$$\begin{aligned} \phi_j(k) &= \frac{C^j}{j^j} k^j \\ &\geq \frac{C^j}{j^j} ((k-1)^j + j(k-1)^{j-1}) \\ &= \frac{C^j}{j^j} (k-1)^j + C \frac{(j-1)^{j-1}}{j^{j-1}} \frac{(k-1)^{j-1}}{k^{j-1}} \frac{C^{j-1}}{(j-1)^{j-1}} k^{j-1} \\ &\geq \phi_j(k-1) + C \frac{(j-1)^{j-1}}{j^{j-1}} \frac{(k-1)^{j-1}}{k^{j-1}} \phi_{j-1}(k) \\ &\geq \phi_j(k-1) + \phi_{j-1}(k-1). \end{aligned}$$

From this we deduce that, for each $j, k \in \mathbb{N}$ with $k \geq j \geq 1$, we have

$$\phi_{j-1}(k) + \phi_j(k-1) \leq \phi_j(k). \quad (4.6)$$

We now *claim* that, for all $j, k \in \mathbb{N}$ with $j, k \geq 0$ we have

$$B_{j+1, k+1}^{(\alpha)} \leq \phi_j(k) \alpha^k + \phi_{j-1}(k-1) \alpha^{k-1} + \cdots + \phi_0(k-j) \alpha^{k-j}. \quad (\dagger_j^k)$$

We prove the claim by induction on j and k . First, if $j = 0$ then we have $B_{j,k}^{(\alpha)} = 0$ for all k and so $(\dagger_0^k)$ holds for all $k \geq 0$. Also, for each $j \in \mathbb{N}$, we have $B_{j,j}^{(\alpha)} = 1$ and so $(\dagger_j^j)$ holds for all j .

Assume now that $(\dagger_{j+1}^k)$ and $(\dagger_j^{k+1})$ hold for some $j, k \in \mathbb{N}_0$ with $k \geq j \geq 0$. Then

$$\begin{aligned} B_{j+1, k+1}^{(\alpha)} &= B_{j, k+1}^{(\alpha)} + \alpha B_{j+1, k}^{(\alpha)} \\ &\leq \phi_{j-1}(k) \alpha^k + \phi_{j-2}(k-1) \alpha^{k-1} + \cdots + \phi_0(k-j+1) \alpha^{k-j+1} \\ &\quad + \phi_j(k-1) \alpha^k + \phi_{j-1}(k-2) \alpha^{k-1} + \cdots + \phi_0(k-j-1) \alpha^{k-j}. \end{aligned}$$

Let $i \in \{0, \dots, j-1\}$. Then, in the above, the coefficient of α^{k-i} is

$$\phi_i(k-i) + \phi_{i-1}(k-i-1).$$

Applying (4.6), for each i , we obtain

$$\begin{aligned} B_{j+1, k+1}^{(\alpha)} &\leq \phi_j(k) \alpha^k + \cdots + \phi_1(k-j+1) \alpha^{k-j+1} + \phi_0(j-i-1) \alpha^{j-i} \\ &= \phi_j(k) \alpha^k + \cdots + \phi_1(k-j+1) \alpha^{k-j+1} + \phi_0(j-i) \alpha^{j-i}. \end{aligned}$$

Thus $(\dagger_{j+1}^{k+1})$ holds. This proves the claim.

Now, for each $j \in \mathbb{N}_0$, we have

$$B_{j+1,j+2}^{(\alpha)} \leq \phi_j(j+1)\alpha^{j+1} + \cdots + \alpha,$$

and, for each $i \in \{1, \dots, j\}$, we have

$$\phi_i(i+1) = \left(\frac{C(i+1)}{i} \right)^i \leq eC^i.$$

Since $\alpha < 1/2C$, we have

$$B_{j+1,j+2}^{(\alpha)} \leq \sum_{i=0}^j \phi_i(i+1)\alpha^{i+1} \leq e\alpha \sum_{i=0}^j C^i \alpha^i \leq \frac{e\alpha}{1-C\alpha} < 2e\alpha.$$

Set $K = 2e$. It is easy to see that, if $j, k \in \mathbb{N}_0$ with $j < k$ then we have $B_{j,k}^{(\alpha)} \leq B_{k-1,k}^{(\alpha)}$. It follows that, for each $j, k \in \mathbb{N}_0$ with $j < k$, we have $B_{j,k}^{(\alpha)} \leq K\alpha$ and hence we obtain the first claim. To obtain the second claim, note that $\alpha < 1/2e^2$ implies $B_{j,k}^{(\alpha)} < 2e/2e^2 < 1/2$ for all $j, k \in \mathbb{N}$ with $j < k$.

For each $k \in \mathbb{N}$, we have $k^k/k! \leq e^k$ so that

$$k! \geq \left(\frac{k}{e} \right)^k,$$

and so

$$\frac{((k+1)\alpha)^k}{k!} \leq \left(\frac{k+1}{k} \right)^k \alpha^k e^k \leq \alpha^k e^{k+1}.$$

Since $\alpha < 1/4e$, for each $n \in \mathbb{N}$ we have

$$\sum_{j=0}^n \frac{((j+1)\alpha)^j}{j!} < \sum_{j=0}^n \frac{e}{4^j} \leq \frac{1}{2} + \frac{e}{3} \left(1 - \frac{1}{4^n} \right) < 2.$$

Since $2 \leq ((n+1)/n)^n$ for all $n \in \mathbb{N}$, we have

$$\sum_{j=0}^{n-1} \frac{((j+1)\alpha)^j}{j!} + 2 \frac{(n\alpha)^n}{n!} \leq \sum_{j=0}^n \frac{((j+1)\alpha)^j}{j!} < 2.$$

This completes the proof. □

Note that, if $\alpha \in (0, 1/2e^2)$, then

$$B_{j+1,k+1}^{(\alpha)} \geq B_{j,k+1}^{(\alpha)} \tag{4.7}$$

for all $j, k \in \mathbb{N}_0$ with $k \geq j$.

We now state and prove our main result. The proof is essentially the one used in [22], adapted for $\mathcal{F}$ -differentiation, and including additional details for the convenience of the reader.

Theorem 4.2.7. *Let $\Gamma : [a, b] \rightarrow \mathbb{C}$ be an admissible path and let $\mathcal{F}$ denote the collection of all subpaths and reverses of subpaths of Γ . Let $(M_n)_{n=0}^\infty$ be a sequence of positive real numbers such that $\sum_{n=1}^\infty M_n^{-1/n} = \infty$. Suppose that $f \in D_{\mathcal{F}}^{(\infty)}(\Gamma^*)$ and $z_0 \in \Gamma^*$ satisfy*

$$|f^{[k]}|_{\Gamma^*} \leq M_k \quad \text{and} \quad f^{[k]}(z_0) = 0 \quad (4.8)$$

for all $k \in \mathbb{N}_0$. Then $f(z) = 0$ for all $z \in \Gamma^*$.

Proof. If $\liminf_{n \rightarrow \infty} M_n^{1/n} < \infty$ then the result follows from Lemma 4.2.5, so suppose that $\liminf_{n \rightarrow \infty} M_n^{1/n} = \infty$. By Lemma 4.2.2, there exist a log-convex sequence $(M_n^c)_{n=0}^\infty$ of positive real numbers and a strictly increasing sequence $(n_j)_{j=0}^\infty$ with $n_0 = 0$ such that:

- (a) $M_k^c \leq M_k$ for all $k \in \mathbb{N}_0$;
- (b) $M_{n_k}^c = M_{n_k}$ for all $k \in \mathbb{N}_0$;
- (c) for each $k \in \mathbb{N}_0$, we have $M_j^c/M_{j+1}^c = M_{n_k}^c/M_{n_k+1}^c$ for all $j \in \mathbb{N}_0$ with $n_k \leq j < n_{k+1}$.

By the comments following Lemma 4.2.2, we have

$$\sum_{n=0}^\infty \frac{M_n^c}{M_{n+1}^c} = \infty.$$

Let $z \in \Gamma^*$ and let $\gamma \in \mathcal{F}$ such that $\gamma^- = z_0$, $\gamma^+ = z$, and $\sigma = \gamma^{\text{pl}}$.

Fix $n \in \mathbb{N}$ such that n is an element in the sequence $(n_p)_{p=0}^\infty$ and such that $\sum_{j=0}^n M_j^c/M_{j+1}^c > 2e^2 \Lambda(\sigma)$. Let $\alpha := \Lambda(\sigma)/(\sum_{j=0}^n M_j^c/M_{j+1}^c)$. Then we have $\alpha \in (0, 1/2e^2)$ and so, by Lemma 4.2.6, $B_{u,v}^{(\alpha)} < 1/2$ for all $u, v \in \mathbb{N}$ with $v > u > 0$.

Define the points $0 = x_0 < x_1 < \dots < x_n \leq \Lambda(\sigma)$ such that

$$x_j - x_{j-1} = \alpha M_{n-j}^c/M_{n-j+1}^c \quad j = 1, 2, \dots, n.$$

For each $j \in \{0, \dots, n\}$ we *claim* that, for each $k \in \mathbb{N}_0$, with $k \leq n - j + 1$,

$$|f^{[k]}(\sigma(t))| \leq B_{j, n-k+1}^{(\alpha)} M_k^c \quad (t \in [0, x_j]). \quad (*_j^k)$$

The proof of the claim is by induction on j . Since $f^{[k]}(\sigma^-) = 0$ for each $k \in \mathbb{N}_0$, $(*_0^k)$ holds for all $k \in \mathbb{N}_0$ with $k \leq n + 1$.

Fix $j \in \{1, \dots, n\}$. Assume now that $(*_j^{k'})$ holds for all $j', k' \in \mathbb{N}_0$ with $j' < j$ and $k' \leq n - j' + 1$. Set $i := n - j + 1$. We now prove $(*_j^k)$ holds for each $k \in \mathbb{N}_0$ with $k \leq i$ by backwards induction on k . We first check the base case. Suppose

that $k = i$. If $i = n_r$ for some $r \in \mathbb{N}_0$, then $B_{j,j}^{(\alpha)} = 1$ and $|f^{[i]}(\sigma(t))| \leq M_i = M_i^c$ for all $t \in [0, x_j]$ by (4.8), and so $(*_j^k)$ holds.

Otherwise, $i \neq n_r$ for all $r \in \mathbb{N}_0$, in which case there exists $r \in \mathbb{N}_0$ such that $n_r < i < n_{r+1} \leq n$. For each $s \in [0, x_{j-1}]$, by $(*_{j-1}^i)$ and (4.7), we have

$$|f^{[i]}(\sigma(s))| \leq B_{j-1,j}^{(\alpha)} M_i^c \leq B_{j,j}^{(\alpha)} M_i^c = M_i^c,$$

and so it remains to show that $|f^{[i]}(\sigma(s))| \leq M_i^c$ for all $s \in [x_{j-1}, x_j]$.

As in [22], let $m := n_{r+1} - i$ and let $R := M_{n_r}^c / M_{n_{r+1}}^c$. Note that

$$j - m = n - n_{r+1} + 1 \geq 1$$

and, for each $p \in \mathbb{N}_0$ with $p \leq m$, we have $M_i^c = M_{i+p}^c R^p$. Also, for each $\ell \in \mathbb{N}$ with $n_r \leq \ell < n_{r+1}$, we have $M_\ell^c / M_{\ell+1}^c = R$ and so $x_{j-p} - x_{j-p-1} = \alpha R$ for each $p \in \mathbb{N}$ with $p \leq m$. In particular, $M_i^c = M_{n_{r+1}}^c R^m$. For each $s \in [x_{j-1}, x_j]$, by Lemma 4.2.4 (applied with $M = M_{n_{r+1}}^c = M_{n_{r+1}}$, $g = f^{[i]}$ and points $s_p = x_{j-m+p}$ for $p = 0, \dots, m-1$), we have

$$\begin{aligned} |f^{[i]}(\sigma(s))| &\leq \sum_{p=0}^{m-1} \frac{|f^{[i+p]}(\sigma(x_{j-p-1}))|(s - x_{j-p-1})^p}{p!} + \frac{M_{n_{r+1}}^c}{m!} (s - x_{j-m})^m \\ &\leq \sum_{p=0}^{m-1} \frac{|f^{[i+p]}(\sigma(x_{j-p-1}))|((p+1)\alpha R)^p}{p!} + \frac{M_{n_{r+1}}^c}{m!} (m\alpha R)^m \end{aligned}$$

and by applying $(*_{j-p-1}^{i+p})$ for each $0 \leq p \leq m-1$, we obtain

$$\begin{aligned} |f^{[i]}(\sigma(s))| &\leq \sum_{p=0}^{m-1} \frac{B_{j-1-p,n-(i+p)+1}^{(\alpha)} M_{i+p}^c ((p+1)\alpha R)^p}{p!} + \frac{M_{n_{r+1}}^c}{m!} (m\alpha R)^m \\ &= M_i^c \left(\sum_{p=0}^{m-1} \frac{B_{j-1-p,j-p}^{(\alpha)} ((p+1)\alpha)^p}{p!} + \frac{(m\alpha)^m}{m!} \right), \end{aligned}$$

Since $B_{u,v}^{(\alpha)} < 1/2$ for all $u, v \in \mathbb{N}_0$ with $v > u > 0$, we have

$$|f^{[i]}(\sigma(s))| \leq \frac{M_i^c}{2} \left(\sum_{p=0}^{m-1} \frac{((p+1)\alpha)^p}{p!} + 2 \frac{(m\alpha)^m}{m!} \right),$$

and so, by Lemma 4.2.6, we have $|f^{[i]}(\sigma(s))| < M_i^c$. This concludes the proof of the base case $k = i$.

Let $k \in \mathbb{N}_0$ with $k \leq i-1$ and assume that $(*_j^{k+1})$ holds; i.e.

$$|f^{[k+1]}(\sigma(t))| \leq B_{j,n-(k+1)+1}^{(\alpha)} M_{k+1}^c$$

for all $t \in [0, x_j]$. Let $s \in [0, x_j]$. If $s \in [0, x_{j-1}]$ then, by applying $(*_j^k)$ and (4.7), we have

$$|f^{[k]}(\sigma(s))| \leq B_{j-1, n-k+1}^{(\alpha)} M_k^c \leq B_{j, n-k+1}^{(\alpha)} M_k^c.$$

Thus we may assume that $s \in [x_{j-1}, x_j]$. By Lemma 4.2.3, we have

$$|f^{[k]}(\sigma(s))| \leq |f^{[k]}(\sigma(x_{j-1}))| + (x_j - x_{j-1}) \sup\{|f^{[k+1]}(\sigma(t))| : t \in [x_{j-1}, x_j]\}.$$

Applying $(*_j^{k+1})$ to the first term and applying $(*_j^k)$ to the second term we obtain

$$|f^{[k]}(\sigma(s))| \leq B_{j-1, n-k+1}^{(\alpha)} M_k^c + \alpha \frac{M_{n-j}^c}{M_{n-j+1}^c} B_{j, n-k}^{(\alpha)} M_{k+1}^c.$$

Since M^c is log-convex and $k \leq n - j$, we have $M_{n-j}^c / M_{n-j+1}^c \leq M_k^c / M_{k+1}^c$, so we obtain

$$|f^{[k]}(\sigma(s))| \leq M_k^c (B_{j-1, n-k+1}^{(\alpha)} + \alpha B_{j, n-k}^{(\alpha)}) = B_{j, n-k+1}^{(\alpha)} M_k^c.$$

Thus $(*_j^k)$ holds, and both inductions may now proceed.

Now, by Lemma 4.2.6, there exists a constant $K > 0$ such that, for all $u, v \in \mathbb{N}$ with $v > u$, we have $B_{u,v} < \alpha K$. Thus $|f(\sigma(t))| \leq K\alpha M_0$ for all $t \in [0, \Lambda(\sigma)]$. It follows that $f(\sigma(t)) = 0$ for all $t \in [0, \Lambda(\sigma)]$. In particular, $|f(z)| = 0$ and hence $f(z) = 0$. Since $z \in \Gamma^*$ was arbitrary, the above holds for all $z \in \Gamma^*$. This completes the proof. $\square$

In the remainder of this chapter we adopt the following convention. Let X be a semi-rectifiable compact plane set, let $\mathcal{F}$ be an effective collection of paths in X , and let $f \in D_{\mathcal{F}}^{(\infty)}(X)$. If there exists $k \in \mathbb{N}_0$ such that $|f^{[k]}|_X = 0$ then we write

$$\sum_{j=1}^{\infty} |f^{[j]}|_X^{-1/j} = \infty.$$

Our first corollary will be used in the next section.

Corollary 4.2.8. *Let γ be an admissible path in $\mathbb{C}$, let $\mathcal{F}$ be an effective collection of paths in γ^* , and let $f \in D_{\mathcal{F}}^{(\infty)}(\gamma^*)$. Suppose that*

$$\sum_{j=1}^{\infty} |f^{[j]}|^{-1/j} = \infty. \quad (4.9)$$

If there exists $z_0 \in \gamma^*$ such that $f^{[k]}(z_0) = 0$ for all $k \in \mathbb{N}_0$ then f is identically zero on γ^* .

Proof. We may assume that we have $|f^{[k]}|_X > 0$ for all $k \in \mathbb{N}_0$, as the degenerate case is easily dealt with using, for example, Lemma 4.2.5. For each $k \in \mathbb{N}_0$, set $M_k = |f^{[k]}|_{\gamma^*}$. By (4.9), the sequence $M = (M_k)_{k=0}^\infty$ satisfies (4.1) and certainly $|f^{[k]}|_{\gamma^*} \leq M_k$ for all $k \in \mathbb{N}_0$. Suppose that there exists $z_0 \in \gamma^*$ such that $f^{[k]}(z_0) = 0$ for all $k \in \mathbb{N}_0$. Then, by Theorem 4.2.7, we have $f(\gamma^*) \subseteq \{0\}$. This completes the proof. $\square$

Our next corollary asserts the existence of a $\mathcal{F}$ -quasianalytic algebra of the form $D_{\mathcal{F}}(X, M)$.

Corollary 4.2.9. *Let X be a semi-rectifiable compact plane set, let $\mathcal{F}$ be an effective collection of paths in X , and let $M = (M_n)_{n=0}^\infty$ be an algebra sequence which satisfies (4.1). Then $D_{\mathcal{F}}(X, M)$ is $\mathcal{F}$ -quasianalytic.*

Proof. Fix $f \in D_{\mathcal{F}}(X, M)$. Then there exists $N \in \mathbb{N}$ large enough so that $|f^{[k]}|_X \leq M_k$ for all $k \in \mathbb{N}_0$ with $k \geq N$. If there exists $k \in \mathbb{N}_0$ such that $|f^{[k]}|_X = 0$, then (4.9) holds, so suppose that $|f^{[k]}|_X > 0$ for all $k \in \mathbb{N}_0$. Then we have

$$\sum_{j=1}^{\infty} |f^{[j]}|_X^{-1/j} \geq \sum_{j=N}^{\infty} |f^{[j]}|_X^{-1/j} \geq \sum_{j=N}^{\infty} M_j^{-1/j} = \infty.$$

Thus, in either case, f satisfies (4.9). Hence, by Corollary 4.2.8, if there exist $\gamma \in \mathcal{F}$ and $z_0 \in \gamma^*$ such that $f^{[k]}(z_0) = 0$ for all $k \in \mathbb{N}_0$, then $f(\gamma^*) \subseteq \{0\}$. It follows that $D_{\mathcal{F}}(X, M)$ is $\mathcal{F}$ -quasianalytic. $\square$

Since $D^{(\infty)}(X) \subseteq D_{\mathcal{F}}^{(\infty)}(X)$, Theorem 4.2.7 generalises [28, Theorem 1.11] to obtain the following.

Corollary 4.2.10. *Let $M = (M_n)_{n=0}^\infty$ be a sequence of positive real numbers such that $\sum_{n=1}^\infty M_n^{-1/n} = \infty$. Let γ be an admissible path in $\mathbb{C}$. Suppose that $f \in D^{(\infty)}(X)$, $f^{(k)}(\gamma^-) = 0$, and $|f^{(k)}| \leq M_k$ for $k = 0, 1, 2, \dots$. Then f is identically zero on γ^* .*

4.2.1 Quasianalyticity and $\mathcal{F}$ -quasianalyticity

We now briefly explain the similarities and differences between $\mathcal{F}$ -quasianalyticity and quasianalyticity (in the sense of Definition 4.1.1). Let X be a pointwise-regular compact plane set, and let $\mathcal{F}$ be the collection of all admissible paths in X . Then $D_{\mathcal{F}}^{(\infty)}(X) = D^{(\infty)}(X)$ and the notions of $\mathcal{F}$ -quasianalyticity and quasianalyticity coincide. This is not the case in general. In this section, we give various examples which show that the conditions for $\mathcal{F}$ -quasianalyticity and

the behaviour of functions in a $\mathcal{F}$ -quasianalytic algebra depend heavily on the collection of paths $\mathcal{F}$.

To avoid restricting our attention to those compact plane sets X and collections $\mathcal{F}$ of paths in X such that $D_{\mathcal{F}}^{(1)}(X) = D^{(1)}(X)$, we extend the notion of quasianalyticity from Definition 4.1.1 to include sets of infinitely $\mathcal{F}$ -differentiable functions.

Definition 4.2.11. Let X be a semi-rectifiable compact plane set, let $\mathcal{F}$ be an effective collection of paths in X , and let $A \subseteq D_{\mathcal{F}}^{(\infty)}(X)$. Then A is *quasianalytic* if, for each $z_0 \in X$, the conditions

$$f \in A \quad \text{and} \quad f^{[k]}(z_0) = 0 \quad (k \in \mathbb{N}_0)$$

together imply that $f(z) = 0$ for all z in the component of X containing z_0 .

Let X be a semi-rectifiable compact plane set, let $M = (M_n)_{n=0}^{\infty}$ be an algebra sequence, and let $\mathcal{F}$ be an effective collection of paths in X . If $A \subseteq D_{\mathcal{F}}^{(\infty)}(X)$ is quasianalytic then A must be $\mathcal{F}$ -quasianalytic. Indeed, if $A \subseteq D_{\mathcal{F}}^{(\infty)}(X)$ is quasianalytic, $f \in A$, $z \in X$ such that $z \in \gamma^*$ for some $\gamma \in \mathcal{F}$, and $f^{[k]}(z) = 0$ for all $k \in \mathbb{N}_0$, then f must vanish on the component of X containing z . This component certainly contains γ^* . The converse fails in general. For example, suppose that $\bigcup \mathcal{F}^* \neq X$. In this case, if $D_{\mathcal{F}}(X, M)$ is $\mathcal{F}$ -quasianalytic then $D(X, M)$ need not be quasianalytic.

In the following examples, we illustrate how our notion of $\mathcal{F}$ -quasianalyticity depends on the collection $\mathcal{F}$.

Example 4.2.12. Let $\mathcal{F}$ be the collection of paths describing horizontal line segments in $\overline{\mathbb{D}}$. For each $n \in \mathbb{N}_0$, let $M_n = (n!)^2$ and let $M = (M_n)_{n=0}^{\infty}$. By [76, Theorem 1], the class $\mathcal{D}$ of functions $f \in D^{(\infty)}(\overline{\mathbb{D}})$ satisfying $|f^{(k)}|_{\overline{\mathbb{D}}} \leq C_f M_k$ (where $C_f > 0$ is a constant depending on f) is quasianalytic if and only if

$$\sum_{n=1}^{\infty} \frac{1}{\inf_{k \geq n} \sqrt{M_k}^{1/k}} = \infty. \quad (4.10)$$

The sequence M satisfies the condition (4.10), and so the Dales–Davie algebra $D(\overline{\mathbb{D}}, M)$ is quasianalytic. Let $\mathcal{G}$ be the collection of all admissible paths in $\overline{\mathbb{D}}$. Then, since $\overline{\mathbb{D}}$ is uniformly regular, we have $D(\overline{\mathbb{D}}, M) = D_{\mathcal{G}}(\overline{\mathbb{D}}, M)$. Thus $D_{\mathcal{G}}(\overline{\mathbb{D}}, M)$ is $\mathcal{G}$ -quasianalytic.

On the other hand, by Proposition 4.4.19 and Theorem 4.4.22 of [27], there exists $f \in D([-1, 1], M)$ such that $\text{supp } f \subseteq [-1/2, 1/2]$ and $f(0) > 0$. Set

$$g(x + iy) := f(x) \quad (x + iy \in \overline{\mathbb{D}}).$$

Then $g \in D_{\mathcal{F}}(\overline{\mathbb{D}}, M)$ and g is not identically zero but has $g^{(k)}(z) = 0$ for all $k \in \mathbb{N}_0$ whenever $z = x + iy$ has $|x| \geq 1/2$. It follows that the algebra $D_{\mathcal{F}}(\overline{\mathbb{D}}, M)$ is not $\mathcal{F}$ -quasianalytic. ■

Example 4.2.13. We define the following collections of paths in $\overline{\mathbb{D}}$:

$$\begin{aligned}\mathcal{F} &:= \{r \exp it \ (t \in [a, b]) : r \in (0, 1], [a, b] \subseteq [-\pi, \pi]\}, \\ \mathcal{G} &:= \{r \exp it \ (r \in [a, b]) : t \in (-\pi, \pi], [a, b] \subseteq [0, 1]\}.\end{aligned}$$

Then both $\mathcal{F}$ and $\mathcal{G}$ are effective collections of paths in $\overline{\mathbb{D}}$. Note that for each pair of distinct points $z, w \in \overline{\mathbb{D}}$, there is a pair of paths $\gamma_1, \gamma_2 \in \mathcal{G}$ such that $\gamma_1^- = z$, $\gamma_1^+ = 0$, $\gamma_2^- = 0$ and $\gamma_2^+ = w$; that is, any pair of distinct points can be joined by the combination of a pair of paths in $\mathcal{G}$. The collection $\mathcal{F}$ does not have this property.

Let $M = (M_k)_{k=0}^\infty$ be an algebra sequence which satisfies (4.1). Suppose that $f \in D_{\mathcal{F}}(\overline{\mathbb{D}}, M)$ and $z_0 \in \overline{\mathbb{D}}$ such that $f^{[k]}(z_0) = 0$ for all $k \in \mathbb{N}_0$. By Theorem 4.2.7, the function f must be identically zero on any path $\gamma \in \mathcal{F}$ such that $z_0 \in \gamma^*$. In particular, f vanishes identically on $\{|z_0| \exp i\theta : \theta \in [-\pi, \pi]\}$. However, f need not vanish identically on $\overline{\mathbb{D}}$. For example, consider the function $h(z) = \psi(|z|)z$ ($z \in \overline{\mathbb{D}}$), where $\psi : [0, 1] \rightarrow [0, 1]$ is a non-zero continuous function such that $\psi(r_0) = 0$ for some $r_0 \in [0, 1]$. Then $h^{[1]}(z) = \psi(|z|)$ ($z \in \overline{\mathbb{D}}$) and hence $h^{[k]}(z_0) = 0$ for all $k \in \mathbb{N}_0$ if $z_0 = r_0 \exp it$ for some $t \in [-\pi, \pi]$.

On the other hand, let $g \in D_{\mathcal{G}}(\overline{\mathbb{D}}, M)$ and suppose that there exists $z_0 \in \overline{\mathbb{D}}$ such that $g^{[k]}(z_0) = 0$ for all $k \in \mathbb{N}_0$. Again, by Theorem 4.2.7, the function g , and each $\mathcal{G}$ -derivative of g , must be identically zero on each path $\gamma \in \mathcal{G}$ such that $z_0 \in \gamma^*$. However, this implies that $g^{[k]}(0) = 0$ for all $k \in \mathbb{N}_0$ and hence g must vanish identically on $\overline{\mathbb{D}}$. ■

4.3 $\mathcal{F}$ -analyticity

We pause to discuss $\mathcal{F}$ -analytic functions. Let X be a semi-rectifiable compact plane set and let $\mathcal{F}$ be an effective collection of paths in X . Recall that a function $f \in D_{\mathcal{F}}^{(\infty)}(X)$ is $\mathcal{F}$ -analytic if

$$\limsup_{k \rightarrow \infty} \left(\frac{|f^{[k]}|_X}{k!} \right)^{1/k} < \infty.$$

We say that a function $g \in D^{(\infty)}(X)$ is *analytic* if

$$\limsup_{k \rightarrow \infty} \left(\frac{|g^{(k)}|_X}{k!} \right)^{1/k} < \infty.$$

In this section, we say that g is *holomorphic on X* if there exists an open neighbourhood U of X , and an analytic function $h : U \rightarrow \mathbb{C}$ (that is, h is complex differentiable on U) such that $h|_X = g$.

In section 3.5 (see also [18]), we showed that $\mathcal{F}$ -analyticity of a function $f \in D_{\mathcal{F}}^{(\infty)}(X)$ (along with the relevant compatibility condition and bound on the norm of f) is sufficient for the map f to induce an algebra homomorphism from $D_{\mathcal{F}}(X, M)$ (where M is a non-analytic algebra sequence). The calculations in Theorem 3.5.3 are effectively those in [43, 70], where analyticity is used in place of $\mathcal{F}$ -analyticity. (See also [44, 45].) Analyticity also appears in [42] but, in this paper, the definition of analyticity is what we call holomorphic on X in the present section.

We wish to illustrate that these notions are quite different for arbitrary compact plane sets. It is quite easy to see that if g is holomorphic on X then g is analytic, and if g is analytic then g is $\mathcal{F}$ -analytic. (This is independent of the choice of collection of paths, since $D^{(\infty)}(X) \subseteq D_{\mathcal{G}}^{(\infty)}(X)$ for all effective collections of paths $\mathcal{G}$ in X .) Clearly if $D_{\mathcal{F}}^{(\infty)}(X) = D^{(\infty)}(X)$ then $\mathcal{F}$ -analytic and analytic are equivalent.

Suppose that X is pointwise regular, and that $g \in D^{(\infty)}(X)$. If g is analytic then g is holomorphic on X ; this is easily deduced from Lemma 1.5 or Theorem 1.11 of [28], with some suitable modifications. Let $\mathcal{G}$ be the collection of all admissible paths in X . Then $D_{\mathcal{G}}^{(\infty)}(X) = D^{(\infty)}(X)$ and so the properties $\mathcal{G}$ -analytic, analytic, and holomorphic on X are equivalent in this case.

We seek an example of a semi-rectifiable compact plane set Y where $\mathcal{G}$ -analytic, analytic, and holomorphic are not equivalent, where $\mathcal{G}$ is an effective collection of paths in Y .

Following [46], we say that a compact plane set X is a *simple set* if there are sequences $(a_n), (b_n)$ of positive real numbers with $b_n > a_n > b_{n+1}$ ($n \in \mathbb{N}$) and $a_n \rightarrow 0, b_n \rightarrow 0$ as $n \rightarrow \infty$ such that

$$X = \{0\} \cup \bigcup_{n=1}^{\infty} [a_n, b_n].$$

If X is a simple set, we let $I_n := [a_n, b_n]$ be the closed intervals associated with X . It is easy to see that any simple compact plane set is semi-rectifiable.

Let X be a simple compact plane set. Then $D(X, M)$ is not complete for any algebra sequence M (see [9, Theorem 3.10]).

The following lemma is an easy modification of [46, Theorem 4.3]. We include the details of the proof for the convenience of the reader.

Lemma 4.3.1. *Let $M = (M_n)_{n=0}^\infty$ be an algebra sequence. There exist a simple compact plane set X , and $f \in D(X, M)$ such that $f^{(k)}(0) = 0$ for all $k \in \mathbb{N}_0$ but $f(x) > 0$ for all $x \in X \setminus \{0\}$.*

Proof. For each $n \in \mathbb{N}$, let $I_n := [2/3^{n+1}, 1/3^n]$. Set $X := \{0\} \cup \bigcup_{n \in \mathbb{N}} I_n$. Set $g_1 = 1$ on X and, for $k \in \mathbb{N}$ with $k \geq 2$, we define inductively

$$g_k(z) = \begin{cases} \frac{1}{4^{k+1}} & z \in \{0\} \cup \bigcup_{j=k}^\infty I_j, \\ g_{k-1}(z) & z \in \bigcup_{j=1}^{k-1} I_j. \end{cases}$$

Then, for each $n \in \mathbb{N}$, g_n is continuous and differentiable on X with derivative 0. For each $x \in X \setminus \{0\}$, the sequence $(g_n(x))$ is eventually constant, and $f_n(0) \rightarrow 0$ as $n \rightarrow \infty$. Let $f(x) := \lim_{n \rightarrow \infty} f_n(x)$ ($x \in X$). It is easy to see that, for each $n \in \mathbb{N}$,

$$|f_n - f|_X = |f_n(0) - f(0)| = \frac{1}{4^{n+1}}$$

and so f_n converges uniformly to f as $n \rightarrow \infty$. Let $x \in X \setminus \{0\}$, then f is constant on a neighbourhood of x , and so $f'(x) = 0$. It remains to show that f is differentiable at 0. Let $x \in X \setminus \{0\}$. There exists (a unique) $k \in \mathbb{N}$ such that $x \in I_k$. It follows that $f(x) = f_k(x) = 1/4^k$. Now

$$\frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x} \leq \frac{1/4^k}{2/3^k} = \frac{1}{2} \left(\frac{3}{4} \right)^k,$$

and so $f'(0)$ exists and is equal to 0.

Now we have $f \in D^{(\infty)}(X)$ and $f^{(k)} = 0$ for all $k \in \mathbb{N}$. It follows that $f \in D(X, M)$ and has the desired properties. $\square$

Remark 4.3.2. It is not hard to modify this proof to show what, in fact, every simple compact plane set X admits a function f which has derivative 0 everywhere on X but is non-constant on every neighbourhood of 0. To see this, choose a sequence (c_n) of positive real numbers such that $c_n/a_n \rightarrow 0$ as $n \rightarrow \infty$ (where the a_n are as in the definition of the set X). Then take $f_n|_{I_n} = c_n$ for each n , and set $f(0) = 0$. Then this function has the desired properties. Some of these details are noted in the proof of [46, Theorem 4.3].

It is not hard to see that the function f constructed in Lemma 4.3.1 extends to be analytic on a neighbourhood U of $X \setminus \{0\}$, but we cannot have $0 \in U$. (Take, for example, the union of a sequence of open disks, each of which contains exactly one of the I_n , and use constant functions on each disk.)

The following is an immediate corollary of Lemma 4.3.1.

Corollary 4.3.3. *There exist a simple compact plane set X and $f \in D^{(\infty)}(X)$ such that f is analytic but not holomorphic on X .*

Let X be a simple compact plane set. The connected subsets of X that are both open and closed in X are precisely the intervals I_n , and each I_n is uniformly regular. Let $\mathcal{G}$ be the collection of all admissible paths in X . Then it is easy to see that, for each $f \in D_{\mathcal{G}}^{(1)}(X)$ and $n \in \mathbb{N}$, we have $f|_{I_n} \in D^{(1)}(I_n)$.

We can use techniques similar to those in the proof of Lemma 4.3.1 to prove the following theorem.

Theorem 4.3.4. *There exist a simple compact plane set X , an effective collection of paths $\mathcal{F}$ in X , and $g \in D_{\mathcal{F}}^{(\infty)}(X)$ such that g is $\mathcal{F}$ -analytic but not analytic.*

Proof. Let C be the Cantor middle-thirds set in $[0, 1]$, and let $V = [0, 1] \setminus C$. For each $n \in \mathbb{N}$, let $I_n := [2/3^{n+1}, 1/3^n]$, and let $\varphi_n : I_n \rightarrow [0, 1]$ be the function $\varphi_n(z) := 3^{n+1}z - 2$. Set $X := \{0\} \cup \bigcup_{n \in \mathbb{N}} I_n$. For each $n \in \mathbb{N}$, let $\mathcal{F}_n$ be the collection of all paths γ in I_n such that $\gamma^* \subseteq \varphi_n^{-1}(V)$. Set $\mathcal{F} := \bigcup_{n=1}^{\infty} \mathcal{F}_n$. It is easy to see that $\mathcal{F}$ is an effective collection of paths in X .

Let ψ be the Cantor function on $[0, 1]$ such that ψ is constant on each component of V . (Such a function is constructed in, for example, [93, pp. 144–145] or [57, Section 8.15].) For each $n \in \mathbb{N}$, set

$$g_n(z) := \frac{1}{4^{n+2}} \left[(\psi \circ \varphi_n)(z) + 1 \right] \quad (z \in I_n),$$

and let g be the function on X obtained by setting $g(0) = 0$ and, for each $n \in \mathbb{N}$, set $g|_{I_n} = g_n$. Then g is continuous on X and, for each $x \in X \setminus \{0\}$ there exists $k \in \mathbb{N}$ with $x \in I_k$ and so

$$\frac{g(x) - g(0)}{x - 0} = \frac{g(x)}{x} \leq \frac{1/4^k}{2/3^k} = \frac{1}{2} \left(\frac{3}{4} \right)^k.$$

It follows that g is complex-differentiable at 0. Note that g is not continuously differentiable on I_n for any n . Let $\gamma \in \mathcal{F}$. Then there exists $k \in \mathbb{N}$ such that $\gamma^* \subseteq \varphi_k^{-1}(V)$. Since γ^* must lie in a connected component of $\varphi_k^{-1}(V)$, it follows that g is constant on γ^* . Hence g is $\mathcal{F}$ -differentiable, with $\mathcal{F}$ -derivative 0. It follows quickly that g is $\mathcal{F}$ -analytic. Since $g \notin D^{(\infty)}(X)$, it cannot be analytic (or holomorphic on X). This completes the proof. $\square$

The function g constructed in Theorem 4.3.4 also illustrates that we cannot extend our definitions of $\mathcal{F}$ -quasianalyticity to include those points in X which do not lie on one of the paths in $\mathcal{F}$.

We now investigate the notion of $\mathcal{F}$ -analyticity in further detail. We aim to show that the restriction to the image of each path in $\mathcal{F}$ of an $\mathcal{F}$ -analytic function is actually holomorphic on the image of that path.

Our main result for this section is the following theorem. We dedicate the remainder of this section to proving this theorem.

Theorem 4.3.5. *Let X be a semi-rectifiable compact plane set, let $\mathcal{G}$ be an effective collection of paths in X , and let $g \in D_{\mathcal{G}}^{(\infty)}(X)$ be $\mathcal{G}$ -analytic. Then, for each $\gamma \in \mathcal{G}$ there exist an open neighbourhood U of γ^* and an analytic function $f : U \rightarrow \mathbb{C}$ such that $f|_{\gamma^*} = g|_{\gamma^*}$.*

Before we can give the proof, we require some preliminary results. The following Lemma is possibly well known, although we have not been able to find the statement in the literature. The proof provided below is due to Langley.

Lemma 4.3.6. *Let X be a compact plane set, and let $h : X \rightarrow \mathbb{C}$ be a function. Suppose that there exist a non-empty indexing set I , a family of open disks D_i ($i \in I$), and a family of analytic functions $f_i : D_i \rightarrow \mathbb{C}$ such that $X \subseteq \bigcup_{i \in I} D_i$ and $f_i|(D_i \cap X) = h|(D_i \cap X)$. Then there exist an open neighbourhood U of X and an analytic function $g : U \rightarrow \mathbb{C}$ such that $g|_X = h$.*

Proof. The result is trivial if X is finite, so suppose not. There exists a function $\rho : X \rightarrow (0, \infty)$ such that, for each $x \in X$, $B(x, 2\rho(x)) \subseteq D_i$ for some $i \in I$. By compactness, there exists $x_1, \dots, x_n \in X$ such that $X \subseteq \bigcup_{j=1}^n B(x_j, \rho(x_j))$. For each $j \in \{1, \dots, n\}$, set $r_j = \rho(x_j)$. Set $\varepsilon = \min\{r_1, \dots, r_n\}/4$. By our choice of the r_j , there is an $i \in I$ such that $B(x_j, 2r_j) \subseteq D_i$ for some $i \in I$. Now, for each $y \in X$, there exists $k \in \{1, \dots, n\}$ such that $y \in B(x_k, r_k)$, and $B(y, \varepsilon) \subseteq B(x_k, 2r_k)$. For each $k \in \{1, \dots, n\}$, there exists $i_k \in I$ such that $B(x_k, 2r_k) \subseteq D_{i_k}$.

By compactness, there exist $y_1, \dots, y_m \in X$ such that $X \subseteq \bigcup_{j=1}^m B(y_j, \varepsilon)$. Relabelling if necessary, let m_0 be an integer with $1 \leq m_0 < m$ such that, for each $1 \leq j < m_0$, y_j is an isolated point of X , and for each $m_0 \leq j \leq m$, y_j is not an isolated point. (If $m_0 = 1$, then X has no isolated points. We must have $m_0 < m$ since X is not finite.)

If $m_0 > 1$ then, for $1 \leq j < m_0$, let $s_j := \text{dist}(y_j, X \setminus \{y_j\}) > 0$ and set $\delta = \min\{s_1, \dots, s_{m_0-1}\}/4$. For each $j \in \{1, \dots, m_0 - 1\}$, let $E_j := B(y_j, \delta)$. Then the E_j ($j \in \{1, \dots, m_0 - 1\}$) are connected open sets in $\mathbb{C}$ and have pairwise disjoint closures. Set $F := \bigcup_{j=1}^{m_0} \overline{E_j}$.

Otherwise, if $m_0 = 1$, set $\delta = 0$ and let $F := \emptyset$.

For each $j \in \{m_0, \dots, m\}$, set $E_j := B(x_j, \varepsilon) \setminus F$. Then each of the E_j ($j \in \{1, \dots, m\}$) are connected open sets in $\mathbb{C}$ and, if $j < m_0$, then $E_j \cap E_k = \emptyset$ for $k \neq j$. For each $j \in \{1, \dots, m_0 - 1\}$, let $g_j(z) = h(y_j)$ for $z \in E_j$. For each $j \in \{m_0, \dots, m\}$, there exists $k \in \{1, \dots, n\}$ such that $B(y_j, \varepsilon) \subseteq B(x_k, 2r_k)$, and there exists a corresponding $i_k \in I$ such that $B(x_k, 2r_k) \subseteq D_{i_k}$. Let g_j be the restriction of f_{i_k} to E_j . Then, for each $j \in \{1, \dots, m\}$, $g_j : E_j \rightarrow \mathbb{C}$ is an analytic function with $g_j|(E_j \cap X) = h|(E_j \cap X)$.

It is clear that g_j is uniquely defined on E_j for each $j \in \{1, \dots, m_0 - 1\}$ in the case where $m_0 > 1$.

Fix $j \in \{m_0, \dots, m\}$. We now *claim* that g_j is uniquely defined on E_j . Let $k, \ell \in \{1, \dots, n\}$ such that $E_j \subseteq B(x_k, 2r_k) \subseteq D_{i_k}$ and $E_j \subseteq B(x_\ell, 2r_\ell) \subseteq D_{i_\ell}$. Then $f_{i_k}|(E_j \cap X) = h|(E_j \cap X) = f_{i_\ell}|(E_j \cap X)$. Since $E_j \cap X$ does not contain any isolated points, y_j is a limit point of $E_j \cap X$, and so $f_{i_k}|E_j = f_{i_\ell}|E_j$ by the identity principle. This proves the claim.

Let $j, k \in \{1, \dots, m\}$ such that $j \neq k$ and $E_j \cap E_k \neq \emptyset$. Let $\ell \in \{1, \dots, n\}$ such that $y_j \in B(x_\ell, r_\ell)$. We have $|y_j - y_k| < 2\varepsilon$, and so $E_j \cup E_k \subseteq B(y_j, 4\varepsilon)$. By our choice of ε , we have $E_j \cup E_k \subseteq B(x_\ell, 2r_\ell) \subseteq D_{i_\ell}$. Thus $g_j|E_j = f_{i_\ell}|E_j$ and $g_k|E_k = f_{i_\ell}|E_k$. In particular $g_j|(E_j \cap E_k) = g_k|(E_j \cap E_k)$.

Let $U = \bigcup_{j=1}^m E_j$, and let g be the analytic function such that $g|E_j = g_j$ for each $j \in \{1, \dots, m\}$. Then g is well defined and analytic on U , and has $g|X = h$. This completes the proof. $\square$

Remark 4.3.7. The author is grateful to Dr. Daniel Nicks and Prof. James Langley for their help with the proof of Lemma 4.3.6, and several useful discussions about complex function theory.

We also require the following Lemma.

Lemma 4.3.8. *Let X be a semi-rectifiable compact plane set, let $\mathcal{F}$ be an effective collection of paths in X , let $\gamma \in \mathcal{F}$, and let $g \in D_{\mathcal{F}}^{(\infty)}(X)$. Suppose that g is $\mathcal{F}$ -analytic. Then there exists $R > 0$ such that, for each $x \in \gamma^*$, the power series*

$$h_x(z) := \sum_{k=0}^{\infty} \frac{f^{[n]}(x)}{k!} (z - x)^k \quad (z \in B(x, R)) \quad (4.11)$$

converges on $B(x, R)$ and $h_x|(B(x, R) \cap \gamma^*) = g|(B(x, R) \cap \gamma^*)$.

Proof. Since g is $\mathcal{F}$ -analytic, there exists $C > 0$ such that $|g^{[k]}|_{\gamma^*} \leq C^k k!$ for all $k \in \mathbb{N}_0$. For each $k \in \mathbb{N}_0$, set $M_k := C^k k!$. Then $M_k \leq C^k k^k$ for each k , and so

$$\sum_{j=1}^{\infty} \frac{1}{M_j^{1/j}} \geq \sum_{j=1}^{\infty} \frac{1}{C^j j} = \infty.$$

Let $R > 0$ such that $1/R = \limsup_{k \rightarrow \infty} (|g^{[k]}|_{\gamma^*}/k!)^{1/k}$.

Fix $x \in \gamma^*$. Then the Taylor series h_x converges on $B(x, R)$ and so h_x is analytic on $B(x, R)$. Moreover, we have $h_x^{(k)}(x) = g^{[k]}(x)$ for all $k \in \mathbb{N}_0$. Let σ be a subpath of γ such that $\sigma^* \subseteq B(x, R)$ and $x \in \sigma^*$. Set $\varphi := g - h_x$. It is easy to see that $\varphi \in D_{\mathcal{F}|\sigma^*}^{(\infty)}(\sigma^*)$ and $\varphi^{[k]}(x) = 0$ for all $k \in \mathbb{N}_0$. For each $k \in \mathbb{N}_0$ and $z \in B(x, R)$, we have

$$h_x^{(k)}(z) = \sum_{j=k}^{\infty} \frac{g^{[j]}(x)}{(j-k)!} (z-x)^{j-k}$$

so that

$$|h_x^{(k)}(z)| \leq \sum_{j=k}^{\infty} \frac{C^j}{(j-k)!} |z-x|^{j-k} \leq C^k \sum_{j=0}^{\infty} \frac{C^j}{j!} |z-x|^j \leq C^k \exp(CR).$$

It follows that $|\varphi^{[k]}|_{\sigma^*} \leq M_k(1 + \exp(CR))$ for all $k \in \mathbb{N}_0$. This implies that (4.9) holds for φ , and hence, by Corollary 4.2.8, $\varphi(z) = 0$ for all $z \in \sigma^*$.

Since σ was an arbitrary subpath of γ with $x \in \sigma^*$ and $\sigma^* \subseteq B(x, R)$, it follows that $h_x(z) = g(z)$ for all $z \in \gamma^* \cap B(x, R)$. This completes the proof. $\square$

We may now prove Theorem 4.3.5.

Proof of Theorem 4.3.5. Let $\gamma \in \mathcal{F}$. By Lemma 4.3.8, there exists $R > 0$ such that, for each $x \in \gamma^*$, the power series h_x (given by (4.11)) converges on $B(x, R)$ and $h_x|_{(B(x, R) \cap \gamma^*)} = g|_{(B(x, R) \cap \gamma^*)}$. For each $x \in \gamma^*$, let $D_x := B(x, R)$. The proof is completed by applying Lemma 4.3.6 to the families $\{D_x : x \in \gamma^*\}$ and $\{h_x : x \in \gamma^*\}$. $\square$

4.4 Sets of quasianalyticity

We wish to explore the properties of an $\mathcal{F}$ -quasianalytic subalgebra of $D_{\mathcal{F}}^{(\infty)}(X)$. To do this, we introduce sets of A -quasianalyticity, which capture the behaviour of the functions in A . In this section, we fix a semi-rectifiable compact plane set X , an effective collection $\mathcal{F}$ of paths in X , and a subalgebra A of $D_{\mathcal{F}}^{(\infty)}(X)$. Unless otherwise specified, all results in this section are new.

Definition 4.4.1. Let $z_0 \in X$ and let $E \subseteq X$ such that $z_0 \in E$. We say that E is a set of A -quasianalyticity for z_0 (with respect to $\mathcal{F}$) if E is closed and whenever $f \in A$ such that $f^{[k]}(z_0) = 0$ for all $k \in \mathbb{N}_0$ we have $f^{[k]}(E) \subseteq \{0\}$ for all $k \in \mathbb{N}_0$.

Note that the sets of A -quasianalyticity depend on the collection of paths $\mathcal{F}$, as illustrated in the previous section. When there is no risk of ambiguity in the collection of paths considered, we shall simply say E is a set of A -quasianalyticity for z_0 . This will be the case throughout most of this section.

Let $z_0 \in X$. Certainly the set $\{z_0\}$ is a set of A -quasianalyticity for z_0 . However, we are interested in when there is a set of A -quasianalyticity E for z_0 such that E is a proper superset of $\{z_0\}$.

The following lemma contains two elementary properties of sets of A -quasianalyticity.

Lemma 4.4.2. *Let $z_0 \in X$.*

- (a) *Let (E_α) be a family of sets of A -quasianalyticity for z_0 . Then $\bigcap_\alpha E_\alpha$ is a set of A -quasianalyticity for z_0 .*
- (b) *Let $(E_\alpha)_\alpha$ be a family of sets of A -quasianalyticity for z_0 . Then $\overline{\bigcup_\alpha E_\alpha}$ is a set of A -quasianalyticity for z_0 .*

Proof. (a) Set $K := \bigcap_\alpha E_\alpha$. First note that $z_0 \in E_\alpha$ for all α , so K is a non-empty closed set. Let $f \in A$ and suppose that $f^{[k]}(z_0) = 0$ for all $k \in \mathbb{N}_0$. Then $f^{[k]}$ vanishes on E_α for all α and all $k \in \mathbb{N}_0$. Thus $f^{[k]}$ vanishes on K for each $k \in \mathbb{N}_0$. Thus K is a set of A -quasianalyticity for z_0 .

(b) Set $S := \bigcup_\alpha E_\alpha$. Then $\overline{S}$ is a closed set containing z_0 . Let $f \in A$ and suppose that $f^{[k]}(z_0) = 0$ for all $k \in \mathbb{N}_0$. Let $z \in S$. Then there exists β such that $z \in E_\beta$. Since E_β is a set of A -quasianalyticity for z_0 , we have $f^{[k]}(E_\beta) \subseteq \{0\}$ for all $k \in \mathbb{N}_0$. Thus $f^{[k]}$ vanishes on S for all $k \in \mathbb{N}_0$. It follows that $\overline{S}$ is a set of A -quasianalyticity for z_0 . $\square$

We obtain the following corollary.

Corollary 4.4.3. *Let $z_0 \in X$. Then there exists a unique set of A -quasianalyticity for z_0 which is maximal with respect to inclusion.*

Proof. Let $\mathcal{S}$ denote the collection of all sets of A -quasianalyticity for z_0 . Certainly $\{z_0\} \in \mathcal{S}$ so that $\mathcal{S}$ is non-empty. Set

$$K := \overline{\bigcup \mathcal{S}}.$$

By Lemma 4.4.2(b), $K \in \mathcal{S}$. This completes the proof. $\square$

Let $z_0 \in X$. Let $Q_{A,\mathcal{F}}(z_0)$ to be the largest set of A -quasianalyticity given by the above corollary. We now investigate the properties of the sets $Q_{A,\mathcal{F}}(z_0)$.

Lemma 4.4.4. *Let $x, y \in X$. If $y \in Q_{A,\mathcal{F}}(x)$ then $Q_{A,\mathcal{F}}(y) \subseteq Q_{A,\mathcal{F}}(x)$.*

Proof. Suppose that $y \in Q_{A,\mathcal{F}}(x)$. Let $f \in A$ such that $f^{[k]}(x) = 0$ for all $k \in \mathbb{N}_0$. Then, since $y \in Q_{A,\mathcal{F}}(x)$, $f^{[k]}(y) = 0$ for all $k \in \mathbb{N}_0$. Thus, for each $k \in \mathbb{N}_0$, we have $f^{[k]}(z) = 0$ for all $z \in Q_{A,\mathcal{F}}(y)$. It follows that $Q_{A,\mathcal{F}}(y)$ is a set of A -quasianalyticity for x , and hence $Q_{A,\mathcal{F}}(y) \subseteq Q_{A,\mathcal{F}}(x)$. $\square$

Note that, if $Q_{A,\mathcal{F}}(x) = \{x\}$ then the above holds since, in this case, we must have $x = y$.

Our first theorem of this section is the following.

Theorem 4.4.5. *Let $x \in X$. Suppose that A is a natural Banach function algebra on X . Then $Q_{A,\mathcal{F}}(x)$ is a connected subset of X .*

Proof. Let $x \in X$ and let F be the component of X containing x . We must show that $Q_{A,\mathcal{F}}(x) \subseteq F$. Let $y \in X \setminus F$, and let F_y denote the component of X containing y . Fix an open neighbourhood U of F_y in X with $U \cap F = \emptyset$. We *claim* that there exists a subset E of X with $F_y \subseteq E \subseteq U$ such that E is both open and closed.

Let $\sim$ be the relation on X such that $w \sim z$ if and only if w, z lie in the same component of X . Let $Y := X/\sim$ with the quotient topology, and let $\pi : X \rightarrow Y$ be the quotient map. Then Y is a compact, totally disconnected topological space, and $\pi(U)$ is an open set containing $\pi(F_y)$. Since Y is compact and totally disconnected, the collection of sets which are both open and closed form a base for the topology on Y . (This fact is mentioned in [27, p. 789]; see also [64, Theorem 2.14].) Thus there exists a set $E_0 \subseteq Y$ containing $\pi(F_y)$, such that $E_0 \subseteq \pi(U)$ and E_0 is both open and closed. Let $E = \pi^{-1}(E_0)$. Then E is both open and closed, and $E \subseteq U$. This proves the claim.

Now by the Shilov idempotent theorem (Proposition 1.4.4), there exists an idempotent $f \in A$ such that $f(E) \subseteq \{1\}$ and $f(X \setminus E) \subseteq \{0\}$. But then $f^{[k]}(x) = 0$ for all $k \in \mathbb{N}_0$, but $f(y) \neq 0$. It follows that $y \notin Q_{A,\mathcal{F}}(x)$. This holds for all $y \in X \setminus F$ and so $Q_{A,\mathcal{F}}(x) \subseteq F$ as required. $\square$

We can characterise quasianalyticity and $\mathcal{F}$ -quasianalyticity of A in terms of the sets $Q_{A,\mathcal{F}}(z_0)$.

Theorem 4.4.6. *The algebra A is quasianalytic if and only if, for each $x \in X$, $Q_{A,\mathcal{F}}(x)$ contains the component of X containing x , and A is $\mathcal{F}$ -quasianalytic if and only if, for each $x \in X$ and $\gamma \in \mathcal{F}$ with $x \in \gamma^*$, we have $\gamma^* \subseteq Q_{A,\mathcal{F}}(x)$.*

Proof. The first statement is obvious.

Suppose that A is $\mathcal{F}$ -quasianalytic. Let $z_0 \in \bigcup \mathcal{F}^*$ and suppose there exists $f \in A$ with $f^{[k]}(z_0) = 0$ for all $k \in \mathbb{N}_0$. Let $\gamma \in \mathcal{F}$ with $z_0 \in \gamma$. Then $f(z) = 0$ for all $z \in \gamma^*$. But then $f^{[k]}(z) = 0$ for all $z \in \gamma^*$ for all $k \in \mathbb{N}_0$. Thus γ^* is a set of A -quasianalyticity for z_0 and so $\gamma^* \subseteq Q_{A,\mathcal{F}}(z_0)$, as required.

Now suppose that, for each $x \in \bigcup \mathcal{F}^*$, and each $\gamma \in \mathcal{F}$ with $x \in \gamma^*$ we have $\gamma^* \subseteq Q_{A,\mathcal{F}}(x)$. Let $\gamma \in \mathcal{F}$ and let $z_0 \in \gamma^*$. Then $\gamma^* \subseteq Q_{A,\mathcal{F}}(z_0)$. Let $f \in A$. Suppose that $f^{[k]}(z_0) = 0$ for all $k \in \mathbb{N}_0$. Then $f^{[k]}(z) = 0$ for all $z \in Q_{A,\mathcal{F}}(z_0)$ and all $k \in \mathbb{N}_0$. In particular, $f(z) = 0$ for all $z \in \gamma^*$. Thus A is $\mathcal{F}$ -quasianalytic. $\square$

By combining Theorem 4.4.5 and Theorem 4.4.6 we obtain the following.

Corollary 4.4.7. *Suppose that A is a natural Banach function algebra on X . Then A is quasianalytic if and only if, for each $x \in X$, $Q_{A,\mathcal{F}}(x)$ is the component of X containing x .*

In fact, if A is $\mathcal{F}$ -quasianalytic then more is true.

Lemma 4.4.8. *Suppose that A is $\mathcal{F}$ -quasianalytic. Let $z_0 \in X$ and $\gamma \in \mathcal{F}$. If $\gamma^* \cap Q_{A,\mathcal{F}}(z_0) \neq \emptyset$ then $\gamma^* \subseteq Q_{A,\mathcal{F}}(z_0)$.*

Proof. Suppose that $\gamma^* \cap Q_{A,\mathcal{F}}(z_0) \neq \emptyset$. Then there exists $x \in Q_{A,\mathcal{F}}(z_0) \cap \gamma^*$. Let $f \in A$ and suppose that $f^{[k]}(z_0) = 0$ for all $k \in \mathbb{N}_0$. Then, for all $k \in \mathbb{N}_0$, we have $f^{[k]}(x) = 0$. Since A is $\mathcal{F}$ -quasianalytic, we have $f(z) = 0$ for all $z \in \gamma^*$. Hence $f^{[k]}(z) = 0$ for all $k \in \mathbb{N}_0$ and $z \in \gamma^*$. Thus $Q_{A,\mathcal{F}}(z_0) \cup \gamma^*$ is a set of A -quasianalyticity for z_0 and so $Q_{A,\mathcal{F}}(z_0) \cup \gamma^* \subseteq Q_{A,\mathcal{F}}(z_0)$. This completes the proof. $\square$

Using this property, we can identify those sets E which are candidates for the sets $Q_{A,\mathcal{F}}(z_0)$. Let $z_0 \in X$. Let $\mathcal{Q}_{\mathcal{F}}(z_0)$ denote the set of all closed sets E of X with the following properties: $z_0 \in E$; for each $\gamma \in \mathcal{F}$, if $\gamma^* \cap E \neq \emptyset$ then $\gamma^* \subseteq E$. Clearly $X \in \mathcal{Q}_{\mathcal{F}}(z_0)$ so that $\mathcal{Q}_{\mathcal{F}}(z_0)$ is non-empty. Note that $\mathcal{Q}_{\mathcal{F}}(z_0)$ does not depend on the subalgebra of $D_{\mathcal{F}}^{(\infty)}(X)$ under consideration.

Lemma 4.4.9. *Let $z_0 \in A$. Let (E_α) be a family of sets belonging to $\mathcal{Q}_{\mathcal{F}}(z_0)$. Then $\bigcap_\alpha E_\alpha \in \mathcal{Q}_{\mathcal{F}}(z_0)$. In particular, $\mathcal{Q}_{\mathcal{F}}(z_0)$ contains a minimum element with respect to inclusion.*

Proof. Let (E_α) be a family of sets in $\mathcal{Q}_{\mathcal{F}}(z_0)$. Set $K := \bigcap_\alpha E_\alpha$. Then K is a non-empty closed subset of X with $z_0 \in K$. Let $\gamma \in \mathcal{F}$ and suppose that $\gamma^* \cap K \neq \emptyset$.

Then $\gamma^* \cap E_\alpha \neq \emptyset$ for all α , and so $\gamma^* \subseteq E_\alpha$ for all α . Thus $\gamma^* \subseteq K$. It follows that $K \in \mathcal{Q}_{\mathcal{F}}(z_0)$.

To show that $\mathcal{Q}_{\mathcal{F}}(z_0)$ contains a minimum element, set $K = \bigcap \mathcal{Q}_{\mathcal{F}}(z_0)$. Then K is the minimum element of $\mathcal{Q}_{\mathcal{F}}(z_0)$. $\square$

Let $z_0 \in X$. It is clear that if $z_0 \in X \setminus \bigcup \mathcal{F}^*$ then we have $\{z_0\} \in \mathcal{Q}_{\mathcal{F}}(z_0)$. Also, by Lemma 4.4.8, if A is $\mathcal{F}$ -quasianalytic then $Q_{A,\mathcal{F}}(z_0) \in \mathcal{Q}_{\mathcal{F}}(z_0)$. We do not know when $Q_{A,\mathcal{F}}(z_0)$ is the minimum element of $\mathcal{Q}_{\mathcal{F}}(z_0)$. However, for each $E \in \mathcal{Q}_{\mathcal{F}}(z_0)$, if $E \subseteq Q_{A,\mathcal{F}}(z_0)$ then E is a set of A -quasianalyticity for z_0 when A is $\mathcal{F}$ -quasianalytic.

Note that, if F is the component of X containing z_0 , then $F \cup \gamma^* = F$ for each $\gamma \in \mathcal{F}$ with $\gamma^* \cap F \neq \emptyset$, and so $F \in \mathcal{Q}_{\mathcal{F}}(z_0)$. It follows that the minimal element of $\mathcal{Q}_{\mathcal{F}}(z_0)$ is always connected.

We now establish the relationships between the sets $Q_{A,\mathcal{F}}(z_0)$ and $Q_{A,\mathcal{G}}(z_0)$ where A is a subalgebra of both $D_{\mathcal{F}}^{(\infty)}(X)$ and $D_{\mathcal{G}}^{(\infty)}(X)$. In particular, we aim to prove that the sets of A -quasianalyticity with respect to $\mathcal{F}$ coincide with the sets of A -quasianalyticity with respect to $\mathfrak{m}(\mathcal{F})$.

Let $\mathcal{G}$ be an effective collection of paths in X . We write

$$\iota_{\mathcal{G}} : D_{\mathcal{G}}^{(\infty)}(X) \rightarrow \prod_{n=0}^{\infty} C(X)$$

for the inclusion given by $\iota_{\mathcal{G}}(f) = (f, f^{[1]}, f^{[2]}, \dots)$, where the derivatives are taken with respect to $\mathcal{G}$. Our final theorem for this section is the following.

Theorem 4.4.10. *Let $\mathcal{G}$ be an effective collection of paths in X , let $E \subseteq X$, and let $z_0 \in X$. Suppose that $A \subseteq D_{\mathcal{G}}^{(\infty)}(X)$ and that $\iota_{\mathcal{G}}(f) = \iota_{\mathcal{F}}(f)$ for all $f \in A$. Then E is set of A -quasianalyticity for z_0 with respect to $\mathcal{F}$ if and only if E is a set of A -quasianalyticity for z_0 with respect to $\mathcal{G}$.*

Proof. Suppose that E is a set of A -quasianalyticity for z_0 with respect to $\mathcal{F}$. The converse is established by a symmetric argument. Fix $f \in A$. Let (f_n) be the sequence in $C(X)$ such that f_1 is the $\mathcal{F}$ -derivative for f and, for each $k \in \mathbb{N}$, f_{k+1} is the $\mathcal{F}$ -derivative for f_k . Let (g_n) be the sequence in $C(X)$ such that g_1 is the $\mathcal{G}$ -derivative for f and, for each $j \in \mathbb{N}$, g_{k+1} is the $\mathcal{G}$ -derivative for g_k . Since $\iota_{\mathcal{G}}(f) = \iota_{\mathcal{F}}(f)$, it follows that $f_k = g_k$ for all $k \in \mathbb{N}$.

Suppose that $f(z_0) = 0$ and, for each $k \in \mathbb{N}$, we $g_k(z_0) = 0$. Then we have $f_k(z_0) = 0$ for all $k \in \mathbb{N}$. Since E is a set of A -quasianalyticity for z_0 with respect to $\mathcal{F}$, it follows that both f and f_k ($k \in \mathbb{N}$) vanish on E . But then g_k vanishes

on E for all $k \in \mathbb{N}$. This implies that E is a set of A -quasianalyticity for z_0 with respect to $\mathcal{G}$. $\square$

The following example shows that the condition $\iota_{\mathcal{G}}(f) = \iota_{\mathcal{F}}(f)$ for all $f \in A$ need not hold in general. We do not know whether this condition is necessary.

Example 4.4.11. Let $X := \{x + iy : x, y \in [0, 1]\}$, let $\mathcal{F}$ denote the collection of all line segments in X which are parallel to the real axis, and let $\mathcal{G}$ denote the collection of all line segments in Y which are parallel to the imaginary axis. (This is the situation in Example 3.3.18.) Then, by Theorem 3.6.11, the complex conjugate function $\bar{Z}$ is both $\mathcal{F}$ - and $\mathcal{G}$ -differentiable. Let $A = \text{alg}\{1, Z, \bar{Z}\}$. Then $A \subseteq D_{\mathcal{F}}^{(\infty)}(X)$ and $A \subseteq D_{\mathcal{G}}^{(\infty)}(X)$. However, $\iota_{\mathcal{G}}(\bar{Z}) = (\bar{Z}, -1, 0, 0, \dots)$ and $\iota_{\mathcal{F}}(\bar{Z}) = (\bar{Z}, 1, 0, 0, \dots)$, and so $\iota_{\mathcal{G}}(\bar{Z}) \neq \iota_{\mathcal{F}}(\bar{Z})$. $\blacksquare$

From Theorem 4.4.10, we obtain the desired relationship between the sets of quasianalyticity with respect to $\mathcal{F}$ and to $\mathbf{m}(\mathcal{F})$.

Corollary 4.4.12. *Let $E \subseteq X$ and $z_0 \in X$. Then E is a set of A -quasianalyticity for z_0 with respect to $\mathbf{m}(\mathcal{F})$ if and only if E is a set of A -quasianalyticity for z_0 with respect to $\mathcal{F}$.*

Proof. We have $D_{\mathbf{m}(\mathcal{F})}^{(\infty)}(X) = D_{\mathcal{F}}^{(\infty)}(X)$ so that $A \subseteq D_{\mathbf{m}(\mathcal{F})}^{(\infty)}(X)$. Moreover, since $\mathbf{m}(\mathcal{F})$ -derivatives coincide with $\mathcal{F}$ -derivatives, it follows that $\iota_{\mathbf{m}(\mathcal{F})}(f) = \iota_{\mathcal{F}}(f)$ for all $f \in A$. Thus the result follows from Theorem 4.4.10. $\square$

Corollary 4.4.13. *Let $\mathcal{G}$ be an effective collection of paths in X such that $\mathcal{G} \subseteq \mathcal{F}$, let $E \subseteq X$, and let $z_0 \in X$. If E is a set of A -quasianalyticity for z_0 with respect to $\mathcal{G}$ then E is set of A -quasianalyticity for z_0 with respect to $\mathcal{F}$.*

Proof. Since $\mathcal{G} \subseteq \mathcal{F}$ we have $A \subseteq D_{\mathcal{F}}^{(\infty)}(X) \subseteq D_{\mathcal{G}}^{(\infty)}(X)$ and $\iota_{\mathcal{G}}(f) = \iota_{\mathcal{F}}(f)$ for all $f \in A$. The result follows from Theorem 4.4.10. $\square$

4.5 Trivial Jensen measures without regularity

In this section we construct a classical Swiss cheese set X such that $R(X)$ has no non-trivial Jensen measures but $R(X)$ is not regular.

We begin with the definition of a Jensen measure.

Definition 4.5.1. Let X be a compact space, let A be a uniform algebra on X , and let $\varphi \in \Phi_A$. A probability measure μ on X is a Jensen measure for φ (with

respect to A) if

$$\log |\varphi(f)| \leq \int_X \log |f| d\mu \quad (f \in A).$$

In the above definition we adopt the convention that $\log(0) = -\infty$. Let X, A, φ be as in the above definition. It is standard that every Jensen measure for φ is a representing measure for φ . Moreover, for each $\varphi \in \Phi_A$, there is a Jensen measure on X for φ ([27, Theorem 4.3.9]). Note that, for $x \in X$, the point-mass measure δ_x is a Jensen measure for ε_x . We say that a Jensen measure μ on X for ε_x is *trivial* if $\mu = \delta_x$. We shall often say that μ is a *Jensen measure for $x \in X$* to mean that μ is a Jensen measure of ε_x .

The following is effectively [52, Lemma 3.1].

Lemma 4.5.2. *Let X be a compact plane set and let $x \in X$. Suppose that there exists a neighbourhood U of x in X such that every point in $U \setminus \{x\}$ is a point of continuity. Then x is an R -point.*

The following lemma is standard (see, for example, [37]) but we include a short proof for the convenience of the reader.

Lemma 4.5.3. *Let X be a compact space, let A be a uniform algebra on X , and let $x \in X$. Suppose that x is a point of continuity for A and μ is a Jensen measure for x . Then μ is trivial.*

Proof. Let $K := \text{supp } \mu \cup \{x\}$. Suppose, towards a contradiction, that $y \in K \setminus \{x\}$. Since x is a point of continuity for A , there exist $f \in A$ and an open neighbourhood V of y in X , with $x \notin V$, such that $f(x) = 1$ and $f(V) \subseteq \{0\}$. Then, since $V \cap K \neq \emptyset$ so that $\mu(V) > 0$ and $f(V) \subseteq \{0\}$, we have

$$\int_V \log |f| d\mu = -\infty, \quad \int_{X \setminus V} \log |f| d\mu < \infty.$$

But μ is a Jensen measure, so

$$0 = \log |f(x)| \leq \int_X \log |f| d\mu = \int_{X \setminus V} \log |f| d\mu + \int_V \log |f| d\mu = -\infty.$$

This is a contradiction, and it follows that $K = \{x\}$. Thus μ is trivial. $\square$

For further details on Jensen measures, we refer the reader to [16, 27, 54, 55].

In [37], Feinstein gave an example of a non-classical Swiss cheese set which has the properties described above. We show that there is a classical Swiss cheese set with these properties; this is the content of the following theorem, which is the main theorem of this section.

Theorem 4.5.4. *There exists a classical abstract Swiss cheese $A = ((a_n, r_n))$ such that $R(X_A)$ is not regular and does not admit any non-trivial Jensen measures.*

Most of the remainder of this section is devoted to the proof of this theorem. We require some preliminary results. The following proposition is from [52, Lemma 4.1].

Proposition 4.5.5. *Let X be a compact plane set, let Y be a non-empty closed subset of X , and let $x \in Y$. Suppose that no bounded component of $\mathbb{C} \setminus Y$ is contained in X , and that there exists a non-trivial representing measure μ for ε_x with respect to $R(X)$ such that $\text{supp } \mu \subseteq Y$. Then μ is a non-trivial representing measure for ε_x with respect to $R(Y)$, and $R(Y) \neq C(Y)$.*

Note that, if $\text{int}(X) = \emptyset$ then the condition on bounded components of $\mathbb{C} \setminus Y$ is automatically satisfied.

Let X be a compact plane set with $\text{int}_{\mathbb{C}} X = \emptyset$, let $x \in X$, and let μ be a non-trivial representing measure for ε_x . Let $Y = \text{supp } \mu \cup \{x\}$. Then, by the above, we must have $R(Y) \neq C(Y)$. In particular, as noted in [52], Y must have positive area. (See also the Hartogs–Rosenthal theorem [54, Corollary II.8.4].) Combining these observations with Proposition 4.5.5 gives the following corollary, which we use below.

Corollary 4.5.6. *Let X be a compact plane set with $\text{int}(X) = \emptyset$, let E be a closed subset of X , and let $x \in E$. Suppose that E has area 0, that μ is a Jensen measure for ε_x with respect to $R(X)$, and μ is supported on E . Then μ is trivial.*

The following is a special case of [52, Lemma 2.1].

Lemma 4.5.7. *Let X be a compact plane set and let $x \in X$. Suppose that μ is a non-trivial Jensen measure for x , and let F be the closed support of μ . Then, for all $y \in F \setminus \{x\}$, we have $J_y \subseteq M_x$. Thus x is not a point of continuity and no point of $F \setminus \{x\}$ is an R -point.*

We also require the following lemma, which is possibly known, but we include details for the convenience of the reader.

Lemma 4.5.8. *Let X, Y be compact plane sets with $Y \subseteq X$. Suppose that $R(Y)$ is regular. Then each point $y \in \text{int}_X(Y)$ is a point of continuity for $R(X)$.*

Proof. Let $y \in \text{int}_X(Y)$, and let $x \in X$ with $x \neq y$. Then there exists $r > 0$ such that $\bar{B}_X(y, r) \subseteq Y$. Choose $\delta \in (0, r/3)$ such that $|y - x| > 2\delta$. Since $R(Y)$ is regular, it follows from Proposition 2.5.2(c) that $R(Y \cap \bar{B}(y, r))$ is regular. Set

$E := \bar{B}_X(y, r) \setminus B_X(y, \delta)$. Then there exists a function $g \in R(Y \cap \bar{B}_X(y, r))$ such that $f(y) = 1$ and $f(x) = 0$ for all $x \in E$. Let $f \in C(X)$ be given by $f(z) = g(z)$ for all $z \in X \cap \bar{B}_X(y, r)$ and $f(z) = 0$ for all $z \in X \setminus \bar{B}_X(y, r)$. It follows from Proposition 2.5.3 that $f \in R(X)$ and clearly f vanishes on a neighbourhood of x . Thus $J_x \not\subseteq M_y$ (where these are the ideals in $R(X)$). It follows that y is a point of continuity for $R(X)$ and so the proof is complete. $\square$

The following estimates on derivatives are standard. See, for example [52, Lemma 4.4]. (This result also appears in [37] but with some typographical errors.) We include a short proof for the convenience of the reader.

Lemma 4.5.9. *Let $A = ((a_n, r_n))_{n=1}^\infty$ be an abstract Swiss cheese, and let $z \in \mathbb{C}$. For each $n \in \mathbb{N}$, let d_n denote the distance from $B(a_n, r_n)$ to z . Let $d_0 = r_0 - |z - a_0|$. Suppose that $d_n > 0$ for all $n \in \mathbb{N}_0$. Then $z \in X_A$ and, for all $f \in R_0(X_A)$ and $k \in \mathbb{N}_0$, we have*

$$|f^{(k)}(z)| \leq k! \sum_{j=0}^{\infty} \frac{r_j}{d_j^{k+1}} |f|_{X_A}.$$

Proof. Set $X := X_A$, and let $U := \mathbb{C} \setminus \bar{B}(a_0, r_0)$. For each $n \in \mathbb{N}$, let γ_n denote the path describing once clockwise the circle $\partial B(a_n, r_n)$, and let γ_0 denote the path describing once anticlockwise the circle $\partial B(a_0, r_0)$. Let $f \in R_0(X)$, let S_f be the set of poles of f , and let $k \in \mathbb{N}_0$. Then there exist $n_1, \dots, n_m \in \mathbb{N}$ such that $S_f \subseteq U \cup \bigcup_{j=1}^m B(a_{n_j}, r_{n_j})$. Let Γ be the set $\gamma_0^* \cup \bigcup_{j=1}^m \gamma_{n_j}^*$. Note that $z \notin \Gamma$.

By Cauchy's integral formula we have

$$f^{(k)}(z) = \frac{k!}{2\pi i} \left(\int_{\gamma_0} \frac{f(\zeta)}{(\zeta - z)^{k+1}} d\zeta + \sum_{j=1}^m \int_{\gamma_{n_j}} \frac{f(\zeta)}{(\zeta - z)^{k+1}} d\zeta \right).$$

We have $|\zeta - z| \geq d_0$ for all $\zeta \in \gamma_0^*$, and $|\zeta - z| \geq d_{n_j}$ for all $\zeta \in \gamma_{n_j}^*$ ($j = 1, \dots, m$). Moreover, for each $n \in \mathbb{N}_0$, we have $\Lambda(\gamma_n) = 2\pi r_n$. Thus, applying Lemma 4.2.3, we have

$$|f^{(k)}(z)| \leq k! |f|_X \left(\frac{r_0}{d_0^{k+1}} + \sum_{j=1}^m \frac{r_{n_j}}{d_{n_j}^{k+1}} \right) \leq k! |f|_X \sum_{n=0}^{\infty} \frac{r_n}{d_n^{k+1}},$$

and this gives the desired result. $\square$

We now give the details of the construction.

Lemma 4.5.10. *Let $0 < r < 1$ be given, let C_r denote the circle of radius r centred at 0, and let $\varepsilon > 0$. Then there exists a classical abstract Swiss cheese $A = ((a_n, r_n))$ such that*

$$(a) \quad \rho(A) < \varepsilon, \text{ int}(X_A) = \emptyset, \text{ and } C_r \subseteq X_A,$$

(b) *there is a dense open subset U of X_A such that $X_A \setminus U$ has area zero and each point $z \in U$ is a point of continuity for $R(X_A)$,*

(c) *for each $f \in R(X)$, we have $f|_{C_r} \in D^{(\infty)}(C_r)$ and $\sum_{k=1}^{\infty} |f^{(k)}|_{C_r}^{-1/k} = \infty$.*

Proof. Our abstract Swiss cheese A will be obtained by combining a certain pair of sequences (A_n) , (B_n) of abstract Swiss cheeses in a suitable way. We first construct the sequences (A_n) , (B_n) .

Choose a positive integer n_0 large enough so that $r + 2^{1-n_0} < 1$ and $r - 2^{1-n_0} > 0$. As in [52], choose a sequence (γ_n) of positive real numbers such that, for each $n, k \in \mathbb{N}$, we have

$$\gamma_n \leq \frac{(2^{1-n_0-n})^{k+1} (\log(k+3))^k}{2^n},$$

and such that $\sum_{n=1}^{\infty} \gamma_n < \varepsilon$. Thus, for each $k \in \mathbb{N}$,

$$\sum_{n=1}^{\infty} \frac{\gamma_n}{(2^{1-n_0-n})^{k+1}} \leq (\log(k+3))^k.$$

Let $A_1 = ((a_n^{(1)}, r_n^{(1)}))$ be the classical abstract Swiss cheese obtained from Corollary 2.2.12 with $\lambda_0 = r - 2^{1-n_0}$, $a = 0$, and $\varepsilon = \gamma_1/2$. Let $B_1 = ((b_n^{(1)}, r_n^{(1)}))$ be the classical abstract Swiss cheese obtained from Proposition 2.6.5 with $a = 0$, $\lambda_0 = 1$, $\lambda_1 = r + 2^{1-n_0}$, and $\varepsilon = \gamma_1/2$.

For each $k \in \mathbb{N}$ with $n \geq 2$: let $A_k = ((a_n^{(k)}, r_n^{(k)}))$ be the classical abstract Swiss cheese obtained from Proposition 2.6.5 with $\lambda_0 = r - 2^{2-n_0-k}$, $\lambda_1 = r - 2^{3-n_0-k}$, $a = 0$, and $\varepsilon = \gamma_k/2$; let $B_k = ((b_n^{(k)}, r_n^{(k)}))$ be the classical abstract Swiss cheese obtained from Proposition 2.6.5 with $a = 0$, $\lambda_0 = r + 2^{3-n_0-k}$, $\lambda_1 = r + 2^{2-n_0-k}$, and $\varepsilon = \gamma_k/2$.

Note that, since $R(X_{A_n})$ and $R(X_{B_n})$ are regular for all $n \in \mathbb{N}$, it follows that $\text{int}(X_{A_n}) = \emptyset$ and $\text{int}(X_{B_n}) = \emptyset$ for all $n \in \mathbb{N}$.

Let $a_0 = 0$ and $r_0 = 1$ and let $((a_n, r_n))_{n=1}^{\infty}$ be an enumeration (without repeats) of the set

$$\bigcup_{\substack{m, n \in \mathbb{N} \\ n \geq 2}} \{(a_n^{(m)}, r_n^{(m)}), (b_n^{(m)}, s_n^{(m)})\}.$$

Then $A = ((a_n, r_n))_{n=0}^{\infty}$ is an abstract Swiss cheese with $C_r \subseteq X_A$. It is not hard to see that A is classical and $\rho(A) < \varepsilon$. It is clear that $\text{int}(X_A) = \emptyset$. Also, for each $n \in \mathbb{N}$, let

$$E_n := \partial B(a_0^{(n)}, r_0^{(n)}) \cup \partial B(a_1^{(n)}, r_1^{(n)}) \cup \partial B(b_0^{(n)}, s_0^{(n)}) \cup \partial B(b_1^{(n)}, s_1^{(n)}),$$

and set $E := C_r \cup \bigcup_{n=1}^{\infty} E_n$. Set $X := X_A$ and $U := X \setminus E$. It is easy to see that E is a closed set and that U is a dense open subset of X . Moreover, E has area zero.

We *claim* that each point $x \in U$ is a point of continuity for $R(X)$. Let $x \in U$. Then there exists a unique $n \in \mathbb{N}$ such that either $x \in X_{A_n}$ or $x \in X_{B_n}$. If $x \in X_{A_n}$, set $Y = X_{A_n}$, and if $x \in X_{B_n}$ then set $Y = X_{B_n}$. By our construction $R(Y)$ is regular and it is not hard to see that $x \in \text{int}_X Y$. Thus, by Lemma 4.5.8, x is a point of continuity for $R(X)$. This proves the claim.

It remains to show that (c) holds. We first consider functions in $R_0(X)$. Let $z \in C_r$ and let $f \in R_0(X)$. For each $n \in \mathbb{N}$, let d_n the distance from $\bar{B}(a_n, r_n)$ to C_r , and let $d_0 = 1 - r$. Since each A_n and each B_n are classical, and since $C_r \cap X_{A_n} = \emptyset$ and $C_r \cap X_{B_n} = \emptyset$ for all $n \in \mathbb{N}$, it follows that $d_n > 0$ for all $n \in \mathbb{N}_0$. By Lemma 4.5.9, for each $k \in \mathbb{N}_0$, we have

$$|f^{(k)}(z)| \leq k! \sum_{j=0}^{\infty} \frac{r_j}{d_j^{k+1}} |f|_X.$$

Fix $m \in \mathbb{N}$. Then there exists a unique $n \in \mathbb{N}$ such that there exists $\ell \in \mathbb{N}$ with $(a_m, r_m) = (a_\ell^{(n)}, r_\ell^{(n)})$ or there exists $\ell \in \mathbb{N}$ with $(a_m, r_m) = (b_\ell^{(n)}, s_\ell^{(n)})$. In either case, since A_n and B_n are classical, we have $d_m > 2^{3-n_0-m}$. Thus

$$k! \sum_{j=0}^{\infty} \frac{r_j}{d_j^{k+1}} |f|_X \leq k! |f|_X \left(\frac{1}{d_0^{k+1}} + \sum_{j=1}^{\infty} \frac{\gamma_j}{(2^{1-n_0-j})^{k+1}} \right).$$

for each $k \in \mathbb{N}_0$. It follows that, for each $k \in \mathbb{N}_0$, we have

$$|f^{(k)}|_{C_r} \leq k! |f|_X \left(\frac{1}{d_0^{k+1}} + (\log(k+3))^k \right).$$

From this we deduce that, for each $f \in R(X)$ (not necessarily in $R_0(X)$), we have $f|_{C_r} \in D^{(\infty)}(C_r)$ and the same estimates hold. (For example, note that C_r is uniformly regular, and apply Lemma 3.3.21.)

Now let $f \in R(X)$. As in [52], choose $N \in \mathbb{N}$ so that $(\log(k+3))^k \geq 1/d_0^{k+1}$ for all $k \in \mathbb{N}$ with $k \geq N$. Then we have

$$\sum_{j=1}^{\infty} \frac{1}{|f^{(j)}|_{C_r}^{1/j}} \geq \sum_{j=N}^{\infty} \frac{1}{(2|f|_X)^{1/j} j \log(j+3)} = \infty,$$

and so (c) holds. This completes the proof. $\square$

We are now ready to prove Theorem 4.5.4.

Proof of Theorem 4.5.4. Apply Lemma 4.5.10 with $r = 1/2$ and $\varepsilon = 1$ to obtain a classical abstract Swiss cheese A which satisfies properties (a)–(c) described in the statement of the lemma, and let $X := X_A$. Then, for each $f \in R(X)$, we have $f|_{C_r} \in D^{(\infty)}(C_r)$ and

$$\sum_{n=1}^{\infty} |f^{(n)}|_{C_r}^{-1/n} = \infty.$$

So, by Corollary 4.2.8, if $f \in R(X)$ and $f^{(k)}(z) = 0$, for some $z \in C_r$ and all $k \in \mathbb{N}_0$, then f is identically zero on C_r . It follows that no point of C_r can be a point of continuity for $R(X)$ and therefore $R(X)$ is not regular. It remains to see that $R(X)$ has no non-trivial Jensen measures on X . By Lemma 4.5.10(b) there is a dense open subset U of X for which every point $z \in U$ is a point of continuity for $R(X)$ and such that $X \setminus U$ has area 0. Set $E := X \setminus U$. By Lemma 4.5.2, every point of U is also an R -point. Let $x \in C_r$ and let μ be a Jensen measure for ε_x . Then since $x \notin U$, by Lemma 4.5.7, $\text{supp } \mu \cap U = \emptyset$ and so $\text{supp } \mu \subseteq E$. Since $\text{int}(X) = \emptyset$ and the area of E is 0, it follows from Corollary 4.5.6 that μ must be trivial. This completes the proof. $\square$

Remark 4.5.11. In [37], Feinstein argued that $R(X)$ admits no non-trivial Jensen measures supported on the interval $[-1/2, 1/2]$ by appealing to the theory of Jensen (fine) interior discussed in [33] and [56]. See also [55].

4.6 Open problems and discussion

We now note some interesting open problems related to the content of this chapter. In this section, let X be a semi-rectifiable compact plane set, let $\mathcal{F}$ be an effective collection of paths in X , and let $M = (M_n)_{n=0}^\infty$ be an algebra sequence.

Question 4.6.1. Is the condition (4.1) necessary in Theorem 4.2.7?

Question 4.6.2. Let X be a perfect compact plane set such that $\text{int } X$ is dense in X . Does it follow that $D(X, M)$ is quasianalytic if and only if the sequence $(M_n^{1/2})_{n=0}^\infty$ satisfies (4.1)?

We are also interested in the connection between $\mathcal{F}$ -quasianalyticity and $\mathbf{m}(\mathcal{F})$ -quasianalyticity.

Question 4.6.3. Let A be a subalgebra of $D_{\mathcal{F}}^{(\infty)}(X)$. Suppose that A is $\mathcal{F}$ -quasianalytic. Does it follow that A is $\mathbf{m}(\mathcal{F})$ -quasianalytic?

It is easy to see that $\mathbf{m}(\mathcal{F})$ -quasianalyticity implies $\mathcal{F}$ -quasianalyticity, so an affirmative answer to this question would imply that $\mathcal{F}$ -quasianalyticity is equivalent to $\mathbf{m}(\mathcal{F})$ -quasianalyticity.

Question 4.6.4. Let A be a subalgebra of $D_{\mathcal{F}}^{(\infty)}(X)$ and let $z_0 \in X$. Suppose that A is quasianalytic. Is $Q_{A, \mathcal{F}}(z_0)$ the minimum element of $\mathcal{Q}_{\mathcal{F}}(z_0)$?

This is certainly the case in some simple examples. For instance, in Example 4.4.11, the minimal elements of $\mathcal{Q}_{\mathcal{F}}(z_0)$ consists of the horizontal line segment in X of maximum length containing z_0 . If A is an $\mathcal{F}$ -quasianalytic subalgebra of $D_{\mathcal{F}}^{(\infty)}(X)$ then the sets of A -quasianalyticity also coincide with the horizontal line segments of maximum length.

Our final problem relates to the content of Section 4.5, and specifically to the construction used in the proof of Theorem 4.5.4 (Lemma 4.5.10).

Question 4.6.5. Is the uniform algebra $R(X_A)$ constructed in Theorem 4.5.4 antisymmetric? If not, can the construction be modified to yield an example with the same properties and such that $R(X_A)$ is antisymmetric?

We refer the reader to [16, 54] for the relevant definitions.

It seems hard, in general, to determine whether $R(X)$ (for a connected compact plane set X with $\text{int } X = \emptyset$) is antisymmetric. In [96], Steen gave an example of a Swiss cheese set X such that $R(X)$ is non-trivial and is not antisymmetric. We do not know whether the constructions in, for example, [85, 99] yield a compact plane set X such that $R(X)$ is antisymmetric.

CHAPTER 5

Extensions of uniform algebras

In this chapter we consider various extensions of a uniform algebra. Let $\mathfrak{A}$ be a Banach algebra. We say that a Banach algebra $\mathfrak{B}$ is a *Banach algebra extension of $\mathfrak{A}$* if there exists an isometric algebra homomorphism $\theta : \mathfrak{B} \rightarrow \mathfrak{A}$. If $\mathfrak{A}$ is a uniform algebra, then we say that a Banach algebra extension $\mathfrak{B}$ of $\mathfrak{A}$ is a *uniform algebra extension of $\mathfrak{A}$* if $\mathfrak{B}$ is also a uniform algebra and θ is unital. We are interested in those uniform algebra extensions which preserve certain properties of the original algebra. In particular, we seek extensions that preserve properties such as non-triviality and naturality.

Many of the extensions that we consider are constructed by adjoining roots of monic polynomials over a given algebra. In [7], Arens and Hoffmann described a construction for obtaining a Banach algebra extension of a commutative Banach algebra A by adjoining roots of monic polynomials with coefficients in A . Their construction has been studied and adapted by several authors. We refer the reader to [31] for a survey of various methods for constructing Banach algebra extensions.

In his thesis [24], Cole constructed a counter-example to the peak point conjecture. This example was obtained by constructing a uniform algebra extension B , of a certain uniform algebra A , such that the set $\{f^2 : f \in B\}$ is dense in B . Similar constructions, which we now call Cole extensions, have been used in [37, 38], for example, to construct various other examples of uniform algebras. Detailed accounts of the properties of Cole extensions are contained in [32] and [61]. The connections between Cole extensions and various other constructions can also be found in [32] (see also [31]).

In this chapter, we discuss Cole extensions and a larger class of uniform algebra extensions, which includes Cole extensions. In particular, we describe the structure of a Cole extension, relative to certain projections, and describe some cases where Cole extensions are equivalent. The theory developed in the present chapter will

be used in Chapter 6. Some of the proofs in this chapter will use results from operator algebra theory. A brief introduction to operator algebras is provided in Appendix A for the convenience of the reader.

The first section gives contains a brief introduction to direct and inverse limits. The second section contains some elementary facts about Banach algebras and uniform algebras. In particular, we prove a general result about the Shilov boundary for uniform algebra extensions. We then give the formal definition of (systems of) Cole extensions, and state some elementary results from the literature. In Section 5.4 we give a new proof of a result about the Shilov boundary for Cole extensions. We then study the structure of simple Cole extensions (see Section 5.3) relative to certain projections. Finally, we describe a more general class of uniform algebra extension, originally defined by Cole and Feinstein (in a private communication [23]), and discuss the properties of such extensions. We conclude the section with some open problems and a short discussion.

5.1 Direct limits and inverse limits

We begin with a discussion of direct and inverse limits in various categories. We refer the reader to [81] for an introduction to the language of category theory. In this section we mostly consider four categories: the category of normed algebras; the category of Banach algebras; the category of uniform algebras; and the category of compact spaces. In the latter, the morphisms are simply continuous functions, and isomorphisms are homeomorphisms. In the other categories, the morphisms are given by continuous algebra homomorphisms. Unless otherwise stated, all results in this section are standard.

Throughout this section, I will be a non-empty directed set. That is, a set with a relation $\leq$ such that, for each $i, j, k \in I$, we have $i \leq i$, $i \leq j \leq k$ implies $i \leq k$, and there exists $\ell \in I$ such that $i \leq \ell$ and $j \leq \ell$. Also, in this section, i, j, k will be elements of I . We also use the notation $(\theta_{i,j})_{i \leq j}$ for the family $\{\theta_{i,j} : i, j \in I, i \leq j\}$.

Definition 5.1.1. Let $\mathcal{C}$ be a category. A *direct system* is a pair $((O_i)_{i \in I}, (\theta_{i,j})_{i \leq j})$ such that O_i is an object in $\mathcal{C}$, and, for each $i \leq j$, $\theta_{i,j} : O_i \rightarrow O_j$ is a morphism with $\theta_{i,i} = \text{id}_{O_i}$ and $\theta_{i,k} = \theta_{j,k} \circ \theta_{i,j}$ for all $i \leq j \leq k$.

An *inverse system* is a pair $((O_i)_{i \in I}, (\theta_{i,j})_{i \leq j})$ such that O_i is an object in $\mathcal{C}$, and, for each $i \leq j$, $\theta_{i,j} : O_j \rightarrow O_i$ is a morphism with $\theta_{i,i} = \text{id}_{O_i}$ and $\theta_{i,k} = \theta_{i,j} \circ \theta_{j,k}$ for all $i \leq j \leq k$.

In either case, we call the morphisms $\theta_{i,j}$ the *connecting maps* of the system.

We shall often consider direct systems in the category of normed algebras, Banach algebras, and uniform algebras (with continuous algebra homomorphisms as morphisms). We consider inverse systems in the category of compact spaces (with continuous maps as morphisms).

We now define a direct limit of a direct system. Discussions of direct limits can be found in, for example, [78, Section III.10], or [90, pp. 42–43]. An explicit construction of a direct limit of a direct system of normed algebras—where the connecting maps are contractive algebra homomorphisms—is given in the latter. A more comprehensive discussion of direct limits of topological algebras is given in Chapters IV and V of [82].

Definition 5.1.2. Let $\mathcal{C}$ be a category. Let $((O_i)_{i \in I}, (\theta_{i,j})_{i \leq j})$ be a direct system in $\mathcal{C}$. The *direct limit* of the system $((O_i)_{i \in I}, (\theta_{i,j})_{i \leq j})$ is a pair $(O, (\theta_i)_{i \in I})$ where O is an object in $\mathcal{C}$ and, for each i , $\theta_i : O_i \rightarrow O$ is a morphism with $\theta_i = \theta_j \circ \theta_{i,j}$ for all $j \geq i$, such that $(O, (\theta_i)_{i \in I})$ satisfies the following universal property: *for each object $C \in \mathcal{C}$ such that there exists morphisms $\psi_i : O_i \rightarrow C$ with $\psi_i = \psi_j \circ \theta_{i,j}$ for all $i \leq j$, there exists a unique morphism $\psi : O \rightarrow C$ such that $\psi_i = \psi \circ \theta_i$ for all $i \in I$.*

The map ψ , from the above, is the unique map which makes the diagram in Figure 5.1(a) commute.

We also define an inverse limit of an inverse system. As above, a discussion of inverse limits can be found in [78] (see also [90, pp. 43–45]). An explicit construction of the inverse limit of a inverse system of sets is given in [27, Section 1.1].

Definition 5.1.3. Let $\mathcal{C}$ be a category. Let $((O_i)_{i \in I}, (\theta_{i,j})_{i \leq j})$ be an inverse system. The *inverse limit* of the system $((O_i)_{i \in I}, (\theta_{i,j})_{i \leq j})$ is a pair $(O, (\theta_i)_{i \in I})$ such that O is an object in $\mathcal{C}$ and, for each i , $\theta_i : O \rightarrow O_i$ is a morphism with $\theta_j = \theta_i \circ \theta_{i,j}$ for all $j \geq i$, such that $(O, (\theta_i)_{i \in I})$ satisfies the following universal property: *for each object $C \in \mathcal{C}$ such that there exists morphisms $\psi_i : C \rightarrow O_i$ with $\psi_j = \psi_i \circ \theta_{i,j}$ for all $i \leq j$, there exists a unique morphism $\psi : C \rightarrow O$ such that $\psi_i = \theta_i \circ \psi$ for all $i \in I$.*

The map ψ , from the above, is the unique map which makes the diagram in Figure 5.1(b) commute.

It is standard that direct limits and inverse limits, where they exist, are unique up to isomorphism (in the category in which they are defined).

We now examine some special cases of direct and inverse limits which will be of use to us. The first summarises the discussion from [90, Chapter 1].

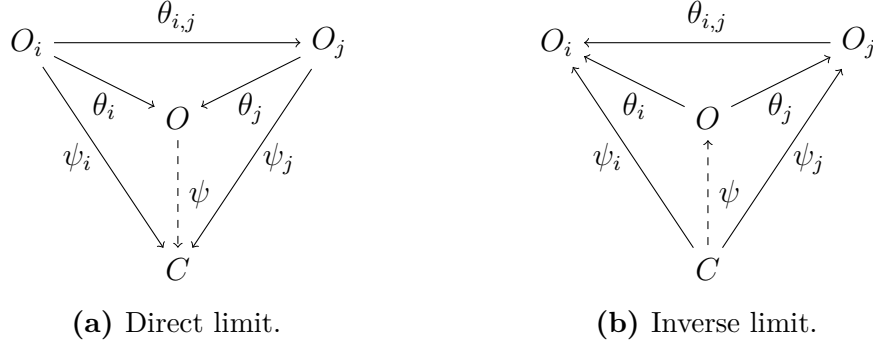


Figure 5.1: Commutative diagrams illustrating the universal property for direct limits and inverse limits.

Proposition 5.1.4. *Let $S = ((\mathfrak{A}_i)_{i \in I}, (\theta_{i,j})_{i \leq j})$ be a direct system in the category of normed algebras. Suppose that $\limsup_{i \in I} \|\theta_{i,j}\| < \infty$. Then the direct limit of S exists.*

Note that, if each $\mathfrak{A}_i$ ($i \in I$) in the above is a Banach algebra, then the uniform boundedness principle shows that the direct limit of S exists if, for each $i \in I$, $\limsup_{j \in I} \|\theta_{i,j}(a)\|_j$ for all $a \in \mathfrak{A}_i$ (where $\|\cdot\|_j$ is the norm on $\mathfrak{A}_j$). In particular, we see that, if S is as in Proposition 5.1.4, and all the connecting maps in S are isometric, then the direct limit of S exists and the norm $\|\theta_i(a)\| = \|a\|_i$ for all $i \in I$ and $a \in \mathfrak{A}_i$. Thus, in this case, the homomorphisms θ_i associated with the direct limit are also isometric.

We also require the following. The statement below is part of a more general categorical statement about adjoints and limits. See, for example, [81, Chapter V].

Proposition 5.1.5. *Let $S = ((X_i)_{i \in I}, (\pi_{i,j})_{i \leq j})$ be an inverse system in the category of compact spaces. Then the system $T = ((C(X_i))_{i \in I}, (\pi_{i,j}^*)_{i \leq j})$ is a direct system in the category of uniform algebras. If $(X, (\pi_i)_{i \in I})$ is the inverse limit of S , then $(C(X), (\pi_i^*)_{i \in I})$ is the direct limit of T .*

Note that we may consider an direct system of (non-trivial) uniform algebras A_i on the X_i in the same way. Also note that the connecting maps $\pi_{i,j}^*$ in T (in the above) are isometric algebra homomorphisms if the connecting maps $\pi_{i,j}$ in S are surjective.

In the remainder of this section we prove the following lemma which will be useful throughout this chapter. This lemma is effectively contained in the proof of [32, Theorem 1.4.2.1] or [61, Theorem 3.1.19], and is well known. We include a proof for the convenience of the reader.

Lemma 5.1.6. *Let $S = ((\mathfrak{A}_i)_{i \in I}, (\theta_{i,j})_{i \leq j})$ be a direct system in the category of Banach algebras and suppose that the connecting maps $\theta_{i,j}$ are isometric for all $i \leq j$. Let $(\mathfrak{A}, (\theta_i)_{i \in I})$ be the direct limit of S . Suppose that I is totally ordered and that, for each $i \leq j$, $T_{i,j} : \mathfrak{A}_j \rightarrow \mathfrak{A}_i$ is a surjective, contractive linear map such that $T_{i,j} \circ \theta_{i,j} = \text{id}_{\mathfrak{A}_i}$ and, for each $k \geq j$, we have $T_{i,j} \circ T_{j,k} = T_{i,k}$. Then:*

- (a) *$((\mathfrak{A}_i)_{i \in I}, (T_{i,j})_{i \leq j})$ is an inverse system of Banach spaces (with contractive linear maps as morphisms);*
- (b) *for each $i \in I$, there exists a surjective, contractive linear map $T_i : \mathfrak{A} \rightarrow \mathfrak{A}_i$ such that $T_i \circ \theta_i = \text{id}_{\mathfrak{A}_i}$ and $T_i = T_j \circ \theta_{i,j}$ ($i \leq j$).*

Proof. (a) This clear from our choice of the $T_{i,j}$.

(b) We consider the inverse system S in the category of Banach spaces with contractive linear maps as morphisms. (Clearly the connecting maps are isometric linear maps.) Fix $\ell \in I$. We define the maps φ_i ($i \in I$) as follows: for $i < \ell$, set $\varphi_i := \theta_{i,\ell}$; for $i = \ell$ set $\varphi_\ell := \text{id}_{\mathfrak{A}_\ell}$; for $i > \ell$ set $\varphi_i := T_{i,\ell}$. By the assumptions in the lemma, the maps φ_i satisfy $\varphi_i = \varphi_j \circ \theta_{i,j}$ for all $i \leq j$. Thus by the universal property of direct limits, there exists a unique, contractive linear map $T : \mathfrak{A} \rightarrow \mathfrak{A}_\ell$ such that $T_i = T \circ \theta_i$ for all $i \in I$. The remaining statements are now clear, and so (b) is proved. $\square$

For a similar result, see [97, Lemma 19.3].

5.2 Some results for Banach algebras

In this section we give some general results about Banach algebras which we will require later.

Let $\mathfrak{X}, \mathfrak{Y}$ be normed spaces, let $E \subseteq \mathfrak{A}$ be a linear subspace, and let $T : \mathfrak{X} \rightarrow \mathfrak{Y}$ be a linear map. We say that T is *bounded below on E* if there exists $C > 0$ such that $\|T(a)\| \geq C\|a\|$ for all $a \in E$. Note that if T is bounded below on E , then $T|_E$ is an open map from E into $\mathfrak{Y}$.

The following follows immediately from Proposition 2.23 and Corollary 3.43 from [5].

Lemma 5.2.1. *Let $\mathfrak{A}$ be a Banach algebra, let $E \subseteq \mathfrak{A}$ be a closed linear subspace, and let $b \in \mathfrak{A}$. Let $T \in \mathcal{B}(\mathfrak{A})$ given by either*

$$T(a) = ba \quad \text{or} \quad T(a) = ab \quad (a \in \mathfrak{A}).$$

If T is bounded below on E then $T(E)$ is a closed linear subspace of $\mathfrak{A}$. Suppose further that b is not a zero divisor in $\mathfrak{A}$. Then $T(E)$ is a closed linear subspace of $\mathfrak{A}$ if and only if T is bounded below on E .

We extend this result to give the following corollary. This result is probably well known but we give details of the proof.

Corollary 5.2.2. *Let $\mathfrak{A}$ be a Banach algebra, let $E \subseteq \mathfrak{A}$ be a non-trivial closed linear subspace, let P be a continuous projection on $\mathfrak{A}$ with $P(\mathfrak{A}) = E$, and let $b \in \mathfrak{A}$ such that b is not a zero-divisor. Let $T \in \mathcal{B}(\mathfrak{A})$ given by either*

$$T(a) = ba \quad \text{or} \quad T(a) = ab \quad (a \in \mathfrak{A}).$$

Fix $m \in \mathbb{N}$ and, for each $k \in \{0, \dots, m\}$, let $E_k = \sum_{j=0}^k T^j(E)$. Suppose that $(P \circ T)(a) = 0$ for all $a \in E_{m-1}$, that T is bounded below on E_m , and that $T^j(E) \cap T^k(E) = \{0\}$ for all $j, k \in \{0, \dots, m\}$ with $j \neq k$. Then $T^k(E)$ and E_k are closed in $\mathfrak{A}$ for each $k \in \{0, 1, \dots, m\}$.

Proof. Let $C > 0$ be such that $\|T(a)\| \geq C\|a\|$ for all $a \in E_m$. First note that

$$E = E_0 \subseteq E_1 \subseteq \dots \subseteq E_m,$$

so that T is bounded below on E_k for each $k \in \{0, \dots, m\}$. Thus, for each $a \in E$ and each $k \in \{0, \dots, m\}$, we have

$$\|T^k(a)\| \geq C^k \|a\|.$$

By Lemma 5.2.1, $T^k(E)$ is closed in $\mathfrak{A}$ for $k = 0, \dots, m$. We now prove that E_k is closed in $\mathfrak{A}$ by induction on k . If $k = 0$ then $E_0 = E$ which is closed in $\mathfrak{A}$, so there is nothing to prove.

Let $k \in \{1, \dots, m\}$ and assume that E_{k-1} is closed in $\mathfrak{A}$. By Lemma 5.2.1, $K = T(E_{k-1})$ is a closed linear subspace of E_k . Since $(P \circ T)(a) = 0$ for all $a \in E_{k-1}$, we have $P(x + T(y)) = P(x) = x$ for all $x \in E$ and $y \in E_{k-1}$, and it follows that $\|x\| \leq \|P\|\|x + T(y)\|$ for all $x \in E$ and $y \in E_{k-1}$. Let (x_n) be a sequence in E and let (y_n) be a sequence in E_{k-1} . Suppose that $(x_n + T(y_n))$ converges to some point $z \in \mathfrak{A}$ as $n \rightarrow \infty$. Then $P(x_n + T(y_n))$ converges to $P(z) \in E$ as $n \rightarrow \infty$. Thus $(T(y_n)) \rightarrow z - P(z)$ as $n \rightarrow \infty$. Since K is closed, $z - P(z) \in K$. Thus $z \in E_k$ and hence E_{k-1} is closed in $\mathfrak{A}$.

It now follows by induction that E_k is closed for each $k \in \{0, \dots, m\}$ and this completes the proof. $\square$

5.2.1 Uniform algebras

We now discuss some necessary results which are specific to uniform algebras. These results are probably well known, but we provide details for the convenience of the reader. For a similar result, see [71, Theorem 3.5.4].

Lemma 5.2.3. *Let X be a compact space, let A be a uniform algebra on X , and let $f \in C(X)$ such that, for all $g \in A \setminus \{0\}$, we have $fg \neq 0$. The following are equivalent:*

- (a) *the map $g \mapsto fg$ from A into $C(X)$ is bounded below;*
- (b) *f has no zeros on the Shilov boundary of A .*

Proof. (a) $\implies$ (b) By scaling, we may assume that $|f|_X = 1$. Suppose that f has a zero on $\Gamma(A)$. Let $x_0 \in \Gamma(A)$ be such that $f(x_0) = 0$. For each $n \in \mathbb{N}$, let V_n be an open neighbourhood of x_0 such that $|f(y)| < 1/4^n$ for all $y \in V_n$, and let $x_n \in V_n$ be a strong boundary point for A . By Proposition 1.6.4, for each $n \in \mathbb{N}$, there exists $g_n \in A$ with $|g_n|_X \leq 1$ such that $|g_n(x_n)| > 3/4$ and $|g_n(y)| < 1/4$ for all $y \in X \setminus V_n$. For each $n \in \mathbb{N}$, set $h_n = g_n^n$ so that $|h_n|_X \leq 1$, $|h_n(x_n)| > (3/4)^n$ and $|h_n(y)| < 1/4^n$ for all $y \in X \setminus V_n$.

Let $n \in \mathbb{N}$. For each $y \in V_n$ we have

$$|fh_n(y)| \leq |f(y)| < \frac{1}{4^n},$$

and for each $y \in X \setminus V_n$, we have

$$|fh_n(y)| \leq |h_n(y)| < \frac{1}{4^n}.$$

It follows that $|fh_n|_X \leq 1/4^n$. But $|h_n(x_n)| > (3/4)^n$ so that $|h_n|_X \geq 3^n|fh_n|$. This holds for all $n \in \mathbb{N}$, and it follows that the map $g \mapsto fg$ is not bounded below on A .

(b) $\implies$ (a) Suppose that f has no zeros on the $\Gamma(A)$. Set $\delta = \inf_{x \in \Gamma(A)} |f(x)|$. Then $\delta > 0$ and, for each $x \in \Gamma(A)$, we have $|f(x)g(x)| \geq \delta|g(x)|$. It follows that $\delta|g(x)| \leq |fg|_{\Gamma(A)}$ for each $x \in \Gamma(A)$ and hence $\delta|g|_{\Gamma(A)} \leq |fg|_{\Gamma(A)}$. But $|fg|_{\Gamma(A)} \leq |fg|_X$ and $|g|_{\Gamma(A)} = |g|_X$. This completes the proof. $\square$

We also require the following well-known facts. We include some the details of the proofs for the convenience of the reader.

Lemma 5.2.4. *Let A be a uniform algebra on X , let $f \in C(X) \setminus \{0\}$, let $g \in A$, and let $n \in \mathbb{N}$.*

- (a) We have $f^n g = 0$ if and only if $fg = 0$.
- (b) If the map $L_f^n : A \rightarrow C(X) : h \mapsto f^n h$ is bounded below then the map $L_f : A \rightarrow C(X) : h \mapsto fh$ is bounded below.
- (c) If $f \in A$ and L_f is bounded below then L_f^n is bounded below.

Proof. (a) This is clear.

(b) Now suppose that L_f^n is bounded below so that there exists $d > 0$ such that $|f^n h|_X \geq d|h|_X$ for all $h \in A$. Fix $h \in A$. Then we have

$$d|h|_X \leq |f^n h|_X \leq |f^{n-1}|_X |fh|_X.$$

Since $f \neq 0$, setting $d' := d/|f^{n-1}|_X$, we have $d'|h|_X \leq |fh|_X$.

(c) Suppose that the map L_f is bounded below. Then there exists $c > 0$ such that $|fh|_X \geq c|h|_X$ for all $h \in A$. Fix $g \in A$. Then

$$c^n |g|_X \leq c^{n-1} |fg|_X \leq \cdots \leq |f^n g|_X.$$

Thus L_f^n is bounded below. □

We now prove some results related to the Shilov boundary for uniform algebra extensions. We shall make repeated use of the following fact, which is also used in [79] (see also [27, Section. 4.3]).

Lemma 5.2.5. *Let A be a uniform algebra on a compact space X and let $x \in X$. Then $x \in \Gamma(A)$ if and only if, for each open neighbourhood U of x , there exist $f \in A$ and $y \in U$ such that $|f(y)| = |f|_X$ and $|f|_{X \setminus U} < |f|_X$.*

The following lemma is well known (see [79]), but we include the details for the convenience of the reader.

Lemma 5.2.6. *Let X and Y be compact spaces, let A be a uniform algebra on X , let B be a uniform algebra on Y , and let $\pi : Y \rightarrow X$ be a continuous map. Suppose that π^* is isometric. Then $\Gamma(A) \subseteq \pi(\Gamma(B))$.*

Proof. Suppose not so that $\Gamma(A) \not\subseteq \pi(\Gamma(B))$. Then there exists $x \in \Gamma(A) \setminus \pi(\Gamma(B))$ and an open neighbourhood U of x in X such that $U \cap \pi(\Gamma(B)) = \emptyset$. Since $x \in \Gamma(A)$, by Lemma 5.2.5, there exist $f \in A$ and $x' \in U$ such that $|f(x')| = |f|_X = 1$ and $|f|_{X \setminus U} < 1$. Since π^* is isometric, we have $|f|_X = |\pi^*(f)|_Y$. But, for each $y \in \Gamma(B)$, we have $|\pi^*(f)(y)| = |f(\pi(y))| < 1 = |\pi^*(f)|_Y$, which is a contradiction. Thus $\Gamma(A) \subseteq \pi(\Gamma(B))$. □

Our main theorem for this section is the following, which generalises several results from Cole extension theory. The remainder of this section is dedicated to proving this theorem.

Theorem 5.2.7. *Let A be a uniform algebra on Φ_A , let B be a uniform algebra on Φ_B , and let $\theta : A \rightarrow B$ be an isometric, unital algebra homomorphism. Set $\tau := \theta^*|_{\Phi_B} : \Phi_B \rightarrow \Phi_A$. Suppose that $\tau(\Gamma(B)) \subseteq \Gamma(A)$ and that, for each $\psi \in \Gamma(A)$, $B|_{\tau^{-1}(\{\psi\})}$ is uniformly dense in $C(\tau^{-1}(\{\psi\}))$. Then $\Gamma(B) = \tau^{-1}(\Gamma(A))$.*

To prove the theorem we shall require the following lemma. We include the details for the convenience of the reader.

Lemma 5.2.8. *Let X be a compact space, let $x \in X$, let U be an open neighbourhood of x in X , and let A be a dense linear subspace of $C(X)$. Then there exists $f \in A$ such that $|f|_X \leq 1$, $|f(x)| > 3/4$, and $|f|_{X \setminus U} < 1/4$.*

Proof. There exists $g_0 \in C(X)$ such that $g_0(x) = 1$ and $|g(y)| < 1$ for all $y \in X \setminus U$. Let $n_0 \in \mathbb{N}$ large enough so that $|g_0^{n_0}|_{X \setminus U} < 1/8$. Set $g := g_0^{n_0}$. Since A is dense in $C(X)$, there exists $g_1 \in A$ such that $|g_1 - g|_X < 1/8$. Then we have $|g_1|_X < 9/8$ and $|g_1|_{X \setminus U} < 1/4$. Set $f := 8g_1/9$. Then $|f|_X \leq 1$ and $|f|_{X \setminus U} < 1/4$. It remains to show that $|f(x)| > 3/4$. Indeed $|g_1(x) - g(x)| < 1/8$ so that $|f(x)| > 7/9 > 3/4$. This completes the proof. $\square$

We now ready to prove the theorem.

Proof of Theorem 5.2.7. We must show that $\tau^{-1}(\Gamma(A)) \subseteq \Gamma(B)$.

Let $\varphi \in \Gamma(A)$, let $E = \tau^{-1}(\{\varphi\})$, and let $\psi \in E$. We show that $\psi \in \Gamma(B)$. Fix an open neighbourhood U of ψ in Φ_B . By Lemma 5.2.8, there exists $f \in B$ such that $|f|_E \leq 1$, $|f(\psi)| > 3/4$, and $|f|_{E \setminus U} < 1/4$. Let U_1 be an open neighbourhood of ψ in Φ_B such that $U_1 \subseteq U$, and $|f(\chi)| > 1/2$ for all $\chi \in U_1$. For each $\chi \in E \setminus U$, let U_χ be an open neighbourhood of χ in Φ_B such that, for all $\chi' \in U_\chi$, we have $|f(\chi') - f(\chi)| < 1/4$. By compactness, there exists $\chi_1, \dots, \chi_n \in E$ such that $E \setminus U \subseteq \bigcup_{j=1}^n U_{\chi_j}$. Set $V_j := U_{\chi_j}$ for $j = 1, \dots, n$, and set $V := \bigcup_{j=1}^n V_j$. Note that $V \cap U_1 = \emptyset$.

Let $W := \tau(U_1) \cap \bigcap_{j=1}^n \tau(V_j)$ so that W is an open neighbourhood of φ in Φ_A , and $U_1 \subseteq \tau^{-1}(W)$. By Lemma 5.2.5, there exist $g \in A$ and $\varphi' \in W$ such that $|g(\varphi')| = |g|_{\Phi_A} = 1$ and $|g|_{\Phi_A \setminus W} < 1$. By replacing g with g^m for large enough m , we may assume that $|g|_{\Phi_A \setminus W} < 1/(2|f|_{\Phi_B})$. Set $h = \theta(g) \in B$.

Let $\psi' \in \Phi_B$. If $\tau(\psi') \notin W$, then we have $|h(\psi')| < 1/(2|f|_{\Phi_B})$ and so $|(hf)(\psi')| \leq 1/2$. If $\tau(\psi') \in W$ and $\psi' \notin U$, then we have $\psi' \in V$. So there exist $j \in \{1, \dots, n\}$ such that $\psi' \in V_j$. It follows that $|f(\psi') - f(\chi_j)| < 1/4$ and

$$|(hf)(\psi')| \leq |f(\psi')| \leq |f(\chi_j)| + |f(\psi') - f(\chi_j)| \leq \frac{1}{2}.$$

If $\tau(\psi') = \varphi'$ and $\psi' \in U_1$, then we have $|(hf)(\psi')| = |f(\psi')| > 1/2$. It now follows that $\{\chi \in \Phi_B : |(hf)(\chi)| = |hf|_{\Phi_B}\} \subseteq U$. Since U was an arbitrary open neighbourhood of ψ , we have $\psi \in \Gamma(B)$ by Lemma 5.2.5. This concludes the proof. $\square$

5.3 Systems of Cole extensions

In this section we discuss Cole extensions and systems of Cole extensions. We refer the reader to [32, 61] for further information and related results. We first recall the definition and some elementary facts about Cole extensions. Unless otherwise stated, the results in this section are standard.

Throughout this section, let X be a compact space and let A be a uniform algebra on X .

Definition 5.3.1. Let U be a family of monic polynomials with coefficients in A . Set

$$X^U := \{(x, \xi) \in X \times \mathbb{C}^U : \varepsilon_x(q)(\xi_q) = 0 \ (q \in U)\},$$

and give X^U the product topology. Let $p_q : X^U \rightarrow \mathbb{C}$ ($q \in U$) and $\pi_U : X^U \rightarrow X$ be the usual coordinate projections.

The space X^U is a compact space (see [24, 31] or [61]) and the continuous projection $\pi_U : X^U \rightarrow X$ induces an isometric algebra homomorphism π_U^* from $C(X)$ into $C(X^U)$ given by $\pi_U^*(f) = f \circ \pi$ for each $f \in C(X)$. If there is no risk of ambiguity, we write π in place of π_U in this setting.

Definition 5.3.2. Let U be a family of monic polynomials with coefficients in A . Let π be the projection $X^U \rightarrow X$ as above. We let A^U be uniform closure of $\pi^*(A) \cup \{p_q : q \in U\}$ taken in $C(X^U)$.

We call the pair (A^U, X^U) the *Cole extension of A (with respect to U)*. We say that a Cole extension (A^U, X^U) of A is *simple* if $U = \{q\}$ for some monic polynomial $q \in A[t]$. In this case, we write A^q , X^q , and π_q for A^U , X^U , and π_U , respectively.

Definition 5.3.3. Let $q \in A[t]$ be a monic polynomial. For each $x \in X$, let $\lambda_{x,j}$ ($j = 1, \dots, n$) be the roots of the polynomial $\varepsilon_x(q)$. For $f \in C(X^q)$, set

$$T(f)(x) := \frac{1}{n} \sum_{j=1}^n f(x, \lambda_{x,j}) \quad (x \in X). \quad (5.1)$$

One of the most important features of Cole extensions is that the map T , defined above, is a surjective, unital linear map from $C(X^q)$ onto $C(X)$, with $\|T\| = 1$ and $T \circ \pi^* = \text{id}_{C(X)}$. In fact, T is an A -module homomorphism when A^q is given the module operations $a \cdot g = g \cdot a = \pi^*(a)g$ ($a \in A, g \in A^q$) making it a commutative, unital A -bimodule. In addition, $T(f)(x) \in \overline{\text{conv}} f(\pi^{-1}\{x\})$ for each $f \in C(Y)$ and each $x \in X$. The details of this can be found in [31, Theorem 1.4.2.1] and [61].

Remark 5.3.4. The construction of the map T given in [31] and [61] appear different, but are equivalent. For another method for constructing the map T in the case where the polynomials are degree 2, see [24].

We sometimes refer to the map T as the *averaging map* for the extension A^q . The following lemma follows immediately from the definition and properties of the map T .

Lemma 5.3.5. *Let q be a monic polynomial in $A[t]$ with $\deg q = n$. Let T be the averaging map for the extension A^q and set $P := \pi^* \circ T$. Then P is a contractive unital projection on A^q and $P(A^q) = \pi^*(A)$. Moreover, P is an $\pi^*(A)$ -module homomorphism.*

Let $\mathfrak{X}$ be a normed space, let $E \subseteq \mathfrak{X}$, and let $x \in \mathfrak{X}$. Recall that $\text{dist}_{\mathfrak{X}}(x, E)$ is defined by

$$\text{dist}_{\mathfrak{X}}(x, E) = \inf_{y \in E} \|x - y\|,$$

and that $\text{dist}_{\mathfrak{X}}(x, E) = 0$ if and only if $x \in \overline{E}$. The following can be distilled from the proof of [31, Theorem 1.4.2.5].

Proposition 5.3.6. *Let q be a monic polynomial in $A[t]$. Then, for all $f \in C(X)$, we have*

$$\text{dist}_{C(X^q)}(\pi^*(f), A^q) = \text{dist}_{C(X)}(f, A).$$

In particular, if A is non-trivial then A^q is non-trivial.

The following characterisation of a simple Cole extension A^q (for some monic polynomial q) in terms of the powers of the coordinate projection p_q will be of use to us. A proof can be found in [31, Proposition 1.3.2.1] (see also [61, Lemma 3.1.14]).

Proposition 5.3.7. *Let $q \in A[t]$ be a monic polynomial of degree n . Then*

$$A^q = \overline{\left(\sum_{j=0}^{n-1} \pi^*(A) \cdot p_q^j \right)},$$

where the closure is taken in $C(X^q)$.

In the next section, we show that a Cole extension of A with respect to a family U of monic polynomials can be obtained by repeated simple Cole extensions. From this, we deduce that many of the properties described above for simple Cole extensions also hold for other Cole extensions.

5.3.1 Systems of Cole extensions

We now discuss systems of Cole extensions. Throughout this section, we let τ be a non-zero ordinal.

Definition 5.3.8. For each $\alpha \leq \tau$, let X_α be a compact space, let A_α be a uniform algebra on X_α and, for each β with $\alpha \leq \beta \leq \tau$, let $\pi_{\alpha,\beta} : X_\beta \rightarrow X_\alpha$ be a continuous surjection. We say that $((X_\alpha, \pi_{\alpha,\beta}), (A_\alpha))_{\alpha \leq \beta \leq \tau}$ is a *system of Cole extensions* if each of the following hold:

- (a) $((X_\alpha, \pi_{\alpha,\beta}))_{\alpha \leq \beta \leq \tau}$ is an inverse system of compact spaces and continuous functions;
- (b) for each α with $\alpha < \tau$ there exists a family U_α of non-constant, monic polynomials in $A_\alpha[t]$ such that $(X_{\alpha+1}, A_{\alpha+1}) = (X_\alpha^{U_\alpha}, A_\alpha^{U_\alpha})$ and the map $\pi_{\alpha,\alpha+1} : X_{\alpha+1} \rightarrow X_\alpha$ is the continuous surjection as in Definition 5.3.1;
- (c) for a non-zero limit ordinal γ with $\gamma \leq \tau$, $(X_\gamma, (\pi_{\alpha,\gamma}))_{\alpha < \gamma}$ is the inverse limit of the inverse system $((X_\alpha, \pi_{\alpha,\beta}))_{\alpha \leq \beta < \gamma}$ and $(A_\gamma, (\pi_{\alpha,\gamma}^*))_{\alpha < \gamma}$ is the corresponding direct limit of the direct system $((A_\alpha, \pi_{\alpha,\beta}))_{\alpha \leq \beta < \gamma}$.

The pair (X_0, A_0) is called the *base of the system* and A_0 is the *base algebra* of the system.

In the above, taking the direct limit is equivalent to setting

$$A_\gamma = \overline{\left(\bigcup_{\alpha < \gamma} \pi_{\alpha,\gamma}^*(A_\alpha) \right)}. \quad (5.2)$$

This is noted in [24] and [61].

We shall often seek to extend a given system of Cole extensions, usually within an inductive construction. Let $S_\tau = ((X_\alpha, \pi_{\alpha,\beta}), A_\alpha)_{\alpha \leq \beta \leq \tau}$ be a system of Cole extensions. Let δ be an ordinal such that $\delta \geq \tau$ and suppose that $S_\delta = ((Y_\alpha, \rho_{\alpha,\beta}), B_\alpha)_{\alpha \leq \beta \leq \delta}$ is a system of Cole extensions. We say that S_δ *extends* S_τ if, for all α, β with $\alpha \leq \beta \leq \tau$, we have $(X_\alpha, \pi_{\alpha,\beta}, A_\alpha) = (Y_\alpha, \rho_{\alpha,\beta}, A_\alpha)$.

For example, with $S_\tau = ((X_\alpha, \pi_{\alpha,\beta}), A_\alpha)_{\alpha \leq \beta \leq \tau}$ as above, we can construct a system of Cole extensions $S_{\tau+1} = ((X_\alpha, \pi_{\alpha,\beta}), A_\alpha)_{\alpha \leq \beta \leq \tau+1}$ which extends S_τ by simply choosing a family U of monic polynomials in $A_\tau[t]$ and setting $A_{\tau+1} := A_\tau^U$, $X_{\tau+1} := X_\tau^U$, $\pi_{\tau+1,\tau+1} := \text{id}_{C(X_{\tau+1})}$, $\pi_{\tau,\tau+1} : X_{\tau+1} \rightarrow X_\tau$ to be a continuous surjection as in Definition 5.3.1, and, for each $\alpha \leq \tau$, set $\pi_{\alpha,\tau+1} := \pi_{\alpha,\tau} \circ \pi_{\tau,\tau+1}$.

If, for each $\alpha < \tau$, $(A_{\alpha+1}, X_{\alpha+1})$ is a simple Cole extension of A , then we say that the system $((X_\alpha, \pi_{\alpha,\beta}), A_\alpha)_{\alpha \leq \beta \leq \tau}$ is a *system of simple Cole extensions*. The following proposition is effectively [61, Lemma 3.1.16].

Recall that, in this section, X is a compact space and A is a uniform algebra on X .

Proposition 5.3.9. *Let U be a family of monic polynomials in $A[t]$. There exist an ordinal τ , a system of simple Cole extensions $((X_\alpha, \pi_{\alpha,\beta}), A_\alpha)_{\alpha \leq \beta \leq \tau}$, and a homeomorphism $\varphi : X_\tau \rightarrow X^U$ such that $\varphi^* : C(X^U) \rightarrow C(X_\tau)$ is an isometric algebra isomorphism with $\varphi^* \circ \pi_U^* = \pi_{0,\tau}^*$, and such that $(A_0, X_0) = (A, X)$.*

The following is a consequence of Proposition 5.3.9 and Lemma 5.1.6.

Corollary 5.3.10. *Let U be a family of monic polynomials in $A[t]$. There exists a surjective linear map $T : C(X^U) \rightarrow C(X)$ such that $\|T\| = 1$, $T(A^U) = A$ and $T \circ \pi^* = \text{id}_{C(X)}$.*

Let $((X_\alpha, \pi_{\alpha,\beta}), A_\alpha)_{\alpha \leq \beta \leq \tau}$ be a system of Cole extensions. Using transfinite induction, Corollary 5.3.10, and Lemma 5.1.6, we obtain the following proposition. This is effectively a combination of Theorem 3.1.19(5) and Proposition 3.1.6 from [61]; the latter is originally due to Dawson [31]. This result can also be obtained by a direct application of Lemma 5.1.6.

Proposition 5.3.11. *Suppose that τ is a limit ordinal. Let $((X_\alpha, \pi_{\alpha,\beta}), A_\alpha)_{\alpha \leq \beta \leq \tau}$ be a system of Cole extensions. Let $\{T_{\alpha,\beta} : \alpha \leq \beta \leq \tau\}$ be the collection of bounded linear maps as above. Then $(A_\alpha, T_{\alpha,\beta})_{\alpha \leq \beta < \tau}$ is an inverse system of Banach spaces and bounded linear maps. Moreover, for each $\alpha < \tau$, there exists contractive, linear maps $T_{\alpha,\tau} : C(X_\tau) \rightarrow C(X_\alpha)$ such that $T_{\alpha,\tau}(A_\tau) = A_\alpha$ and $T_{\alpha,\tau} \circ \pi_{\alpha,\tau} = \text{id}_{C(X_\alpha)}$.*

The following proposition is [32, Proposition 1.3.1.3] and is central to our study of systems of Cole extensions.

Proposition 5.3.12. *Let τ be a non-zero ordinal and let $((X_\alpha, \pi_{\alpha,\beta}), A_\alpha)_{\alpha \leq \beta \leq \tau}$ be a system of Cole extensions. If A_0 is natural on X_0 then A_α is natural on X_α for each $\alpha \leq \tau$.*

5.4 The Shilov boundary for Cole extensions

It was proved in [32] that if τ is an ordinal and $((X_\alpha, \pi_{\alpha,\beta}), A_\alpha)_{\alpha \leq \beta \leq \tau}$ is a system of Cole extensions then, for each $\alpha \leq \tau$, we have $\Gamma(A_\alpha) = \pi_{0,\alpha}^{-1}(\Gamma(A_0))$. The original proof ([32, Section 1.5]) of this result uses [80, Theorem 1.6], which relies on unpublished work of Björk. In this section we prove the following result.

Proposition 5.4.1. *Let τ be a non-zero ordinal and let $((X_\alpha, \pi_{\alpha,\beta}), A_\alpha)_{\alpha \leq \beta \leq \tau}$ be a system of Cole extensions. Then, for each $\alpha \leq \tau$, we have $\Gamma(A_\alpha) = \pi_{0,\alpha}^{-1}(\Gamma(A_0))$.*

We prove the proposition using a sequence of lemmas. The following is stated in [79], and proved in [32, Lemma 1.5.3.4]. (The proof in the latter is for the Arens–Hoffmann extension with respect to a monic polynomial q over A . However, this proof can be easily adapted to obtain the below.)

Lemma 5.4.2. *Let X be a compact space, let A be a uniform algebra on X , and let $q(t) \in A[t]$ be monic. If $x \in X$ and $\pi_q^{-1}(\{x\}) = \{(x, \zeta_j) : j = 1, \dots, m\}$ for some $m \in \mathbb{N}$ and $\zeta_1, \dots, \zeta_m \in \mathbb{C}$ then there exist pairwise disjoint neighbourhoods U_j of (x, ζ_j) ($j = 1, \dots, m$) in X^q such that $\pi_q(U_j) = \pi_q(U_1)$ for $j = 2, \dots, m$ and $\pi_q^{-1}(\pi_q(U_1)) = \bigcup_{j=1}^m U_j$.*

The following is a modification of [79, Theorem 3.1]. This lemma will act as our replacement for [80, Theorem 1.6].

Lemma 5.4.3. *Let A be a uniform algebra on a compact space X and let $q(t)$ be a monic polynomial in $A[t]$ of degree $n \in \mathbb{N}$. Then $\Gamma(A^q) = \pi_q^{-1}(\Gamma(A))$.*

Proof. Fix $x \in X$. Then the set $K := \pi_q^{-1}(\{x\})$ consists of finitely many points (corresponding to the distinct roots of $\varepsilon_x(q)(t)$), and so $p_q(K)$ is a compact plane set with only finitely many points. By the Hartogs–Rosenthal theorem (or otherwise), $P(p_q(K)) = C(p_q(K))$. The map $(p_q|K)^* : C(p_q(K)) \rightarrow C(K)$ maps $P_0(p_q(K))$ into $A^q|K$, and so $A^q|K$ is uniformly dense in $C(K)$. This holds for each $x \in X$.

Let $T : C(X^q) \rightarrow C(X)$ be the averaging map for the extension by q . Let $(x, \zeta) \in \Gamma(A^q)$ and fix a neighbourhood W of x in X . As above, let $\zeta_1, \dots, \zeta_m$ denote the distinct roots of the polynomial $\varepsilon_x(q)(t) \in \mathbb{C}[t]$ so that

$$\pi_q^{-1}(\{x\}) = \{(x, \zeta_j) : j = 1, \dots, m\}.$$

Without loss of generality, we may assume that $\zeta_1 = \zeta$.

If $m = 1$ then let $V_1 := \pi^{-1}(W)$. Otherwise, by Lemma 5.4.2, there are pairwise disjoint neighbourhoods U_j of (x, ζ_j) in X^q ($j = 1, \dots, m$) such that $\pi_q^{-1}(\pi(U_1)) = \bigcup_{j=1}^m U_j$ and $\pi_q(U_j) = \pi_q(U_1)$ for $j = 2, \dots, m$. For $j \in \{1, \dots, m\}$, let $V_j := \pi^{-1}(W) \cap U_j$. Then the V_j are pairwise disjoint and open.

In either case, by Lemma 5.2.5 there exist $g \in A^q$ and $(y, \lambda) \in V_1$ such that $|g(y, \lambda)| = |g|_{X^q} = 1$ and $|g|_{X^q \setminus V_1} < 1$. Let n_0 be an integer large enough so that $|g^{n_0}|_{X^q \setminus V_1} < 1/2n$.

Let $f := T(g^{n_0}) \in A$. Then, for each $z \in X \setminus \pi(V_1) \subseteq X \setminus W$, we have that $|f(z)| < 1/2n$. Let $\lambda_1, \dots, \lambda_k$ denote the distinct roots of the polynomial $\varepsilon_y(q)(t) \in \mathbb{C}[t]$, and let μ_j denotes the multiplicity of the root λ_j ($j = 1, \dots, k$). Without loss of generality, we may assume that $\lambda = \lambda_1$. We have that

$$f(y) = \frac{1}{n} \sum_{j=1}^k \mu_j g(y, \lambda_j).$$

If $k = 1$ then we have $|f(y)| = |g(y, \lambda_1)| = 1$. Otherwise, we have

$$|f(y)| \geq \frac{\mu_1}{n} |g(y, \lambda_1)| - \frac{1}{n} \sum_{j=2}^k \mu_j |g(y, \lambda_j)|.$$

But since $(y, \lambda_j) \notin V_1$ for $j = 2, \dots, k$, it follows that

$$|f(y)| \geq \frac{\mu_1}{n} - \frac{1}{2n^2} \sum_{j=2}^k \mu_j > \frac{1}{2n}.$$

In either case, we have $y \in W$ such that $|f(y)| = |f|_X$ and $|f|_{X \setminus W} < |f|_X$. Since W was arbitrary, by Lemma 5.2.5, we have that $x \in \Gamma(A)$. This proves that $\pi_q(\Gamma(A^q)) \subseteq \Gamma(A)$.

The result now follows from Theorem 5.2.7. □

We now deal with systems of Cole extensions. The following lemma is from [32, Lemma 1.5.2.1].

Lemma 5.4.4. *Let X be a compact space, let A be a uniform algebra on X , let F be a closed subset of X , and let E be a dense subset of A . Suppose that, for each $f \in E$, there exists $x \in F$ such that $|f(x)| = |f|_X$. Then $\Gamma(A) \subseteq F$.*

The following is adapted from the limit ordinal case of [32, Proposition 1.5.2.2].

Lemma 5.4.5. *Let τ be a non-zero limit ordinal and let $((X_\alpha, \pi_{\alpha,\beta}), A_\alpha)_{\alpha \leq \beta \leq \tau}$ be a system of Cole extensions. Suppose that, for each $\alpha < \tau$, $\Gamma(A_\alpha) = \pi_{0,\alpha}^{-1}(\Gamma(A_0))$. Then $\Gamma(A_\tau) = \pi_{0,\tau}^{-1}(\Gamma(A_0))$.*

Proof. We first show that $\pi_{0,\tau}^{-1}(\Gamma(A_0)) \subseteq \Gamma(A_\tau)$. Let $\xi \in \pi_{0,\tau}^{-1}(\Gamma(A_0))$, and let U be a basic open neighbourhood of ξ in X_τ . Then there exist $\alpha_1, \dots, \alpha_n < \tau$ and open sets $U_j \subseteq X_{\alpha_j}$ ($j = 1, \dots, n$) such that $U = \bigcap_{j=1}^n \pi_{\alpha_j,\tau}^{-1}(U_j)$. Set $\beta := \max\{\alpha_1, \dots, \alpha_n\}$. Note that $\beta < \tau$. Put $W := \bigcap_{j=1}^n \pi_{\alpha_j,\beta}^{-1}(U_j)$. Then W is an open set in X_β and we have $\pi_{\beta,\tau}^{-1}(W) = U$.

Set $\xi_0 := \pi_{0,\tau}(\xi)$ and set $\xi_\beta := \pi_{\beta,\tau}(\xi)$. Since $\pi_{0,\tau} = \pi_{0,\beta} \circ \pi_{\beta,\tau}$, we have $\pi_{0,\beta}(\xi_\beta) = \xi_0$. By our choice of ξ , and the assumption that $\Gamma(A_\beta) = \pi_{0,\beta}^{-1}(\Gamma(A_0))$, it follows that $\xi_\beta \in \Gamma(A_\beta)$. By Lemma 5.2.5, there exist $\zeta_0 \in W$ and $f \in A_\beta$ such that $|f(\zeta_0)| = |f|_{X_\beta} = 1$ and $|f|_{X_\beta \setminus W} < |f|_{X_\beta}$. Let $\zeta \in \pi_{\beta,\tau}^{-1}(\{\zeta_0\})$. Then we have $\pi_{\beta,\tau}(\zeta) = \zeta_0 \in W$ so that $\zeta \in U$. Put $g := \pi_{\beta,\tau}^*(f)$. Then $|g(\zeta)| = |g|_{X_\tau} = 1$ and $|g|_{X_\tau \setminus U} < 1$. It now follows from Lemma 5.2.5 that $\xi \in \Gamma(A_\tau)$. Thus we have $\pi_{0,\tau}^{-1}(\Gamma(A_0)) \subseteq \Gamma(A_\tau)$.

We must now show that $\Gamma(A_\tau) \subseteq \pi_{0,\tau}^{-1}(\Gamma(A_0))$. Set $E = \bigcup_{\alpha < \tau} \pi_{\alpha,\tau}^*(A_\alpha)$. Then E is a dense subset of A_τ . Let $\alpha < \tau$ and let $f \in A_\alpha$. There exists $\xi_\alpha \in \pi_{0,\alpha}^{-1}(\Gamma(A_0))$ such that $|f(\xi_\alpha)| = |f|_{X_\alpha}$. Let $\xi \in \pi_{\alpha,\tau}^{-1}(\{\xi_\alpha\})$. Then $\pi_{0,\tau}(\xi) = \pi_{0,\alpha}(\xi_\alpha) \in \Gamma(A_0)$. Also, we have $|\pi_{\alpha,\tau}^*(f)(\xi)| = |\pi_{\alpha,\tau}^*(f)|_{X_\tau}$. It follows that, for each $f \in E$, there exists $\zeta \in \pi_{0,\tau}^{-1}(\Gamma(A_0))$ such that $|f(\zeta)| = |f|_{X_\tau}$. Thus, by Lemma 5.4.4, we have $\Gamma(A_\tau) \subseteq \pi_{0,\tau}^{-1}(\Gamma(A_0))$. Equality holds and this completes the proof. $\square$

By combining Lemma 5.4.5 and Proposition 5.3.9, we obtain the following immediate corollary.

Corollary 5.4.6. *Let X be a compact space, let A be a uniform algebra on X , and let U be a non-empty collection of monic polynomials in $A[t]$. Then we have $\Gamma(A^U) = \pi_U^{-1}(\Gamma(A))$.*

Using Lemma 5.4.5 and Corollary 5.4.6, we obtain Proposition 5.4.1 by transfinite induction.

5.5 Properties of Simple Cole extensions

In this section, we discuss the properties of simple Cole extensions of a given uniform algebra. Unless otherwise stated, all results in this section are new.

Throughout this section, let X be a compact space and let A be a uniform algebra on X .

We first aim to prove a theorem about the structure of a simple Cole extension of a uniform algebra. We require the following lemma.

Lemma 5.5.1. *Let $h \in A$, let $q(t)$ be a monic polynomial in $A[t]$, and let $n := \deg q$. Let $\rho(t) = q(t+h)$. Then there exists an isometric isomorphism $\theta : C(X^q) \rightarrow C(X^\rho)$ such that $\theta(p_q) = p_\rho$, $\theta \circ \pi_q^* = \pi_\rho^*$, and $P(A^q) = A^\rho$.*

Proof. Let $x \in X$. Let $\lambda_{1,x}, \dots, \lambda_{n,x}$ be the roots of the polynomial $\varepsilon_x(q)(t)$ in $\mathbb{C}$. Then, by Proposition 1.2.13, the roots of the polynomial $\varepsilon_x(\rho)(t)$ are

$$\lambda_{1,x} - h(x), \dots, \lambda_{n,x} - h(x).$$

Let $\varphi : X^q \rightarrow X^\rho$ be given by $\varphi(x, \xi) = (x, \xi - h(x))$. By the above φ is a bijection and $\varphi^{-1}(x, \zeta) = (x, \zeta + h(x))$. It is also easy to see that φ is continuous and hence is a homeomorphism. It follows that $\theta := \varphi^*$ is an isometric isomorphism from $C(X^q)$ onto $C(X^\rho)$. Clearly $\theta(p_q) = p_\rho$, $\theta \circ \pi_q^* = \pi_\rho^*$, and $\theta(A^q) = A^\rho$. This completes the proof. $\square$

Our first theorem of this section is the following.

Theorem 5.5.2. *Let $q(t) = f_0 + f_1 t + \dots + f_{n-1} t^{n-1} + t^n \in A[t]$. Then there exists a contractive, unital projection $P : A^q \rightarrow A^q$ such that $P(A^q) = \pi_q^*(A)$ and $P(p_q) = 0$. Moreover, P is a $\pi_q^*(A)$ -module homomorphism.*

Proof. It is easy to see, from (5.1) and standard facts about polynomials, that if $f_{n-1} = 0$ then $T(p_q) = 0$. The result follows immediately in this case, so suppose that $f_{n-1} \neq 0$.

Let $h = f_{n-1}/n$ and set $\rho(t) = q(t - h)$. By Lemma 5.5.1, there exists an isometric isomorphism $\theta : C(X^q) \rightarrow C(X^\rho)$ such that $\theta(p_q) = p_\rho$, $\theta \circ \pi_q^* = \pi_\rho^*$ and $\theta(A^q) = A^\rho$. We have $\rho(t) = g_0 + g_1 t + \dots + g_{n-2} t^{n-2} + t^n$ so that, for each $x \in X$, the sum of the roots of the polynomial $\varepsilon_x(\rho)(t)$ is zero. Let $T_\rho : A^\rho \rightarrow A$ be the root averaging map for A^ρ . Then $T_\rho(p_\rho) = 0$ and $P_0 := \pi_\rho^* \circ T_\rho$ is a contractive, unital projection from A^ρ into A with $P_0(A^\rho) = \pi_\rho^*(A)$ and $P_0(p_\rho) = 0$. Set $P = \theta^{-1} \circ P_0 \circ \theta$. It is trivial to show that P is indeed a contractive, unital projection on A^q with $P(A^q) = \pi_q^*(A)$ and $P(p_q) = 0$. It is also easy to see that P is a $\pi_q^*(A)$ -module homomorphism. $\square$

We have already seen that given a monic polynomial $q(t) \in A[t]$ of degree $n \in \mathbb{N}$, we have

$$A^q = \overline{\left(\sum_{j=0}^{n-1} \pi^*(A) \cdot p_q^j \right)}.$$

We now seek to characterise those simple Cole extension in which we are not required to take any closures in the above. To do this, we will give a new decomposition of A^q and describe those monic polynomials q such that $\pi^*(A) \cdot p_q^j$ is closed in A^q . We require the following lemmas.

Lemma 5.5.3. *Let $q(t)$ be a monic polynomial in $A[t]$ and let $x \in X$. Then $(x, 0) \in X^q$ if and only if $\varepsilon_x(q)(0) = 0$.*

Proof. We have $q(t) = f_0 + f_1 t + \cdots + f_{n-1} t^{n-1} + t^n$, with $f_0, \dots, f_{n-1} \in A$ and $n = \deg q$. Let $x \in X$. Then

$$\varepsilon_x(q)(t) = \prod_{j=1}^n (t - \zeta_{j,x}) = f_0(x) + f_1(x)t + \cdots + f_{n-1}(x)t^{n-1} + t^n,$$

where $\zeta_{1,x}, \dots, \zeta_{n,x}$ are the roots of $\varepsilon_x(q)(t)$, and

$$f_0(x) = (-1)^n \prod_{j=1}^n \zeta_{j,x}.$$

Thus $f_0(x) = 0$ if and only if there is a $j \in \{1, \dots, n\}$ such that $\zeta_{j,x} = 0$. The result follows. $\square$

Lemma 5.5.4. *Let $q(t) \in A[t]$ with $q(t) = f_0 + f_1 t + \cdots + f_{n-1} t^{n-1} + t^n$. Let $E \subseteq X$ and let $F = \pi^{-1}(E)$. Then p_q has no zeros on F if and only if f_0 has no zeros on E . In particular p_q has no zeros on $\Gamma(A^q)$ if and only if f_0 has no zeros on $\Gamma(A)$.*

Proof. Let $x \in E$. By Lemma 5.5.3, $(x, 0) \in F$ if and only if $f_0(x) = 0$. Thus p_q has a zero on F if and only if f_0 has a zero on E .

By Proposition 5.4.3, we have $\Gamma(A^q) = \pi_q^{-1}(\Gamma(A))$. It follows that p_q has a zero on $\Gamma(A^q)$ if and only if f_0 has a zero on $\Gamma(A)$. $\square$

Lemma 5.5.5. *Let $q(t)$ be a monic polynomial in $A[t]$. If p_q is a zero divisor in A^q then $q(0)$ is a zero divisor in A .*

Proof. We have $q(t) = f_0 + f_1 t + \cdots + f_{n-1} t^{n-1} + t^n$, with $f_0, \dots, f_{n-1} \in A$ and $n = \deg q$. Suppose that p_q is a zero divisor in A^q . Then there exists $h \in A^q$ such that $hp_q = 0$. Thus, for each $(x, \xi) \in X^q$, either $h(x, \xi) = 0$ or $\xi = 0$. In the first

case, we have $h(x, \xi)\pi^*(f_0)(x, \xi) = 0$. In the second case $f_0(x) = 0$, by Lemma 5.5.3, and so $\pi^*(f_0)(x, \xi)h(x, \xi) = 0$. Thus $\pi^*(f_0)$ is a zero divisor on A^q . But now

$$0 = T(\pi^*(f_0)h) = f_0T(h),$$

and it follows that f_0 is a zero divisor in A . (In the above, T is the averaging map associated with the extension by q .) This completes the proof. $\square$

The next theorem of the section is the following.

Theorem 5.5.6. *Let X be a compact space, let A be a uniform algebra on X , and let $q(t) \in A[t]$ be a monic polynomial of degree $n \in \mathbb{N}$.*

- (a) *Let $P \in \mathcal{B}(A^q)$ be a continuous, unital projection such $P(A^q) = \pi^*(A)$ and such that P is a $\pi^*(A)$ -module homomorphism. Set $Q := \text{id}_{A^q} - P$. Then*

$$A^q = \pi^*(A) + \overline{\sum_{j=1}^{n-1} Q(p_q^j) \cdot \pi^*(A)}.$$

- (b) *Suppose that $q(0)$ is not a zero divisor in A . The following are equivalent.*

- (i) *The function $q(0)$ has no zeros on the Shilov boundary for A .*
- (ii) *The map $L_{p_q} \in \mathcal{B}(C(X^q))$ given by $L_{p_q}(f) = p_q f$ is bounded below on A^q .*
- (iii) *The subspace $p_q^j \cdot \pi^*(A)$ is closed in A^q for $j = 0, \dots, n-1$.*

- (c) *Suppose that $q(0)$ has no zeros on $\Gamma(A)$ and that there exists a continuous, projection P on A^q with $P(A^q) = \pi^*(A)$ such that P is an $\pi_q^*(A)$ -module homomorphism and $(\text{id}_{A^q} - P)(p_q^j) = p_q^j$ for $j = 1, \dots, n-1$. Then*

$$A^q = \pi^*(A) + \sum_{j=1}^{n-1} p_q^j \cdot \pi^*(A).$$

Proof. (a) Let P, Q be as in the statement of the theorem. By Proposition 5.3.7, we have

$$A^q = \sum_{j=0}^{n-1} \pi^*(A) \cdot p_q^j.$$

and clearly $A^q = P(A^q) + Q(A^q)$. Since $P(A^q) = \pi^*(A)$, we have $Q(f) = 0$ for all $f \in \pi^*(A)$. Set

$$E := \sum_{j=1}^{n-1} \pi^*(A) \cdot Q(p_q^j)$$

Let $f_0, \dots, f_{n-1} \in \pi^*(A)$. Then

$$Q(f_0 + f_1 p_q + \dots + f_{n-1} p_q^{n-1}) = Q(f_1 p_q) + \dots + Q(f_{n-1} p_q^{n-1})$$

and, since P is a $\pi^*(A)$ -module homomorphism, Q is also a $\pi^*(A)$ -module homomorphism. Thus $Q(f_j p_q^j) = f_j Q(p_q^j)$ for $j = 1, \dots, n-1$. It follows that

$$Q\left(\sum_{j=0}^{n-1} \pi^*(A) \cdot p_q^j\right) = E$$

and so $Q(A^q) \subseteq \overline{E}$. But now if $f \in \overline{E}$ then $P(f) = 0$, by the continuity of P , so that $f \in Q(A^q)$. Thus $Q(A^q) = \overline{E}$. This proves (a).

(b) Now suppose that $q(0)$ is not a zero divisor in A . By Lemma 5.5.5, p_q is not a zero divisor in A^q . We can write $q(t) = f_0 + f_1 t + \dots + f_{n-1} t^{n-1} + t^n$, where $f_0, \dots, f_{n-1} \in A$, so that $q(0) = f_0$. Set $F := \pi^*(A)$.

(i) $\iff$ (ii) This follows from Lemma 5.5.4 and Lemma 5.2.3.

(ii) $\implies$ (iii) This is Corollary 5.2.2.

(iii) $\implies$ (ii) Let $k \in \{0, \dots, n\}$. Since $L_{p_q}^k(F)$ is closed in A^q , it follows from Lemma 5.2.1 that $L_{p_q}^k$ is bounded below on E . By Lemma 5.2.4, L_{p_q} is bounded below on A^q . The proof of (b) is complete.

(c) Set $Q := \text{id}_{A^q} - P$. By (b), L_{p_q} is bounded below on A^q . By (a), we have

$$A^q = \pi^*(A) + \overline{\sum_{j=1}^{n-1} Q(p_q^j) \cdot \pi^*(A)}.$$

If $Q(p_q^j) = p_q^j$ for each j , then we have $P(p_q^j) = 0$ for each j . Thus $(P \circ L_{p_q})(f) = 0$ for each $f \in \sum_{j=0}^{n-1} p_q^j \cdot \pi^*(A)$. It is easy to see that, under these conditions, we have $L_{p_q}^j(\pi^*(A)) \cap L_{p_q}^k(\pi^*(A)) = \{0\}$ for each $j, k \in \{0, \dots, n-1\}$. By Corollary 5.2.2, $\sum_{j=0}^{n-1} p_q^j \cdot \pi^*(A)$ is closed in A^q , and (c) follows. $\square$

Remark 5.5.7. The equivalences given in (b) are similar to the main result in [62] for Arens–Hoffmann extensions of a uniform algebra. Another related result, due to Arens, appears in [27, Theorem 2.1.20].

We do not know, in general, whether there is a continuous projection P such that $P(p_q^j) = 0$ for $j = 1, \dots, n-1$ (with p_q and n as in the theorem). However, if $n = 2$ then Theorem 5.5.2 gives a projection P such that $P(p_q) = 0$ so, in this case, we obtain the following corollary.

Corollary 5.5.8. *Let X be a compact space, let A be a uniform algebra on X , and let $q(t) \in A[t]$ be a monic polynomial of degree 2. Suppose that $q(0)$ has no zeros on $\Gamma(A)$. Then $A^q = \pi^*(A) + p_q \cdot \pi^*(A)$.*

Remark 5.5.9. By considering symmetric polynomials, it is possible to show that, for $n \in \mathbb{N}$, given the monic polynomial $q(t) = t^n - f$, where $f \in A$ has no zeros on $\Gamma(A)$, we have $A^q = \sum_{j=0}^{n-1} p_q^j \cdot \pi^*(A)$. This is beyond the scope of this thesis so we omit the details.

5.6 Generalised Cole extensions

We now discuss a larger class of uniform algebra extensions which we call generalised Cole extensions. The following definition is essentially due to Feinstein and Cole (in [23]).

Definition 5.6.1. Let X, Y be compact spaces, let A be a uniform algebra on X , and let B be a uniform algebra on Y . We say that B is a *generalised Cole extension* of A if the following conditions are satisfied:

- (a) there exists a continuous surjection $\pi : Y \rightarrow X$ which induces an isometric algebra homomorphism $\pi^* : C(X) \rightarrow C(Y)$;
- (b) there exists a contractive, unital linear map $T : C(Y) \rightarrow C(X)$ such that $T \circ \pi^* = \text{id}_{C(X)}$ and $T(B) = A$.

We say a generalised Cole extension B of A is *natural* if B is natural on Y if and only if A is natural on X .

Note that, with π and T as in the definition, we automatically have $\pi^*(A) \subseteq B$. We do not require that the contractive linear map T , associated with a generalised Cole extension, to have any additional properties. The following lemma lists some additional properties of the map T .

Lemma 5.6.2. *Let X, Y be compact spaces, let A be a uniform algebra on X , and let B be a uniform algebra on Y . Suppose that B is a generalised Cole extension of A and let π, T be the maps associated with this extension. Then $P := \pi^* \circ T$ is a contractive, unital projection on $C(Y)$ such that $P(B) = \pi^*(A)$. Moreover, P is a $\pi(A)$ -module homomorphism, and T is an A -module homomorphism when B is given the operation $f \cdot g = g \cdot f = \pi^*(f)g$ ($f \in A, g \in B$).*

Proof. We regard B as a commutative, unital Banach A -bimodule with the operations $f \cdot g = g \cdot f = \pi^*(f)g$. It is clear that P is a contractive, unital linear operator on B and clearly $P(B) = \pi^*(A)$. Moreover,

$$(\pi^* \circ T) \circ (\pi^* \circ T) = \pi^* \circ (T \circ \pi^*) \circ T = \pi^* \circ T$$

so that P is a projection. By Lemma A.2.6, P is an A -module homomorphism.

We must now show that T is an A -module homomorphism. Let $f \in A$ and $g \in B$. Then $P(f \cdot g) = f \cdot P(g)$. Hence

$$T(f \cdot P(g)) = T(P(f \cdot g)) = ((T \circ \pi^*) \circ T)(f \cdot g) = T(f \cdot g),$$

and $\pi^*(f), P(g) \in \pi^*(A)$, so that there exists $h \in A$ with $\pi^*(h) = P(g)$. Thus $f \cdot P(g) = \pi^*(f)\pi^*(h) = \pi^*(fh)$ and so $T(f \cdot P(g)) = fh$. It follows that $T(f \cdot g) = fh$. The proof is complete by noting that $h = T(P(g))$. $\square$

Remark 5.6.3. The full strength of Lemma A.2.6 is not used in the above, and we could have instead used a famous result due to Tomiyama to obtain the result. However, the above proof works if we instead consider projections on B . See Section 5.7 for a related open problem.

We also require the following lemma, which is effective the same as the proof of [31, Theorem 1.4.2.5], and is analogous to Proposition 5.3.6. We include the details of the proof for the convenience of the reader.

Lemma 5.6.4. *Let X, Y be compact spaces, let A be a uniform algebra on X , and let B be a uniform algebra on Y . Suppose that B is a generalised Cole extension of A and let π, T be the maps associated with this extension. Then, for all $f \in C(X)$, we have*

$$\text{dist}_{C(Y)}(\pi^*(f), \pi^*(A)) = \text{dist}_{C(X)}(f, A).$$

Proof. Let $f \in C(X)$. On one hand, since π^* is isometric, we have

$$|\pi^*(f) - \pi^*(g)|_Y = |f - g|_X$$

for all $g \in A$. Since $\pi^*(A) \subseteq B$, it follows that $|f - g|_X \geq \text{dist}_{C(Y)}(\pi^*(f), B)$ and so $\text{dist}_{C(X)}(f, A) \geq \text{dist}_{C(Y)}(\pi^*(f), B)$. On the other hand, since T is contractive, we have

$$|\pi^*(f) - h|_Y \geq |f - T(h)|_X$$

for all $h \in B$. Since $T(h) \in A$ for all $h \in B$, it follows that

$$|\pi^*(f) - h|_Y \geq \text{dist}_{C(X)}(f, A).$$

Thus $\text{dist}_{C(Y)}(\pi^*(f), B) \geq \text{dist}_{C(X)}(f, A)$ and equality holds. This completes the proof. $\square$

The following corollary is central to our study of generalised Cole extensions, and will be used implicitly throughout this chapter.

Corollary 5.6.5. *Let X, Y be compact spaces, let A be a uniform algebra on X and let B be a uniform algebra on Y . Suppose that B is a generalised Cole extension of A . If A is non-trivial then B is non-trivial.*

Remark 5.6.6. This is related to the fact that a non-trivial uniform algebra on a compact space X cannot be complemented in $C(X)$. (See, for example, [27, p. 457].)

Clearly Cole extensions, of a given uniform algebra, are also generalised Cole extensions of that uniform algebra. The following is an example of a generalised Cole extension which is not a Cole extension.

Example 5.6.7. Let X be a compact space, and let A be a uniform algebra on X . Let $B := A \check{\otimes} A$, the injective tensor product of A with itself (see Section 1.4.4). Then B is a uniform algebra on $X \times X$. Let $\pi : X \times X \rightarrow X$ be the projection $(x, y) \mapsto x$. Then π^* is an isometric, unital algebra embedding from A into B . Fix $y_0 \in X$. Then, for each $f \in C(X \times X)$, let $T(f)(x) = f(x, y_0)$ ($x \in X$) for each $f \in B$. Then T is a surjective, unital linear map from B onto A with $\|T\| = 1$ and $T \circ \pi^* = \text{id}_A$. (In fact, T is an algebra homomorphism.) So B is a generalised Cole extension of A . $\blacksquare$

Before we continue, we demonstrate that, in certain cases, generalised Cole extensions must in fact be Cole extensions. The proof of the following theorem uses some operator algebra theory. The necessary facts are contained in Appendix A.

Theorem 5.6.8. *Let X be a compact space, let A be a uniform algebra on X , and let $P \in \mathcal{B}(A)$ be a contractive, unital projection. Suppose that $\text{id}_A - P$ is also contractive. Set $B := P(A)$ and set $E := \ker P$. Suppose that B is a uniform algebra on a compact space Y , that E is a Banach B -bimodule, and that there exists $h_0 \in E$ such that $E = \overline{h_0 \cdot B}$. Then there exists $h \in B$ and an isometric, unital algebra isomorphism $\theta : B^q \rightarrow A$, where $q(t) = t^2 - h$, such that $\theta \circ \pi_q^* = \text{id}_B$ and $\theta(p_q) = h_0$.*

Proof. Set $Q := \text{id}_A - P$. By Proposition A.2.7, we have $E^2 \subseteq B$. In particular $h_0^2 \in B$. Set $h := h_0^2$ and let q be as in the statement of the theorem. By Proposition

5.3.7, we have

$$B^q = \overline{\pi^*(B) + p_q \cdot \pi^*(B)}.$$

For each $\pi^*(f_1) + p_q \pi^*(f_2) \in B^q$, let

$$\theta(\pi^*(f_1) + p_q \pi^*(f_2)) = f_1 + h_0 f_2.$$

Clearly θ is a unital algebra homomorphism from $\pi^*(B) + p_q \cdot \pi^*(B)$ into A . It is easy to see that $\theta(\pi^*(B) + p_q \cdot \pi^*(B)) = B + h_0 \cdot B$ and that $\theta \circ \pi^* = \text{id}_B$ so that $\theta|_{\pi^*(B)}$ is a isometric, unital algebra isomorphism from $\pi^*(B)$ onto B . The map $\pi^*(f_1) + p_q \pi^*(f_2) \mapsto \pi^*(f_1) - p_q \pi^*(f_2)$ ($\pi^*(f_1) + p_q \pi^*(f_2) \in B^q$) defines an isometric algebra isomorphism on B^q (it is the adjoint of the map $(x, \xi) \mapsto (x, -\xi)$ from Y^q onto Y^q). The map $f_1 + h_0 f_2 \mapsto f_1 - h_0 f_2$ ($f_1 + h_0 f_2 \in A$) gives an automorphism of A , and so is also isometric. Let $\pi^*(f_1) + p_q \pi^*(f_2) \in B^q$. Then

$$|\pi^*(f_1) + p_q \pi^*(f_2)|_{Y^q}^2 = |(\pi^*(f_1) + p_q \pi^*(f_2))(\pi^*(f_1) - p_q \pi^*(f_2))|_{Y^q},$$

and

$$(\pi^*(f_1) + p_q \pi^*(f_2))(\pi^*(f_1) - p_q \pi^*(f_2)) = \pi^*(f_1)^2 - \pi^*(h) \pi^*(f_2)^2 \in \pi^*(B).$$

It follows that

$$|\pi^*(f_1) + p_q \pi^*(f_2)|_{Y^q}^2 = |\theta(\pi^*(f_1) + p_q \pi^*(f_2))|_X^2.$$

Now

$$\theta(\pi^*(f_1)^2 - \pi^*(h) \pi^*(f_2)^2) = f_1^2 - h^2 f_2^2 = (f_1 + h_0 f_2)(f_1 - h_0 f_2)$$

and so

$$|\theta(\pi^*(f_1)^2 - \pi^*(h) \pi^*(f_2)^2)|_X = |f_1 + h_0 f_2|_X^2.$$

Thus

$$|\pi^*(f_1) + p_q \pi^*(f_2)|_{Y^q}^2 = |\theta(\pi^*(f_1) + p_q \pi^*(f_2))|_X^2$$

so that θ is isometric. Since $B + h_0 \cdot B$ is dense in A and $\pi^*(B) + p_q \cdot \pi^*(B)$ is dense in B^q , θ extends to a isometric, unital isomorphism from B^q onto A . This completes the proof. $\square$

We do not know to what extent Theorem 5.6.8 generalises to when E is not generated by a single element.

The observation that the map $f_1 + h_0 f_2 \mapsto f_1 - h_0 f_2$ is isometric is used in [11] (where it is credited to Feinstein) to show that all projections P on a uniform algebra A such that both P and $\text{id}_A - P$ are contractive are given by $P = (\text{id}_A + \theta)/2$, where θ is an automorphism of A with $\theta \circ \theta = \text{id}_A$.

Remark 5.6.9. The author is grateful to Prof. David Blecher for several helpful discussions about operator algebras and projections.

5.6.1 Systems of generalised Cole extensions

We shall usually consider systems of generalised Cole extensions. Unlike systems of Cole extensions we must include the existence of the linear maps $T_{\alpha,\beta}$ in the definition.

Throughout this section, fix a non-zero ordinal τ .

Definition 5.6.10. Let τ be a fixed ordinal. A *system of generalised Cole extensions* is a system $((X_\alpha, \pi_{\alpha,\beta}), (A_\alpha, T_{\alpha,\beta}))_{\alpha \leq \beta \leq \tau}$ with the following properties:

- (a) for each $\alpha \leq \tau$, A_α is a uniform algebra on X_α ;
- (b) $(X_\alpha, \pi_{\alpha,\beta})_{\alpha \leq \beta \leq \tau}$ is an inverse system of compact spaces and continuous maps;
- (c) for each $\alpha < \tau$, $A_{\alpha+1}$ is a generalised Cole extension of A_α with associated maps $T_{\alpha,\alpha+1} \in \mathcal{B}(C(X_{\alpha+1}); C(X_\alpha))$ and $\pi_{\alpha,\alpha+1} : X_{\alpha+1} \rightarrow X_\alpha$;
- (d) for each $\alpha \leq \beta \leq \gamma \leq \tau$, we have $T_{\alpha,\gamma} = T_{\alpha,\beta} \circ T_{\beta,\gamma}$;
- (e) if $\gamma \leq \tau$ is a limit ordinal then
 - (i) $(X_\gamma, (\pi_{\alpha,\gamma})_{\alpha < \gamma})$ is the inverse limit of the inverse system $(X_\alpha, \pi_{\alpha,\beta})_{\alpha \leq \beta < \gamma}$,
 - (ii) $(A_\gamma, (\pi_\alpha^*)_{\alpha < \gamma})$ is the direct limit of the system $((A_\alpha)_{\alpha < \gamma}, (\pi_{\alpha,\beta}^*)_{\alpha \leq \beta < \gamma})$,
 - (iii) for each $\alpha < \gamma$, $T_{\alpha,\gamma} \circ \pi_{\alpha,\gamma}^* = \text{id}_{C(X_\alpha)}$.

We say that a system of generalised Cole extensions $((X_\alpha, \pi_{\alpha,\beta}), (A_\alpha, T_{\alpha,\beta}))_{\alpha \leq \beta \leq \tau}$ is *natural* if, for each $\alpha \leq \tau$, A_α is natural on X_α if and only if A_β is natural on X_β for all $\beta < \alpha$.

Note that our definition of X_γ and A_γ , where γ is a limit ordinal, does indeed yield a generalised Cole extension by Lemma 5.1.6. Note also that $(A_\alpha, T_{\alpha,\beta})_{\alpha \leq \beta \leq \tau}$ is an inverse system of Banach spaces with contractive linear maps.

Remark 5.6.11. If $((X_\alpha, \pi_{\alpha,\beta}), (A_\alpha, T_{\alpha,\beta}))_{\alpha \leq \beta \leq \tau}$ is a system of generalised Cole extensions then, for each $\alpha \leq \beta \leq \tau$, A_β is a generalised Cole extension A_α .

We also introduce the following terminology for extensions of systems of generalised Cole extensions.

Definition 5.6.12. Let σ be an ordinal with $\tau \leq \sigma$. Let

$$C = ((X_\alpha, \pi_{\alpha,\beta}), (A_\alpha, T_{\alpha,\beta}))_{\alpha \leq \beta \leq \tau} \quad \text{and} \quad D = ((Y_\alpha, \rho_{\alpha,\beta}), (B_\alpha, S_{\alpha,\beta}))_{\alpha \leq \beta \leq \sigma}$$

be systems of generalised Cole extensions. We say that D *extends* C if, for all $\alpha \leq \beta \leq \tau$, we have $(X_\alpha, \pi_{\alpha,\beta}, A_\alpha, T_{\alpha,\beta}) = (Y_\alpha, \rho_{\alpha,\beta}, B_\alpha, S_{\alpha,\beta})$.

While we do not know sufficient conditions for a generalised Cole extension to be natural, in general, the following—presumably well-known—lemma shows that no problems can occur at limit ordinals when constructing natural systems of generalised Cole extensions. We include the details of the proof for the convenience of the reader. (In fact, this is an easy modification of [32, Proposition 1.3.1.2].)

Lemma 5.6.13. *Let $((X_\alpha, \pi_{\alpha,\beta}), (A_\alpha, T_{\alpha,\beta}))_{\alpha \leq \beta \leq \tau}$ be a system of generalised Cole extensions. Suppose that τ is a limit ordinal and, for each $\alpha < \tau$, the uniform algebra A_α is natural on X_α . Then A_τ is natural on X_τ .*

Proof. We may assume that τ is non-zero. Let $\varphi \in \Phi_{A_\tau}$ and, for each $\alpha < \tau$, set $\varphi_\alpha := \varphi \circ \pi_{\alpha,\tau}^*$. Then $\varphi_\alpha \in \Phi_{A_\alpha}$ ($\alpha < \tau$) and so there exists $x_\alpha \in X_\alpha$ such that $\varphi_\alpha = \varepsilon_{x_\alpha}$ for each $\alpha < \tau$. We have that $\pi_{\alpha,\beta}(x_\beta) = x_\alpha$ for each $\alpha \leq \beta < \tau$. Indeed, let $\alpha \leq \beta < \tau$ then $\pi_{\alpha,\tau}^* = \pi_{\beta,\tau}^* \circ \pi_{\alpha,\beta}^*$ so that

$$\varphi_\alpha = \varphi \circ \pi_{\alpha,\tau}^* = \varphi \circ (\pi_{\beta,\tau}^* \circ \pi_{\alpha,\beta}^*) = \varphi_\beta \circ \pi_{\alpha,\beta}^*.$$

It follows quickly that $x_\alpha = \pi_{\alpha,\beta}(x_\beta)$.

Now there exists $x_\tau \in X_\tau$ such that $\pi_{\alpha,\tau}(x_\tau) = x_\alpha$ for each $\alpha < \tau$. The proof is complete by showing that $\varepsilon_{x_\tau} = \varphi$. Let $\alpha < \tau$ and let $f \in A_\alpha$. Then $\varphi(\pi_{\alpha,\tau}^*(f)) = f(x_\alpha) = \pi_{\alpha,\tau}^*(f)(x_\tau)$. This holds for all $\alpha < \tau$ and all $f \in A_\alpha$. But since $\bigcup_{\alpha < \tau} \pi_{\alpha,\tau}^*(A_\alpha)$ is dense in A_τ , it follows that $\varphi(f) = f(x_\tau)$ for all $f \in A_\tau$. This completes the proof. $\square$

5.6.2 Maps back from extensions

In this section we examine maps back from a generalised Cole extension of a uniform algebra A . By the definition of a generalised Cole extension, there is a linear contraction from the extension to the original uniform algebra. Of course, it is likely that there are a large number of choices for such a map. We are particularly interested in those extensions such that there is an algebra homomorphism onto the original algebra.

We quickly recall the properties that are required by such a map. Let X, Y be compact spaces such that there exists a continuous surjection $\pi : Y \rightarrow X$. Let A be a uniform algebra on X and let B be a uniform algebra on Y . We are interested in those unital linear contractions $S : C(Y) \rightarrow C(X)$ such that $S(B) = A$ and $S \circ \pi^* = \text{id}_{C(X)}$.

We begin with the following lemma. The proof is trivial and contained in Example 5.6.7.

Lemma 5.6.14. *Let X be a compact space and let A be a uniform algebra on X . Let $Y = X \times X$, let $B = A \check{\otimes} A$ and let $\pi : Y \rightarrow X$ be the coordinate projection $\pi((x_1, x_2)) = x_1$ ($(x_1, x_2) \in Y$). There exists a contractive, surjective, unital algebra homomorphism $T : C(Y) \rightarrow C(X)$ such that $T \circ \pi^* = \text{id}_{C(X)}$ and $T(B) = A$.*

We can construct many algebra homomorphisms from the injective tensor product $C(X) \check{\otimes} C(Y)$ onto $C(X)$. The following lemma gives a method for constructing such algebra homomorphisms.

Lemma 5.6.15. *Let X, Y be compact spaces, let π_X and π_Y be the coordinate projections from $X \times Y$ onto X and Y respectively, let Z be a closed subset of $X \times Y$, and let $\varphi : X \rightarrow Y$ be a continuous function. Set $\tilde{\varphi}(x) = (x, \varphi(x))$ ($x \in X$). Suppose that $\tilde{\varphi}(x) \in Z$ for all $x \in X$. Then there exists a contractive, unital algebra homomorphism $\theta : C(Z) \rightarrow C(X)$ such that $\theta(f) = f \circ \tilde{\varphi}$ for all $f \in C(Z)$, and $\theta \circ \pi_X^* = \text{id}_{C(X)}$.*

Proof. Set $\theta_0 := \tilde{\varphi}^*$. Then θ_0 is a contractive, surjective, unital linear map from $C(X \times Y)$ onto $C(X)$ satisfying $\theta_0 \circ \pi_X^* = \text{id}_{C(X)}$. Let $f \in C(Z)$. Let $f_1, f_2 \in C(X \times Y)$ be such that $f_j|_Z = f$ for $j = 1, 2$. Then we have $f_j \circ \tilde{\varphi} = f \circ \tilde{\varphi}$, for $j = 1, 2$, since $\tilde{\varphi}(x) \in Z$ for all $x \in X$. Thus we have $\theta_0(f_1) = \theta_0(f_2)$. Set $\theta := \theta_0|_{C(Z)}$. Then θ has the desired properties. $\square$

We now wish to refine the above lemma to find those homomorphisms which have range contained within a given uniform algebra. We obtain the following theorem.

Theorem 5.6.16. *Let X, Y be compact spaces, let π_X and π_Y be the coordinate projections from $X \times Y$ onto X and Y respectively, let Z be a closed subset of $X \times Y$, let A be a uniform algebra on X , let $S \subseteq C(Z)$ such that $\pi_X^*(A) \subseteq S$, and let $\varphi : X \rightarrow Y$ be a continuous function. Set $\tilde{\varphi}(x) = (x, \varphi(x))$ ($x \in X$). Suppose that, for all $x \in X$, we have $\tilde{\varphi}(x) \in Z$ and that, for all $f \in S$, we have $f \circ \tilde{\varphi} \in A$. Then there exists a contractive, unital algebra homomorphism $\theta : C(Z) \rightarrow C(X)$ such that $\theta \circ \pi_X^* = \text{id}_{C(X)}$ and $\theta(f) = f \circ \tilde{\varphi}$ for all $f \in C(Z)$. Moreover, we have $\theta(\overline{\text{alg}_{C(Z)} S}) = A$.*

Proof. Let θ be a contractive, surjective, unital algebra homomorphism from $C(Z)$ onto $C(X)$ such that $\theta \circ \pi_X^* = \text{id}_{C(X)}$ given by Lemma 5.6.15. Set $B := \overline{\text{alg}_{C(Z)} S}$. We need only show that $\theta(B) \subseteq A$. Since $f \circ \tilde{\varphi} \in A$ for all $f \in S$, it follows that

$f \circ \tilde{\varphi} \in A$ for all $f \in \text{alg}_{C(Z)} S$. By continuity, $\theta(f) \in A$ for all $f \in B$. Since $\pi_X^*(A) \subseteq S$, it follows that $\theta(B) = A$. This completes the proof. $\square$

In the above, if B separates the points of Z then B is a generalised Cole extension of A .

We also obtain the following corollary.

Corollary 5.6.17. *Let X be a compact space, let A be a uniform algebra on X , and let $q(t)$ be a monic polynomial in $A[t]$. Suppose that there exists $h \in A$ such that $q(h) = 0$. Then there exists a contractive, surjective, unital algebra homomorphism $\theta : C(X^q) \rightarrow C(X)$ such that $\theta(p_q) = h$, $\theta(A^q) = A$, and $\theta \circ \pi^* = \text{id}_{C(X)}$.*

Proof. We have $A^q = \overline{\text{alg}}_{C(X^q)}\{\pi^*(A) \cup \{p_q\}\}$, and $p_q(x, h(x)) = h(x)$ for all $x \in X$. By Corollary 5.6.16, there exists an algebra homomorphism $\theta : C(X^q) \rightarrow C(X)$ with the desired properties. $\square$

Let X , A and q be as in the above corollary. If there exists $h \in C(X)$ such that $q(h) = 0$, then the above shows that there is a contractive, surjective, unital algebra homomorphism $\theta : C(X^q) \rightarrow C(X)$ such that $\theta(A^q) = \overline{\text{alg}}_{C(X)}(A \cup \{h\})$. We refer the reader to [55, Section III.9] for related results.

5.7 Open problems and discussion

We now discuss some open problems related to the content of this chapter. Throughout this section let X be a compact space and let A be a uniform algebra on X .

Our first question is regarding the nature of certain generalised Cole extensions.

Question 5.7.1. Let Y be a compact space and let B be a uniform algebra on Y . Suppose there exists a continuous surjection $\pi : Y \rightarrow X$ and a contractive, unital linear map $T : B \rightarrow A$ such that $T \circ \pi^* = \text{id}_A$. Does it follow that B is a generalised Cole extension of A ?

An equivalent question to the above is: *does the map T extend to a contractive, unital map from $C(Y)$ into $C(X)$?*

Question 5.7.2. Let Y be a compact space and let B be a uniform algebra on Y . Suppose that B is a generalised Cole extension of A and that A is natural on X . Does it follow that B is natural on Y ?

It does not seem likely that the answer to the above is affirmative in general. However, we do not know any explicit counter-examples.

The following questions concern Cole extensions and generalised Cole extensions. Below Theorem 5.5.6, we noted that if $q(t) = t^n - h$ for some $h \in A$, we noted (by considering symmetric polynomials) that there is a continuous, unital projection $P : A^q \rightarrow A^q$ such that $P(p_q^j) = 0$ for $j = 1, \dots, n-1$.

Question 5.7.3. Let $q(t)$ be a monic polynomial in $A[t]$. Does there exist a continuous, unital projection $P : A^q \rightarrow A^q$ such that $P(p_q^j) = 0$ for $j = 1, \dots, n-1$?

Assume that such a projection P exists, that P is an A -module homomorphism, and that $q(t) = h_0 + h_1 t + \dots + h_{n-1} t^{n-1} + t^n$. Then elementary calculations show that we must have $P(p_q^n) = -\pi_q^*(h_0)$, and

$$Q(p_q^n) = -p_q(\pi_q^*(h_1) + \pi_q^*(h_2)p_q + \dots + \pi_q^*(h_{n-1})p_q^{n-2}).$$

If $h_j \neq 0$ for some $j \in \{1, \dots, n-1\}$ then $Q(p_q^n) = 0$ implies that p_q is a zero divisor and, by Lemma 5.5.5, h_0 must also be a zero divisor.

CHAPTER 6

Extensions of derivations

In this chapter we discuss the extension of derivations from Banach algebras. We give results towards an open problem concerning weakly amenable uniform algebras, which is the motivation for the content of this chapter. Recall, from Chapter 1, that a Banach algebra $\mathfrak{A}$ is amenable if every continuous derivation from $\mathfrak{A}$ into a $\mathfrak{X}^*$ is inner (where $\mathfrak{X}$ is a Banach $\mathfrak{A}$ -bimodule). A Banach algebra $\mathfrak{A}$ is weakly amenable if every continuous derivation from $\mathfrak{A}$ into $\mathfrak{A}^*$ is inner. If $\mathfrak{A}$ is commutative, then $\mathfrak{A}$ is weakly amenable if every continuous derivation into $\mathfrak{A}^*$ is necessarily zero.

Amenability for Banach algebras was introduced by Johnson [69], where he proved that the group algebra $L^1(G)$, for a locally compact group G , is amenable if and only if G is amenable. (We refer the reader to [69] for the relevant definitions.) Weak amenability for commutative Banach algebras was introduced by Bade, Curtis, and Dales [8]. This definition was later extended to non-commutative Banach algebras by Johnson. In fact, a commutative Banach algebra A is weakly amenable if and only if there are no non-zero, continuous derivations from A into any commutative Banach A -bimodule. (This was given as the definition of weak amenability in [8].) An alternative characterisation of weak amenability was given by Grønbæk [58] via the projective tensor product.

Let A be a uniform algebra on a compact space X . Šeĭnberg [98] showed that A is amenable if and only if $A = C(X)$ (see also [27, Chapter 5]). The problem of whether the a uniform algebra A is weakly amenable if and only if $A = C(X)$ is still open. It is this question which provides the motivation for the content of this chapter. A survey of known results of amenability and weak amenability for uniform algebras can be found in [39].

In his thesis [36], Feinstein showed that, for a compact plane set X , if $R(X)$ is non-trivial then there is a non-zero, continuous derivation from A into a locally

bounded (F) - $R(X)$ -bimodule, thus concluding that $R(X) = C(X)$ if and only if $R(X)$ is not weakly (F) -amenable. (We refer the reader to [36] for the relevant definitions.) In [38], Feinstein constructed a non-trivial uniform algebra on a compact space X such that A is not weakly amenable, but does not admit any non-zero, bounded point derivations. (See also [48].)

Cole and Feinstein investigated extensions of derivations in an attempt to construct a non-trivial, weakly amenable uniform algebra. Their strategy was to find generalised Cole extensions of a uniform algebra such that if a continuous derivation had an extension then the operator norm of the extension must strictly increase. If such constructions are possible then, by constructing systems of these extensions, we can construct a non-trivial, weakly amenable uniform algebra. Unfortunately, no such constructions are known.

In this chapter, we discuss several results regarding the extension of derivations from uniform algebras (Section 6.1). We give the details of a construction due to Cole and Feinstein [23] which leads to an interesting dichotomy (Section 6.2). We then discuss a specific example of the extension of derivations from the disk algebra (Section 6.3). We conclude the chapter with some open problems and a short discussion.

6.1 Extensions of derivations

In this section, we discuss the ways in which we can extend a derivation from a uniform algebra into its dual to a derivation from a Cole extension of that algebra.

We begin with a lemma which is surely well known, but we have not been able to find a reference. The proof is trivial and omitted.

Lemma 6.1.1. *Let $\mathfrak{B} \subseteq \mathfrak{A}$ be algebras, let $S \subseteq \mathfrak{A}$, and let $\mathfrak{X}$ be a $\mathfrak{A}$ -bimodule. Let $D : \mathfrak{A} \rightarrow \mathfrak{X}$ be a linear map. Suppose that, for each $a, b \in S$, we have*

$$D(ab) = a \cdot D(b) + D(a) \cdot b.$$

If $\text{lin}_{\mathfrak{A}} S = \mathfrak{B}$ then $D|_{\mathfrak{B}}$ is a derivation.

The next lemma extends Lemma 6.1.1 for Banach algebras, and is similar to [20, Lemma 4.1]. Again this lemma is probably well known but we have not been able to locate an explicit reference.

Lemma 6.1.2. *Let $\mathfrak{A}$ be a Banach algebra, let $S \subseteq \mathfrak{A}$, and let $\mathfrak{X}$ be a Banach $\mathfrak{A}$ -bimodule. Let $D \in \mathcal{B}(\mathfrak{A}; \mathfrak{X})$. Suppose that, for each $a, b \in S$, we have*

$$D(ab) = a \cdot D(b) + D(a) \cdot b.$$

If $\text{lin}_{\mathfrak{A}} S$ is a subalgebra of $\mathfrak{A}$ and $\overline{\text{lin}_{\mathfrak{A}} S} = \mathfrak{A}$ then D is a continuous derivation.

Proof. Let $\mathfrak{B} = \text{lin}_{\mathfrak{A}} S$. By Lemma 6.1.1, $D|_{\mathfrak{B}}$ is a continuous derivation. Thus D extends to be a continuous derivation on $\mathfrak{A}$. $\square$

The next lemma provides a useful method for extending derivations from one algebra to another by using algebra homomorphisms and module homomorphisms. The result is certainly well known. The proof is routine and we omit the details.

Lemma 6.1.3. *Let $\mathfrak{A}, \mathfrak{B}$ be algebras, let $\mathfrak{X}$ be an $\mathfrak{A}$ -bimodule and let $\mathfrak{Y}$ be a $\mathfrak{B}$ -bimodule. Let $\theta : \mathfrak{A} \rightarrow \mathfrak{B}$ be an algebra homomorphism and let $T : \mathfrak{Y} \rightarrow \mathfrak{X}$ be a linear map. Suppose that T is a $\mathfrak{A}$ -module homomorphism. Let $D : \mathfrak{B} \rightarrow \mathfrak{Y}$ be a derivation. Then $T \circ D \circ \theta : \mathfrak{A} \rightarrow \mathfrak{X}$ is a derivation.*

Note that if $\mathfrak{A}$ and $\mathfrak{B}$ are Banach algebras, $\mathfrak{X} = \mathfrak{A}^*$, $\mathfrak{Y} = \mathfrak{B}^*$ and $T = \theta^*$ in the above, then T is a $\mathfrak{A}$ -module homomorphism from $\mathfrak{X}$ into $\mathfrak{Y}$.

Remark 6.1.4. In the above, we give $\mathfrak{Y}$ the $\mathfrak{A}$ -module actions of $a \cdot y = \theta(a) \cdot y$ and $y \cdot a = y \cdot \theta(a)$ for each $y \in \mathfrak{Y}$ and $a \in \mathfrak{A}$. These are the module operations induced by θ .

The following is the continuous analogue of the above. This provides the simplest means for extending derivations to a Banach algebra extension.

Corollary 6.1.5. *Let $\mathfrak{A}, \mathfrak{B}$ be Banach algebras, let $\mathfrak{X}$ be a Banach $\mathfrak{A}$ -bimodule and let $\mathfrak{Y}$ be a Banach $\mathfrak{B}$ -bimodule. Let $\theta : \mathfrak{A} \rightarrow \mathfrak{B}$ be a contractive algebra homomorphism and let $T \in \mathcal{B}(\mathfrak{Y}; \mathfrak{X})$. Suppose that T is a $\mathfrak{A}$ -module homomorphism. Let $D : \mathfrak{B} \rightarrow \mathfrak{Y}$ be a continuous derivation. Then $\tilde{D} := T \circ D \circ \theta : \mathfrak{A} \rightarrow \mathfrak{X}$ is a continuous derivation and $\|\tilde{D}\| \leq \|T\| \|\theta\| \|D\|$.*

In particular, if $\|T\| \leq 1$ and $\|\theta\| \leq 1$ then we have $\|\tilde{D}\| \leq \|D\|$. If, in the above, $\mathfrak{X} = \mathfrak{A}^*$, $\mathfrak{Y} = \mathfrak{B}^*$, and $T = \theta^*$, then the conditions of this corollary are automatically satisfied.

We now examine how derivations can be extended to the direct limit of a direct system of Banach algebras. This following lemma is probably well known; we include the details for the convenience of the reader.

Lemma 6.1.6. *Let $S = ((\mathfrak{A}_i)_{i \in I}, (\theta_{i,j})_{i \leq j})$ be an direct system in the category of Banach algebras and suppose that the connecting maps $\theta_{i,j}$ are isometric for all $i \leq j$. Let $(\mathfrak{A}, (\theta_i)_{i \in I})$ be the direct limit of S . Let $\mathfrak{X}$ be a Banach $\mathfrak{A}$ -bimodule and suppose that, for each $i \in I$, $D_i : \mathfrak{A}_i \rightarrow \mathfrak{X}$ is a continuous derivation such that $D_i = D_j \circ \theta_{i,j}$ for all $i \leq j$. Then there exists a unique continuous derivation $D : \mathfrak{A} \rightarrow \mathfrak{X}$ such that $D_i = D \circ \theta_i$ for all $i \in I$. Moreover, $\|D\| = \limsup_i \|D_i\|$.*

Proof. For each $i \in I$, we give $\mathfrak{X}$ the module actions of $a \cdot x = \theta_i(a) \cdot x$ and $x \cdot a = x \cdot \theta_i(a)$ ($x \in \mathfrak{X}, a \in \mathfrak{A}_i$). The $\mathfrak{X}$ is a Banach $\mathfrak{A}_i$ -module for each $i \in I$. Set $\mathfrak{B} := \mathfrak{A} \oplus_1 \mathfrak{X}$ and, for each $i \in I$, let $\psi_i : \mathfrak{A}_i \rightarrow \mathfrak{B}$ be the map given by $\psi_i(a) = (\theta_i(a), D_i(a))$ ($a \in \mathfrak{A}_i$). By Proposition 1.4.9, $\mathfrak{B}$ is a Banach algebra with multiplication given by $(a, x)(a', x') = (aa', a \cdot x' + x \cdot a')$ for $(a, x), (a', x') \in \mathfrak{B}$.

Let $i, j \in I$ with $i \leq j$ and let $a \in \mathfrak{A}_i$. Then we have

$$\psi_i(a) = (\theta_i(a), D_i(a)) = ((\theta_j \circ \theta_{i,j})(a), (D_j \circ \theta_{i,j})(a)) = (\psi_j \circ \theta_{i,j})(a).$$

This holds for all $a \in \mathfrak{A}_i$, so we have $\psi_i = \psi_j \circ \theta_{i,j}$. The maps ψ_i ($i \in I$) are continuous algebra homomorphisms.

By the universal property of direct limits, there exists a unique, continuous algebra homomorphism $\psi : \mathfrak{A} \rightarrow \mathfrak{B}$ such that $\psi_i = \psi \circ \theta_i$ for all $i \in I$. Let $\pi : \mathfrak{B} \rightarrow \mathfrak{X}$ be the natural projection, and set $D = \pi \circ \psi$. Note also that $D_i = \pi \circ \psi_i$ for all $i \in I$. It is not hard to see that D is indeed a derivation and, since π is contractive, D is continuous.

It is easy to see that, for all $i \in I$, we have $\|\psi_i\| = 1 + \|D_i\|$. Also, since the connecting maps $\theta_{i,j}$ ($i \leq j$) and the maps θ_i ($i \in I$) are isometric, we have $\|\psi_i\| = \|\psi_j \circ \theta_{i,j}\| \leq \|\psi_j\|$ ($i \leq j$) and $\|\psi_i\| \leq \|\psi\|$ ($i \in I$). Thus

$$\sup_{i \in I} \|\psi_i\| = \lim_{i \in I} \|\psi_i\| = \limsup_{i \in I} \|\psi_i\| \leq \|\psi\|.$$

On the other hand, since $\bigcup_{i \in I} \theta_i(\mathfrak{A}_i)$ is dense in $\mathfrak{A}$, we cannot have $\|\psi\| > \sup_i \|\psi_i\|$. Thus $\|\psi\| = \sup_{i \in I} \|\psi_i\|$. It now follows easily that $\|D\| = \limsup_{i \in I} \|D_i\|$. This completes the proof. $\square$

Remark 6.1.7. In particular, if all of the norms of the derivations D_i in Lemma 6.1.6 are equal, then the unique derivation D given by the lemma has $\|D\| = \|D_i\|$ for all $i \in I$.

In the remainder of this section, we aim to prove that, under certain technical assumptions, in order to extend a continuous derivation it is sometimes sufficient to have a contractive linear map rather than a continuous algebra homomorphism.

We shall require the following elementary lemma; we include the details for the convenience of the reader.

Lemma 6.1.8. *Let $\mathfrak{A}$ be an algebra, and let $P : \mathfrak{A} \rightarrow \mathfrak{A}$ be a projection. Suppose that $\mathfrak{B} := P(\mathfrak{A})$ is an algebra and that $P, Q := \text{id}_{\mathfrak{A}} - P$ are $\mathfrak{B}$ -module homomorphisms. Set $E := Q(\mathfrak{A})$. Suppose further that $Q(E^2) = \{0\}$. Then*

$$Q(ax) = P(a)x \quad \text{and} \quad Q(xa) = xP(a) \quad (x \in E, a \in \mathfrak{A}).$$

Proof. Let $a \in \mathfrak{A}$ and let $x \in E$. Then

$$Q(ax) = Q((P(a) + Q(a))x) = Q(P(a)x) + Q(Q(a)x).$$

But, since $Q(a), x \in E$, we have $Q(Q(a)x) = 0$ and, since Q is a $\mathfrak{B}$ -module homomorphism, we have $Q(P(a)x) = P(a)Q(x) = P(a)x$. Thus $Q(ax) = P(a)x$. Similarly $Q(xa) = xP(a)$. $\square$

The main result of this section is the following theorem.

Theorem 6.1.9. *Let $\mathfrak{B} \subseteq \mathfrak{A}$ be Banach algebras. Let P be a continuous, unital projection on $\mathfrak{A}$ such that $P(\mathfrak{A}) = \mathfrak{B}$ and set $Q := \text{id}_{\mathfrak{A}} - P$. Suppose that P is a $\mathfrak{B}$ -module homomorphism, $E := Q(\mathfrak{A})$ is a Banach $\mathfrak{B}$ -bimodule, and $Q(E^2) = \{0\}$. Suppose further that there exist bounded $\mathfrak{B}$ -module homomorphisms $\delta, \mu : E \rightarrow \mathfrak{B}$ such that*

$$P(xy) = xy = \delta(x)\mu(y) = \mu(x)\delta(y) \tag{6.1}$$

for all $x, y \in E$. Let $D_1 \in \mathcal{D}(\mathfrak{B})$ and set

$$D_2 = (\mu^* \circ D_1 \circ \delta + \delta^* \circ D_1 \circ \mu)/2.$$

Then $D := P^* \circ D_1 \circ P + Q^* \circ D_2 \circ Q$ is a continuous derivation from $\mathfrak{A}$ into $\mathfrak{A}^*$. Moreover, for all $a \in \mathfrak{A}$, we have

$$\|D(a)\| \leq \|D_1\|(\|P\|\|P(a)\| + \|\delta\|\|\mu\|\|Q\|\|Q(a)\|).$$

Proof. The maps $\mu, \delta, \mu^*, \delta^*$ are bounded linear maps, and D_1 is a derivation so clearly D_2 is a bounded linear map from E into E^* . By Lemma 6.1.1, it is sufficient to show that D satisfies the derivation identity on a set S such that $\mathfrak{A} = \text{lin}_{\mathfrak{A}} S$. In particular, it is sufficient to show that D satisfies the derivation identity on the set $S = \mathfrak{B} \cup E$. It is clear that $D|_{\mathfrak{B}}$ is a derivation. The bounds on the norm of D are immediate.

Since $Q(E^2) = \{0\}$ we have $E^2 = P(E^2)$ and, by Lemma 6.1.8, we have

$$\delta(P(a)x) = \delta(Q(ax)), \quad \delta(xP(a)) = \delta(Q(xa)), \quad (6.2)$$

$$\mu(P(a)x) = \mu(Q(ax)), \quad \mu(xP(a)) = \mu(Q(xa)), \quad (6.3)$$

for all $x \in E$ and $a \in \mathfrak{A}$.

Let $a, b \in E$ and let $c \in \mathfrak{A}$. Then, applying (6.1), we have

$$\begin{aligned} D(ab)(c) &= D_1(P(ab))(P(c)) + D_2(Q(ab))(Q(c)) \\ &= D_1(ab)(P(c)) \\ &= \frac{1}{2}(D_1(\delta(a)\mu(b))(P(c)) + D_1(\mu(a)\delta(b))(P(c))) \\ &= \frac{1}{2}(\delta(a) \cdot D_1(\mu(b))(P(c)) + D_1(\delta(a))(P(c)) \cdot \mu(b) \\ &\quad + \mu(a) \cdot D_1(\delta(b))(P(c)) + D_1(\mu(a))(P(c)) \cdot \delta(b)) \\ &= \frac{1}{2}(D_1(\mu(b))(\delta(P(c)a)) + D_1(\delta(a))(\mu(bP(c))) \\ &\quad + D_1(\delta(b))(\mu(P(c)a)) + D_1(\mu(a))(\delta(bP(c)))). \end{aligned}$$

Now, applying (6.2) and (6.3), we obtain

$$\begin{aligned} D(ab)(c) &= \frac{1}{2}(D_1(\mu(b))(\delta(P(c)a)) + D_1(\delta(a))(\mu(bP(c))) \\ &\quad + D_1(\delta(b))(\mu(P(c)a)) + D_1(\mu(a))(\delta(bP(c)))) \\ &= \frac{1}{2}(D_1(\mu(b))(\delta(Q(ca))) + D_1(\delta(a))(\mu(Q(bc))) \\ &\quad + D_1(\delta(b))(\mu(Q(ca))) + D_1(\mu(a))(\delta(Q(bc)))) \\ &= (Q^* \circ D_2 \circ Q)(b)(ca) + (Q^* \circ D_2 \circ Q)(a)(bc) \end{aligned}$$

Similar, but slightly easier, calculations show that $D(ab) = a \cdot D(b) + D(a) \cdot b$ and that $D(ba) = b \cdot D(a) + D(b) \cdot a$ for all $a \in E$ and all $b \in \mathfrak{B}$.

Thus D is a derivation from $\mathfrak{A}$ into $\mathfrak{A}^*$. □

Note that the proof of Theorem 6.1.9 holds without the topological assumptions (continuity and norm structure) if we replace $\mathfrak{B}^*$ and $\mathfrak{A}^*$ with a subspace of the algebraic dual of $\mathfrak{B}$ and $\mathfrak{A}$, respectively. (The algebraic dual is the linear space of all linear functionals.)

We can apply this theorem directly in the case where we adjoin the square roots of certain functions to obtain the following corollary.

Corollary 6.1.10. *Let X be a compact space, let A be a uniform algebra on X , and let $h \in A$. Set $q(t) := t^2 - h$. Suppose that h has no zeros on $\Gamma(A)$.*

Let $D \in \mathcal{D}(A)$. Then there exists $\tilde{D} \in \mathcal{D}(A^q)$ with $\tilde{D}(\pi_q^*(f))(\pi_q^*(g)) = D(f)(g)$ ($f, g \in A$).

Proof. By Theorem 5.5.6, $A^q = \pi_q^*(A) + p_q \cdot \pi_q^*(A)$ and the bounded linear map $\mu : C(X^q) \rightarrow C(X^q)$, given by $\mu(f) = p_q f$ ($f \in C(X^q)$), is bounded below on A^q . (Note also that μ is injective.) Let $\delta : p_q \cdot A^q \rightarrow A^q$ be the map $\delta(p_q f) = f$ ($f \in A^q$). It is easy to see that both μ and δ are $\pi_q^*(A)$ -module homomorphisms.

By Theorem 5.5.2, there exists a contractive, unital projection $P : A^q \rightarrow A^q$ with $P(A^q) = \pi_q^*(A)$, $P(p_q) = 0$, and P is a $\pi_q^*(A)$ -module homomorphism. Set $Q := \text{id}_{A^q} - P$. Then Q is a continuous projection and is also a $\pi_q^*(A)$ -module homomorphism. Set $E := Q(A^q)$. Clearly $E^2 \subseteq \pi_q^*(h) \cdot \pi_q^*(A) \subseteq \pi_q^*(A)$.

Let $f \in A^q$. Then there exists $f_1, f_2 \in \pi_q^*(A)$ such that $f = f_1 + p_q f_2$. Moreover, $P(f) = f_1$ and $Q(f) = p_q f_2$. Let $g := p_q(f_1 + p_q f_2) = \pi_q^*(h)f_2 + p_q f_1$. Thus $P(g) = \pi_q^*(h)f_2$ and, since P is contractive, $|\pi_q^*(h)f_2|_{X^q} \leq |g|_{X^q}$. We have $|g|_{X^q} \leq |p_q|_{X^q}|f|_{X^q}$ and, for each $(x, \xi) \in X^q$, we have

$$|p_q^2(x, \xi)f_2(x, \xi)| \leq |\pi_q^*(h)f_2|_{X^q} \leq |g|_{X^q} \leq |p_q|_{X^q}|f|_{X^q}.$$

But this implies that, for each $(x, \xi) \in X^q$, we have

$$|p_q(x, \xi)| |p_q(x, \xi)f(x, \xi)| \leq |p_q|_{X^q}|f|_{X^q},$$

and it follows that $|p_q|_{X^q}|p_q f_2|_{X^q} \leq |p_q|_{X^q}|f|_{X^q}$. Since $|p_q|_{X^q} \neq 0$, we have $|Q(f)|_{X^q} \leq |f|_{X^q}$ and hence Q is also a contractive projection.

Let $\psi : \pi_q^*(A) \rightarrow A$ be the inverse of the restriction to $\pi_q^*(A)$ of π_q^* . Then ψ is an isometric isomorphism from $\pi_q^*(A)$ onto A . Set $D_1 := \psi^* \circ D \circ \psi$. Then D_1 is a continuous derivation from $\pi_q^*(A)$ into $(\pi_q^*(A))^*$.

Let $f_1, f_2 \in E$. Then there exists $h_1, h_2 \in A$ such that $f_j = h_j p_q$ for $j = 1, 2$. Thus

$$P(f_1 f_2) = f_1 f_2 = h_1 p_q h_2 p_q = p_q^2 h_1 h_2,$$

from which we obtain $P(f_1 f_2) = \mu(f_1)\delta(f_2) = \delta(f_1)\mu(f_2)$.

Now, by Theorem 6.1.9, there exists a continuous derivation $\tilde{D} : A^q \rightarrow (A^q)^*$, which extends D_1 , and such that

$$\|D\| \leq \|\tilde{D}\| \leq \|D\|(1 + \|\mu\|\|\delta\|).$$

It is easy to see that $\tilde{D}$ has the desired properties. □

Remark 6.1.11. The author is grateful to Dr. Yemon Choi for several helpful discussions about derivations and the homology of Banach algebras.

6.2 An interesting dichotomy

Let A be a commutative Banach algebra. In this section, $\mathcal{D}_0(A)$ denotes the collection of non-zero, continuous derivations from A into A^* . In this section we give the full details of a construction, due to Cole and Feinstein ([23]), that leads to an interesting dichotomy.

We require the following lemmas. The first is certainly well known; we include the details for the convenience of the reader (and because it is very nice proof).

Lemma 6.2.1. *Let ω_1 be the first uncountable ordinal. Let $(x_\alpha)_{\alpha < \omega_1}$ be a net of positive real numbers such that if $\alpha < \beta < \omega_1$ then $x_\alpha < x_\beta$. Then $(x_\alpha)_{\alpha < \omega_1}$ does not converge in $[0, \infty)$.*

Proof. Suppose, towards a contradiction, that there exists $x \in [0, \infty)$ such that $x_\alpha \rightarrow x$. For each $n \in \mathbb{N}$, there exists an ordinal $\alpha_n < \omega_1$ such that, for each $\alpha \geq \alpha_n$, we have $|x_\alpha - x| < 1/n$. Set $\beta = \sup_{n \in \mathbb{N}} \alpha_n$. Then $\beta < \omega_1$ and, for each $\alpha < \omega_1$ with $\alpha \geq \beta$ we have $|x_\alpha - x| < 1/n$ for all $n \in \mathbb{N}$ so that $x_\alpha = x$. This is a contradiction. $\square$

The next lemma is clear.

Lemma 6.2.2. *Let X, Y be compact spaces, let A be a uniform algebra on X , and let B be a uniform algebra on Y . Suppose that B is a generalised Cole extension of A with associated maps $\pi : Y \rightarrow X$ and $T : C(Y) \rightarrow C(X)$. Let $D \in \mathcal{D}(B)$. Then there exists $D_0 \in \mathcal{D}(A)$ such that $D(\pi^*(f))(\pi^*(g)) = D_0(f)(g)$ for all $f, g \in A$.*

The next lemma provides the construction in this section.

Lemma 6.2.3. *Suppose that, for every non-trivial uniform algebra A_0 on a compact space X_0 , and each $D \in \mathcal{D}_0(A)$ there exist a compact space X_1 and a uniform algebra A_1 on X_1 with the following properties:*

- (a) A_1 is a generalised Cole extension of A_0 with associated maps $\pi_1 : X_1 \rightarrow X_0$ and $T_1 : C(X_1) \rightarrow C(X_0)$;
- (b) if $\tilde{D} \in \mathcal{D}_0(A_1)$ such that $\tilde{D}(\pi_1^*(f))(\pi_1^*(g)) = D(f)(g)$, for all $f, g \in A_0$, then $\|\tilde{D}\| > \|D\|$.

Let A be a uniform algebra on a compact space X . Then there exists a uniform algebra B on a compact space Y such that B is a generalised Cole extension of A , with associated maps $\pi : Y \rightarrow X$ and $T : C(Y) \rightarrow C(X)$, and such that, for

each $D \in \mathcal{D}_0(A)$ and each $\tilde{D} \in \mathcal{D}_0(B)$, we have $\tilde{D}(\pi^*(f))(\pi^*(g)) = D(f)(g)$, for all $f, g \in A$, we have $\|\tilde{D}\| > \|D\|$.

Proof. We may assume, without loss of generality, that $\mathcal{D}(A) \neq \{0\}$. Fix an ordinal τ such that there exists a bijection $\psi : \{\alpha : \alpha < \tau\} \rightarrow \mathcal{D}_0(A)$. We construct a system of generalised Cole extensions $((X_\alpha, \pi_{\alpha,\beta}), (A_\alpha, T_{\alpha,\beta}))_{\alpha \leq \beta \leq \tau}$ satisfying the following condition: for each $\alpha < \beta \leq \tau$, if there exists $\tilde{D} \in \mathcal{D}(A_\beta)$

$$\tilde{D}(\pi_{0,\beta}^*(f))(\pi_{0,\beta}^*(g)) = D(f)(g) \quad (f, g \in A_0), \quad (6.4)$$

where $D = \psi(\alpha) \in \mathcal{D}_0(A_0)$, then $\|\tilde{D}\| > \|D\|$.

Set $X_0 := X$, $A_0 := A$, $\pi_{0,0} := \text{id}_X$, and $T_{0,0} := \text{id}_{C(X)}$. The condition above holds for $\beta = 0$.

Assume now that, for some ordinal $\gamma \leq \tau$, we have constructed the system of generalised Cole extensions $((X_\alpha, \pi_{\alpha,\beta}), (A_\alpha, T_{\alpha,\beta}))_{\alpha \leq \beta < \gamma}$ satisfying the condition above. We extend this system as follows.

If γ is a successor ordinal, say $\gamma = \delta + 1$, then A_δ is a non-trivial uniform algebra. Let $D = \psi(\delta)$. If there is no $\tilde{D} \in \mathcal{D}_0(A_\delta)$ satisfying (6.4) (with $\beta = \delta$), then set $X_\gamma := X_\delta$, $A_\gamma := A_\delta$, $\pi_{\delta,\gamma} := \text{id}_{X_\delta}$, and $T_{\delta,\gamma} := \text{id}_{C(X_\delta)}$.

Otherwise, there exists $D_\delta \in \mathcal{D}_0(A_\delta)$ satisfying (6.4). By the assumptions of the lemma, there exists a uniform algebra A_γ on a compact space X_γ such that A_γ is a generalised Cole extension of A_δ , with associated maps $\pi_{\delta,\gamma} : X_\gamma \rightarrow X_\delta$ and $T_{\delta,\gamma} : C(X_\gamma) \rightarrow C(X_\delta)$ and such that, if $\tilde{D} \in \mathcal{D}_0(A_\gamma)$ such that

$$\tilde{D}(\pi_{\delta,\gamma}^*(f))(\pi_{\delta,\gamma}^*(g)) = D_\delta(f)(g),$$

for all $f, g \in A_\delta$, then $\|\tilde{D}\| > \|D_\delta\|$.

In either case set $\pi_{\gamma,\gamma} := \text{id}_{X_\gamma}$, $T_{\gamma,\gamma} := \text{id}_{C(X_\gamma)}$ and, for each $\alpha < \delta$, set $\pi_{\alpha,\gamma} := \pi_{\alpha,\delta} \circ \pi_{\delta,\gamma}$ and $T_{\alpha,\gamma} := T_{\alpha,\delta} \circ T_{\delta,\gamma}$. Then $((X_\alpha, \pi_{\alpha,\beta}), (A_\alpha, T_{\alpha,\beta}))_{\alpha \leq \beta \leq \gamma}$ is the desired extension of the system of generalised Cole extensions.

If γ is a limit ordinal then let $(X_\gamma, (\pi_{\alpha,\gamma})_{\alpha < \gamma})$ be the inverse limit of the inverse system $(X_\alpha, (\pi_{\alpha,\beta})_{\alpha \leq \beta < \gamma})$, and let $(A_\gamma, (\pi_{\alpha,\gamma}^*)_{\alpha < \gamma})$ be the corresponding direct limit of direct system $(A_\alpha, \pi_{\alpha,\beta}^*)_{\alpha \leq \beta < \gamma}$. Set $\pi_{\gamma,\gamma} := \text{id}_{X_\gamma}$ and $T_{\gamma,\gamma} := \text{id}_{C(X_\gamma)}$. By Lemma 5.1.6, there exist bounded linear maps $T_{\alpha,\gamma} : C(X_\gamma) \rightarrow C(X_\alpha)$ ($\alpha < \gamma$) such that $T_{\alpha,\gamma}(1) = 1$, $\|T_{\alpha,\gamma}\| = 1$, $T_{\alpha,\gamma} \circ \pi_{\alpha,\gamma}^* = \text{id}_{C(X_\alpha)}$, and $T_{\alpha,\gamma}(A_\gamma) = A_\alpha$ for all $\alpha < \gamma$.

Let $\alpha < \gamma$, let $D = \psi(\alpha)$, and let $\delta = \alpha + 1$. If there exists $\tilde{D} \in \mathcal{D}_0(A_\gamma)$ such that (6.4) holds (with $\beta = \gamma$), then—by Lemma 6.2.2—there exists $D_\delta \in \mathcal{D}_0(A_\delta)$ such that

$$\tilde{D}(\pi_{0,\gamma}^*(f))(\pi_{0,\gamma}^*(g)) = D_\delta(\pi_{0,\delta}^*(f))(\pi_{0,\delta}^*(g)) = D(f)(g),$$

for all $f, g \in A_0$. Thus $\|\tilde{D}\| \geq \|D_\delta\| > \|D\|$.

It follows that $((X_\alpha, \pi_{\alpha,\beta}), (A_\alpha, T_{\alpha,\beta}))_{\alpha \leq \beta \leq \gamma}$ is the desired extension of the system of generalised Cole extensions.

The induction proceeds and we obtain a system of generalised Cole extensions $((X_\alpha, \pi_{\alpha,\beta}), (A_\alpha, T_{\alpha,\beta}))_{\alpha \leq \beta \leq \tau}$ satisfying the specified conditions. Let $B := A_\tau$, $Y := X_\tau$, $\pi := \pi_{0,\tau}$, and $T := T_{0,\tau}$. Then B is the desired generalised Cole extension of A . $\square$

The main result of this section is the following.

Theorem 6.2.4. *There exists a non-trivial uniform algebra A on a compact space X with the following property: for every $D \in \mathcal{D}_0(A)$ and every uniform algebra B on a compact space Y , such that B is a generalised Cole extension of A with associated maps $\pi : Y \rightarrow X$ and $T : C(Y) \rightarrow C(X)$, if there exists $\tilde{D} \in \mathcal{D}_0(B)$ such that $\tilde{D}(\pi^*(f))(\pi^*(g)) = D(f)(g)$ for all $f, g \in A$ then $\|\tilde{D}\| = \|D\|$.*

Proof. Assume, towards a contradiction, that no such uniform algebra exists. In particular, this implies that there does not exist non-trivial, weakly amenable uniform algebras. Now, for every non-trivial uniform algebra A_0 on a compact space X_0 and every $D \in \mathcal{D}_0(A_0)$, there exists a uniform algebra A_1 on a compact space X_1 satisfying (a) and (b) in Lemma 6.2.3. Let ω_1 denote the first uncountable ordinal. We apply Lemma 6.2.3 inductively to construct a non-trivial, weakly amenable uniform algebra, and hence obtain a contradiction. To do this, we construct a system of generalised Cole extensions $((X_\alpha, \pi_{\alpha,\beta}), (A_\alpha, T_{\alpha,\beta}))_{\alpha \leq \beta \leq \omega_1}$ such that, for each $\alpha < \omega_1$, if $D_\alpha \in \mathcal{D}_0(A_\alpha)$ and $\tilde{D} \in \mathcal{D}_0(A_{\alpha+1})$ with

$$\tilde{D}(\pi_{\alpha,\alpha+1}^*(f))(\pi_{\alpha,\alpha+1}^*(g)) = D_\alpha(f)(g) \quad (f, g \in A_\alpha), \quad (6.5)$$

then $\|\tilde{D}\| > \|D_\alpha\|$. Fix a non-trivial uniform algebra A_0 on a compact space X_0 . Let $\pi_{0,0} := \text{id}_{X_0}$ and $T_{0,0} := \text{id}_{C(X_0)}$.

Assume that, for some ordinal $\gamma \leq \omega_1$, we have constructed a system of generalised Cole extensions $((X_\alpha, \pi_{\alpha,\beta}), (A_\alpha, T_{\alpha,\beta}))_{\alpha \leq \beta < \gamma}$ such that, for each $\alpha < \gamma$, if $D_\alpha \in \mathcal{D}_0(A_\alpha)$ and $\tilde{D} \in \mathcal{D}_0(A_{\alpha+1})$ satisfies (6.5) then $\|\tilde{D}\| > \|D_\alpha\|$. We extend the system as follows.

First suppose that γ is a successor ordinal, say $\gamma = \delta + 1$. Then, by Lemma 6.2.3, there exists a uniform algebra A_γ on a compact space X_γ such that A_γ is a generalised Cole extension of A_δ , with associated maps $\pi_{\delta,\gamma} : X_\gamma \rightarrow X_\delta$ and $T_{\delta,\gamma} : C(X_\gamma) \rightarrow C(X_\delta)$, and, for each $D \in \mathcal{D}_0(A_\delta)$ and each $\tilde{D} \in \mathcal{D}_0(A_\gamma)$ satisfying (6.5) (with $\alpha = \delta$), we have $\|\tilde{D}\| > \|D\|$. Set $\pi_{\gamma,\gamma} := \text{id}_{X_\gamma}$, $T_{\gamma,\gamma} := \text{id}_{C(X_\gamma)}$

and, for each $\alpha < \delta$, set $\pi_{\alpha,\gamma} := \pi_{\alpha,\delta} \circ \pi_{\delta,\gamma}$ and $T_{\alpha,\gamma} := T_{\alpha,\delta} \circ T_{\delta,\gamma}$. Then $((X_\alpha, \pi_{\alpha,\beta}), (A_\alpha, T_{\alpha,\beta}))_{\alpha \leq \beta \leq \gamma}$ is the desired extension of the system of generalised Cole extensions.

Now suppose that γ is a limit ordinal. Let $(X_\gamma, (\pi_{\alpha,\gamma})_{\alpha < \gamma})$ be the inverse limit of the inverse system $(X_\alpha, (\pi_{\alpha,\beta}))_{\alpha \leq \beta < \gamma}$, and let $(A_\gamma, (\pi_{\alpha,\gamma}^*)_{\alpha < \gamma})$ be the corresponding direct limit of direct system $(A_\alpha, \pi_{\alpha,\beta}^*)_{\alpha \leq \beta < \gamma}$. Set $\pi_{\gamma,\gamma} := \text{id}_{X_\gamma}$ and $T_{\gamma,\gamma} := \text{id}_{C(X_\gamma)}$. By Lemma 5.1.6, there exist bounded linear maps $T_{\alpha,\gamma} : C(X_\gamma) \rightarrow C(X_\alpha)$ ($\alpha < \gamma$) such that, for all $\alpha < \gamma$, we have $T_{\alpha,\gamma}(1) = 1$, $\|T_{\alpha,\gamma}\| = 1$, $T_{\alpha,\gamma} \circ \pi_{\alpha,\gamma}^* = \text{id}_{C(X_\alpha)}$, and $T_{\alpha,\gamma}(A_\gamma) = A_\alpha$. It is now clear that the system of generalised Cole extensions $((X_\alpha, \pi_{\alpha,\beta}), (A_\alpha, T_{\alpha,\beta}))_{\alpha \leq \beta \leq \gamma}$ has the desired properties.

The induction may now proceed, and we obtain a system of generalised Cole extensions $((X_\alpha, \pi_{\alpha,\beta}), (A_\alpha, T_{\alpha,\beta}))_{\alpha \leq \beta \leq \omega_1}$ with the property described above. We *claim* that A_{ω_1} is weakly amenable.

We prove the claim by contradiction. Suppose that $D \in \mathcal{D}_0(A_{\omega_1})$. Then, by Lemma 6.2.2, for each $\alpha < \omega_1$ there exists $D_\alpha \in \mathcal{D}(A_\alpha)$ such that

$$D(\pi_{\alpha,\omega_1}^*(f))(\pi_{\alpha,\omega_1}^*(g)) = D_\alpha(f)(g) \quad (f, g \in A_\alpha).$$

Since D is non-zero, there exists an ordinal $\alpha_0 < \omega_1$ such that $D_{\alpha_0} \in \mathcal{D}_0(A_{\alpha_0})$. By the construction, for each ordinal α with $\alpha_0 \leq \alpha < \omega_1$, we have

$$\|D\| \geq \|D_{\alpha+1}\| > \|D_\alpha\| \geq \|D_{\alpha_0}\|.$$

In particular, $(\|D_\alpha\|)_{\alpha=\alpha_0}^{\omega_1}$ is a strictly increasing net of positive real numbers indexed by the ordinals up to the first uncountable ordinal. This net cannot have a finite limit, by Lemma 6.2.1, so we conclude D cannot be continuous. This is a contradiction, so no such D can exist. Hence A_{ω_1} is weakly amenable. This proves the claim and completes the proof. $\square$

Note that, the above construction yields a non-trivial weakly amenable uniform algebra. So we obtain the following corollary.

Corollary 6.2.5. *Exactly one of the following occurs:*

- (a) *there exists a non-trivial, weakly amenable uniform algebra;*
- (b) *there exists a non-trivial uniform algebra A with $\mathcal{D}(A) \neq \{0\}$ such that, for every generalised Cole extension B of A , with associated maps π and T , and every derivation $D \in \mathcal{D}(A)$, if there exists a derivation $\tilde{D} \in \mathcal{D}(B)$ such that*

$$\tilde{D}(\pi^*(f))(\pi^*(g)) = D(f)(g),$$

for all $f, g \in A$, then $\|\tilde{D}\| = \|D\|$.

6.3 An example: the disk algebra

In this section we discuss a specific example of the extension of a derivation from the disk algebra, to the Cole extension of the disk algebra. This example will illustrate that finding uniform algebra extensions such that derivations extend with a strictly larger norm is not trivial. In this section $A = A(\overline{\mathbb{D}})$ is the disk algebra, and let $q(t) = t^2 - Z \in A[t]$ (where $Z \in A$ is the coordinate function $z \mapsto z$ ($z \in \overline{\mathbb{D}}$)). The content of this section is discussed in [23], we give the full details of the constructions discussed there.

We begin with the following lemma. We include the details for the convenience of the reader.

Lemma 6.3.1. *Let $B := \overline{\text{alg}}_A\{Z^{2n} : n \in \mathbb{N}_0\} \subseteq A$. Then A^q is isometrically isomorphic to A , via the algebra isomorphism θ , and $\theta(\pi_q^*(A)) = B$.*

Proof. Let $\varphi : \overline{\mathbb{D}} \rightarrow \overline{\mathbb{D}}$ be the map $\varphi(z) = z^2$ ($z \in \overline{\mathbb{D}}$). Then φ is an analytic, surjective map and so φ^* is an isometric, unital algebra homomorphism from A into A , such that $\varphi^*(A) = B$.

The function Z has no zeroes on $\Gamma(A)$ (which is the unit circle $\mathbb{T}$) so, by Theorem 5.5.6(b), we have $\overline{p_q \cdot \pi_q^*(A)} = p_q \cdot \pi_q^*(A)$ and, by Corollary 5.5.8, we have

$$A^q = \pi_q^*(A) + p_q \cdot \pi_q^*(A).$$

Let $\theta : C((\overline{\mathbb{D}})^q) \rightarrow C(\overline{\mathbb{D}})$ be the map given by

$$\theta(f)(\xi) = f(\xi^2, \xi) \quad (\xi \in \overline{\mathbb{D}}, f \in C((\overline{\mathbb{D}})^q)).$$

Clearly $\theta \circ \pi_q^* = \varphi^*$, and $\theta(p_q) = Z$, by definition, so that $\theta(A^q) \subseteq A$. In fact, since A^q is polynomially generated by p_q , it follows that $\theta(A^q) = A$. It is easy to see that θ is indeed a homomorphism and is isometric. This completes the proof. $\square$

Note that B is not a uniform algebra on $\overline{\mathbb{D}}$, since it fails to separate the points of $\overline{\mathbb{D}}$.

Remark 6.3.2. For a similar result, see [89, Lemma 3.2.9]. Note also that θ in the above is simply the adjoint of the map $\xi \mapsto (\xi^2, \xi) : \overline{\mathbb{D}} \rightarrow (\overline{\mathbb{D}})^q$.

For the remainder of the section, we write $\iota : A \rightarrow A$ for the map $f \mapsto f \circ Z^2$, and θ is the isometric algebra isomorphism from Lemma 6.3.1. Let $\mu : A \rightarrow A$ be the function $\mu(f) = fZ$ ($f \in A$) and let $\delta : \mu(A) \rightarrow A$ be the function $\delta(Zf) = f$ ($f \in A$). It is easy to see that both of these maps are well-defined, continuous B -module homomorphisms such that $\|\delta\| = \|\mu\| = 1$.

We now consider extensions of derivations from the disk algebra into its dual.

Let $P : A \rightarrow A$ be the projection given by $P = \theta \circ \pi_q^* \circ T \circ \theta^{-1}$, and let $Q : A \rightarrow A$ be the projection $Q = \text{id}_A - P$. Let B be as in Lemma 6.3.1. Then $P(A) = B$ and both P and Q are contractive. It is easy to verify that the maps μ, δ, P and Q satisfy the conditions of Theorem 6.1.9, see also Corollary 6.1.10. The following lemma is an easy consequence of Theorem 6.1.9.

Lemma 6.3.3. *Let $D : A \rightarrow A^*$ be a non-zero, continuous derivation. Then*

$$\tilde{D}(\iota(f_1) + Z\iota(f_2))(\iota(g_1) + Z\iota(g_2)) := D(f_1)(g_1) + \frac{1}{2} \left(D(Zf_2)(g_2) + D(f_2)(Zg_2) \right), \quad (6.6)$$

for each $f_1, f_2, g_1, g_2 \in A$, defines a continuous derivation from A into A^* such that $(\theta \circ \pi_q^*)^* \circ \tilde{D} \circ (\theta \circ \pi_q^*) = D$. Moreover, $\|D\| \leq \|\tilde{D}\| \leq 2\|D\|$.

We investigate, in particular, the extensions of the derivation $D_0 \in \mathcal{D}(A)$ given by $D_0(f)(g) = f'(0)g(0)$ for each $f, g \in A$. Clearly D_0 is a non-zero, continuous derivation with $\|D_0\| = 1$.

Let $A_0 = A$. For each $n \in \mathbb{N}$, we define the uniform algebras $A_n = \theta(A_{n-1}^q)$. Then $A_n = A$ for all n , and A_{n-1} is isometrically embedded into A_n as the algebra B . As in [23], for each $n \in \mathbb{N}$, let $D_n \in \mathcal{D}(A)$ be the map

$$D_n(f)(g) = \frac{(f'g)^{(n)}(0)}{(n+1)!} \quad (f, g \in A_n).$$

Note that D_0 is also of the form above, when $n = 0$, and that, for each $n \in \mathbb{N}_0$, D_n is a continuous derivation from A into A^* . This sequence of derivations is also investigated in [61, Section 5.6].

Lemma 6.3.4. *Fix $n \in \mathbb{N}$. Let $\tilde{D}_n \in \mathcal{D}(A)$ be obtained by applying (6.6) to $D_n \in \mathcal{D}(A)$. Then $\tilde{D}_n = D_{2n+1}$.*

Proof. Fix $n \in \mathbb{N}_0$. Since the polynomials are dense in A and $\tilde{D}_n, D_{2n+1}$ are linear, we need only check that $\tilde{D}_n$ and D_{2n+1} agree on the powers of Z .

Let $j, k \in \mathbb{N}_0$. First note that $\tilde{D}_n(Z^{2j+1})(Z^{2k}) = \tilde{D}_n(Z^{2j})(Z^{2k+1}) = 0$. If $j = 0$, then $\tilde{D}_n(Z^{2j}) = 0$, since $\tilde{D}_n$ is a derivation, so, in the first case, we may assume that $j > 0$. We have

$$\tilde{D}_n(Z^{2j})(Z^{2k}) = \frac{1}{(n+1)!} (jZ^{j+k-1})^{(n)}(0) = \begin{cases} 0 & j+k-1 \neq n, \\ \frac{j}{j+k} & j+k-1 = n. \end{cases}$$

Similarly,

$$\tilde{D}_n(Z^{2j+1})(Z^{2k+1}) = \frac{1}{2(n+1)!} \left(((j+1)Z^{j+k})^{(n)}(0) + (jZ^{j+k})^{(n)}(0) \right)$$

and it follows that

$$\tilde{D}_n(Z^{2j+1})(Z^{2k+1}) = \begin{cases} 0 & j+k \neq n, \\ \frac{2j+1}{2(j+k+1)} & j+k = n. \end{cases}$$

On the other hand, we have

$$D_{2n+1}(Z^j)(Z^k) = \frac{1}{(2n+2)!} (jZ^{j+k-1})^{(2n+1)}(0) = \begin{cases} 0 & j+k-1 \neq 2n+1, \\ \frac{j}{j+k} & j+k-1 = 2n+1. \end{cases}$$

Note that $2n+1$ is odd so that $j+k$ must be even. It follows that

$$D_{2n+1}(Z^{2j+1})(Z^{2k}) = D_{2n+1}(Z^{2j})(Z^{2k+1}) = 0,$$

for all $j, k \in \mathbb{N}_0$. It now follows that $\tilde{D}_n(Z^j)(Z^k) = D_{2n+1}(Z^j)(Z^k)$ for all $j, k \in \mathbb{N}_0$ and so $\tilde{D}_n = D_{2n+1}$ by linearity and continuity. This completes the proof. $\square$

We now aim to show that, for each $n \in \mathbb{N}$, the norm of D_n is 1 so that the derivation D_0 extends to the Cole extension A^q and the norm is preserved. This is the content of Proposition 6.3.5, which is due to Michael White. We will give the details of the proof for the convenience of the reader. Before stating and proving the result, we first recall some standard facts about Fourier series of disk algebra functions.

Let $f \in A$. For each $n \in \mathbb{Z}$, we define

$$\hat{f}(n) := \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) e^{-int} dt.$$

We call $\hat{f}(n)$ the n th Fourier coefficient of f . By Cauchy's theorem, we have $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}$ with $n < 0$, and

$$\hat{f}(n) = \frac{f^{(n)}(0)}{n!} \quad (n \in \mathbb{Z}, n \geq 0).$$

By standard results of Fourier series (see, for example, [5, 72, 93]), we have

$$f(e^{it}) = \sum_{n=0}^{\infty} \hat{f}(n) e^{int} \quad (t \in [0, 2\pi])$$

and $\hat{f} \in \ell^2$, where ℓ^2 denotes the Hilbert space of all square-summable complex sequences and $\hat{f} = (\hat{f}(n))_{n=0}^{\infty}$. We also have $\hat{f}'(n) = in\hat{f}(n)$ for all $n \in \mathbb{N}_0$. We

also note that $|f|_{\mathbb{T}} = |f|_{\overline{\mathbb{D}}}$ (where $\mathbb{T}$ is the unit circle) and $\|\hat{f}\|_2 = \|f\|_2$ where the latter norm is taken in $L^2(\mathbb{T})$, with respect to the normalised Lebesgue measure on $\mathbb{T}$. Note also that $\|f\|_2 \leq |f|_{\mathbb{T}}$.

We can now prove the following.

Proposition 6.3.5 (White). *For each $n \in \mathbb{N}_0$, we have $\|D_n\| = 1$.*

Proof. Fix $f, g \in A$ and $n \in \mathbb{N}$. We have

$$\begin{aligned} \left| \frac{(f'g)^{(n)}(0)}{(n+1)!} \right| &= \left| \frac{1}{(n+1)!} \sum_{j=0}^n \binom{n}{j} (f')^{(j)}(0) g^{(n-j)}(0) \right| \\ &\leq \sum_{j=0}^n \left| \frac{1}{(n+1)j!} (f')^{(j)}(0) \right| \left| \frac{1}{(n-j)!} g^{(n-j)}(0) \right| \\ &= \sum_{j=0}^n \frac{j}{n+1} |\hat{f}(j)| |\hat{g}(n-j)| \\ &\leq \sum_{j=0}^n |\hat{f}(j)| |\hat{g}(n-j)| \\ &\leq \|\hat{f}\|_2 \|\hat{g}\|_2. \end{aligned}$$

Now we have $\|\hat{f}\|_2 = \|f\|_2$ and $\|\hat{g}\|_2 = \|g\|_2$, where $\|f\|_2, \|g\|_2$ are the norms in $L^2(\mathbb{T})$. But now $\|f\|_2 \leq |f|_{\overline{\mathbb{D}}}$ and $\|g\|_2 \leq |g|_{\overline{\mathbb{D}}}$. Thus $\|D_n\| \leq 1$.

The result is proved by observing that $D_n(Z^{n+1})(1) = 1$ so that $\|D_n\| \geq 1$. $\square$

We now examine the extension of arbitrary derivations from the disk algebra when we adjoin square roots to a function of the form $Z - \alpha$ ($\alpha \in \mathbb{C}$). We shall demonstrate that if $\alpha \in \mathbb{C} \setminus \mathbb{D}$ then there is a contractive, surjective, unital algebra homomorphism $\theta : A^q \rightarrow A$ such that $\theta \circ \pi_q^* = \text{id}_A$. The results in the remainder of this section are all new.

Theorem 6.3.6. *Let $\alpha \in \mathbb{C} \setminus \mathbb{D}$, let $h = Z - \alpha$, and let $\rho(t) = t^2 - h \in A[t]$. Then there exists a contractive, surjective, unital algebra homomorphism $\theta : A^\rho \rightarrow A$ with $\theta \circ \pi_\rho^* = \text{id}_A$.*

Proof. Suppose that $\alpha \in \mathbb{C} \setminus \mathbb{D}$. Then $Z - \alpha$ has no zeros on $\mathbb{D}$ so that there exists an analytic function g on $\mathbb{D}$ such that $g^2 = Z - \alpha$ on $\mathbb{D}$. Then g extends to a continuous function $\tilde{g}$ on $\overline{\mathbb{D}}$ so that $\tilde{g} \in A$. By Corollary 5.6.17, there exists a contractive, surjective, unital algebra homomorphism $\theta : A^\rho \rightarrow A$ with $\theta \circ \pi_\rho^* = \text{id}_A$. This completes the proof. $\square$

As a consequence of Theorem 6.3.6, Corollary 6.1.10, and Corollary 6.1.5 we obtain the following.

Corollary 6.3.7. *Let $D \in \mathcal{D}(A)$, let $\alpha \in \mathbb{C}$, and let $\rho(t) = t^2 - (Z - \alpha)$. Then there exists $\tilde{D} \in \mathcal{D}(A^\rho)$ such that $D(f)(g) = \tilde{D}(\pi_\rho^*(f))(\pi_\rho^*(g))$ for all $f, g \in A$. Moreover, if $\alpha \in \mathbb{C} \setminus \mathbb{D}$ then $\tilde{D}$ can be chosen such that $\|\tilde{D}\| = \|D\|$.*

6.4 Open problems and discussion

In this section, let X be a compact space and let A be a uniform algebra on X . The first problem is a famous open problem regarding weak amenability of uniform algebras.

Question 6.4.1. Suppose that A is weakly amenable. Does it follow that A is trivial?

Our results in Chapter 6 show that an interesting dichotomy exists. Either there exists a non-trivial, weakly amenable uniform algebra, or there exists a uniform algebra B such that for every generalised Cole extension of B , every continuous derivation from B into B^* extends to a continuous derivation from the extension, and this extension is norm preserving.

We do not know whether an integrally closed¹ uniform algebra (see, for example, [31]) necessarily has the latter property described above. Indeed, any simple Cole extension of such a uniform algebra will yield a contractive unital algebra homomorphism from the Cole extension to the original algebra (Corollary 5.6.17); thus every derivation extends and norms are preserved. However, this Cole extension need not be integrally closed.

Question 6.4.2. Let Y be a compact space and let B be a uniform algebra on Y . Suppose that there is an isometric algebra homomorphism $\theta : A \rightarrow B$. Let $D \in \mathcal{D}(A)$. When is it true that D extends via θ to $\tilde{D} \in \mathcal{D}(B)$ such that $\|\tilde{D}\| = \|D\|$?

The results discussed in the present chapter give some special cases of when it is possible to extend a continuous derivation to the larger uniform algebra. We do not have sharp estimates for the norm of an extension in general, or have an example of a derivation which extends with a strictly larger norm.

Let K be the kernel of the map $\pi_A : A \hat{\otimes} A \rightarrow A$ given by $a \otimes b \mapsto ab$. In [58], Grønbæk shows that $\mathcal{D}(A)$ is isomorphic to $(K/\overline{K^2})^*$, where the closure of K^2 is

¹Let A be a commutative algebra. We say that A is *integrally closed* if, for each monic polynomial $q(t) \in A[t]$, there exists an element $a \in A$ such that $q(a) = 0$.

taken in the projective tensor product $A \hat{\otimes} A$. (See also [95].) The following is essentially a rewording of the previous question.

Question 6.4.3. Let A_1 be a closed subalgebra of A . Let K_1 be the kernel of the map $a \otimes b \mapsto ab : A_1 \hat{\otimes} A_1 \rightarrow A_1$, and let K be as above. When is $K_1/\overline{K_1^2}$ isomorphic to a closed linear subspace of $K/\overline{K^2}$?

The following question is related to the content of Section 6.3.

Question 6.4.4. Let $\alpha \in \mathbb{D}$. Let $D : A(\overline{\mathbb{D}}) \rightarrow A(\overline{\mathbb{D}})^*$ be the derivation given by $D(f)(g) = f'(\alpha)g(\alpha)$ for all $f, g \in A(\overline{\mathbb{D}})$. If we construct a similar sequence of extensions of $A(\overline{\mathbb{D}})$ and a sequence of derivations $D_n : A(\overline{\mathbb{D}}) \rightarrow A(\overline{\mathbb{D}})^*$, as in Section 6.3, do we necessarily have $\|D_n\| = \|D\|$ for all $n \in \mathbb{N}$?

Note that there is a complete characterisation of the derivations from the disk algebra into its dual (due to Choi and Heath [21]).

APPENDIX A

Operator spaces and operator algebras

In this section we briefly discuss operator spaces, operator algebras, C^* -algebras, and maps between these objects. We refer the reader to [12, 34, 91] for further details on operator spaces and operator algebras, and to [27, 87] for details on C^* -algebras.

A.1 C^* -algebras

Before we can introduce operator spaces and operator algebras, we must recap some elementary theory related to C^* -algebras.

Definition A.1.1. Let $\mathfrak{A}$ be an algebra. An *involution* on $\mathfrak{A}$ is a map $\cdot^* : \mathfrak{A} \rightarrow \mathfrak{A}$ such that $(a^*)^* = a$ ($a \in \mathfrak{A}$), $(ab)^* = b^*a^*$ ($a, b \in \mathfrak{A}$), and $(\alpha a + \beta b)^* = \overline{\alpha}a^* + \overline{\beta}b^*$ ($a, b \in \mathfrak{A}, \alpha, \beta \in \mathbb{C}$). An algebra with an involution is called a *$*$ -algebra*.

Let $\mathfrak{A}$ be a $*$ -algebra, and let $a \in \mathfrak{A}$. We say that a^* is the *adjoint* of a . We say that a is *self-adjoint* if $a^* = a$. We denote the set of all self-adjoint elements of $\mathfrak{A}$ by $\mathfrak{A}_{\text{sa}}$. A *$*$ -subalgebra* of $\mathfrak{A}$ is a subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ such that $\mathfrak{B}$ is a $*$ -algebra when given the restricted involution from $\mathfrak{A}$.

Note that, if $\mathfrak{A}$ is unital, then $1_{\mathfrak{A}}^* = 1_{\mathfrak{A}}$, and if $a \in \text{inv } \mathfrak{A}$ then $(a^{-1})^* = (a^*)^{-1}$. Let $\mathfrak{A}, \mathfrak{B}$ be $*$ -algebras, and let $\theta : \mathfrak{A} \rightarrow \mathfrak{B}$ be an algebra homomorphism. We say that θ is a *$*$ -homomorphism* if, for all $a \in \mathfrak{A}$, we have $\theta(a^*) = \theta(a)^*$.

Definition A.1.2. Let $\mathfrak{A}$ be a $*$ -algebra. We say that $\mathfrak{A}$ is a *Banach $*$ -algebra* if $\mathfrak{A}$ is a Banach algebra such that the involution on $\mathfrak{A}$ is isometric. We say

that $\mathfrak{A}$ is a C^* -algebra if $\mathfrak{A}$ is a Banach $*$ -algebra and, for all $a \in \mathfrak{A}$, we have $\|aa^*\| = \|a^*a\| = \|a\|^2$.

Let $\mathfrak{A}$ be a C^* -algebra. A C^* -subalgebra of $\mathfrak{A}$ is a closed subalgebra $\mathfrak{B}$ such that $\mathfrak{B}$ is a C^* -algebra when given the restricted involution from $\mathfrak{A}$.

It is standard that if $\mathfrak{A}$ is a C^* -algebra then the norm on $\mathfrak{A}$ is unique. In addition, C^* -algebras have several useful properties, including the following automatic continuity result. (See Corollary 3.2.4 and Theorem 3.2.23 of [27].)

Proposition A.1.3. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^* -algebras, and let $\theta : \mathfrak{A} \rightarrow \mathfrak{B}$ be a $*$ -homomorphism. Then θ is contractive and $\theta(\mathfrak{A})$ is a C^* -subalgebra of $\mathfrak{B}$. Moreover, if θ is injective then θ is isometric.*

Let X be a compact space and let $A = C(X)$. Then A is a C^* -algebra when given the involution $f \mapsto \bar{f}$ ($f \in A$).

A.2 Operator spaces and operator algebras

Let $\mathfrak{X}$ be a linear space, and let $m, n \in \mathbb{N}$. We denote by $\mathbb{M}_{m,n}(\mathfrak{X})$ the linear space of all $m \times n$ matrices

$$\begin{bmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{m,1} & \cdots & x_{m,n} \end{bmatrix}$$

where $x_{i,j} \in \mathfrak{X}$ for $i = 1, \dots, m$ and $j = 1, \dots, n$. We shall often write $[x_{i,j}]_{m,n}$ for an element of $\mathbb{M}_{m,n}(\mathfrak{X})$ with entries $x_{i,j}$ for $i = 1, \dots, m$ and $j = 1, \dots, n$. We write $\mathbb{M}_n(\mathfrak{X}) := \mathbb{M}_{n,n}(\mathfrak{X})$, and $[x_{i,j}]_n$ for an arbitrary element of $\mathbb{M}_n(\mathfrak{X})$. We shall often identify $\mathfrak{X}$ with $\mathbb{M}_1(\mathfrak{X})$. It is easy to see that $\mathbb{M}_{m,n}(\mathfrak{X})$ is a linear space. We write $\mathbb{M}_{m,n} := \mathbb{M}_{m,n}(\mathbb{C})$. We give $\mathbb{M}_n(\mathbb{C})$ the (unique) norm by regarding $\mathbb{M}_n$ as $\mathcal{B}(\mathbb{C}^n)$, which we shall always denote by $\|\alpha\|$ for $\alpha \in \mathbb{M}_n$. (It should be clear from context which norm we require.)

Definition A.2.1. An (abstract) operator space is pair $(\mathfrak{X}, (\|\cdot\|_n))$ consisting of a linear space $\mathfrak{X}$ and norms $\|\cdot\|_n$ on $\mathbb{M}_n(\mathfrak{X})$ ($n \in \mathbb{N}$) such that

- (a) for all $n \in \mathbb{N}$, $\alpha, \beta \in \mathbb{M}_n$, and $\mathbf{x} \in \mathbb{M}_n(\mathfrak{X})$ we have $\|\alpha\mathbf{x}\beta\| \leq \|\alpha\|\|\mathbf{x}\|_n\|\beta\|$,
- (b) for all $m, n \in \mathbb{N}$, $\mathbf{x} \in \mathbb{M}_m(\mathfrak{X})$, and $\mathbf{y} \in \mathbb{M}_n(\mathfrak{X})$ we have

$$\left\| \begin{bmatrix} \mathbf{x} & \mathbf{0} \\ \mathbf{0} & \mathbf{y} \end{bmatrix} \right\|_{m+n} = \max\{\|\mathbf{x}\|_m, \|\mathbf{y}\|_n\}.$$

We shall often write $\mathfrak{X}$ is a operator space and omit the sequence of associated norms.

It is standard that a C^* -algebra $\mathfrak{A}$ automatically inherits an operator space structure by considering an embedding of $\mathfrak{A}$ into $\mathcal{B}(\mathcal{H})$, for some Hilbert space $\mathcal{H}$, and giving $\mathbb{M}_n(\mathfrak{A})$ the norm from $\mathcal{B}(\mathcal{H}^n)$ for each $n \in \mathbb{N}$. (See, for example, [34, Chapter 2], [12, Chapter 1], or numerous books on C^* -algebra theory.)

Definition A.2.2. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be operator spaces, and let $T : \mathfrak{X} \rightarrow \mathfrak{Y}$ be a linear map. We write $T^{(n)} : \mathbb{M}_n(\mathfrak{X}) \rightarrow \mathbb{M}_n(\mathfrak{Y})$ for the linear map such that $T^{(n)}([x_{i,j}]_n) = [T(x_{i,j})]_n$ for $[x_{i,j}]_n \in \mathbb{M}_n(\mathfrak{X})$. We say that T is *completely bounded* if

$$\|T\|_{\text{cb}} := \sup\{\|T^{(n)}(\mathbf{x})\|_n : \mathbf{x} \in \mathbb{M}_n(\mathfrak{X}), \|\mathbf{x}\|_n \leq 1, n \in \mathbb{N}\} < \infty.$$

We say that T is *completely contractive* if each map $T^{(n)}$ is a contractive linear map from $\mathbb{M}_n(\mathfrak{X})$ into $\mathbb{M}_n(\mathfrak{Y})$ ($n \in \mathbb{N}$). We say that T is *completely isometric* if each map $T^{(n)}$ ($n \in \mathbb{N}$) is an isometry.

It is standard in the theory of operator spaces that a pair $(\mathfrak{X}, (\|\cdot\|_n))$ is an operator space if and only if there exist Hilbert spaces $\mathcal{H}, \mathcal{K}$ and a completely isometric linear map $\rho : \mathfrak{X} \rightarrow \mathcal{B}(\mathcal{H}; \mathcal{K})$.

Definition A.2.3. An *operator algebra* is an operator space $\mathfrak{A}$ which is also a Banach algebra such that there exist a Hilbert space $\mathcal{H}$ and a completely isometric algebra homomorphism $\rho : \mathfrak{A} \rightarrow \mathcal{B}(\mathcal{H})$.

We may regard a uniform algebra as a commutative, unital operator algebra with a certain choice of matrix norms. In particular, we can use the results from operator algebra theory in a study of uniform algebras. In the remainder of this section, we describe various results which will be useful in the study of uniform algebras.

The first proposition is [34, Proposition 2.2.6].

Proposition A.2.4. Let $\mathfrak{X}$ be an operator space, let A be a commutative C^* -algebra, and let $T \in \mathcal{B}(\mathfrak{X}; A)$. Then T is completely bounded and $\|T\|_{\text{cb}} = \|T\|$.

In particular, if T (as above) is contractive then T is completely contractive. Note that, if T maps $\mathfrak{X}$ into a uniform algebra A (on a compact space X), then T maps $\mathfrak{X}$ into $C(X)$, and so we can apply the above proposition.

Our second proposition is effectively [12, Corollary 4.2.9].

Proposition A.2.5. *Let $\mathfrak{A}$ be a unital operator algebra, let $\mathfrak{B}$ be a unital subalgebra of $\mathfrak{A}$, and let P be a completely contractive projection on $\mathfrak{A}$ such that $P(\mathfrak{A}) = \mathfrak{B}$. Then*

$$P(b_1ab_2) = b_1P(a)b_2 \quad (a \in \mathfrak{A}, b_1, b_2 \in \mathfrak{B}).$$

By combining the above propositions, we obtain the following Lemma.

Lemma A.2.6. *Let X be a compact space, let A be a uniform algebra on X , let B be a unital subalgebra of A , and let P be a contractive, unital projection on A such that $P(A) = B$. Then*

$$P(b_1ab_2) = b_1P(a)b_2 \quad (a \in A, b_1, b_2 \in B).$$

The following proposition is part of [11, Lemma 4.1].

Proposition A.2.7. *Let $\mathfrak{A}$ be a unital operator algebra, let P be a unital projection on $\mathfrak{A}$, and let $Q := \text{id}_{\mathfrak{A}} - P$. Suppose that Q is completely contractive. Let $C := \ker P$. Then $C^2 \subseteq P(A)$.*

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Index of Symbols

Common

$\hat{A}, \hat{a}$	20
$A[t]$	17
$\text{alg}_{\mathfrak{A}} S$	12
$\overline{\text{alg}_{\mathfrak{A}}} S$	19
$A(X)$	26
$\bar{B}_X(x, r), \bar{B}(x, r)$	11
$B_X(x, r), B(x, r)$	11
$\mathcal{B}(\mathfrak{X}; \mathfrak{Y}), \mathcal{B}(\mathfrak{X})$	15
$\text{conv } S, \overline{\text{conv}} S$	12
$\partial_X E, \partial E$	11
δ_x	16
$\text{dist}_X(x, E), \text{dist}(x, E)$	12
$\mathbb{D}, \overline{\mathbb{D}}, \mathbb{T}$	10
$\overline{E}$	10
ε_x	24
$f E$	10
$\bar{f}$	10
$f^{-1}(F)$	10
$\Gamma(A)$	27
$\Gamma_0(A)$	26
id_X	10
$\text{int}_X(E), \text{int}(E)$	10
$J_x(A), J_x$	25
$\text{lin}_{\mathfrak{X}} S$	12
$\overline{\text{lin}_{\mathfrak{X}}} S$	15
$M_x(A), M_x$	25
ω_1	10

$\Phi_{\mathfrak{A}}$	13
φ^*	10
$P_0(X), P(X)$	26
$R_0(X), R(X)$	26
$S \cdot T, a \cdot T$	12
$S + T$	12
$\text{supp } f$	11
$\text{supp } \mu$	16
$\mathfrak{X} \otimes \mathfrak{Y}$	18
$\mathfrak{X} \check{\otimes} \mathfrak{Y}$	23
$\mathfrak{X} \hat{\otimes} \mathfrak{Y}$	23
$ f _E, f _X$	11
$\mathfrak{X} \oplus_1 \mathfrak{Y}$	15
$\mathfrak{X} \oplus \mathfrak{Y}$	12
Z	10

Chapter 2

$\mathcal{A}$	52
δ_{α}	31
$E(A)$	43
$\mathcal{F}$	30
$F(\mathcal{C})$	43
$H_A(E)$	31
K_A	49
$\mathcal{L}_A(\mathcal{C})$	43
$\mathcal{L}_A^*(\mathcal{C})$	45
$\mathcal{N}, \mathcal{N}(M, R)$	31
π_n	31
S_A	30

$U(K, M)$	47
$V(\mathcal{C})$	43

Chapters 2 and 4

$\delta_{\text{ann}}(A)$	31
$\mu(A)$	31
$\rho(A), \rho_E(A), \rho_{\text{ann}}(A)$	31
X_A	30

Chapter 3

$\text{BVC}([0, 1])$	76
γ^{no}	65
$-\gamma$	74
$\gamma_1 \dot{+} \gamma_2$	74
$\mathfrak{p}(S)$	74
V_c^d	64

Chapters 3 and 4

$D_{\mathcal{F}}^{(1)}(X)$	71
$D_{\mathcal{F}}^{(n)}(X), D_{\mathcal{F}}^{(\infty)}(X)$	72
$D_{\mathcal{F}}(X, M)$	73
$D(X, M)$	69
$d(M)$	69
$f^{(n)}$	68
$D^{(n)}(X), D^{(\infty)}(X)$	68
$f^{[n]}$	72
$\mathcal{F} E$	89
$\mathcal{F}^*$	70
γ^-, γ^+	64
γ^{pl}	65
$\Lambda(\gamma)$	64
$\mathfrak{m}(\mathcal{F})$	74
$S_{\mathcal{F}}$	73

Chapter 4

M^c	102
$Q_{A, \mathcal{F}}(z_0)$	119
$\mathcal{Q}_{\mathcal{F}}(z_0)$	121

Chapter 5

A^U, A^q	140
π_U, π_q	140
$((X_{\alpha}, \pi_{\alpha, \beta}), (A_{\alpha}))_{\alpha \leq \beta \leq \tau}$	142
X^U, X^q	140

Chapters 5 and 6

$((X_{\alpha}, \pi_{\alpha, \beta}), (A_{\alpha}, T_{\alpha, \beta}))_{\alpha \leq \beta \leq \tau}$	155
--	-----

Chapter 6

$\mathcal{D}_0(A)$	167
--------------------------	-----

Index

- abstract Swiss cheese, 30
 - classical, 30
 - delete disk, 34
 - error set, 43
 - insert disk, 34
 - replace disks, 34
- admissible path, 65
- algebra, 12
 - inverse, 12
 - unital, 12
- algebra homomorphism, 13
- algebra sequence, 69
 - naturalisable, 96
- amenable, 22, 160
- annular discrepancy function, 31
- annular Swiss cheese, 30
- averaging map, 141
- Banach algebra, 19
 - semisimple, 20
- Banach algebra extension, 131
- Banach function algebra, 24
- Banach module, 21
- Bland–Feinstein algebra, 73
- bounded linear map, 15
 - bounded below, 135
 - open, 16
- bounded variation, 76
- character, 13
- character space, 13, 19
- Choquet boundary, 26
- classicalisation, 29
- Cole extension, 140
 - system, 142
- compact plane set, 10
- compact space, 10
- completely contractive, 179
- completely isometric, 179
- continuous derivation, 22
- contractive linear map, 16
- convex hull, 12
- C^* -algebra, 178
- Dales–Davie algebra, 69
- derivation, 14
- direct limit, 133
- direct system, 132
- discrepancy function, 31
- dual Banach module, 21
- effective collection, 70
- essential set, 27
- evaluation character, 24
- $\mathcal{F}$ -analytic, 84, 112
- $\mathcal{F}$ -derivative, 71
- $\mathcal{F}$ - $\mathcal{G}$ -compatible, 78
- $\mathcal{F}$ -quasianalytic, 101
- $\mathcal{F}$ -regular, 70

- Fourier series, 173
- γ -derivative, 73
- Gelfand topology, 19
- Gelfand transform, 20
- generalised Cole extension, 151
 - system, 155
 - system extension, 155
- geodesic distance, 65
- geodesically bounded, 66
- higher point derivation, 14
 - continuous, 22
 - degenerate, 15
- initial segment, 10
- injective tensor product, 23
- inverse limit, 133
- inverse system, 132
- isometric, 16
- Jensen measure, 123
- Jordan path, 64
- linear homeomorphism, 16
- log-convex minorant, 102
- logarithmically convex, 69
- lower-semicontinuous, 11
- maximal collection, 74
- measure, 16
- module, 13
 - commutative, 14
 - submodule, 13
- module homomorphism, 14
- natural algebra, 24
- non-analytic sequence, 69
- normal algebra, 25
- operator algebra, 179
- operator space, 178
- partially above, 36
- path, 64
 - join, 74
 - length, 64
 - reverse, 74
- path length parametrisation, 65
- peak point, 26
- point derivation, 14
- point mass measure, 16
- point of continuity, 25
- pointwise regular, 65
- polynomial, 17
 - evaluation homomorphism, 17
 - monic, 17
- probability measure, 16
- projection, 16
- projective tensor product, 23
- pushforward measure, 66
- quasianalytic, 100, 111
- R -point, 25
- rectifiable path, 64
- rectifiably connected, 65
- redundancy-free, 33
- regular algebra, 25
- representing measure, 26
- Riemann–Stieltjes integral, 66
- semi-rectifiable, 65
- set of quasianalyticity, 118
- Shilov boundary, 27
- simple Cole extension, 140
- simple set, 113
- strong boundary point, 26
- subalgebra, 12
- Swiss cheese set, 28

- classical, 30
- semiclassical, 30
- tensor product, 18
- total variation, 64
- uniform algebra, 24
- uniform algebra extension, 131
- uniform norm, 11
- uniformly regular, 65
- upper-semicontinuous, 11
- weakly amenable, 22, 160
- zero divisor, 12