# International Portfolio Optimisation under Uncertainty




**Nonthachote Chatsanga**

School of Computer Science
University of Nottingham

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# Declaration

I hereby declare that except where specific reference is made to the work of others, the contents of this dissertation are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other university. This dissertation is my own work and contains nothing which is the outcome of work done in collaboration with others, except as specified in the text and Acknowledgements. This dissertation contains fewer than 100,000 words including appendices, bibliography, footnotes, tables and equations and has fewer than 150 figures.

Nonthachote Chatsanga
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# Abstract

Portfolio optimisation problems are generally concerned with allocating funds to investments. The goal is to find an allocation that minimises risk subject to some certain constraints. To attain robust solutions from the optimisation, it is vital to ensure that the model is able to properly represent the underlying uncertainty in portfolio management.

The main source of uncertainty in managing portfolios is from asset returns fluctuation. Typically, it is depicted through scenarios or return distributions which are commonly assumed to be normal. Such assumption, however, does not illustrate the true characteristics of financial asset returns and thus distorts the representation of returns, risks and interrelationship of assets in a portfolio.

This thesis presents novel approaches to represent the uncertainty in international portfolio management in order to mitigate risks associated. Three main issues are covered in our studies. The first issue deals with an approach to hedge risk from exchange rates of a multi-currency portfolio. A hedging mechanism is incorporated into a portfolio optimisation model to produce a portfolio that is optimal in terms of asset and currency exposure. Costs associated to exchange rate hedging are also included into the model to improve accuracy in calculating risk and return of the portfolio.

The second issue is about representing the uncertainty of multiple assets in a portfolio through scenarios. Under the assumption of normality, some risk and return characteristics of assets are omitted from standard scenario generation method. We present a novel approach to extract more accurate information from assets to generate realistic scenarios. A two-stage stochastic international portfolio optimisation model is formulated accordingly to create portfolios that are expected to be more efficient than portfolios optimised with scenarios generated from the standard approach.

The last issue is associated with an approach to keep portfolio's efficiency over its investment horizon through continual reallocation, known as portfolio rebalancing. Since the uncertainty regarding changes in financial market conditions diminishes the efficiency of conventional rebalancing strategies, we propose a new rebalancing method and formulate it as a stochastic optimisation problem. The proposed rebalancing strategy aims to reallocate the portfolio to attain a high probability of positive returns and low probability of negative returns under second-order stochastic dominance criteria. A portfolio implementing the new rebalancing strategy is expected to outperform a portfolio that follows traditional rebalancing methods in terms of portfolio returns.

Overall, the thesis aims to create a richer portfolio optimisation model as well as to extract the correct information from financial markets to better cope with the underlying uncertainty in portfolio management.

# Table of contents

**List of figures** **xiii**

**List of tables** **xvii**

**1 Introduction** **1**

1.1 Motivation . . . . . 4
1.2 Research Questions . . . . . 7
1.3 Methodology . . . . . 10
1.4 Thesis Overview . . . . . 11
1.5 Publications . . . . . 12

**2 Background and Literature Review** **15**

2.1 Preliminary Background . . . . . 15
2.1.1 International Portfolios . . . . . 15
2.1.2 Modifying Currency Exposure with A Currency Overlay . . . 17
2.1.3 Foreign Exchange Forward Contracts . . . . . 23
2.1.4 Cost of Carry of Foreign Exchange Forwards . . . . . 24
2.2 Literature Review . . . . . 26
2.2.1 International Portfolio Optimisation . . . . . 28
2.2.2 Portfolio Risk Measure . . . . . 30
2.2.3 Solving Cardinality Constrained Optimisation Problems . . . 31
2.2.4 Application of Copulas in Portfolio Optimisation . . . . . 32
2.2.5 Application of Second-Order Stochastic Dominance in Portfolio Optimisation . . . . . 33
2.2.6 Portfolio Rebalancing . . . . . 34

**3 International Portfolio Optimisation with Integrated Currency Overlay Costs and Constraints** **37**

3.1 Introduction . . . . . 37

3.1.1 Current Advancements in The Literature . . . 39
3.1.2 Chapter Overview . . . 40
3.2 Background . . . 41
3.2.1 Mean-Variance Optimisation . . . 41
3.3 Methodology . . . 43
3.3.1 Incorporating Cost of Carry to Risk and Return Calculation of Portfolios . . . 43
3.3.2 Incorporating Transaction Costs of Forward Contracts . . . 45
3.3.3 Optimisation Problem with Overlay Constraints . . . 47
3.4 Experimental Results and Discussions . . . 48
3.4.1 Data . . . 49
3.4.2 Experimental Studies . . . 51
3.5 Conclusions . . . 69

**4 Two-Stage Stochastic International Portfolio Optimisation under Regular-Vine-Copula-Based Scenarios 73**
4.1 Introduction . . . 73
4.1.1 Current Advancements in The Literature . . . 75
4.1.2 Chapter Overview . . . 76
4.2 Background . . . 77
4.2.1 Two-Stage Stochastic Programming with Recourse . . . 77
4.2.2 Copulas . . . 80
4.2.3 Dependency Measures . . . 82
4.2.4 Copula Families . . . 86
4.2.5 Pair-Copula Construction . . . 90
4.2.6 Regular-Vine Copula . . . 93
4.2.7 Selecting Regular-Vine Distributions . . . 102
4.2.8 Scenario Generation for Stochastic Programming . . . 105
4.2.9 Conditional Value-at-Risk (CVaR) . . . 112
4.3 Methodology . . . 114
4.3.1 Scenario Generation with Regular-Vine Copula Dependence Structure . . . 115
4.3.2 Two-Stage Stochastic International Portfolio Optimisation Model . . . 120
4.4 Algorithm . . . 127
4.4.1 Genetic Algorithm . . . 128
4.5 Experimental Results and Discussions . . . 132

4.5.1 Data . . . 132
4.5.2 Experimental Studies . . . 134
4.6 Conclusions . . . 154

**5 Portfolio Rebalancing under A Second-Order Stochastic Dominance Criterion 155**
5.1 Introduction . . . 155
5.1.1 Current Advancements in The Literature . . . 157
5.1.2 Chapter Overview . . . 158
5.2 Background . . . 159
5.2.1 Portfolio Rebalancing . . . 159
5.2.2 Stochastic Dominance . . . 161
5.3 Methodology . . . 168
5.3.1 Deterministic Portfolio Rebalancing Model . . . 168
5.3.2 Stochastic Portfolio Rebalancing Model . . . 171
5.4 Experimental Results and Discussions . . . 175
5.4.1 Experiment Setting . . . 175
5.4.2 Performances of Rebalancing Strategies . . . 178
5.5 Conclusions . . . 182

**6 Conclusions 185**
6.1 Results Achieved . . . 185
6.1.1 Answers to The Research Questions . . . 189
6.2 Contributions . . . 191
6.3 Future Research Directions . . . 192

**References 195**

**Appendix A Financial Terminologies 205**
A.1 Portfolio . . . 205
A.2 Position . . . 205
A.2.1 Long Position . . . 206
A.2.2 Short Position . . . 206
A.3 Exposure . . . 206
A.4 Exchange Rate Pair . . . 206
A.4.1 Spot Exchange Rate . . . 207
A.4.2 Forward Exchange Rate . . . 207
A.5 Foreign Exchange Rate Forward Contract . . . 208

A.6 Currency Overlay . . . . . 208

**Appendix B Additional information for Chapter 4** **211**

B.1 Data Description . . . . . 211

B.2 Sequential Method to Select an R-Vine Model . . . . . 215

B.3 Scenario Generation using R-Vine Copula . . . . . 215

B.4 Optimal Portfolio Allocations . . . . . 217

**Appendix C Additional information for Chapter 5** **229**

C.1 Rebalancing Allocation Deviation . . . . . 229

**Index** **239**

# List of figures


1.1 Uncertainty treatment framework. . . . . 3
1.2 Example of symmetric and asymmetric dependences. . . . . 6

2.1 An illustration of how interest rates of two countries affect the corresponding forward exchange rates. . . . . 25

3.1 An illustration of an efficient frontier. . . . . 42
3.2 Adjusted expected returns and volatilities of assets and currencies. . . 50
3.3 Efficient frontiers of portfolios optimised by unified and two-stage strategies. . . . . 55
3.4 Total currency overlay, $V$, of portfolios optimised by the unified and two-stage strategies. . . . . 56
3.5 Efficient frontiers from different hedging policies. . . . . 58
3.6 Total currency overlay, $V$, at each return level. . . . . 59
3.7 Efficient frontiers at different levels of margin requirement. . . . . 60
3.8 Percentage volatility increase relative to volatilities of the portfolio with no margin requirement ($M = 0\%$). . . . . 61
3.9 Total currency overlay, $V$, at each return level. . . . . 62
3.10 Efficient frontiers of optimal portfolios at different restriction of total currency overlay $V_u$. . . . . 64
3.11 Percentage volatility increase relative to volatility of portfolios with no restriction on currency overlay ($V_u = 100\%$). . . . . 65
3.13 Composition of total bond and total equity holdings at at each return level. . . . . 65
3.12 Total currency overlay, $V$, at each return level. . . . . 66
3.14 Efficient frontiers produced under different limits on the number of forward contracts allowed. . . . . 67

3.15 Percentage volatility increase relative to volatilities of the portfolio that allows up to 6 forward contracts to construct currency overlay ($G = 6$). . . . . 68
3.16 Total currency overlay at each return level. . . . . 69

4.1 Various relationships between two variables. . . . . 84
4.2 An example of a regular-vine on 4 variables. . . . . 94
4.3 An example of a regular-vine on 7 variables. . . . . 96
4.4 An R-Vine matrix $M^*$, accompanied by a copula type matrix $T^*$ and a copula parameter matrix $P^*$. . . . . 100
4.5 Scenario generation methodologies for asset return distributions. . . . 108
4.6 Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR). . . . . 114
4.7 Histogram estimates with small bandwidth (left) and large bandwidth (right). . . . . 116
4.8 The evolution of fitness values (CVaR) over generations when running GA on portfolio under RVC scenarios at the target returns of 1.20%. . 130
4.9 The evolution of fitness values (CVaR) over generations when running GA on portfolio under RVC scenarios at the target returns of 1.40%. . 131
4.10 The evolution of fitness values (CVaR) over generations when running GA on portfolio under MVN scenarios at the target returns of 1.20%. . 131
4.11 The evolution of fitness values (CVaR) over generations when running GA on portfolio under MVN scenarios at the target returns of 1.40%. . 132
4.12 The first tree ($T_1$) structure of the estimated R-Vine tree. . . . . 140
4.13 Comparison of efficient portfolios under different environments. . . . 146
4.14 Equity allocation and foreign currencies exposure of portfolios. . . . . 147
4.15 Currency Overlays of MVN and RVC portfolios. . . . . 148
4.16 Comparison of portfolios without cardinality constraints and those with the cardinality constraint on the number of currencies ($K_C$). . . . 150
4.17 Comparison of portfolios without cardinality constraints and those with the cardinality constraint on the number of forwards ($K_G$). . . . . 151
4.18 Comparison of portfolios without cardinality constraints and those with both the cardinality constraints on the number of currencies ($K_C$) and the number of forwards ($K_G$). . . . . 152
4.19 Cumulative wealth over the out-of-sample period (Jan-12 to May-15) of optimal portfolios from two scenario generation methods; RVC and MVN approaches. . . . . 153

4.20 Comparison of performances of efficient portfolios under RVC and MVN scenarios. . . . . 153

5.1 An illustration of the first-order stochastic dominance criterion. . . . . 163
5.2 An illustration of the second-order stochastic dominance criterion. . . 164
5.3 An example of two portfolios that satisfy the SSD condition but not the FSD. . . . . 166
5.4 A diagram showing an application of second-order stochastic dominance in a rebalancing routine. . . . . 171
5.5 Monthly Returns of the low-risk portfolio subject to each rebalancing strategy. . . . . 179
5.6 Monthly Returns of the medium-risk portfolio subject to each rebalancing strategy. . . . . 180
5.7 Monthly Returns of the high-risk portfolio subject to each rebalancing strategy. . . . . 181

B.1 The box plot of government bond indices and currency returns. . . . . 212
B.2 The box plot of stock indices, corporate bond indices and gold returns. 213
B.3 Marginal distribution fitting of US treasury index, S&P500 index and EURUSD monthly returns over Jan-02 to Dec-11. . . . . 216
B.4 The regular vine copula of US government bond index, S&P500 index and EURUSD. . . . . 216
B.5 The return histograms of raw data (left panels) and simulated data (right panels). . . . . 217
B.6 The joint return distributions by asset pairs. . . . . 217

C.1 Currency allocation deviation between the SSD and BHD rebalancing strategies implemented on the low-risk portfolio. . . . . 230
C.2 Bond allocation deviation between the SSD and BHD rebalancing strategies implemented on the low-risk portfolio. . . . . 231
C.3 Equity allocation deviation between the SSD and BHD rebalancing strategies implemented on the low-risk portfolio. . . . . 232
C.4 Currency allocation deviation between the SSD and BHD rebalancing strategies implemented on the medium-risk portfolio. . . . . 233
C.5 Bond allocation deviation between the SSD and BHD rebalancing strategies implemented on the medium-risk portfolio. . . . . 234
C.6 Equity allocation deviation between the SSD and BHD rebalancing strategies implemented on the medium-risk portfolio. . . . . 235

C.7 Currency allocation deviation between the SSD and BHD rebalancing strategies implemented on the high-risk portfolio. . . . . 236
C.8 Bond allocation deviation between the SSD and BHD rebalancing strategies implemented on the high-risk portfolio. . . . . 237
C.9 Equity allocation deviation between the SSD and BHD rebalancing strategies implemented on the high-risk portfolio. . . . . 238

# List of tables


2.1 A sample international portfolio funded with USD, investing in the US, the UK and Germany by 50%, 30% and 20%. . . . . 16
2.2 International portfolios with and without a currency overlay. . . . . 18
2.3 Sample choices to construct the same currency overlay. . . . . 19
2.4 Structure of a multi-currency portfolio. . . . . 20

3.1 Costs of carry associated with foreign exchange rate forward contracts. 44
3.2 Average yields to maturity (as a proxy of interest rate) of 1-year treasury bills over Jun-11 to Jun-12. . . . . 49
3.3 Adjusted expected returns used in the optimisation problems. . . . . 50
3.4 Average percent spreads . . . . . 51
3.5 The structure of international portfolio investing in four countries with two asset classes. . . . . 52
3.6 Parameters of constraints associated to currency overlay in international portfolio optimisation problem. . . . . 53
3.7 Selected optimal portfolios at low (0.5%), medium (1.2%) and high (1.8%) return targets. . . . . 57
3.8 Example of currency overlay which needs three or more forward contracts to produce. . . . . 68

4.1 Example of widely-used kernel functions. . . . . 118
4.2 Notations for the optimisation model parameters. . . . . 121
4.3 Notations for the transaction costs. . . . . 122
4.4 Notations for the auxiliary variables of the optimisation model. . . . . 122
4.5 Notations for the decision variables of the optimisation model. . . . . 123
4.6 Currencies and asset classes to be invested in a portfolio. . . . . 133
4.8 List of bivariate copula families and ranges of associated parameters. . 136

4.9 In-sample stability of mean-CVaR objective value for varying sizes of the scenario set. . . . 137
4.7 List of assets and currencies in a portfolio. . . . 139
4.10 Matrix representation of the R-Vine tree from the estimated R-Vine specification. . . . 141
4.11 Matrix representation of bivariate copula families from the estimated R-Vine specification. . . . 142
4.12 Matrix representation of parameters corresponding to bivariate copula families of the estimated R-Vine specification. . . . 143
4.13 Matrix representation of additional parameters corresponding to bivariate copula families of the estimated R-Vine specification. . . . 144
4.14 Descriptive statistics of raw return data, MVN return data and RVC return data. . . . 145
4.15 The average runtime (minutes) taken in solving the optimisation problem at a given level of return under different cardinality constraints configurations. . . . 150

5.1 Implementation of the buy-and-hold rebalancing strategy on a two-asset portfolio. . . . 160
5.2 Rebalancing a two-asset portfolio with the periodic strategy to maintain constant allocation over time. . . . 161
5.3 Notations for a deterministic portfolio rebalancing model. . . . 170
5.4 Representative portfolios for implementing rebalancing strategies. . . . 177
5.5 Average, minimum and maximum returns of portfolios under different rebalancing strategies. . . . 182

B.1 p-values of the Jarque-Bera test for normal distribution. . . . 214
B.2 Optimal portfolio allocations under regular-vine-copula-based scenarios at return targets 0.55% to 0.75% monthly. . . . 218
B.3 Optimal portfolio allocations under regular-vine-copula-based scenarios at return targets 0.80% to 1.00% monthly. . . . 219
B.4 Optimal portfolio allocations under regular-vine-copula-based scenarios at return targets 1.05% to 1.25% monthly. . . . 220
B.5 Optimal portfolio allocations under regular-vine-copula-based scenarios at return targets 1.30% to 1.50% monthly. . . . 221
B.6 Optimal portfolio allocations under regular-vine-copula-based scenarios at return targets 1.55% to 1.60% monthly. . . . 222

B.7 Optimal portfolio allocations under multivariate normal distribution scenarios at return targets 0.55% to 0.75% monthly. . . . . 223
B.8 Optimal portfolio allocations under multivariate normal distribution scenarios at return targets 0.80% to 1.00% monthly. . . . . 224
B.9 Optimal portfolio allocations under multivariate normal distribution scenarios at return targets 1.05% to 1.25% monthly. . . . . 225
B.10 Optimal portfolio allocations under multivariate normal distribution scenarios at return targets 1.30% to 1.50% monthly. . . . . 226
B.11 Optimal portfolio allocations under multivariate normal distribution scenarios at return targets 1.55% to 1.60% monthly. . . . . 227

# Chapter 1

# Introduction

A portfolio[1] is basically a collection of investments. Optimal portfolio allocation was introduced by Harry Markowitz in the 1950's [91]. The idea is basically to allocate a given capital to each investment to strike a good balance between expected return and risk (which is originally represented by variance of portfolio returns). This so-called mean-variance framework is the foundation for later developments in the area of portfolio optimisation and mathematical programming techniques have become essential tools in financial management since then.

Since the establishment of the mean-variance optimisation, there has been several developments in financial markets across the world. Past decades have seen increasing diversity of complex financial instruments and recognition of multiple uncertain factors have a combined effect on the risk-performance characteristics of securities. In particular, financial derivatives[2] have brought a new dimension in managing risk and, at the same time, more speculation in financial markets. Managing risk and return of a portfolio with derivatives has become a common practice for portfolio management. In addition, the increased integration of financial markets across the world allows easier transferring of shocks from one country to others. These two prominent factors play

[1] More details on a definition and a purpose of constructing a portfolio can be found in Section A.1 of Appendix A.

[2] A derivative is a contract between two or more parties that derives its value from the performance of an underlying security such as an equity, bond and currency. The key advantage of financial derivatives, e.g., forwards, futures, options is that they offer an opportunity to investors to make profit in both market upturns and downturns. An investor can hence devise a wide range of investment strategies including hedging – reducing risk of an investment by betting that a derivative and an underlying security will move in different directions.

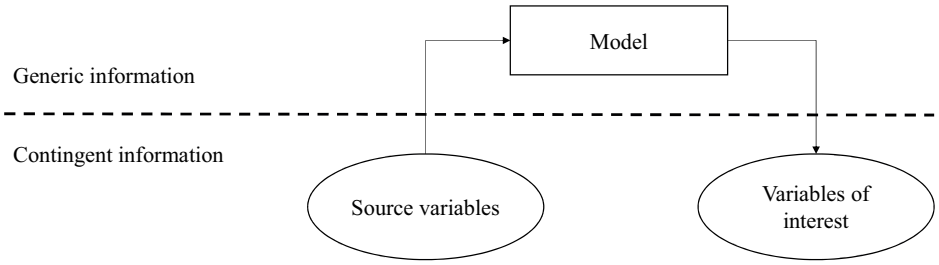
an important role in intensifying extreme events[3] emerged in the recent years (e.g., the sub-prime mortgage crisis in 2008 and the European debt crisis in 2011) and ongoing.

The aftermath of recent financial crises included the collapse of several investment banks and hedge funds. This reveals flaws in existing risk management framework and seriously casts doubt on tools widely employed in portfolio management. Extreme events become an unignorable issue in portfolio allocation. Although they could happen only once in ten years, the severity can, in one day, wipe out all the accumulated profit. It is thus crucial for portfolio managers to be able to capture such uncertainty that could affect risk and return of a portfolio and manage it effectively. Any inadequacy in addressing relevant uncertainty in a portfolio may result in underestimating underlying risk and could cause severe negative outcomes to the portfolio. Advanced mathematical models of financial asset characteristics, along with elegant applications of optimisation methods, are needed to construct a reliable portfolio which is also resilient to a turbulent market environment.

In the context of portfolio optimisation, the main source of uncertainty lies in asset prices whose exact values are not perfectly known. This lack of knowledge is due either to the aleatory nature of some phenomena influencing asset prices or to a lack of precise and fully reliable information concerning them. The components involved in a treatment of uncertainty are illustrated in Figure 1.1. Generic information refers to theoretical background or general beliefs about the process. Contingent information, on the other hand, corresponds to information, beliefs or knowledge concerning idiosyncratic situations. Source variables are observations for which we have some information on them but not complete. A model processes generic information about the relationship existing between source variables and variables of interest. Variables of interest capture refined information which is deemed reliable in supporting decision making.

[3] If the assumption of normal distribution holds, events that occur outside the range of plus and minus 3 standard deviation (6 sigma) are considered extreme as a standard practice, Sage and Haimes [121]. 6 sigma covers the probability of 99.99% which hence implies that extreme events are rarely to occur. However, their effects are immense.

**Figure 1.1 Uncertainty treatment framework.** Adapted from Sebastien [123]




Uncertainty may exist in source variables (e.g., inputs of an optimisation problem) or on the model itself, or both. To cope with underlying uncertainty, a careful representation of uncertainty is vital. Existing approaches employed to represent uncertainty in portfolio management still lack the power to capture some important characteristics of financial markets. Specifically, normal distributions are often assumed to make models and computations simpler, however, it comes with the expense of poorer representation of uncertainty and potential risk underestimation.

Combining models that can properly capture the stochastic evolution of the underlying financial processes and stochastic optimisation models inevitably results in challenging and computationally demanding problems. There is a large body of research that focuses on developing algorithms to solve portfolio optimisation problems more effectively but that somewhat overlook deficiencies in representing financial fundamentals. Rather than focusing on solving misspecified problems more efficiently, this thesis puts the emphasis on cultivating portfolio optimisation problems that reflect realistic issues in international portfolio management.

Accordingly, we develop a novel portfolio optimisation model which incorporates state-of-the-art techniques to capture extreme uncertainty of financial assets and also demonstrate a solving methodology for the respective optimisation problems. Resulting portfolios are expected to improve risk-return trade-off from those produced from standard portfolio construction methods and be more resilient to financial market fluctuations and, in particular, to extreme events that are not captured in many models of the uncertainty.

## 1.1 Motivation

A portfolio in our study particularly focuses on international investment in order to earn broader range of diversification benefit and higher chance of returns from markets in different economies, comparing with an investment in a single market. The benefits from investing globally are clearly exhibited in the works of Levy and Sarnat [84]; the more assets with low correlation among each other are added to the portfolio, the lower the overall portfolio risk. However, investing in markets with different currencies entails an additional risk factor, that is, foreign exchange risk.

In general, risk of a portfolio emerges from uncertainty in the value of the portfolio due to unforeseen and unpredictable future events. For the case of a portfolio investing in international assets, risk emanates from two sources which are market risk and currency risk. Market risk is associated with changes in the market prices of assets over time. Currency risk involves with variation of exchange rates[4] which affects portfolio valuation when converting investments in foreign currencies to a funding currency of the portfolio. Risk from exchange rate fluctuation is immense, potential benefits reaped from international diversification could possibly be offset by loss from adverse movement of exchange rates [44]. In consequence, diversification alone is insufficient to satisfy the investor's risk management objectives. Exchange rate hedging products are thus employed to mitigate risk of international portfolios.

Hedging is an approach aimed to minimise the potential effects of various risk factors, thereby minimising uncertainty in a future value of investments and stabilise estimated earnings and cash flows. In essence, hedging is equivalent to an insurance against risk. The basic idea behind constructing a hedge is to identify exposures[5] of a portfolio to various risks and then establish positions[6] with the opposite exposure. The core of portfolio construction is to attain appropriate decision strategies so that the portfolio achieves a desirable combination between expected return and risk. Consequently, hedging decisions become an integral part of the overall investment structuring decision so as to achieve a balance between expected return and risk of an international portfolio.

Typically, the decision on currency hedging is treated as subordinate to asset allocation as pointed out in Topaloglou et al. [134]. Allocation of funds to international assets is decided first, then investors decide if and how to hedge the exposure to foreign

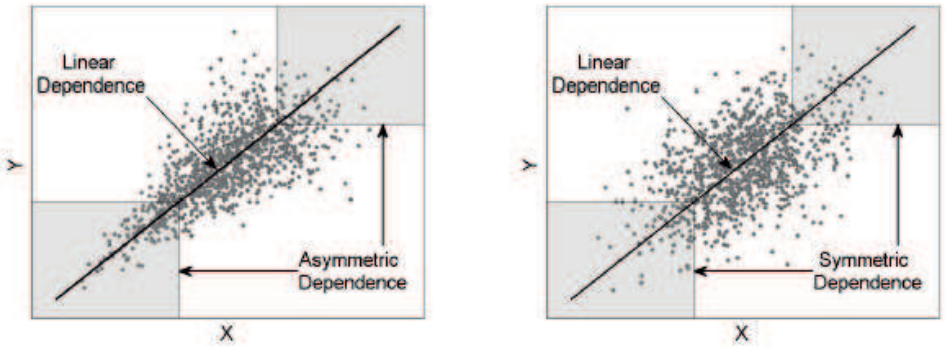
[4] See more details and related terminologies of exchange rates in Section A.4 of Appendix A.
[5] More details of "exposure" in the context of investment are given in Section A.3 of Appendix A.
[6] See Section A.2 of Appendix A for more details on "position" in the context of investment.

investments where an ad-hoc rule is usually applied (e.g., each currency is 50% hedged). This simple practice, however, disregards cross hedging effects between assets and currencies which does not constitute a portfolio which jointly optimises asset and currency holdings.

The first motivation of our study is hence to create a portfolio optimisation model with a hedging instrument incorporated which determines both optimal asset allocation and optimal positions of the hedging instrument. In addition, to obtain as accurate as possible return and risk calculation, associated hedging costs are taken into account and integrated into the optimisation model.

It is mentioned earlier that ability to capture extreme events largely determines the quality of portfolio optimisation solutions. While describing extreme events for an asset is not trivial, capturing extreme events for multiple assets is even more difficult as interaction between assets must be taken into account. When studying dependence among random variables, linear correlation is by far the most commonly used measure in portfolio construction model. However, as it is a measure of linear dependence; linear correlation is limited to those of multivariate normal and elliptical distributions. However, returns of financial assets such as bond indices, exchange rates and equity prices are known for non-normal behaviour [19]. The distributions of such returns are often skewed with heavier left tails than a theoretical normal distribution, implying a greater likelihood of large losses than gains. Hence, measures based on multivariate normal assumption cannot fully account for the dependence structure of financial assets in practice [95]. In addition, deviation from normality, particularly in a multivariate case, can be seen in the form of asymmetric dependence (see Figure 1.2). That is, two asset returns exhibit greater correlation during market downturns than market upturns (as a consequence of investors having greater uncertainty about the state of the market), as reported in Longin and Solnik [87] and Ang and Chen [6]. Most importantly, such dual characteristics of asset dependence cannot be explained with linear correlation.

**Figure 1.2 Example of symmetric and asymmetric dependences.** Asymmetric dependence (left) indicates unequal chances for co-movements in good and bad times. In contrast, symmetric dependence (right) exhibits relatively equal probabilities for co-movements in both good and bad market conditions. This figure is produced with artificial data for the purpose of illustration only.




Thus the second motivation is to develop a portfolio optimisation model that can capture non-linear dependence structure of assets in order to better handle portfolio uncertainty. In our study, a copula is chosen as a tool to model time-varying dependence between assets when generating scenarios for a stochastic programming model. A copula is basically a distribution function which indicates a probability associated to a mutual event of two (or more) variables. The advantage of copula-based approach is that it provides a flexibility to model a dependence structure between random variables independently of their marginal distributions. That is, a multivariate distribution can be decomposed into marginal distributions that are selected individually and then linked through suitable copulas. A fitted marginal distribution provides a good description of risk and returns for an individual asset while fitted copulas describe an association among assets by not limiting the dependence structure to merely a linear fashion as when using a correlation.

It is also known that, even if a sophisticated portfolio optimisation model is developed, it cannot perfectly anticipate financial markets in the future. Portfolios that are expected to be risk-return optimal based on in-sample data, most of the time, become nonoptimal after implementation due to unexpected changes in market environment. Furthermore, when time passes, portfolio allocation is most likely to deviate from an original position due to different market condition of each asset. Portfolio rebalancing is a practice commonly applied to restore the efficiency of a portfolio. Typical rebalancing strategy is a constant mix which reallocates a deviated portfolio back to target allocation (derived from optimisation) in every rebalancing period (e.g., monthly or quarterly). Nonetheless, such rebalancing strategy may cause unnecessary transactions and waste transaction costs in some instances (e.g., selling off performing assets in order to keep

their original proportion). Additionally, the underlying assumption in the constant mix strategy is that rebalancing could maintain the risk-return profile of a portfolio throughout an investment horizon. Such an objective, however, is not always feasible. A return target of a portfolio achieved in optimisation might not be attainable in a real situation as a result of evolving financial market conditions. Between 1985 and 2014, the US stock market delivered returns of 7.9% annually, on average; the bond market rewarded investors with an average annual return of 5%. It is forecasted that, over the next 20 years [93], those figures would shrink to as little as 4% to 5% for stocks and zero to 0.1% for bonds. It is widely perceived that, currently, financial markets have entered a "new normal". It follows that a conventional rebalancing approach is too rigid and not adaptive to varying market environment.

The last motivation of our study is hence to propose a new rebalancing strategy with new efficiency criterion which is adaptable to changing market condition. We adopt "second-order stochastic dominance" as an objective in developing a portfolio rebalancing model. The optimal rebalancing yields a resulting portfolio that is at least as good as a portfolio with floating allocation (buy-and-hold) in terms of probability of return. The new strategy is expected to save unnecessary transaction costs and generate more stable risk-return profile than conventional rebalancing methods.

## 1.2 Research Questions

This thesis develops a novel modelling framework for addressing problems of optimally managing international portfolios and controlling the associated uncertainty. The main goal of this study is to investigate the relative effectiveness of an alternative hedging strategy and dependence structure modelling in controlling risks, particularly for extreme cases, that affect the performance of international portfolios.

From this point onwards, the thesis refers to a handful of financial terminologies. Readers who are unfamiliar with financial terminologies are strongly recommended to go through Chapter 2 and Appendix A to familiarise the terms that are frequently used throughout the thesis.

The thesis begins with developing a hedging strategy and incorporate it into an optimisation model. Foreign exchange forward contracts[7] (FX forwards) are selected as a

[7]The contracts are obligation to exchange two currencies in the future at a rate agreed today. Thus, uncertainty regarding fluctuating exchange rate in the future is reduced, allowing trading strategies to be

hedging instrument due to its flexibility. Since forward contracts are generally traded outside exchanges by banks and other financial institutions, they are customisable to individual preferences in terms of maturity and trading size. Hence, some operational constraints such as the contract length[8] can be discarded. A combination of forward contracts from distinct currency pairs constitutes a "currency overlay"[9] which individually adjusts exposure of each currency. We incorporate the currency overlay into a mean-variance portfolio optimisation model to find optimal asset allocation and optimal holdings of foreign exchange forward contracts for an international portfolio. In addition, we include costs and constraints related to holding forward contracts into the optimisation model in order to obtain most accurate solutions. The questions consequently arise:

- Does exchange rate hedging on an international portfolio result in more efficient portfolio?
- How could the incorporation of a currency overlay affect the risk-return reward of a resulting portfolio?
- How would the risk-return compensation of a portfolio be affected when constraints associated to forward contracts holding are altered?

These questions lead to interesting experiments in Chapter 3 which help establish understanding of the importance of managing asset and currency exposure and how the currency overlay is used to improve portfolio efficiency.

Once the optimisation model with a proper hedging mechanism is constructed and thoroughly examined, we turn our attention to the issue of addressing uncertainty of an international portfolio. Basically, uncertainty regarding risk and return of a portfolio is usually expressed in a form of scenarios. The scenarios represent discrete, joint distributions of the random variables and constitute the primary inputs to the stochastic optimisation models. Reliability of optimisation solutions is thus determined by the quality of scenarios generated. Typically, correlated scenarios for multiple

planned beforehand. Further details on the foreign exchange forward contracts can be found in Section A.5 of Appendix A.

[8] Hedging instruments that are traded over exchanges such as futures and options have specific maturities, e.g., three or six months. A portfolio that needs to hold such products longer than the specified period must take into account an extension of hedging positions before expiry.

[9] A currency overlay on each currency is an aggregate currency exposure from forward contracts. With different combination of FX forwards, currency exposure can be adjusted according to portfolio mangers' preferences. See more details of a currency overlay in Section A.6 of Appendix A.

assets are generated assuming that all returns are normally distributed[10] and linearly correlated[11]. In contrast, our approach does not impose any specific distribution function on individual asset and models a dependence structure of assets using a copula-based approach. Hence, the key question to the work becomes

- Whether the proposed scenario generation method produces portfolios with better risk-return compensation than those produced by standard scenario generation approach?

The verification is made on out-of-sample data where more details are given in Chapter 4.

Problems related to international investment management are complex as they involve many interrelated sources of uncertainty (currencies and assets). Models that can effectively capture the stochastic evolution of the underlying financial processes and suitable stochastic optimisation models inevitably become challenging and computationally demanding. In terms of computational complexity, optimisation problems using a stochastic model are much more difficult to solve compared to using a deterministic model. Furthermore, the optimisation problem can be more challenging if integer constraints are added into the model. Moreover, the more scenarios in the scenario tree provides closer approximation of the "true" stochastic problem corresponding to continuous processes (distributions) for the underlying random variables. However, the search space of the corresponding optimisation problem grows exponentially with the number of decision stages, variables and constraints. An appropriate balance is needed between accuracy in the scenario set, for effectiveness of the solutions, and problem size so as to maintain computational tractability. For our international portfolio optimisation model, we apply a genetic algorithm to solve the optimisation problems. It is thus interesting that

- Would genetic algorithms provide reliable solutions for our international portfolio optimisation model?

The answer to this question is covered in Chapter 4 where the runtimes taken for solving international portfolios under different conditions are also reported.

[10]The risk of assuming that return distributions are normal while in fact they are not is a false sense of security. That is, underestimating the chance and severity of extreme events that could occur, which is eventually leading people to take more risk than they are supposed to.

[11]If two variables are linearly correlated; visually, the relationship between them can be depicted as a straight line when plotted out next to each other in a graph. Statistically, the correlation between the two variables is not zero, can be either positive or negative.

With the stochastic international portfolio established, we apply it along with the second-order stochastic dominance criterion in developing a new rebalancing strategy. The aim is that the model can yield more stable returns than conventional rebalancing methods. It is therefore a question that

- Would the new rebalancing strategy outperform traditional rebalancing strategies in terms of portfolio return?

The details of this study is provided in Chapter 5 in which the new rebalancing model is presented and results from the rebalancing strategy proposed are compared with those from the standard methods.

The key questions mentioned above are the guideline in shaping the structure of this thesis. In addition to these core questions, there are more interesting findings discovered along the studies which are described later in the thesis.

## 1.3 Methodology

Uncertainty in financial market conditions is a vital determinant of portfolio performance. The thesis develops the models to support decision making in the face of uncertainty in dynamic setting. As such, decisions on portfolio allocation and hedging strategies need to be made simultaneously. The framework of two-stage stochastic programming with recourse is deemed suitable to handle this problem.

Stochastic programmings exhibit numerous attractive features applicable to international portfolio allocation and portfolio rebalancing problems. Uncertainty in input parameters of these models is represented through discrete distributions (scenarios) that allow describing both individual and joint characteristics of risks and returns of financial assets. Scenarios are not restricted to follow any specific distribution or stochastic process. Thus, any joint distribution of the random variables can be used to model risk-return characteristics of assets. This is a clear advantage over classical mean-variance quadratic programming which is restrictively compatible with assets whose returns are normally distributed.

In order to accommodate non-linear relationship among multiple assets, we apply a "regular-vine copula"[12] in generating scenarios. The technique allows us the flexibility to model marginal distributions and a dependence structure among assets separately.

[12]More details are given in Section 4.2.6 of Chapter 4.

This ensures that individual extreme events of each asset and mutual extreme events of multiple assets can be included in scenarios which help produce a resilient portfolio.

In addition, we include costs associated to hedging and transaction costs to improve accuracy of risk and return calculation of an international portfolio. Practical constraints concerning currency exposure management and asset transaction are also integrated in an optimisation model to reflect operational burden in reality. These features are incorporated into international portfolio selection and portfolio rebalancing models through mathematical modelling and theoretical background in finance.

The resulting optimisation models (international portfolio selection and portfolio rebalancing) are considered a large-scale stochastic programming due to a large number of scenarios. Moreover, with the inclusion of cardinality constraints, the optimisation problems become intractable hence genetic algorithms are employed to solve the problems.

## 1.4 Thesis Overview

The rest of this thesis is organised as follows:

Chapter 2 gives a fundamental background of portfolio construction and relevant issues, along with reviews of the past literature that are similar or related to works done in the thesis. The aim is to provide readers a background how other researchers have dealt with similar issues in the thesis. Limitations and drawbacks of existing methodologies and theoretical frameworks are raised to uncover the gaps this thesis attempts to fill. The reviews of similar literature are gathered in the same section to help see the evolution of each area in our study. Readers who are unfamiliar with financial terminologies are strongly recommended to go through this chapter along with Appendix A before advancing to the later chapters.

Chapter 3 demonstrates a deterministic mean-variance international portfolio optimisation model which uses a currency overlay in managing currency exposure of the international portfolio. We include the costs that are related to exchange rate hedging into the model to make the risk-return calculation more accurate. The experiments aim to show how different settings of constraints related to a currency overlay could affect risk-return compensation of an international portfolio.

Chapter 4 extends the optimisation model constructed in Chapter 3 in several aspects to capture uncertainty in managing an international portfolio. The deterministic international portfolio optimisation model of Chapter 3 is reformulated as a stochastic programming problem to take into account uncertainty regarding financial asset returns and risks. The risk measurement is changed from variance to "conditional value at risk" (CVaR). Costs concerning a currency overlay and asset transactions (See details in Chapter 3) are all included into the model. A new copula-based scenarios generation approach is introduced to capture complex characteristics of financial assets. The resulting sophisticated model is solved with the genetic algorithm where we also prove reliability of solutions obtained. The experiments are primarily conducted to examine if the new scenario generation methods, comparing to the standard approach, produces more efficient international portfolios in terms of risk and return.

Chapter 5 applies the stochastic international portfolio model in Chapter 4 to develop a new rebalancing model. The goal of rebalancing is to reallocate deviated allocation so that a portfolio is "second-order stochastic dominant". We show the rebalancing model based on the second-order stochastic dominance criterion, and results from solving the optimisation problem with the genetic algorithm. The experiments compare results from implementing the new rebalancing strategy and conventional rebalancing methods on international portfolios. The aim is to investigate if the new strategy produces more stable returns than existing rebalancing approaches along the given investment horizon.

Lastly, Chapter 6 summarises results and findings of each chapter. Also, limitations of the works are elaborated and directions for future developments are outlined.

## 1.5 Publications

As of the time of thesis submission, the following publications which are related to the works herein have been produced. They are presented below in chronological order.

1. Chatsanga, Nonthachote, and Andrew J. Parkes. International Portfolio Optimisation with Integrated Currency Overlay Costs and Constraints. *Expert Systems with Applications*. (Accepted).

2. Chatsanga, Nonthachote, and Andrew J. Parkes. Two-Stage Stochastic International Portfolio Optimisation under Regular-Vine-Copula-Based Scenarios. *European Journal of Operational Research*. (Under review).

3. Chatsanga, Nonthachote, and Andrew J. Parkes. Portfolio Rebalancing under A Second-Order Stochastic Dominance Criterion. (In final preparation).

# Chapter 2

# Background and Literature Review

This chapter aims to provide details on necessary background before going through more complicated issues in later chapters along with details of how other scholars, based on literature to date, have tackled similar problems that appear in the thesis. Section 2.1 gives a foundation of the structure of an international portfolio and relevant issues involving in managing exchange rate risk. A review of literature in Section 2.2 presents details, topic-by-topic, on recent advancements that other researchers have implemented in managing an international portfolio.

## 2.1 Preliminary Background

This section provides necessary foundations for an international portfolio construction that are used throughout Chapters 3 to 5. This section contains the background from existing works, common practices in financial industry and novel contributions from the thesis. New developments arise in the thesis are explicitly indicated, otherwise they are taken from existing literature or standard usages in finance.

### 2.1.1 International Portfolios

An international portfolio is a collection of investments over multiple countries. Basically, a portfolio can be funded by one or more currencies but in this thesis, for the sake of simplicity, the international portfolio has a "base currency" (funding currency)

in US dollar (USD)[1]. When an international portfolio invests in different countries, it is very likely that such countries have different currencies. In that case, exchange rate plays an important role in converting investment values from different countries to USD. Exchange rate can both increase or decrease portfolio return. With some financial instruments, effect of exchange rates can even be "hedged" (removed) away from a portfolio. To make the scope of this thesis more clearer, an international portfolio refers only to a portfolio that invests in countries of different currencies.

Generally, return and risk of a portfolio that invests in one currency (can be either from one or more countries) are driven by changes in asset prices in corresponding countries. For an international portfolio, apart from changes in asset prices, there are also changes of exchange rates that affects risk and return of an international portfolio. By the definition of asset exposure and currency exposure (in Section A.4 of Appendix A), it can be seen that a single-currency portfolio has only asset exposure while an international portfolio has both asset and currency exposure.

In what follows in this thesis, an international portfolio is presented as a table. Columns denote currencies corresponding to countries a portfolio invests (some columns may share the same currency as others, e.g., investment in Eurozone countries). Rows denote exposure associated to investment in each country, e.g., asset and currency exposure. An example of a portfolio investing in the US, the UK and Germany by 50%, 30% and 20% can be illustrated as in Table 2.1.

**Table 2.1 A sample international portfolio funded with USD, investing in the US, the UK and Germany by 50%, 30% and 20%.** Valuation of all investments outside the US are subject to exchange rates against USD. Considering the investment in the UK which exhibits asset exposure and currency exposure at 30% each, it indicates that return and risk of the portfolio is influenced by the UK asset for 30% of the portfolio value and also by the GBPUSD exchange rate for 30% of the portfolio value.

| | USD | GBP | EUR |
|---|---|---|---|
| asset exposure (%) | 50 | 30 | 20 |
| currency exposure (%) | 50 | 30 | 20 |

The number in each cell specifies the asset or currency exposure as a percentage of a portfolio dollar value. For example, the investment in the UK exhibits asset exposure and currency exposure at 30% each, it indicates that return and risk of the portfolio is influenced by the UK asset for 30% of the portfolio value and also by the GBPUSD

[1]USD is chosen as a base currency of portfolios throughout the thesis as it is regarded as a world currency. Changing from USD to other currencies does not affect the complexity of an optimisation problem but it will change final solutions from currency to currency.

exchange rate for 30% of the portfolio value. The summation of asset exposure across all investments is 100%, and so is for currency exposure. Initially, asset and currency exposure are equal unless there is an intention to adjust currency exposure by holding foreign exchange forward contracts[2]. This is known as an "unhedged" portfolio.

Next, it is shown that if portfolio managers have a view on exchange rates and desires to adjust existing currency exposure, they can do so by holding forward contracts and creating a currency overlay[3] on an international portfolio.

### 2.1.2 Modifying Currency Exposure with A Currency Overlay

A portfolio manager can hedge or even speculate on exchange rates by reducing exposure of one currency and increase it on another. Since an exchange rate is a pair of currencies, the exposure increase must be equal to the exposure decrease.

Clarification of how currency overlay helps manage exchange rate risk in multi-currency portfolios is exemplified in Table 2.2. Suppose that a portfolio manager invests in three stock markets, the United States, the United Kingdom and Japan with an allocation of 35%, 45% and 20% respectively. If the base currency of a portfolio is USD and they anticipate that GBPUSD will appreciate while USDJPY will depreciate, they can adjust the final currency exposure to align to their favour – more exposure in British Pound and less exposure on Japanese Yen. In contrast, if they have no view on exchange rate movements, the portfolio remains unhedged and there exists no currency overlay position.

A currency overlay values or a currency overlay position specifies currency exposure that increase or decrease from asset exposure. A positive value signifies an increase while a negative value indicates a decrease. Since the exposure increase is equal to the exposure decrease, the summation of currency overlay positions is thus equal to zero. The total overlay value in the last line of Table 2.2 represents the overall adjustment of currency exposure in the portfolio. The value is calculated from the summation of all absolute values of overlay positions divided by two. Two portfolios with the same total overlay values can thus have different overlay positions.

In this thesis, a currency overlay is constructed from a combination of multiple forward contracts. An overlay position of each currency is therefore a sum of one or more

[2]See more details of foreign exchange forward contracts in Section A.5 of Appendix A.
[3]See an example of overlay construction in A.6 of Appendix A.

foreign exchange forward positions on that currency. It is shown later in Table 2.3 that portfolios with the same overlay positions can be created from various combination of foreign exchange forward contracts.

**Table 2.2 International portfolios with and without a currency overlay.**

| | Hedged | | | Unhedged | | |
|---|---|---|---|---|---|---|
| | USD | GBP | JPY | USD | GBP | JPY |
| asset exposure (%) | 35 | 45 | 20 | 35 | 45 | 20 |
| currency exposure (%) | 27 | 55 | 18 | 35 | 45 | 20 |
| overlay position (%) | -8 | 10 | -2 | | - | |
| total overlay (%) | | 10 | | | - | |

The given example treats choices of currency overlay as a discretionary decision and made as a separate practice after obtaining optimal asset allocation. This relies considerably on personal judgment and experience. To avoid personal discretion, there has been an improvement to incorporate hedging decisions into optimisation process, for instance, in the studies of Adjaoute and Tuchschmid [2], Beltratti et al. [15], Brown et al. [21], Glen and Jorion [48], Larsen and Resnick [78], Rudolf and Zimmerman [120], Topaloglou et al. [134] and Topaloglou et al. [136]. The currency overlay positions in the past literature are, however, defined as a difference between asset and currency allocation and are not associated to individual pairs of foreign exchange forward contracts (resulting in an unknown combination of foreign exchange forwards that should be held). This limitation also disallows implementation of different transaction costs on each forward pairs and imposition of constraints related to transaction costs reduction. The approaches on the structure of portfolios and the inclusion of cost of carry and associated transaction costs of foreign exchange forwards are thus introduced to bridge those gaps, which will make an optimisation model able to accommodate practical constraints and specify right forward contracts that should be held in a portfolio.

#### 2.1.2.1 Structure of A Portfolio with A Currency Overlay

This section presents a structure of an international portfolio with a currency overlay which is later used in developing an optimisation model. The proposed structure is introduced nowhere before, hence it is a direct contribution to the literature.

As shown in Section A.6 of Appendix A, a currency overlay can be created from a combination of foreign exchange forwards. There can be many possible choices that result in a common currency overlay position by holding various number of forward pairs as shown in Table 2.3.

**Table 2.3 Sample choices to construct the same currency overlay.**

| | | USD | GBP | JPY |
|---|---|---|---|---|
| overlay position (%) | | -8 | 10 | -2 |
| choice 1 | USDJPY | 2 | | -2 |
| | GBPUSD | -10 | 10 | |
| choice 2 | GBPJPY | | 2 | -2 |
| | GBPUSD | -8 | 8 | |
| choice 3 | USDJPY | 1 | | -1 |
| | GBPUSD | -9 | 9 | |
| | GBPJPY | | 1 | -1 |

In fact, for an investment of $C$ currencies, there exists at most $\binom{C}{2}$ different forward contracts. To optimally choose forward pairs from all available choices, we introduce a new structure of currency overlay and incorporate it in a portfolio optimisation problem. Suppose that an investment plan is to invest in $A$ asset classes from $C$ countries, exposure on assets and currencies in a portfolio can be characterised as in Table 2.4 with the following notations:

i Index of asset classes; $i = 1,...,A$.

j Index of countries, or synonymously currencies; $j = 1,...,C$.

k Index of forward contracts $k = 1,...,K$.

$a_{ij}$ Exposure to asset class $i$ of country $j$.

$f_{kj}$ Forward position of contract $k$ on country (currency) $j$.

The forward position $f_{kj}$ represents how much additional exposure is added into or taken off from a pair of currencies. Since each forward contract contains only a pair of currencies, given that there are $C$ countries to invest, then there are $K = \binom{C}{2}$ different forward contracts in total. Considering exposure on each country $C$ from a forward

contract $K$, since the exposure from a pair of currencies are equal when being valued in a portfolio's base currency, then it is strictly required that $\sum_j f_{kj} = 0$ for all $k = 1, ..., K$.

Denoting $\mathbf{f}_k = (f_{k1}, ..., f_{kC})$ a vector of exposure from a forward contract $k$, the previous requirement implies that only two elements of $\mathbf{f}_k$ represent the exposure with one being equal to a negative value of another, while the rest of the elements only takes a value of zero. To avoid putting those requirements into constraints of an optimisation problem, we define $f_{kj}$ as an element of a matrix $\mathbf{F}$ in which

$$\mathbf{F} \stackrel{\text{def}}{=} \mathbf{T} \circ (\mathbf{1}^{\mathrm{T}} \otimes \mathbf{q}) \tag{2.1}$$

where $\circ$ is the Hadamard product operator, $\otimes$ is the Kronecker product operator, $\mathbf{T}$ is a $K \times C$ combinatorial matrix of $\{-1, 0, 1\}$, $\mathbf{1}$ is a $C \times 1$ column vector of ones and $\mathbf{q}$ is a $K \times 1$ column vector determining the size of exposure.

**Table 2.4 Structure of a multi-currency portfolio.**

| | Country 1 | $\cdots$ | Country j | $\cdots$ | Country C |
|---|---|---|---|---|---|
| Asset class 1 | $a_{11}$ | $\cdots$ | $a_{1j}$ | $\cdots$ | $a_{1C}$ |
| $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ | $\ddots$ | $\vdots$ |
| Asset class $i$ | $a_{i1}$ | $\cdots$ | $a_{ij}$ | $\cdots$ | $a_{iC}$ |
| $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ | $\ddots$ | $\vdots$ |
| Asset class $A$ | $a_{A1}$ | $\cdots$ | $a_{Aj}$ | $\cdots$ | $a_{AC}$ |
| Forward position 1 | $f_{11}$ | $\cdots$ | $f_{1j}$ | $\cdots$ | $f_{1C}$ |
| $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ | $\ddots$ | $\vdots$ |
| Forward position $k$ | $f_{k1}$ | $\cdots$ | $f_{kj}$ | $\cdots$ | $f_{kC}$ |
| $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ | $\ddots$ | $\vdots$ |
| Forward position $K$ | $f_{K1}$ | $\cdots$ | $f_{Kj}$ | $\cdots$ | $f_{KC}$ |
| Asset exposure | $\sum_{i=1}^{A} a_{i1}$ | $\cdots$ | $\sum_{i=1}^{A} a_{ij}$ | $\cdots$ | $\sum_{i=1}^{A} a_{iC}$ |
| Overlay position | $\sum_{k=1}^{K} f_{k1}$ | $\cdots$ | $\sum_{k=1}^{K} f_{kj}$ | $\cdots$ | $\sum_{k=1}^{K} f_{kC}$ |
| Currency exposure | $\sum_{i=1}^{A} a_{i1} + \sum_{k=1}^{K} f_{k1}$ | $\cdots$ | $\sum_{i=1}^{A} a_{ij} + \sum_{k=1}^{K} f_{kj}$ | $\cdots$ | $\sum_{i=1}^{A} a_{iC} + \sum_{k=1}^{K} f_{kC}$ |
| Total overlay | $\frac{1}{2} \sum_{j=1}^{C} \left\lvert \sum_{k=1}^{K} f_{kj} \right\rvert$ | | | | |

The combinatorial matrix $\mathbf{T}$ of ternary variables is constructed by first specifying $\mathbf{T}$ as a matrix of zeros of size $K \times C$, then we denote a set $\mathbf{D}$ containing all combinatoric pairs from $\binom{C}{2}$. At each row of $\mathbf{T}$, the first member of each pair in $\mathbf{D}$ specifies which element to take the value of 1 and the second member of the pair specifies the element that take value of -1. For example, if there are 4 countries to invest, that is, $C = 4$, then $K = \binom{4}{2} = 6$ and so:

$$\mathbf{D} = \{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\}.$$

The matrix $\mathbf{T}$ whose elements are specified according to $\mathbf{D}$ is

$$\mathbf{T} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}. \tag{2.2}$$

Consequently, referring to (2.1), $\mathbf{F}$ is equal to

$$\begin{aligned} \mathbf{F} &= \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \circ \left( \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \otimes \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{bmatrix} \right) \\ &= \begin{bmatrix} q_1 & -q_1 & 0 & 0 \\ q_2 & 0 & -q_2 & 0 \\ q_3 & 0 & 0 & -q_3 \\ 0 & q_4 & -q_4 & 0 \\ 0 & q_5 & 0 & -q_5 \\ 0 & 0 & q_6 & -q_6 \end{bmatrix} \end{aligned}$$

and $f_{kj}$ is defined accordingly as an element of $\mathbf{F}$. Subsequently, all requirements regarding the characteristics of forward contracts are completely fulfilled.

If a portfolio holds no forward contract (i.e., $f_{kj} = 0$ for all $k$ and $j$), then there is no adjustment on currency exposure and the final currency exposure is equal to asset

exposure (e.g., as in Table 2.1). In contrast, when the portfolio holds one or more forward contracts (i.e., $f_{kj} \neq 0$ for some $k$ and $j$), then the final currency exposure of each currency deviates from asset exposure by the sum of all forward positions in that currency. Specifically, the final currency exposure of a currency $j$ is defined by

$$\text{Overlay position of a currency } j = \sum_{i=1}^{A} a_{ij} + \sum_{k=1}^{K} f_{kj}. \tag{2.3}$$

Accordingly, the total overlay of a portfolio is defined by the total increase (or decrease) of currency exposure on all currencies, which is equivalent to the sum of absolute overlay positions divided by two as follows:

$$\text{Total currency overlay of a portfolio} = \frac{1}{2} \sum_{j=1}^{C} \left| \sum_{k=1}^{K} f_{kj} \right|. \tag{2.4}$$

With all necessary variables being defined, the constraints associated with overlay positions and forward contracts can be of the following examples:

1. *Limited total overlay positions* - total overlay specifies how much currency exposure can deviate from asset exposure. Let $V$ denote the total overlay of a portfolio and $V_u$ the total overlay limit allowed on a portfolio, the constraint is defined as

$$V \leq V_u \tag{2.5}$$

where $V = \frac{1}{2} \sum_{j=1}^{C} \left| \sum_{k=1}^{K} f_{kj} \right|$. Notice that $V_u = 1$ allows total currency exposure deviate from total asset exposure up to 100% of the portfolio while $V = 0$ implies an unhedged portfolio which disallows any use of forward contract. More room for currency and asset exposures deviation provides more opportunity to shift from less-performing to better-performing currencies which hence better improve the risk-return profile of the portfolio.

2. *Bounded currency exposure* - this constraint is imposed, for example, when the net short currency positions are not permitted on specific countries. the constraint can be used in tandem with restrictions of short positions on assets. As a consequence, a specific bound $E_j^l$ and $E_j^u$ of currency exposure can be imposed on each country or currency. The constraint is thus specified as

$$E_j^l \leq \sum_{i=1}^{A} a_{ij} + \sum_{k=1}^{K} f_{kj} \leq E_j^u; \quad j = 1,..,C. \tag{2.6}$$

3. *Limited number of forward contracts* - this can be viewed as a cardinality constraint[4] on the number of forward contracts. In practice, a larger number of forward contracts means more operational burden which can be lessened by specifying a limit on number of contracts allowed. The constraint can be formulated as

$$l_k b_k \leq q_k \leq u_k b_k; \quad k = 1,..,K, \tag{2.7}$$

$$\sum_{k=1}^{K} b_k \leq G, \tag{2.8}$$

$$b_k \in \{0,1\}, \tag{2.9}$$

where $l_k$ and $u_k$ are respectively lower bound and upper bound for exposure sizes and $G$ specifies the total allowance on number of forward contracts. This constraint can be regarded as a cardinality constraint.

It is worth noting that the minimum contract size of forward contracts is customisable with counterparties, the constraint like minimum holding size is hence unnecessary.

The next two subsections provide details of forward contracts and the cost of carry which determines gain and loss of each individual foreign exchange forward contract held in a portfolio.

### 2.1.3 Foreign Exchange Forward Contracts

When a portfolio invests in multiple currencies, there is currency risk entailed from exchange rates fluctuation. A wide range of financial derivatives can be used to mitigate currency risk such as foreign exchange forward contracts, foreign exchange futures and foreign exchange options.

A foreign exchange forward contract is basically an obligation to buy or sell securities at a future date with the price agreed today. The agreed price or a forward rate is slightly different from today's market price so as to prevents an investor from making a riskless profit by exploiting the interest rate differential.

The similar tool to forward contracts and that can be used to hedge foreign exchange rates risk is foreign exchange futures. The key difference is that a currency forward

[4]Typically, a cardinality constraint appearing in portfolio optimisation literature restricts the number of assets (e.g., individual stocks) in a portfolio such as in Soleimani et al. [128].

contract is a private over-the-counter transaction between counterparties known to each other, on terms agreed between themselves, while a currency futures contract is traded on a public exchange, e.g., the International Money Market (IMM) division of the Chicago Mercantile Exchange (CME). The futures contracts thus have standardised features such as units of trading, delivery and settlement dates and minimum price increments which could cause several constraints. Forward contracts are therefore preferred to model portfolio optimisation over futures contracts in terms of customisation flexibility.

Foreign exchange rate options is structured to serve the same purpose as forward contracts but with the main distinction that they offer the right to buy or sell at future date in contrast of obligation. As there is a choice not to exercise the right, there is a cost attached (details on options pricing can be found in Wilmott et al. [139]) as opposed to forward contracts. Hence in terms of transaction cost, hedging a portfolio with forward contracts is more desirable.

In terms of valuation, when there is a change in the prices of underlying assets, gain and loss on forwards and options are impacted differently. Since valuation of futures and forwards positions are linear in the price of underlying securities, computation of gain and loss is therefore straightforward and similar to that of other assets in the portfolio. In contrast, for options their prices move in non-linear fashion with underlying assets so they need sophisticated predictions to calculate gain and loss. Therefore, in order to keep the portfolio valuation simple to study the benefit of currency overlay constraint, forward contracts are chosen as the hedging instrument in the study.

In the thesis, holding forward contracts in a portfolio causes gain and loss due to the difference between a spot[5] and a forward[6] exchange rates. This characteristic of FX forwards provides flexibility in modifying exposure of currency pairs (increasing exposure in performing currency while decreasing exposure of non-performing currency) in a portfolio.

### 2.1.4 Cost of Carry of Foreign Exchange Forwards

The relationship between a spot exchange rate and a forward exchange rate is determined by the difference in the interest rates earned on the respective currency pairs, known as the “cost of carry”. The idea is that buying a forward contract is equivalent

[5] See Section A.4.1 of Appendix A for more details.
[6] See Section A.4.2 of Appendix A for more details.

to buying an underlying asset now and pay the carry until the end of the contract. To exemplify how interest rates between two countries affect the forward rates and gain or loss on holding foreign exchange forwards, we consider Figure 2.1 for a US investor who:

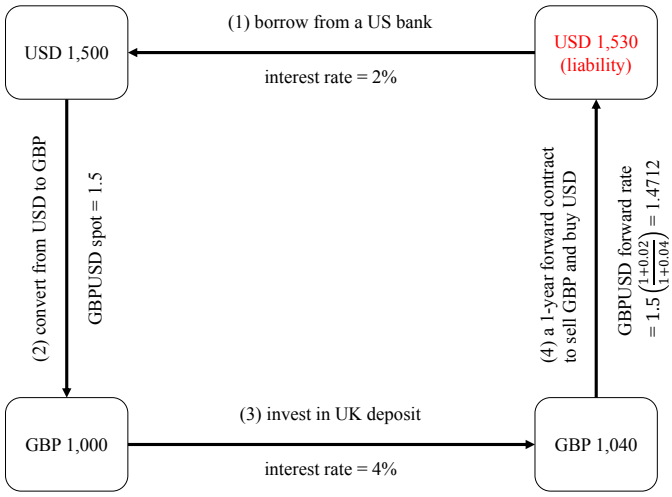
1) Borrows USD 1,500 from a US bank under the annual interest rate of 2%. the investor is now obliged to pay back USD 1,530 to the bank one year later.

2) With the spot exchange rate of GBPUSD = 1.500, the investor converts USD 1,500 to GBP 1,000.

3) The investor invests in a deposit in the UK to earn a profit of 4% per year for the sum of GBP 1,040. The final amount will be converted to USD one year later to pay the obligation to the US bank.

4) To avoid the loss from exchange rates, the investor enters into a forward contract to sell GBP and buy USD in one year later.

**Figure 2.1 An illustration of how interest rates of two countries affect the corresponding forward exchange rates.**




It is observed in step 4) that if the forward rate to sell GBP and buy USD in one year ahead is equal to the spot exchange rate in step 2), the investor can then make a round-tripping with the profit equal to the interest rate differential of 4% - 2% = 2% per year. To eliminate such arbitrage opportunity, the 1-year forward rate for GBPUSD is priced as

$$F_{GBPUSD} = S_{GBPUSD}\frac{(1+i_{GBP})}{(1+i_{USD})} \tag{2.10}$$

where $i_{GBP}$ and $i_{USD}$ are respectively the interest rates of UK and US, $S_{GBPUSD}$ is the spot exchange rate and $F_{GBPUSD}$ is the forward exchange rate of GBPUSD. More details on how the forward rates are priced are given in [73].

The fair price of the forward rate in this example provides a reduced amount of USD comparing to the one converted by the spot rate (1.4712 to 1.5000) to offset a profit from borrowing from low to invest in high interest rates. Equivalently, in this case, holding a foreign exchange forward contract incurs a cost of carry by 2% - 4% = -2% which matches the profit from exploiting the interest rate differential. Therefore, the cost of carry from holding a foreign exchange forward contract can be computed by

$$\text{Cost of Carry} = i_{buy} - i_{sell} \tag{2.11}$$

where $i_{sell}$ is an interest rate of a country that one wants to sell the currency off so as to buy another currency and $i_{buy}$ is an interest rate of a country that one desires to buy.

Holding a foreign exchange forward contract creates a positive position[7] on one currency and a negative position[7] on another. The cost of carry associated with is consequently calculated from the difference between interest rates of a country whose forward position is positive ($i_{buy}$) and a country whose forward position is negative ($i_{sell}$).

The cost of carry is an important feature of a foreign exchange forward contract which distinguishes return of forward from spot exchange rates. It is thus crucial for an international portfolio optimisation model to incorporate such cost when calculating return and risk associated with forward positions. Past literature generally ignore this, so an international portfolio optimisation model with cost of carry incorporated is introduced in this thesis. Details of the model are given in Section 3.3.1 of Chapter 3.

## 2.2 Literature Review

This section provides a review of literature related to the studies in this thesis. To give an idea on the coverage of relevant topics, a brief overview of the thesis is given below.

The thesis particularly focuses on developing an optimisation model and applying optimisation techniques to cope with practical problems arising in managing an international portfolio. The key factors to successfully manage the international portfolio,

[7] See Section A.6 of Appendix A for more details.

apart from good asset allocation, is managing risk from exchange rates. Many scholars have proposed different approaches as discussed in Section 2.2.1. In Chapter 3, a mean-variance optimisation model is applied to optimally allocated fund to invest in assets across countries as well as manage risk from exchange rates. In Chapter 4, a deterministic optimisation model is extended by a two-stage stochastic programming to take into account uncertainty of portfolio returns. Scenarios are deliberately modelled to capture complex relationships among assets using a regular-vine-copula-based approach. Then in Chapter 5 the attention is on managing an international portfolio under uncertainty along an investment horizon. This involves a rebalancing of portfolio allocation over time where the second-order stochastic dominance criterion is applied as an objective of a portfolio rebalancing routine.

To see the evolution of literature more clearly, we start from the earliest idea of a portfolio construction model which was first introduced by Markowitz [91]. The concept is to allocate capital over a number of available financial securities in order to minimise risk associated (evaluated by a risk measure) with the investment at a desired level of return. Originally, the return is measured by the expected value of random portfolio returns while the risk is quantified by the variance of the portfolio. The Markowitz's formulation is thus also known as a mean-variance model or MV(see Section 3.2.1 for more details).

In order to imitate limitations and restrictions of portfolio allocation problems in practice, several "real-world" constraints (see, for example, Cesarone et al. [25] or Soleimani et al. [128]) were subsequently introduced to portfolio selection models. For instance, the buy-in threshold which sets upper and lower bounds on the amount of capital to be invested in each asset to avoid too small or too large investments in an asset, Beale and Forrest [11]. The cardinality constraint allows investors to specify the number of distinct assets in a portfolio to limit transaction costs or to follow some regulatory issues; for instance, in Chang et al. [26], Soleimani et al. [128] and Cui et al. [29]. Fixed and variable transaction costs [86] are included in a portfolio return calculation to improve accuracy of an expected return. Among the real-world constraints imposed on a portfolio optimisation model, an optimisation problem becomes hardest to solve when a cardinality constraint is added as it causes a discrete search space. Exact search methods are deemed inappropriate Martí and Reinelt [94] to deal with cardinality-constrained problems and heuristics are often introduced to find approximate solutions. Example of heuristic search methods used in the literature to solve cardinality-constrained portfolio optimisation problems are given in Section 2.2.3.

Despite its strong theoretical background and ease of implementation, the mean-variance model contains some serious practical drawbacks. For example, estimations of expected returns and variance are assumed to be deterministic regardless of uncertainty of inputs such as asset prices, exchange rates and interest rates. Such a stochastic feature of financial securities is vital for investment decisions and needs to be included in a portfolio optimisation problem as pointed out in Baldacci et al. [9] and Barro and Canestrelli [10]. Stochastic programming is one of the techniques employed to handle parameter uncertainty. The idea has been proposed and well studied since the late 1950s by, for example, Dantzig [31], Beale [12], Charnes and Cooper [27]. In the context of portfolio optimisation problems, uncertainty of input such as asset prices or returns are taken into account by representing it in form of probability distributions.

The departure from normality assumptions causes inevitable complexity in modelling risk and return of a portfolio. The first concern is on the choice of a risk measure. The Markowitz's mean-variance model represents risk by variance which cannot distinguish between variation of profits and losses and thus penalises them alike. Besides, variance is unable to capture extreme losses in the case that returns of financial securities are not Gaussian. Thus, in the thesis, CVaR is chosen as a risk measure of an international portfolio. Other possible risk measures including CVaR employed in portfolio optimisation problems are discussed in Section 2.2.2. The next concern is on the representation of uncertainty through scenarios that are capable to elicit extreme events and complex relationships among multiple financial securities. A copula-based approach is introduced to construct realistic scenarios for optimisation. Typically, bivariate or multivariate copulas are used to model return distribution of a portfolio in literature (see Section 2.2.4 for more details). In this thesis, however, a new representation of multivariate copulas called a regular-vine copula is applied to model asset dependencies to generate realistic scenarios. The advantage of a regular-vine copula over typical multivariate copulas and how it is applied in scenario generation method are discussed in Section 4.2.5 of Chapter 4.

### 2.2.1 International Portfolio Optimisation

Literature related to portfolio optimisation are numerous but most of them limit their investment in a single country (although in several asset classes) and relatively a few focus on international asset allocation. Theoretically, international investment provides

more opportunities for gaining higher returns and more diversified portfolios than investing in only one country.

Studies of international portfolio allocation are inspired by the work of Levy and Sarnat [84] which concludes that low correlation between emerging and developed economies improve portfolio diversification benefit. Recently, Khaoula and Slaheddine [68] study impacts of an integration of emerging stock market indices to a portfolio of developed stock market indices. Their results show that an efficient frontier[8] composed of developed and emerging markets dominates an efficient frontier composed of only developed markets alone.

However, Eun and Resnick [44] point out the impact of exchange rate risk which, in adverse case, may override asset gains. They hence proposed a hedging strategy by short selling foreign currencies at forward rates. Their results show that the hedging strategy outperforms the unhedged one. Subsequently, approaches to effectively hedge currency risk are actively studied. Glen and Jorion [48] use forward contracts to hedge against foreign currency depreciation and optimise simultaneously. They discovered that portfolios that are hedged limited to the size of foreign asset holdings perform better than portfolios with unitary hedging strategies.

In practice, and more often than not in the literature as well, managing exchange rate risk of international portfolios is treated as a currency overlay which is often viewed as a supplementary decision, and is carried out separately from asset allocation to cover specific exposure of foreign asset positions that have been decided precedently. Jorion [60] argues that this overlay approach is suboptimal as interaction between assets and hedging positions are not unified. Filatov and Rappoport [45] introduce a selective hedging approach in which the each currency can be hedged unequally. The empirical results show that their hedging strategy dominates unitary hedging (all currency exposures are fully hedged) and partial hedging strategies. Besides, optimal hedging positions are varied with base currencies of portfolios. Beltratti et al. [15] incorporate selective hedging decisions to a single-stage portfolio optimisation model with the mean-absolute-deviation objective. The results confirm the superiority of selective hedging to unitary hedging. Topaloglou et al. [134] extended this model for international asset allocation using the conditional-value-at-risk (CVaR) objective. Topaloglou et al. [135, 136] formulate a multi-stage stochastic programming model using CVaR as a risk measure. The model jointly determines asset weights and corresponding hedge ratios for an international portfolio. Foreign exchange forward

[8] An efficient frontier is a set of optimal portfolios at different levels of return target.

contracts and quanto options[9] are used as hedging instruments for exchange rate risk. Their findings show that only "in-the-money"[10] put options have a comparable performance with forward contracts. Fonseca and Rustem [47] present a hedging strategy that combines robust optimisation with the investment in quanto options and forward contracts which allows for a combined protection of both the asset and the currency risk. It is found that portfolios with adjusted currency positions is generally more efficient than those without. In addition, the backtesting results show better performance of the robust model when compared to the classical mean-variance model.

### 2.2.2 Portfolio Risk Measure

Several methodologies have been proposed as alternatives to the variance risk measure. For example, Yitzhaki [142] use the Gini's Mean Difference (GMD)[11] as a measure of risk and proposed the so-called GMD model, Konno and Yamazaki [72] use the Mean Absolute Deviation (MAD)[12] as a measure of risk and propose the MAD model for portfolio optimisation. For a more comprehensive review of the various risk measures, we refer the reader to the work of Mansini et al. [90] and references therein.

In 1996, JP Morgan [114] introduced the value-at-risk (VaR)[13] which marks a new era of risk management. As opposed to previous risk measures which focus on some theoretical models that relate risk to portfolio return, VaR allows quantification of risk in terms of possible losses. Nonetheless, VaR has received a lot of criticism as a sub-additivity property could be violated when asset returns are not normally

[9] Quanto options allow an investor to hold an underlying asset denominated in one currency to receive the payout settled in another currency at a predefined spot exchange rate. This lets the investor take a view on the path of a foreign currency pair or investment without worrying about the conversion of any future profits back into the domestic currency.

[10] A put option is in-the-money when the strike price of the option is above the market price of the underlying asset. Conversely, a call option is in-the-money when the option's strike price is below the market price of the underlying asset. Note that being in-the-money means the option is worth exercising, not in profit unless the option premium (cost) is taken into account.

[11] Suppose that there are two variables, $X_1$ and $X_2$, the GMD is defined as the expected absolute difference between two randomly drawn observations:

$$GMD = \mathbb{E}\left[|X_1 - X_2|\right].$$

[12] Given a random variable $X$, the MAD is defined as

$$MAD = \mathbb{E}\left[|X - m(X)|\right]$$

where $m(X)$ is a measure of central tendency, e.g., mean, median and mode.

[13] More details of VaR and CVaR are given in Section 4.2.9 of Chapter 4.

distributed. This means means that VaR of a portfolio can be greater than the sum of individual VaRs of assets. This consequently makes VaR a non-coherent risk measure, Artzner et al. [8]. A modification of VaR, the conditional-value-at-risk (CVaR)[13], was subsequently introduced by Rockafellar and Uryasev [115].

### 2.2.3 Solving Cardinality Constrained Optimisation Problems

An imposition of cardinality constraints generally transforms an MV optimisation from a quadratic programming to a mixed-integer quadratic programming which is NP-hard[14]. As a result, many researchers opt to find approximate solutions rather than exact ones by means of heuristic algorithms.

Bertsimas and Shioda [17] attempt to find an exact solution with Lemke's pivoting algorithm [81] and CPLEX but both of them fail to deliver any optimal solutions within the computational time limit (two minutes or one hour depending on the size of the problem). Shaw et al. [124] present a lagrangean relaxation based procedure for finding an exact solution. They decompose a covariance matrix into a diagonal asset risk matrix and a covariance matrix to reduce size of a problem. Computational results are reported for eight test problems. It is shown that CPLEX[15] is unable to solve any test problems within the time limit while their approach solves seven of the eight test problems.

Chang et al. [26] find an approximate solution with three heuristic algorithms, i.e., genetic algorithm[16], tabu search[17] and simulated annealing[18]. They also show that resulting efficient frontiers are discontinuous due to cardinality restrictions. Performance comparison among the three algorithms indicates that no one heuristic outperforms the others. Cura [30] apply a particle swarm based algorithm where each particle represents a portfolio. If a portfolio does not contain an appropriate number of assets, then some assets are added or deleted from the portfolio. Soleimani et al. [128] employ a genetic algorithm to solve a cardinality constrained portfolio optimisation problem

[14]Informally, "NP" problems have a "large but shallow" solution space. That is, there could have numerous solutions but it doesn't take long to verify a solution. For example, a chess game. Accordingly, an NP-hard problem is defined as a problem that is at least as hard as the hardest problems in NP.

[15]An optimisation package developed by IBM, typically used to solve linear programming, integer programming, convex and non-convex quadratic programming problems, and convex quadratically constrained problems.

[16]Formal details for genetic algorithms are given in Section 4.4.1 of Chapter 4.

[17]See more details in Glover [49, 50].

[18]See more details in Kirkpatrick et al. [70].

which includes a constraint limiting an investment in each industry sector. Pai and Michel [108] apply a clustering approach in order to choose assets into a portfolio thereby eliminating the cardinality constraint. An evolutionary strategy is used to decide the proportion to be invested in each of the assets.

### 2.2.4 Application of Copulas in Portfolio Optimisation

Copulas are introduced to portfolio optimisation primarily to enhance the ability of capturing complex characteristics of financial assets, e.g., asymmetric dependence and nonlinear relationship which cannot be described by correlation. Besides, it is confirmed in the work of Kaut and Wallace [66] that correlation can lead to sub-optimal solutions for a stochastic model.

Patton [110] finds significant asymmetry in the marginal distributions and dependence structure of financial returns. Jondeau and Rockinger [59] employ Gaussian and Student's t copulas to model the dependence structure between the returns of four major international stock indices (i.e., the S&P500, Financial Times 100 stock index, Deutsche Aktien Index, and the French Cotation Automatique Continue index). They suggest that their approach can be used in asset allocations or Value-at-Risk (VaR) computations in a non-normal framework. However, their model is only able to capture elliptical dependence, not asymmetric dependence. Kole et al. [71] apply Gaussian, Student's t and Gumbel copulas in risk management. They find that Gaussian copulas underestimate the probability of joint extreme downward movements whereas Gumbel copulas tend to overestimate this risk. They conclude that Student's t copulas are best suited to minimise risk upon their portfolio of stocks, bonds and real estates. Morton and Popova [101] model the return distribution of hedge funds using a copula-based procedure. The optimal allocation from the copula-based model exhibits less proportion of risky assets resulting in higher Sharpe ratios[19]. Ortobelli et al. [107] employ Student's t copula to model co-movements of assets and compare the performance of this portfolio to the one built from traditional approach (without copula). The comparison displays the better performance in terms of cumulative wealth for the copula-based portfolio.

The application of copula in modelling asset returns is also possible for creating scenarios for stochastic programming. However, since copulas are quite new to the

[19]The Sharpe ratio is the average return earned in excess of the risk-free rate per unit of volatility or total risk.

community, relevant literature of this area is scarce. Kaut [63] develops a heuristic for generating scenarios for a two-stage stochastic programming using empirical copulas to describe the dependence between marginal distributions of assets. He finds that his method is better than scenarios generated with both sampling and a moment-matching method using correlations. Specifically, the new method needs only fifty scenarios to obtain better solutions than the other two methods produced with one thousand scenarios.

The use of non-parametric copula gives flexibility in fitting a copula function to raw data by not restricting to a specific copula family. In our point of view, however, the quality of estimation depends largely on the number of observations. Insufficient observations lead to often lead to false description of asset dependence. Therefore, in our study, we estimate copulas parametrically.

In most of the existing literature, multivariate copulas such as multivariate Gaussian and Archimedean copulas are commonly employed in modelling a dependence structure of multiple assets. Such methods, restrict that the dependence structure of all assets are mutually identical which is, however, unrealistic. In contrast, we adopt the notion of a vine-copula which describes multiple asset dependency by a tree-like combination of bivariate copulas. The idea to systematically organise multiple bivariate copulas is introduced in Bedford and Cooke [13, 14]. The approaches used in extracting and sampling a vine-copula from raw data are given in Dissmann et al. [36]. Such techniques are used as a foundation in developing our scenario generation method.

### 2.2.5 Application of Second-Order Stochastic Dominance in Portfolio Optimisation

The application of stochastic dominance was first brought to the area of portfolio optimisation in the works of Ogryczak and Ruszczyński [105] and Ogryczak and Ruszczynski [106]. They propose an optimisation problem that identifies mean-risk efficient frontiers of stochastically non-dominated portfolios. Dentcheva and Ruszczyński [33] develop an approach to verify if a portfolio that stochastically dominates a given benchmark among an infinite number of diversified portfolios by imposing stochastic dominance constraints in an optimisation problem.

An extension to the standardised framework such as the second-order stochastic dominance (SSD) has created new researches regarding portfolio construction. It is

pointed out in the work of Fishburn [46] that optimal portfolios under SSD objective are consistent with the efficiency in the perspective of rational and risk-averse investors. Roman et al. [118] formulate the portfolio optimisation problem with SSD constraints. The objective function of the respective optimisation problem can be recast as a function of CVaR.

It can be seen from literature that the prominent characteristic of stochastic dominance is to identify stochastically dominant portfolios from an efficient frontier. This is because mean-risk models (e.g., mean-variance) may produce portfolios that are dominated with respect to second order stochastic dominance and consequently not optimal for rational and risk-averse[20] investors [118]. Such feature inspires the idea of applying the SSD criterion in a portfolio rebalancing model in this thesis. Additionally, the finding of Roman et al. [118] that SSD-objective portfolio optimisation problem can be alternatively expressed by CVaR facilitates a formulation of a rebalancing model.

### 2.2.6 Portfolio Rebalancing

Portfolio managers generally construct optimal portfolios to match their desired risk-return profile. However, since different assets exhibit different rates of return, a manager cannot maintain a target allocation over time without active rebalancing [132]. The most conventional approach is a periodic rebalancing [37, 96]. The drawback of this method is that trading decisions are independent of market behaviour.

Optimal rebalancing strategies are typically formulated in a form of dynamic programming in order to identify when and how much to rebalance a portfolio. Previous studies of Perold and Sharpe [112] and Leland [80] have established the existence of a no-trade region around the optimal target portfolio weights. The strategy shows that transaction costs can be reduced by approximately 50%. Mulvey and Simsek [102] formulate a multi-stage rebalancing problem including transaction costs which are modelled using a generalised network. An algorithm based on Benders decomposition [16] is developed for solving the resulting problem. Donohue and Yip [37] confirm the results of Leland [80] and have characterized the shape and size of the no-trade region and compare the performances of different rebalancing strategies. Sun et al. [132]

[20] It is generally assumed that investors are risk-averse, that is, reluctant to take risks. Thus when facing with two investments with a similar expected return (but different risks), will prefer the one with the lower risk.

formulate a dynamic programming to develop a rebalancing policy that trades only when the cost of trading is less than the cost of doing nothing. The results show that their strategy performs better than periodic and buy-and-hold rebalancing methods.

Alternatively, a rebalancing model can be formulated using a single-period optimisation. Dybvig [38] apply a single-period mean-variance optimisation with various types of transaction costs. The results show that when fixed and variable costs are included, there is a non-trading region within which trading does not pay off (since trading further would incur additional costs that are not justified). When only variable costs are included, any trading is to the boundary of the non-trading region while fixed costs induce trading to the interior. Woodside-Oriakhi [140] study the importance of investment horizons when rebalancing a portfolio through a mixed-integer quadratic programming with fixed and variable transaction costs. The empirical results show that efficient frontiers of longer investment horizons dominate efficient frontiers of shorter investment horizons.

In a later study (Chapter 5), the second-order stochastic dominance (SSD) criterion is applied as an efficiency measure instead of a traditional risk-return target. This is because, in some instances, a desired risk-return reward could not be satisfied when there is a structural change in financial markets, portfolio efficiency cannot be sustained throughout an investment horizon with typical rebalancing strategies. We hence develop a new rebalancing strategy using a probability-based criterion providing the advantage that portfolio efficiency is independent of financial market conditions.

# Chapter 3

# International Portfolio Optimisation with Integrated Currency Overlay Costs and Constraints

## 3.1 Introduction

This chapter primarily deals with the foundation of an international portfolio for subsequent chapters. Specifically, the way it is constructed, what the constraints associated are, how the optimisation problem is formulated and how the problem is solved. The fundamentals established in this chapter are expanded and applied in later chapters.

Typically, a portfolio[1] is referred to a collection of investments that are usually different in terms of return and risk. Most likely, assets with higher return have higher risk (generally represented by standard deviation of returns from their expected values). Moreover, when assets are held together, there can arise diversification benefits from imperfect correlation between the assets which subsequently reduces the risk of portfolios. The key purpose of constructing a portfolio is hence to take the most advantage from asset correlations and achieve the lowest risk at the desirable level of return [91].

To widen the scope of diversification rather than diversifying asset holdings across asset classes in only one single country, international investment is therfore considered.

[1]The linkages between a portfolio and other terminologies are given in Section A.1 of Appendix A.

Theoretically, investment in multiple countries could reduce exposure regarding systematic risk in specific markets and offer more opportunities to gain higher profits from promising foreign markets. The key advantage of multi-currency portfolios arises from potentially larger extent of risk diversification; as it is more likely that assets in similar economies are more correlated to their peers than those in different economies. Levy and Sarnat [84] documented that low correlation between investment returns from developed and developing countries leads to the reduction of the portfolio variance.

However, such benefits induce extra exposure to exchange rate variation which could raise, reduce or even ruin the portfolio values at the same time. The importance of managing exchange rate risk is vital for business as addressed in [69] and applied widely throughout various industries. Eun and Resnick [44] discuss the effect of exchange rates on multi-currency portfolios and that, in an adverse case, exchange rate losses could possibly override asset gains. They hence proposed a hedging strategy by short selling foreign currencies at forward rates. Their results show that such hedging strategy outperforms the unhedged one. Subsequently, Glen and Jorion [48] employed forward contracts to hedge against foreign currency depreciation and formulated an optimisation problem that allowed asset allocation and forward contract positions to be optimised together at the same time. They discovered that portfolios that are hedged limited to the size of foreign asset holdings perform better than portfolios employing the unitary hedging strategy[2].

Although there is a strong linkage between security markets and exchange rates, much of the time assets and currencies are not moving in the same direction. In order to manage currency exposure of international portfolios separately from asset allocation, a "currency overlay"[3] is employed in devising a hedged portfolio.

In early development, currency hedging is applied to portfolios after obtaining optimal asset allocation, then the adjustment on initial currency exposure is "overlaid" on the portfolios [103]. The most significant benefit of using currency overlay is that currency exposure management is separated from managing underlying assets. The resulting flexibility becomes even more prominent when investing in a country whose asset markets negatively correlate to its exchange rates. The portfolio can be hedged so that it has limited exposure to exchange rate risk, but still maintains its exposure in asset

[2]Also known as a fully-hedged portfolio which removes all exposure in foreign currencies to only exposure in local currency. For instance, if a USD-funded portfolio invests in US, UK and Japan, then the fully-hedged portfolio will have only currency exposure in USD.

[3]More details on a currency overlay are given in Subsection 2.1.2 of Chapter 2 and in Section A.6 of Appendix A.

markets abroad. Furthermore, a well-managed currency overlay can also be used to manipulate currency positions for speculation on exchange rate markets which could enhance the return on the portfolios [83].

### 3.1.1 Current Advancements in The Literature

Although portfolio optimisation studies are prevalent, a large body of researches focuses on investing in different asset classes (e.g., equities, bonds) in only one country (and hence one currency). There are just a few papers that study international portfolio thoroughly.

At the beginning, an incorporation of a currency overlay constructed from forward contracts[4] in a portfolio optimisation model was introduced by Glen and Jorion [48]. Later, Filatov and Rappoport [45] and Beltratti et al. [15] apply their idea to create selective hedging decisions[5] for an international portfolio. It is also shown in their works that a hedged portfolio is more efficient than an unhedged one, and that a portfolio under selective hedging strategy is more efficient than a portfolio under fully-hedge strategy.

Nevertheless, none of the previous works has included the costs associated to foreign exchange forward holdings. The primary and novel contribution of this chapter is consequently to develop an optimisation model that includes a realistic and comprehensive cost model of forward contracts. The importance of transaction costs are widely examined in financial and portfolio management literature. In essence, transaction costs affect the net gain to investments and actual performance of portfolios, [117]. Arnott and Wagner [7] mentioned that ignoring the transaction costs would lead to inefficient portfolios. Based on existing literature, the implementation of transaction costs on portfolio currency overlay and its effects on risk and return have not been investigated before.

[4]There are other studies which use foreign exchange options rather than foreign exchange forwards to adjust portfolio currency exposure. In the thesis, forward contracts are preferred to options. A discussion over this issue is given in Subsection 2.1.3 of Chapter 2.

[5]The strategy hedges exchange rate risk of each currency independently, unlike, for example, the "fully-hedge" strategy which removes exposure of all foreign currencies away from a portfolio.

### 3.1.2 Chapter Overview

This chapter develops a methodology to formulate the currency overlay by combining foreign exchange forwards and incorporating it into the multi-currency portfolio optimisation model. Crucially, the study introduces an approach to include the cost of carry of forward contracts and associated transaction costs into the process of risk and return calculation which improves the accuracy in determining total returns and volatilities of portfolios. The side benefit is an opportunity to study the effects of particular constraints imposed to reduce transaction costs from holding currency overlay. Since currency overlay utilisation could differ by composition of assets and currencies in a portfolio, different impacts from transaction costs reduction should affect the risk-return compensation of portfolios differently at each return target.

The benefits of the introduced model are twofold. Firstly, the resulting portfolios are optimal in both asset and currency points of view. With the currency overlay allowed, if an adjustment on currency exposure provides no significant improvement on risk and return of a portfolio then the result is exactly the same as the one obtained with no currency overlay allowed. On the other hand, if in some circumstances that extra currency exposure helps improve risk-return reward of the portfolio, then the portfolio holds forward contracts to form the currency overlay. Secondly, by constructing currency overlay from a collection of foreign exchange forwards, it provides flexibility to represent different transaction costs on individual forward contract which also improves accuracy of optimal solutions.

The rest of the chapter is organised as follows: the next section provides a brief summary of a deterministic mean-variance portfolio optimisation model which the thesis aims to improve upon, followed by the approach demonstrating the inclusion of the cost of carry and transaction costs of forward contracts in the return and risk calculation and then formulating an international portfolio optimisation model with associated currency overlay constraints. Section 3.4 reports the experiment results and Section 3.5 provides the summary of this study.

## 3.2 Background

This section gives a background of a mean-variance portfolio optimisation model which is a foundation of more advanced optimisation models introduced in later chapters.

### 3.2.1 Mean-Variance Optimisation

The mean-variance portfolio optimisation is introduced by Markowitz [92] with the aim to determine the allocation $x_i$ of a given capital to be invested in each asset $i$ in a portfolio. The criterion employed to justify an "optimal" allocation is that the portfolio has minimum risk at a desired level of expected return. Typically, an expected return of the portfolio is represented by the historical average of portfolio returns while risk is identified by the variance of portfolio returns.

Suppose that there are $n$ assets available in the portfolio, let $r_i$ denote the expected return of asset $i$ and $\sigma_{ij}$ be the covariance of returns of asset $i$ and asset $j$ for $i, j = 1, \ldots, n$. Also, let $\mu$ be the required level of return for the portfolio. Then, the classical mean-variance model is:

$$\text{minimise} \quad \sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij} x_i x_j \tag{3.1}$$

subject to

$$\sum_{i=1}^{n} r_i x_i = \mu, \tag{3.2}$$

$$\sum_{i=1}^{n} x_i = 1, \tag{3.3}$$

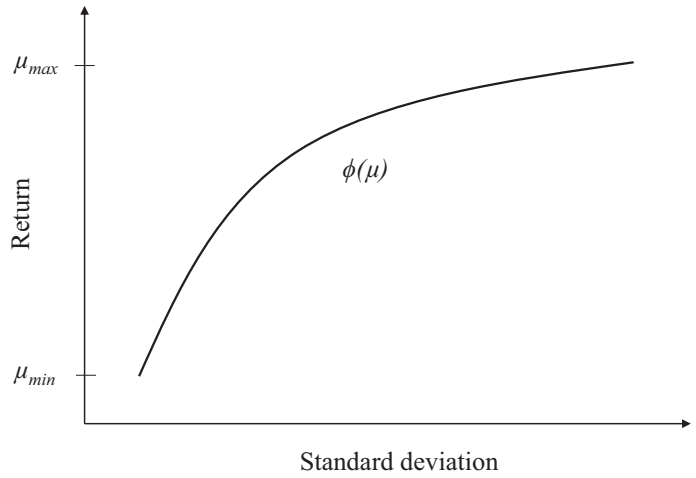
$$x_i \geq 0; i = 1, \ldots, n. \tag{3.4}$$

The objective function (3.1) is to minimise the portfolio variance, given that the portfolio expected return satisfies its required target in (3.2) and the total capital allocation is summed to 1 as in (3.3). The constraint (3.4) disallows the short selling[6] of any assets in the portfolio. The optimisation problem above is a convex quadratic programming problem which can be solved by a number of efficient algorithms with a moderate computational effort even for large instances [24].

[6]Short selling is the practice of selling securities or other financial instruments that are not currently owned, and subsequently repurchasing them. The repurchase is usually performed when the market price is lowered than the initial price when entering a short position.

Let $\phi(\mu)$ denote the resulting optimal portfolio from (3.1) at a target return $\mu$, then the plot of $\phi(\mu)$ over a range of return targets $[\mu_{min}, \mu_{max}]$ establishes the set of all non-dominated (or efficient) portfolios, which is also known as an efficient frontier. The efficient frontier is typically constructed by solving (3.1) over several (equally spaced) values of $\mu$ in $[\mu_{min}, \mu_{max}]$. An example of an efficient frontier is shown in Figure 3.1.

**Figure 3.1 An illustration of an efficient frontier.** This figure is produced with artificial data for the purpose of illustration only.




The convexity of (3.1) implies that, for $\mu > \mu_{min}$, $\phi(\mu)$ is increasing and convex. Hence, the solution of (3.1) does not change if we replace the first constraint with $\sum_{i=1}^{n} r_i x_i > \mu$.

Commonly, some constraints can be added to the classical mean-variance model to make it more realistic. For example, there could be a limit that no more than $K$ assets should be held in the portfolio (a cardinality constraint) [26] and, additionally, that the allocation $x_i$ of each asset that is included in the portfolio should be bounded within a given interval $[l_i, u_i]$ (a quantity constraint or buy-in threshold) [130]. The mean-variance model with the two additional constraints mentioned earlier is formulated as:

$$\text{minimise} \quad \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_i x_j \tag{3.5}$$

subject to

$$\sum_{i=1}^{n} r_i x_i \geq \mu, \tag{3.6}$$

$$\sum_{i=1}^{n} x_i = 1, \tag{3.7}$$

$$\sum_{i=1}^{n} y_i \leq K, \tag{3.8}$$

$$l_i y_i \leq x_i \leq u_i y_i; i = 1, \ldots, n, \tag{3.9}$$

$$x_i \geq 0; i = 1, \ldots, n, \tag{3.10}$$

$$y_i \in \{0, 1\}; i = 1, \ldots, n \tag{3.11}$$

where the cardinality constraint is represented by (3.8) and (3.11) while the buy-in threshold is represented by (3.9). Notice that there are additional $n$ binary variables ($y_i$) when the cardinality constraint is imposed. The optimisation problem (3.5) is classified as the mixed integer quadratic programming and is no longer a convex optimisation problem because of the non-convexity of its feasible region.

## 3.3 Methodology

Based on the structure of an international portfolio with currency overlay given earlier in Section 2.1.2.1 of Chapter 2, this section shows an approach to include a cost of carry (when holding forward contracts) in calculating portfolio risk and return. Also, the proposed portfolio structure allows an easy incorporation of transaction cost associated with an individual forward contract. All these improvements are made with the aim to deliver a more realistic solution of an international portfolio optimisation problem than that of the existing literature.

### 3.3.1 Incorporating Cost of Carry to Risk and Return Calculation of Portfolios

It was mentioned earlier that entering into forward contracts incurs a cost of carry which can be positive or negative depending on interest rate differential. Consider a portfolio holding three foreign exchange forwards as given in Table 3.1. The cost of carry of each forward contract depends on which currency to sell or buy, corresponding interest rates, and a position taken on the portfolio. For instance, selling JPY for USD at 1% of the portfolio amounts a positive carry of $1\% \times 2\% - 1\% \times 1\% = 0.01\%$ to the portfolio. Selling GBP for JPY, however, generates a negative carry of $-2\% \times 4\% + 2\% \times 1\% = -0.06\%$ as a result of shifting the exposure from country with high interest rate to

the country with lower interest rate. The total overlay position is 8% of the portfolio bearing the positive carry of 0.13% from the three forward contracts combined. This amount of carry is subsequently added to the total return of the portfolio.

**Table 3.1 Costs of carry associated with foreign exchange rate forward contracts.**

| | USD | GBP | JPY | Cost of Carry |
|---|---|---|---|---|
| interest rate (%) | 2 | 4 | 1 | |
| sell JPY, buy USD (%) | 1 | | -1 | 0.01 |
| sell USD, buy GBP (%) | -9 | 9 | | 0.18 |
| sell GBP, buy JPY (%) | | -2 | 2 | -0.06 |
| overlay (%) | -8 | 7 | 1 | 0.13 |

Recall that a cost of carry of each forward pair is calculated by summing the product of an interest rate and a forward position over two respective currency pairs of an FX forward. The total cost of carry is consequently found by aggregating all costs of carry of a portfolio. Equivalently, the total cost of carry can be calculated by summing over the product of a net forward position (overlay position) and a corresponding interest rate. Specifically, in Table 3.1, by using the overlay positions, the total cost of carry of the portfolio is equal to

$$(-8\% \times 2\%) + (7\% \times 4\%) + (1\% \times 1\%) = -0.16\% + 0.28\% + 0.01\% = 0.13\%$$

Accordingly, a total return of an investment in a country $j$ contributing to a portfolio is

$$r_j = a_j r_j^a + c_j r_j^c + v_j i_j \tag{3.12}$$

where $r_j$ is total return from investment in country $j$; $a_j$, $c_j$ and $v_j$ are respectively asset exposure, currency exposure and overlay position on country $j$; $r_j^a$, $r_j^c$ and $i_j$ are respectively expected asset return, expected currency return and expected interest rate of country $j$.

Since an overlay position can be viewed as the difference in currency and asset exposures (see A.6 of Appendix A), equation (3.12) can be equivalently expressed as

$$\begin{aligned} r_j &= a_j r_j^a + c_j r_j^c + (c_j - a_j) i_j \\ &= a_j (r_j^a - i_j) + c_j (r_j^c + i_j). \end{aligned} \tag{3.13}$$

$r_j^a - i_j$ and $r_j^c + i_j$ are defined as adjusted return of asset and adjusted return of currency, respectively. Equation (3.13) demonstrates that portfolio total return (return from

assets, currencies and costs of carry of foreign exchange forwards) is equal to the sum between the product of adjusted return and asset exposure and the product of adjusted return and currency exposure.

This implies that the expression of overlay positions is not explicitly required to calculate total returns of a portfolio. In addition, if a portfolio holds no forward contract, the interest rate terms in equation (3.13) will be cancelled out, showing that the formulation in equation (3.13) generalises total return calculation of international portfolios.

Similarly to asset and currency returns, interest rates are not constant over time, and hence volatility of interest rates needs to be included in the calculation of portfolio risk. In accordance with return calculation of international portfolios, equation (3.13) is implemented to adjust return time series for variance and covariance calculation. For example, denotes the covariance between S&P500 index and GBPUSD by $\omega$, the adjusted-return covariance is calculated by

$$\omega = \text{cov}\left((X - Z_{US}), (Y + Z_{UK})\right) \tag{3.14}$$

where $\text{cov}(\cdot)$ is a covariance function, $X$ is the time series of S&P500 index returns, $Y$ is the time series of GBPUSD returns, $Z_{US}$ is the time series of US short-term interest rates and $Z_{UK}$ is the time series of UK short-term interest rates.

### 3.3.2 Incorporating Transaction Costs of Forward Contracts

Generally, transaction costs associated to holding forward contracts are categorised as fixed operating cost, bid-ask spread[7] and opportunity cost from margin requirement. All the costs involve in portfolio return calculation by means of a cost function reducing expected returns of portfolios. The optimisation problem can still be cast as single-period as the initial portfolio is the one without currency overlay implemented.

[7]The spread represents the difference between the highest price that a buyer is willing to pay (bid) for a security and the lowest price that a seller is willing to accept for it (ask). It is considered a cost as a security is purchased at an ask price while it is valued at a bid price which is always lower. As an example, suppose that stock A is quoted as bid = \$5.0 and ask = \$5.5 per unit (bid-ask spread = \$0.5), if investors buy 100 units of stock A, they need to pay \$550 for \$500 worth of stock A. Thus the wider the spread, the higher one needs to pay to acquire a security and the less the net profit.

#### 3.3.2.1 Fixed and Variable Transaction Costs

The fixed cost[8] could be attributed to operational overheads charged from entering into a contract and maintaining the position until maturity date, the cost thus occurs when there is a forward position and does not increase proportionally with exposure. Bid-ask spreads act like break-even costs on holding securities and are attached to sizes of transaction, thus larger positions carry more costs on portfolios. Basically, determinants of bid-ask spreads are volatility and liquidity of respective forward pairs, thus each foreign exchange forward pair carries different charge unlike the fixed costs. Bid-ask spreads and fixed costs can be modelled as a cost function as follows,

$$\phi_k(q_k) = \alpha + \beta_k |q_k|$$

where $\phi(\cdot)$ is a transaction cost function, $\alpha$ is a fixed operating cost per contract, $\beta_k$ is a percent spread of each forward pair calculated from $(\text{ask price} - \text{bid price})/\text{ask price}$ and $q_k$ is exposure on each forward contract as defined by equation (2.1). Note that, in practice, buying and selling securities experience unequal percent spread but for liquid instruments like exchange rate forwards, the difference in buying and selling percent spreads could be insignificant, therefore it can be assumed that buying or selling encounter the same costs. In addition, it is worth noting that the cost function is non-convex due to the presence of fixed operating costs. Hence, the resulting optimisation problem is also non-convex.

#### 3.3.2.2 Margin Requirement

The last component of transaction costs associated to forward contracts is considered an opportunity cost arising due to a "margin requirement" which requires cash to be set aside as a collateral. Since the contract is marked to market daily, cash (or liquid assets) of values equivalent to some percentage of contract values must be reserved to ensure that all contract participants are able to meet the claims from continuous settlement process, [56]. Because forwards are not traded on exchanges, the contract terms are not standardised and thus the percentages required on margin vary by counterparties. The existence of a maintenance margin requires a portfolio to set aside its portion to

[8]The fixed cost for a forward contract is referred to a labour cost such as person-hours spent on dealing with counterparties. The cost occurs once to establish a new contract and does not vary with the contract size. Nonetheless, it is assumed to be proportionated to the portfolio size to reflect difficulty in negotiating contracts when the portfolio value is large.

cash and earn zero or nearly-zero return which diminishes attractiveness of holding forwards in portfolios. The relationship between cash portion and margin requirement is expressed by the following equation,

$$a_{mn} = M \sum_{k=1}^{K} |q_k|$$

where $a_{mn}$ represents cash allocation in a portfolio, $M$ is a percentage of margin requirement and $q_k$ is exposure on each forward contract. This is regarded as a cost since there is a lost opportunity of investment of the cash.

### 3.3.3 Optimisation Problem with Overlay Constraints

Based on the above overlay constraints and the portfolio structure as in Table 2.4, the mean-variance portfolio optimisation problem can be set up with following additional notations:

- $\mathbf{a}$ Vector of asset exposure; $\mathbf{a} = [a_{11}, ..., a_{ij}, ..., a_{AC}]^{\mathrm{T}}$.
- $\mathbf{c}$ Vector of currency exposure; $\mathbf{c} = [c_1, ..., c_j, ..., c_C]^{\mathrm{T}}$ where $c_j = \sum_{i=1}^{A} a_{ij} + \sum_{k=1}^{K} f_{kj}$; $j = 1, ..., C$.
- $\mathbf{x}$ Vector of decision variables; $\mathbf{x} = [\mathbf{a}, \mathbf{c}]^{\mathrm{T}}$.
- $\mathbf{r}$ Vector of expected returns; $\mathbf{r} \in \mathscr{R}^{C(A+1)}$.
- $\mu$ Target return of a portfolio.
- $\Omega$ Variance-covariance matrix of asset and currency returns.

Note that $\mathbf{f}$ is a function of $\mathbf{q}$ as in equation (2.1), thus the total number of decision variables is $AC + K$ and $\mathbf{x} \in \mathscr{R}^{AC+K}$. The vector $\mathbf{r}$ contains adjusted expected returns of assets and currencies according to equation (3.13) by subtracting expected interest rates from asset returns and adding expected interest rates to currency returns. Each element of the covariance matrix $\Omega$ is calculated from adjusted return time series as exhibited in equation (3.14). The mean-variance portfolio optimisation problem with overlay constraints is subsequently formulated as:

$$\text{minimise} \quad \mathbf{x}^{\mathrm{T}} \Omega \mathbf{x} \tag{3.15}$$

$$\text{subject to} \quad \mathbf{x}^{\mathrm{T}}\mathbf{r} - \sum_{k=1}^{K} \phi_k(q_k) = \mu, \tag{3.16}$$

$$\mathbf{x} = [\mathbf{a}, \mathbf{c}]^{\mathrm{T}}, \tag{3.17}$$

$$\mathbf{F} = \mathbf{T} \circ (\mathbf{1}^{\mathrm{T}} \otimes \mathbf{q}), \tag{3.18}$$

$$c_j = \sum_{i=1}^{A} a_{ij} + \sum_{k=1}^{K} f_{kj}; f_{kj} = \mathbf{F}_{kj}, \tag{3.19}$$

$$\frac{1}{2} \sum_{j=1}^{C} \left| \sum_{k=1}^{K} f_{kj} \right| \leq V_u, \tag{3.20}$$

$$E_j^l \leq \sum_{i=1}^{A} a_{ij} + \sum_{k=1}^{K} f_{kj} \leq E_j^u, \tag{3.21}$$

$$l_k b_k \leq q_k \leq u_k b_k, \tag{3.22}$$

$$\sum_{k=1}^{K} b_k \leq G, \tag{3.23}$$

$$a_{mn} = M \sum_{k=1}^{K} |q_k|, \tag{3.24}$$

$$\mathbf{1}^{\mathrm{T}}\mathbf{a} = 1, \tag{3.25}$$

$$\mathbf{1}^{\mathrm{T}}\mathbf{c} = 1, \tag{3.26}$$

$$0 \leq a_{ij} \leq 1, \tag{3.27}$$

$$b_k \in \{0, 1\}. \tag{3.28}$$

## 3.4 Experimental Results and Discussions

This section provides details on data sets used in the experiments and studies the impact of how a constrained overlay influences the return and risk of portfolios. Portfolios in the tests are constructed based on the formulation given in earlier section. We set USD as the base currency of a portfolio, meaning that we try to maximise profit or minimise risk in only USD.

Portfolio investment is scoped to four major countries with four major currencies; the United States (USD), Germany (EUR), the United Kingdom (GBP) and Japan (JPY). The selection of major currencies ensure that foreign exchange forward contracts are available for all possible currency crosses. The number of asset classes are set as three, i.e., bonds and stocks to represent low and high risk investments and operating cash to cover margin requirement of forward contracts. With this setting, there are 14 decision

variables from $AC + K = (2 \times 4) + 6$ where $A = 3$ (number of asset classes), $C = 4$ (number of currencies) and $K = \binom{4}{2} = 6$ (number of all distinct forward contracts).

### 3.4.1 Data

Government bond returns are collected from MorganMarkets[9] by J.P. Morgan while stock and currency returns are retrieved from Bloomberg. Data frequency is monthly, spanning from Jan-00 to Jun-12. The interest rate of each country is proxied by its yield to maturity of 1-month treasury bill to reflect the risk-free return over a month. The yields to maturity of treasury bills are retrieved from Bloomberg.

Expected returns of assets in each country are adjusted by subtracting with corresponding expected interest rate while an expected return of each currency is added with an expected interest rate of that country. Expected returns of assets, exchange rates are calculated using average historical returns. For expected interest rates, since the study period covers the severe financial crisis in 2008 that forced central banks in large economies to keep interest rates historically low, the historical averages over Jan-00 to Jun-12 of yield to maturity are thus implausible to reflect the forward-looking expected interest rates. We therefore use average yields to maturity over Jun-11 to Jun-12 to represent recent information on expected interest rates. The estimated values are shown in Table 3.2.

**Table 3.2 Average yields to maturity (as a proxy of interest rate) of 1-year treasury bills over Jun-11 to Jun-12.** Note that the yield to maturity is generally given in terms of Annual Percentage Rate (A.P.R.), the figures shown are therefore converted to monthly rates by dividing the annual yields by 12 so that the resulting yields to maturity are comparable with other monthly returns.

| | average yield to maturity (% monthly) | | | |
|---|---|---|---|---|
| | US | Germany | UK | Japan |
| 1-month treasury bill | 0.004 | 0.012 | 0.053 | 0.008 |

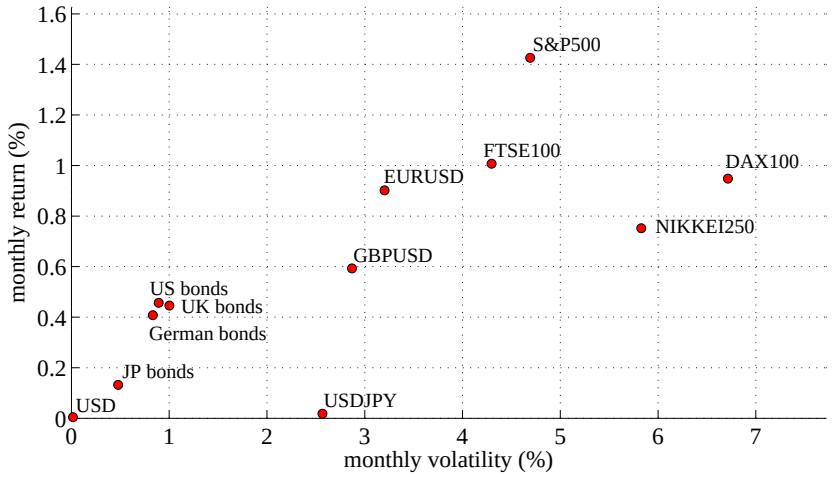
The covariance matrix is constructed from adjusted time series of asset returns and exchange rates as stated in equation (3.14). The adjusted expected returns and volatilities are shown in Table 3.3. The plot of risk (volatility) versus return of each security in the portfolio is illustrated in Figure 3.2.

[9] https://markets.jpmorgan.com/

**Table 3.3 Adjusted expected returns used in the optimisation problems.** Note that asset returns are in local currencies and exchange rate returns are measured against USD. The adjusted return of 0.004% monthly on USD exchange rate is therefore solely from the expected US interest rate presented in Table 3.2.

| | adjusted expected return (% monthly) | | | | adjusted volatility (% monthly) | | | |
|---|---|---|---|---|---|---|---|---|
| | US | Germany | UK | Japan | US | Germany | UK | Japan |
| government bond | 0.446 | 0.408 | 0.457 | 0.132 | 1.003 | 0.833 | 0.893 | 0.478 |
| stock index | 1.426 | 0.948 | 1.007 | 0.752 | 4.692 | 6.714 | 4.297 | 5.828 |
| exchange rate (against USD) | 0.004 | 0.902 | 0.593 | 0.018 | 0.017 | 3.203 | 2.870 | 2.567 |

**Figure 3.2 Adjusted expected returns and volatilities of assets and currencies.**




For the transaction costs of forward contracts, the fixed operating cost of each forward contract is arbitrarily set as 0.0001%[10] of portfolio value. The margin requirement of each forward contract is set as 10% of the position size (the margin requirement of forward contracts is generally charged around 5 to 10% on contract values, [82]). The percent spreads of six possible forward pairs are collected from Bloomberg and are shown in Table 3.4.

[10]Raising the fixed cost value affects a portfolio by diminishing attractiveness of holding forward contracts, i.e., smaller overlay positions. If the cost is so high that holding forwards provides a net loss, then there is no currency overlay on a portfolio.

Regarding the value chosen for the fixed cost; generally, small international portfolios value around USD 1 billion, which incurs a corresponding fixed cost of USD 1,000 per forward contract which is reasonable for a one-off task. Although the fund could be subsequently split out to small portfolios (by country, for example) with smaller size, a currency overlay is usually decided at an aggregate level as a global portfolio.

**Table 3.4 Average percent spreads** (ask price − bid price)/ask price, of 1-month foreign exchange forward contracts during Jan-00 to Jun-12.

| | average percent spread (% of ask price) |
|---|---|
| USDEUR | 0.0036 |
| USDGBP | 0.0051 |
| USDJPY | 0.0050 |
| EURGBP | 0.0042 |
| EURJPY | 0.0068 |
| GBPJPY | 0.0122 |

The percent spreads are calculated from (ask price − bid price)/ask price. The values shown in Table 3.4 are the averages over the period of Jan-00 to Jun-12. The percent spreads are applied on the forward positions selected by portfolios. Thus a larger number of forward pairs and larger overlay positions generate more cost and lower return to portfolios. Since buying and selling forward contracts encounter indistinguishable percent spreads, we assume that all transactions in the experiments hereafter are subject to common bid-ask spreads shown in Table 3.4.

### 3.4.2 Experimental Studies

We perform various experiments to investigate the benefit of incorporating currency overlay to international portfolios and effects of overlay constraints imposition on risk and return of portfolios. The first study exhibits a comparison of different approaches to the way that a currency overlay is implemented. The second experiment demonstrates how currency hedging affects the risk and return of portfolios. The third experiment focuses on the impact of margin requirement of foreign exchange forwards on risk-return compensation of portfolios. Then we study approaches to reduce transaction costs within overlay construction.

Generally, smaller currency overlay positions reduces transaction costs but at the same time results in deterioration of the portfolio efficiency. We therefore aim to explore if there is any situation that saving transaction costs does not much affect risk-return reward of portfolios. The fourth experiment hence inspects if different total overlay limits affects risk of portfolios similarly at each return target, then the last experiment examines further how limiting maximum numbers of forward contracts

allowed (which is one way to reduce the fixed-cost) in creating currency overlay could impact risk-return profile of portfolios.

The structure of the international portfolio is presented in Table 3.5. Note that the forward positions are defined as in equation (2.1), therefore they are expressed in terms of $q$. For each forward contract, a minus sign indicates selling and a plus sign indicates buying. For instance, the forward contract 1 (USDEUR) represents entering into a contract to sell EUR for USD at a proportion of $q_1$ of a portfolio. If the value of $q_1$ is negative, the contract is made reverse to sell USD for EUR instead. Since the portfolio is funded in USD, the operating cash is reserved only in USD and $a_{mn}$ in the constraint (3.24) is thus $a_{31}$. In consequence, the decision variables are a vector of 15 elements, i.e., $(a_{11},\ldots,a_{31},q_1,\ldots,q_6)$.

Note that, however, the calculation of return and variance of a portfolio requires a vector of asset exposure and currency exposures (the vector $\mathbf{x}$ defined in equation (3.17)). The vector of asset and currency exposures thus consists of the first three rows (asset classes 1, 2 and 3) and the last row (currency exposure) of Table 3.5.

**Table 3.5 The structure of international portfolio investing in four countries with two asset classes.** All the variables $(a_{11},\ldots,a_{31},q_1,\ldots,q_6)$ are decision variables in the portfolio optimisation problem.

| | US | Germany | UK | Japan |
|---|---|---|---|---|
| asset class 1 (bond) | $a_{11}$ | $a_{12}$ | $a_{13}$ | $a_{14}$ |
| asset class 2 (equity) | $a_{21}$ | $a_{22}$ | $a_{23}$ | $a_{24}$ |
| asset class 3 (cash) | $a_{31}$ | | | |
| forward 1 (USDEUR) | $q_1$ | $-q_1$ | | |
| forward 2 (USDGBP) | $q_2$ | | $-q_2$ | |
| forward 3 (USDJPY) | $q_3$ | | | $-q_3$ |
| forward 4 (EURGBP) | | $q_4$ | $-q_4$ | |
| forward 5 (EURJPY) | | $q_5$ | | $-q_5$ |
| forward 6 (GBPJPY) | | | $q_6$ | $-q_6$ |
| asset exposure | $a_{11}+a_{21}$ | $a_{12}+a_{22}$ | $a_{13}+a_{23}$ | $a_{14}+a_{24}$ |
| overlay position | $q_1+q_2+q_3$ | $-q_1+q_4+q_5$ | $-q_2-q_4+q_6$ | $-q_3-q_5-q_6$ |
| currency exposure | $a_{11}+a_{21}+$ $q_1+q_2+q_3$ | $a_{12}+a_{22}-$ $q_1+q_4+q_5$ | $a_{13}+a_{23}-$ $q_2-q_4+q_6$ | $a_{14}+a_{24}-$ $q_3-q_5-q_6$ |

For associated constraints, the key parameters are an overlay limit $V$, lower bound and upper bound of currency exposure $E_j^l$ and $E_j^u$, lower bound and upper bound of forward positions $l_k$ and $u_k$ and a number of forward contracts allowed $G$. The

constraint parameters in the experiments are imposed as in Table 3.6 if they are not stated otherwise. For other components of transaction costs, the bid-ask spreads are fixed at market averages as in Table 3.4 and the fixed operating cost per forward contract is set at 0.0001% of portfolio value.

**Table 3.6 Parameters of constraints associated to currency overlay in international portfolio optimisation problem.** In some experiments, some parameters are varied from these default values to study their effects on portfolios.

| Constraint | Parameter | Description |
|---|---|---|
| constraint (3.20) | $V_u = 100\%$ | Total currency exposure can deviate from total asset exposure up to 100% of a portfolio. |
| constraint (3.21) | $E_j^l = 0$ and $E_j^u = 1$ | Currency exposure of each country is always positive. |
| constraint (3.22) | $l_k = -1$ and $u_k = 1$ | Upper and lower bounds of each forward position is respectively 100% and -100% of portfolio value. |
| constraint (3.23) | $G = 6$ | A portfolio can hold up to 6 different forward contracts. |
| constraint (3.24) | $M = 10\%$ | Cash equivalent to 10% of the value of each forward position must be held in a portfolio. |

To compare risk and return of optimal portfolios with different constraints, an efficient frontier (Pareto front) is rendered accordingly. Comparison among frontiers is possible by visualisation. The frontier that is more north-western is considered more efficient as with the same level of risk, it offers higher return. An efficient frontier is constructed by setting a return target (constraint (3.16)), then minimises the optimisation problem (3.3.3) for an optimal portfolio and proceed to the next return target. We start from the monthly return level of 0.5% to the maximum of 1.8%. The return increment of each return level is set to 0.01% which results in total 130 portfolios on each efficient frontier.

All the experiments are run on PC (8GB RAM, CPU 2.10GHz) using `cplexmiqpex` package from CPLEX on MATLAB as a solver for mixed-integer quadratic programming. It is worth noting that although the optimisation model (3.3.3) is originally

convex, incorporating fixed and linear transaction costs of forward contracts eventually results in non-convex programming, [86].

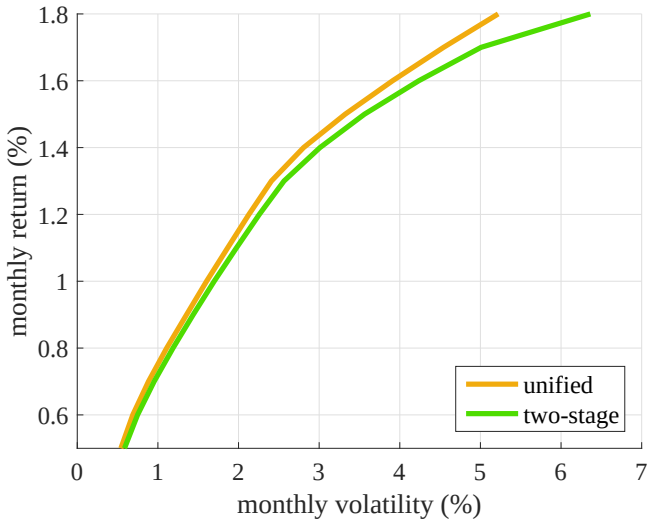
#### 3.4.2.1 Different Approaches to Implement Currency Overlay

This analysis aims to study the effects of two strategies of currency overlay implementation on risk return reward of corresponding portfolios. Since currency overlay is devised to allow currency exposure to deviate from asset exposure to achieve better risk-return compensation and more efficient portfolios, there are basically two ideas to implement currency overlay.

- The unified approach – optimises asset and overlay positions in a portfolio simultaneously for optimal asset and currency exposures.
- The two-stage approach – begins with optimising portfolio allocation with no currency overlay and make an adjustment later. At the first step, exposure in asset and currency of investment in each country will be equal. The second step is primarily optimising currency exposure to make adjustments on original currency positions. This strategy, despite being sub-optimal comparing to the unified approach, is practical for fine-tuning final currency exposure of existing portfolios. It could save significant transaction costs on hedging currency risk as all the alteration is made by entering new foreign exchange forward contracts while asset allocation remains unchanged.

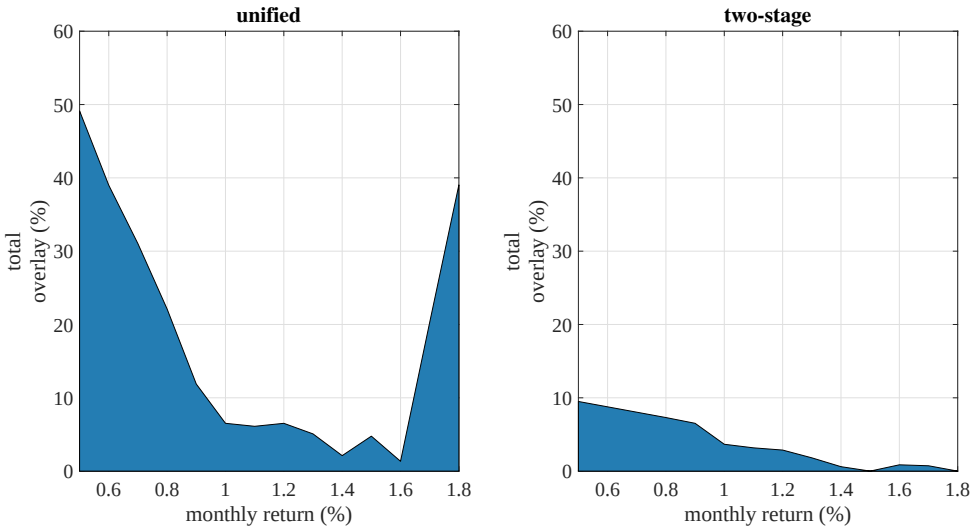
The constraint parameters of each currency overlay implementation approach are set as in Table 3.6. The resulting efficient frontiers are exhibited in Figure 3.3. As expected, the two-step optimisation generates inferior risk-return rewards for portfolios. For each portfolio with equal volatility, the differences in portfolio returns obtained from the two approaches can be narrowed if there are further adjustments made on asset allocations.

**Figure 3.3 Efficient frontiers of portfolios optimised by unified and two-stage strategies.** The unified approach optimises currency overlay and asset allocation at the same time while the two-stage approach optimises asset allocation first and then optimises the currency overlay on top of existing asset allocation to adjust the currency exposure. For currency overlay constraints, the key parameters are given as $V_u = 100\%$ (unrestricted total overlay), $G = 6$ (unrestricted number of forwards for currency overlay construction) and $M = 10\%$ (margin requirement for each forward contract is 10%).




The difference between the two frontiers points out that modification of the currency exposure alone is insufficient to significantly improve risk and return to portfolios. A larger deviation between the frontiers is spotted in higher returns region. This is because high return portfolios tend to concentrate their investments in only one or two countries giving high pay-offs (see Table 3.7). Given that those asset allocations are fixed in the first place, then there is limited possibility to adjust original currency exposure to better position. This argument is supported by Figure 3.4.

**Figure 3.4 Total currency overlay, $V$, of portfolios optimised by the unified and two-stage strategies.** For both cases, the key parameters controlling currency overlay constraints are $V_u = 100\%$ (unrestricted total overlay), $G = 6$ (unrestricted number of forwards for currency overlay construction) and $M = 10\%$ (margin requirement for each forward contract is 10%).




Currency adjustments on two-stage optimised portfolios are clearly less than those of portfolios optimised by the unified strategy. Focusing on the currency overlay of portfolios with the two-stage approach (right panel of Figure 3.4), it is observed that the overlay positions gradually decrease while portfolio returns increase.

This is because exposure of forward contracts is not allowed to be optimised along with asset allocation, thus at the first-stage the portfolios need to invest in stocks only in order to achieve high returns. Consequently, given that currencies generate lower risk and lower return compared to stocks, the portfolios need to have no additional currency exposure, as otherwise the return target could not be satisfied.

**Table 3.7 Selected optimal portfolios at low (0.5%), medium (1.2%) and high (1.8%) return targets.** The constraint parameters are set as shown in Table 3.6.

| Return | Risk | Allocation | USD | EUR | GBP | JPY | Total |
|---|---|---|---|---|---|---|---|
| | | | US | Germany | UK | Japan | |
| 0.50 | 0.56 | Bond | 33.09 | 28.15 | 7.23 | 14.10 | 82.58 |
| | | Equity | 5.10 | 1.34 | 2.67 | 3.40 | 12.51 |
| | | Cash | 4.91 | | | | 4.91 |
| | | Total asset | **43.11** | **29.49** | **9.90** | **17.50** | **100.00** |
| | | Total Currency | **92.24** | **4.46** | **3.30** | **0.00** | **100.00** |
| | | Overlay | **49.13** | **-25.03** | **-6.60** | **-17.50** | **49.13** |
| 1.20 | 2.15 | Bond | 4.98 | 23.51 | 37.45 | 0.00 | 65.95 |
| | | Equity | 23.31 | 1.04 | 2.71 | 6.34 | 33.40 |
| | | Cash | 0.65 | | | | 0.65 |
| | | Total asset | **28.94** | **24.55** | **40.17** | **6.34** | **100.00** |
| | | Total Currency | **31.29** | **28.73** | **39.98** | **0.00** | **100.00** |
| | | Overlay | **2.35** | **4.17** | **-0.18** | **-6.34** | **6.52** |
| 1.80 | 6.06 | Bond | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| | | Equity | 0.00 | 56.93 | 39.16 | 0.00 | 96.10 |
| | | Cash | 3.90 | | | | 3.90 |
| | | Total asset | **3.90** | **56.93** | **39.16** | **0.00** | **100.00** |
| | | Total Currency | **3.90** | **95.97** | **0.12** | **0.00** | **100.00** |
| | | Overlay | **0.00** | **39.04** | **-39.04** | **0.00** | **39.04** |

#### 3.4.2.2 Effects of Currency Hedging on Risk and Return of Portfolios

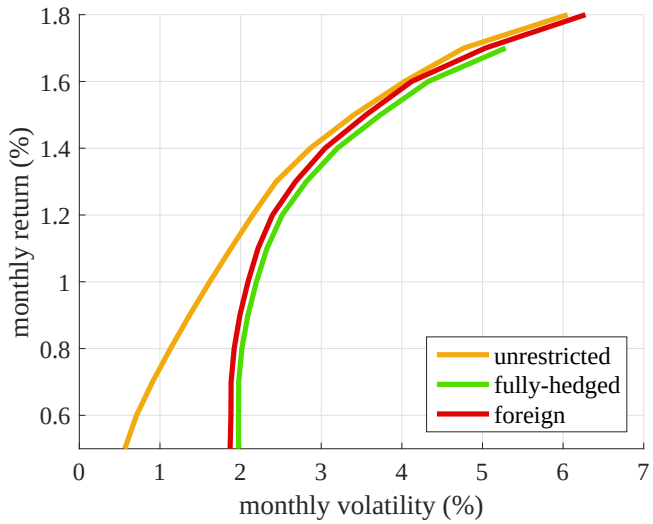
The aim of this experiment is to emphasise the benefit of currency hedging on risk-return profile of portfolios. As described earlier, currency overlay is introduced to enable a portfolio to manage asset and currency exposures separately. In one extreme, a portfolio can be fully-hedged to have only USD exposure or, in another extreme, the portfolio is made fully-exposed to only foreign currencies. A balance between these two provides a mix of exposures on local and foreign currencies which is expected to deliver the best risk-return profile to portfolios.

We compare the results from three hedging policies. The first one is the "fully-hedged" policy where foreign exchange forwards are employed to remove entire foreign currency exposures (EUR, GBP and JPY) and remain only the exposure in USD (base currency of the portfolio). Referring to Table 3.5, this is achieved by

enforcing $a_{11}+a_{21}+q_1+q_2+q_3=1$. The second policy is the "foreign" which uses forwards to remove USD exposure to foreign currencies, in this case the sum of foreign currency exposure is 100% while USD exposure is 0%, i.e., $a_{11}+a_{21}+q_1+q_2+q_3=0$. The last hedging policy is the "unrestricted" which makes no restriction on specific currency exposure. Note that, in all the hedging policies presented, no constraint is imposed on asset exposure, so the portfolio can hold assets in Germany, UK and Japan while exposing to only USD exposure in the fully-hedged case. Other than the restriction on currency exposure, other constraints are as given in Table 3.6.

The resulting efficient frontiers from the experiment are displayed in Figure 3.5. It can be seen that portfolios under the "fully-hedged" and "foreign" policies are clearly dominated by those from the unrestricted currency exposure policy, particularly at returns under 1.4% monthly.

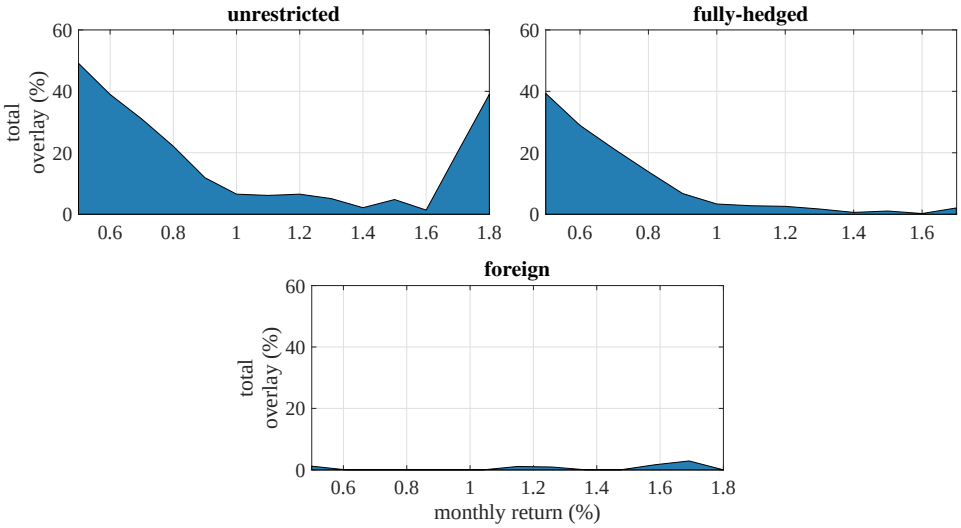
**Figure 3.5 Efficient frontiers from different hedging policies.** The "fully-hedged" restricts portfolios to have only currency exposure in USD. The "foreign" allows portfolios to expose only to foreign currencies. The "unrestricted" permits portfolios to expose to any currency. For portfolios under the "fully-hedged" policy, the highest return achievable is around 1.7% monthly because foreign exchange gain from foreign currencies is not allowed. The efficient frontiers produced are subject to specific constraint according to each hedging policy. Apart from that, other constraints parameters are $V_u = 100\%$ (unrestricted total overlay), $G = 6$ (unrestricted number of forwards for currency overlay construction) and $M = 10\%$ (margin requirement for forwards is 10%).




Referring to the adjusted expected returns in Table 3.3, foreign currencies supply higher return than the local currency (USD), hence limiting portfolios to only expose to USD blocks the opportunity to reach high returns (in particular, approximately over 1.7% monthly). The advantage of holding foreign currencies exposure also contributes to more efficient portfolios comparing to those holding only exposure in USD. However, the absence of USD means that portfolios are obligated to take more

risk unnecessarily in some situations. For instance, at low returns where portfolios are supposed to hold more USD to preserve low volatility, disallowing USD exposure dramatically elevates higher volatilities at the lower end of the only-foreign-currencies efficient frontier.

**Figure 3.6 Total currency overlay, $V$, at each return level.** Each panel represents each hedging policy. The total currency overlays in the "foreign" case are sparse as portfolios invest only in foreign countries. The fully-hedged portfolios require some currency overlay at low returns to remove foreign currency exposures from asset investments in German, UK or Japan while at higher returns, the portfolios tend to invest more in US equities which hence require only slight currency overlay.




It is also observed that the efficient frontiers lie closer when portfolios approach high return targets (above 1.4%). Generally, at high returns, portfolios tend to hold risky assets in one or two countries (as diversification is second to return). For the fully-hedged policy, high returns could be achieved by holding a large portion of US equities and expose to only USD, while for the "foreign" case, portfolios invest in German and UK stocks and expose to only corresponding currencies. Such actions result in comparable returns and volatilities and are less inclined to currency overlay as portfolios expose to only currencies of the countries invested as shown in Figure 3.6. For the unrestricted case, freedom of choosing currency exposure along with the currency overlay helps reduce volatilities and thus its resulting portfolios are the most efficient among all hedging policies.

#### 3.4.2.3 Effects of Forward Contract Margin Requirements

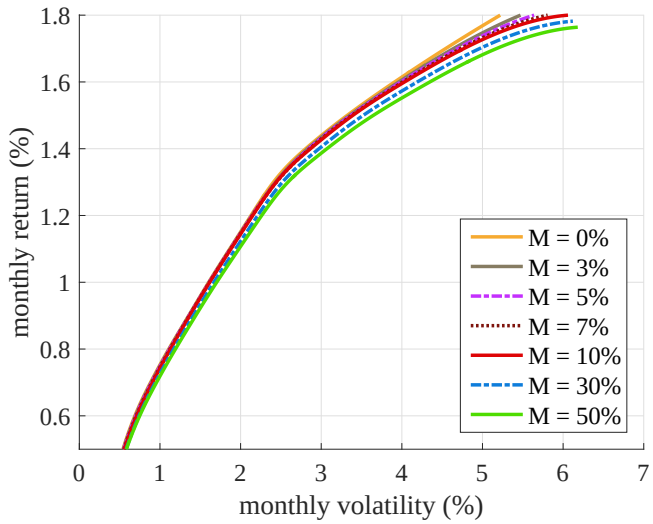
In the study, the transaction costs associated to holding forward contracts are the fixed costs, bid-ask spreads and margin requirement. The fixed costs depend on the number

of forward contract pairs and the variable costs depend on bid-ask spreads which are associated to both size and number of forwards. Margin requirement is also considered a transaction cost as it requires portfolio to reserve more cash (with zero return) to cover forward positions.

In general, higher transaction cost is expected to shift efficient frontiers south-eastern and lower risk-return compensation of portfolios. However, different return levels require different additional currency exposure from foreign exchange forwards. Thus at some return levels, increasing transaction costs might not worsen the risk-return reward as much as others. This study aims to see which parts of the efficient frontier are most and least affected by the variation of margin requirements in particular.

We vary $M$ arbitrarily by 0%, 3%, 5%, 7%, 10%, 30% and 50%. Other constraint parameters are set as in Table 3.6. The efficient frontiers of optimal portfolios under different margin requirements exhibited in Figure 3.7 show that higher margins results in reduction of return per risk particularly when portfolio returns are high (greater than 1% monthly approximately).

**Figure 3.7 Efficient frontiers at different levels of margin requirement.** $M$ determines how much cash is needed to be reserved for every 1% of forward position on a portfolio. $M = 0\%$ indicates that forward positions can be held without cash reserves while $M = 5\%$ means that cash equivalent to 5% of portfolio is allocated to cash in order to cover a forward position of 1% on a portfolio. Since the experiment particularly studies impact of different $M$, major overlay constraint parameters remain constant, i.e., $V_u = 100\%$ (unrestricted total overlay) and $G = 6$ (unrestricted number of forwards for currency overlay construction).




To highlight the effect of margin requirement levels, we plot the percentage increases of volatilities from the base case ($M = 0\%$) at each return level in Figure 3.8. The graph shows that a portfolio is more risky when margin requirement is higher since

the advantage of holding forward contracts is offset by cash portion in the portfolio. This also implies that total currency overlay should be lower when margin requirement increases.

**Figure 3.8 Percentage volatility increase relative to volatilities of the portfolio with no margin requirement ($M = 0\%$).** At each $M$, other key overlay constraint parameters are $V_u = 100\%$ (unrestricted total overlay) and $G = 6$ (unrestricted number of forwards for currency overlay construction).

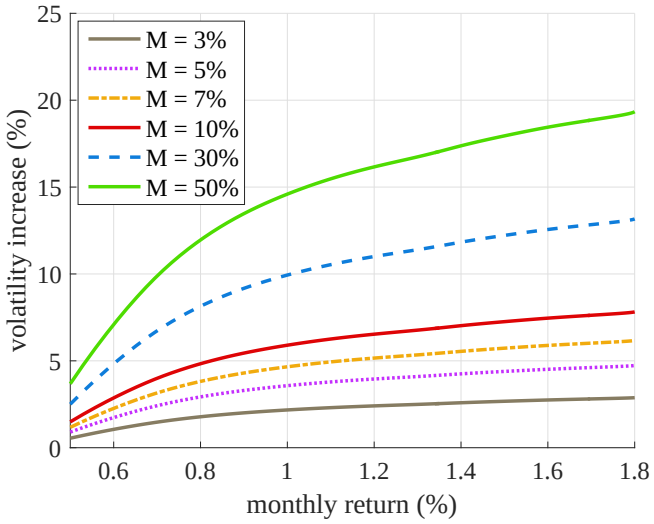
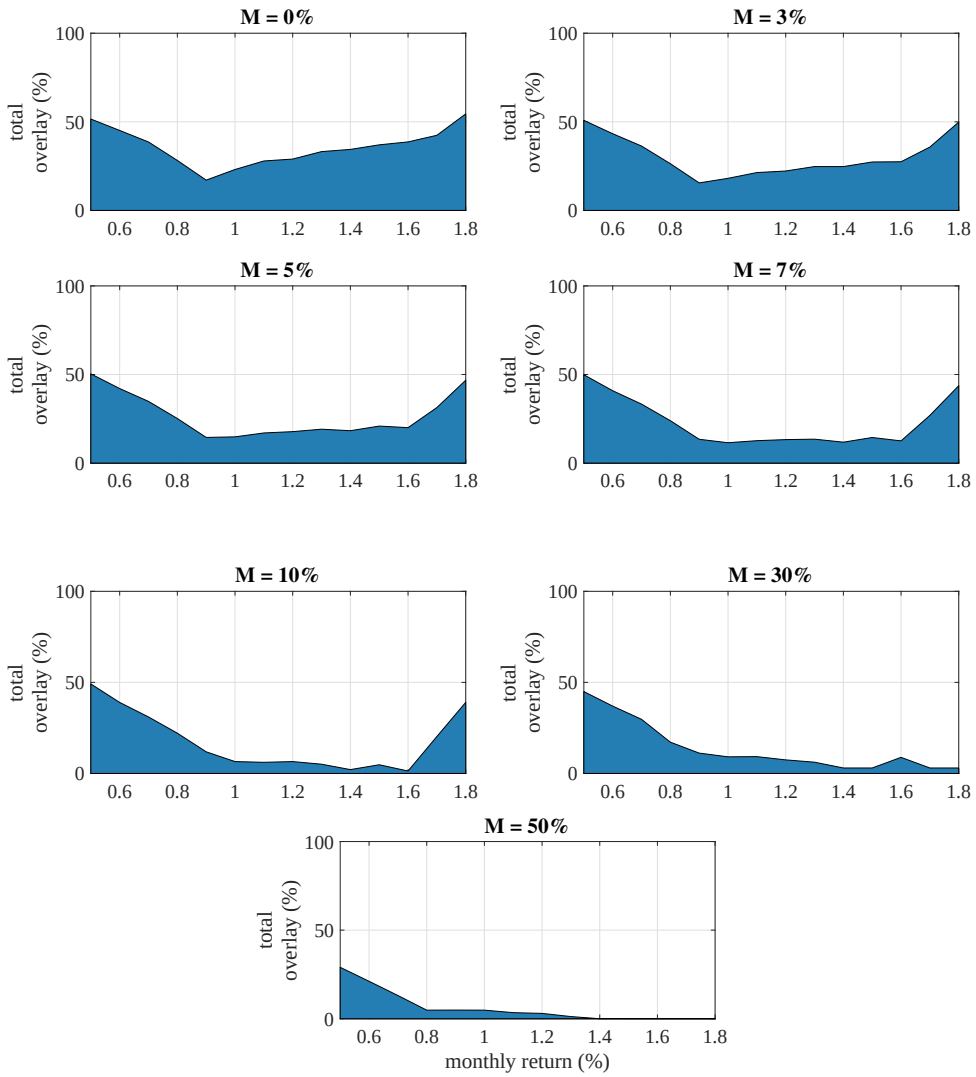



Figure 3.9 plots total currency overlays of portfolios at each level of margin requirement. When there is no margin requirement ($M = 0\%$), no cash is needed in a portfolio, the efficient frontier under this condition is hence the most effective comparing to the others.

Basically, holding forward contracts generates extra exposure on desired currencies without any purchase of physical assets. Given that currencies have comparable return and have lower volatilities than equities (see Figure 3.2), forwards are more favourable than stocks to raise portfolio returns. However, the benefit of competitive returns of forward contracts are offset with zero return on cash when margin requirement comes into effect. When the maintenance margin of holding forward contracts grows larger, the attractiveness of forwards gradually diminishes as portfolios need to reserve more cash (with zero return) for forward positions. Higher margin requirements simply reduce return on holding forward contracts and leave them only beneficial for risk reduction. Therefore, when portfolios require higher return, they tend to give up forward contracts and hold risky assets instead.

It can be concluded that margin requirement of forward contracts affects portfolios most at the high return regions as the losing benefit of forward contracts causes

**Figure 3.9 Total currency overlay, $V$, at each return level.** Each panel represents different margin requirement indicated by values of $M$ over each panel. At each panel, other overlay constraint parameters are kept constant; $V_u = 100\%$ (unrestricted total overlay) and $G = 6$ (unrestricted number of forwards for currency overlay construction). At high returns, portfolios hold forwards in place of equities as they provide comparable returns with lower volatilities. However, such benefit of forwards diminishes when $M$ is large as returns from holding forwards are offset by large cash position which provides zero return. A large drop of total currency overlay at high returns is thus observed when $M$ increases.



portfolios to hold more risky assets to earn higher returns. In contrast, at lower return targets where risk reduction is a top priority, forwards are still in use as part of volatility reduction making low returns least affected.

In following experiments, we try different approaches to reduce transaction costs occur from holding foreign exchange forwards. The first approach limits total currency overlay and another approach limits the number of forward contracts to construct currency overlay.

#### 3.4.2.4 Limiting Total Currency Overlay

The total currency overlay limit $V_u$ corresponds to the constraint (3.20) which limits how much total currency exposure can be deviated from asset exposure by holding foreign exchange forward contracts. If $V_u$ is set as 0%, currency exposure and asset exposure in each country are identical while $V_u = 100\%$ implies full flexibility of deviation in asset and currency exposures thereby the best option to improve risk-return reward to portfolios.

Confining total currency overlay limits forward exposure on portfolios which reduces transaction costs from bid-ask spreads and margin requirement. Since portfolios adjust currency exposure differently at each return level, the impact of limiting $V_u$ could also be dissimilar at different return targets.

In the experiment we compare the optimal portfolios from five arbitrary values of $V_u$ which are 0%, 10%, 30%, 50% and 100%. Other constraints are applied as in Table 3.6. Figure 3.10 exhibits efficient frontiers obtained from optimisation. The one produced by setting $V_u = 0\%$ demonstrates lowest risk-return compensation while the others generated from higher values of $V_u$ show improvement on the risk-return trade-off. It is observed that in all settings of $V_u$, total currency overlays $V$ on the portfolios never exceed 50% (see Figure 3.12), hence optimal portfolios obtained from $V_u = 50\%$ and $V_u = 100\%$ are identical and their efficient frontiers coincide.

Figure 3.11 shows percentage volatility increase relative to the case of $V_u = 100\%$ at each return level. It appears that at low returns, portfolio volatilities increase most when total currency overlays are bounded and this effect is lessened when portfolio return increases. The least affected regions are when returns are approximately between 1-1.6% monthly where portfolios need only modest adjustments on currency exposure to achieve optimal portfolios.

**Figure 3.10 Efficient frontiers of optimal portfolios at different restriction of total currency overlay $V_u$.** Larger $V_u$ allows flexibility for currency exposure to deviate from asset exposure, implying larger positions of forwards and higher transaction costs. Restricting $V_u$ hence reduces transaction costs associated to forwards at the expense of less efficient portfolios. Each efficient frontier is generated with different values of $V_u$ while other overlay constraint parameters remain constant; $G = 6$ (unrestricted number of forwards for currency overlay construction) and $M = 10\%$ (margin requirement for forwards is 10%). The frontiers when $V_u = 50\%$ and $V_u = 100\%$ coincide as total currency overlays never exceed 50%.

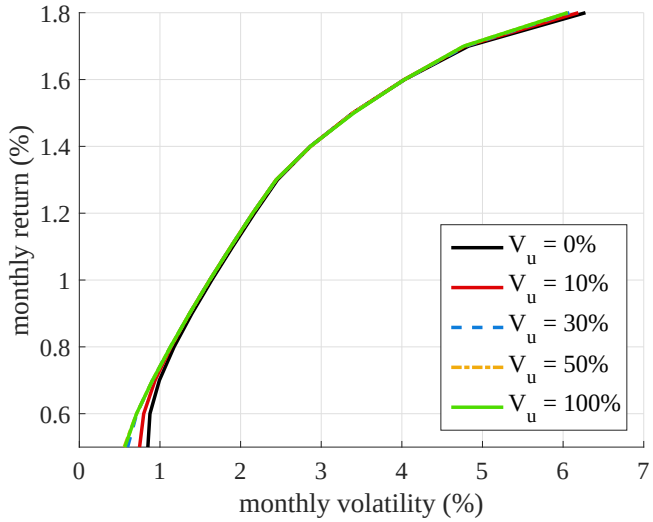



Figure 3.12 illustrates the total overlay at each return level of a portfolio. Since $V_u = 0\%$ indicates currency exposure and asset exposure are identical, there is no overlay position presented. When $V_u = 10\%$ and $V_u = 30\%$, the constraint (3.20) is binding suggesting that the risk-return profile can be further improved if the total overlay limit is loosened and when $V_u = 50\%$ and $V_u = 100\%$, total currency overlay usages are identical as portfolios never require total currency exposure adjustment over 50% of portfolio values.

Generally, a portfolio gains higher return by reducing allocation of low-risk investments for riskier investments. Figure 3.13 demonstrates aggregate holdings in bonds and equities across countries at different values of $V_u$. It shows that bond allocations gradually decrease along with increasing returns. However, the decrease in bond allocation is slower when $V_u$ is more relaxed. This signifies that additional currency exposure from overlay positions contributes the holding of risky investments in tandem with equities.

**Figure 3.11 Percentage volatility increase relative to volatility of portfolios with no restriction on currency overlay ($V_u = 100\%$).** For each $V_u$, portfolios are optimised under currency overlay constraints with following parameters, $G = 6$ (unrestricted number of forwards for currency overlay construction) and $M = 10\%$ (margin requirement for each forward contract is 10%). Since $V_u = 50\%$ and $V_u = 100\%$ produce identical optimal portfolios, the relative volatility increase for $V_u = 50\%$ is 0.

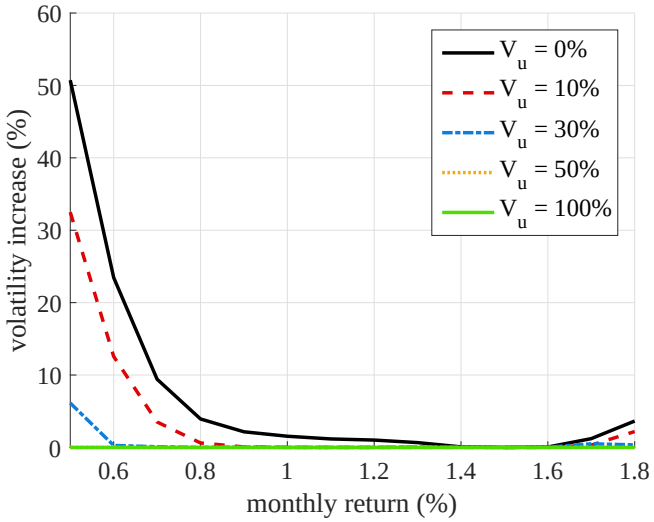
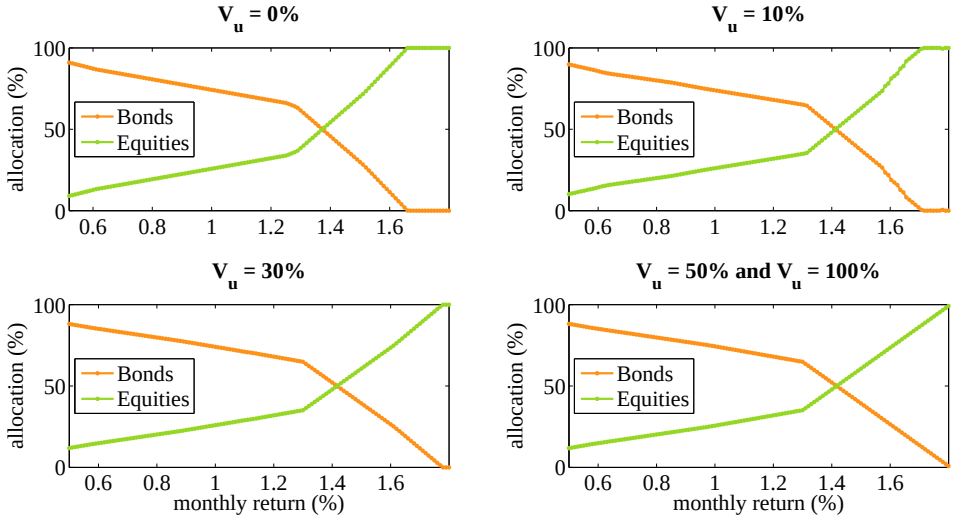
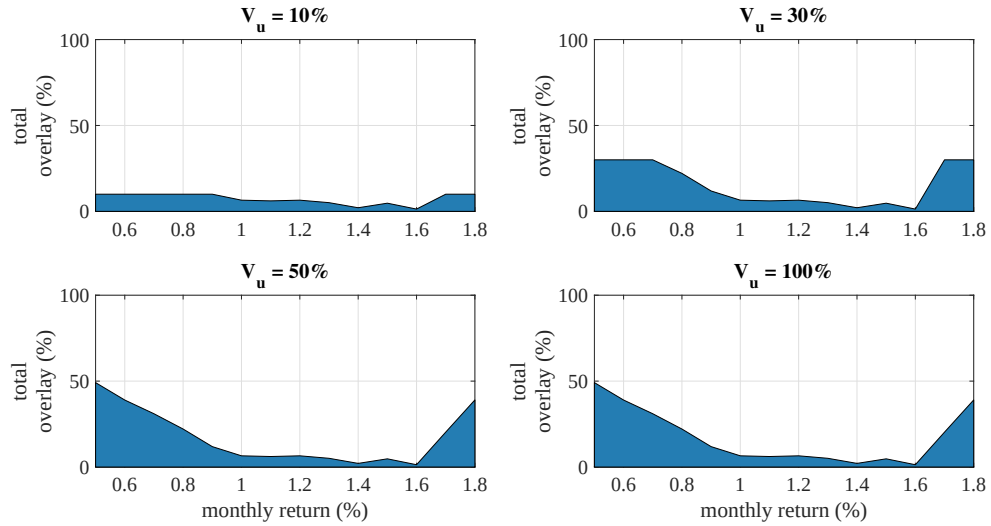



**Figure 3.13 Composition of total bond and total equity holdings at at each return level.** Each panel represents different total currency overlay limit $V_u$. At each panel, apart from the varied $V_u$, other constraint parameters are fixed as $G = 6$ (unrestricted number of forwards for currency overlay construction) and $M = 10\%$ (margin requirement for each forward contract is 10%).




The plot of volatilities and returns in Figure 3.2 shows that EURUSD gives comparable return to equities while having significantly lower volatility. Holding substantial portion of EUR-cross forward contracts along with equities assists risk reduction of

**Figure 3.12 Total currency overlay, $V$, at each return level.** Each panel represents different total overlay limit indicated by values of $V_u$ over each panel. Other currency overlay constraint parameters are $G = 6$ (unrestricted number of forwards for currency overlay construction) and $M = 10\%$ (margin requirement for each forward contract is 10%). The plot of $V_u = 0\%$ is not included in the figure as it presents no currency overlay.



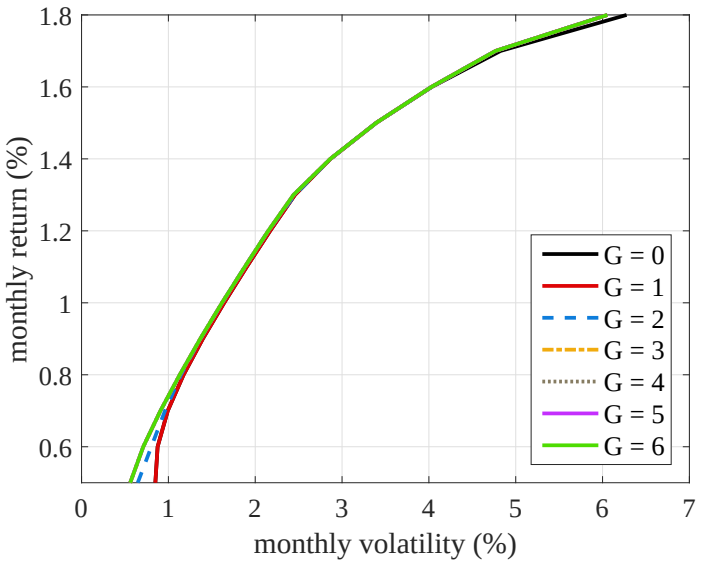

portfolio rather than investing in stocks alone and results in better risk-return trade-off for portfolios.

#### 3.4.2.5 Limiting The Number of Forward Contracts

For an investment in four currencies, currency overlay can be constructed from up to six different foreign exchange forwards which are shown in Table 3.5. In this experiment, referring to the constraint (3.23), we vary values of $G$ from 1 to 6 to limit the number of forward contracts spent in overlay construction. Fewer forward contracts cuts down the fixed operating costs along with variable costs from bid-ask spreads and margin requirement. This is opposed to limiting the total currency overlay in which transaction costs are lessened primarily from reduced exposure while the fixed costs might remain the same.

In terms of computation, values of $G$ between 1 and 5 create a cardinality constraint on the selection of forward pairs constituting currency overlay. For example, $G = 3$ forces a portfolio to choose only three or fewer forwards from Table 3.5 that can minimise variance of the portfolio at a given return level. The other constraint parameters are applied as in Table 3.6.

**Figure 3.14 Efficient frontiers produced under different limits on the number of forward contracts allowed.** $G$ is the maximum number of forward contracts allowed to construct the currency overlay. Fewer number of contracts reduce the fixed operating cost but also impair risk-return profile of portfolios. For the case of four currencies, $G = 0$ implies the currency overlay is not permitted while $G = 6$ implies all possible forward contracts can be used to create the currency overlay. At each $G$, parameters of other currency overlay constraints are $V_u = 100\%$ (unrestricted total overlay) and $M = 10\%$ (margin requirement for forwards is 10%).

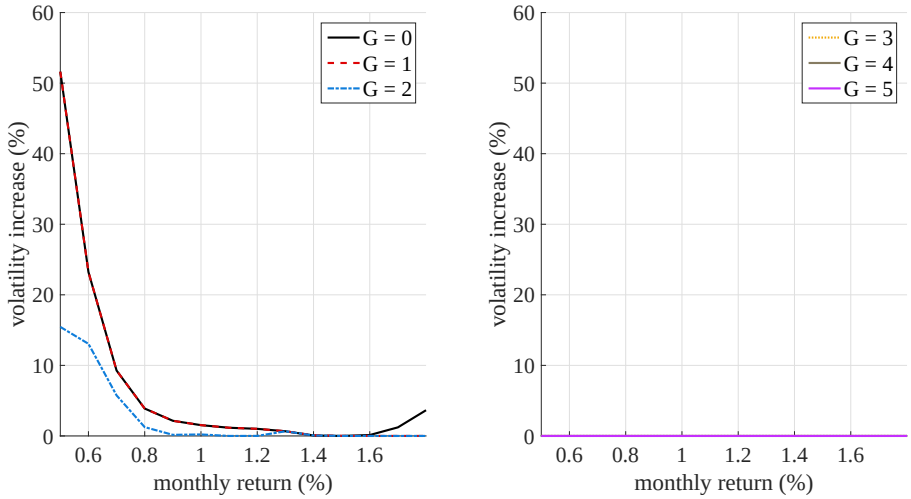



In Figure 3.14, the significant difference of efficient frontiers is visible when increasing $G$ from 0 to 1 and 2 in which the efficient frontiers shift north-west when $G$ increases. Figure 3.15 displays percentage volatility increase relative to the most relaxed case ($G = 6$). It is obvious that more freedom to choose forward contracts to construct currency overlay produces better return per risk ratio although the increment of improvement is less noticeable when $G \geq 3$ than when $G$ increases from 0 to 2. In addition, returns lower than 1% monthly experience larger volatility increase than higher returns as flexibility to modify currency exposure is diminished with limited number of available forward contracts.

When stepping from $G = 0$ to $G = 1$, the risk-return reward is improved as portfolios are allowed to hold forward contracts rather than stocks thereby lowering total volatility. Stepping from $G = 1$ to $G = 2$ gives a substantial progress as $G = 1$ permits an adjustment of currency exposure on only two out of four currencies while $G = 2$ can cover all four currencies (with USDEUR and GBPJPY) but with limited flexibility[11] to adjust exposure of each currency. $G = 3$ makes further flexibility to modify currency

[11] With two forwards, only two pairs of currency exposure can be modified. For example, with USDEUR and GBPJPY, exposures on EUR and GBP depend on USD and JPY, respectively and cannot be altered separately until more number of forward contracts are allowed.

**Figure 3.15 Percentage volatility increase relative to volatilities of the portfolio that allows up to 6 forward contracts to construct currency overlay ($G = 6$).** At each $G$, parameters of other currency overlay constraints are set as $V_u = 100\%$ (unrestricted total overlay) and $M = 10\%$ (margin requirement for forwards is 10%). For the case of 4 currencies, only 3 forward contracts are sufficient to create all overlay positions. Imposing $G \geq 3$ (right panel) therefore produces similar portfolios and their increase in relative volatility from the case of $G = 6$ is zero.




exposure as some patterns of currency overlays cannot be produced by two forward contracts. For instance, the following overlay is impossible to be replicated by two forward contracts but possible with three or more.

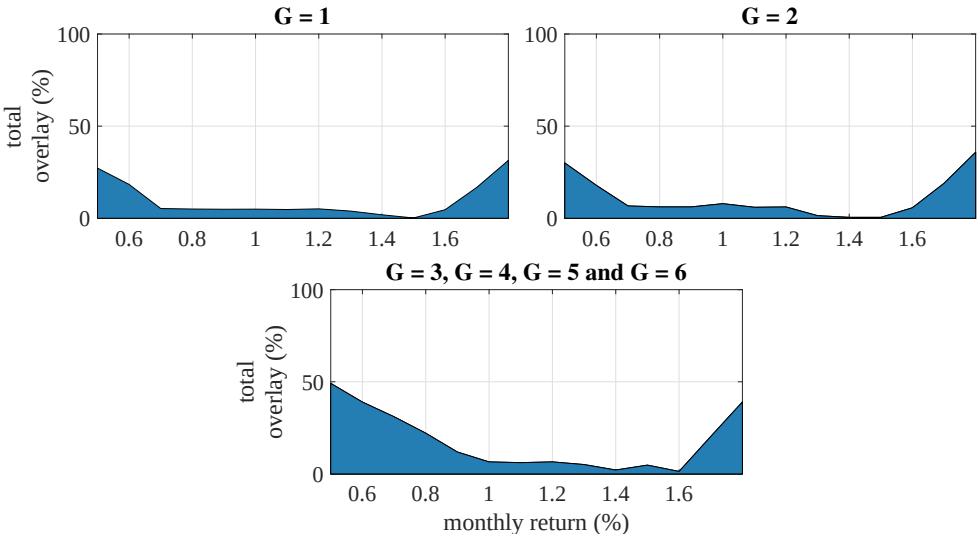
**Table 3.8 Example of currency overlay which needs three or more forward contracts to produce.**

| | USD | EUR | GBP | JPY |
|---|---|---|---|---|
| overlay (%) | 1 | 5 | -2 | -4 |

Thus shifting from $G = 2$ to $G = 3$ or more significantly provides improvement to risk-return reward of portfolios. In fact, for $C$ currencies, there needs $C - 1$ different forward pairs to cover all possible currency overlays. Therefore allowing $G \geq 3$ generates the same optimal portfolios.

Figure 3.16 shows total currency overlay at each return target. Comparing the top panel ($G = 1$ and $G = 2$) and bottom panel ($G \geq 3$), we notice a large drop of total currency overlay when returns are less than 1% monthly while only a slight decrease in total overlay position is observed at higher returns. These results coincide with relative volatility increase in Figure 3.15. It can be explained that at high returns, currency overlays are generally composed of one or two forward contracts associated to EUR or GBP to raise target returns. On the other hand, at low returns and low risks, more

**Figure 3.16 Total currency overlay at each return level.** Each panel represents maximum number of forward contracts used in currency overlay. At each $G$, other currency overlay constraint parameters are $V_u = 100\%$ (unrestricted total overlay) and $M = 10\%$ (margin requirement for forwards is 10%). The plot of $G = 0$ is not included in the figure as there is no forward contract allowed and hence no currency overlay. For the case of four currencies, three forward contracts are sufficient to replicate all possible currency overlays, the resulting total overlays for $G \geq 3$ are therefore similar.




forward contracts are required to reduce volatility. For example, in order to hedge risk from all foreign currencies, the pairs USDEUR, USDGBP and USDJPY are needed to modify currency exposure to solely USD. Thus saving transaction cost by restricting the number of forward contracts affects portfolios most when target returns are low (1% monthly).

## 3.5 Conclusions

A currency overlay is created to modify currency exposure of an international portfolio. In this study, a currency overlay is formed by holding foreign exchange forward contracts. A novel portfolio optimisation model that incorporates a currency overlay to allow flexibility in investing asset and exchange rate in each country is introduced in this chapter. The key feature is that overlay positions are designed to be structured by different forward contract pairs, which allows an inclusion of different transaction costs on each forward pair. Also, the study presents the approach to incorporate the cost of carry of forward contracts into the process of return and risk calculation of the portfolios which reflects more accurate gain and loss of forward contracts held by portfolios and helps complete calculating total return and volatility.

This chapter conducts four experiments to investigate effects on risk-return compensation of portfolios when different types of currency overlay constraints are imposed. The first experiment (3.4.2.1 Different Approaches to Implement Currency Overlay) studies the two different strategies in implementing a currency overlay, the unified and the two-stage approaches. The results support that allowing a portfolio to optimise currency overlay and asset allocation simultaneously yields better risk-return reward than optimising asset allocation first and adjusting the currency exposure afterwards.

The second experiment (3.4.2.2 Effects of Currency Hedging on Risk and Return of Portfolios) shows that a fully-hedged portfolio (only USD exposure) or a portfolio that exposes to only foreign currencies are not as efficient as a portfolio whose currency exposure is from local and foreign currencies combined.

The third experiment (3.4.2.3 Effects of Forward Contract Margin Requirements) concludes that large margin requirement diminishes benefit of holding currency overlay and induces a portfolio to hold only risky assets and ignore forward contracts in order to achieve high return (above 1% monthly approximately).

Subsequently, the chapter focuses on imposing constraints to reduce transaction costs occurred from holding currency overlay. The fourth experiment (3.4.2.4 Limiting Total Currency Overlay) enforces the limits of total currency overlay which directly reduces costs from bid-ask spread and opportunity lost from margin requirement. The results show that the higher total overlay limit is allowed, the better the risk-return compensation on portfolios. In addition, the findings show that a portfolio utilises more currency overlay when its expected return target is lower than 1%. This low return region therefore experiences significantly high relative-volatility increase comparing to the case that expected returns are higher.

The last experiment (3.4.2.5 Limiting The Number of Forward Contracts) limits the maximum number of forward contracts used in constructing a currency overlay with the purpose to reduce the fixed operating cost per forward contract. The experiment results show that for a portfolio investing in four currencies, three different forward contract pairs are sufficient to replicate the benefits of all currency overlay positions. For the impact on risk-return trade-off, the results are similar to the case of limiting total currency overlay. At returns lower than 1%, a portfolio shows the highest relative-volatility increase as it tends to hold more forward contracts to hedge currency risk at low return; while at higher returns, only a few forward pairs that deliver high returns are required to achieve high expected return targets.

In all the experiments, it is noticed that the total currency overlay position is differed by expected return target. That is, a currency overlay is used more when a portfolio seeks very low or very high expected returns (see, for example, Figures 3.9 and 3.12). This indicates that an international portfolio adjusts its currency exposure differently at each return target. Generally, currency overlay positions are large at both ends of efficient frontiers where returns are either low or high, while smaller overlay positions are observed at moderate return targets.

At low-return level, a portfolio requires different forward contracts to hedge foreign currencies to keep a low-risk-low-return profile. On the other hand, when a portfolio requires higher return, forwards are held instead of equities as they provide competitive returns with lower volatilities. Such advantage of forwards, however, is fading away if there requires a margin requirement for sustaining forward positions. This also suggests that counterparties that offer lower margin requirement are more favourable as long as their credibility is not a concern (otherwise credit risk of counterparties becomes an issue).

Lastly, all the experiment results indicate that reducing forwards transaction costs through constraints imposition affects portfolios differently as overlay positions are varied by return targets. The middle-range returns (where a small currency exposure adjustment is required) on an efficient frontier are therefore the least affected area from imposing tightening constraints.

# Chapter 4

# Two-Stage Stochastic International Portfolio Optimisation under Regular-Vine-Copula-Based Scenarios

## 4.1 Introduction

This chapter presents an improvement of the deterministic mean-variance international portfolio optimisation model given in Chapter 3. A deterministic optimisation model has a limitation that any decisions are made on a single fixed possibility, which is too rigid for volatile financial markets. Accordingly, the previous optimisation model requires an extension to take into account uncertainty of asset and currency returns.

Stochastic programming is an alternative that is generally employed to cope with uncertainty [18] and [61]. Rather than representing input as a point (single) estimation, many possible estimations (scenarios) are represented. Typically, an empirical probability distribution is employed to approximate a list of realisations or scenarios and corresponding probability [66]. The process to transform the given distributions to scenarios is known as scenario generation. A discussion of modelling and applications is well presented in Wallace and Ziemba [138].

Since the quality of optimisation solution is largely determined by the quality of scenarios, this thesis introduces a new scenario generation method that can alleviate

limitations in conventional methods. Hence this chapter mainly focuses on developing a state-of-the-art scenario generation approach and formulating a corresponding stochastic international portfolio optimisation model.

In general, scenario generation methods parameterise marginal distributions by the first four statistical moments[1]. The relationship among marginal distributions is generally described by a correlation matrix (see Høyland et al. [54] and Kaut et al. [64] for examples). Although it may suffice to capture the shape of marginal distributions through the first four moments, there exists limitation in describing the relationship among distributions by correlation. Indeed, correlation measures linear dependence between two random variables and hence cannot capture non-linear dependence. Also, the correlation is sensitive[2] to non-normal distribution [74]. Nonetheless, return distributions are conventionally assumed normally-distributed when using correlation for the ease of implementation.

In reality, returns of most financial securities are non-Gaussian; and also they present asymmetric dependence (see, for instance, Erb et al. [43], Longin and Solnik [87], Ang and Bekaert [5], Ang and Chen [6], Campbell et al. [23], Mitchell and Pulvino [98] and [110]) in which returns are more strongly correlated in bear markets than in flat and bull markets. Thus scenarios generated under normality and linear dependence assumptions do not reflect realistic events and substantially affect the quality of optimisation solutions. In particular, downturns can be more frequent and extreme than given by normal distributions. In consequence, a tool that is able to capture asymmetric dependence among return distributions is needed to model realistic scenarios.

Copulas have gained popularity for generating scenarios in recent years, for example, in the works of Kaut [63], Kaut and Wallace [66] and Sutiene and Pranevicius [133]. Sklar [127] describes a copula as a function that links a multidimensional distribution to its one-dimensional margins. The mathematical formulations of copulas are given in Sklar [126] and Nelsen [104]. The major advantage of copulas is that they allow a separation of marginal distributions and their dependence structure and hence these components can be modelled independently. The feature offers flexibility to combine marginal distributions from different families to copulas, unlike a multivariate normal distribution where all the marginal distributions are assumed Gaussian with linear

[1]The four moments, from the first to the fourth, are: mean, variance (or standard deviation), skewness and kurtosis, respectively. See Larson [79] for more details.

[2]The correlation coefficient (or Pearson's correlation) tends to give a misleading measurement when the measured data contain outliers or highly-skewed. In such cases, an alternative treatment is to employ a rank correlation such as Kendall's tau (see Section 4.2.3.2 for its descriptions).

dependence. In consequence, the resulting scenarios generated take into account non-normality such as fat tail and asymmetric dependence. Such improvements help avoid risk underestimation in standard methods such as generating scenarios from multivariate normal distributions.

### 4.1.1 Current Advancements in The Literature

In most of the literature employing stochastic programming for portfolio optimisation, scenarios are generally created from multivariate normal distributions. Since copulas are new for portfolio construction literature, to date, there are only a few papers (about portfolio optimisation) that applies copula-based scenarios in portfolio optimisation. For instance, Kaut [63] and Kaut and Wallace [66] use empirical copulas whilst Sutiene and Pranevicius [133] employ Gaussian and Student's t multivariate copulas in generating scenarios.

However, in modelling multivariate distribution for multiple asset return scenarios, there arise some limitations for the use of aforementioned methods. The drawback of empirical copulas is that the solutions could be unreliable when they are estimated from small samples, while applying multivariate copulas implies that a dependence structure of multiple assets must be described by a common copula family which lacks flexibility in high-dimensional modelling. To deal with such deficiencies, vine copulas were introduced to model a dependence structure of multiple variables [14]. In essence, a vine copula is a collection of bivariate copulas which can be represented by a nested set of trees that fulfils some certain conditions. Vine copulas benefit from being able to use a wide range of bivariate copula families. The dependence structure of random variables is then estimated pair-by-pair which is more flexible than being characterised by single copula family. The resulting scenarios generated with vine copulas should therefore better model the dependencies.

In this thesis, a new scenario generation method which uses a regular-vine copula is introduced. Then a stochastic portfolio optimisation is performed based on such scenarios. The developments (a new scenario generation method and a stochastic international portfolio optimisation model) have not done elsewhere before, hence they are a significant contribution.

### 4.1.2 Chapter Overview

This chapter begins with creating scenarios with a new technique and then apply them in a stochastic portfolio optimisation problem. In generating scenarios, asset and currency returns are modelled with empirical distributions (hence no need for normality assumption) and, instead of using a correlation coefficient, the relationship among asset and currency returns is modelled with a regular-vine copula.

Subsequently, a two-stage stochastic international portfolio optimisation problem is formulated using conditional value-at-risk (CVaR)[3] as a risk measure. The formulation of the optimisation model is novel and it is a contribution of this thesis. A currency overlay constructed with foreign exchange forwards is incorporated to allow currency exposure adjustments on the portfolio. The constraints (Section 2.1.2.1 of Chapter 2) associated to currency overlay and transactions are included into the optimisation model. Costs related to exchange rates hedging that affect risk and return of the portfolio are also taken into account (Section 3.3.1 of Chapter 3). Effects of the new scenario generation method and related constraints on risk-return profile of a portfolio are examined to investigate if the new approach produces portfolios that are more resilient to extreme events than a standard approach.

Specifically, portfolios are optimised with two types of scenarios. One from regular-vine copula based scenarios as outlined earlier, hereafter referred to as an "RVC portfolio". Another is from a conventional method assuming that asset returns are normally distributed and relationship between asset returns is described by a correlation coefficient; generally, the sampling is performed on multivariate normal distribution; portfolios optimised with these scenarios are henceforth referred to as an "MVN portfolio". Experiments are conducted accordingly to evaluate portfolio performances under two types of scenarios.

The rest of this paper is organised as follow: Section 4.2 gives details on the background of copulas, regular-vine copulas and CVaR. Section 4.3 demonstrates an approach utilising a regular-vine copula to generate scenarios and a formulation of the optimisation model. Section 4.4 describes an algorithm used in solving the optimisation problem. Section 4.5 exhibits experiment results with discussions, and section 4.6 provides a conclusion of this chapter.

[3] See Section 4.2.9 for more details.

## 4.2 Background

This section aims to provide necessary existing background in formulating a two-stage international portfolio optimisation under regular-vine-copula-based scenarios. Section 4.2.1 shows the formulation of a two-stage stochastic programming. Section 4.2.2 gives definition of copulas and its role in modelling multivariate distributions. Section 4.2.4 summarises widely-used copula families in multivariate modelling. Section 4.2.5 demonstrates an approach to construct a multivariate copula from a combination of bivariate copulas (pair-copula construction) which gives more flexibility in modelling multivariate distribution. Section 4.2.3 exhibits dependence measures commonly used in representing relationship among variables. Section 4.2.9 provides definition of the conditional-value-at-risk and its advantage for risk measurement.

### 4.2.1 Two-Stage Stochastic Programming with Recourse

Stochastic programming models have been proposed and well studied since the late 1950s by, for example, Dantzig [31], Beale [12] and Charnes and Cooper [27]. They propose to represent unknown coefficients or parameters with an assumed probability distribution that is independent of decision variables. Stochastic programming provides a general purpose-modelling framework, especially for portfolio optimisation area. It captures the real-world features such as turnover constraints, transaction costs, risk aversion, limits on groups of assets and other consideration. Uncertainty underlying in asset returns is typically described through a discrete distribution (scenarios) which captures risks and returns of individual assets as well as interdependence (e.g. correlation) among assets in various situations.

Among the variants of stochastic programming models, the most widely-applied is a recourse model which is built from a combination of the anticipative and the adaptive models [143]. Therefore, before proceeding to a background of a two-stage stochastic programming with recourse, we first introduce the two cornerstones of the model.

#### 4.2.1.1 Anticipative Model

An anticipative model [28] is also known as a static model since decisions made do not depend on future observations. The prudent planning is required to take into account

all possible future realisations since there is no opportunity to amend decisions later on, thereby leading to restrictive decisions.

The model express feasibility in terms of probabilistic (or chance) constraints. For example, given a confidence level $\beta$, $0 < \beta < 1$, constraints can be formulated as

$$\Pr[\xi \,|\, f_r(\mathbf{x}, \xi) = 0, r = 1, 2, \ldots, M] \geq \beta \tag{4.1}$$

where $\mathbf{x}$ is an $n$-dimensional vector of decision variables, $\xi$ is a parameter denoting uncertainty and $f_r(\cdot) : \mathbb{R}^n \times \Xi \to \mathbb{R}$, $r = 1, 2, \ldots, M$.

An anticipative model selects a policy that satisfies desirable characteristics of the constraints and an objective function. In the example above, it is desirable that the probability of a constraint violation is less than the prespecified threshold value of $\beta$. Note that the choice of $\beta$ depends on individual's preference regarding the cost of constraint violation. As a rule of thumb, $\beta = 0.05$ is commonly used.

#### 4.2.1.2 Adaptive Model

In the notion of an adaptive model [28], information on uncertainty is partially available before decision making is performed. Accordingly, optimisation takes place in a learning environment which is opposed to the framework of an anticipative model. Let $\mathscr{A}$ be the collection of all the relevant information available through observation which is a subfield of all possible events and a decision $x$ be dependent on events being observed, an adaptive stochastic programming can be formulated as

$$\begin{aligned} \text{minimise} \quad & \mathbb{E}[f_0(\mathbf{x}(\xi), \xi) | \mathscr{A}] \\ \text{subject to} \quad & \mathbb{E}[f_r(\mathbf{x}(\xi), \xi) | \mathscr{A}] = 0, r = 1, 2, \ldots, M, \\ & \mathbf{x} \in X. \end{aligned} \tag{4.2}$$

where $X$ represents the set of all feasible decisions. Solutions of the above problem can be obtained by solving for every $\xi$ the following deterministic problem:

$$\begin{aligned} \text{minimise} \quad & \mathbb{E}[f_0(x, \cdot) | \mathscr{A}](\xi) \\ \text{subject to} \quad & \mathbb{E}[f_r(x, \cdot) | \mathscr{A}](\xi) = 0, r = 1, 2, \ldots, M, \\ & \mathbf{x} \in X. \end{aligned} \tag{4.3}$$

A combination of these two extreme cases, complete information and no information at all, constitutes a recourse model.

#### 4.2.1.3 Two-Stage Stochastic Programming with Recourse

A recourse model [28] seeks a policy that not only anticipates future observations but also takes into account partially available information for associate recourse decisions. For instance, a portfolio manager considers both fluctuations of stock prices (anticipation) as well as reallocating portfolio positions as prices change (adaptation).

The two-stage stochastic programming problem with recourse can be written as follows:

$$\begin{aligned} \text{minimise} \quad & f(x) + \mathbb{E}[Q(\mathbf{x}, \xi)] \\ \text{subject to} \quad & A\mathbf{x} = b, \\ & \mathbf{x} \in \mathbb{R}^n, \end{aligned} \tag{4.4}$$

where $\mathbf{x}$ contains the first-stage decisions which are made before the random variables are observed and $Q(\mathbf{x}, \xi)$ is the optimal value, for any given $\Xi$, of the following nonlinear programming problem:

$$\begin{aligned} \text{minimise} \quad & q(\mathbf{y}, \xi) \\ \text{subject to} \quad & W(\xi)\mathbf{y} = h(\xi) - T(\xi)\mathbf{x}, \\ & \mathbf{y} \in \mathbb{R}^n, \end{aligned} \tag{4.5}$$

where $y$ represents second-stage adaptive, or "recourse", decisions which depends on the realisation of the first-stage random vector, $q(\mathbf{y}, \xi)$ denotes the second-stage cost function and $\{W(\xi), T(\xi), h(\xi) | \xi \in \Xi\}$ are model parameters which are random. $T$ is a technology matrix containing coefficients that convert the first-stage decision $\mathbf{x}$ into initial positions for the second-stage problem. $W$ is a recourse matrix and $h$ is a second-stage resource vector.

In general, a two-stage stochastic programming with recourse model is formulated as

$$\begin{aligned} \text{minimise} \quad & f(x) + \mathbb{E}[\min_{y}\{q(\mathbf{y}, \xi) | T(\xi)\mathbf{x} + W(\xi)\mathbf{y} = h(\xi)\}] \\ \text{subject to} \quad & A\mathbf{x} = b, \\ & \mathbf{x} \in \mathbb{R}^n. \end{aligned} \tag{4.6}$$

#### 4.2.1.4 Deterministic-Equivalent Formulation of A Recourse Model

Now consider the case that the random vector $\xi$ has a discrete and finite distribution with support $\Xi = \{\xi^1, \xi^2, \ldots, \xi^M\}$ demonstrating a scenario set. Probability of realisation of the $r^{th}$ scenario of $\xi$ is denoted by $p^r$. It is required that $p^r > 0$ for all $\xi^r \in \Xi$ and $\sum_{r=1}^{M} p^r = 1$. The expected value of the second-stage optimisation problem [28] can be expressed as

$$\mathbb{E}[Q(\mathbf{x}, \xi)] = \sum_{r=1}^{M} p^r Q(\mathbf{x}, \xi^r). \tag{4.7}$$

For each realisation of the random vector $\xi^r \in \Xi$, a different second-stage decision denoted by $\mathbf{y}^r$ is made. The resulting second-stage problem can thus be written as

$$\begin{aligned} \text{minimise} \quad & q(\mathbf{y}^r, \xi^r) \\ \text{subject to} \quad & W(\xi^r)\mathbf{y}^r = h(\xi^r) - T(\xi^r)\mathbf{x}, \\ & \mathbf{y}^r \in \mathbb{R}^n. \end{aligned} \tag{4.8}$$

Combining (4.7) and (4.8) allows reformulating the stochastic programming in (4.6) to the following large-scale deterministic-equivalent model [28]:

$$\begin{aligned} \text{minimise} \quad & f(x) + \sum_{r=1}^{M} p^r q(\mathbf{y}^r, \xi^r) \\ \text{subject to} \quad & A\mathbf{x} = b, \\ & T(\xi^r)\mathbf{x} + W(\xi^r)\mathbf{y}^r = h(\xi^r), \forall \xi^r \in \Xi, \\ & \mathbf{x} \in \mathbb{R}^n, \\ & \mathbf{y}^r \in \mathbb{R}^n. \end{aligned} \tag{4.9}$$

### 4.2.2 Copulas

Nelsen [104] describes copulas as functions that join or "couple" multivariate distribution functions to their one-dimensional marginal distribution functions. Copulas are useful to separate the modelling of the marginal distributions from the modelling of the dependence structure of multiple random variables.

Let $X = (X_1, \ldots, X_n)$ be an $n$-dimensional random variable with a joint density function $f(x_1, \ldots, x_n)$ and a cumulative distribution function (CDF) $F(x_1, \ldots, x_n)$. In addition,

let $F_1(x_1),\ldots,F_n(x_n)$ be a corresponding (strictly increasing, continuous) marginal distributions of $X_1,\ldots,X_n$. A copula, that is itself a multivariate CDF, is the link that connects marginal distributions to a CDF.

**Definition 1 (Copula).** *An n-dimensional copula C is a multivariate CDF on* $[0,1]^n$*,*

$$C : [0,1]^n \to [0,1],$$

*with univariate uniform distributed margins.*

An application of copulas in multivariate modelling was introduced by Sklar [127] showing that a multivariate distribution can be decomposed into marginal distributions which are linked by copulas. Consequently, a given copula can result in various multivariate distributions by selecting different marginal distribution functions and vice versa. Formally, Sklar's theorem is given by

**Theorem 1 (Sklar's theorem).** *Let F be an n-dimensional distribution function with margins* $F_1,\ldots,F_n$*. Then there exists a unique copula C such that*

$$F(x_1,\ldots,x_n) = C\left(F_1(x_1),\ldots,F_n(x_n)\right) \tag{4.10}$$

*for all* $x = (x_1,\ldots,x_n) \in \mathbb{R}^n$*. If* $F_1,\ldots,F_n$ *are continuous, then C is unique; otherwise, C is uniquely determined on* $Ran(F_1)\times\cdots\times Ran(F_n)$ *which is the cartesian product of the ranges of the marginal cumulative distribution functions. Conversely, given a copula* $C : [0,1]^n \to [0,1]$ *and margins* $F_i(x)$ *then* $C\left(F_1(x_1),\ldots,F_n(x_n)\right)$ *defines an n-dimensional cumulative distribution function.*

The multivariate density function $f$ can be generated by differentiating (4.10) using chain rule as follows:

$$f(x_1,\ldots,x_n) = c_{12\ldots n}\left(F_1(x_1),\ldots,F_n(x_n)\right) f_1(x_1)\ldots f_n(x_n) \tag{4.11}$$

and consequently,

$$c_{12\ldots n}(u_1,\ldots,u_n) = \frac{f(F_1^{-1}(u_1),\ldots,F_n^{-1}(u_n))}{f_1(F_1^{-1}(u_1))\cdot\ldots\cdot f_n(F_n^{-1}(u_n))} \tag{4.12}$$

where $u_n \in [0,1]$ and $F_n^{-1}(u_n) = x_n$.

In what follows, let $i = 1,\ldots,n$, $r,l = 1,\ldots,n$ and $i_1,\ldots,i_r,j_1,\ldots,j_l \in \{1,\ldots,n\}$, we use the following notations unless stated otherwise:

| | |
|---|---|
| $F_i$ | a marginal cumulative distribution function of $X_i$. |
| $f_i$ | a marginal density function of $X_i$. |
| $F_{i_1 \dots i_r}$ | a joint cumulative distribution function of $X_{i_1}, \dots, X_{i_r}$. |
| $f_{i_1 \dots i_r}$ | a joint density function of $X_{i_1}, \dots, X_{i_r}$. |
| $C_{i_1 \dots i_r}$ | a copula cumulative distribution function of $X_{i_1}, \dots, X_{i_r}$. |
| $c_{i_1 \dots i_r}$ | a copula density function of $X_{i_1}, \dots, X_{i_r}$. |
| $F_{i_1 \dots i_r \mid j_1 \dots j_l}$ | a joint conditional cumulative distribution function of $X_{i_1}, \dots, X_{i_r}$ given $X_{j_1}, \dots, X_{j_l}$. |
| $f_{i_1 \dots i_r \mid j_1 \dots j_l}$ | a joint conditional density function of $X_{i_1}, \dots, X_{i_r}$ given $X_{j_1}, \dots, X_{j_l}$. |
| $c_{i_1 \dots i_r \mid j_1 \dots j_l}$ | a conditional copula density function of $X_{i_1}, \dots, X_{i_r}$ given $X_{j_1}, \dots, X_{j_l}$. |

### 4.2.3 Dependency Measures

We start with the Pearson correlation which is widely used in representing relationship among variables which also involves in constructing a covariance matrix in a classical mean-variance portfolio optimisation. Then we present the Kendall's tau as an alternative dependence measure which can cope with some deficiencies of the Pearson correlation. Lastly, the tail dependence is introduced for measuring probability of mutual extreme events of two random variables.

#### 4.2.3.1 Pearson Correlation

The Pearson correlation, also known as the product moment correlation or the correlation coefficient, is the most widely-used measure of dependencies in finance and portfolio selection literature. It is basically a scalar measure for the linear correlation of two random variables. Definitions and characteristics are comprehensively provided by Kurowicka and Cooke [77].

**Definition 2** (**Pearson Correlation**). *The Pearson correlation of two random variables X, Y with finite expectations* $\mathbb{E}[X]$, $\mathbb{E}[X]$ *and corresponding finite variances* $\sigma_X^2$,

$\sigma_Y^2$ *is*

$$\rho(X,Y) = \frac{\operatorname{cov}(X,Y)}{\sqrt{\sigma_X^2 \sigma_Y^2}} = \frac{\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]}{\sqrt{\sigma_X^2 \sigma_Y^2}}$$

*where* $\operatorname{cov}(X,Y)$ *is a covariance of* $X$ *and* $Y$. $\rho(X,Y) = 0$ *indicates that* $X$ *and* $Y$ *are uncorrelated.*

The values Pearson correlation are between +1 and -1. For normally distributed random variables the Pearson correlation can fully specifies the dependence between them while, for general distributions, it can only measures the linear dependence. For example, let $X$ be a standard normal distributed random variable, then all moments of $X$ are:

$$\begin{aligned} \mathbb{E}[X] &= 0, \\ \mathbb{E}[X^2] &= 1, \\ \mathbb{E}[X^3] &= 0, \\ \mathbb{E}[X^4] &= 3 \end{aligned}$$

where $\mathbb{E}[X]$, $\mathbb{E}[X^2]$, $\mathbb{E}[X^3]$ and $\mathbb{E}[X^4]$ are the mean, variance, skewness and kurtosis of $X$, respectively.

Suppose that $Y$ depends on $X$ as $Y = X^2$ but the Pearson correlation shows that

$$\begin{aligned} \rho(X,Y) &= \rho(X,X^2) \\ &= \frac{\mathbb{E}[XX^2] - \mathbb{E}[X]\mathbb{E}[X^2]}{\sqrt{\sigma_X^2 \sigma_Y^2}} \\ &= \frac{0-0}{\sqrt{\sigma_X^2 \sigma_Y^2}} \end{aligned}$$

which implies that $X$ and $Y$ are uncorrelated. Thus, in case that variables are nonlinearly dependent, the linear correlation is incapable to capture the relationship. An example of different correlation readings on different dependence structure of variables is given in Figure 4.1.

Kurowicka and Cooke [77] list some drawbacks of the correlation coefficient as follows:

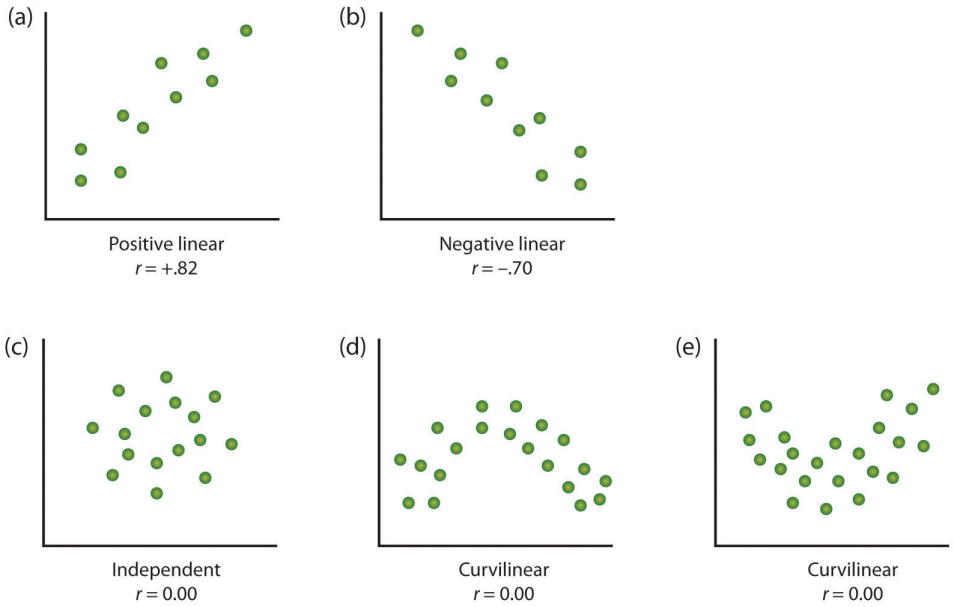
(a) The Pearson correlation is not defined if an expectation or variance of $X$ and $Y$ are not finite (e.g. Cauchy distribution).

(b) The Pearson correlation is not invariant under non-linear strictly increasing transformations.

(c) The possible values of the Pearson correlation depend on marginal distributions.

These deficiencies are problematic when modelling a dependence structure and marginal distributions separately as they influence each other which give biased results. An alternative dependence measure such as the rank correlation or Kendall's tau does not have such downsides.

**Figure 4.1 Various relationships between two variables.** (a) and (b) show linearly dependent variables with positive and negative linear correlations. (c) shows independent variables and thus linear correlation = 0. (d) and (e) show nonlinearly dependent variables, hence the linear correlations are zero. Source: Stangor [129].




#### 4.2.3.2 Kendall's Tau

Kendall's tau is a rank correlation coefficient; a statistic used to measure the ordinal association between two quantities. Definitions and characteristics are provided in [67].

**Definition 3 (Kendall's Tau).** *Let $(X_1, Y_1)$ and $(X_2, Y_2)$ be two independent pairs of random variables with joint CDF $F$ and marginal distributions $F_X$ and $F_Y$, then the Kendall's tau is defined as*

$$\tau(X,Y) = \Pr[(X_1 - X_2)(Y_1 - Y_2) > 0] - \Pr[(X_1 - X_2)(Y_1 - Y_2) < 0],$$

*or alternatively,*

$$\tau(X,Y) = \mathbb{E}[sgn((X_1 - X_2)(Y_1 - Y_2))]$$

*where*

$$\mathbb{E}[sgn((X_1 - X_2)(Y_1 - Y_2))] = \begin{cases} -1 \cdot \Pr[(X_1 - X_2)(Y_1 - Y_2) < 0] \\ 0 \cdot \Pr[(X_1 - X_2)(Y_1 - Y_2) = 0] \\ 1 \cdot \Pr[(X_1 - X_2)(Y_1 - Y_2) > 0]. \end{cases}$$

The value of Kendall's tau is in the range $-1 \leq \tau \leq 1$. Intuitively, the value Kendall's tau of two variables is high when observations have a similar (or identical for $\tau = 1$) rank between the two variables, and low when observations have a dissimilar (or fully different for $\tau = -1$) rank between the two variables. Essentially, the Kendall's tau is not affected by how far from each other ranks are but only by whether the ranks between observations are equal or not, it is thus only appropriate for discrete variables but not defined for continuous variables[4].

To clarify how Kendall's tau works in practice, let $(x_i, y_i)$, $i = 1, \ldots, N$ be data points denoting realisations of $X$ and $Y$, then two data points $(x_i, y_i)$ and $(x_j, y_j)$ with $i, j = 1, \ldots, N$ are:

- concordant, if $(x_i - x_j)(y_i - y_j) > 0$, i.e, the ordering of the two $x$'s is the same as the ordering of the two $y$'s.
- discordant, if $(x_i - x_j)(y_i - y_j) < 0$, i.e, the ordering of the two $x$'s is opposite from the ordering of the two $y$'s.

Additionally, let $N_{conc}$ and $N_{disc}$ be the number of concordant pairs and the number of discordant pairs, respectively. A tie in $x$'s is labelled as an *extra-y* pair, a tie in $y$'s is labelled as an *extra-x* pair where $N_{extra-x}$ and $N_{extra-y}$ are the numbers of extra-x and extra-y pairs respectively. If there are ties in both $x$'s and $y$'s, all of them are excluded. Subsequently, the empirical version of Kendall's tau [35] is given by

$$\hat{\tau}(X,Y) = \frac{N_{conc} - N_{disc}}{\sqrt{N_{conc} + N_{disc} + N_{extra-y}}\sqrt{N_{conc} + N_{disc} + N_{extra-x}}}. \tag{4.13}$$

A high value in the numerator means that most pairs are concordant, indicating that the two rankings are corresponding and $\hat{\tau}$ will take a value close to 1. Conversely, if most pairs are discordant, $\hat{\tau}$ will take a value close to -1.

[4] In practice, continuous variables can be discretised, hence the Kendall's tau can be estimated.

#### 4.2.3.3 Tail Dependence

The Kendall's tau is applied to measure dependence on the whole event space $u \in [0,1]$ of a copula. In contrast, a bivariate tail dependence particularly measure dependence of extreme values, i.e., an upper tail or a lower tail of a bivariate distribution. The following definition is given by Joe [58].

**Definition 4 (Upper / Lower Tail Dependence).** *If a bivariate copula C with a property*

$$\lim_{u \nearrow 1} \frac{1-2u+C(u,u)}{1-u} = \lambda_U \tag{4.14}$$

*exists, then C has upper tail dependence if* $\lambda_U \in (0,1]$ *and has no upper tail dependence if* $\lambda_U = 0$*. Similarly, if*

$$\lim_{u \searrow 1} \frac{C(u,u)}{u} = \lambda_L$$

*exists, then C has lower tail dependence if* $\lambda_L \in (0,1]$ *and has no lower tail dependence if* $\lambda_L = 0$*.*

If $U_1$ and $U_2$ are two random variables with a joint distribution function $C$, then from theorem 5.4.2 of Nelsen [104], it shows that (4.14) can also be expressed as

$$\lambda_U = \lim_{u \nearrow 1} \Pr[U_1 > u | U_2 > u] = \lim_{u \nearrow 1} \Pr[U_2 > u | U_1 > u], \tag{4.15}$$

$$\lambda_L = \lim_{u \searrow 1} \Pr[U_1 < u | U_2 < u] = \lim_{u \searrow 1} \Pr[U_2 < u | U_1 < u]. \tag{4.16}$$

The equation (4.15) and (4.16) show that $\lambda_U$ and $\lambda_L$ measure a probability of an extreme event for $U_1$ if an extreme event occurs for $U_2$.

### 4.2.4 Copula Families

This section provides an overview of widely-used bivariate copula families for later applications. Copula functions are largely classified as elliptical and Archimedean. The class of elliptical distributions provides a rich source of multivariate distributions which share many of the tractable properties of elliptical distribution such as standard normal distribution and Student's t which enables modelling of multivariate extremes and other forms of non-normal dependences. Another class of copulas is Archimedean copulas which are created from generator functions. They are popular because the ease of modelling dependence in arbitrarily high dimensions with only one parameter,

governing the strength of dependence. Examples of Archimedean copulas are Clayton, Gumbel and Frank. A more comprehensive list of copula families can be found in [104].

#### 4.2.4.1 Gaussian or Normal Copula

The Gaussian copula [141] is defined based on the multivariate normal distribution function. For the case of $n$-dimension, given a correlation matrix $R \in [-1,1]^{n\times n}$, the Gaussian copula with parameter matrix $R$ can be written as

$$C(u_1,\ldots,u_n) = \Phi_R\left(\Phi^{-1}(u_1),\ldots,\Phi^{-1}(u_1)\right) \tag{4.17}$$

where $\Phi^{-1}$ is an inverse cumulative distribution function of a standard normal and $\Phi_R$ is a joint cumulative distribution function[5] of a multivariate normal distribution with mean vector zero and covariance matrix equal to the correlation matrix $R$. The Gaussian copula density function can be expressed as

$$c(u_1,\ldots,u_n) = \frac{1}{\sqrt{\det(R)}}\exp\left(-\frac{1}{2}\begin{pmatrix}\Phi^{-1}(u_1)\\ \vdots \\ \Phi^{-1}(u_d)\end{pmatrix}^{\mathrm{T}}\cdot\left(R^{-1}-\mathbf{I}\right)\cdot\begin{pmatrix}\Phi^{-1}(u_1)\\ \vdots \\ \Phi^{-1}(u_d)\end{pmatrix}\right) \tag{4.18}$$

where $\mathbf{I}$ is an identity matrix and $\det(R)$ is a determinant of the correlation matrix $R$.

For the case of bivariate, given a correlation coefficient $\rho \in (-1,1)$, the bivariate Gaussian copula density function is

$$c(u_1,u_2) = \frac{1}{\sqrt{1-\rho^2}}\exp\left(-\frac{\rho^2(x_1^2+x_2^2)-2\rho x_1 x_2}{2(1-\rho^2)}\right) \tag{4.19}$$

where $x_1 = \Phi^{-1}(u_1)$ and $x_2 = \Phi^{-1}(u_2)$. For $\rho \to 1$, the Gaussian copula exhibits complete positive dependence and, for $\rho \to -1$, it exhibits complete negative dependence.

[5]Note that there are no closed forms for the CDFs of both univariate and multivariate normal distributions, they can be acquired only by numerical computation.

#### 4.2.4.2 Student's t Copula

The Student's t or t copula [32] is created based on the multivariate Student's t distribution. For the case of $n$-dimension, let $R$ be a correlation matrix, $T_{R,\nu}$ be a standardised multivariate Student's t distribution[6] with $\nu$ degrees of freedom and a correlation matrix $R$. A multivariate t copula is then defined as follows:

$$C(u_1,\ldots,u_n) = T_{R,\nu}\left(T_\nu^{-1}(u_1),\ldots,T_\nu^{-1}(u_n)\right) \tag{4.21}$$

where $T_\nu^{-1}(\cdot)$ is an inverse of a univariate Student's t distribution function with $\nu$ degrees of freedom. The corresponding density function for t copula is

$$c(u_1,\ldots,u_n) = \frac{1}{\sqrt{\det(R)}} \frac{\Gamma\left(\frac{\nu+n}{2}\right)\left[\Gamma\left(\frac{\nu}{2}\right)\right]^n}{\Gamma\left(\frac{\nu}{2}\right)\left[\Gamma\left(\frac{\nu+1}{2}\right)\right]^n} \frac{\left(1+\frac{1}{\nu}\mathbf{x}^{\mathrm{T}}R^{-1}\mathbf{x}\right)^{-\frac{\nu+n}{2}}}{\prod_{i=1}^{n}\left(1+\frac{x_i^2}{\nu}\right)^{-\frac{\nu+1}{2}}} \tag{4.22}$$

where $x_i = T_\nu^{-1}(u_i)$ and $\mathbf{x} = [x_1,\ldots,x_n]$.

For a bivariate t copula, given a correlation coefficient $\rho \in (-1,1)$ and a degree of freedom $\nu > 0$, the density function is given by

$$c(u_1,u_2) = \frac{2}{2\pi \cdot dT_\nu(x_1)\cdot dT_\nu(x_2)\cdot\sqrt{1-\rho^2}}\left(1+\frac{x_1^2+x_2^2-2\rho x_1 x_2}{\nu(1-\rho^2)}\right) \tag{4.23}$$

where

$$\mathrm{d}T_\nu(x) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi\nu}}\left(1+\frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}} \tag{4.24}$$

is a density function of a univariate t distribution with $\nu$ degrees of freedom. For $\rho \to 1$, the t copula exhibits complete positive dependence and, for $\rho \to -1$, it exhibits complete negative dependence. Note that, for a degree of freedom higher than 30, the Student's t copula converges to Gaussian copula and exhibits similar characteristics.

[6] The closed form is:

$$T_{R,\nu}(x_1,\ldots,x_n) = \int_{-\infty}^{x_1}\cdots\int_{-\infty}^{x_n} \frac{\Gamma\left(\frac{\nu+n}{2\sqrt{\det(R)}}\right)}{\Gamma\left(\frac{\nu}{2}(\nu\pi)^{\frac{N}{2}}\right)}\left(1+\frac{1}{\nu}\mathbf{x}^{\mathrm{T}}R^{-1}\mathbf{x}\right)^{-\frac{\nu+n}{2}} d\mathbf{x} \tag{4.20}$$

where $\Gamma(\cdot)$ is the gamma function.

#### 4.2.4.3 Clayton Copula

The Clayton copula [104] is a bivariate copula function with the distribution function defined as

$$C(u_1,u_2) = \max\left\{\left(u_1^{-\delta}+u_2^{-\delta}-1\right)^{-\frac{1}{\delta}},0\right\} \tag{4.25}$$

with the corresponding density function

$$c(u_1,u_2) = (1+\delta)(u_1u_2)^{-1-\delta}\left(u_1^{-\delta}+u_2^{-\delta}-1\right)^{-\frac{1}{\delta}-2} \tag{4.26}$$

where $\delta \in [-1,\infty)\backslash\{0\}$. However, Dissmann [35] suggests to confine to $\delta \in (0,\infty)$ to avoid numerical problems when $\delta \in [-1,0)$. For $\delta \to 0$, the Clayton copula exhibits independence and, for $\delta \to \infty$, it exhibits complete positive dependence.

#### 4.2.4.4 Gumbel Copula

The Gumbel copula [104] is a bivariate copula function with the distribution function

$$C(u_1,u_2) = \exp\left(-\left((-\log(u))^{\delta}+(-\log(v))^{\delta}\right)^{\frac{1}{\delta}}\right) \tag{4.27}$$

and the density function

$$\begin{aligned} c(u_1,u_2) = \frac{1}{C(u_1,u_2)}&\left(-\left((-\log(u_1))^{\delta}+(-\log(u_2))^{\delta}\right)^{\frac{2}{\delta}-2}\right) \\ &\cdot(\log(u_1)\log(u_2))^{\delta-1} \\ &\cdot\left(1+(\delta-1)\left((-\log(u_1))^{\delta}+(-\log(u_2))^{\delta}\right)^{\frac{1}{\delta}}\right) \end{aligned} \tag{4.28}$$

where $\delta \in [1,\infty)$. The Gumbel copula is only applicable to model positive dependence. For $\delta = 1$, the Gumbel copula exhibits independence and, for $\delta \to \infty$, it exhibits complete positive dependence.

#### 4.2.4.5 Frank Copula

The Frank copula [104] is a bivariate copula function with the distribution function

$$C(u_1,u_2) = -\frac{1}{\delta}\log\left(1+\frac{(\mathrm{e}^{-\delta u_1}-1)(\mathrm{e}^{-\delta u_2}-1)}{\mathrm{e}^{-\delta}-1}\right) \tag{4.29}$$

and the density function

$$c(u_1,u_2) = \frac{\delta(1-\mathrm{e}^{-\delta})\mathrm{e}^{-\delta(u_1+u_2)}}{1-\mathrm{e}^{-\delta}-(1-\mathrm{e}^{-\delta u_1})(1-\mathrm{e}^{-\delta u_2})} \tag{4.30}$$

where $\delta \in (-\infty,\infty)\backslash\{0\}$. For $\delta \to -\infty$, the Frank copula exhibits complete negative dependence and, for $\delta \to \infty$, it exhibits complete positive dependence.

### 4.2.5 Pair-Copula Construction

Theoretically, construction of higher-dimensional copulas with more than two variables is viable. The simplest choice is to employ a multivariate copula which unintentionally assumes that a dependence structure of all assets are characterised by the same copula family. This is somewhat inflexible as, in practice, dependencies among assets tend to be heterogeneous. That is, some pairs of assets could have different dependency from others (e.g., have no tail independent while the others have). Thus, applying a multivariate copula might not be an appropriate choice to model a dependence structure of multiple assets.

One solution is to model the dependence structure pair-by-pair, or a pair-copula construction (PCC) as introduced in Aas et al. [1]. With a wide range of bivariate copulas available, the idea of PCC is to decompose a multivariate distribution into bivariate copulas and marginal distributions. The structure of the pair-copula decomposition is then illustrated by vines, a graphical model of representing the construction steps and a dependence structure introduced by Bedford and Cook [14]. The pair-copula constructions starts with decomposing an $n$-dimensional joint density function $f$ as

$$f(x_1,\ldots,x_n) = f_1(x_1)f_{2|1}(x_2|x_1)f_{3|21}(x_3|x_2,x_1)\ldots f_{n|n-1\ldots 21}(x_n|x_{n-1},\ldots,x_2,x_1). \tag{4.31}$$

Note that, to keep the presentation tidy, a joint density is represented without subscripts from this point onwards. For instance, from the above expression, a joint density of $x_1,\ldots,x_n$ is expressed as $f(x_1,\ldots,x_n)$.

By the definition of conditional densities,

$$f_{j|1\ldots j-1}(x_j|x_1,\ldots,x_{j-1}) = \frac{f(x_1,\ldots,x_{j-1},x_j)}{f(x_1,\ldots,x_{j-1})}. \tag{4.32}$$

According to Sklar's theorem, the joint density function $f$ can be expressed as a product of marginal density functions and their corresponding copula densities as in (4.11). Combining (4.11) and (4.32) allows the conditional densities to be expressed in terms of a pair-copula density and a marginal density. For instance,

$$f_{1|2}(x_1|x_2) = c_{12}\left(F_1(x_1),F_2(x_2)\right) f_2(x_2) \tag{4.33}$$

where $c_{12}\left(F_1(x_1),F_2(x_2)\right)$ is a pair-copula density of $X_1$ and $X_2$, and $f_2(x_2)$ is a marginal density of $X_2$.

The factorisation of a 2-dimensional conditional density in (4.33) is a building block to higher-dimensional cases which can be generalised to

$$f(x_i|\mathbf{v}) = c_{x_ix_j|\mathbf{v}_{-j}}\left(F(x_i|\mathbf{v}_{-j}),F(x_j|\mathbf{v}_{-j})\right) f(x_i|\mathbf{v}_{-j}) \tag{4.34}$$

for $i,j = 1,\ldots,n$ and $\mathbf{v}$ denotes an arbitrary set of $x_1,\ldots,x_n$ including $x_j$ but not $x_i$ while $\mathbf{v}_{-j}$ denotes all the elements from $\mathbf{v}$ excluding $x_j$. Subsequently to (4.34), Joe [57] shows that the conditional cumulative distribution function can be expressed in the following form:

$$F(x_i|\mathbf{v}) = \frac{\partial C_{x_ix_j|\mathbf{v}_{-j}}\left(F(x_i|\mathbf{v}_{-j}),F(x_j|\mathbf{v}_{-j})\right)}{\partial F(x_j|\mathbf{v}_{-j})} \tag{4.35}$$

where $C_{x_ix_j|\mathbf{v}_{-j}}$ is the bivariate copula distribution function to $F_{x_ix_j|\mathbf{v}_{-j}}$. Applying (4.34) to all conditional densities in (4.31) obtains the following decomposition

$$f(x_1,\ldots,x_n) = \prod_{i=1}^{n} f_i(x_i) \prod_{i=2}^{n}\prod_{j=1}^{i-1} c_{ij|(j+1)\ldots(i-1)}\left(F(x_i|x_{j+1},\ldots,x_{i-1}),F(x_j|x_{j+1},\ldots,x_{i-1})\right) \tag{4.36}$$

and there can be numerous possibilities to decompose (4.36).

For a case of 3 random variables, $X_1$, $X_2$ and $X_3$, a 3-dimensional PCC can be found as follows:

Suppose that $f(x_1,x_2,x_3)$ is factorised as

$$f(x_1,x_2,x_3) = f_{3|12}(x_3|x_1,x_2)f_{2|1}(x_2|x_1)f_1(x_1). \tag{4.37}$$

Applying Sklar's theorem (equation 4.11) on $f(x_1,x_2)$, $f_{13|2}(x_1,x_3|x_2)$ and $f(x_2,x_3)$ yields

$$\begin{aligned}
f_{2|1}(x_2|x_1) &= c_{12}\left(F_1(x_1),F_2(x_2)\right) f_2(x_2),\\
f_{3|12}(x_3|x_1,x_2) &= f_{13|2}(x_1,x_3|x_2)\frac{1}{f_{1|2}(x_1|x_2)},\\
&= c_{13|2}\left(F_{1|2}(x_1|x_2),F_{3|2}(x_3|x_2)\right) \cancel{f_{1|2}(x_1|x_2)} f_{3|2}(x_3|x_2)\frac{1}{\cancel{f_{1|2}(x_1|x_2)}},\\
&= c_{13|2}\left(F_{1|2}(x_1|x_2),F_{3|2}(x_3|x_2)\right) f_{3|2}(x_3|x_2),\\
&= c_{13|2}\left(F_{1|2}(x_1|x_2),F_{3|2}(x_3|x_2)\right) c_{23}\left(F_2(x_2),F_3(x_3)\right) f_3(x_3).
\end{aligned}$$

Therefore, the factorisation in equation (4.37) is equivalent to

$$\begin{aligned}
f(x_1,x_2,x_3) =& c_{13|2}\left(F_{1|2}(x_1|x_2),F_{3|2}(x_3|x_2)\right) c_{23}\left(F_2(x_2),F_3(x_3)\right) c_{12}\left(F_1(x_1),F_2(x_2)\right)\\
&\times f_3(x_3)f_2(x_2)f_1(x_1),
\end{aligned} \tag{4.38}$$

which is identical to the formula given in (4.36). However, the factorisation of $f(x_1,x_2,x_3)$ can also be

$$f(x_1,x_2,x_3) = f_{1|23}(x_1|x_2,x_3)f_{3|2}(x_3|x_2)f_2(x_2), \tag{4.39}$$

which yields another PCC by:

$$\begin{aligned}
f(x_1,x_2,x_3) =& c_{12|3}\left(F_{1|3}(x_1|x_3),F_{2|3}(x_2|x_3)\right) c_{13}\left(F_1(x_1),F_3(x_3)\right) c_{23}\left(F_2(x_2),F_3(x_3)\right)\\
&\times f_3(x_3)f_2(x_2)f_1(x_1).
\end{aligned} \tag{4.40}$$

Thus, it is shown that different factorisations result in different PCCs. It is also obvious that the number of decompositions increases with the dimension of random variables. An evaluation on which PCC is better to characterise the data is discussed later in

Section 4.2.7.1. Most importantly, before reaching that point, it is necessary to create a tool for organising a large number of pair-copula constructions for later analyses. Hence, a vine copula is introduced.

### 4.2.6 Regular-Vine Copula

It is shown in Bedford and Cooke [14] that it requires $\frac{n(n-1)}{2}$ bivariate copulas to specify a distribution of $n$ variables. It needs $n-1$ unconditional bivariate copulas, $n-2$ bivariate copulas that are conditioned on one variable, $n-3$ bivariate copulas conditioned on two and so on. They organise a PCC using graph theory and illustrate it as a nested set of trees or a vine.

In a vine, random variables will be nodes of a tree and edges that connect nodes that represent bivariate copulas. The first tree shows all pairs that are not conditioned on other variables, the second tree shows all pairs conditioned on one other variable, the third tree shows all pairs conditioned on two other variables and so on. The formal definition of a vine or a regular-vine (R-Vine) is given in Bedford and Cooke [14] and Kurowicka and Cooke [77] as

**Definition 5 (Vine, R-Vine).** *$\mathcal{V} = (T_1, \ldots, T_{n-1})$ is an R-Vine on n elements if*

*(i)* *$T_1$ is a tree with nodes $N_1 = \{1, \ldots, n\}$ and a set of edges denoted by $E_1$.*

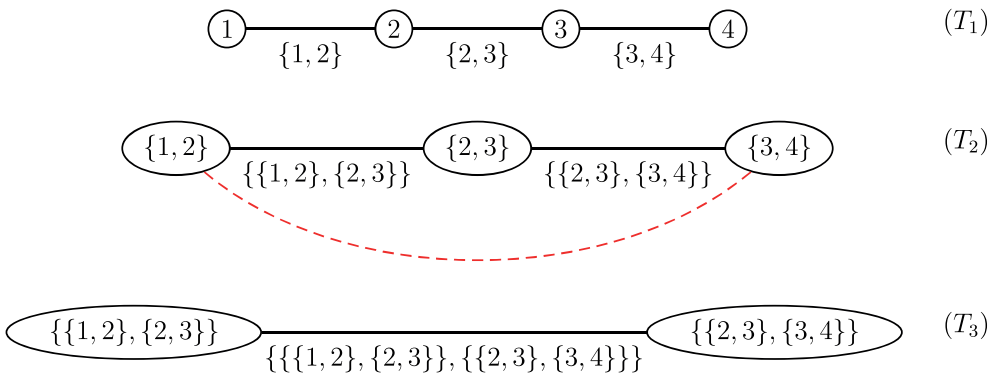
*(ii)* *For $i = 2, \ldots, n-1$, $T_i$ is a tree with nodes $N_i = E_{i-1}$ and an edge set $E_i$ whereby $E_0 = N_1$*

*(iii)* *For $i = 2, \ldots, n-1$ and $\{a, b\} \in E_i$ with $a = \{a_1, a_2\}$ and $b = \{b_1, b_2\}$ it must hold that $\#(a \cap b) = 1$ (proximity condition).*

Briefly speaking, regular-vines are a graph-based tool for specifying conditional bivariate constraints. An R-Vine of $n$-dimensions is a nested set of $n-1$ trees and $\frac{n(n-1)}{2}$ edges such that the nodes of tree $i+1$ are the edges of tree $i$ and two nodes of tree $i+1$ are connected by an edge only if they share a common node in tree $i$ (proximity condition).

An example of an R-Vine of 4 variables is given in Figure 4.2. The R-Vine is defined by 3 trees $T_j$ , $j = 1, 2, 3$ whereas each tree $T_j$ has $5-j$ nodes and $4-j$ edges.

**Figure 4.2 An example of a regular-vine on 4 variables.** The red dashed line prevent the connection between that two nodes since they do not share a common node in $T_1$ according to the proximitiy condition. The figure is adapted from Dissmann [35].




By the Definition 5, it follows from Figure 4.2 that

$$
\begin{aligned}
N_1 &= E_0 = \{1,2,3,4\},\\
N_2 &= E_1 = \Big\{\{1,2\},\{2,3\},\{3,4\}\Big\},\\
N_3 &= E_2 = \Big\{\Big\{\{1,2\},\{2,3\}\},\{\{2,3\},\{3,4\}\Big\}\Big\},\\
N_4 &= E_3 = \Big\{\Big\{\Big\{\{1,2\},\{2,3\}\},\{\{2,3\},\{3,4\}\Big\}\Big\}\Big\}.
\end{aligned}
$$

The edge names are shown under the edges in the trees and the node names are shown in the circles of the trees in Figure 4.2. Since an edge is a set of two nodes, an edge in the third tree is set of a set of a set. The proximity condition restricts how nodes can be connected. For example, in $T_2$, The first edge connects the nodes $\{1,2\}$ and $\{2,3\}$ since they share the node 2 in the tree $T_1$ while the nodes $\{1,2\}$ and $\{3,4\}$ cannot be connected as they do not share a common node in $T_1$.

In order to apply R-Vine in constructing a PCC, Bedford and Cooke [14] define three following sets associated with tree edges of an R-Vine to facilitate labelling unconditional and conditional bivariate copulas in an R-Vine. The *complete union* of an edge is a set of all indices that this edge contains. If two nodes $a$ and $b$ are joined by an edge, then the *conditioned* and *conditioning sets* of this edge are the symmetric difference and the intersection of the complete unions of $a$ and $b$, respectively.

**Definition 6** (**Complete Union, Conditioning Set and Conditioned Set**). *For* $e_i = \{a,b\} \in E_i; a,b \in E_{i-1}; i = 1,\ldots,n-1$,

*(i) The* ***complete union*** *of an edge* $e_i \in E_i$ *is the set* $U_{e_i} = \{n \in N_1 | \exists e_j \in E_j, j = 1,\ldots,i-1$ with $n \in e_1 \in e_2 \in \ldots \in e_{i-1} \in e_i\} \subset N$.

*(ii) The* ***conditioning set*** *of an edge* $e_i$ *is* $D_{e_i} = U_a \cap U_b$.

*(iii) The* ***conditioned set*** *of an edge* $e_i$ *is* $C_{e_i,a} = U_a \backslash D_{e_i}, C_{e_i,b} = U_b \backslash D_{e_i}$ *and* $C_{e_i} = C_{e_i,a} \cup C_{e_i,b} = U_a \triangle U_b$, *where* $A \triangle B := (A \backslash B) \cup (B \backslash A)$ *denotes the symmetric difference of two sets.*

To exemplify an implication of Definition 6, consider the tree $T_2$ of Figure 4.2, the complete union of the edge $a$ between $\{1,2\}$ and $\{2,3\}$ is $\{1,2,3\}$ and the complete union of the edge $b$ between $\{2,3\}$ and $\{3,4\}$ is $\{2,3,4\}$. Consequently, the conditioning and the conditioned sets of the edge joining $a$ and $b$ are $\{2,3\}$ and $\{1,4\}$, respectively.

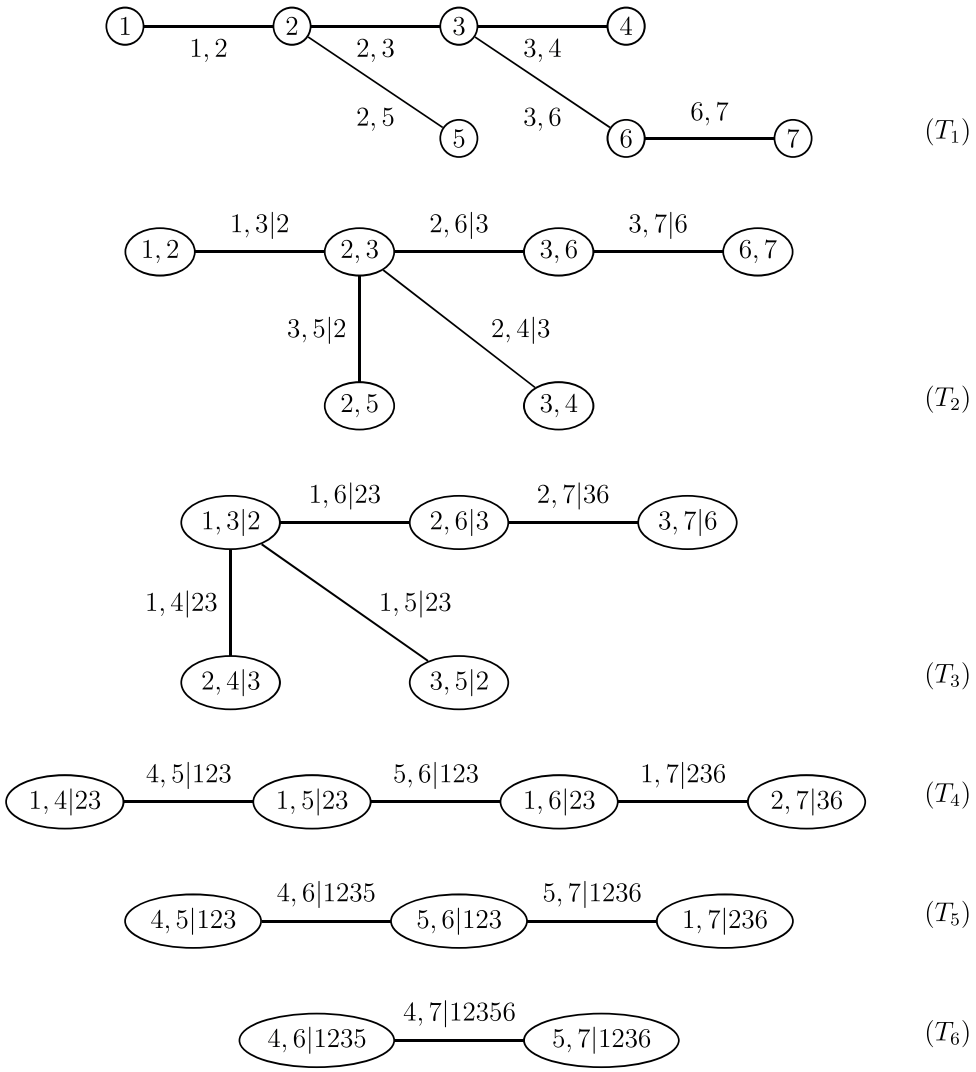
The conditioned and conditioning sets of all edges of $\mathscr{V}$ are collected in a set called the *constraint set*. Each element of this set is composed of a pair of indices corresponding to the conditioned set and a set containing indices corresponding to the conditioning set.

**Definition 7** (**Constraint Set**). *The constraint set for an R-Vine* $\mathscr{V}$ *is a set:*

$$\mathscr{CV} = \left\{ \left\{ \{C_{e,a}, C_{e,b}\}, D_e \right\} | e \in E_i, e = \{a,b\}, i = 1,\ldots,n-1 \right\}.$$

Figure 4.3 shows an R-Vine on 7 variables which employs the conditioning and conditioned sets in enumerating nodes of the trees that are associated to conditional and unconditional bivariate copulas. Each edge of the R-Vine is assigned with its conditioned sets shown before '|' and the conditioning set shown after '|'. Moreover, it is noticed that the constraint set $\mathscr{CV}$ of this R-vine contains all necessary information needed to distinguish it from others.

**Figure 4.3 An example of a regular-vine on 7 variables.** At each edge $e = \{a, b\} \in E_i$, the terms $C_{e,a}$ and $C_{e,b}$ are separated by a comma and given to the left of the 'l' sign, while $D_e$ appears on the right. The figure is adapted from Dissmann [35].




In order to use an R-Vine to specify the structure of a PCC, Bedford and Cooke [14] define the following specification:

**Definition 8 (R-Vine Copula Specification).** *($\boldsymbol{F}$, $\mathscr{V}$, $B$) is an R-Vine copula specification if*

*(i)* $\boldsymbol{F} = (F_1, \ldots, F_n)$ *is a vector of continuous invertible distribution functions.*

*(ii)* $\mathscr{V}$ *is an R-Vine on n variables.*

*(iii)* $B = \{B_e | i = 1, \ldots, n-1; e \in E_i\}$ *is a set of copulas with* $B_e$ *being a bivariate copula (or a pair-copula).*

According to Aas et al. [1], a joint distribution $F$ of a random vector $(X_1, \ldots, X_n)$ is said to realise an R-Vine copula specification $(\mathbf{F}, \mathscr{V}, B)$ or exhibit R-Vine dependence if, for each $e \in E_i; i = 1, \ldots, n-1; e = \{a, b\}$; $B_e$ is the bivariate copula of $X_{C_{e,a}}$ and $X_{C_{e,b}}$ given $\mathbf{X}_{D_e} = \{X_i | i \in D_e\}$ where it is assumed that this conditional copula is independent of the conditioning variables $\mathbf{X}_{D_e}$. Such distribution is called an *R-Vine distribution*. Additionally, the marginal distribution of $X_j$ has to be $F_j$ for $j = 1, \ldots, n$. We denote the copula density of the copula $B_e$ for the edge $e = \{a, b\}$ as $c_{C_{e,a}, C_{e,b}|D_e}$.

In an R-Vine, all copulas can be selected from different families and their parameters can be specified independently from each other. Additionally, since copulas specified in a tree will affect conditioned variables used in later trees, the choices of bivariate copula connecting each node inevitably influence each other. The density of an R-Vine copula is specified by assigning appropriate bivariate copulas to edges of the R-Vine which is demonstrated in Bedford and Cooke [14].

By applying an R-Vine copula, the pair-copula construction (PCC) of $n$ variables in (4.36) can be written as:

$$f(x_1, \ldots, x_n) = \prod_{k=1}^{n} f(x_k) \prod_{i=1}^{n-1} \prod_{e \in E_i} c_{C_{e,a}, C_{e,b}|D_e} \left( F_{C_{e,a}|D_e}(x_{C_{e,a}} | \mathbf{x}_{D_e}), F_{C_{e,b}|D_e}(x_{C_{e,b}} | \mathbf{x}_{D_e}) \right) \tag{4.41}$$

where $e = \{a, b\}$ and $\mathbf{x}_{D_e}$ denotes the variables in $D_e$, i.e., $\mathbf{x}_{D_e} = \{x_i | i \in D_e\}$. The copulas in (4.41) are indexed according to the constraint set $\mathscr{CV}$ (see Definition 7).

To obtain the conditional distributions $F_{C_{e,a}|D_e}(x_{C_{e,a}} | \mathbf{x}_{D_e})$ and $F_{C_{e,b}|D_e}(x_{C_{e,b}} | \mathbf{x}_{D_e})$, let $e \in E_i; e = \{a, b\}, a = \{a_1, a_2\}, b = \{b_1, b_2\}$ be the edge which connects $C_{e,a}$ with $C_{e,b}$ given the variables in the conditioning set $D_e$, Joe [57] shows that

$$\begin{aligned} F_{C_{e,a}|D_e}(x_{C_{e,a}} | \mathbf{x}_{D_e}) &= \frac{\partial C_{C_a|D_a} \left( F_{C_{a,a_1}|D_a}(x_{C_{a,a_1}} | \mathbf{x}_{D_a}), F_{C_{a,a_2}|D_a}(x_{C_{a,a_2}} | \mathbf{x}_{D_a}) \right)}{\partial F_{C_{a,a_2}|D_a}(x_{C_{a,a_2}} | \mathbf{x}_{D_a})} \\ &=: h \left( F_{C_{a,a_1}|D_a}(x_{C_{a,a_1}} | \mathbf{x}_{D_a}), F_{C_{a,a_2}|D_a}(x_{C_{a,a_2}} | \mathbf{x}_{D_a}) \right) \end{aligned} \tag{4.42}$$

where $F_{C_{a,a_1}|D_a}(x_{C_{a,a_1}} | \mathbf{x}_{D_a})$ and $F_{C_{a,a_2}|D_a}(x_{C_{a,a_2}} | \mathbf{x}_{D_a})$ are obtained by recursive calculation which is described in Algorithm 1. $F_{C_{e,b}|D_e}(x_{C_{e,b}} | \mathbf{x}_{D_e})$ can be obtained similarly. The notation of the $h$-function in (4.42) is introduced for convenience. Dissmann et

al. [36] name the conditional distributions $F_{C_{e,a}|D_e}(x_{C_{e,a}}|\mathbf{x}_{D_e})$ and $F_{C_{e,b}|D_e}(x_{C_{e,b}}|\mathbf{x}_{D_e})$ as *transformed variables*.

#### 4.2.6.1 Matrix Representation of Regular-Vines

To develop an algorithm to fit an R-Vine copulas to given data, a convenient way of representing an R-Vine is needed. Storing the nested set of trees is too expensive and difficult to establish an inference algorithm. Morales-Napoles [100] introduces an approach to store an R-vine in a lower triangular matrix to count number of different R-Vines. Dissmann et al. [36] apply the technique for likelihood computation and a sampling procedure.

**Definition 9** (**Matrix Constraint Set**). *Let $M = (m_{i,j})_{i,j=1,\ldots,n}$ be a lower triangular matrix, the i-th constraint set for M is*

$$\mathscr{C}_M(i) = \left\{ \left\{ \{m_{i,i}, m_{k,i}\}, D \right\} | k = i+1, \ldots, n; D = \{m_{k+1,i}, \ldots, m_{n,i}\} \right\}$$

*for $i = 1, \ldots, n-1$. If $k = n$, we set $D = \emptyset$. The constraint set for the matrix M is the union $\mathscr{C}M = \mathscr{C}_M(1) \cup \ldots \cup \mathscr{C}_M(n-1)$. For the elements of the constraint set $\left\{ \{m_{i,i}, m_{k,i}\}, D \right\} \in \mathscr{C}M$, $\{m_{i,i}, m_{k,i}\}$ is the conditioned set and D is the conditioning set.*

Elements of the matrix constraint set comprises a diagonal entry $m_{i,i}$, an entry in the same column below the diagonal $m_{k,i}$ and the remaining elements in that column $\{m_{k+1,i}, \ldots, m_{n,i}\}; k = i+1, \ldots, n; i = 1, \ldots, n$.

The R-Vine copulas in Figure 4.3 can be represented in a matrix form as

$$M^* = \begin{bmatrix} 7 & & & & & & \\ 4 & 4 & & & & & \\ 5 & 6 & 6 & & & & \\ 1 & 5 & 5 & 5 & & & \\ 2 & 1 & 1 & 1 & 1 & & \\ 3 & 2 & 2 & 3 & 3 & 3 & \\ 6 & 3 & 3 & 2 & 2 & 2 & 2 \end{bmatrix}. \tag{4.43}$$

To demonstrate how the structure of the R-Vine copulas in Figure 4.3 is stored in $M^*$, consider, for example, the first column of $M^*$ where the diagonal entry $m_{1,1} = 7$ and the element in the fourth row $m_{4,1} = 1$. It follows from Definition 9 that $\Big\{\{7,1\},\{2,3,6\}\Big\} \in \mathscr{C}M^*$ which corresponds to the constraint set of the rightmost edge of $T_4$ in the R-Vine in Figure 4.3.

To formally define an R-Vine matrix, two additional sets that helps characterise the matrix form and ensure that the proximity condition (Definition 5) is satisfied are needed. For a lower triangular matrix $M = (m_{i,j})_{i,j=1,\dots,n}$ the two sets are

$$B_M(i) := \Big\{\{m_{i,i}, D\} | k = i+1,\dots,n; D = \{m_{k,i},\dots,m_{n,i}\}\Big\}, \tag{4.44}$$

$$\tilde{B}_M(i) := \Big\{\{m_{i,i}, D\} | k = i+1,\dots,n; D = \{m_{i,i}\} \cup \{m_{k+1,i},\dots,m_{n,i}\}\Big\}. \tag{4.45}$$

Dissmann et al. [36] define an R-Vine matrix as follows:

**Definition 10 (R-Vine Matrix).** *A lower triangular matrix $M = (m_{i,j})_{i,j=1,\dots,n}$ is an R-Vine matrix if, for $i = 1,\dots,n-1$ and for all $k = i+1,\dots,n-1$, there exists $j \in \{i+1,\dots,n-1\}$ such that*

$$\Big\{m_{k,i}, \{m_{k+1,i},\dots,m_{n,i}\}\Big\} \in B_M(j) \text{ or } \in \tilde{B}_M(j). \tag{4.46}$$

It follows from Definition 10 that

1. $\{m_{i,i},\dots,m_{n,i}\} \subset \{m_{j,j},\dots,m_{n,j}\}$ for $1 \le j < i \le n$,
2. $m_{i,i} \notin \{m_{i+1,i+1},\dots,m_{n,i+1}\}$ for $i = 1,\dots,n-1$.

The property 1 states that every column contains all the entries that a column to the right contains and the property 2 ensures that there is a new entry on the diagonal in every column. The proof that there is an equivalent R-Vine matrix with the same constraint set for every R-Vine and vice versa can be found in Dissmann [35]. Note, however, that the matrix corresponding to an R-Vine is not unique. For example, consider the matrix obtained after swapping the elements 2 and 3 in the lower right 2-by-2 corner of $M^*$ which still represents the same R-Vine as $M^*$.

Recall the PCCs of $n$ variables in (4.36) and (4.41), by applying an R-Vine matrix, the pair-copula construction becomes

$$f(x_1,\ldots,x_n) = \prod_{j=1}^{n} f(x_j) \cdot \prod_{k=n-1}^{1}\prod_{i=n}^{k+1} c_{m_{k,k},m_{i,k}|m_{i+1,k},\ldots,m_{n,k}}\left(F_{m_{k,k}|m_{i+1,k},\ldots,m_{n,k}}, F_{m_{i,k}|m_{i+1,k},\ldots,m_{n,k}}\right). \tag{4.47}$$

Note that arguments of $F$'s are omitted to shorten the notation.

To specify parameters and families associated with bivariate copulas in an R-Vine matrix, two additional matrices are introduced. $T^* = (t_{i,j})_{i,j=1,\ldots,n}$ indicates a type (or a family) of a bivariate copula and $P^* = (p_{i,j})_{i,j=1,\ldots,n}$ contains parameters of each bivariate copula. Note that some copulas need more than one parameter (such as Student's t copula which requires $\rho$ and $\nu$), Dissmann et al. [36] suggest to store them using two matrices $P_1$ and $P_2$.

**Figure 4.4 An R-Vine matrix $M^*$, accompanied by a copula type matrix $T^*$ and a copula parameter matrix $P^*$.** The copula $c_{4,5|123}$ (red circles) has conditioned variables indexed by $\{4,5\}$ and conditioning variables indexed by $\{1,2,3\}$ belongs to the copula type $t_{4,1}$ with parameter $p_{4,1}$. The copula $c_{7,6}$ (green circles) is of the type $t_{7,4}$ and has the parameter $p_{7,4}$. Source: The figure is adapted from [36].

$M^* =$

| | | | | | | |
|---|---|---|---|---|---|---|
| 4 | | | | | | |
| 7 | 5 | | | | | |
| 6 | 7 | 1 | | | | |
| 5 | 6 | 7 | 7 | | | |
| 1 | 1 | 6 | 2 | 6 | | |
| 2 | 3 | 3 | 3 | 2 | 2 | |
| 3 | 2 | 2 | 6 | 3 | 3 | 3 |

$T^* =$

| | | | | | | |
|---|---|---|---|---|---|---|
| $t_{2,1}$ | | | | | | |
| $t_{3,1}$ | $t_{3,2}$ | | | | | |
| $t_{4,1}$ | $t_{4,2}$ | $t_{4,3}$ | | | | |
| $t_{5,1}$ | $t_{5,2}$ | $t_{5,3}$ | $t_{5,4}$ | | | |
| $t_{6,1}$ | $t_{6,2}$ | $t_{6,3}$ | $t_{6,4}$ | $t_{5,5}$ | | |
| $t_{7,1}$ | $t_{7,2}$ | $t_{7,3}$ | $t_{7,4}$ | $t_{7,5}$ | $t_{7,6}$ | |

$P^* =$

| | | | | | | |
|---|---|---|---|---|---|---|
| $p_{2,1}$ | | | | | | |
| $p_{3,1}$ | $p_{3,2}$ | | | | | |
| $p_{4,1}$ | $p_{4,2}$ | $p_{4,3}$ | | | | |
| $p_{5,1}$ | $p_{5,2}$ | $p_{5,3}$ | $p_{5,4}$ | | | |
| $p_{6,1}$ | $p_{6,2}$ | $p_{6,3}$ | $p_{6,4}$ | $p_{6,5}$ | | |
| $p_{7,1}$ | $p_{7,2}$ | $p_{7,3}$ | $p_{7,4}$ | $p_{7,5}$ | $p_{7,6}$ | |

To obtain a density associated to an R-Vine which involves calculating conditional distributions as in (4.42), Dissmann et al. [36] develop a recursive algorithm as outlined in Algorithm 1.

**Algorithm 1** Density of an R-Vine specification. Source: Dissmann et al. [36]

**Input:** An R-Vine specification in matrix form, i.e., $M$, $T$ and $P$ where $m_{k,k} = n-k+1$ and $k = 1,\dots,n$.

**Output:** A density of the R-Vine on $x_1,\dots,x_n$ under the given R-Vine specification.

1: Set F = 1.
2: Let $V^{direct} = (v_{i,k}^{direct} | i,k = 1,\dots,n)$.
3: Let $V^{indirect} = (v_{i,k}^{indirect} | i,k = 1,\dots,n)$.
4: Set $(v_{n,1}^{direct}, v_{n,2}^{direct}, \dots, v_{n,n}^{direct}) = (F_n(x_n), F_{n-1}(x_{n-1}), \dots, F_1(x_1))$.
5: Let $\mathbb{M} = (\mathbf{m}_{i,k} | i,k = 1,\dots,n)$ with $\mathbf{m}_{i,k} = \max\{m_{i,k},\dots,m_{n,k}\}$ for all $k = 1,\dots,n$ and $i = k,\dots,n$.
6: **for** $k = n-1,\dots,1$ **do**
7:  **for** $i = n,\dots,k+1$ **do**
8:  Set $z_{i,k}^{(1)} = v_{i,k}^{direct}$.
9:  **if** $\mathbf{m}_{i,k} = m_{i,k}$ **then**
10:  Set $z_{i,k}^{(2)} = v_{i,(n-\mathbf{m}_{i,k}+1)}^{direct}$.
11:  **else**
12:  Set $z_{i,k}^{(2)} = v_{i,(n-\mathbf{m}_{i,k}+1)}^{indirect}$.
13:  **end if**
14:  Set $F = F \cdot c(z_{i,k}^{(1)}, z_{i,k}^{(2)} | t_{i,k}, p_{i,k})$.
15:  Set $v_{i-1,k}^{direct} = h(z_{i,k}^{(1)}, z_{i,k}^{(2)} | t_{i,k}, p_{i,k})$ and $v_{i-1,k}^{indirect} = h(z_{i,k}^{(2)}, z_{i,k}^{(1)} | t_{i,k}, p_{i,k})$.
16:  **end for**
17: **end for**
18: **return** F

Algorithm 1 shows how to compute the density for a given R-Vine copula specification stored in a matrix format. $h(\cdot,\cdot|t_{i,k},p_{i,k})$ in Line 15 denotes the h-function (4.42) for the copula of type $t_{i,k}$ with parameters $p_{i,k}$. The matrices $V^{direct}$ and $V^{indirect}$ are introduced to store the arguments of the bivariate copulas in (4.47).

The outer for-loop of Algorithm 1 iterates over the columns of $M$ from right to left, starting with $n-1$. The inner for-loop iterates over the rows from the bottom up to one element below the diagonal entry of $M$. Therefore, Line 14 of Algorithm 1 is executed once for every edge of the R-Vine with the corresponding copula type and parameters.

The essence of the algorithm is how the conditional distributions that are arguments of bivariate copulas in ((4.47)), denoted as $z_{i,k}^{(1)}$ and $z_{i,k}^{(2)}$, are selected. It is shown from

Dissmann et al. [36] that

$$z_{i,k}^{(1)} = F_{m_{k,k}|m_{i+1,k},\ldots,m_{n,k}}(x_{m_{k,k}}|x_{m_{i+1,k}},\ldots,x_{m_{n,k}}),$$
$$z_{i,k}^{(2)} = F_{m_{i,k}|m_{i+1,k},\ldots,m_{n,k}}(x_{m_{i,k}}|x_{m_{i+1,k}},\ldots,x_{m_{n,k}}),$$

for $k = n-1,\ldots,1$ and $i = n,\ldots,k+1$.

### 4.2.7 Selecting Regular-Vine Distributions

Dissmann et al. [36] outline a procedure to fit an R-vine copula specification to a given dataset as follows:

(a) Selection of the R-Vine (structure), i.e., selecting which unconditional and conditional copula pairs should be connected in each tree.

(b) Selection of a bivariate copula family for each copula pair selected in (a).

(c) Estimation of the corresponding parameter(s) for the selected copula family in (b).

In order to find the best specification, an exhaustive approach is to explore all possible R-Vine structures then perform steps (b) and (c). Nonetheless, Morales-Napoles [100] shows that there are $\frac{n!}{2} \times 2^{\binom{n-2}{2}}$ possible R-Vines on $n$ variables and hence the exhaustive method becomes intractable. Dissmann et al. [36] thus develop a heuristic method to select the tree structure of an R-Vine.

#### 4.2.7.1 Sequential Method to Select A Regular-Vine Copula Specification Based on Kendall's Tau

Dissmann et al. [36] find the structure of an R-Vine sequentially tree-by-tree, starting from the first tree $T_1$. The R-Vine structure (which nodes are connected) are selected in such a way that the chosen pairs exhibit the strongest pairwise dependencies. The method is thereby referred as the *sequential method* with details given in Algorithm 2. It is noted that, since all trees are examined separately, it is not guaranteed to find a global optimum where global optimum is meant in terms of model fit, e.g., higher likelihood, smaller AIC/BIC[7] or superior in terms of the likelihood-ratio based test.

[7] Akaike Information Criterion (AIC) [3] and Bayesian Information Criterion (BIC) [62] are goodness-of-fit measures of model selection among a finite set of models.

**Algorithm 2** Sequential method to select an R-vine specification based on Kendall's tau. Source: Dissmann et al. [36]

**Input:** Realisations of $n$ independent identically distributed (i.i.d.) random vectors $(x_{l1},\ldots,x_{ln})$ where $l = 1,...,N$.

**Output:** R-vine copula specification, i.e., $(\mathbf{F}, \mathscr{V}, B)$.

1: Calculate the empirical Kendall's tau $\hat{\tau}_{j,k}$ (according to (4.13)) for all possible variable pairs $\{j,k\}, 1 \leq j < k \leq n$.

2: Select the tree that maximises the sum of absolute empirical Kendall's taus, i.e.,

$$\sum_{e=\{j,k\}} |\hat{\tau}_{j,k}|.$$

3: For each edge $\{j,k\}$ in the selected tree, select a copula family and estimate the corresponding parameter(s). Then transform $\hat{F}_{j|k}(x_{lj}|x_{lk})$ and $\hat{F}_{k|j}(x_{lk}|x_{lj})$, $l = 1,\ldots,N$ according to (4.42).

4: **for** $i = 2,\ldots,n-1$ **do**

5: Calculate the empirical Kendall's tau $\hat{\tau}_{j,k|D}$ for all conditional variable pairs $\{j,k|D\}$ that can be part of tree $T_i$ with respect to the proximity condition (Definition 5).

6: Among these edges, select the tree that maximises the sum of absolute empirical Kendall's taus, i.e.,

$$\sum_{e=\{j,k|D\}} |\hat{\tau}_{j,k|D}|.$$

7: For each edge $\{j,k|D\}$ in the selected tree, select a conditional copula and estimate the corresponding parameter(s). Then transform $\hat{F}_{j|k\cup D}(x_{lj}|x_{lk},\mathbf{x}_{lD})$ and $\hat{F}_{k|j\cup D}(x_{lk}|x_{lj},\mathbf{x}_{lD})$, $l = 1,\ldots,N$ according to (4.42).

8: **end for**

Dissmann et al. [36] justify that the sequential method is appropriate for R-Vine selection because

- Copula families specified in the first tree of the R-Vine often have the greatest influence on the model fit.
- It is more important to model the dependence structure between random variables that have high dependencies correctly, because most copula families can be

modelled as independence and the copula distribution functions for parameters close to independence are very similar.

- The approach minimises the influence of rounding errors in later trees in which pairs with strong pairwise dependence are most prone to, e.g., when assessing the joint tail behaviour of two variables. For pairs of variables that are close to independence, such issues are less relevant.

- For practical applications, it is natural to assume that randomness is driven by the dependence of only some variables and not all. Therefore, if copulas with high dependence are located in the first trees, the transformed variables (conditional distribution functions, see (4.42)) for the later trees will often be rather independent. This is a desirable feature especially for datasets with a large number of variables because it reduces computational time by assuming that the last $k$ trees are independence. More details of R-Vine truncation are given in Brechmann et al. [20].

Kendall's tau is selected as a dependence measure since it measures dependence independently of the assumed distribution (unlike the correlation coefficient) which is particularly useful when combining different non-Gaussian copula families.

Kurowicka [76] proposes another method to generate R-Vines by starting with the last tree resulting in an R-Vine with the lowest dependencies in the top tree. The method, however, employs the partial correlation to allow calculation without knowing the exact R-Vine structure of the first trees but it is problematic when dealing with non-Gaussian copulas.

To select the tree that maximises the sum of absolute empirical Kendall's taus (Lines 2 and 6 of Algorithm 2), the optimisation problem is formulated as a maximum spanning tree[8] which is considered an NP-equivalent problem. Dissmann et al. [36] apply Prim Algorithm[9] [113] in order to solve it.

In selecting a bivariate copula family and parameter(s) to each edge of an R-Vine (Lines 3 and 7 of Algorithm 2), the Akaike Information Criterion (AIC) [3] is chosen as a goodness-of-fit measure. It circumvents the drawback of log-likelihood which unintentionally penalises Student's t copula for having two parameters. Bivariate copula selection using the AIC is extensively investigated in [89] and [20]. Copula

[8]The optimisation problem maximises the sum of weights (empirical Kendall's tau values) on edges of a tree.

[9]Typically, the algorithm is developed in a way to find a minimum spanning tree.

family selection proceeds by computing the AIC values for each possible family and then choosing the one with the smallest AIC value.

### 4.2.8 Scenario Generation for Stochastic Programming

Mitra et al. [99] suggest that decision making under risk should be concerned with the following issues:

1. **Models for decision making under risk:** These allow to state a preference among random variables (representing results of decisions) and lead to a clearly defined optimisation model. These models are normally formulated, given a multivariate distribution for the uncertain parameters.

2. **Representation of parameter uncertainty (scenario generation):** Basically, it is a way to determine distributional representations for the uncertain parameters such as dynamics and uncertainty of asset prices. In particular, it is how to describe the multivariate distribution of asset random returns. The models used to characterise these distributions are known as descriptive models.

3. **Timing:** Since decisions are made at the present in order to obtain a good outcome at a specified future time point. These decisions thus needs to be updated according to new information which arrives as time unfolds. There is also the case when only partial decisions have to be taken at the present time (first stage decisions) and the rest of the decisions (recourse actions) taken after the uncertainty is revealed. These lead to the notion of multi-stage stochastic programming problems. For thorough details for scenario generation in a multi-stage perspective, see Di Domenica et al. [34].

Most applied stochastic programming models are formulated with discrete distributions. In the cases where the stochastic parameters are described by continuous distributions, a discrete approximation is employed to discretise the continuous distributions.

Scenario generation is the process of creating a finite collection of scenarios which describe (in many cases approximately) the distribution of (relevant) random parameters of optimisation models for ex-ante decisions. Also, scenario sets are applied in ex-post evaluation, e.g., simulation for risk quantification of portfolios. A scenario is a single possible realisation of all uncertain parameters. Let $\xi$ denotes a vector of random variables which represent the stochastic model parameters, scenarios describe

the possible values of $\xi$ at a future point in time and can be generalised to multiple future time points by discrete time stochastic processes.

Scenario generation construct plausible outcomes, both pessimistic and optimistic. Each scenario is weighted by its probability of occurrence. Scenarios are not forecasts, while forecasting is a prediction of the most likely value for a random variable (or a vector of random variables), a scenario set is rather viewed as a density forecast. Let $\xi_t, t = 1, \ldots, T$ be a stochastic process which specifies the evolution of the random parameters over time, scenario generation requires

1. Identification of plausible events, that is possible future values of $\xi_t^{\omega}$ for $\tilde{\xi}$ where $\omega \in \{1, \ldots, M\}$ denotes number of plausible events for each time period.
2. An assignment of probabilities to these events $\Pr[\tilde{\xi} = \xi_t^{\omega}] = p_{\omega}$ where $\sum_{\omega=1}^{M} p_{\omega} = 1$.

For financial decision problems, we require forward looking parameter distributions which account for uncertainty in the economy and financial markets. Mitra et al. [99] introduce three desirable properties for scenario generation methods based on the works of Kaut et al. [64] and Zenios [143].

- **Correctness:** Ideally, we wish to use scenario sets which are correct representations of asset random returns but it is difficult in practice to know the correct return distribution. A model which derives from prevalent theory which captures those aspects of asset price dynamics is always preferable. For example, if return volatility is believed to vary over time, a Generalised Autoregressive Conditional Heteroskedasticity (GARCH) model developed by Engle [41] could be used to describe the volatility dynamics.
- **Consistency:** Some certain characteristics of financial assets should be preserved. When considering multiple related random variables, the values of these, under any particular scenario, should be consistent with each other. For instance, bond prices are driven by a few common sources of uncertainty (risk factors), e.g., short term yields, long term yields, credit rating migrations. Bonds of different maturities, cash flows (coupon payments) and credit ratings have differing prices but these are related by particular relationships which are expected to hold under financial theory (no-arbitrage requirements for example).
- **Stability:** The combination of a good scenario generator and a good decision model leads to good solutions. Consequently, a resulting good decision should

at least be a stable decision, that is, the optimal solutions for different scenario sets should not vary significantly.

It is important to note that a scenario generator should be chosen with consideration of the stochastic program for which it is used. As an example, if the problem is to minimise the loss that could be incurred under extremely unfavourable events, then the scenario generators should account for extreme events (e.g., copula-based). In addition, the conditions of correctness and consistency are rather, in practice, subjective conditions while the stability conditions can be tested and are usually regarded as necessary conditions for a scenario generators. Mitra et al. [99] test stability of scenario sets by comparing descriptive statistics of objective values obtained from an optimisation problem under scenarios of different size.

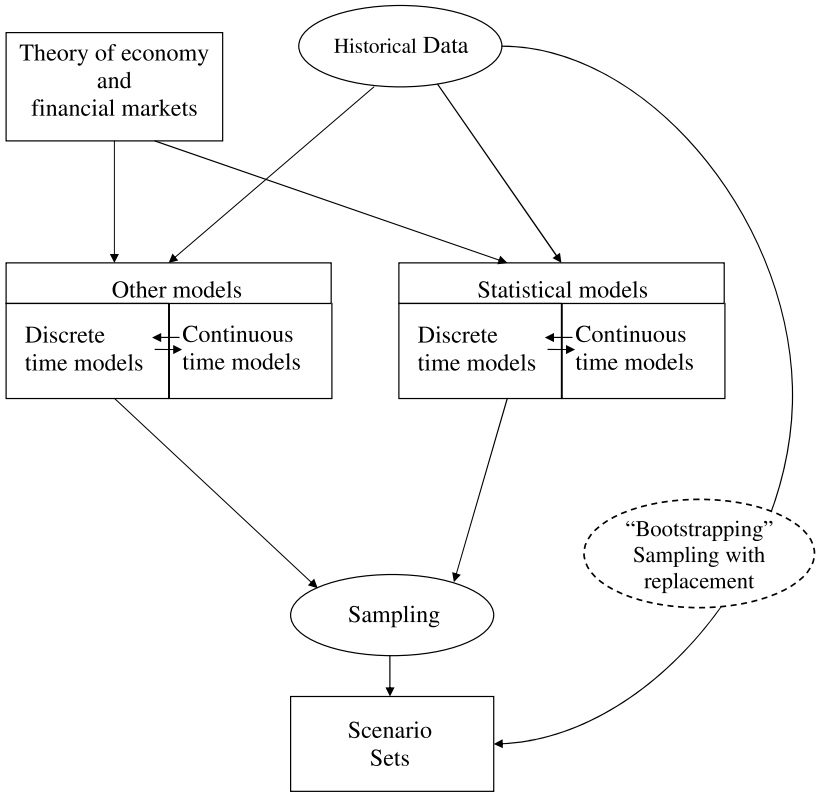
The simplest way to determine scenarios is by using historical observations as scenarios or by bootstrapping[10] the observations. Otherwise, a model developed from economic and financial theories could be employed to describe returns and prices of assets. An outline for scenario generation is illustrated in Figure 4.5.

Kaut and Wallace [65] point out that a good decision should at least be a stable decision, *"We are not concerned about how well the distribution is approximated, as long as the scenario tree leads to good decisions"*. That is, the decisions should be good approximation of the optimal decisions. Nevertheless, it is impossible to tell exactly how good is the approximation we have obtained - if we knew the "true" optimal value and decisions, we would not need scenario generation in the first place. Testing the quality of a scenario is not a precise task and has subjective elements. The quality of the decisions can be assessed by backtesting, in-sample, and out-of-sample stability testing and other out-of-sample analysis, such as stress testing.

In our study, we introduce a new scenario generation method based on regular-vine copula. To investigate the quality of the new scenario generation approach over a standard method, the obtained solutions of an optimisation problem with the new method is subsequently compared against the optimisation solutions obtained from a benchmark scenario set. Next, we give details on the construction of benchmark scenarios which is based on multivariate normal distribution and the constant expected return model.

[10]Bootstrapping [40] is considered as random sampling with replacement. It is generally used in estimating properties of an estimator (such as its mean or variance) by measuring those properties when sampling from an approximating distribution.

**Figure 4.5 Scenario generation methodologies for asset return distributions.** Discrete and continuous time models can be overlapping in some cases. Many discrete time models in their limit (e.g., as the time step becomes of infinitesimal length) lead to continuous time representations which are approximations of their continuous counterparts. The figure is adapted from Mitra et al. [99].




#### 4.2.8.1 Simulating Asset Returns with The Constant Expected Return Model

The constant expected return (CER) model [22] assumes that an asset return is independent and identically normally distributed (i.i.d.) with a constant (time invariant) mean and variance over time. The model allows for the returns on different assets to be contemporaneously correlated but that the correlations are constant over time. The CER model is widely used in finance for modelling asset returns and asset prices, generally served as a benchmark for comparison and evaluation against more complicated models. For instance, it is used in mean-variance portfolio analysis, the Capital Asset Pricing model and the Black-Scholes option pricing model.

Let $r_{it} = \ln\left(\frac{P_{it}}{P_{i,t-1}}\right)$ denotes the continuously compounded return on asset $i = 1,\ldots,n$ at time $t = 1,\ldots,T$ with respect to its prices at time $t$ and $t-1$, $P_{it}$ and $P_{i,t-1}$ respectively, the CER model assumes that

(i) **Covariance stationarity and ergodicity:** $r_{it}$ is a covariance stationary and ergodic stochastic process with $\mathbb{E}[r_{it}] = \mu_i$, $\text{var}(r_{it}) = \sigma_i^2$, $\text{cov}(r_{it}, r_{jt}) = \sigma_{ij}$ and $\text{cor}(r_{it}, r_{jt}) = \rho_{ij}$.

(ii) **Normality:** $r_{it} \sim N(\mu_i, \sigma_i^2)$ for all $i$ and $t$.

(iii) **No serial correlation:** $\text{cov}(r_{it}, r_{js}) = \text{cor}(r_{it}, r_{js}) = 0$ for $t \neq s$ and $i = 1, \ldots, n$.

The mathematical expression of the CER model for asset returns is

$$\begin{aligned} r_{it} &= \mu_i + \varepsilon_{it}, \\ \varepsilon_{it} &\sim N(0, \sigma_i^2), \\ \text{cov}(\varepsilon_{it}, \varepsilon_{js}) &= \begin{cases} \sigma_{ij}, & t = s \\ 0, & t \neq s \end{cases} \end{aligned} \tag{4.48}$$

where the residual $\varepsilon_{it}$ is a Gaussian white noise process with $\mathbb{E}[\varepsilon_{it}] = 0$, $\text{var}(\varepsilon_{it}) = \sigma_i^2$ and has no serial correlation.

The CER model states that each asset return is equal to a constant $\mu_i$ (the expected return) plus a normally distributed random error $\varepsilon_{it}$ with mean zero and constant variance. The random error $\varepsilon_{it}$ can be interpreted as the unexpected news concerning the value of the asset that arrives between time $t-1$ and time $t$ causing the random return deviates from its expected value. Good news drives the realised value of $\varepsilon_{it}$ positive and the observed return above its expected value $\mu_i$ while bad news leads to negative $\varepsilon_{it}$ and take the observed return lower than expected. The assumption that $\varepsilon_{it} = 0$ means that news is neutral on the average, $\text{var}(\varepsilon_{it}) = \sigma_i^2$ indicates that volatility or magnitude of news arrival is constant over time. The random news affecting asset $i$, $\varepsilon_{it}$, is allowed to contemporaneously correlate with the random news variable affecting asset $j$, $\varepsilon_{jt}$ demonstrating that news about one asset can affect another asset.

Alternatively, the CER model (4.48) can be expressed as

$$\begin{aligned} r_{it} &= \mu_i + \varepsilon_{it} = \mu_i + \sigma_i z_{it}, \\ z_{it} &\sim N(0,1). \end{aligned} \tag{4.49}$$

In this form, the random news shock is the i.i.d. standard normal random variable $z_{it}$ scaled by the news volatility $\sigma_i$. This form is particularly convenient for value-at-risk and conditional value-at-risk calculations.

To model asset returns on multiple assets, it is assumed that all returns follow a multivariate normal distribution (MVN). Let $\mathbf{r}_t = [r_{1t}, \ldots, r_{nt}]^{\mathrm{T}}$, $\boldsymbol{\mu} = [\mu_1, \ldots, \mu_n]^{\mathrm{T}}$, $\boldsymbol{\varepsilon}_t = [\varepsilon_{1t}, \ldots, \varepsilon_{nt}]^{\mathrm{T}}$ and a positive definite covariance matrix

$$\mathrm{var}(\boldsymbol{\varepsilon}_t) = \Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{12} & \sigma_2^2 & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1n} & \sigma_{2n} & \cdots & \sigma_n^2 \end{bmatrix}$$

in which $\mathbf{r}_t \sim N(\boldsymbol{\mu}, \Sigma)$. The multivariate CER model is hence expressed as

$$\begin{aligned} \mathbf{r}_t &= \boldsymbol{\mu} + \boldsymbol{\varepsilon}_t, \\ \boldsymbol{\varepsilon}_t &\sim N(\mathbf{0}, \Sigma). \end{aligned} \tag{4.50}$$

In $n$-dimension, the density of multivariate normal is

$$f(\mathbf{r}_t) = \frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{\det(\Sigma)}} \mathrm{e}^{-\frac{1}{2}(\mathbf{r}_t - \boldsymbol{\mu})^{\mathrm{T}} \Sigma^{-1} (\mathbf{r}_t - \boldsymbol{\mu})}. \tag{4.51}$$

To generate normally distributed asset returns from the multivariate CER model, Monte Carlo simulation is employed with details given in Algorithm 3.

**Algorithm 3** Monte Carlo simulation for asset returns from the multivariate CER model. Source: adapted from Campbell et al. [22].

**Input:** An $n \times 1$ vector of expected returns $\boldsymbol{\mu}$ and an $n \times n$ covariance matrix $\Sigma$.

**Output:** A simulated returns matrix $\tilde{\mathbf{R}}$.

1: Set the number of simulations $M$.
2: **for** $i = 1, \ldots, M$ **do**
3: Sampling from the multivariate normal distribution $N(\mathbf{0}, \Sigma)$ for an $n \times 1$ random error vector $\boldsymbol{\varepsilon}_i$.
4: Create an $n \times 1$ simulated return vector $\tilde{\mathbf{r}}_i = \boldsymbol{\mu} + \boldsymbol{\varepsilon}_i$.
5: **end for**
6: Create the simulated returns matrix $\tilde{\mathbf{R}} = [\tilde{\mathbf{r}}_1, \ldots, \tilde{\mathbf{r}}_M]$.

To specify the multivariate normal model, a simple proxy for an expected return is an average of historical asset returns while a covariance matrix can be calculated from variance and covariance of historical asset returns. Alternatively, sophisticated

statistical models can be employed to estimate the expected return and the covariance matrix.

#### 4.2.8.2 The Random Walk Model of Asset Prices

Asset returns from the CER model can be transformed to asset prices by the random walk (RW) model. Recall that the continuously compounded return $r_{it}$ is defined from asset prices by $r_{it} = \ln\left(\frac{P_{it}}{P_{i,t-1}}\right) = \ln(P_{it}) - \ln(P_{i,t-1})$. Let $p_{it} = \ln(P_{it})$, we can express the CER model in terms of log-prices as

$$p_{it} = p_{i,t-1} + \mu_i + \varepsilon_{it} \tag{4.52}$$

which is known as the RW model for log-prices or a random walk with drift.

In (4.52) $\mu_i$ represents the expected change in the log-price (continuously compounded return) between time $t-1$ and $t$ and $\varepsilon_{it}$ represents an unexpected change in the log-price. That is

$$\mathbb{E}[\Delta p_{it}] = \mathbb{E}[r_{it}] = \mu_i$$

and hence

$$\varepsilon_{it} = \Delta p_{it} - \mathbb{E}[\Delta p_{it}]$$

where $\Delta p_{it} = p_{it} - p_{i,t-1}$. In addition, in the RW model, the unexpected changes in log-price, $\varepsilon_{it}$ are assumed to be uncorrelated over time, so future changes in log-price cannot be predicted from past changes.

To understand the evolution of log-prices, let $p_{i0}$ denote the initial log price of an asset $i$ at time $t = 0$, then the log-price at time $t = 1$ is

$$p_{i1} = p_{i0} + \mu_i + \varepsilon_{i1}$$

where $\varepsilon_{i1}$ is the random value of news occurring between times 0 and 1. Similarly, by substitution of $p_{i1}$, the log-price at time $t = 2$ is

$$\begin{aligned} p_{i2} &= p_{i1} + \mu_i + \varepsilon_{i2}, \\ &= p_{i0} + \mu_i + \mu_i + \varepsilon_{i1} + \varepsilon_{i2}, \\ &= p_{i0} + 2\mu + \sum_{t=1}^{2} \varepsilon_{it}. \end{aligned}$$

Thus, by repeated recursive substitution, the log-price at time $t = T$ is

$$p_{iT} = p_{i0} + T\mu_i + \sum_{t=1}^{T} \varepsilon_{it}. \tag{4.53}$$

The RW log-price model in (4.53) can be expressed in terms of an asset price at time $t$ as

$$\begin{aligned} P_{it} &= \mathrm{e}^{p_{it}}, \\ &= P_{i0}\mathrm{e}^{\mu_i t + \sum_{s=1}^{t} \varepsilon_{is}}, \\ &= P_{i0}\mathrm{e}^{\mu_i t}\mathrm{e}^{\sum_{s=1}^{t} \varepsilon_{is}}. \end{aligned} \tag{4.54}$$

The term $\mathrm{e}^{\mu_i t}$ represents the expected exponential growth rate in prices between time 0 and time $t$ while the term $\mathrm{e}^{\sum_{s=1}^{t} \varepsilon_{is}}$ represents the unexpected exponential growth in prices.

### 4.2.9 Conditional Value-at-Risk (CVaR)

A good selection of appropriate risk measures is vital to portfolio optimisation problems. The Basel III regulatory framework formulates risk measures as percentiles of a loss distribution, i.e., the Value-at-Risk (VaR). It represents maximum loss under specified probability (confidence level) over a certain time period. In order to define Value-at-Risk and Conditional Value-at-Risk, we first need to define the loss function which represents negative returns of a portfolio, where the portfolio return is given by the summation of individual asset return $\xi_i$ weighted by an allocation $w_i$.

$$f_L(w, \xi) = -\sum_i w_i \xi_i.$$

Then, the probability that the loss $f_L(w, \xi)$ does not exceed a threshold $\alpha$ is

$$\Phi(w, \alpha) = \int_{f_L(w,\xi) \leq \alpha} p(\xi) d\xi$$

where $p(\xi)$ is the joint density function of the random returns and $\Phi(w, \alpha)$ is the cumulative distribution loss function associated with $w$ which is continuous and non-decreasing with respect to $\alpha$. Formally, VaR with respect to the portfolio weights $w$ at a confidence level $\beta \in (0, 1)$ is given by the smallest $\alpha$ such that the probability of the

loss $f_L(w,\xi)$ exceeding $\alpha$ is at most $1-\beta$ as follows:

$$\text{VaR}_\beta(w) = \inf\{\alpha : \Phi(\xi,\alpha) \geq \beta\}.$$

However, by its definition, VaR does not distinguish the extent of losses beyond the threshold. Besides, the non-convex characteristic of VaR implies that minimising VaR does not guarantee a global minimum. In addition, Rockafellar and Uryasev [116] address the drawback of VaR that it is unstable and difficult to handle numerically when dealing with non-normal distributions. All these shortcomings imply that VaR is not an appropriate risk measure for portfolio optimisation problems.

Artzner et al. [8] suggest the desirable properties of risk measures leading up to the notion of coherent risk measures. Conditional Value-at-Risk (CVaR) which estimates the expected loss greater than VaR satisfies all the criteria for coherence. Formally, CVaR with respect to a portfolio allocation $w$ at a confidence level $\beta \in (0,1)$ is defined by

$$\text{CVaR}_\beta(w) = \frac{1}{1-\beta}\int_{f_L(w,\xi)\geq \text{VaR}_\beta(w)} f_L(w,\xi)p(\xi)d\xi. \tag{4.55}$$

Note that the probability that $f_L(w,\xi)$ exceeding VaR accumulates to $1-\beta$. The relationship between VaR and CVaR is illustrated in Figure 4.6. Rockafellar and Uryasev [115] introduce the auxiliary function to compute VaR and CVaR as follows:

$$\begin{aligned} F_\beta(w,\alpha) &= \alpha + \frac{1}{1-\beta}\int_{f_L(w,\xi)\geq\alpha} \big(f_L(w,\xi)-\alpha\big)\, p(\xi)d\xi, \\ &= \alpha + \frac{1}{1-\beta}\mathbb{E}[\big(f_L(w,\xi)-\alpha\big)^+] \end{aligned} \tag{4.56}$$

where $\mathbb{E}[\cdot]$ is the expectation operator and $\big(f_L(w,\xi)-\alpha\big)^+ = \max\big(f_L(w,\xi)-\alpha, 0\big)$. Practically, the true joint density $p(\xi)$ is often unknown and needed to be estimated. A discrete approximation of $p(\xi)$ are generally used to represent the joint density of portfolio returns. Accordingly, the corresponding approximation of $F_\alpha(w,\alpha)$ in (4.56) given scenarios $s = 1,\ldots,N$ is

$$\widetilde{F}_\beta(w,\alpha) = \alpha + \frac{1}{(1-\beta)N}\sum_{s=1}^{N}\big(f_L(w,\xi_s)-\alpha\big)^+ \tag{4.57}$$

The function approximating CVaR value $\widetilde{F}_\beta(w,\alpha)$ can be transformed into a linear expression by replacing $\left(f_L(w,\xi_s)-l\right)^+$ with an artificial variable $e_s$ as

$$\widetilde{F}_\beta(w,\alpha) = \alpha + \frac{1}{(1-\beta)N}\sum_{s=1}^{N} e_s \tag{4.58}$$

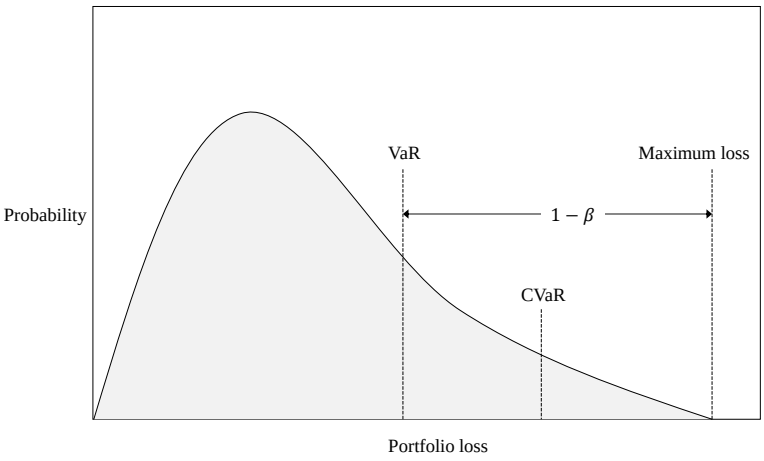
where

$$e_s \geq f_L(w,\xi_s) - \alpha, \tag{4.59}$$

$$e_s \geq 0. \tag{4.60}$$

When applying CVaR in optimisation problems, it is shown in [115] that mean-CVaR and mean-variance portfolios produce the same efficient frontier if the loss function is normally distributed. The difference between the two approaches, however, occurs when the underlying distribution is non-normal and asymmetric.

**Figure 4.6 Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR).** CVaR represents the expected loss exceeding VaR. This figure is produced with artificial data for the purpose of illustration only.




## 4.3 Methodology

This section presents a new approach used in generating scenarios for a stochastic programming problem and a new stochastic optimisation model for an international portfolio.

### 4.3.1 Scenario Generation with Regular-Vine Copula Dependence Structure

The proposed methodology to generate scenarios can be summarised into the following three steps:

1. Modelling marginal distributions of asset returns.
2. Estimating an R-Vine copula specification.
3. Sampling from an R-Vine density.

The results are a scenario set of asset returns which are subsequently converted to asset prices with the random walk model (see Subsubsection 4.2.8.2). Details of each step are given below.

#### 4.3.1.1 Modelling marginal distributions

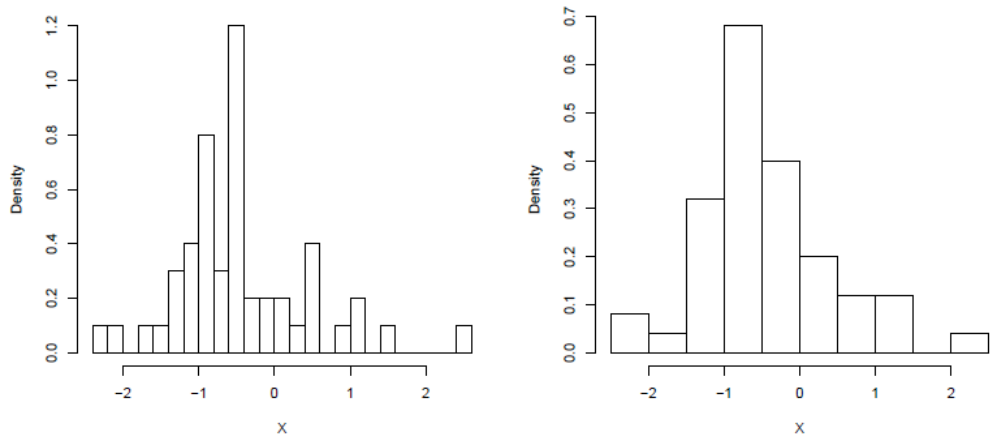
Given multiple time series of financial returns, each is separately fit to an invertible empirical distribution, then its marginal probability distribution function (PDF) is subsequently estimated. Basically, a return series can be fitted parametrically using univariate normal distribution or univariate Student's t distribution. However, in order to capture extreme returns of financial assets, a kernel density estimation is employed to estimate an empirical PDF of a return series non-parametrically. The advantage of an empirical distribution is that it does not restrict the shape of return distribution to some restrictive distribution functions (e.g., Gaussian) but conforms with given observations. Hence, empirical distributions can express asymmetric shape and fat tails of return distribution more accurately. Its reliability, however, depends on the number of observations available.

Kernel density estimation (KDE) is an approach to approximate a density $f(x)$ from observations on $x$ which is similar to how it works on a histogram. Technically, a histogram itself is a kind of kernel density estimate. The data are divided into non-overlapping intervals, and counts are made of the number of data points within each interval. Histograms are bar graphs that represents these frequency counts in which the bar is centred at the midpoint of each interval and its height reflects the average number of data points in the interval.

For general kernel density estimates, the range is still divided into intervals and estimates of the density at the centre of intervals are produced. The first difference of the KDE from a histogram is that the intervals are allowed to overlap. The overlapping intervals can be thought of sliding the interval (called as a window) along the range of the data and collecting the centre-point density estimates. The second difference is that, rather than just counting the number of observations in a window, a kernel density estimator assigns a weight between 0 and 1, based on the distance from the centre of the window, and sums the weighted values. The function that determines these weights is called the kernel.

The smoothness of the histogram estimate is controlled by a bandwidth (width of each bar) and a starting point of each bin edge (locations of bar edges). These choices can produce different impressions of the shape, and hence different estimates as shown in Figure 4.7. Kernel density estimates have the advantages of being smooth and of being independent of the choice of origin (corresponding to the location of the bins in a histogram).

**Figure 4.7 Histogram estimates with small bandwidth (left) and large bandwidth (right).** This figure is produced with artificial data for the purpose of illustration only.




The kernel estimate [119] approximates the probability density function $f$ of a random variable $X_i$ for $i = 1,\ldots,n$ assuming that we have $m$ independent observations $x_{i1},\ldots,x_{im}$ on each random variable, the kernel density estimator for the estimation of the density value at point $x$ is defined as

$$\hat{f}(x_i) = \frac{1}{mh}\sum_{j=1}^{m} K\left(\frac{x_{ij}-x_i}{h}\right) \tag{4.61}$$

where $K$ denotes a so-called kernel function, and $h$ is a bandwidth.

A kernel function has a requirement that it satisfies $\int_{-\infty}^{\infty} K(x)$ and it is assumed that $K$ is a unimodal probability density function that is symmetric about 0 to ensure that the resulting $\hat{f}(x)$ is a density. Let $\mathbf{1}_{\{|x|\leq 1\}}$ be the indicator function which is equal to 1 if $|x| \leq 1$ and 0 otherwise, the example of widely-used kernel functions are shown in Table 4.1.

The optimal kernel function is selected by minimising the Mean Integrated Square Error (MISE) [109]

$$MISE(\hat{f}) = \mathbb{E}\left[\int (\hat{f}(x) - f(x))^2 dx\right]$$

which is a measure of discrepancy between the estimated and true densities in the kernel density estimation. The kernel function that minimises the MISE is the Epanechnikov kernel [42]. It is regarded as the most efficient kernel and efficiency of other kernel functions are measured relative to the Epanechnikov kernel. Therefore, the Epanechnikov kernel is selected as a kernel function for estimating marginal distributions of asset returns in this thesis. Correspondingly, Silverman [125] shows that the optimal bandwidth $h^*$ which minimises the MISE is approximately

$$h^* = 1.06\hat{\sigma}m^{-\frac{1}{5}}$$

where $\hat{\sigma}$ the sample standard deviation of $x_{i1}, \ldots, x_{im}$.

A marginal distribution estimated by KDE represents an empirical probability distribution. An empirical cumulative distribution function (CDF) of each return series can be subsequently produced from an estimated empirical probability distribution as follows:

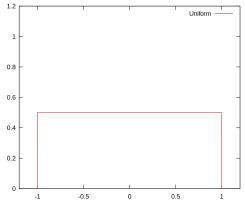
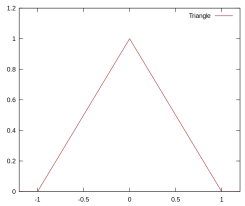
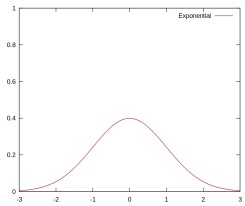
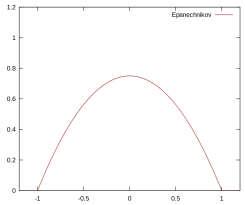
$$F(x_i) = \sum_{x_{ij} \leq x_i} \hat{f}(x_{ij}). \tag{4.62}$$

The resulting CDF is uniform on the interval $[0, 1]$ and is an input argument to a copula function. In what follows, the CDF of a random variable $i$ is denoted by $u_i$, which is also known as a "copula observation".

#### 4.3.1.2 Estimating an R-Vine copula specification

To fit an R-Vine copula to a given data, Dissmann et al. [36] outline the following procedure:

**Table 4.1 Example of widely-used kernel functions.** Source: adapted from Vermeesch [137].

| Kernel function | $K(x)$ | Illustration |
|---|---|---|
| Uniform | $\frac{1}{2}\mathbf{1}_{\{\lvert u\rvert\leq 1\}}$ |  |
| Triangular | $(1-\lvert x\rvert)\mathbf{1}_{\{\lvert x\rvert\leq 1\}}$ |  |
| Gaussian | $\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}x^2}$ |  |
| Epanechnikov | $\frac{3}{4}(1-x^2)\mathbf{1}_{\{\lvert x\rvert\leq 1\}}$ |  |

(a) Selection of the R-Vine structure, i.e. selecting which unconditioned and conditioned pairs to use for the PCC.

(b) Fitting a pair-copula family to each pair selected in (a).

(c) Estimation of the corresponding parameters for each copula.

The inputs of this step are copula observations obtained from marginal distributions modelled earlier. The sequential method (Algorithm 2) which fits an R-Vine tree-by-tree is employed to estimate an R-Vine copula. The bivariate copula families involving in the model estimation include elliptical (Gaussian and Student-t) as well as Archimedean (Clayton, Gumbel, Frank, Joe, BB1, BB6, BB7 and BB8) copulas to cover a broad range of possible dependence structures. For the Archimedean copula families, rotated versions (e.g., rotated Clayton and rotated Gumbel copulas) are

included to cover negative dependence too. Thorough details of the bivariate copula families other than those presented in Section 4.2.4 are provided in [58] and [104].

In the study, the R-Vine copula specification is estimated using the `VineCopula` package [122] in the R programming language. The estimation results are stored in an R-Vine matrix (see Section 4.2.6.1). The results of this step provide the exact combination of unconditional bivariate copulas and conditional bivariate copulas that constitutes an R-Vine and associated parameters of those pair copulas.

#### 4.3.1.3 Sampling from an R-Vine density

This step starts by creating an R-Vine density according to Algorithm 1, then sampling for $u_1, \ldots, u_n$ which are independent and uniform on $[0,1]$ and set

$$\begin{aligned}
\xi_1 &= u_1, \\
\xi_2 &= F^{-1}_{2|1}(u_2|\xi_1), \\
\xi_3 &= F^{-1}_{3|12}(u_3|\xi_1,\xi_2), \\
&\vdots \\
\xi_n &= F^{-1}_{n|12\ldots n-1}(u_n|\xi_1,\ldots,\xi_{n-1})
\end{aligned} \tag{4.63}$$

where the transformed variables $F^{-1}_{i|12\ldots i-1}(u_n|\xi_1,\ldots,\xi_{i-1})$ for $i = 1,\ldots,n$ can be calculated using the inverse of the $h$ function (4.42). (4.63) produces dependent $\xi_1,\ldots,\xi_n$ which is equivalent to one scenario corresponding to $n$ random variables. To generate $M$ scenarios, the random sampling of $u_1,\ldots,u_n$ and (4.63) are repeated $M$ times. The process described is summarised in Algorithm 4.

After obtaining the generated asset returns, the corresponding asset prices are calculated with the following formula

$$P_{ij} = P_{i0}\mathrm{e}^{\xi_{ij}} \tag{4.64}$$

where $P_{i0}$ is a current market price of an asset $i$, $P_{ij}$ is a price of the asset $i$ in a scenario $j$ and $\xi_{ij}$ is a return of the asset $i$ in the scenario $j$ generated with Algorithm 4.

**Algorithm 4** R-Vine Sampling

**Input:** An R-Vine specification ($\mathbf{F}$, $\mathscr{V}$, $B$).

**Output:** Generated return series of $n$ assets for $M$ scenarios.

1: Create an R-Vine density from Algorithm 1 based on the given R-Vine specification.
2: Set the number of simulations $M$.
3: **for** $j = 1, \ldots, M$ **do**
4: Perform a random sampling for random observations $u_{1j}, \ldots, u_{nj}$ from $[0,1]^n$.
5: Set $\xi_{1j} = u_{1j}$.
6: **for** $i = 1, \ldots, n$ **do**
7: Calculate $\xi_{ij} = F^{-1}_{i|12\ldots i-1}(u_{ij}|\xi_{1j}, \ldots, \xi_{i-1,j})$ using the inverse of the $h$ function defined in equation (4.42) based on the constructed R-Vine density in Line 1.
8: **end for**
9: **end for**
10: **return** $\xi_{11}, \ldots, \xi_{nM}$.

An example of scenario generation on three variables using the proposed method is shown in Appendix B.3. Estimation of asset prices and asset returns from generated scenarios are subsequently used as the inputs for an optimisation problem.

### 4.3.2 Two-Stage Stochastic International Portfolio Optimisation Model

This section extends the the deterministic mean-variance international portfolio optimisation model given in Section 3.3.3 of Chapter 3 to accommodate uncertainty of asset and currency returns. Scenarios generated according to Section 4.3.1.3 are also used to represent uncertain returns of assets and currencies in the introduced optimisation model.

#### 4.3.2.1 Notations

The notations used in formulating the optimisation model are categorised by their types and presented in separate tables. Table 4.2 provides the notations for parameters in the optimisation model. The values are set arbitrarily and can be adjusted to individual

preferences. Table 4.3 shows the notations of variables related to transaction costs. The variable costs are retrieved from market data while the fixed costs are set arbitrarily. Table 4.4 displays auxiliary variables that facilitate calculations of the fitness function and constraints. Table 4.5 exhibits the associated decision variables.

Note that in Table 4.4, $a_{ij}$ denotes the final units of an asset class $i$ in country $j$ which differs from its corresponding exposure $\tilde{a}_{ij}$ defined earlier in Table 2.4. To calculate the exposure $\tilde{a}_{ij}$, let $P^0_{ij}$ be a market price of $a_{ij}$, then $\tilde{a}_{ij} = a_{ij}P^0_{ij}$.

**Table 4.2 Notations for the optimisation model parameters.**

| Notation | Range | Description |
|---|---|---|
| $A$ | | A set of assets. |
| $C$ | | A set of currencies. |
| $G$ | | A set of foreign exchange forwards. |
| $N_r$ | | A set of recourse nodes, each node corresponds to each recourse portfolio. |
| $i$ | $\mathbb{N}$ | An index of asset classes, $i \in A$. |
| $j$ | $\mathbb{N}$ | An index of currencies, $j \in C$. |
| $k$ | $\mathbb{N}$ | An index of foreign exchange forwards, $k \in G$. |
| $\mu$ | $\mathbb{R}^+$ | A return target of a portfolio. |
| $\beta$ | $[0,1]$ | The quantile for CVaR. |
| $a^{min}_{ij}$, $a^{max}_{ij}$ | $\mathbb{R}^+$ | Minimum and maximum holding positions of an asset $a_{ij}$, respectively. |
| $c^{min}_{j}$, $c^{max}_{j}$ | $\mathbb{R}^+$ | Minimum and maximum currency exposures of currency $j$ allowed on a portfolio. |
| $t^{min}_{ij}$, $t^{max}_{k}$ | $\mathbb{R}^+$ | A minimum trading size of an asset $a_{ij}$ and a forward pair $q_k$. |
| $V_u$ | $[0,1]$ | A total currency overlay limit on a portfolio. |
| $K_C$, $K_G$ | $\mathbb{N}$ | The number of countries allowed to invest and the number of forward pairs allowed to hold in a portfolio (cardinality). |
| $M$ | $[0,1]$ | A percentage of cash margin required to hold a forward contract. |
| $B$ | $\mathbb{R}^+$ | An arbitrary big constant. |
| $h^0$ | $\mathbb{R}^+$ | An initial cash amount in a portfolio. |
| $a^0_{ij}$ | $\mathbb{R}^+$ | An initial position of an asset class $i$ in currency $j$. |
| $q^0_k$ | $\mathbb{R}^+$ | An initial position of a forward pair $k$. |
| $W^0$ | $\mathbb{R}^+$ | An initial wealth of a portfolio. |
| $p^r$ | $[0,1]$ | A probability of a recourse node $r$ in the second stage. |

**Table 4.3 Notations for the transaction costs.**

| Notation | Range | Description |
|---|---|---|
| $\pi_{ij}$, $\pi_k$ | $\mathbb{R}^+$ | Fixed costs for buying an asset $a_{ij}$ and a forward pair $q_k$, respectively. |
| $\psi_{ij}$, $\psi_k$ | $\mathbb{R}^+$ | Fixed costs for selling an asset $a_{ij}$ and a forward pair $q_k$, respectively. |
| $\rho_{ij}$, $\rho_k$ | $\mathbb{R}^+$ | Variable costs for buying an asset $a_{ij}$ and a forward pair $q_k$, respectively. |
| $\lambda_{ij}$, $\lambda_k$ | $\mathbb{R}^+$ | Variable costs for selling an asset $a_{ij}$ and a forward pair $q_k$, respectively. |
| $P^0_{ij}$, $P^0_k$ | $\mathbb{R}^+$ | Prices (in base currency) per unit of an asset $a_{ij}$ and a forward pair $q_k$ in the first stage, respectively. |

**Table 4.4 Notations for the auxiliary variables of the optimisation model.**

| Notation | Range | Description |
|---|---|---|
| $\mathbf{F}$, $\mathbf{F}^r$ | | A matrix specifying currency exposure adjustment corresponding to market values of forward positions held in the first stage and at a recourse node $r$, respectively. |
| $\mathbf{T}$ | | A ternary matrix containing 0,1 and -1. |
| $\mathbf{q}$, $\mathbf{q}^r$ | | A vector containing market values of forward positions in the first stage and at a recourse node $r$, respectively. |
| $f_{kj}$, $f^r_{kj}$ | $\mathbb{R}$ | A matrix element of $\mathbf{F}$ in the first stage and at a recourse node $r$, respectively. |
| $W^r$ | $\mathbb{R}$ | Wealth of a portfolio at a recourse node $r$. |
| $c_j$, $c^r_j$ | $\mathbb{R}$ | Currency exposure of currency $j$ in the first stage and at a recourse node $r$, respectively. |
| $e^r$ | $\mathbb{R}$ | Expected shortfall (expected loss exceeding VaR) at a recourse node $r$. |
| $\alpha$ | $\mathbb{R}$ | VaR value. |
| $a_{ij}$, $q_k$ | $\mathbb{R}$ | Final units of an asset $a_{ij}$ and a forward pair $q_k$ in the first stage. |
| $a^r_{ij}$, $q^r_k$ | $\mathbb{R}$ | Final units of an asset $a_{ij}$ and a forward pair $q_k$ at a recourse node $r$, respectively. |
| $a^{csh}_{ij}$, $a^{csh,r}_{ij}$ | $\mathbb{R}$ | An amount of cash reserved for margin requirement of forwards (considered as an asset class denominated in base currency) in the first stage and at a recourse node $r$, respectively. |

**Table 4.5 Notations for the decision variables of the optimisation model.**

| Notation | Range | Description |
|---|---|---|
| $b_{ij}$, $b_k$ | $\mathbb{R}^+$ | The number of units of an asset $a_{ij}$ and the number of units of a forward pair $q_k$ that are bought in the first stage, respectively. |
| $b^r_{ij}$, $b^r_k$ | $\mathbb{R}^+$ | The number of units of an asset $a^r_{ij}$ and the number of units of a forward pair $q^r_k$ that are bought at a recourse node $r$, respectively. |
| $s_{ij}$, $s_k$ | $\mathbb{R}^+$ | The number of units of an asset $a_{ij}$ and the number of units of a forward pair $q_k$ that are sold in the first stage, respectively. |
| $s^r_{ij}$, $s^r_k$ | $\mathbb{R}^+$ | The number of units of an asset $a_{ij}$ and the number of units of a forward pair $q_k$ that are sold at a recourse node $r$, respectively. |
| $x_{ij}$, $x_k$ | $\mathbb{B}$ | The binary decision variables for buying an asset $a_{ij}$ and a forward pair $q_k$ in the first stage, respectively. |
| $x^r_{ij}$, $x^r_k$ | $\mathbb{B}$ | The binary decision variables for buying an asset $a_{ij}$ and a forward pair $q_k$ at a recourse node $r$, respectively. |
| $y_{ij}$, $y_k$ | $\mathbb{B}$ | The binary decision variables for selling a forward pair $q_k$ in the first stage, respectively. |
| $y^r_{ij}$, $y^r_k$ | $\mathbb{B}$ | The binary decision variables for selling a forward pair $q_k$ at a recourse node $r$, respectively. |
| $z_j$, $z^r_j$ | $\mathbb{B}$ | The binary decision variable for investing in country $j$ in the first stage and at a recourse node $r$, respectively. |

#### 4.3.2.2 The Two-Stage Stochastic Portfolio Optimisation Model with Recourse

The proposed stochastic portfolio optimisation model comprises two stages, the first stage makes decision to buy or sell assets and forwards based on existing information before the realisation of the uncertain asset and forward prices (equations (4.66) to (4.85)). Then when prices become realised, further decisions are made in the second stage to avoid constraints violations (equations (4.86) to (4.109)). A decision at this stage generally depends on a particular realisation (scenario) of the uncertain variables.

$$\text{minimise} \quad \left(\alpha + \frac{1}{1-\beta}\sum_{r\in N_r} p^r e^r\right) \tag{4.65}$$

subject to

*First Stage: Portfolio Selection*

$$a_{ij} = a^0_{ij} + b_{ij}x_{ij} - s_{ij}y_{ij}, \tag{4.66}$$

$$q_k = q^0_k + b_k x_k - s_k y_k, \tag{4.67}$$

$$
\begin{aligned}
& h^0 + \sum_{i=1}^{A}\sum_{j=1}^{C}\left(s_{ij}P_{ij}^0\right)y_{ij} - \sum_{i=1}^{A}\sum_{j=1}^{C}\left(\psi_{ij}+\lambda_{ij}s_{ij}P_{ij}^0\right)y_{ij} \\
& \quad + \sum_{k=1}^{G}\left(s_kP_k^0\right)y_k - \sum_{k=1}^{G}\left(\psi_k+\lambda_k s_k P_k^0\right)y_k \\
& \quad = \sum_{i=1}^{A}\sum_{j=1}^{C}\left(b_{ij}P_{ij}^0\right)x_{ij} - \sum_{i=1}^{A}\sum_{j=1}^{C}\left(\pi_{ij}+\rho_{ij}b_{ij}P_{ij}^0\right)x_{ij} \\
& \quad + \sum_{k=1}^{G}\left(b_kP_k^0\right)x_k - \sum_{k=1}^{G}\left(\pi_k+\rho_k b_k P_k^0\right)x_k, \qquad (4.68)
\end{aligned}
$$

$$\mathbf{q} = \left[q_k \circ P_k^0\right], \qquad (4.69)$$

$$\mathbf{F} = \mathbf{T} \circ \left(\mathbf{1}^{\mathrm{T}} \otimes \mathbf{q}\right), \qquad (4.70)$$

$$a_{ij}^{csh} = M\sum_{k=1}^{G}|q_k|P_k^0, \qquad (4.71)$$

$$c_j = \sum_{i=1}^{A}a_{ij}P_{ij}^0 + \sum_{k=1}^{G}f_{kj} + a_{ij}^{csh}, \qquad (4.72)$$

$$\frac{1}{2}\sum_{j=1}^{C}\left|\sum_{k=1}^{G}f_{kj}\right| \leq V_u\sum_{j=1}^{C}c_j, \qquad (4.73)$$

$$x_{ij} + y_{ij} \leq 1, \qquad (4.74)$$

$$x_k + y_k \leq 1, \qquad (4.75)$$

$$t_{ij}^{min}x_{ij} \leq b_{ij} \leq B, \qquad (4.76)$$

$$t_{ij}^{min}y_{ij} \leq s_{ij} \leq B, \qquad (4.77)$$

$$t_k^{min}x_k \leq b_k \leq B, \qquad (4.78)$$

$$t_k^{min}y_k \leq s_k \leq B, \qquad (4.79)$$

$$c_j^{min}z_j \leq c_j \leq c_j^{max}, \qquad (4.80)$$

$$\sum_{i=1}^{A}x_{ij} + \sum_{i=1}^{A}y_{ij} \geq z_j; \quad \forall j \in C, \qquad (4.81)$$

$$\sum_{j=1}^{C}z_j \leq K_C, \qquad (4.82)$$

$$\sum_{k=1}^{G}x_k + \sum_{k=1}^{G}y_k \leq K_G, \qquad (4.83)$$

$$b_{ij}, b_k, s_{ij}, s_k \in \mathbb{R}^+, \qquad (4.84)$$

$$x_{ij}, x_k, y_{ij}, y_k, z_j \in \{0,1\}, \qquad (4.85)$$

*Second Stage: Recourse*

$$a_{ij}^r = a_{ij} + b_{ij}^r x_{ij}^r - s_{ij}^r y_{ij}^r, \qquad (4.86)$$

$$q_k^r = q_k + b_k^r x_k^r - s_k^r y_k^r, \qquad (4.87)$$

$$\sum_{i=1}^{A}\sum_{j=1}^{C}\left(s_{ij}^{r}P_{ij}^{r}\right)y_{ij}^{r}-\sum_{i=1}^{A}\sum_{j=1}^{C}\left(\psi_{ij}+\lambda_{ij}s_{ij}^{r}P_{ij}^{r}\right)y_{ij}^{r}$$
$$+\sum_{k=1}^{G}\left(s_{k}^{r}P_{k}^{r}\right)y_{k}^{r}-\sum_{k=1}^{G}\left(\psi_{k}+\lambda_{k}s_{k}^{r}P_{k}^{r}\right)y_{k}^{r}$$
$$=\sum_{i=1}^{A}\sum_{j=1}^{C}\left(b_{ij}^{r}P_{ij}^{r}\right)x_{ij}^{r}-\sum_{i=1}^{A}\sum_{j=1}^{C}\left(\pi_{ij}+\rho_{ij}b_{ij}^{r}P_{ij}^{r}\right)x_{ij}^{r}$$
$$+\sum_{k=1}^{G}\left(b_{k}^{r}P_{k}^{r}\right)x_{k}^{r}-\sum_{k=1}^{G}\left(\pi_{k}+\rho_{k}b_{k}^{r}P_{k}^{r}\right)x_{k}^{r};\quad\forall r\in N_{r} \tag{4.88}$$

$$\mathbf{q}^{r}=\left[q_{k}^{r}\circ P_{k}^{r}\right], \tag{4.89}$$

$$\mathbf{F}^{r}=\mathbf{T}\circ\left(\mathbf{1}^{\mathrm{T}}\otimes\mathbf{q}^{r}\right), \tag{4.90}$$

$$a_{ij}^{csh,r}=M\sum_{k=1}^{G}\left|q_{k}^{r}\right|P_{k}^{r}, \tag{4.91}$$

$$c_{j}^{r}=\sum_{i=1}^{A}a_{ij}^{r}P_{ij}^{r}+\sum_{k=1}^{G}f_{kj}^{r}+a_{ij}^{csh,r}, \tag{4.92}$$

$$\frac{1}{2}\sum_{j=1}^{C}\left|\sum_{k=1}^{G}f_{kj}^{r}\right|\leq V_{u}\sum_{j=1}^{C}c_{j}^{r};\quad\forall r\in N_{r}, \tag{4.93}$$

$$x_{ij}^{r}+y_{ij}^{r}\leq 1;\quad\forall r\in N_{r}, \tag{4.94}$$

$$x_{k}^{r}+y_{k}^{r}\leq 1;\quad\forall r\in N_{r}, \tag{4.95}$$

$$t_{ij}^{min}x_{ij}^{r}\leq b_{ij}^{r}\leq B;\quad\forall r\in N_{r}, \tag{4.96}$$

$$t_{ij}^{min}y_{ij}^{r}\leq s_{ij}^{r}\leq B;\quad\forall r\in N_{r}, \tag{4.97}$$

$$t_{k}^{min}x_{k}^{r}\leq b_{k}^{r}\leq B;\quad\forall r\in N_{r}, \tag{4.98}$$

$$t_{k}^{min}y_{k}^{r}\leq s_{k}^{r}\leq B;\quad\forall r\in N_{r}, \tag{4.99}$$

$$c_{j}^{min}z_{j}^{r}\leq c_{j}^{r}\leq c_{j}^{max};\quad\forall r\in N_{r}, \tag{4.100}$$

$$\sum_{i=1}^{A}x_{ij}^{r}+\sum_{i=1}^{A}y_{ij}^{r}\geq z_{j}^{r};\quad\forall j\in C,\forall r\in N_{r}, \tag{4.101}$$

$$\sum_{j=1}^{C}z_{j}^{r}\leq K_{C};\quad\forall r\in N_{r}, \tag{4.102}$$

$$\sum_{k=1}^{G}x_{k}^{r}+\sum_{k=1}^{G}y_{k}^{r}\leq K_{G};\quad\forall r\in N_{r}, \tag{4.103}$$

$$W^{r}=\sum_{i=1}^{A}\sum_{j=1}^{C}a_{ij}^{r}P_{ij}^{r}+\sum_{k=1}^{G}q_{k}^{r}P_{k}^{r}+a_{ij}^{csh,r}, \tag{4.104}$$

$$e^{r}\geq-\left(\frac{W^{r}}{W^{0}}-1\right)-\alpha;\quad\forall r\in N_{r}, \tag{4.105}$$

$$e^{r}\geq 0;\quad\forall r\in N_{r}, \tag{4.106}$$

$$\sum_{r\in N_{r}}p^{r}\left(\frac{W^{r}}{W^{0}}-1\right)\geq\mu, \tag{4.107}$$

$$b_{ij}^r, b_k^r, s_{ij}^r, s_k^r \in \mathbb{R}^+, \tag{4.108}$$

$$x_{ij}^r, x_k^r, y_{ij}^r, y_k^r, z_j^r \in \{0,1\}. \tag{4.109}$$

The details of the proposed optimisation model are given as follows:

- *Objective function* - (4.65) determines the expected $\beta$-percentile CVaR of portfolio returns at the end of the second stage where $\alpha$ is the corresponding VaR value.
- *Unit balance* - the constraints (4.66) and (4.67) in the first stage and (4.86) and (4.87) in the second stage. The final position is calculated from initial units plus the bought-in units deducted by the sold-out units.
- *Cash balance* - the constraints (4.68) in the first stage and (4.88) in the second stage. Money for purchasing securities is strictly from initial cash plus cash received from selling securities.
- *Currency overlay construction* - the constraints (4.69) and (4.70) in the first stage and (4.89) and (4.90) in the second stage. The currency overlay is a combination of foreign exchange forwards.
- *Margin requirement of forwards* - the constraints (4.71) in the first stage and (4.91) in the second stage. The portfolio is required to reserve some cash to retain forwards positions.
- *Currency Exposure* - the constraints (4.72) in the first stage and (4.92) in the second stage. The exposure on each currency is the sum of total market values of all investments (assets and forwards) denominated in that currency.
- *Total currency overlay* - the constraints (4.73) in the first stage and (4.93) in the second stage. The total currency overlay of a portfolio cannot exceed the predefined limit.
- *Either buy or sell* - the constraints (4.74) and (4.75) in the firs stage and (4.94) and (4.95) in the second stage. A security is not allowed to be bought or sold at the same time in order to prevent unnecessary transaction cost.
- *Bounded trading size* - the constraints (4.76), (4.77), (4.78) and (4.79) in the first stage and (4.96), (4.97), (4.98) and (4.99) in the second stage. The size of each transaction is required to be in the predefined limits.

- *Bounded curency exposure* - the constraints (4.80) in the first stage and (4.100) in the second stage. Exposure of each currency is bounded to the predefined limits.
- *Cardinality constraint on number of currencies* - the constraints (4.81) and (4.82) in the first stage and (4.101) and (4.102) in the second stage. The number of currencies the portfolio exposes to cannot exceed $K_C$.
- *Cardinality constraint on number of forward contracts* - the constraints (4.83) in the first stage and (4.103) in the second stage. The number of forwards in the currency overlay cannot exceed $K_G$.
- *CVaR evaluation* - the constraints (4.104), (4.105) and (4.106). CVaR is calculated as the expected loss beyond the VaR value $\alpha$.
- *Portfolio return target* - the constraint (4.107). The target return of the portfolio.

## 4.4 Algorithm

The optimisation problem outlined earlier is a two-stage stochastic programming which is generally more challenging than a deterministic model due to an increasing search space from additional variables in the second stage. It is proved in the work

of Dyer and Stougie [39] that two-stage linear stochastic programming problems are #P-hard[11] which is at least as hard as the corresponding NP problems[12].

Cardinality constraints are also included into our optimisation model which makes the problem integer-mixed and the search space discontinuous resulting in more challenging optimisation problem [131]. We initially solve our two-stage stochastic portfolio optimisation with the exact search method using CPLEX which fails to give any feasible solution within a time limit of three hours. We therefore employ a genetic algorithm to solve this problem.

### 4.4.1 Genetic Algorithm

Genetic algorithm (GA) is a stochastic search algorithms developed by Holland [53]. The idea is inspired by the basic principles of biological evolution and natural selection. GA imitates the evolution of living organisms, where the fittest individuals dominate over the weaker ones, by mimicking the biological mechanisms of evolution, such as selection, crossover and mutation.

In GA, an initial population which consists of several feasible solutions (individuals). Based on the fitness values, some individuals are probabilistically selected to remain unchanged to the next generation and some individuals are probabilistically selected to engage in the genetic operations to produce children (offspring) for the next generation. The cross over and mutation genetic operators take part in the process of "generation

[11] The complexity class #P is the set of the counting problems associated with the decision problems in the set NP. The problems ask "how many" rather than "are there any" as in NP problems. #P problems are at least as hard as the corresponding NP problem because if counting answers is easy, then it must be easy to tell whether there are any answers.

[12] Suppose that a decision problem is a problem with a yes or no answer, then brief descriptions for complexity classes are given as:

- P is a complexity class that represents the set of all decision problems that can be solved in polynomial time ("easy to solve").
- NP (non-deterministic polynomial-time) is a complexity class that represents the set of all decision problems for which the instances where the answer is "yes" have proofs that can be verified in polynomial time. An NP problem is often of the form "Are there any solutions that satisfy certain constraints?".
- NP-complete is a complexity class which represents the set of all problems $X$ in both NP and NP-hard for which it is possible to reduce any other NP problem $Y$ to $X$ in polynomial time. This implies that we can solve $Y$ quickly if we know how to solve $X$ quickly.
- NP-hard problems are the problems that are at least as hard as the NP-complete problems. But NP-hard problems do not have to be in NP and they do not have to be decision problems.

production" and in successive generations the fitness values are evaluated again. This process is repeated until acceptable solutions are found or a termination criterion is satisfied. The comprehensive descriptions of GA can be found in [51].

Let $\Theta$ denote the population size, $\Gamma$ denote the number of generations, $\Delta$ denote the selection probability, $\Lambda$ denote the crossover rate and $\Upsilon$ denote the mutation rate. The details of genetic algorithm applied to solve our optimisation problem is provided in Algorithm 5 and the parameter values for GA and the optimisation model are given in Section 4.5.2.1.

**Algorithm 5** Genetic Algorithm for the optimisation problem.

0: Initialise the $0^{th}$ generation from a randomly-weighted portfolio.
1: **for** each individual in the initial generation **do**
2: Evaluate the fitness values.
3: **end for**
4: **while** termination criterion is not satisfied **do**
5: **Elitism:** select the fittest individuals in the current generation in the current generation that are guaranteed to survive to the next generation.
6: **Selection:** select $\Delta(\Theta-1)$ individuals to become parents for the next generation using the Stochastic Uniform Selection Method.
7: **Crossover:** pairing the selected parents to generate $\Lambda(\Theta-1)$ children for the next generation other than the elite child using the Arithmetic Mean method.
8: **Mutation:** $(1-\Lambda)(\Theta-1)$, or the rest of the next generation, are generated through mutation. The Adaptive Feasible Mutation is employed to add a randomly generated number to each gene (a decision variable) of the selected individuals.
9: **end while**

In our study, the genetic algorithm is implemented via GA Toolbox in MATLAB R2015b. A chromosome contains all the decision variables presented in Table 4.5. We specify the selection function as the Stochastic Uniform Selection Method. The approach lays out a line in which each parent corresponds to a section of the line of length proportional to its scaled value. The algorithm moves along the line in steps of equal size. At each step, the algorithm allocates a parent from the section it lands on. The first step is a uniform random number less than the step size.

For the crossover function, children are created from the arithmetic mean of two parents. For the mutation function, since our optimisation problem is constrained, the Adaptive Feasible Mutation is selected. The idea is to randomly generates directions (positive or negative random numbers) that are adaptive with respect to the last successful or

unsuccessful generation. Other GA Toolbox configurations to solve our optimisation problem are set to the default.

Due to the probabilistic development of the solutions, GA cannot guarantee optimality even when it may be reached. To verify the reliability of solutions from GA under our settings, we examine the plot of fitness values at each generation for portfolios at target returns 1.20% and 1.40%. In Figure 4.8 to Figure 4.11, the top panel shows the results from vine-copula-based scenarios and the bottom panel displays the results from multivariate normal scenarios. The blue hollow circles represent the average of fitness values over an entire population and the red dots indicate the best fitness values (minimum CVaRs). All the optimisations (both shown and not shown in Figure 4.8 to Figure 4.11) converge within the maximum number of generations (500). The runtime spent on solving each portfolio is approximately 50 minutes (see more details in Table 4.15). For most cases, the fitness value reaches the minimum at around $250^{th}$ generation which more or less proves reliability of solutions obtained from GA in solving our optimisation problem.

**Figure 4.8 The evolution of fitness values (CVaR) over generations when running GA on portfolio under RVC scenarios at the target returns of 1.20%.** The blue hollow circles represent the average of fitness values over an entire population and the red dots indicate the best fitness values (minimum CVaRs).

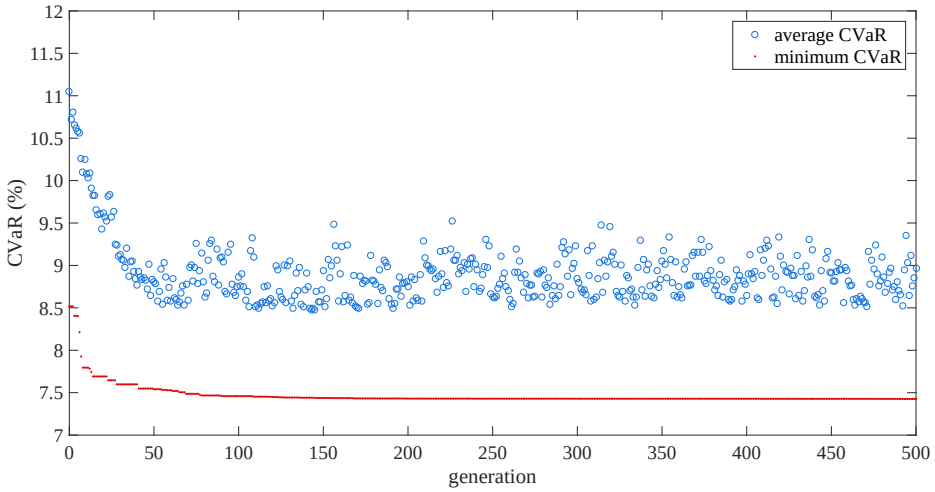


**Figure 4.9 The evolution of fitness values (CVaR) over generations when running GA on portfolio under RVC scenarios at the target returns of 1.40%.** The blue hollow circles represent the average of fitness values over an entire population and the red dots indicate the best fitness values (minimum CVaRs).

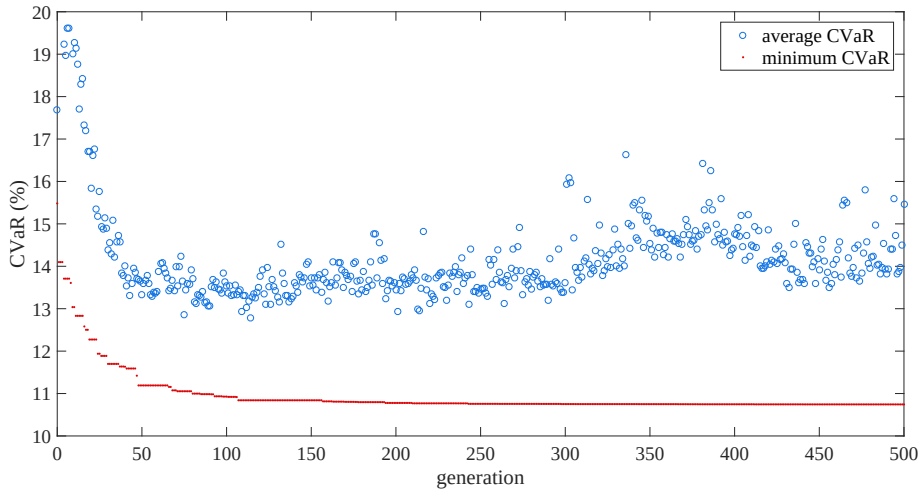



**Figure 4.10 The evolution of fitness values (CVaR) over generations when running GA on portfolio under MVN scenarios at the target returns of 1.20%.** The blue hollow circles represent the average of fitness values over an entire population and the red dots indicate the best fitness values (minimum CVaRs).

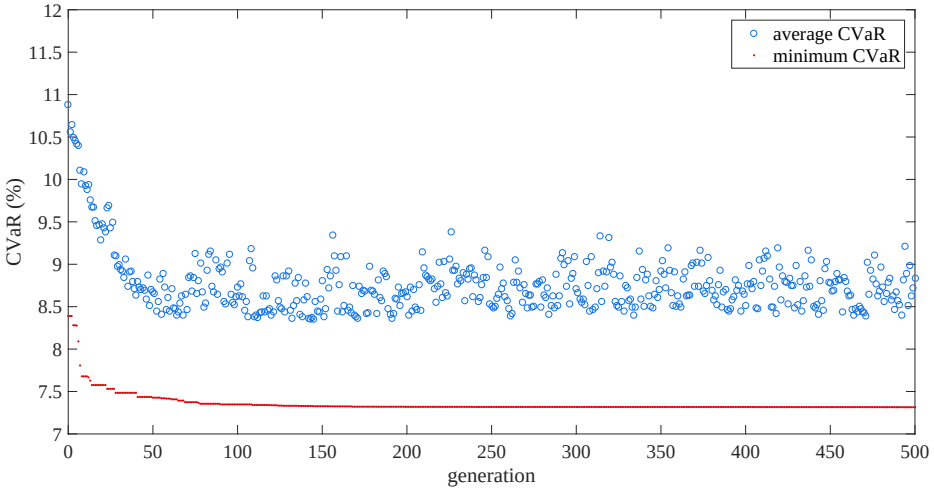
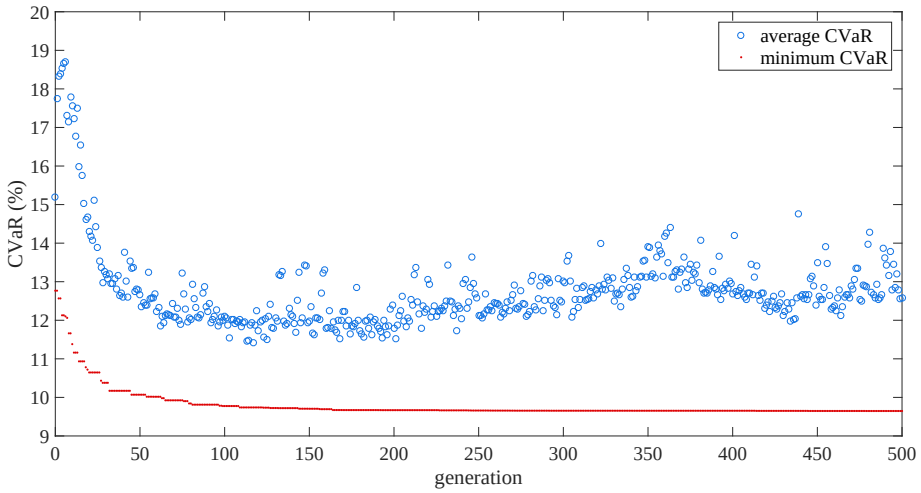


**Figure 4.11 The evolution of fitness values (CVaR) over generations when running GA on portfolio under MVN scenarios at the target returns of 1.40%.** The blue hollow circles represent the average of fitness values over an entire population and the red dots indicate the best fitness values (minimum CVaRs).




## 4.5 Experimental Results and Discussions

This section presents results of experiments designed to investigate the benefit of applying the proposed stochastic portfolio optimisation model on regular-vine-copula based scenarios.

### 4.5.1 Data

In this study, the investments of interest are in four asset classes; blue-chip stock[13] indices, government bond indices (with 1-10 years maturity), corporate bond indices (with 1-10 years maturity), commodity (gold) and currencies. There are in total eighteen countries under fourteen currencies. A portfolio invests in equity indices in all countries but invest in government bond indices in some countries (some are omitted due to small market sizes and lack of liquidity). For corporate bonds, only those issued in EUR, GBP and USD are selected as those denominated in other currencies are not actively traded. The only commodity invested in the portfolio is gold which is quoted in USD per troy ounce. All investments in a portfolio are summarised in Table 4.6.

[13] A blue-chip is stock in a corporation with a national reputation for quality, reliability, and the ability to operate profitably in good and bad times.

**Table 4.6 Currencies and asset classes to be invested in a portfolio.** The black dots indicate available investment. Investments in equities, government bonds and corporate bonds are on indices where returns are in local currency. The gold spot price (XAU) is quoted in USD.

| Currency | Country | Asset Class | | | |
|---|---|---|---|---|---|
| | | equities | government bonds | corporate bonds | commodity |
| ARS | Argentina | • | | | |
| AUD | Australia | • | • | | |
| BRL | Brazil | • | | | |
| CAD | Canada | • | • | | |
| CHF | Switzerland | • | | | |
| CLP | Chile | • | | | |
| EUR | Austria | • | • | | |
| EUR | Belgium | • | • | | |
| EUR | Germany | • | • | | |
| EUR | France | • | • | | |
| EUR | The Netherlands | • | • | | |
| EUR | all EU | | | • | |
| GBP | United Kingdom | • | • | • | |
| INR | India | • | | | |
| JPY | Japan | • | • | | |
| KRW | Korea | • | • | | |
| MXN | Mexico | • | • | | |
| SGD | Singapore | • | • | | |
| USD | United States | • | • | • | |
| XAU | | | | | • |

The description of assets and currencies invested in a portfolio is given in Table 4.7. The first column denotes an ID of each asset (used for referencing when presenting a result from R-Vine copula estimation), the second column shows a notation of each asset for brevity of presentation, The last column provides a description of all financial securities invested in our portfolio.

The monthly return data are from Jan-02 to May-15. The in-sample period used for estimating scenarios, expected returns and risks covers Jan-02 to Dec-11. The out-of-sample takes the rest of the data, from Jan-12 to May-15.

A return horizon of assets is generally set to coincide with an investment horizon of a portfolio. In this study, the investment horizon is set to cover an entire range of the out-of-sample data (Jan-12 to May-15), the return horizon for each asset in a portfolio is thus set to 3.5 years.

For scenario generation, raw return data of government and corporate bond indices are collected from MorganMarkets, raw return data of currencies, stock indices and gold are collected from Bloomberg. The box plots illustrating median and dispersion of all return data are given in Figures B.1 and B.2 in Appendix B.1.

### 4.5.2 Experimental Studies

This section aims to present the effects of different scenario generation methods and different types of cardinality constraints on risk-return profile of portfolios. All the experiments are run on Intel Core i7-5500U 3.0 GHz processor with 12.00 GB RAM under Windows 10 operating system.

#### 4.5.2.1 Parameter Settings

Key parameters for GA are set as below while the rest of GA toolbox parameters take the default values. For the optimisation model in all experiments, if not stated otherwise, the parameters are set as follows:

(a) *GA parameters:* The population size $\Theta = 500$, the number of generations $\Gamma = 500$, selection probability $\Delta = 10\%$ and crossover rate $\Lambda = 80\%$.

(b) *Model parameters:*

- An efficient frontier is created with 22 portfolios[14], with the target returns $\mu = 0.55, 0.60, \ldots, 1.60$ percent monthly.
- The confidence level of CVaR $\beta = 95\%$.
- Fixed costs for buying assets and forwards $\pi_{ij} = 0.001\%$ and $\pi_k = 0.001\%$ of portfolio values. Fixed costs for selling assets and forwards $\psi_{ij} = 0.001\%$ and $\psi_k = 0.001\%$ of portfolio values.
- Variable costs for buying assets and selling assets denominated in USD, EUR, GBP and JPY; $\rho_{ij} = 0.01\%$, $\lambda_{ij} = 0.01\%$ while $\rho_{ij} = 0.05\%$, $\lambda_{ij} = 0.05\%$ when buying or selling assets denominated in other currencies. Variable costs for buying and selling each forward pair ($\rho_k$ and $\lambda_k$) are the historical average of bid-ask spreads collected from Bloomberg.
- The minimum and maximum holding positions of each asset $a_{ij}^{min} = 0.01\%$ and $a_{ij}^{max} = 100\%$, respectively.
- The minimum and maximum currency exposures for each currency $c_j^{min} = 0.01\%$ and $c_j^{max} = 100\%$, respectively.

[14]When cardinality constraints are imposed, some return targets may not be achievable, the resulting efficient frontier could thus be comprised of fewer portfolios.

- The minimum trading size of an asset and a forward pair $a_{ij} = 0.1\%$ and $q_k = 0.1\%$. The maximum total overlay limit $V_u = 100\%$.
- The cash margin requirement to hold a forward contract $M = 10\%$. At its inception, a portfolio holds only cash of an initial amount $h^0 = 10,000,000$ US dollars.
- The cardinality constraint parameters on the number of currencies $K_C = 14$. The cardinality constraint parameters on the number of forwards $K_G = \binom{14}{2} = 91$.
- The number of scenarios (recourse nodes) $N_r = 1,000$.

#### 4.5.2.2 Scenarios Generation Results

In the study, scenarios are generated from two methods, RVC and MVN. The first one assumes no parametric form on return distribution and models asset dependence structure with R-Vine copula (see Section 4.3.1 for details). The latter assumes that asset returns are normally distributed and the dependence structure is described through linear correlation (see Section 4.2.8.1 for details). The resulting scenarios generated from the two methods are likely to be different due to different assumptions.

When generating RVC scenarios, a vine copula is constructed from a combination of 5 bivariate copula families including the rotated version of Clayton and Gumbel copulas in order to capture broader range of asset dependence structure. Table 4.8 lists all bivariate copula families used in RVC scenario construction where the ID of each copula family is given to identify fitted copula families in an R-Vine matrix.

**Table 4.8 List of bivariate copula families and ranges of associated parameters.** Student's t copula requires an additional parameter to specify its degree of freedom.

| ID | Copula family | Range of parameters | Range of additional parameters |
|---|---|---|---|
| 1 | Gaussian | (-1,1) | - |
| 2 | Student's t | (-1,1) | $(2, \infty)$ |
| 3 | Clayton | $(0, \infty)$ | - |
| 4 | Gumbel | $[1, \infty)$ | - |
| 5 | Frank | $\mathbb{R}\backslash\{0\}$ | - |
| 13 | 180-degree rotated Clayton copula | $(0, \infty)$ | - |
| 14 | 180-degree rotated Gumbel copula | $[1, \infty)$ | - |
| 23 | 90-degree rotated Clayton copula | $(-\infty, 0)$ | - |
| 24 | 90-degree rotated Gumbel copula | $(-\infty, -1]$ | - |
| 33 | 270-degree rotated Clayton copula | $(-\infty, 0)$ | - |
| 34 | 270-degree rotated Gumbel copula | $(-\infty, -1]$ | - |

The results from extracting an R-Vine from the raw data contain the R-Vine trees (the dependence structure of assets), the corresponding bivariate copula families and associated parameters. The estimation results are all stored in a matrix format.

The structure of trees in the estimated R-Vine is presented in Table 4.10. Figure 4.12 illustrates the first tree of the R-Vine which demonstrates the structure that variables are most dependent (in terms of Kendall's tau). The nodes in the first tree denotes assets and currencies in a portfolio while the edges represent bivariate copula families for the pairs of variables. Information of the first tree is stored in the diagonal and bottom rows of the R-Vine structure matrix. For example, in the first column of the R-Vine structure matrix, the diagonal element is 12 (KRW) and the bottom row is 13 (SGD), thus KRW and SGD must be connected in the first tree. Structures of the remaining trees (which denote connection of unconditional and conditional bivariate copulas) can be obtained according to the instruction given in Section 4.2.6.1. One can observe the dependence clusters of fixed-income assets (government bonds and corporate bonds), stock indices and currencies implying that assets of the same type tends to be more dependent on each other than those of different types. Among all asset classes, stock indices are connected to both the fixed-income assets and currencies. This indicate that stock indices are affected by both shocks from fixed-income and currencies and hence they are the most volatile asset class in a portfolio.

The bivariate copula families for each tree of the estimated R-Vine are shown in Table 4.11. The bivariate copula family that corresponds to each edge in each tree is located

where the second term of the conditioned variables reside (see Figure 4.4 for details). For instance, the copula family that links KRW and SGD is located at the bottom row of the first column of Table 4.11 which has an ID of 2 (Student's t copula). The parameter values of each bivariate copula family are exhibited in Tables 4.12 and 4.13. The location of parameter value in the matrix is located at the same place as the copula family ID. For example, the Student's t copula that connects KRW and SGD has $\rho = 0.47$ and a degree of freedom $\nu = 3.73$.

In order to justify if the number of scenarios specified ($N_r = 1000$ in Section 4.5.2.1) can capture the necessary characteristics of assets, the in-sample stability inspection methodology in Mitra et al. [99] and Kaut and Wallace [65] is applied. The in-sample stability of scenario sets of size 500, 1000, 1500 and 2000 are put into the test. For each scenario set size, the descriptive statistics of the 22 optimised objective values (the number of portfolios on an efficient frontier as specified in in Section 4.5.2.1) are calculated. Table 4.9 compares the descriptive statistics of CVaRs across different sizes of the scenario set for the two scenario generation methods.

**Table 4.9 In-sample stability of mean-CVaR objective value for varying sizes of the scenario set.** The statistics reported of each scenario set are calculated from 22 observations. The objective value is the value of minimum CVaR of a portfolio at each return target.

| | number of scenarios (RVC) | | | | number of scenarios (MVN) | | | |
|---|---|---|---|---|---|---|---|---|
| | 500 | 1000 | 1500 | 2000 | 500 | 1000 | 1500 | 2000 |
| average | 0.086 | 0.070 | 0.071 | 0.070 | 0.072 | 0.067 | 0.067 | 0.068 |
| standard deviation | 0.056 | 0.032 | 0.032 | 0.032 | 0.034 | 0.027 | 0.027 | 0.027 |
| range | 0.103 | 0.101 | 0.102 | 0.102 | 0.082 | 0.081 | 0.081 | 0.082 |
| min | 0.038 | 0.033 | 0.032 | 0.033 | 0.040 | 0.031 | 0.030 | 0.030 |
| max | 0.141 | 0.134 | 0.134 | 0.135 | 0.122 | 0.112 | 0.111 | 0.112 |

It can be seen that when the number of scenarios reaches 1000 or more, the objective values (minimum CVaRs) are effectively identical, as opposed to when the number of scenarios is 500. This is consistent with the notion of decision stability of Kaut and Wallace [65] that "good" scenarios should lead to stable decision. In our context, a stable decision is that portfolio allocation does not change when the number of scenarios varies. The results from Table 4.9 point out that the number of scenarios fewer than 1000 demonstrates unstable outcomes comparing to the case of larger than 1000 scenarios. Therefore, our earlier setting of 1000 scenarios should convincingly provide stable portfolio decisions.

The comparison of descriptive statistics of asset returns under the scenario set of size 1000 is given in Table 4.14. Note that returns shown in the table are produced with interest-rate-adjusted to take into account the cost-of-carry of FX forwards (see equation (3.13) in Chapter 3). By comparing the minimum, mean and maximum returns of RVC and MVN scenarios to the raw data, it is noticed that assuming normal distribution on scenarios somewhat distorts the original characteristics of asset returns. On the contrary, in our approach where asset return distribution is not assumed and asset dependence structure is modelled by vine copula, the minimum, mean and maximum returns of raw data are relatively well preserved. The difference in generated scenarios causes different solutions for the optimisation problem. Comparison of resulting portfolios from RVC and MVN scenarios is presented in Section 4.5.2.5.

**Table 4.7 List of assets and currencies in a portfolio.**

| ID | Notation | Description |
|---|---|---|
| 1 | ARS | Argentinian Peso |
| 2 | BRL | Brazillian Lira |
| 3 | CAD | Canadian Dollar |
| 4 | CLP | Chilean Peso |
| 5 | MXN | Mexican Peso |
| 6 | EUR | Euro |
| 7 | GBP | Sterling Pound |
| 8 | CHF | Swiss Franc |
| 9 | AUD | Australian Dollar |
| 10 | INR | Indian Rupee |
| 11 | JPY | Japanese Yen |
| 12 | KRW | South Korean Won |
| 13 | SGD | Singapore Dollar |
| 14 | govt.AU | Australian government bond index (1-10 years) |
| 15 | govt.AT | Austria government bond index (1-10 years) |
| 16 | govt.BE | Belgium government bond index (1-10 years) |
| 17 | govt.CA | Canada government bond index (1-10 years) |
| 18 | govt.FR | France government bond index (1-10 years) |
| 19 | govt.DE | Germany government bond index (1-10 years) |
| 20 | govt.KR | South Korean government bond index (1-10 years) |
| 21 | govt.JP | Japan government bond index (1-10 years) |
| 22 | govt.MX | Mexico government bond index (1-10 years) |
| 23 | govt.NL | The Netherlands government bond index (1-10 years) |
| 24 | govt.SG | Singapore government bond index (1-10 years) |
| 25 | govt.UK | UK government bond index (1-10 years) |
| 26 | govt.US | US government bond index (1-10 years) |
| 27 | corp.EU | European corporate bond index (1-10 years) |
| 28 | corp.UK | UK corporate bond index (1-10 years) |
| 29 | corp.US | US corporate bond index (1-10 years) |
| 30 | stock.NL | The Netherlands stock index (AEX) |
| 31 | stock.AT | Austria stock index (ATX) |
| 32 | stock.AU | Australia stock index top-200 (ASX200) |
| 33 | stock.BE | Belgium stock index top-20 (BEL20) |
| 34 | stock.IN | India stock index top-50 (NIFTY50) |
| 35 | stock.BR | Brazil stock index (BOVESPA) |
| 36 | stock.FR | France stock index top-40 (CAC40) |
| 37 | stock.UK | UK stock index top-100 (FTSE100) |
| 38 | stock.DE | Germany stock index (DAX) |
| 39 | stock.US | US stock index top-500 (S&P500) |
| 40 | stock.CA | Canada stock index top-60 (S&P/TSX 60) |
| 41 | stock.CL | Chile stock index (IPSA) |
| 42 | stock.KR | South Korea stock index (KOSPI) |
| 43 | stock.AR | Argentina stock index (MERVAL) |
| 44 | stock.MX | Mexico stock index (IPC) |
| 45 | stock.JP | Japan stock index top-225 (NIKKEI225) |
| 46 | stock.CH | Switzerland stock index (SMI) |
| 47 | stock.SG | Singapore stock index (STI) |
| 48 | XAU | Gold (USD per troy ounce) |

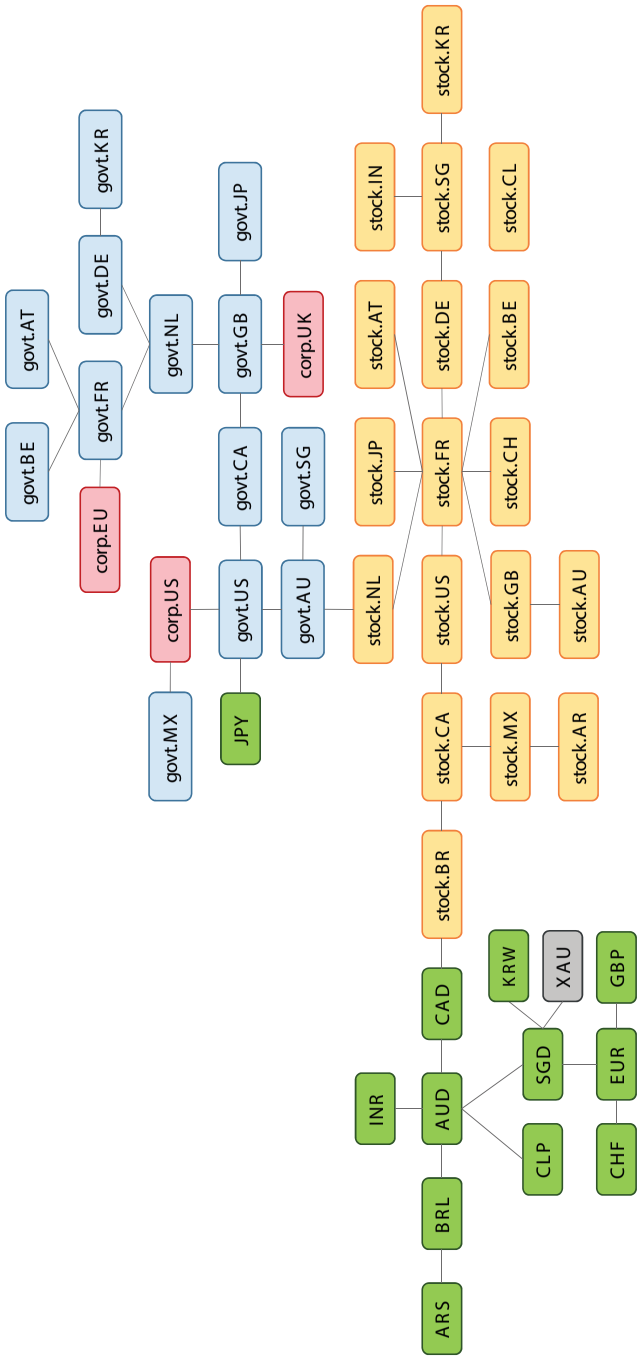



**Figure 4.12 The first tree ($T_1$) structure of the estimated R-Vine tree.** Currencies are in green boxes, government bond indices are in blue boxes, corporate bond indices are in pink boxes, stock indices are in orange boxes and gold is in a grey box. Structure of the other trees can be extracted from the R-Vine matrix shown in Table 4.10.

**Table 4.10 Matrix representation of the R-Vine tree from the estimated R-Vine specification.** Numbers in parentheses indicate rows and columns of the matrix. Numbers in the matrix are ID of variables as listed in Table 4.7.

| | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | (11) | (12) | (13) | (14) | (15) | (16) | (17) | (18) | (19) | (20) | (21) | (22) | (23) | (24) | (25) | (26) | (27) | (28) | (29) | (30) | (31) | (32) | (33) | (34) | (35) | (36) | (37) | (38) | (39) | (40) | (41) | (42) | (43) | (44) | (45) | (46) | (47) | (48) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| (1) | 12 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (2) | 20 | 7 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (3) | 21 | 20 | 8 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (4) | 11 | 21 | 20 | 6 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (5) | 24 | 11 | 21 | 20 | 43 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (6) | 22 | 24 | 11 | 21 | 20 | 48 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (7) | 28 | 22 | 24 | 11 | 21 | 20 | 13 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (8) | 45 | 28 | 22 | 24 | 11 | 21 | 20 | 10 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (9) | 31 | 45 | 28 | 22 | 24 | 11 | 21 | 20 | 4 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (10) | 41 | 31 | 45 | 28 | 22 | 24 | 11 | 21 | 20 | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (11) | 42 | 41 | 31 | 45 | 28 | 22 | 24 | 11 | 21 | 20 | 2 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (12) | 34 | 42 | 41 | 31 | 45 | 28 | 22 | 24 | 11 | 21 | 20 | 9 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (13) | 16 | 34 | 42 | 41 | 31 | 45 | 28 | 22 | 24 | 11 | 21 | 20 | 20 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (14) | 27 | 16 | 34 | 42 | 41 | 31 | 45 | 28 | 22 | 24 | 11 | 21 | 3 | 21 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (15) | 15 | 27 | 16 | 34 | 42 | 41 | 31 | 45 | 28 | 22 | 24 | 11 | 35 | 3 | 11 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (16) | 18 | 15 | 27 | 16 | 34 | 42 | 41 | 31 | 45 | 28 | 22 | 24 | 44 | 35 | 3 | 24 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (17) | 29 | 18 | 15 | 27 | 16 | 34 | 42 | 41 | 31 | 45 | 28 | 22 | 5 | 44 | 35 | 3 | 22 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (18) | 19 | 29 | 18 | 15 | 27 | 16 | 34 | 42 | 41 | 31 | 45 | 28 | 45 | 5 | 44 | 35 | 3 | 28 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (19) | 23 | 19 | 29 | 18 | 15 | 27 | 16 | 34 | 42 | 41 | 31 | 45 | 31 | 45 | 5 | 44 | 35 | 3 | 3 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (20) | 25 | 23 | 19 | 29 | 18 | 15 | 27 | 16 | 34 | 42 | 41 | 31 | 41 | 31 | 45 | 5 | 44 | 35 | 45 | 35 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (21) | 47 | 25 | 23 | 19 | 29 | 18 | 15 | 27 | 16 | 34 | 42 | 41 | 21 | 41 | 31 | 45 | 5 | 44 | 31 | 45 | 44 | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (22) | 33 | 47 | 25 | 23 | 19 | 29 | 18 | 15 | 27 | 16 | 34 | 42 | 42 | 42 | 41 | 31 | 45 | 5 | 41 | 31 | 45 | 5 | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (23) | 17 | 33 | 47 | 25 | 23 | 19 | 29 | 18 | 15 | 27 | 16 | 34 | 34 | 34 | 42 | 41 | 31 | 45 | 42 | 41 | 31 | 45 | 16 | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (24) | 26 | 17 | 33 | 47 | 25 | 23 | 19 | 29 | 18 | 15 | 27 | 16 | 47 | 47 | 34 | 42 | 41 | 31 | 34 | 42 | 41 | 31 | 45 | 27 | | | | | | | | | | | | | | | | | | | | | | | | |
| (25) | 14 | 26 | 17 | 33 | 47 | 25 | 23 | 19 | 29 | 18 | 15 | 27 | 11 | 33 | 47 | 34 | 42 | 41 | 16 | 34 | 42 | 41 | 31 | 45 | 15 | | | | | | | | | | | | | | | | | | | | | | |
| (26) | 5 | 14 | 26 | 17 | 33 | 47 | 25 | 23 | 19 | 29 | 18 | 15 | 33 | 40 | 33 | 47 | 34 | 42 | 27 | 16 | 34 | 42 | 41 | 31 | 45 | 18 | | | | | | | | | | | | | | | | | | | | | |
| (27) | 30 | 5 | 14 | 26 | 17 | 33 | 47 | 25 | 23 | 19 | 29 | 18 | 40 | 28 | 40 | 33 | 47 | 34 | 15 | 27 | 16 | 34 | 42 | 41 | 31 | 45 | 29 | | | | | | | | | | | | | | | | | | | | |
| (28) | 46 | 30 | 5 | 14 | 26 | 17 | 33 | 47 | 25 | 23 | 19 | 29 | 24 | 16 | 28 | 40 | 33 | 47 | 18 | 15 | 27 | 16 | 34 | 42 | 41 | 31 | 45 | 19 | | | | | | | | | | | | | | | | | | | |
| (29) | 38 | 46 | 30 | 5 | 14 | 26 | 17 | 33 | 47 | 25 | 23 | 19 | 22 | 27 | 16 | 28 | 40 | 33 | 29 | 18 | 15 | 27 | 47 | 34 | 42 | 41 | 31 | 45 | 23 | | | | | | | | | | | | | | | | | | |
| (30) | 32 | 38 | 46 | 30 | 5 | 14 | 26 | 17 | 33 | 47 | 25 | 23 | 32 | 15 | 27 | 16 | 28 | 40 | 19 | 29 | 18 | 15 | 33 | 47 | 34 | 42 | 41 | 31 | 45 | 25 | | | | | | | | | | | | | | | | | |
| (31) | 37 | 32 | 38 | 46 | 30 | 5 | 14 | 26 | 17 | 33 | 47 | 25 | 37 | 18 | 15 | 27 | 16 | 32 | 23 | 19 | 29 | 18 | 40 | 33 | 47 | 34 | 42 | 41 | 31 | 45 | 40 | | | | | | | | | | | | | | | | |
| (32) | 43 | 37 | 32 | 38 | 46 | 30 | 5 | 14 | 26 | 17 | 33 | 47 | 39 | 32 | 18 | 15 | 27 | 37 | 25 | 23 | 19 | 29 | 32 | 40 | 33 | 47 | 34 | 42 | 41 | 31 | 45 | 17 | | | | | | | | | | | | | | | |
| (33) | 36 | 43 | 37 | 32 | 38 | 46 | 30 | 5 | 14 | 26 | 17 | 33 | 38 | 37 | 32 | 18 | 15 | 39 | 47 | 25 | 23 | 19 | 37 | 32 | 40 | 33 | 47 | 34 | 42 | 41 | 31 | 45 | 26 | | | | | | | | | | | | | | |
| (34) | 39 | 36 | 43 | 37 | 32 | 38 | 46 | 30 | 5 | 14 | 26 | 17 | 46 | 39 | 37 | 32 | 18 | 38 | 33 | 47 | 25 | 23 | 39 | 37 | 32 | 40 | 33 | 47 | 34 | 42 | 41 | 31 | 45 | 14 | | | | | | | | | | | | | |
| (35) | 44 | 39 | 36 | 43 | 37 | 32 | 38 | 46 | 30 | 5 | 14 | 26 | 36 | 38 | 39 | 37 | 32 | 46 | 17 | 33 | 47 | 25 | 38 | 39 | 37 | 32 | 40 | 33 | 47 | 34 | 42 | 41 | 31 | 45 | 30 | | | | | | | | | | | | |
| (36) | 40 | 44 | 39 | 36 | 36 | 37 | 32 | 38 | 46 | 30 | 5 | 14 | 30 | 46 | 38 | 39 | 37 | 36 | 26 | 17 | 33 | 47 | 46 | 38 | 39 | 37 | 32 | 40 | 33 | 47 | 34 | 42 | 41 | 31 | 45 | 45 | | | | | | | | | | | |
| (37) | 35 | 40 | 44 | 39 | 39 | 36 | 37 | 32 | 38 | 46 | 30 | 5 | 14 | 36 | 46 | 38 | 39 | 30 | 14 | 26 | 17 | 33 | 36 | 46 | 38 | 39 | 37 | 32 | 40 | 33 | 47 | 34 | 42 | 41 | 31 | 46 | 31 | | | | | | | | | | |
| (38) | 7 | 35 | 40 | 44 | 48 | 39 | 36 | 37 | 32 | 38 | 46 | 30 | 29 | 30 | 36 | 46 | 38 | 14 | 5 | 14 | 26 | 17 | 30 | 36 | 46 | 38 | 39 | 37 | 32 | 40 | 33 | 47 | 34 | 42 | 41 | 41 | 46 | 46 | | | | | | | | | |
| (39) | 8 | 3 | 35 | 40 | 13 | 44 | 39 | 36 | 37 | 32 | 38 | 46 | 26 | 24 | 30 | 36 | 46 | 29 | 30 | 5 | 14 | 26 | 14 | 30 | 36 | 46 | 38 | 39 | 37 | 32 | 17 | 33 | 47 | 34 | 42 | 42 | 41 | 41 | 33 | | | | | | | | |
| (40) | 3 | 1 | 3 | 35 | 10 | 40 | 44 | 39 | 36 | 37 | 32 | 38 | 17 | 22 | 24 | 30 | 36 | 26 | 46 | 30 | 5 | 14 | 29 | 14 | 30 | 36 | 46 | 38 | 39 | 37 | 26 | 32 | 33 | 47 | 34 | 34 | 42 | 42 | 41 | 32 | | | | | | | |
| (41) | 1 | 4 | 1 | 3 | 4 | 35 | 40 | 44 | 39 | 36 | 37 | 32 | 16 | 29 | 22 | 14 | 30 | 17 | 38 | 46 | 30 | 30 | 26 | 29 | 14 | 30 | 36 | 46 | 38 | 39 | 14 | 37 | 32 | 33 | 47 | 47 | 34 | 34 | 42 | 41 | 37 | | | | | | |
| (42) | 4 | 2 | 4 | 1 | 1 | 3 | 35 | 40 | 44 | 39 | 36 | 37 | 27 | 19 | 29 | 19 | 14 | 16 | 32 | 38 | 46 | 46 | 17 | 26 | 29 | 14 | 30 | 36 | 46 | 38 | 30 | 39 | 37 | 32 | 33 | 38 | 47 | 47 | 34 | 42 | 41 | 39 | | | | | |
| (43) | 2 | 10 | 2 | 4 | 2 | 1 | 3 | 35 | 40 | 44 | 39 | 36 | 15 | 14 | 19 | 23 | 19 | 27 | 37 | 32 | 38 | 38 | 25 | 17 | 26 | 29 | 14 | 30 | 36 | 46 | 46 | 38 | 39 | 37 | 32 | 39 | 38 | 33 | 47 | 34 | 42 | 41 | 36 | | | | |
| (44) | 10 | 48 | 10 | 2 | 9 | 4 | 1 | 3 | 35 | 40 | 44 | 39 | 18 | 23 | 14 | 25 | 23 | 15 | 36 | 37 | 32 | 32 | 27 | 25 | 17 | 26 | 19 | 14 | 30 | 36 | 38 | 46 | 38 | 39 | 37 | 32 | 39 | 32 | 38 | 47 | 34 | 42 | 41 | 34 | | | |
| (45) | 48 | 8 | 48 | 10 | 3 | 2 | 4 | 1 | 3 | 35 | 40 | 44 | 28 | 11 | 23 | 17 | 25 | 18 | 39 | 36 | 37 | 37 | 19 | 19 | 25 | 17 | 23 | 26 | 14 | 30 | 32 | 36 | 46 | 38 | 39 | 37 | 32 | 37 | 39 | 38 | 47 | 34 | 42 | 41 | 38 | | |
| (46) | 9 | 9 | 9 | 48 | 35 | 10 | 2 | 4 | 1 | 3 | 35 | 40 | 25 | 26 | 25 | 22 | 17 | 19 | 44 | 39 | 36 | 40 | 23 | 15 | 19 | 25 | 25 | 17 | 26 | 14 | 37 | 30 | 36 | 46 | 38 | 33 | 37 | 39 | 32 | 39 | 38 | 47 | 34 | 42 | 41 | 41 | |
| (47) | 6 | 13 | 13 | 9 | 40 | 9 | 10 | 2 | 2 | 9 | 3 | 35 | 23 | 17 | 17 | 29 | 26 | 23 | 40 | 44 | 39 | 36 | 15 | 23 | 23 | 19 | 17 | 25 | 17 | 26 | 36 | 14 | 30 | 36 | 46 | 31 | 33 | 38 | 37 | 36 | 39 | 38 | 47 | 38 | 42 | 42 | 47 | |
| (48) | 13 | 6 | 6 | 13 | 44 | 13 | 9 | 9 | 9 | 2 | 9 | 3 | 19 | 25 | 26 | 26 | 29 | 25 | 35 | 40 | 40 | 39 | 18 | 18 | 18 | 23 | 26 | 23 | 25 | 17 | 39 | 26 | 14 | 30 | 36 | 36 | 36 | 36 | 36 | 37 | 36 | 36 | 38 | 47 | 47 | 47 | 42 | 42 |

**Table 4.11 Matrix representation of bivariate copula families from the estimated R-Vine specification.** Numbers in parentheses indicate rows and columns of the matrix. Numbers in the matrix are ID of bivariate copula families as listed in Table 4.8.

| | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | (11) | (12) | (13) | (14) | (15) | (16) | (17) | (18) | (19) | (20) | (21) | (22) | (23) | (24) | (25) | (26) | (27) | (28) | (29) | (30) | (31) | (32) | (33) | (34) | (35) | (36) | (37) | (38) | (39) | (40) | (41) | (42) | (43) | (44) | (45) | (46) | (47) | (48) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| (1) | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (2) | 33 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (3) | 1 | 5 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (4) | 23 | 5 | 34 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (5) | 5 | 23 | 5 | 2 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (6) | 5 | 3 | 24 | 3 | 23 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (7) | 14 | 2 | 3 | 1 | 23 | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (8) | 23 | 13 | 5 | 14 | 5 | 13 | 14 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (9) | 23 | 1 | 5 | 5 | 33 | 3 | 14 | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (10) | 14 | 14 | 1 | 14 | 5 | 5 | 4 | 13 | 23 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (11) | 23 | 5 | 5 | 5 | 3 | 13 | 3 | 13 | 4 | 24 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (12) | 5 | 3 | 23 | 23 | 34 | 23 | 23 | 5 | 2 | 5 | 13 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (13) | 23 | 24 | 1 | 13 | 5 | 33 | 13 | 3 | 13 | 4 | 1 | 23 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (14) | 5 | 2 | 33 | 5 | 5 | 13 | 3 | 13 | 23 | 34 | 14 | 1 | 14 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (15) | 14 | 5 | 1 | 5 | 3 | 3 | 5 | 5 | 3 | 5 | 5 | 5 | 5 | 3 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (16) | 33 | 14 | 14 | 1 | 5 | 34 | 24 | 2 | 5 | 1 | 5 | 13 | 4 | 1 | 13 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (17) | 3 | 5 | 34 | 3 | 3 | 1 | 14 | 1 | 5 | 5 | 4 | 23 | 4 | 5 | 3 | 13 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (18) | 24 | 34 | 34 | 24 | 5 | 5 | 1 | 33 | 1 | 14 | 13 | 5 | 23 | 1 | 4 | 13 | 3 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (19) | 3 | 14 | 33 | 33 | 33 | 14 | 33 | 3 | 14 | 2 | 5 | 5 | 5 | 4 | 13 | 33 | 4 | 33 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (20) | 14 | 24 | 13 | 5 | 3 | 33 | 33 | 3 | 14 | 23 | 5 | 3 | 4 | 23 | 34 | 23 | 14 | 5 | 5 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (21) | 5 | 4 | 33 | 5 | 2 | 3 | 1 | 24 | 1 | 1 | 2 | 14 | 1 | 4 | 1 | 34 | 23 | 3 | 1 | 5 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (22) | 4 | 23 | 14 | 14 | 5 | 5 | 33 | 3 | 2 | 23 | 3 | 1 | 5 | 2 | 3 | 13 | 24 | 5 | 1 | 5 | 33 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (23) | 5 | 24 | 1 | 33 | 1 | 23 | 4 | 5 | 2 | 1 | 4 | 1 | 14 | 3 | 4 | 5 | 23 | 5 | 5 | 3 | 13 | 5 | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (24) | 1 | 5 | 5 | 14 | 34 | 24 | 1 | 5 | 13 | 23 | 1 | 23 | 5 | 5 | 1 | 1 | 4 | 5 | 13 | 33 | 1 | 5 | 13 | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (25) | 14 | 34 | 14 | 5 | 1 | 5 | 33 | 4 | 4 | 5 | 33 | 5 | 1 | 3 | 24 | 3 | 13 | 5 | 5 | 24 | 1 | 5 | 24 | 3 | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (26) | 33 | 14 | 4 | 23 | 1 | 5 | 14 | 13 | 23 | 33 | 5 | 3 | 24 | 1 | 33 | 5 | 13 | 3 | 13 | 5 | 5 | 1 | 5 | 5 | 24 | | | | | | | | | | | | | | | | | | | | | | | | |
| (27) | 34 | 5 | 5 | 3 | 23 | 33 | 5 | 3 | 4 | 1 | 34 | 33 | 13 | 34 | 13 | 3 | 23 | 1 | 24 | 13 | 23 | 5 | 5 | 33 | 3 | 3 | | | | | | | | | | | | | | | | | | | | | | |
| (28) | 5 | 34 | 1 | 33 | 23 | 1 | 23 | 5 | 13 | 14 | 33 | 3 | 4 | 1 | 23 | 13 | 23 | 3 | 2 | 5 | 13 | 5 | 34 | 1 | 1 | 33 | 5 | | | | | | | | | | | | | | | | | | | | | |
| (29) | 33 | 33 | 5 | 2 | 2 | 14 | 33 | 33 | 13 | 5 | 5 | 3 | 33 | 3 | 3 | 33 | 13 | 5 | 1 | 5 | 3 | 5 | 34 | 1 | 1 | 34 | 5 | 33 | | | | | | | | | | | | | | | | | | | | |
| (30) | 34 | 1 | 3 | 5 | 14 | 5 | 13 | 23 | 24 | 23 | 23 | 5 | 3 | 4 | 5 | 1 | 13 | 5 | 34 | 33 | 3 | 5 | 3 | 3 | 2 | 3 | 23 | 5 | 13 | | | | | | | | | | | | | | | | | | | |
| (31) | 23 | 5 | 14 | 13 | 13 | 24 | 24 | 1 | 13 | 24 | 5 | 23 | 5 | 33 | 5 | 13 | 4 | 33 | 1 | 1 | 23 | 3 | 24 | 14 | 33 | 4 | 34 | 24 | 4 | 1 | | | | | | | | | | | | | | | | | | |
| (32) | 33 | 34 | 1 | 3 | 3 | 3 | 1 | 13 | 33 | 1 | 4 | 5 | 33 | 1 | 5 | 3 | 5 | 1 | 33 | 5 | 33 | 3 | 14 | 1 | 1 | 5 | 1 | 13 | 34 | 3 | 5 | | | | | | | | | | | | | | | | | |
| (33) | 1 | 5 | 1 | 34 | 5 | 5 | 23 | 5 | 1 | 5 | 33 | 2 | 3 | 34 | 23 | 5 | 1 | 5 | 5 | 13 | 2 | 13 | 1 | 1 | 2 | 3 | 34 | 5 | 5 | 3 | 5 | 24 | | | | | | | | | | | | | | | | |
| (34) | 5 | 34 | 5 | 5 | 23 | 24 | 3 | 5 | 5 | 14 | 13 | 4 | 5 | 23 | 4 | 13 | 5 | 2 | 33 | 5 | 4 | 5 | 3 | 1 | 23 | 14 | 5 | 4 | 3 | 1 | 1 | 23 | 3 | | | | | | | | | | | | | | | |
| (35) | 5 | 23 | 1 | 14 | 3 | 1 | 14 | 13 | 34 | 34 | 3 | 14 | 24 | 23 | 3 | 3 | 14 | 2 | 14 | 33 | 5 | 1 | 5 | 23 | 1 | 5 | 2 | 5 | 13 | 34 | 5 | 1 | 5 | 4 | | | | | | | | | | | | | | |
| (36) | 1 | 5 | 5 | 33 | 24 | 33 | 5 | 4 | 14 | 13 | 33 | 5 | 1 | 5 | 5 | 2 | 5 | 23 | 34 | 2 | 13 | 1 | 3 | 23 | 1 | 33 | 3 | 5 | 5 | 2 | 3 | 5 | 34 | 13 | 33 | | | | | | | | | | | | | |
| (37) | 13 | 5 | 5 | 13 | 13 | 1 | 2 | 13 | 5 | 33 | 1 | 1 | 24 | 1 | 1 | 3 | 24 | 3 | 3 | 5 | 13 | 4 | 13 | 4 | 5 | 5 | 1 | 3 | 5 | 13 | 3 | 1 | 5 | 5 | 5 | 5 | | | | | | | | | | | | |
| (38) | 13 | 1 | 23 | 4 | 33 | 3 | 23 | 33 | 24 | 33 | 1 | 34 | 23 | 14 | 1 | 5 | 13 | 23 | 13 | 5 | 5 | 24 | 23 | 33 | 1 | 24 | 34 | 1 | 13 | 23 | 13 | 5 | 23 | 14 | 33 | 3 | 13 | | | | | | | | | | | |
| (39) | 4 | 33 | 33 | 5 | 5 | 5 | 33 | 24 | 3 | 23 | 2 | 33 | 5 | 3 | 13 | 5 | 5 | 13 | 5 | 3 | 23 | 5 | 34 | 5 | 23 | 5 | 1 | 3 | 1 | 14 | 1 | 5 | 5 | 4 | 5 | 3 | 4 | 23 | | | | | | | | | | |
| (40) | 4 | 5 | 5 | 14 | 14 | 3 | 23 | 3 | 33 | 14 | 34 | 3 | 13 | 5 | 3 | 1 | 5 | 1 | 5 | 1 | 14 | 34 | 33 | 1 | 1 | 3 | 23 | 33 | 33 | 5 | 5 | 2 | 3 | 23 | 3 | 5 | 14 | 5 | 34 | | | | | | | | |
| (41) | 5 | 14 | 4 | 5 | 14 | 5 | 14 | 14 | 5 | 5 | 1 | 13 | 1 | 4 | 3 | 5 | 24 | 5 | 5 | 2 | 2 | 1 | 13 | 4 | 34 | 1 | 1 | 13 | 1 | 34 | 34 | 13 | 5 | 1 | 5 | 14 | 5 | 24 | 13 | 13 | | | | | | | |
| (42) | 33 | 1 | 5 | 24 | 33 | 4 | 33 | 13 | 14 | 24 | 4 | 13 | 14 | 4 | 24 | 2 | 2 | 3 | 3 | 14 | 13 | 4 | 5 | 5 | 14 | 24 | 33 | 5 | 5 | 23 | 1 | 5 | 2 | 5 | 1 | 4 | 3 | 4 | 24 | 3 | 4 | | | | | | |
| (43) | 5 | 5 | 24 | 14 | 14 | 5 | 24 | 23 | 33 | 2 | 3 | 5 | 24 | 13 | 34 | 4 | 14 | 33 | 2 | 33 | 4 | 5 | 4 | 5 | 3 | 24 | 5 | 5 | 3 | 23 | 5 | 24 | 2 | 34 | 13 | 33 | 13 | 5 | 4 | 13 | 13 | 3 | | | | | |
| (44) | 23 | 2 | 33 | 5 | 5 | 2 | 23 | 24 | 13 | 2 | 3 | 5 | 5 | 3 | 34 | 4 | 5 | 4 | 5 | 4 | 5 | 2 | 13 | 5 | 2 | 5 | 34 | 23 | 5 | 5 | 4 | 5 | 23 | 23 | 3 | 3 | 1 | 23 | 13 | 23 | 34 | 5 | 3 | | | | |
| (45) | 2 | 13 | 14 | 13 | 34 | 13 | 34 | 3 | 2 | 2 | 3 | 3 | 2 | 3 | 4 | 23 | 5 | 24 | 2 | 23 | 2 | 5 | 33 | 3 | 34 | 4 | 4 | 33 | 5 | 5 | 23 | 24 | 1 | 5 | 5 | 5 | 5 | 33 | 1 | 1 | 5 | 3 | 4 | 14 | | | |
| (46) | 5 | 1 | 13 | 5 | 5 | 1 | 24 | 3 | 5 | 3 | 2 | 3 | 4 | 5 | 5 | 14 | 13 | 2 | 1 | 2 | 3 | 1 | 5 | 23 | 33 | 4 | 14 | 4 | 13 | 2 | 2 | 33 | 13 | 13 | 4 | 13 | 1 | 23 | 1 | 14 | 3 | 13 | 14 | 33 | 4 | | |
| (47) | 3 | 14 | 1 | 5 | 4 | 4 | 4 | 5 | 14 | 23 | 33 | 5 | 5 | 3 | 2 | 4 | 14 | 2 | 2 | 2 | 2 | 3 | 3 | 2 | 2 | 1 | 1 | 2 | 5 | 13 | 1 | 1 | 3 | 2 | 5 | 5 | 3 | 5 | 3 | 4 | 5 | 14 | 1 | 14 | 14 | 1 | |
| (48) | 2 | 2 | 2 | 14 | 2 | 1 | 2 | 14 | 14 | 5 | 14 | 1 | 2 | 2 | 4 | 14 | 4 | 2 | 5 | 2 | 2 | 2 | 5 | 2 | 2 | 2 | 2 | 1 | 14 | 2 | 1 | 24 | 1 | 14 | 14 | 1 | 14 | 14 | 14 | 14 | 14 | 14 | 2 | 2 | 1 | 14 | 14 |

**Table 4.12 Matrix representation of parameters corresponding to bivariate copula families of the estimated R-Vine specification.** Numbers in parentheses indicate rows and columns of the matrix. Numbers in the matrix are parameter values.

| | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | (11) | (12) | (13) | (14) | (15) | (16) | (17) | (18) | (19) | (20) | (21) | (22) | (23) | (24) | (25) | (26) | (27) | (28) | (29) | (30) | (31) | (32) | (33) | (34) | (35) | (36) | (37) | (38) | (39) | (40) | (41) | (42) | (43) | (44) | (45) | (46) | (47) | (48) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| (1) | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (2) | -0.09 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (3) | 0.02 | 0.54 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (4) | -0.03 | 1.29 | -1.04 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (5) | -0.37 | -0.11 | 0.69 | 0.00 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (6) | -0.33 | 0.13 | -1.03 | 0.05 | -0.34 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (7) | 1.04 | -0.04 | 0.05 | -0.22 | -0.01 | 0.07 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (8) | -0.03 | 0.12 | 0.25 | 1.03 | 0.48 | 0.08 | 1.02 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (9) | -0.13 | 0.20 | -0.60 | 0.35 | -0.35 | 0.05 | 1.12 | -0.03 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (10) | 1.02 | 1.04 | -0.08 | 1.14 | 0.57 | -0.29 | 1.07 | 0.12 | -0.04 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (11) | -0.05 | -0.45 | 0.99 | -0.23 | 0.09 | 0.08 | 0.14 | 0.16 | 1.02 | -1.05 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (12) | 0.47 | 0.20 | -0.04 | -0.11 | -1.02 | -0.23 | -0.03 | 0.51 | 0.04 | 0.29 | 0.06 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (13) | -0.07 | -1.05 | -0.11 | 0.15 | 0.25 | -0.01 | 0.10 | 0.11 | 0.19 | 1.08 | -0.17 | -0.09 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (14) | -0.93 | -0.05 | -0.03 | -0.44 | 0.55 | 0.06 | 0.06 | 0.03 | -0.14 | -1.05 | 1.06 | -0.11 | 1.09 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (15) | 1.16 | 0.19 | 0.02 | 0.56 | 0.25 | 0.13 | 0.54 | -0.63 | 0.25 | -0.70 | 0.80 | 0.12 | 1.30 | 0.09 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (16) | -0.16 | 1.11 | 1.11 | 0.13 | 1.20 | -1.07 | -1.01 | -0.23 | 0.05 | -0.02 | -0.28 | 0.29 | 1.13 | 0.16 | 0.16 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (17) | 0.09 | -0.46 | -1.01 | 0.08 | 0.17 | 0.05 | 1.08 | 0.07 | 0.38 | 0.49 | 1.07 | -0.26 | 1.07 | -1.45 | 0.20 | 0.13 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (18) | -1.03 | -1.11 | -1.06 | -1.00 | -2.18 | 0.54 | 0.06 | -0.01 | -0.18 | 1.05 | 0.13 | -0.40 | -0.05 | 0.01 | 1.06 | 0.09 | 0.18 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (19) | 0.18 | 1.02 | -0.22 | -0.07 | -0.19 | 1.06 | -0.15 | 0.12 | 1.02 | 0.04 | -0.65 | 0.16 | 2.38 | 1.05 | 0.16 | 0.00 | 1.04 | -0.04 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (20) | 1.08 | -1.04 | 0.20 | 0.34 | 0.04 | -0.13 | -0.01 | 0.13 | 1.04 | -0.06 | 0.12 | 0.28 | 1.02 | -0.05 | -1.27 | -0.23 | 1.08 | -0.24 | 1.11 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (21) | -0.29 | 1.04 | -0.12 | 0.20 | -0.04 | 0.05 | -0.18 | -1.05 | -0.20 | -0.05 | -0.07 | 1.05 | 0.15 | 1.09 | 0.02 | -1.04 | -0.10 | 0.01 | 0.09 | 0.38 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (22) | 1.04 | -0.04 | 1.03 | 1.03 | 0.40 | -0.55 | -0.12 | 0.11 | -0.04 | -0.09 | 0.07 | 0.14 | -0.40 | -0.02 | 0.16 | 0.12 | -1.04 | 0.48 | 0.02 | -0.61 | -0.12 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (23) | -0.47 | -1.14 | -0.13 | -0.07 | 0.05 | -0.04 | 1.02 | -0.68 | -0.01 | 0.04 | 1.03 | -0.10 | 1.11 | 0.13 | 1.05 | -0.51 | -0.05 | 0.80 | -0.16 | 0.07 | 0.18 | 0.59 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (24) | -0.10 | -0.31 | 0.96 | 1.05 | -1.02 | -1.04 | -0.03 | 2.23 | 0.14 | -0.06 | -0.16 | -0.02 | 0.75 | 0.58 | 0.02 | -0.06 | 1.14 | 0.75 | 0.07 | -0.01 | -0.21 | -0.18 | 0.09 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (25) | 1.12 | -1.09 | 1.07 | -0.21 | -0.16 | 0.67 | -0.02 | 1.06 | 1.06 | 0.96 | -0.03 | 0.53 | 0.04 | 0.14 | -1.02 | 0.04 | 0.01 | 1.01 | -0.11 | -1.05 | 0.10 | -0.93 | -1.06 | 0.09 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (26) | -0.19 | 1.06 | 1.10 | -0.12 | -0.25 | 0.66 | 1.06 | 0.07 | -0.07 | -0.03 | 0.64 | 0.08 | -1.05 | 0.24 | -0.15 | 0.64 | 0.07 | 0.12 | 0.07 | 0.90 | -0.62 | 0.02 | -1.01 | 0.30 | -1.00 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (27) | -1.10 | 0.80 | 0.52 | 0.06 | -0.36 | -0.08 | 0.50 | 0.16 | 1.09 | 0.16 | -1.05 | -0.13 | 0.08 | -1.01 | 0.19 | 0.02 | -0.02 | -0.11 | -1.10 | 0.05 | -0.05 | 0.21 | -0.85 | -0.09 | 0.09 | 0.09 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (28) | 0.68 | -1.04 | 0.11 | -0.02 | -0.22 | -0.03 | -0.21 | -0.67 | 0.16 | 1.03 | -0.11 | 0.21 | 1.05 | 0.01 | -0.24 | 0.10 | -0.01 | 0.17 | 0.01 | 0.44 | 0.18 | 0.65 | -1.15 | -0.03 | 0.01 | -0.09 | 0.64 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (29) | -0.09 | -0.06 | 0.38 | -0.07 | -0.22 | 1.05 | -0.19 | -0.20 | 0.07 | 0.73 | -0.34 | 0.15 | -0.07 | 0.11 | 0.06 | -0.12 | 0.08 | -0.58 | 0.17 | 0.95 | 0.18 | 0.92 | -1.11 | 0.02 | -0.04 | -1.00 | -0.28 | -0.32 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (30) | -1.06 | -0.02 | 0.23 | 0.22 | 1.02 | -0.27 | 0.12 | -0.05 | -1.02 | -0.05 | -0.19 | -0.55 | 0.08 | 1.11 | -0.66 | -0.08 | 0.14 | 0.60 | -1.02 | -0.12 | 0.09 | 0.92 | 0.09 | 0.23 | 0.02 | 0.10 | -0.09 | -0.32 | 0.05 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (31) | -0.17 | -0.40 | 1.11 | 0.05 | 0.02 | -1.01 | -1.02 | 0.12 | 0.17 | -1.05 | -0.14 | -0.01 | 1.70 | -0.16 | 1.55 | 0.09 | 1.05 | -0.10 | -0.04 | -0.02 | -0.05 | 0.10 | -1.02 | 1.04 | -0.10 | 1.07 | -1.04 | -1.03 | 1.06 | -0.02 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (32) | -0.20 | -1.03 | 0.08 | 0.07 | 0.03 | 0.10 | -0.18 | 0.27 | -0.03 | 0.04 | 1.07 | -0.38 | -0.17 | -0.02 | 0.65 | 0.21 | -0.44 | 0.02 | -0.05 | 1.12 | -0.09 | 0.28 | 1.02 | -0.09 | -0.10 | -0.08 | -0.02 | 0.25 | -1.01 | 0.18 | 0.63 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (33) | -0.28 | -0.12 | -0.43 | -1.03 | -0.28 | 0.69 | -0.05 | 0.59 | 0.14 | 0.26 | -0.07 | 0.01 | 0.19 | -1.09 | -0.08 | -0.96 | -0.03 | -0.21 | 0.45 | 0.15 | 0.20 | 0.13 | -0.21 | 0.08 | -0.06 | 0.09 | -1.06 | 0.12 | 0.80 | 0.05 | 1.24 | -1.04 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (34) | -0.21 | -1.10 | -0.43 | -0.92 | -0.03 | -1.05 | 0.09 | 0.62 | 0.43 | 1.04 | 0.16 | 1.03 | 0.25 | -0.03 | 1.02 | 0.16 | 0.10 | -0.07 | -0.04 | 0.74 | 1.10 | 0.95 | 0.15 | -0.04 | -0.02 | 1.05 | 0.34 | 1.04 | 0.16 | 0.18 | 0.05 | -0.13 | 0.16 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (35) | -0.80 | -0.08 | -0.01 | 1.14 | 0.09 | 0.16 | 1.04 | 0.19 | -1.06 | -1.07 | 0.07 | 1.09 | -1.05 | -0.04 | 0.11 | 0.14 | 1.12 | -0.01 | 1.18 | -0.02 | -0.43 | 0.03 | 0.72 | -0.07 | 0.07 | -0.86 | 0.09 | 0.63 | 0.21 | -1.09 | -0.23 | 0.04 | -1.61 | 1.06 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (36) | 0.12 | -0.71 | 1.00 | -0.03 | -1.09 | -0.11 | 1.03 | 1.22 | 1.06 | 0.07 | -0.02 | 0.71 | 0.22 | -0.40 | 0.20 | 0.01 | 0.11 | -0.01 | -1.06 | 0.02 | 0.26 | -0.09 | 0.09 | -0.07 | -0.17 | -0.02 | 0.08 | -0.72 | 0.58 | -0.02 | 0.05 | -0.60 | -1.01 | 0.23 | -0.17 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (37) | 0.07 | -0.62 | -0.79 | 0.07 | 0.20 | 0.18 | -0.01 | 0.04 | 0.56 | -0.03 | 0.15 | -0.10 | -1.06 | 0.04 | -0.03 | 0.16 | -1.06 | 0.03 | 0.22 | -1.42 | 0.20 | 1.06 | 0.03 | 1.03 | 0.77 | 0.45 | 0.06 | 0.02 | 0.53 | 0.26 | 0.15 | -0.15 | 0.99 | 1.40 | 1.28 | -0.44 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (38) | 0.14 | -0.17 | -0.12 | 1.03 | -0.07 | 0.29 | -0.21 | -0.07 | -1.02 | -0.04 | 0.17 | -1.01 | -0.06 | 1.13 | -0.09 | -0.10 | 0.11 | -0.02 | 0.16 | -0.35 | 0.22 | -1.01 | -0.08 | -0.14 | -0.01 | -1.11 | -1.12 | 0.13 | 0.13 | -0.11 | 0.24 | -1.31 | -0.01 | 1.06 | -0.17 | 0.20 | 0.18 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (39) | 1.03 | -0.02 | -0.10 | -0.75 | -0.62 | 0.46 | -0.13 | -1.01 | 0.07 | -0.06 | -0.05 | -0.04 | 0.62 | 0.40 | 0.08 | 1.15 | -0.32 | 0.07 | -0.15 | 0.02 | -0.02 | -0.16 | -1.01 | -0.76 | -0.08 | 1.39 | 0.02 | 0.07 | 0.16 | 1.05 | -0.03 | 1.16 | 0.34 | 1.09 | 0.96 | 0.10 | 1.18 | -0.11 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (40) | 1.07 | -0.14 | 0.25 | 1.17 | 1.01 | 0.24 | -0.05 | 0.28 | -0.20 | 1.08 | -1.06 | 0.17 | 0.08 | 0.93 | 0.17 | 0.18 | 0.24 | -0.15 | 0.36 | 0.06 | 1.15 | -1.05 | -0.09 | -0.15 | 0.12 | 0.05 | -0.05 | -0.04 | -0.01 | 0.31 | 1.12 | -0.01 | 0.18 | -0.06 | 0.06 | 0.35 | 1.04 | 0.95 | -1.01 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (41) | 0.16 | 1.03 | 1.02 | -0.69 | 1.09 | 0.20 | 1.03 | 1.06 | 1.25 | 0.43 | 0.29 | 0.15 | 0.02 | 1.04 | 0.05 | 0.69 | -1.04 | -0.45 | 0.25 | 0.00 | 0.03 | 0.17 | 0.13 | 1.08 | -1.08 | -0.01 | 0.04 | 0.16 | 0.16 | -1.00 | -1.01 | 0.16 | -1.01 | 0.07 | 1.17 | 1.06 | 0.47 | -1.01 | 0.06 | 0.18 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (42) | -0.03 | -0.06 | -0.25 | -1.11 | -0.01 | 1.06 | -0.10 | 0.07 | 1.06 | -1.08 | 1.38 | 0.16 | 1.05 | 1.02 | -1.02 | -0.03 | -0.01 | 0.15 | 0.13 | 1.05 | 0.27 | 1.07 | -0.13 | 0.71 | 1.15 | -1.03 | -0.07 | 0.19 | 0.95 | -0.03 | -0.03 | -0.15 | 0.06 | 0.49 | 0.04 | 1.07 | 0.21 | 1.04 | -1.01 | 0.29 | 1.05 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (43) | -0.78 | -0.27 | -1.02 | 1.07 | 1.07 | -0.83 | -1.03 | -0.02 | -0.10 | 0.10 | 0.19 | -0.52 | -1.12 | 0.16 | -1.04 | 1.08 | 1.25 | -0.09 | 0.14 | -0.04 | 1.06 | 0.37 | 1.03 | 0.39 | 0.26 | -1.13 | 1.01 | -1.11 | 0.03 | -0.04 | 0.71 | -1.02 | 0.13 | -1.02 | 0.12 | -0.12 | 0.21 | 0.38 | 1.12 | 0.38 | 0.18 | 0.19 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (44) | -0.09 | -0.03 | -0.14 | 1.71 | -0.78 | -0.09 | -0.13 | -1.02 | 0.12 | -0.03 | 0.11 | -0.28 | -0.19 | 0.20 | -1.06 | 1.04 | -0.38 | 1.09 | -1.56 | 1.12 | 2.64 | -0.02 | 0.05 | -0.20 | -0.01 | -1.23 | -1.03 | -0.07 | 0.11 | -0.57 | 1.18 | 0.34 | -0.15 | -0.12 | 0.15 | 0.21 | 0.29 | -0.16 | 0.10 | -0.04 | -1.01 | 0.33 | 0.11 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (45) | 0.13 | 0.15 | 1.07 | 0.13 | -1.02 | 0.12 | -1.00 | 0.08 | 0.00 | 0.03 | 0.06 | 0.08 | 0.02 | 0.17 | 1.05 | 0.00 | -0.31 | -1.18 | 0.13 | -0.07 | -0.08 | 1.22 | -0.15 | 0.14 | -1.02 | 1.18 | 1.08 | -0.04 | -0.29 | 1.48 | -0.16 | -1.07 | 0.23 | 1.02 | -0.35 | 0.17 | 1.01 | -0.20 | 0.01 | 0.35 | 0.17 | 0.32 | 1.06 | 1.12 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (46) | 1.84 | -0.12 | 0.21 | 0.84 | 0.74 | 0.24 | -1.07 | 0.17 | 0.21 | 0.10 | 0.03 | 0.51 | 1.10 | 0.36 | -0.60 | 1.12 | 0.07 | -0.09 | 0.23 | -0.07 | 0.11 | 0.12 | 0.48 | -0.05 | -0.09 | 1.20 | 1.20 | 1.27 | 0.31 | -0.05 | -0.09 | -0.08 | 0.06 | 0.12 | 1.14 | 0.28 | 0.26 | -0.09 | -0.11 | 1.14 | 0.17 | 0.37 | 1.15 | 0.00 | 1.07 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (47) | 0.07 | 1.22 | 0.21 | 1.29 | 1.29 | 1.16 | 1.13 | 1.40 | 1.13 | -0.07 | -0.08 | 1.19 | 1.68 | 0.16 | -0.09 | 1.07 | 1.05 | 0.34 | 0.24 | 0.50 | 0.12 | 0.10 | 0.24 | 0.18 | 0.07 | 0.28 | 0.26 | 0.25 | 1.61 | 0.14 | -0.25 | 0.05 | 0.20 | 0.37 | 1.27 | 1.07 | 0.23 | 1.54 | 0.42 | 1.24 | 1.53 | 1.20 | 0.40 | 1.12 | 1.22 | 0.32 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (48) | 0.47 | 0.89 | 0.69 | 1.52 | 0.66 | 0.71 | 0.75 | 1.64 | 1.58 | 2.58 | 1.76 | 0.63 | 0.72 | 0.50 | 1.44 | 1.67 | 1.46 | 0.97 | 14.75 | 0.98 | 0.87 | 0.93 | 3.05 | 0.98 | 0.98 | 0.81 | 0.79 | 0.87 | 1.79 | 0.64 | 0.73 | -1.42 | 0.62 | 2.14 | 2.46 | 0.91 | 1.98 | 3.59 | 1.71 | 2.06 | 3.21 | 2.11 | 0.88 | 0.84 | 0.71 | 2.18 | 1.85 | |

**Table 4.13 Matrix representation of additional parameters corresponding to bivariate copula families of the estimated R-Vine specification.** Numbers in parentheses indicate rows and columns of the matrix. Numbers in the matrix are parameter values.

| | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | (11) | (12) | (13) | (14) | (15) | (16) | (17) | (18) | (19) | (20) | (21) | (22) | (23) | (24) | (25) | (26) | (27) | (28) | (29) | (30) | (31) | (32) | (33) | (34) | (35) | (36) | (37) | (38) | (39) | (40) | (41) | (42) | (43) | (44) | (45) | (46) | (47) | (48) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| (1) | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (2) | 0.00 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (3) | 0.00 | 0.00 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (4) | 0.00 | 0.00 | 0.00 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (5) | 0.00 | 0.00 | 0.00 | 7.73 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (6) | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (7) | 0.00 | 4.25 | 0.00 | 0.00 | 0.00 | 0.00 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (8) | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (9) | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (10) | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (11) | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (12) | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 4.80 | 0.00 | 0.00 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (13) | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (14) | 0.00 | 8.53 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (15) | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (16) | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 5.80 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (17) | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (18) | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (19) | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 7.65 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (20) | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (21) | 0.00 | 0.00 | 0.00 | 0.00 | 4.97 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 2.57 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (22) | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 4.31 | 0.00 | 0.00 | 0.00 | 0.00 | 6.05 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (23) | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 5.70 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (24) | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (25) | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (26) | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (27) | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (28) | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 7.85 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (29) | 0.00 | 0.00 | 0.00 | 4.19 | 4.57 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (30) | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (31) | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (32) | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (33) | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 5.05 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 3.58 | 0.00 | 0.00 | 0.00 | 4.30 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (34) | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 7.23 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (35) | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 5.94 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 5.31 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (36) | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 7.30 | 0.00 | 0.00 | 0.00 | 5.16 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 5.12 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (37) | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 5.85 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (38) | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (39) | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 5.30 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (40) | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (41) | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 6.12 | 6.19 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (42) | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 5.97 | 7.72 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (43) | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 4.96 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 3.27 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (44) | 0.00 | 8.15 | 0.00 | 0.00 | 0.00 | 3.48 | 0.00 | 0.00 | 0.00 | 3.13 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 4.37 | 0.00 | 0.00 | 3.35 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (45) | 2.93 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 6.54 | 3.08 | 0.00 | 0.00 | 4.31 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 8.68 | 0.00 | 7.93 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (46) | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 4.94 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 5.98 | 0.00 | 3.73 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 6.00 | 3.28 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (47) | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 6.00 | 0.00 | 0.00 | 0.00 | 3.83 | 6.00 | 2.52 | 3.96 | 0.00 | 0.00 | 2.37 | 3.94 | 0.00 | 0.00 | 3.35 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 6.90 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (48) | 3.73 | 2.00 | 2.28 | 0.00 | 5.36 | 0.00 | 5.49 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 6.26 | 3.10 | 0.00 | 0.00 | 0.00 | 2.00 | 0.00 | 2.00 | 2.46 | 2.06 | 0.00 | 2.00 | 2.00 | 3.87 | 3.26 | 0.00 | 0.00 | 2.80 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 3.80 | 5.35 | 0.00 | 0.00 | 0.00 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |

**Table 4.14 Descriptive statistics of raw return data, MVN return data and RVC return data.** All return data are interest-rate-adjusted to take into account the cost-of-carry of FX forwards in a portfolio. The statistics for raw data are calculated on 120 observations of in-sample return data. The statistics for MVN and RVC returns are calculated on 1,000 observations from generated scenarios.

| | Raw data | | | MVN | | | RVC | | |
|---|---|---|---|---|---|---|---|---|---|
| | min (%) | mean (%) | max(%) | min (%) | mean (%) | max(%) | min (%) | mean (%) | max(%) |
| USD | 0.000 | 0.001 | 0.008 | 0.000 | 0.001 | 0.008 | 0.000 | 0.001 | 0.008 |
| ARS | -49.38 | -1.17 | 7.71 | -21.96 | -1.13 | 19.60 | -49.38 | -1.05 | 7.70 |
| BRL | -20.05 | -0.03 | 15.18 | -19.66 | 0.01 | 17.15 | -20.01 | -0.04 | 15.18 |
| CAD | -12.56 | 0.21 | 8.42 | -12.51 | 0.24 | 10.33 | -12.43 | 0.20 | 8.40 |
| CLP | -18.08 | 0.11 | 7.10 | -13.64 | 0.15 | 12.52 | -18.08 | 0.16 | 7.09 |
| MXN | -14.66 | -0.28 | 7.86 | -10.86 | -0.22 | 11.52 | -14.66 | -0.24 | 7.84 |
| EUR | -9.73 | 0.20 | 11.42 | -11.08 | 0.22 | 13.07 | -9.72 | 0.20 | 11.42 |
| GBP | -9.35 | 0.07 | 8.78 | -11.24 | 0.06 | 9.71 | -9.34 | 0.05 | 8.78 |
| CHF | -11.03 | 0.41 | 15.56 | -12.29 | 0.38 | 12.14 | -11.02 | 0.40 | 15.56 |
| AUD | -16.36 | 0.34 | 9.20 | -14.73 | 0.38 | 13.88 | -16.33 | 0.35 | 9.20 |
| INR | -8.07 | -0.14 | 7.40 | -9.03 | -0.13 | 8.25 | -8.06 | -0.11 | 7.40 |
| JPY | -8.22 | 0.09 | 7.93 | -10.68 | 0.04 | 11.27 | -8.22 | 0.10 | 7.91 |
| KRW | -12.12 | 0.18 | 16.63 | -11.51 | 0.16 | 12.22 | -12.12 | 0.21 | 16.63 |
| SGD | -7.22 | 0.22 | 4.56 | -5.61 | 0.23 | 7.78 | -7.20 | 0.25 | 4.56 |
| govt.AU | -2.47 | 0.54 | 4.12 | -3.74 | 0.51 | 4.45 | -2.46 | 0.54 | 4.12 |
| govt.AT | -2.35 | 0.51 | 4.58 | -4.13 | 0.49 | 6.20 | -2.35 | 0.49 | 4.58 |
| govt.BE | -3.74 | 0.52 | 6.83 | -4.61 | 0.51 | 6.21 | -3.73 | 0.50 | 6.83 |
| govt.CA | -2.09 | 0.48 | 4.72 | -4.22 | 0.46 | 5.96 | -2.09 | 0.46 | 4.72 |
| govt.FR | -2.65 | 0.48 | 4.41 | -4.21 | 0.46 | 5.55 | -2.64 | 0.46 | 4.41 |
| govt.DE | -1.89 | 0.47 | 3.95 | -4.16 | 0.44 | 5.72 | -1.88 | 0.45 | 3.95 |
| govt.KR | -2.98 | 0.52 | 6.82 | -3.45 | 0.50 | 5.07 | -2.97 | 0.50 | 6.82 |
| govt.JP | -2.22 | 0.16 | 1.89 | -2.01 | 0.16 | 2.17 | -2.22 | 0.15 | 1.89 |
| govt.MX | -5.31 | 0.78 | 6.26 | -7.02 | 0.77 | 7.09 | -5.31 | 0.77 | 6.26 |
| govt.NL | -2.15 | 0.49 | 4.48 | -4.22 | 0.46 | 5.99 | -2.15 | 0.47 | 4.48 |
| govt.SG | -3.15 | 0.29 | 3.71 | -3.38 | 0.26 | 4.44 | -3.15 | 0.27 | 3.71 |
| govt.UK | -4.71 | 0.52 | 5.48 | -6.13 | 0.48 | 7.25 | -4.71 | 0.51 | 5.48 |
| govt.US | -4.68 | 0.42 | 5.63 | -5.18 | 0.40 | 5.82 | -4.66 | 0.41 | 5.63 |
| corp.EU | -2.78 | 0.40 | 2.69 | -3.10 | 0.39 | 3.80 | -2.75 | 0.39 | 2.69 |
| corp.UK | -4.96 | 0.52 | 7.95 | -5.60 | 0.48 | 7.03 | -4.96 | 0.50 | 7.95 |
| corp.US | -8.06 | 0.03 | 8.45 | -7.03 | 0.01 | 7.13 | -7.99 | 0.01 | 8.45 |
| stock.NL | -22.62 | -0.02 | 14.57 | -23.19 | 0.09 | 23.86 | -22.60 | 0.04 | 14.54 |
| stock.AT | -32.59 | 0.51 | 13.55 | -25.35 | 0.65 | 23.33 | -32.41 | 0.60 | 13.54 |
| stock.AU | -13.54 | 0.33 | 7.06 | -13.47 | 0.40 | 13.83 | -13.52 | 0.37 | 7.05 |
| stock.BE | -24.09 | 0.17 | 13.51 | -18.61 | 0.25 | 19.19 | -23.93 | 0.24 | 13.48 |
| stock.IN | -27.30 | 1.32 | 24.89 | -23.67 | 1.43 | 29.73 | -27.05 | 1.40 | 24.86 |
| stock.BR | -28.50 | 0.89 | 16.48 | -29.83 | 1.03 | 30.20 | -28.33 | 0.93 | 16.44 |
| stock.FR | -19.23 | 0.05 | 12.59 | -19.67 | 0.14 | 19.78 | -19.00 | 0.10 | 12.57 |
| stock.UK | -13.95 | 0.18 | 8.30 | -15.16 | 0.28 | 16.50 | -13.93 | 0.24 | 8.30 |
| stock.DE | -29.33 | 0.50 | 19.37 | -24.20 | 0.62 | 23.21 | -29.09 | 0.58 | 19.33 |
| stock.US | -18.56 | 0.37 | 10.23 | -18.29 | 0.48 | 15.22 | -18.35 | 0.43 | 10.20 |
| stock.CA | -18.55 | 0.43 | 10.62 | -18.55 | 0.54 | 15.21 | -18.48 | 0.46 | 10.45 |
| stock.CL | -10.07 | 0.77 | 14.92 | -15.16 | 0.79 | 19.71 | -10.07 | 0.77 | 14.91 |
| stock.KR | -26.31 | 0.70 | 12.68 | -21.92 | 0.77 | 22.31 | -26.26 | 0.83 | 12.68 |
| stock.AR | -45.81 | 2.32 | 39.67 | -33.43 | 2.48 | 38.24 | -45.70 | 2.38 | 39.64 |
| stock.MX | -19.67 | 1.22 | 12.38 | -27.23 | 1.35 | 20.52 | -19.54 | 1.27 | 12.33 |
| stock.JP | -27.22 | 0.39 | 12.09 | -23.55 | 0.52 | 23.67 | -26.98 | 0.52 | 12.09 |
| stock.CH | -14.03 | 0.22 | 10.61 | -16.16 | 0.31 | 16.77 | -13.97 | 0.28 | 10.58 |
| stock.SG | -27.36 | 0.48 | 19.30 | -17.75 | 0.60 | 18.84 | -27.29 | 0.58 | 19.27 |
| XAU | -16.27 | 0.91 | 11.77 | -16.98 | 0.94 | 19.10 | -14.97 | 0.90 | 11.45 |

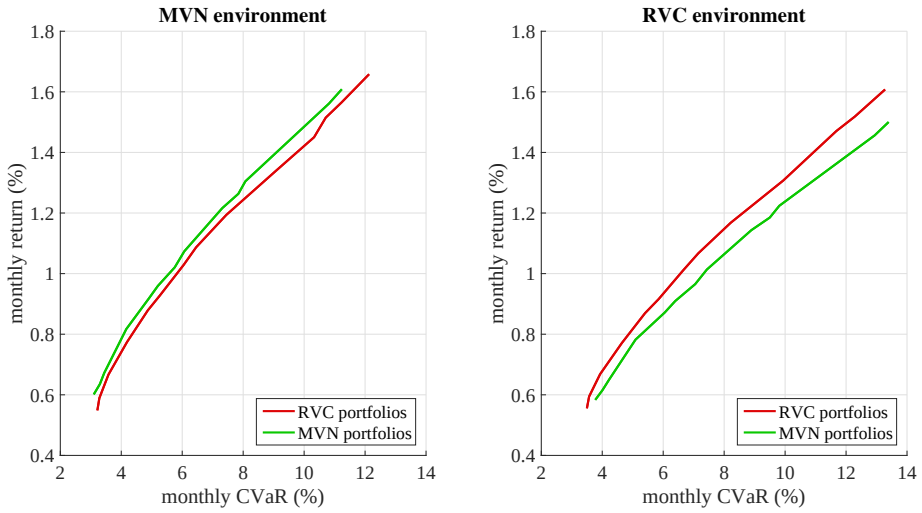
#### 4.5.2.3 Efficient Portfolios from Different Scenario Generation Methods

This study highlights the effects of different assumptions made on generating scenarios. The first approach generating scenarios by assuming that return distributions of

securities are normally distributed and the co-movement between assets is represented by correlation (or linear relationship). Another approach makes no assumption on distribution family, rather, it takes the empirical distribution from available historical data. The inter-relationship between assets is described in terms of dependence and is captured by a regular-vine copula. The solutions obtained from two types of generated scenarios are thus differed by the assumption on the shape of return distributions and linear or non-linear relationship between securities.

Basically, the optimal portfolios are most efficient when evaluated in their own "environment". Here the environment is defined as a state that returns are generated under some specific assumptions (e.g., assuming that asset returns are normal). Hence an optimal portfolios under scenarios generated from a multivariate normal return distribution (MVN) could not be the most efficient when its risk and return are evaluated with return realisations generated from other methods such as a regular-vine copula based scenario construction (RVC). This is evidently demonstrated in Figure 4.13.

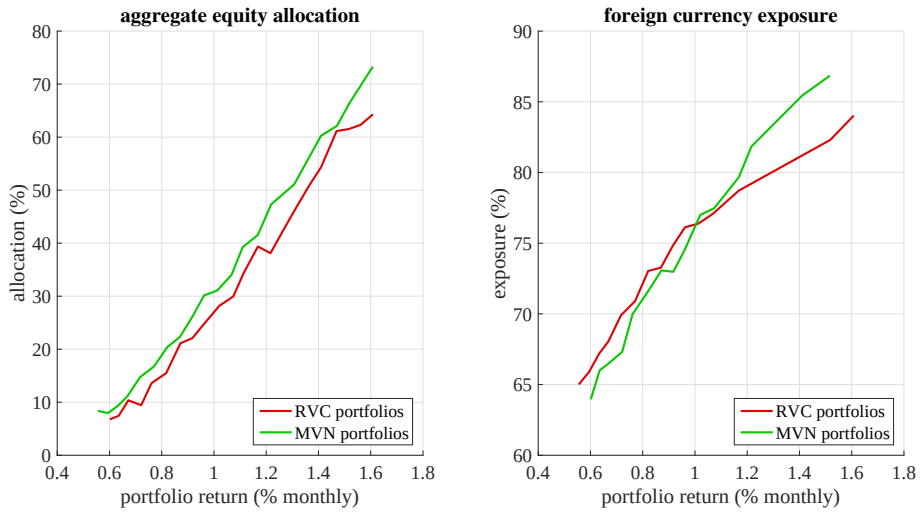
**Figure 4.13 Comparison of efficient portfolios under different environments.** The RVC portfolios are optimal under scenarios that do not assume normal distribution of returns. The MVN portfolios, on the other hand, are optimal under normal-distribution assumption. The left panel demonstrates the risk-return of efficient portfolios if returns are normally distributed and the right panel illustrates the risk-return profile of same portfolios if return distributions are not assumed Gaussian. Naturally, optimal portfolios in one environment are less efficient when evaluated in other environments.




The efficient frontiers in Figure 4.13 is created by applying optimal allocations on returns produced from scenario generators. Naturally, optimal portfolios in one environment are less efficient when evaluated in other environments. This finding coincides with the work of Krokhmal et al. [75] which compare portfolios optimised under different risk measures (variance and CVaR). It is reported that CVaR optimal

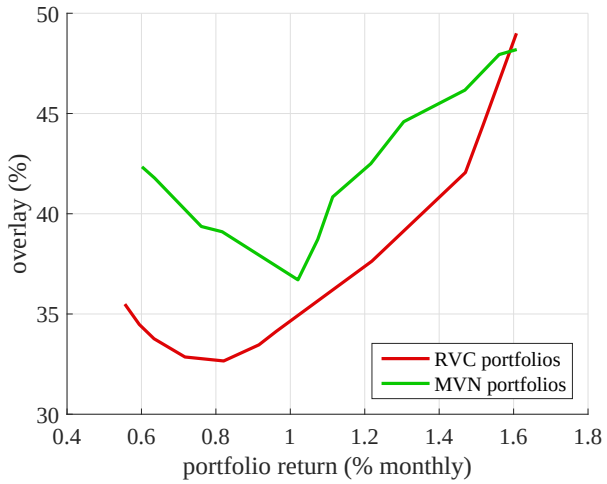
portfolios have higher standard deviation than that of the efficient mean-variance portfolios and the mean-variance optimal portfolios have higher CVaR than that of CVaR portfolios.

**Figure 4.14 Equity allocation and foreign currencies exposure of portfolios.** The left panel shows proportion of equity (from all markets) in portfolios and the right panel presents aggregate exposure in foreign currencies (non-USD) of portfolios. It is noticed that the MVN portfolios tend to hold more equities and foreign currencies (when return is over 1%) than the RVC counterpart.




Consequently, the optimal allocations obtained from two scenario generation methods are compared to investigate if different assumptions cause deviation in optimal allocations. The left panel of Figure 4.14 illustrates equity allocations and the right panel shows foreign currencies (non-USD) exposures of portfolios. Generally, high equity holding and foreign currency exposure constitute risky portfolios. It is observed that optimal portfolios from MVN scenarios hold more equities than those optimised under RVC scenarios. For exposures on foreign currencies, although optimal portfolios from RVC scenarios more expose to foreign currencies at low-range returns, the MVN portfolios demonstrate a steep increase in foreign currencies exposure when the target return is over 1% monthly approximately.

**Figure 4.15 Currency Overlays of MVN and RVC portfolios.** The MVN portfolios demonstrate higher currency exposure adjustment through foreign exchange forwards than the RVC portfolios.




Foreign currencies exposure is basically an aggregation of investments in foreign assets and forward positions. To investigate further on the source of foreign currencies exposure, the total currency overlays are plotted in Figure 4.15. It is shown that optimal portfolios from multivariate normal assumptions hold more forward contracts to adjust their currency exposures than optimal portfolios from copula-based scenarios for almost all return targets. The findings in Figures 4.14 and 4.15 conclude that portfolios optimised assuming that asset returns are normally distributed tend to hold more risky investments than optimal portfolios from copula-based scenarios.

However, it should be noted that proportion of risky investments in optimal allocations cannot predict anything about portfolio performances. The reason that optimal portfolios in normal-distribution assumption hold more equities and foreign currencies could be that risks from extreme events which cannot be captured by normal distribution are overlooked and thus the exact risks could be underestimated. The proportion of risky investments is hence higher than the case that uncertainty is better modelled. To evaluate optimal portfolios from different scenario generation methods, an out-of-sample test is performed in Section 4.5.2.5.

#### 4.5.2.4 Effects of Cardinality Constraints on Risk and Return of Portfolios

The study limits the range of currencies and the number of foreign exchange forwards to be invested in a portfolio. Although investing in multiple countries offers diversification on asset and currency exposures, investing in a small set of countries could save

operational costs and transaction costs. Limiting the number of currencies and forwards associated in a portfolio can be implemented through cardinality constraints. In our optimisation model, the number of currencies and foreign exchange forwards in a portfolio are determined by $K_C$ and $K_G$. In this study, portfolios are optimised using RVC scenarios.

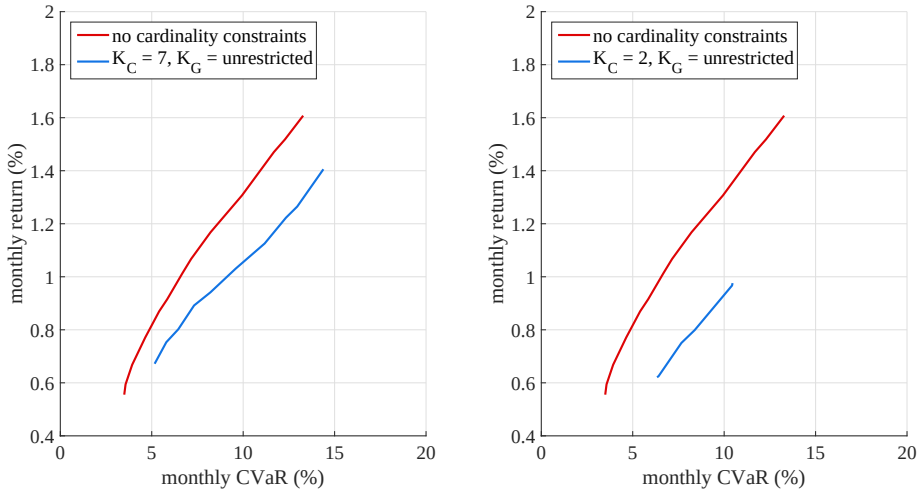
$K_C$ indicates the number of currencies in a portfolio. Since the base currency of the portfolio is USD which is also the funding currency of forward positions, the smallest number of currencies in the portfolio is therefore two, i.e., USD and another currency. $K_G$ limits the number of foreign exchange forwards constituted in a currency overlay. Because there are minimum two currencies in the portfolio, the minimum number of forwards in a portfolio is one. The plot of efficient frontiers at different levels of $K_C$ and $K_G$ are given in Figures 4.16, 4.17 and 4.18.

The unrestricted case allows that a portfolio can invest in all currencies ($K_C = 14$) and that the currency overlay can be created from all forward pairs possible $K_G = \binom{14}{2} = 91$. Restricting the number of currencies that a portfolio can invest in or limiting the number of forward contracts in currency overlay make portfolios less efficient. Efficient frontiers of portfolios with cardinality constraints are truncated, some return targets are not achievable due to restricted choices of investments.

The empirical results show that imposing limits on $K_C$ causes more deviation than on $K_G$. The reason is that limiting $K_C$ affects both investment decisions in assets and currencies while limiting $K_G$ affects only choices in adjusting currency exposure. Focusing on the case that $K_G$ is restricted (Figure 4.17), the deviation between efficient frontiers are widened with the return targets. This is possibly due to limited choices to hedge foreign currency exposure. Basically, portfolios seek higher returns by investing in risky assets, e.g., emerging markets where their currency returns are volatile. Hedging such exposure to local currency is an option to lower risk to portfolio but with the constraints imposed, the capacity of currency hedging is confined and hence the portfolio risk is not well reduced in such cases.

Imposing both restrictions on $K_C$ and $K_G$ (Figure 4.18) at the same time greatly impacts the risk and return of portfolios. The case that portfolios can invest in two currencies and do currency overlay with a single forward contract ($K_C = 2$ and $K_G = 1$) significantly limits the return ranges and flattens the efficient frontiers (implying that taking much more risk to increase the target returns).

**Figure 4.16 Comparison of portfolios without cardinality constraints and those with the cardinality constraint on the number of currencies ($K_C$).** When unrestricted, $K_C = 14$ which allows portfolios to invest in all currencies. $K_C = 7$ and $K_C = 2$ then limits that portfolios can invest in 7 and 2 currencies, respectively. Although saving transaction costs, tighter restriction on the number of currencies results in lower flexibility to modify currency exposure, less efficient portfolios.

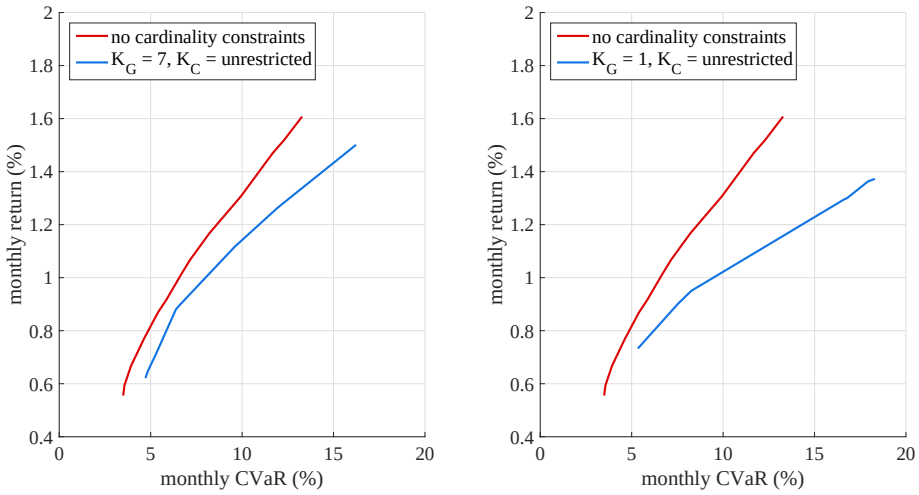



In terms of computational time, the average runtimes (minutes) in running each optimal portfolio are reported in Table 4.15. Note that the sample sizes for average runtime are varied. The number of optimal portfolios are reduced when cardinality constraints become tighter. The cardinality constraints also result in search space reduction which lower the solving time taken accordingly.

**Table 4.15 The average runtime (minutes) taken in solving the optimisation problem at a given level of return under different cardinality constraints configurations.** $K_C = 14$ and $K_G = 91$ indicate no restriction on the number of currencies and no restriction on the number of foreign exchange forwards, respectively. Each portfolio is run under 1,000 scenarios.

| Cardinality Constraints | Sample Size | Average Runtime (mins) |
|---|---|---|
| $K_C = 14, K_G = 91$ | 22 | 49.4 |
| $K_C = 14, K_G = 7$ | 19 | 46.7 |
| $K_C = 14, K_G = 1$ | 14 | 44.3 |
| $K_C = 7, K_G = 91$ | 15 | 48.1 |
| $K_C = 2, K_G = 91$ | 10 | 43.2 |
| $K_C = 7, K_G = 7$ | 13 | 39.2 |
| $K_C = 2, K_G = 1$ | 4 | 38.7 |

**Figure 4.17 Comparison of portfolios without cardinality constraints and those with the cardinality constraint on the number of forwards ($K_G$).** When $K_G$ is unrestricted, it implies that all forward pairs can be used to construct a currency overlay. $K_G = 7$ and $K_G = 1$ then limits that the currency overlay is a combination of 7 and 1 forward pair(s), respectively. Although saving transaction costs, tighter restriction on the number of forwards results in lower flexibility to modify currency exposure, less efficient portfolios.

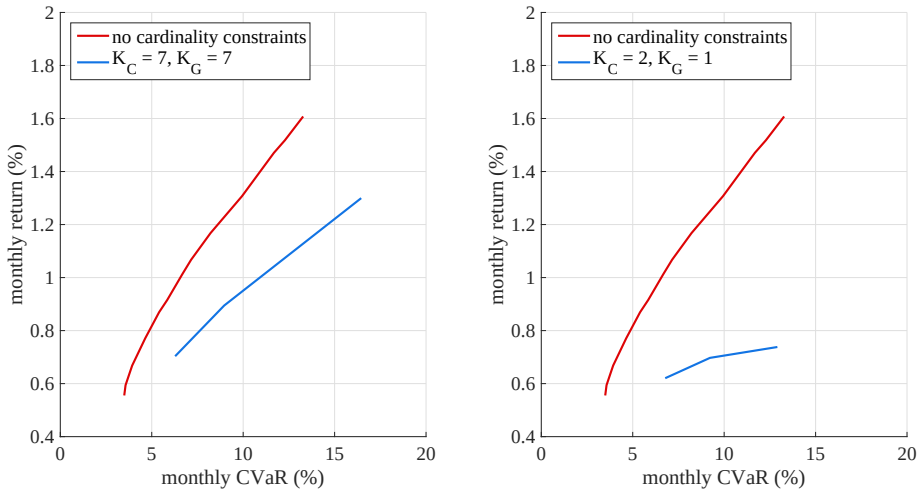



#### 4.5.2.5 Out-of-Sample Performance

This study presents cumulative returns of optimal portfolios from different scenario generation methods over the out-of-sample period (Jan-12 to May-15). Since our data frequency is monthly, there are in total 41 observations for returns and prices of all assets, currencies and forwards. The cumulative return index is constructed to compare the cumulative return by setting an initial wealth of 100 in Jan-12 then accumulating the monthly returns until May-15.

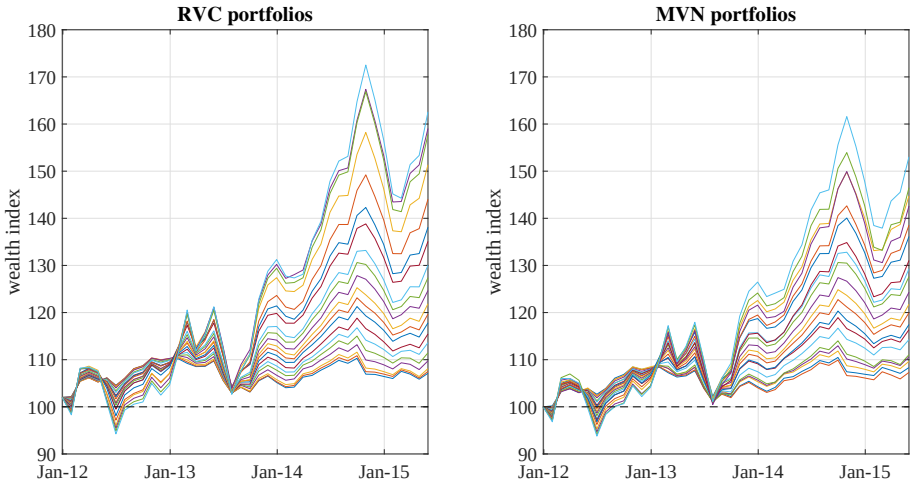
The plot of cumulative wealths is exhibited in Figure 4.19. Each curve represents the cumulative wealth of each optimal portfolio. The top curves are from the portfolios with high expected returns showing highly volatile paths but ending with high final wealths. The bottom curves are from the portfolios with low return targets which demonstrate more stable wealths along the period with decent wealth in May-15. It is noticed that, in the first year of the test period, portfolios with low to moderate expected return experience negative wealth while the risky portfolios do not. In addition, the gaps between portfolio wealths are tight at the beginning and become widening after late 2013. This is mainly due to stock markets rallying as a results of several stimulus packages from the central banks after consecutive financial crises.

**Figure 4.18 Comparison of portfolios without cardinality constraints and those with both the cardinality constraints on the number of currencies ($K_C$) and the number of forwards ($K_G$).** $K_C = 7$ and $K_G = 7$ restrict that portfolios can invest in 7 currencies and adjust currency exposures using 7 forward pairs. $K_C = 2$ and $K_G = 1$ limit further that portfolios can invest in only 7 currencies and adjust currency exposures with only single forward pair. Although transaction costs can be reduced, tighter restriction on the number of countries and the number of forwards results in even less efficient portfolios.



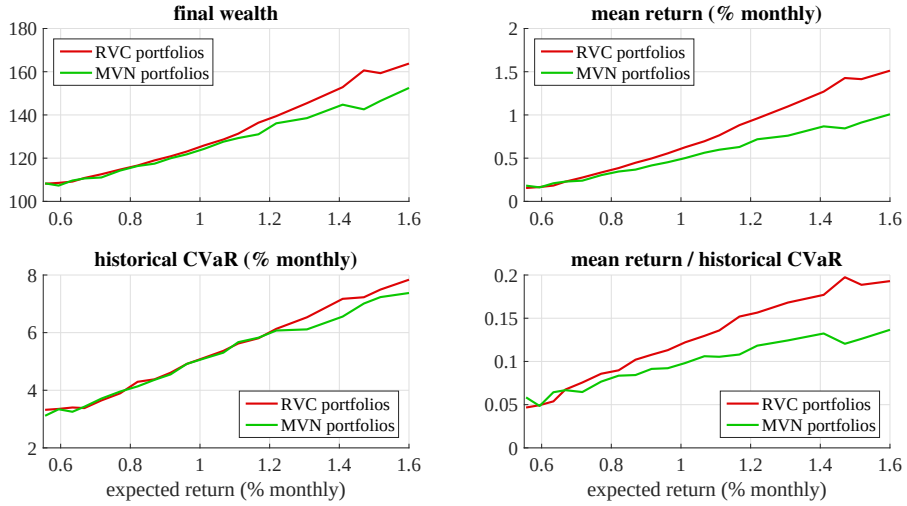

To compare the performances of efficient portfolios from RVC and MVN scenarios, various performance evaluation metrics are reported, as displayed in Figure 4.20. The top-left panel of Figure 4.20 exhibits the final wealths at different levels of expected return target. It is shown that final wealths from two generation methods coincide with each other until when the target return is greater than 1%, the RVC portfolios then shows significantly higher final wealths. The top-right panel displays the average monthly return over the test period where the copula-based portfolios demonstrate higher average return at almost all return targets. The bottom-left panel plots the historical CVaR of portfolio returns which is calculated by averaging returns lower than the $5^{th}$-percentile of portfolio returns in the test period. It is noticed that the CVaRs over the out-of-sample period for RVC and MVN portfolios are relatively the same for low to medium return targets. Then the CVaRs of copula-based portfolios are significantly higher when portfolios reach higher returns. The plots of mean return and historical CVaR reveal that RVC portfolios show higher CVaRs but with higher returns than MVN portfolios. To see the risk-return compensation, the ratio of average return over CVaR is calculated and illustrated in the bottom-right panel. The higher value of the ratio indicates higher average return is rewarded per unit of risk taken. It is observed that RVC portfolios clearly show better risk-return reward although having higher CVaRs when return targets are medium (0.8-1.2% monthly) to high

**Figure 4.19 Cumulative wealth over the out-of-sample period (Jan-12 to May-15) of optimal portfolios from two scenario generation methods; RVC and MVN approaches.** Each curve represents cumulative wealth of an optimal portfolio with respect to a given target return. Portfolios with higher risk-return profile exhibits higher final wealth with higher volatility along the period.




(1.2% monthly or more). Only at low return targets (less than 0.8% monthly) that the results are mixed. This points out the benefit of employing a regular-vine copula in scenario generations which better capture risk-return characteristics of financial assets and results in better portfolio performance, particularly when portfolios hold a substantial portion of risky assets which do not generally follow normality assumption.

**Figure 4.20 Comparison of performances of efficient portfolios under RVC and MVN scenarios.** *final wealth* presents the wealth of a portfolio on May-15 given that the starting wealth is 100 on Jan-12. *mean return* shows average return over the period of portfolios. *historical CVaR* exhibits the average portfolio returns below $5^{th}$-percentile. *mean return / historical CVaR* reports the ratio of average portfolio return over the historical CVaR.



## 4.6 Conclusions

In this study, an optimisation model for international portfolio with currency overlay is proposed. The portfolio is integrated with a currency overlay constructed from foreign exchange forwards providing flexibility for hedging and speculating currency exposure. The portfolio structure allows asset allocation and forward positions to be optimised at the same time for portfolio optimality in terms of asset and currency exposure. The optimisation problem is set up with a stochastic programming model under realistic scenarios.

In generating scenarios for the stochastic optimisation problem, the key improvement is not assuming normal distribution to return series and modelling the dependence structure of returns by a regular-vine copula (RVC). The aim is to better capture underlying risk of securities held by the portfolio. The results show that RVC scenarios can capture characteristic of financial security return distribution better than MVN scenarios. Specifically, mean, minimum and maximum returns for each individual asset under RVC scenarios are relatively identical to those of raw data (see Table 4.14 on page 145).

The resulting portfolios optimised under scenarios generated from our proposed method is compared to those obtained from a traditional approach, i.e., assuming that returns are multivariate normally distributed (MVN). The experiment results show that an efficient portfolio tends to hold a lower proportion of risky assets foreign currency when optimised using copula-based scenarios (see Figure 4.14 on page 147). In addition, the out-of-sample performances also show that the RVC portfolios demonstrate higher risk-return reward than the MVN portfolios (see Figure 4.20 on page 153). The difference in risk-return compensation is insignificant when portfolio return targets are under 1% monthly but becomes more pronounced when the return targets are higher. This finding highlights the benefit of employing a regular-vine copula, particularly when a portfolio holds substantial allocation of risky assets which do not generally follow normality assumption (e.g., equities).

The proposed optimisation problem can be solved within reasonable time using genetic algorithms. However, if the problem scale grows larger from expanding the set of countries and asset classes or investing in individual stocks rather than stock indices, there might need other heuristic algorithms to solve the problem.

# Chapter 5

# Portfolio Rebalancing under A Second-Order Stochastic Dominance Criterion

## 5.1 Introduction

This chapter is a modification of the previous stochastic international portfolio optimisation model to construct a portfolio rebalancing model. As described in Chapters 3 and 4, a portfolio optimisation problem typically aims to find an optimal allocation over a predetermined investment horizon by trading off expected return against risk.

An investment horizon is basically the total length of time that an investor expects to hold a security or a portfolio. Generally, investments of financial asset requires some time to pay off; long or short depending on the type of assets. An investment horizon in bonds or equities, for instance, might take around one to three years while some other types of investment such as real estates may take much longer time to break even. Thus, along the way of an investment duration, it is important to assure that a portfolio can maintain its competency until the end of the investment horizon.

Typically, portfolio efficiency is defined relatively to a mean-variance efficient portfolio set (efficient frontier). The three most common measures in literature are a utility function estimation, a (long-term) geometric mean and short- and long-term probability

objectives[1] [97]. Each has important theoretical or practical limitations. An expected utility approach is generally not a practical investment solution for an optimal portfolio choice since investors do not exactly know their utility function. A geometric mean [52] particularly focuses on the long-term expected return which naturally overlooks risk associated. A probability objective which is generally formulated as minimising the probability of a negative return over some investment horizons has a limitation that resulting portfolios are usually too conservative. A more appropriate measure is thus sought after.

In order to sustain portfolio efficiency, apart from finding an appropriate choice of efficiency measure, another concern is raised on the dynamics of financial market environment. For the case of mean-risk portfolio objectives (e.g., mean-variance or mean-CVaR), asset price fluctuation is a critical factor determining if an expected return and risk of an efficient portfolio could be held along an investment horizon. It would be very likely that a portfolio optimised under bull market in-sample data could not achieve its target expected return and risk during a prolonged period of economic downturn. As such, an adaptive efficiency measure is more favourable.

This thesis investigates the concept of the second-order stochastic dominance (SSD) to establish an efficiency criterion that is varying with ever-changing financial market conditions. Given two return distributions, one " second-order stochastically dominates" another if, on the average, that return distribution is more likely to have positive return and less likely to have negative return than another. It therefore follows that, on the average, an SSD distribution should have greater return than its counterpart in any situations.

The main advantage of applying the SSD criterion as an objective for portfolio rebalancing is that the goal is independent of changing financial market conditions. Whether in economic downturn or upturn, one can still find a portfolio that is SSD, compared to another one. Thus to keep a portfolio profitable along the time, an initial allocation (derived from a portfolio optimisation model such as one proposed in Chapter 4) is set as a reference portfolio and is put into a test against a contender portfolio (derived from the same optimisation model with updated data) to find an SSD allocation in each time period.

The rebalancing routine set up above is expected to deliver a higher and more stable return along the management duration of a portfolio than conventional rebalancing

[1] For example, an investor may seek a strategy that minimises the probability of having a negative return over a 10-year investment horizon.

strategies, i.e., the buy-and-hold and the periodic rebalancing methods. The performance comparison is demonstrated later in Section 5.4.

### 5.1.1 Current Advancements in The Literature

Portfolio rebalancing is an approach commonly employed to alleviate the effect of market fluctuation on portfolio efficiency. Rebalancing is generally performed to restore a portfolio that has drifted (due to financial market movements) from its target asset allocation (optimised allocation). Typically, there arise three important questions concerning rebalancing [4]:

1. Should rebalancing be done periodically (at fixed intervals) or only when some certain condition is triggered (such as deviation from original allocation is over 5%)?

2. If it is done periodically, what is an appropriate interval, i.e., monthly versus quarterly versus annually?

3. Is it possible to identify assets which provide potential extra return?

Among several strategies, periodic rebalancing (constant mix) is the most widely-used [4]. Conventionally, a portfolio is reallocated to its target allocation in every month assuming that market dynamics do not affect expected return and risk of the portfolio and hence the efficiency is attained. As opposed to the continual rebalancing, the buy-and-hold rebalancing strategy lets a portfolio allocation deviate along the way from its target without intervention. The advantage is saving transaction costs while the downside is that inflating the proportion of performing assets could cause a severe loss when the market trend abruptly reverses (more details are given later in Table 5.1).

Other than the periodic and buy-and-hold strategies, optimal rebalancing strategy is one of the widely-studied topics in portfolio rebalancing literature (a review of such studies is given in Section 2.2.6 of Chapter 2.1). The idea is that a portfolio will rebalance its allocation only when a certain condition is met (e.g., a deviation from an original allocation is greater than 5%). The benefit is that it saves a transaction cost as a rebalancing is not performed in every time period. Nevertheless, the portfolio is still obliged to reallocate its allocation to an original positions which might not sustain the same level of expected return and risk when market condition is changed. The proposed SSD criterion is seemingly an appropriate objective in such environment.

The SSD criterion has been applied in several researches in the field of portfolio optimisation as discussed in Section 2.2.5 of Chapter 2.1. The criterion takes part in either a constraint or an objective for a portfolio optimisation problem. However, the SSD criterion has not been applied for an objective of portfolio rebalancing problem before. The development in this chapter thus marks a novel contribution of this thesis.

### 5.1.2 Chapter Overview

This chapter presents an approach of modifying a stochastic international portfolio optimisation model into a portfolio rebalancing model. The stochastic portfolio optimisation model has an objective of minimising CVaR, so the first task is to express the SSD criterion as a function of CVaR. Once a CVaR representation of SSD criterion is formulated, an optimisation model for portfolio rebalancing can be established.

A portfolio rebalancing model is set up as an optimisation problem (either deterministic or stochastic). An application of the SSD criterion on a rebalancing model finds a potentially more superior allocation than an existing one (a portfolio that was implemented at the inception, which has now deviated from its original allocation due to market movements). The optimisation model is formulated to find the number of units for buying or selling each asset relative to an existing allocation to achieve an SSD portfolio (net of transaction costs). In addition, the in-sample data of the optimisation problem are updated with new observations in every rebalancing period so that an SSD allocation can digest new information from new market environment. The results of the proposed methodology is expected to generate more resilient returns and reduced transaction costs compared to conventional rebalancing approaches. Performance of the new strategy is consequently compared with those of the buy-and-hold and the periodic rebalancing techniques at different risk-return profiles.

The rest of this chapter is organised as follows: Section 5.2 elaborates theoretical concept of the second-order stochastic dominance, Section 5.3 demonstrates the construction of SSD rebalancing model, Section 5.4 exhibit the results from comparing performances of the SSD, buy-and-hold and periodic rebalancing strategies and Section 5.5 summarises the findings obtained from the experiment.

## 5.2 Background

This section provides a foundation to the development of a new portfolio rebalancing model. First, common practices of portfolio rebalancing are demonstrated. Then, the concept of stochastic dominance is examined for later use as a criterion to evaluate if a portfolio needs a reallocation or not. Then, we learn how to transform a CVaR objective function of the stochastic portfolio optimisation model in Chapter 4 to an objective function of a rebalancing model under the second-order stochastic dominance criterion.

### 5.2.1 Portfolio Rebalancing

Portfolio management generally involves asset allocation and rebalancing [132]. The process typically begins with setting an investment horizon determining a finite period of management until termination or liquidation, then an optimisation problem is set up to find an efficient portfolio (in terms of risk and return). An allocation obtained is expected to remain efficient until the end of investment horizon. However, market fluctuations usually lead to a deterioration of portfolio efficiency along the investment period and hence there arises a need for portfolio rebalancing to retain the efficiency [4].

Rebalancing is an action to bring a deviated allocation of a portfolio back to a target which is typically an efficient allocation derived from optimisation. After an implementation of an initial portfolio, the allocation may drift from original proportion due to unequal return of each security when time passes. To restore the portfolio's original risk and return characteristics, we need a rebalancing strategy on portfolio allocation.

As part of the portfolio construction process, it is important for portfolio managers to develop a rebalancing strategy that formally addresses "how often, how far, and how much": that is, how frequently the portfolio should be rebalanced (monthly, quarterly or yearly); how far an asset allocation can be allowed to deviate from its target before it is rebalanced; and whether periodic rebalancing should restore a portfolio to its target or to a close approximation of the target.

Several rebalancing strategies are developed accordingly. In the study, the following two widely-used strategies [111] are the candidates in comparison with the new rebalancing strategy.

1. *Buy-and-Hold (BHD)* - this strategy leaves portfolio allocation drifts with market movements and changes nothing from original allocation. The example is given in Table 5.1.

2. *Periodic (PRD)* - also known as constant mix, this strategy pegs portfolio allocation to its original by selling off outperforming assets and buying in underperforming assets at predefined frequency such as monthly or yearly. The example is given in Table 5.2.

To demonstrate how the rebalancing strategies are implemented, consider a portfolio of two assets, namely $A$ and $B$, as follows:

**Table 5.1 Implementation of the buy-and-hold rebalancing strategy on a two-asset portfolio.** Notice that no transaction is needed to execute this strategy and the units of assets $A$ and $B$ are constant over time while monthly allocations deviate from the original proportion due to market fluctuations.

| | | month-0 | month-1 | month-2 | month-3 |
|---|---|---|---|---|---|
| asset A | price | 10.2 | 11.6 | 10.7 | 12.2 |
| | unit | 50 | 50 | 50 | 50 |
| | allocation (%) | 47.2 | 52.3 | 49.3 | 55.5 |
| asset B | price | 5.7 | 5.3 | 5.5 | 4.9 |
| | unit | 100 | 100 | 100 | 100 |
| | allocation (%) | 52.8 | 47.7 | 50.7 | 44.5 |
| market value | | 1,080 | 1,110 | 1,085 | 1,100 |
| portfolio return (%) | | | 2.78 | -2.25 | 1.38 |

We can see from Table 5.1 that the buy-and-hold strategy lets the portfolio allocation float with financial markets. Thus, the strategy produces superior returns in markets with a prolonged upward (or downward) bias but at the expense of compounding portfolio risk (since the positions of performing assets are accumulated along the time and there could occur a large loss on a portfolio when those assets begin to fall). In contrast, the periodic rebalancing requires buying and selling assets in every period to maintain the original allocation. It, therefore, involves transaction costs (which are not shown in Table 5.2). This strategy could unintentionally reduce return by selling performing assets to buy non-performing ones, it thus pays off well in stagnant as well as oscillating markets [55].

**Table 5.2 Rebalancing a two-asset portfolio with the periodic strategy to maintain constant allocation over time.** In every month, a portfolio needs to sell the performing asset (*A*) off and buy in the non-performing asset (*B*) to keep the allocation equal to its original. Note that transaction costs are not included in this example.

| | | month-0 | month-1 | month-2 | month-3 |
|---|---|---|---|---|---|
| asset A | price | 10.2 | 11.6 | 10.7 | 12.2 |
| | unit | 50 | 41 | 50 | 39 |
| | allocation (%) | 47.2 | 47.2 | 47.2 | 47.2 |
| asset B | price | 5.7 | 5.3 | 5.5 | 4.9 |
| | unit | 100 | 100 | 109 | 109 |
| | allocation (%) | 52.8 | 52.8 | 52.8 | 52.8 |
| market value | | 1,080 | 1,003 | 1,135 | 1,011 |
| portfolio return (%) | | | -7.15 | 13.14 | -10.88 |

### 5.2.2 Stochastic Dominance

Stochastic dominance (SD) criteria are analytical tools for decision making under risk and uncertainty [85]. They are a form of stochastic ordering usually applied in situations where one investment (represented by a probability distribution) can be ranked as superior to another investment. In the context of stochastic dominance, an investment is a portfolio with known return distribution.

Stochastic dominance criteria rank random variables by a point-wise comparison subject to some general conditions for an investor's risk preferences. Suppose that two random variables $X_0$ and $X_1$ represent returns of portfolios $P_0$ and $P_1$, respectively. Further, let $F_0(t)$ and $F_1(t)$ denote cumulative return distribution functions (CDFs) corresponding to $\Pr[X_0 < t]$ and $\Pr[X_1 < t]$, respectively. $F(t)$ and $G(t)$ are formally expressed by

$$F_0(t) = \int_{-\infty}^{t} f_0(x)dx, \tag{5.1}$$

$$F_1(t) = \int_{-\infty}^{t} f_1(x)dx \tag{5.2}$$

where $f_0$ and $f_1$ are probability density functions (PDFs) of $X_0$ and $X_1$, respectively. Note that when dealing with discrete random variables, integrals can be replaced with

finite summations. With defined CDFs, stochastic dominance criteria can be developed accordingly.

#### 5.2.2.1 First-Order Stochastic Dominance (FSD)

The first-order stochastic dominance (FSD) criterion [88] requires that $P_1$ dominates $P_0$ with the notation of $P_1 \succeq_{FSD} P_0$ if and only if

$$\begin{aligned} F_1(t) &\leq F_0(t); \quad \forall t \in \mathbb{R}, \\ F_1(t) &< F_0(t); \quad \exists t \in \mathbb{R}. \end{aligned} \tag{5.3}$$

An illustration of the FSD criterion above is exhibited in Figure 5.1. For an application in evaluating the superiority between two portfolios, the requirement (5.3) implies that, at every return target $t$, $P_1$ always has lower chance to have return lower than $t$ and so higher chance to have return higher than $t$ comparing to $P_0$.

It follows that the FSD condition in (5.3) implies $\mathbb{E}[F_1(t)] > \mathbb{E}[F_0(t)]$. However, the converse does not hold, i.e., $X_1 \succeq_{FSD} X_0 \Rightarrow \mathbb{E}[X_1] > \mathbb{E}[X_0]$. See the proof below.

*Proof.* Let $[l,u]$ be a common support of any distribution where $l \simeq -\infty$ and $u \simeq \infty$, an expectation of a random variable $X$ is defined as

$$\begin{aligned} \mathbb{E}[X] &= \int_l^u x f(x) dx, \\ &= [-x(1-F(x))]_{x=l}^{x=u} + \int_l^u [1-F(x)] dx, \\ &= l + \int_l^u [1-F(x)] dx. \end{aligned}$$

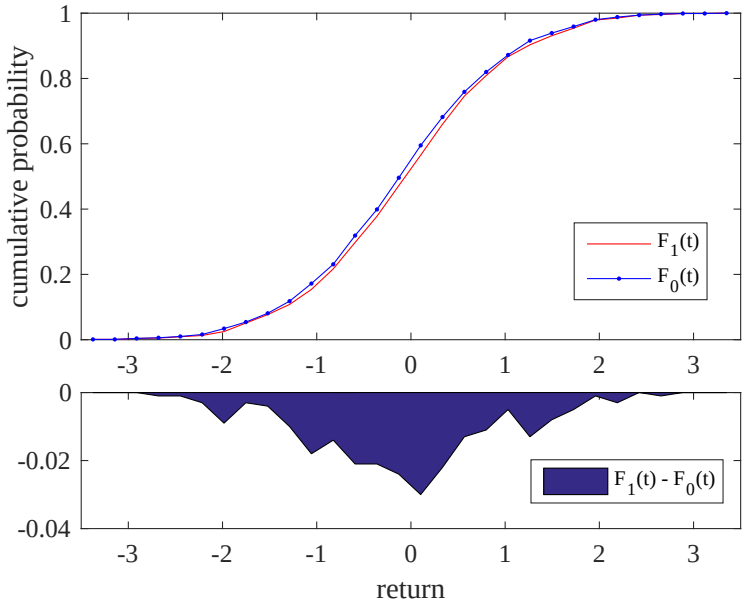
Hence,

$$\begin{aligned} \mathbb{E}[X_0] &= l + \int_l^u [1-F_0(x)] dx, \\ \mathbb{E}[X_1] &= l + \int_l^u [1-F_1(x)] dx. \end{aligned}$$

Since, by FSD condition, $F_1(t) \leq F_0(t)$ for all $t \in \mathbb{R}$ and $F_1(t) < F_0(t)$ for some $t \in \mathbb{R}$, it therefore follows that, $\mathbb{E}[X_1] > \mathbb{E}[X_0]$. □

The shortcoming of FSD is thus if there are two portfolios with the same expected return, we cannot distinguish them by the FSD condition. In such cases, we need some adjustments on the stochastic dominance condition.

**Figure 5.1 An illustration of the first-order stochastic dominance criterion.** $F_1(t) \leq F_0(t)$ for all $t \in \mathbb{R}$ is shown by $F_1(t) - F_0(t) \leq 0$ in the lower panel. That is, for any value of return value $t$, $F_1$ always has lower chance to have a return lower than $t$, comparing to $F_0$. This figure is produced with artificial data for the purpose of illustration only.




#### 5.2.2.2 Second-Order Stochastic Dominance (SSD)

The FSD criterion assumes that investors generally prefer "more to less" (of expected returns). It does not, however, take into account risk in choosing investments. In addition, the FSD condition in (5.3) is unlikely to be satisfied in practice (see an example in Figure 5.3). An alternative is to weaken the condition and include risk aversion in a criterion. The concept of second-order stochastic dominance (SSD) was therefore developed accordingly [88].

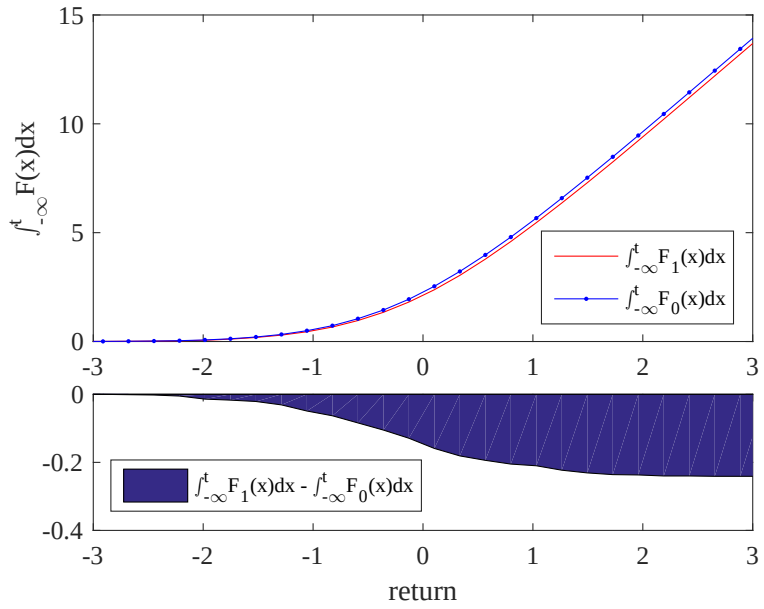
Rather than focusing on each $F(t)$ along the support of a return distribution, we consider the "average value up to $t$" of $F(t)$ instead. That is,

$$\begin{aligned} &\mathbb{E}[F_1(t)] \leq \mathbb{E}[F_0(t)]; \quad \forall t \in \mathbb{R}, \\ &\mathbb{E}[F_1(t)] < \mathbb{E}[F_0(t)]; \quad \exists t \in \mathbb{R} \end{aligned} \tag{5.4}$$

where $\mathbb{E}[\cdot]$ is the expectation operator. The SSD condition in (5.4) is equivalent to

$$\begin{aligned} \int_{-\infty}^{t} F_1(x)dx \leq \int_{-\infty}^{t} F_0(x)dx; \quad \forall t \in \mathbb{R}, \\ \int_{-\infty}^{t} F_1(x)dx < \int_{-\infty}^{t} F_0(x)dx; \quad \exists t \in \mathbb{R} \end{aligned} \tag{5.5}$$

**Figure 5.2 An illustration of the second-order stochastic dominance criterion.** $\int_{-\infty}^{t} F_1(x)dx \leq \int_{-\infty}^{t} F_0(x)dx$ for all $t \in \mathbb{R}$ is shown by $\int_{-\infty}^{t} F_1(x)dx - \int_{-\infty}^{t} F_0(x)dx \leq 0$ in the lower panel. That is, on the average (as represented by an integral over $-\infty$ to $t$ where a denominator is omitted as $F_1$ and $F_0$ share the same support), for any value of return value $t$, $F_1$ always has lower chance to have a return lower than $t$, comparing to $F_0$. This figure is produced with artificial data for the purpose of illustration only.




Therefore, by (5.5), it follows that if $X_1$ is first-order stochastic dominant to $X_0$, then $X_1$ also second-order stochastically dominates $X_0$, i.e., $X_1 \succeq_{FSD} X_0 \Rightarrow X_1 \succeq_{SSD} X_0$.

*Proof.* Given the FSD condition,

$$\begin{aligned} F_1(t) \leq F_0(t); \quad \forall t \in \mathbb{R}, \\ F_1(t) < F_0(t); \quad \exists t \in \mathbb{R}. \end{aligned}$$

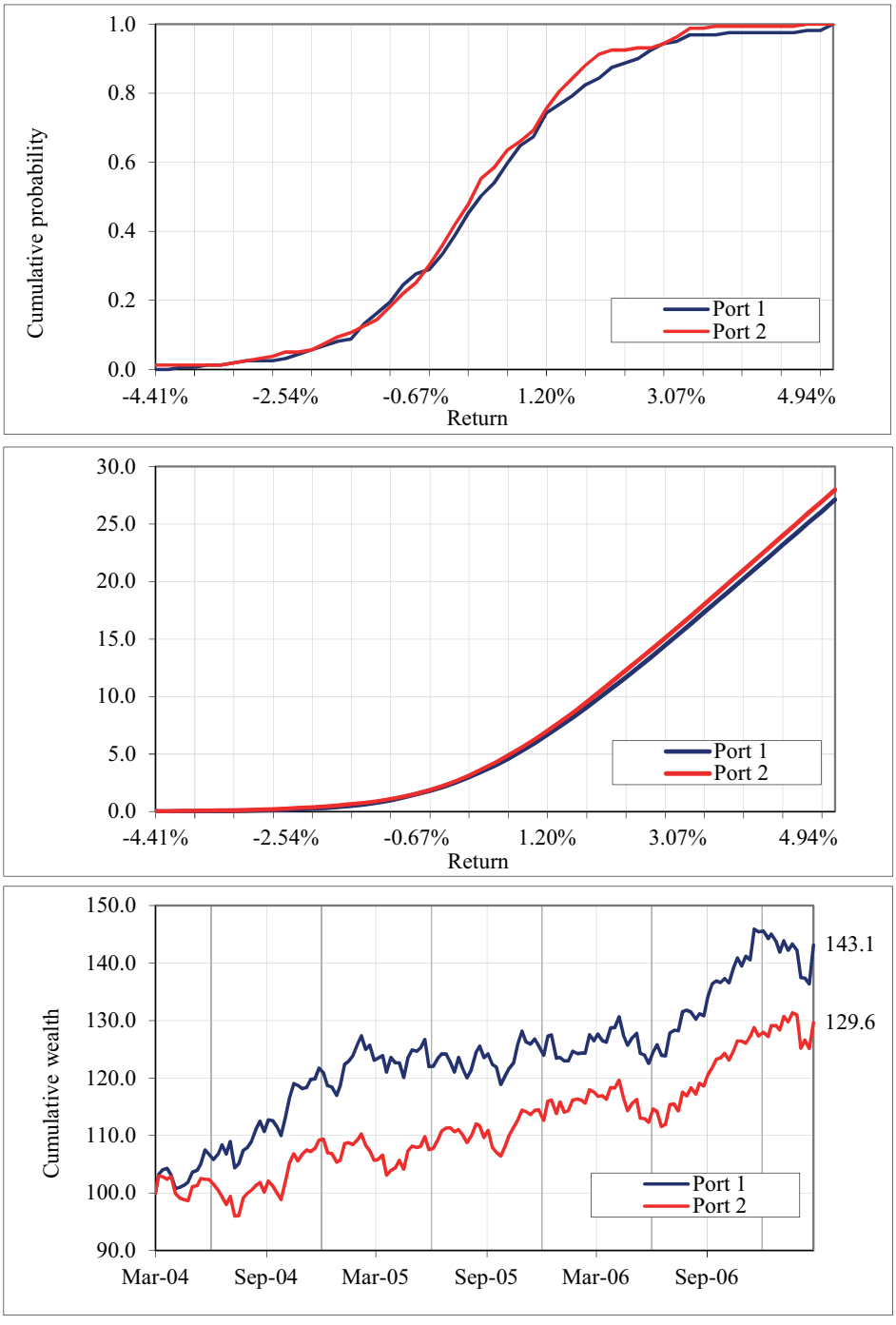
An integration of the above expressions provides the SSD condition similar to (5.5). □

Unlike the FSD, the second-order stochastic dominance can still hold when two portfolios have the same mean return. Hence, we can use the SSD criterion to distinguish between investments or portfolios with the same mean. However, SSD does not always prefer an investment with lower expected return in favour of being less risky. This is inevitable since we have made no assumptions as to how risk-averse an investor is.

In general, when comparing two investments $X_0$ and $X_1$ by FSD and SSD (altogether), possible answers will be [85]

1. Nothing can be concluded.
2. $X_1$ is first- and second-order stochastic dominant to $X_0$ (or $X_0$ to $X_1$).
3. $X_1$ is second-order stochastic dominant to $X_0$ (or $X_0$ to $X_1$) but not first-order stochastic dominant. This is exemplified in Figure 5.3.

**Figure 5.3 An example of two portfolios that satisfy the SSD condition but not the FSD.** The first portfolio (Port 1) contains 54% of General Electric, 21% of Exxon Mobil and 25% of Microsoft stocks. The second portfolio (Port 2) contains 100 % of Microsoft stocks. The FSD condition (top panel) is not satisfied (neither Port 1 nor Port 2 dominates another, e.g., Port 2 mostly dominates but not at returns around -0.67%) while the SSD condition (middle panel) is satisfied (Port 1 second-order stochastically dominates Port 2). The historical stock returns cover the period of Mar-04 to Mar-07. The cumulative returns over the sample period for Port 1 and Port 2 are 43.1% and 29.6%, respectively (bottom panel).



#### 5.2.2.3 CVaR Representation of Second-Order Stochastic Dominance

The second-order stochastic dominance criterion can be expressed as a function of CVaR risk measure [116]. The CVaR representation of SSD criterion is useful in formulating a model to compare CVaRs of two portfolios. Details on how CVaR can represent the SSD condition are given below.

Recall that CVaR is an expected loss greater than VaR and let $X$ be a random variable denoting return of a portfolio. Rockafellar and Uryasev [116] provide a definition of CVaR at a confidence level $\beta$ by

$$\text{CVaR}_\beta(X) = \text{VaR}_\beta(X) + \frac{1}{1-\beta}\mathbb{E}[(\text{VaR}_\beta(X) - x)^+] \tag{5.6}$$

where $[\cdot]^+ = \max(x, 0)$.

From (5.5), an integration by parts on $\int_{-\infty}^{t} F(x)dx$ yields

$$\begin{aligned}\int_{-\infty}^{t} F(x)dx &= tF(t) - \int_{-\infty}^{t} xdF(x), \\ &= \int_{-\infty}^{t} (t-x)dF(x), \\ &= \mathbb{E}[(t-x)^+]. \end{aligned} \tag{5.7}$$

Therefore, the inequalities in (5.5) can be alternatively written as

$$\begin{aligned}&\mathbb{E}[(t-x)_1^+] \leq \mathbb{E}[(t-x)_0^+]; \quad \forall t \in \mathbb{R}, \\ &\mathbb{E}[(t-x)_1^+] < \mathbb{E}[(t-x)_0^+]; \quad \exists t \in \mathbb{R}. \end{aligned} \tag{5.8}$$

From (5.8) and the definition of CVaR in (5.6), it is shown in Roman et al. [118] that

$$\begin{aligned}&\text{CVaR}_\beta(X_1) \leq \text{CVaR}_\beta(X_0); \quad \forall \beta \in (0,1), \\ &\text{CVaR}_\beta(X_1) < \text{CVaR}_\beta(X_0); \quad \exists \beta \in (0,1). \end{aligned} \tag{5.9}$$

Thus, the SSD condition can be verified by comparing CVaRs of two portfolios at every confidence level $\beta$ [118]. From (5.4) and (5.9), we can also see that second-order stochastic dominance takes into account both expected return and risk (in terms of CVaR) which is suitable in comparing risk-return characteristics of portfolios at the same time. Thus SSD is selected as a measure to rank portfolios in the study.

## 5.3 Methodology

This section introduces an approach to formulate a novel portfolio rebalancing strategy using the second-order stochastic dominance criterion. The performance of the proposed strategy is subsequently compared with the buy-and-hold and periodic rebalancing strategies.

### 5.3.1 Deterministic Portfolio Rebalancing Model

From (5.9), it is known that evaluating SSD criterion on two random variables can also be achieved by comparing their CVaRs at every confidence level. By its definition in (5.6), CVaR is continuous and non-decreasing in $\beta$. Therefore, the SSD property in (5.9) is equivalent to, for continuous distributions [118],

$$\int_0^1 \text{CVaR}_\beta(X_1)d\beta \leq \int_0^1 \text{CVaR}_\beta(X_0)d\beta \tag{5.10}$$

and for discrete distributions (as an approximation of a continuous distribution) Roman et al. [118],

$$\sum_{\beta\in B} \text{CVaR}_\beta(X_1) \leq \sum_{\beta\in B} \text{CVaR}_\beta(X_0). \tag{5.11}$$

where $B$ is a set containing the range of confidence levels. In what follows, we specify $B = \{0.01, 0.02, \ldots, 0.99\}$. Note that $\beta = 0$ and $\beta = 1$ are unspecified.

Suppose that a rebalancing routine is set to monthly from month-1 onwards and a portfolio is implemented in month-0, then from month-1 onwards the portfolio is subject to reallocation. It is known that the buy-and-hold and periodic rebalancing are appropriate for trending and oscillating markets respectively [55]. An alternative strategy is to combine their advantages through the SSD criterion.

For instance, suppose that a portfolio is set up in month-0 with allocation obtained from optimisation, then in month-1, the decision is to let the allocation float as in buy-and-hold strategy or to reallocate the allocation to an SSD portfolio. Hence, there are two candidates, the current portfolio with drifted allocation $Port_0$ and the "newly optimised"[2] portfolio $Port_1$. Note that $Port_0$ is an efficient portfolio derived from optimisation in month-0 and in month-1 its allocation has drifted due to market fluctuation. In calculating CVaRs of the two portfolios, asset returns and asset prices are

[2] A portfolio is optimised again in an upcoming month with updated data.

updated from month-0 with new realisations, the in-sample dataset is thus continuously growing when time passes. With the notations given in Table 5.3, a deterministic optimisation model is formulated accordingly.

$$\text{maximise} \quad \sum_{\beta\in B} \text{CVaR}_\beta(Y_0) - \sum_{\beta\in B} \text{CVaR}_\beta(Y_1) \tag{5.12}$$

subject to

$$\text{CVaR}_\beta(Y_0) = \text{VaR}_\beta(Y_0) + \frac{1}{1-\beta}\sum_{r=1}^{N_r} p_0^r e_0^r, \tag{5.13}$$

$$e_0^r \geq y_0^r - \text{VaR}_\beta(Y_0), \tag{5.14}$$

$$e_0^r \geq 0, \tag{5.15}$$

$$Y_0 = \{y_0^1, y_0^2, \ldots, y_0^{N_r}\}, \tag{5.16}$$

$$y_0^r = -\sum_{i\in A} w_i^0 x_i^r, \tag{5.17}$$

$$\text{CVaR}_\beta(Y_1) = \text{VaR}_\beta(Y_1) + \frac{1}{1-\beta}\sum_{r=1}^{N_r} p_1^r e_1^r, \tag{5.18}$$

$$e_1^r \geq y_1^r - \text{VaR}_\beta(Y_1), \tag{5.19}$$

$$e_1^r \geq 0, \tag{5.20}$$

$$Y_1 = \{y_1^1, y_1^2, \ldots, y_1^{N_r}\}, \tag{5.21}$$

$$y_1^r = -\sum_{i\in A} w_i^1 x_i^r, \tag{5.22}$$

$$w_i^1 = \frac{u_i^1 Z_i}{\sum\limits_{i\in A}\left(u_i^1 Z_i - \pi_i - \psi_i - b_i\rho_i - s_i\lambda_i\right)}, \tag{5.23}$$

$$u_i^1 = u_i^0 + b_i - s_i, \tag{5.24}$$

$$b_i, s_i \in \mathbb{R}^+, \tag{5.25}$$

$$B = \{0.01, 0.02, \ldots, 0.99\}. \tag{5.26}$$

**Table 5.3 Notations for a deterministic portfolio rebalancing model.**

| Notation | Range | Description |
|---|---|---|
| $Y_0$ | | A set of portfolio returns for $P_0$. |
| $Y_1$ | | A set of portfolio returns for $P_1$. |
| $A$ | | A set of assets in a portfolio. |
| $B$ | | A set of confidence levels. In this study $B = \{0.01, 0.02, \ldots, 0.99\}$. |
| $r$ | $\mathbb{N}$ | An index of realisations in a probability distribution (discrete). |
| $N_r$ | | The number of realisations in a probability distribution. |
| $i$ | $\mathbb{N}$ | An index of assets, $i \in A$. |
| $x_i^r$ | $\mathbb{R}$ | A simulated return time series of $r$ realisations of an asset $i$. |
| $w_i^0$, $w_i^1$ | $[0,1]$ | Allocations of portfolios $P_0$ and $P_1$, respectively. |
| $u_i^0$, $u_i^1$ | $\mathbb{R}^+$ | The number of units of asset $i$ in a portfolio $P_0$ and $P_1$, respectively. |
| $b_i$ | $\mathbb{R}^+$ | The number of units an asset $i$ are bought. |
| $s_i$ | $\mathbb{R}^+$ | The number of units an asset $i$ are sold. |
| $p_0^r$, $p_1^r$ | $[0,1]$ | Probabilities associated with each realisation $r$ of $Y_0$ and $Y_1$, respectively. |
| $e_0^r$, $e_1^r$ | $\mathbb{R}$ | Auxiliary variables for calculating CVaRs of $Y_0$ and $Y_1$, respectively. |
| $\beta$ | $(0,1)$ | A confidence level for CVaR. $\beta \in B$. |
| $Z_i$ | $\mathbb{R}^+$ | A market price of an asset $i$. |
| $\pi_i$ | $\mathbb{R}^+$ | A fixed cost for buying an asset $i$. |
| $\psi_i$ | $\mathbb{R}^+$ | A fixed cost for selling an asset $i$. |
| $\rho_i$ | $\mathbb{R}^+$ | A variable cost for buying an asset $i$. |
| $\lambda_i$ | $\mathbb{R}^+$ | A variable cost for selling an asset $i$. |

The objective function (5.12) aims to verify if $Port_1 \succeq_{SSD} Port_0$ by comparing their summations of CVaRs across all confidence levels. The decision variables of the problem are $b_i$ and $s_i$ indicating if there are any adjustments on the drifted allocation of the portfolio $Port_0$. Consequently, the possible solutions of the optimisation problem are:

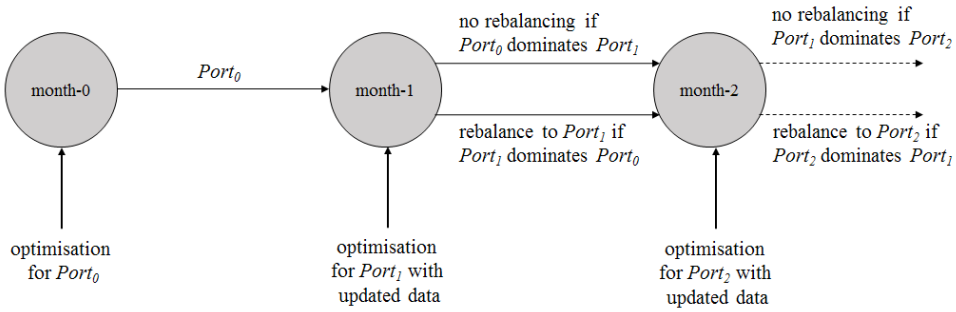
1. $b_i = 0$ and $s_i = 0$ for all $i$, this suggests that the floated allocation is still the most efficient in new market environment (month-1) and an efficient portfolio obtained from the optimisation in month-1, $Port_1$, is the same as the drifted portfolio $Port_0$. Thus, there is no need to rebalance the portfolio in this case.

2. $b_i \neq 0$ and $s_i \neq 0$ for some $i$, this indicates that $Port_1 \succeq_{SSD} Port_0$ or the new allocation second-order stochastically dominates the existing one. Rebalancing is made by buying and selling assets according to the values of $b_i$ and $s_i$.

In short, the portfolio rebalances to a new allocation only if its deviated allocation is not SSD (this allocation is dominated by some other allocations according to the second-order stochastic dominance criterion); otherwise, the allocation remains floated with market fluctuations.

The optimisation problem (5.12) is performed repeatedly along the investment horizon of the portfolio. The example in the optimisation model given earlier demonstrates rebalancing decision occurring in month-1. Suppose that, in month-1, $Port_1 \succeq_{SSD} Port_0$ then the portfolio is carried over to month-2 and repeat the same optimisation problem with updated data where $Port_1$ becomes the existing portfolio in the optimisation model of month-2 and the routine goes on until the end of portfolio management horizon. The SSD rebalancing routine can be illustrated as in Figure 5.4.

**Figure 5.4 A diagram showing an application of second-order stochastic dominance in a rebalancing routine.** The rebalancing decisions occur from month-1 onwards deciding to let the allocation drifted further or to rebalance it to an allocation obtained from a new optimisation.




## 5.3.2 Stochastic Portfolio Rebalancing Model

In reality, asset prices are not known in advance. The rebalancing strategy should thus take into account price uncertainty apart from associated transaction costs. A stochastic portfolio rebalancing model is, therefore, now developed accordingly. By extending the two-stage stochastic portfolio optimisation model in 4.3.2.2 with the associated transaction costs[3] taken into account, the stochastic rebalancing model is

[3]The transaction costs included in the model are: 1) fixed and variable costs when buying or selling assets and foreign exchange forwards. 2) the cost of carry representing gain and loss due to interest rate differential when holding foreign exchange forwards.

presented here as follows:

$$\text{maximise} \quad \sum_{\beta\in B}\left(\alpha_{\beta_0}+\frac{1}{1-\beta}\sum_{r\in N_r}p^r e_0^r\right)-\sum_{\beta\in B}\left(\alpha_{\beta_1}+\frac{1}{1-\beta}\sum_{r\in N_r}p^r e_1^r\right) \tag{5.27}$$

subject to

*First Stage: Portfolio Selection*

$$a_{ij}=a_{ij}^0+b_{ij}x_{ij}-s_{ij}y_{ij}, \tag{5.28}$$

$$q_k=q_k^0+b_k x_k-s_k y_k, \tag{5.29}$$

$$\begin{aligned}h^0&+\sum_{i=1}^{A}\sum_{j=1}^{C}\left(s_{ij}P_{ij}^0\right)y_{ij}-\sum_{i=1}^{A}\sum_{j=1}^{C}\left(\psi_{ij}+\lambda_{ij}s_{ij}P_{ij}^0\right)y_{ij}\\&+\sum_{k=1}^{G}\left(s_k P_k^0\right)y_k-\sum_{k=1}^{G}\left(\psi_k+\lambda_k s_k P_k^0\right)y_k\\&=\sum_{i=1}^{A}\sum_{j=1}^{C}\left(b_{ij}P_{ij}^0\right)x_{ij}-\sum_{i=1}^{A}\sum_{j=1}^{C}\left(\pi_{ij}+\rho_{ij}b_{ij}P_{ij}^0\right)x_{ij}\\&+\sum_{k=1}^{G}\left(b_k P_k^0\right)x_k-\sum_{k=1}^{G}\left(\pi_k+\rho_k b_k P_k^0\right)x_k,\end{aligned} \tag{5.30}$$

$$\mathbf{q}=\left[q_k\circ P_k^0\right], \tag{5.31}$$

$$\mathbf{F}=\mathbf{T}\circ\left(\mathbf{1}^{\mathrm{T}}\otimes\mathbf{q}\right), \tag{5.32}$$

$$a_{ij}^{csh}=M\sum_{k=1}^{G}|q_k|\,P_k^0, \tag{5.33}$$

$$c_j=\sum_{i=1}^{A}a_{ij}P_{ij}^0+\sum_{k=1}^{G}f_{kj}+a_{ij}^{csh}, \tag{5.34}$$

$$\frac{1}{2}\sum_{j=1}^{C}\left|\sum_{k=1}^{G}f_{kj}\right|\le V_u\sum_{j=1}^{C}c_j, \tag{5.35}$$

$$x_{ij}+y_{ij}\le 1, \tag{5.36}$$

$$x_k+y_k\le 1, \tag{5.37}$$

$$t_{ij}^{min}x_{ij}\le b_{ij}\le B, \tag{5.38}$$

$$t_{ij}^{min}y_{ij}\le s_{ij}\le B, \tag{5.39}$$

$$t_k^{min}x_k\le b_k\le B, \tag{5.40}$$

$$t_k^{min}y_k\le s_k\le B, \tag{5.41}$$

$$c_j^{min}z_j\le c_j\le c_j^{max}, \tag{5.42}$$

$$\sum_{i=1}^{A}x_{ij}+\sum_{i=1}^{A}y_{ij}\ge z_j;\quad \forall j\in C, \tag{5.43}$$

$$\sum_{j=1}^{C}z_j\le K_C, \tag{5.44}$$

$$\sum_{k=1}^{G} x_k + \sum_{k=1}^{G} y_k \leq K_G, \tag{5.45}$$

$$b_{ij}, b_k, s_{ij}, s_k \in \mathbb{R}^+, \tag{5.46}$$

$$x_{ij}, x_k, y_{ij}, y_k, z_j \in \{0,1\}, \tag{5.47}$$

*Second Stage: Recourse*

$$a_{ij}^r = a_{ij} + b_{ij}^r x_{ij}^r - s_{ij}^r y_{ij}^r, \tag{5.48}$$

$$q_k^r = q_k + b_k^r x_k^r - s_k^r y_k^r, \tag{5.49}$$

$$\begin{aligned}
&\sum_{i=1}^{A}\sum_{j=1}^{C}\left(s_{ij}^r P_{ij}^r\right) y_{ij}^r - \sum_{i=1}^{A}\sum_{j=1}^{C}\left(\psi_{ij} + \lambda_{ij} s_{ij}^r P_{ij}^r\right) y_{ij}^r \\
&\quad + \sum_{k=1}^{G}\left(s_k^r P_k^r\right) y_k^r - \sum_{k=1}^{G}\left(\psi_k + \lambda_k s_k^r P_k^r\right) y_k^r \\
&\quad = \sum_{i=1}^{A}\sum_{j=1}^{C}\left(b_{ij}^r P_{ij}^r\right) x_{ij}^r - \sum_{i=1}^{A}\sum_{j=1}^{C}\left(\pi_{ij} + \rho_{ij} b_{ij}^r P_{ij}^r\right) x_{ij}^r \\
&\quad + \sum_{k=1}^{G}\left(b_k^r P_k^r\right) x_k^r - \sum_{k=1}^{G}\left(\pi_k + \rho_k b_k^r P_k^r\right) x_k^r; \quad \forall r \in N_r
\end{aligned} \tag{5.50}$$

$$\mathbf{q}^r = \left[q_k^r \circ P_k^r\right], \tag{5.51}$$

$$\mathbf{F}^r = \mathbf{T} \circ \left(\mathbf{1}^{\mathrm{T}} \otimes \mathbf{q}^r\right), \tag{5.52}$$

$$a_{ij}^{csh,r} = M \sum_{k=1}^{G} \left|q_k^r\right| P_k^r, \tag{5.53}$$

$$c_j^r = \sum_{i=1}^{A} a_{ij}^r P_{ij}^r + \sum_{k=1}^{G} f_{kj}^r + a_{ij}^{csh,r}, \tag{5.54}$$

$$\frac{1}{2}\sum_{j=1}^{C}\left|\sum_{k=1}^{G} f_{kj}^r\right| \leq V_u \sum_{j=1}^{C} c_j^r; \quad \forall r \in N_r, \tag{5.55}$$

$$x_{ij}^r + y_{ij}^r \leq 1; \quad \forall r \in N_r, \tag{5.56}$$

$$x_k^r + y_k^r \leq 1; \quad \forall r \in N_r, \tag{5.57}$$

$$t_{ij}^{min} x_{ij}^r \leq b_{ij}^r \leq B; \quad \forall r \in N_r, \tag{5.58}$$

$$t_{ij}^{min} y_{ij}^r \leq s_{ij}^r \leq B; \quad \forall r \in N_r, \tag{5.59}$$

$$t_k^{min} x_k^r \leq b_k^r \leq B; \quad \forall r \in N_r, \tag{5.60}$$

$$t_k^{min} y_k^r \leq s_k^r \leq B; \quad \forall r \in N_r, \tag{5.61}$$

$$c_j^{min} z_j^r \leq c_j^r \leq c_j^{max}; \quad \forall r \in N_r, \tag{5.62}$$

$$\sum_{i=1}^{A} x_{ij}^r + \sum_{i=1}^{A} y_{ij}^r \geq z_j^r; \quad \forall j \in C, \forall r \in N_r, \tag{5.63}$$

$$\sum_{j=1}^{C} z_j^r \leq K_C; \quad \forall r \in N_r, \tag{5.64}$$

$$\sum_{k=1}^{G} x_k^r + \sum_{k=1}^{G} y_k^r \leq K_G; \quad \forall r \in N_r, \tag{5.65}$$

$$W_0^r = \sum_{i=1}^{A}\sum_{j=1}^{C} a_{ij}^0 P_{ij}^r + \sum_{k=1}^{G} q_k^0 P_k^r + M \sum_{k=1}^{G} \left| q_k^0 \right| P_k^r, \tag{5.66}$$

$$W_1^r = \sum_{i=1}^{A}\sum_{j=1}^{C} a_{ij}^r P_{ij}^r + \sum_{k=1}^{G} q_k^r P_k^r + a_{ij}^{csh,r}, \tag{5.67}$$

$$\alpha_{\beta 0} = \mathrm{VaR}_{\beta 0}\left(\frac{W_0^r}{W^0} - 1\right), \tag{5.68}$$

$$\alpha_{\beta 1} = \mathrm{VaR}_{\beta 1}\left(\frac{W_1^r}{W^0} - 1\right), \tag{5.69}$$

$$e_0^r \geq -\left(\frac{W_0^r}{W^0} - 1\right) - \alpha_{\beta 0}; \quad \forall r \in N_r, \tag{5.70}$$

$$e_0^r \geq 0; \quad \forall r \in N_r, \tag{5.71}$$

$$e_1^r \geq -\left(\frac{W_1^r}{W^0} - 1\right) - \alpha_{\beta 1}; \quad \forall r \in N_r, \tag{5.72}$$

$$e_1^r \geq 0; \quad \forall r \in N_r, \tag{5.73}$$

$$b_{ij}^r, b_k^r, s_{ij}^r, s_k^r \in \mathbb{R}^+, \tag{5.74}$$

$$x_{ij}^r, x_k^r, y_{ij}^r, y_k^r, z_j^r \in \{0,1\}, \tag{5.75}$$

$$B = \{0.01, 0.02, \ldots, 0.99\}. \tag{5.76}$$

Most of notations used in the model are the same as given in Tables 4.2, 4.3, 4.4 and 4.5. The new notations introduced in the model from (5.66) to (5.73) are to represent CVaRs at a confidence level $\beta$ of $Port_0$ and $Port_1$. $\alpha_{\beta_0}$ and $\alpha_{\beta_1}$ denote VaRs at a confidence level $\beta$ for $Port_0$ and $Port_1$, respectively. $W_0^r$ and $W_1^r$ are market values for for $Port_0$ and $Port_1$, respectively. $e_0^r$ and $e_0^r$ are expected shortfalls used as auxiliary variables in CVaR calculation for $Port_0$ and $Port_1$, respectively. The decision variables are the units of buying or selling assets and forward contracts in (5.46) and (5.74) and the binary variables in (5.47) and (5.75) if cardinality constraints are imposed.

The summation of confidence levels in (5.12) can be intuitively viewed as a robust version of CVaR minimisation with respect to $\beta$. When there is no particular choice of the confidence level $\beta$, we take the average of them (or summing over a prespecified range of $\beta's$) to obtain a $\beta$-neutral minimum CVaR.

The SSD rebalancing strategy can be alternatively viewed as a comparison of a once-efficient portfolio in new environment $Port_0$ and a current-efficient portfolio $Port_1$. Although, based on updated market data, $Port_1$ is the most efficient in terms of risk and return but it could be dominated by the drifted portfolio using a second-order stochastic dominance point of view.

Note that the optimisation problem (5.27) is not established to attain the risk-return profile[4] of a portfolio but to instigate a reallocation that takes consideration of new information and avoids redundant, too frequent, portfolio adjustments. The key objective of the proposed rebalancing strategy is that the reallocation frequency could be reduced more or less from adopting periodic rebalancing thereby decreasing unnecessary transactions. In addition, since the strategy accounts for new market data, it therefore allows a portfolio to adjust the allocation to avoid potential risk from holding too large portion of performing assets as in the buy-and-hold strategy.

## 5.4 Experimental Results and Discussions

This section presents experiment results of applying a stochastic rebalancing strategy on an international portfolio with integrated currency overlay. The portfolio rebalancing model is as exhibited earlier in 5.3.2.

### 5.4.1 Experiment Setting

We use the results of RVC efficient portfolios with no cardinality constraints imposed, i.e., $K_C = 14$ and $K_G = 91$ (see Section 4.5.2.5 in Chapter 4 for portfolio performances and Table B.2 to Table B.6 in Appendix B.4 for allocations) to test the performance of our proposed rebalancing strategy at different levels of risk.

1. *Low-risk portfolio* - the portfolio with an expected return of 0.55% monthly in Table B.2.
2. *Medium-risk portfolio* - the portfolio with an expected return of 1.05% monthly in Table B.4.
3. *High-risk portfolio* - the portfolio with an expected return of 1.60% monthly in Table B.6.

The out-of-sample data cover the period of Feb-12 to Jun-15. Monthly returns of our SSD rebalancing approach are compared with those of the other two contenders, i.e., the buy-and-hold (BHD) and periodic (PRD) rebalancing strategies.

[4]If the risk-return profile needs to be preserve, the constraints on expected return and CVaRs must be imposed.

In verifying the SSD portfolio according to (5.27), we use $B = \{0.01, 0.02, \ldots, 0.99\}$. That is, the SSD condition is established on the summation of 99 CVaRs. In addition, it is mentioned earlier that, in every new period, asset prices and return data are updated with new observations. Thus, an input for the optimisation problem is simulated with updated data in every new period. This also implies that expected prices, expected returns as well as a dependence structure of assets that involve in SSD rebalancing strategy could vary over time with new market information.

Recall Section 4.5.2.5 of Chapter 4, an investment horizon of each RVC portfolio is set to cover all the out-of-sample period whereby its cumulative wealth is calculated from monthly returns in which no portfolio rebalancing is involved. Those monthly returns thus correspondingly reflect results from the buy-and-hold rebalancing strategy along the out-of-sample data. For periodic rebalancing strategy, each month a portfolio restores its deviated allocation back to the target allocation derived from optimisation. The methodology is as presented in Table 5.2 with transaction costs from monthly rebalancing included.

The summary of the selected representative portfolios are given in Table 5.4. When optimising for an SSD portfolio, the solving algorithm is GA where parameters values are the same as when optimising two-stage stochastic international portfolios in Chapter 4. Details for parameter settings are as displayed in Section 4.5.2.1 of Chapter 4. All portfolios are set to be optimised with the in-sample data on Jan-12 and Feb-12 to Jun-15 is the performance evaluation period for all rebalancing strategies.

**Table 5.4 Representative portfolios for implementing rebalancing strategies.** The low-risk portfolio has an expected return of 0.55% monthly, the medium-risk portfolio has an expected return of 1.05% monthly and the high-risk portfolio has an expected return of 1.60% monthly.

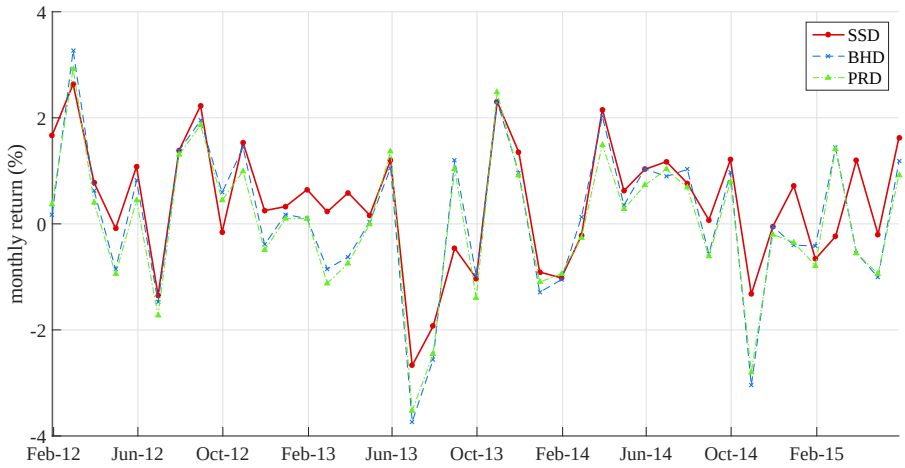
| Return | CVaR | Allocation | USD | ARS | BRL | CAD | CLP | MXN | EUR | | | | | GBP | CHF | AUD | INR | JPY | KRW | SGD | XAU | Total |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | US | Argentina | Brazil | Canada | Chile | Mexico | Austria | Belgium | Netherlands | France | Germany | UK | Switzerland | Australia | India | Japan | South Korea | Singapore | | |
| **Low-risk** | | | | | | | | | | | | | | | | | | | | | | |
| | | Govt bond | 5.73 | | | 4.33 | | 4.94 | 6.37 | 4.80 | 6.68 | 5.79 | 6.05 | 4.85 | | 4.52 | | 3.67 | 3.32 | 3.95 | | 65.00 |
| | | Corp bond | 2.78 | | | | | | 5.73 | | | 3.55 | | | | | | | | | | 12.06 |
| | | Equity | 0.69 | 0.10 | 0.17 | 0.01 | 1.99 | 0.85 | 0.15 | 0.64 | 0.16 | 0.50 | 0.11 | 0.32 | 0.90 | 1.17 | 0.13 | 0.02 | 0.34 | 0.15 | | 8.38 |
| 0.55 | 3.32 | Cash | 9.84 | | | | | | | | | | | | | | | | | | | 9.84 |
| | | Total Asset | **19.03** | **0.10** | **0.17** | **4.33** | **1.99** | **5.79** | **12.25** | **5.44** | **6.84** | **9.84** | **6.16** | **5.16** | **0.90** | **5.69** | **0.13** | **3.69** | **3.66** | **4.10** | **4.73** | **100.00** |
| | | Total Currency | **30.26** | **2.18** | **2.38** | **4.86** | **0.96** | **3.98** | | | **10.65** | | | **6.57** | **4.50** | **1.53** | **1.79** | **11.52** | **4.41** | **9.70** | | **100.00** |
| | | Overlay | **11.23** | **2.08** | **2.20** | **0.52** | **-1.03** | **-1.82** | | | **-29.89** | | | **1.40** | **3.60** | **-4.16** | **1.66** | **7.83** | **0.75** | **5.60** | **0.00** | **36.90** |
| **Medium-risk** | | | | | | | | | | | | | | | | | | | | | | |
| | | Govt bond | 4.73 | | | 3.44 | | 5.31 | 5.11 | 3.31 | 5.26 | 4.24 | 4.40 | 3.36 | | 4.80 | | 0.88 | 3.59 | 2.69 | | 51.11 |
| | | Corp bond | 1.15 | | | | | | 3.65 | | | 1.99 | | | | | | | | | | 6.80 |
| | | Equity | 0.22 | 10.36 | 1.29 | 0.09 | 3.90 | 5.88 | 0.34 | 1.59 | 0.21 | 0.15 | 0.07 | 0.63 | 0.16 | 0.17 | 4.02 | 0.59 | 0.38 | 0.08 | | 30.14 |
| 1.05 | 6.17 | Cash | 5.08 | | | | | | | | | | | | | | | | | | | 5.08 |
| | | Total Asset | **11.18** | **10.36** | **1.29** | **3.53** | **3.90** | **11.19** | **9.10** | **4.91** | **5.46** | **6.39** | **4.47** | **3.99** | **0.16** | **4.98** | **4.02** | **1.47** | **3.97** | **2.76** | **6.87** | **100.00** |
| | | Total Currency | **17.00** | **2.86** | **2.53** | **7.39** | **2.52** | **4.46** | | | **12.50** | | | **6.44** | **9.03** | **5.68** | **2.02** | **6.86** | **6.09** | **7.74** | | **100.00** |
| | | Overlay | **5.82** | **-7.50** | **1.23** | **3.86** | **-1.39** | **-6.72** | | | **-17.83** | | | **2.45** | **8.87** | **0.71** | **-2.00** | **5.39** | **2.12** | **4.98** | **0.00** | **35.43** |
| **High-risk** | | | | | | | | | | | | | | | | | | | | | | |
| | | Govt bond | 1.13 | | | 0.79 | | 2.90 | 1.60 | 1.67 | 1.77 | 1.14 | 2.16 | 0.13 | | 2.03 | | 0.00 | 1.85 | 1.59 | | 18.75 |
| | | Corp bond | 0.00 | | | | | | 1.44 | | | 0.11 | | | | | | | | | | 1.54 |
| | | Equity | 0.55 | 33.71 | 1.16 | 0.26 | 3.41 | 12.49 | 0.61 | 0.61 | 0.39 | 0.43 | 0.21 | 0.22 | 0.00 | 0.33 | 14.32 | 0.25 | 0.40 | 0.59 | | 69.94 |
| 1.60 | 13.40 | Cash | 3.28 | | | | | | | | | | | | | | | | | | | 3.28 |
| | | Total Asset | **4.96** | **33.71** | **1.16** | **1.05** | **3.41** | **15.39** | **3.64** | **2.28** | **2.16** | **1.67** | **2.37** | **0.35** | **0.00** | **2.36** | **14.32** | **0.25** | **2.25** | **2.19** | **6.49** | **100.00** |
| | | Total Currency | **13.04** | **11.95** | **6.28** | **7.54** | **1.98** | **1.53** | | | **10.32** | | | **4.44** | **8.69** | **6.18** | **4.51** | **3.94** | **6.03** | **7.10** | | **100.00** |
| | | Overlay | **8.08** | **-21.76** | **5.12** | **6.49** | **-1.42** | **-13.86** | | | **-1.81** | | | **4.09** | **8.69** | **3.82** | **-9.81** | **3.68** | **3.78** | **4.91** | **0.00** | **48.67** |

### 5.4.2 Performances of Rebalancing Strategies

This subsection presents the results of implementing different rebalancing strategies on low-risk, medium-risk and high-risk portfolios in separate plots. In terms of computational time, since the optimisation problem for an SSD rebalancing model is slightly extended from a two-stage international portfolio optimisation model in Chapter 4, the runtime taken on obtaining each rebalance solution is around the same as the time spent in Table 4.15. For the case of non-cardinality constrained portfolios as in the following experiment, it takes approximately 50 minutes to solve each portfolio rebalancing problem using a GA.

Figure 5.5 shows monthly returns of the low-risk portfolio after applying the second-order stochastic dominance (SSD), buy-and-hold (BHD) and periodic (PRD) rebalancing strategies. It is observed that monthly returns from BHD and PRD are relatively similar suggesting that the buy-and-hold allocation of the low-risk portfolio is only slightly affected by market movements. Consequently, a drifted allocation (BHD) does not deviate much from the original allocation (PRD) in each month. In contrast, when the low-risk portfolio adopts the SSD rebalancing strategy, monthly returns are significantly improved and different from BHD and PRD strategies. It is observed that the SSD strategy capitalises on updated financial market data to mitigate monthly losses at the expense of lowered returns in some periods comparing to the other two strategies. This could possibly be the merit of employing a regular-vine copula in capturing extreme events (through scenarios which are input of an optimisation problem, see Sections 4.3.1 and 4.3.2 of Chapter 4 for details). In addition, the SSD strategy also reduces rebalancing activity in the months that a drifted allocation is already an SSD portfolio, i.e., $\sum_{\beta=0}^{1} \text{CVaR}_{\beta}(Port_0) = \sum_{\beta=0}^{1} \text{CVaR}_{\beta}(Port_1)$. For the case of low-risk portfolio, the SSD strategy omits a reallocation in Oct-12, Jun-13 and Aug-13.

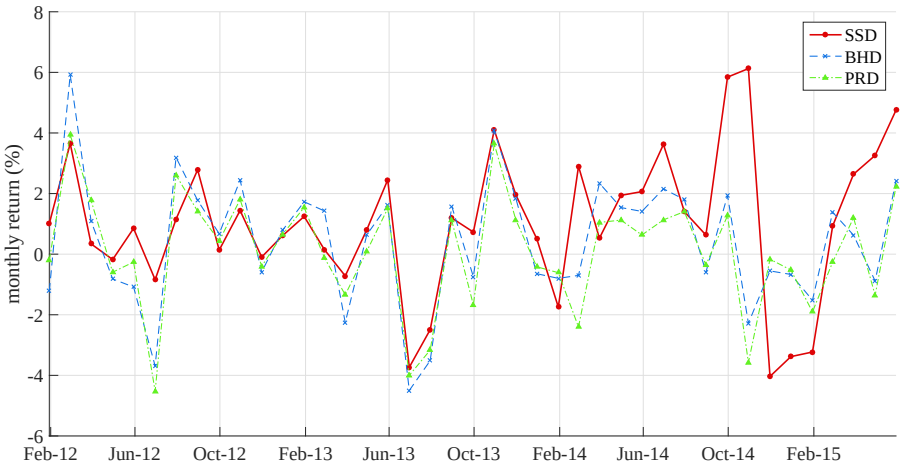
**Figure 5.5 Monthly Returns of the low-risk portfolio subject to each rebalancing strategy.** SSD denotes the second-order stochastic dominance rebalancing strategy. BHD and PRD denote the buy-and-hold and periodic rebalancing strategies. SSD rebalancing routine is omitted in Jul-12 and Oct-13 as BHD allocations in those periods are already second-order stochastic dominant.




What the SSD strategy does to adjust the portfolio allocation in response to new market information is reported in Tables C.1 to C.3 of Appendix C.1. It is noticed that, comparing to drifted allocations in the buy-and-hold strategy, SSD strategy underweighs currencies in Europe and overweighs USD and SGD in order to evade losses from the Eurozone-related currencies depreciation due to the aftermath of chronic sovereign debt crises in the Eurozone during 2010–2012. Besides, the SSD strategy also overweighs the safe-haven bonds in Germany and France and shifts away from the declining Eurozone stocks for Canada and Japan equities instead to earn upside gains for the portfolio.

The results of applying three rebalancing strategies on the medium-risk portfolio illustrated in Figure 5.6 show clearer monthly return divergence among rebalancing approaches. Performances of BHD and PRD are more distinguishable with the PRD becoming more fluctuating. The SSD strategy is still the best among all the strategies although suffering consecutive large losses from Dec-14 to Feb-15. The portfolio rebalancing routine is skipped in Oct-12, Jun-13 and Aug-13 as BHD allocations in those periods are already second-order stochastic dominant.

**Figure 5.6 Monthly Returns of the medium-risk portfolio subject to each rebalancing strategy.** SSD denotes the second-order stochastic dominance rebalancing strategy. BHD and PRD denote the buy-and-hold and periodic rebalancing strategies. SSD rebalancing routine is omitted in Oct-12, Jun-13 and Aug-13 as BHD allocations in those periods are already second-order stochastic dominant.



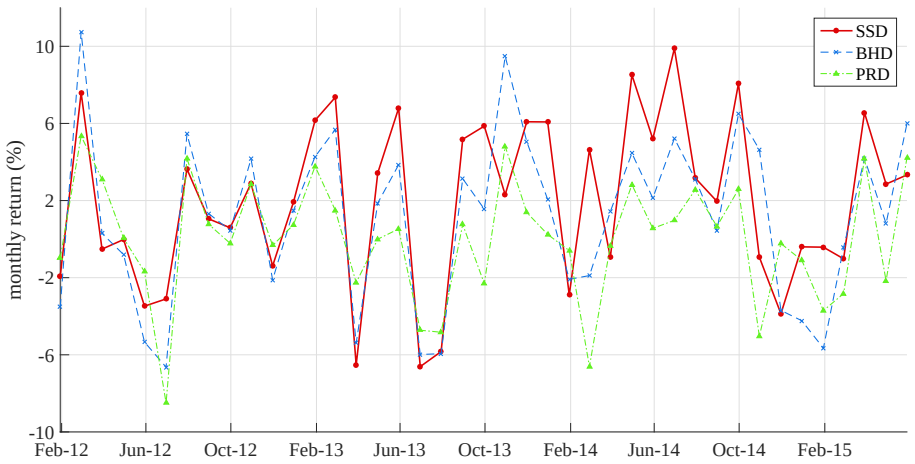

Additionally, it is shown in Tables C.4 to C.6 that the SSD strategy reallocates the medium-risk portfolio similarly to the low-risk portfolio. For currencies, SGD and JPY are overweighted against the drifted allocation. For assets, the SSD strategy still pays attention to high-quality government bonds in US, Germany and France and sells off European for Asian stocks (Korea and Japan) instead.

For performances on the high-risk portfolio, Figure 5.7 presents distinctive characteristics of monthly returns of each strategy. In terms of monthly return stability, the buy-and-hold is the steadiest whilst the SSD is the most oscillating among the three. Nonetheless, the SSD strategy provides healthiest monthly return on the average by achieving the highest return over other strategies in most periods despite some dismal performances in 2013. In terms of rebalancing frequency, there are four months (Sep-12, Oct-13, Dec-14 and Jan-15) that the SSD strategy decides not to rebalance the portfolio since the drifted allocations are already second-order stochastic dominant.

Tables C.7 to C.9 report details when the SSD strategy reallocates asset in the high-risk portfolio. It still shuffles the holdings of countries affected from European-debt crisis to countries in non- or less-affected areas. For currencies, major shifting in currency position is on CHF, followed by currencies in non-affected areas such as USD, GBP, JPY and SGD. An overweighting of CHF, however, takes a sharp reversal after the first quarter of 2014 and turns to be an underweighting eventually. For bonds, we observe inconclusive pattern on either government or corporate bonds reallocation which is

partly due to naturally low composition of low-risk assets in the high-risk portfolio. For equities, the SSD strategy firmly overweighs KRW especially from the first to the third quarter of 2014. The strategic moves on KRW stock reallocation, along with an abrupt CHF-USD switching pay off considerably well with frequent highest returns among the three strategies thereafter.

**Figure 5.7 Monthly Returns of the high-risk portfolio subject to each rebalancing strategy.** SSD denotes the second-order stochastic dominance rebalancing strategy. BHD and PRD denote the buy-and-hold and periodic rebalancing strategies. SSD rebalancing routine is omitted in Sep-12, Oct-13, Dec-14 and Jan-15 as BHD allocations in those periods are already second-order stochastic dominant.




Resulting monthly returns of applying different rebalancing strategies on the low-risk, medium risk and high-risk portfolios are summarised in Table 5.5. It is shown that the SSD strategy outperforms other strategies in terms of average, minimum and maximum returns in all levels of risk preference. The PRD strategy, in contrast, shows the poorest performance among the three which implies that the lack of flexibility in adapting allocation from adhering to original allocation generally leads to adverse asset allocation in fluctuating market environment.

From all the experiments, empirical results reveal the different performances among the rebalancing strategies which is also dependent on the risk preference of a portfolio. Fluctuation of monthly returns is increasing with the riskiness of a portfolio which is consistent for all rebalancing strategies. The benefit of learning new market information allows the SSD strategy to reallocate the portfolio promptly. Furthermore, through an evaluation of SSD criteria, rebalancing routine in some periods can also be discarded if the SSD condition on an existing portfolio still holds and hence save unnecessary transaction costs. Nonetheless, although the SSD strategy is proved to deliver higher

monthly returns, the occasional large losses are also worth considering, especially when implementing the SSD method on high-risk portfolios.

**Table 5.5 Average, minimum and maximum returns of portfolios under different rebalancing strategies.** The statistics are calculated over the rebalancing period of Feb-12 to May-15.

| Portfolio | Rebalancing strategy | Monthly return (%) | | |
|---|---|---|---|---|
| | | mean | min | max |
| Low-risk | SSD | 0.37 | -2.67 | 2.64 |
| | BHD | 0.13 | -3.73 | 3.27 |
| | PRD | 0.02 | -3.52 | 2.93 |
| Medium-risk | SSD | 1.01 | -4.03 | 6.12 |
| | BHD | 0.46 | -4.50 | 5.92 |
| | PRD | 0.10 | -4.52 | 3.93 |
| High-risk | SSD | 2.08 | -6.61 | 9.89 |
| | BHD | 1.24 | -6.64 | 10.75 |
| | PRD | 0.03 | -8.50 | 5.35 |

## 5.5 Conclusions

In this study, we propose a novel rebalancing model for an international portfolio which employs the notion of the second-order stochastic dominance (SSD). By its definition, the SSD criterion seeks a portfolio that is, on the average, more likely to have positive returns and less likely to have negative returns than a reference portfolio. In the study, the two portfolios of interest are a carried-over portfolio from the previous period and a newly-optimised portfolio with updated data. The carried-over (or existing) portfolio then reallocates to an allocation that is SSD. In case that the existing portfolio is already SSD, then rebalancing is not needed - saving unnecessary transaction costs.

Since it is shown that the SSD criterion can be expressed as a function of conditional value-at-risk (CVaR), a transformation of the optimisation model proposed in Chapter 4 can thus be done seamlessly. Solutions from the rebalancing model suggest how much to buy or sell each asset in a portfolio so that all the positions align with those of an SSD portfolio.

Performance of the SSD rebalancing strategy is subsequently compared with two conventional approaches, i.e., the buy-and-hold (BAH) and the periodic (PRD) re-

balancing strategies. All the rebalancing strategies are implemented on low-risk, medium-risk and high-risk portfolios in order to distinguish the impacts of different rebalancing methods at different levels of risk preference. The empirical results show that; although the SSD strategy exhibits occasional extreme negative returns in some periods (particularly for a high-risk portfolio), the SSD strategy outperforms the others in terms of average, minimum and maximum monthly returns at all risk levels.

# Chapter 6

# Conclusions

This thesis studied the problems of international portfolio construction where uncertainty underlying in financial asset characteristics is a major concern. The aims of this thesis were to improve the way uncertainty related to portfolio management is represented. It is shown in the empirical results that our developments in optimisation model formulation and scenarios generation results in better portfolio performance comparing to standard approaches.

The thesis exploits a wide range of disciplines, from finance, statistics, mathematics to computer science. The main theme is most relevant to general issues in robust optimisation under uncertainty. To capture relevant costs and complex interaction of assets, a sophisticated portfolio optimisation model and a regular-vine-based scenario generation method were created, and so depart from the standard notion of portfolio construction that typically relies on normality assumption. In terms of computation, since the resulting optimisation problem cannot be solved with an exact search method, solutions need to be instead approximated from a meta-heuristic. Nonetheless, it is strongly believed that the cost from the lack of perfect solution from a proper model is much smaller than the cost of having exact solutions from a wrong model.

## 6.1 Results Achieved

In Chapter 3, an incorporation of a hedging strategy into an international portfolio optimisation model is demonstrated. The hedging strategy combines various foreign exchange forward contracts in managing exposure of each currency which constitutes

a currency overlay for an international portfolio. Forward contracts allow a portfolio to reduce exposure of unfavourable currencies (in terms of risk and return) and increase exposure of favourable ones. We propose a structure of a portfolio model which facilitates an inclusion of different hedging costs and transaction costs with respect to different forward contract pairs.

The structure introduced allows us to incorporate hedging costs underlying in holding forward contracts which have not been taken into account in the existing literature. The costs associated to holding forwards included in our optimisation model are the cost-of-carry and the margin requirement. The first one accounts for gains and losses in a portfolio due to fluctuation in forward exchange rates while the latter one reflects a proportion of cash required to be reserved in a portfolio to hold a forward contract. Holding more contracts thus lowers portfolio returns. Including these two costs in the model consequently improves accuracy to risk and return calculation of an international portfolio.

Our international portfolio optimisation problem thus greatly extends the classical mean-variance portfolio optimisation which is originally a quadratic programming and convex. However, with fixed and variable transaction costs and cardinality constraints included, the final optimisation problem then becomes non-convex and integer-mixed. Nonetheless, CPLEX can still be used to solve the optimisation problem, at least on those with equal or smaller sizes.

The empirical results show that tight restrictions on the total overlay limit, margin requirement of forward contracts and the number of forward contracts in a currency overlay result in less-efficient portfolios comparing to the case of relaxed restrictions. When considering portfolios along the efficient frontier, it is observed that currency overlays are highly employed at both ends of the frontier. At low returns, portfolios require different forward contracts to hedge foreign currencies to keep low-risk-low-return profile. On the other hand, when portfolios require higher returns, forwards are held instead of equities as they provide competitive returns with lower volatilities.

However, it should be noted that the merit of currency overlay in improving risk-return profile of an international portfolio is more or less dependent on market conditions. Having higher exposure in some currencies in replacement of equities is sensible as long as there are currencies that generate competitive returns to equities but exhibiting lower risk. Otherwise, currency overlay is useful only for mitigating exchange rate risk.

In Chapter 4, we extend the deterministic international portfolio optimisation model in Chapter 3 to a two-stage stochastic programming with recourse and also expand the investment universe to four asset classes in 18 countries (14 currencies). All constraints related to currency overlay and costs concerning buy and sell of assets and forward contracts are included in the optimisation model.

Scenarios for the stochastic international portfolio optimisation model are constructed with a new methodology which applies a regular-vine copula in modelling the dependence structure of assets. The crucial advantage is that asymmetric dependence of financial assets can be expressed in the scenarios generated. More specifically, if two assets are asymmetrically dependent, then we can observe scenarios that two assets mutually having negative returns more frequently than the two mutually showing positive returns, or conversely. Such scenarios are, however, not viable if multivariate normal distribution is assumed. Empirically, it is observed that the scenarios generated from our approach exhibit means, minima and maxima which are closer to those of raw data than scenarios generated from a standard method. This shows that the new scenario generation approach performs well at capturing extreme events and risk-return characteristics of financial assets.

The objective of the optimisation problem is to minimise conditional value-at-risk which can be expressed as a linear function. Nevertheless, with the inclusion of fixed and variable transaction costs and cardinality constraints, the originally convex optimisation problem becomes non-convex and mixed-integer, and the resulting optimisation problem can no longer be practically solved with CPLEX. Instead, we solve the stochastic optimisation problem with a genetic algorithm implemented in MATLAB. It is shown that fitness values converge before reaching the maximum number of generations, demonstrating reliability of efficient portfolios obtained. The runtime taken on solving each portfolio is around 50 minutes.

Performances of portfolios optimised under Multivariate Normal (MVN) scenarios (a standard approach which assumes that asset returns are normally distributed) are compared with those optimised under Regular-Vine-Copula (RVC) scenarios (our approach which does not restrict asset returns to normal distribution and models asset interdependence by means of a regular-vine copula). The empirical results show that the return per risk ratio of RVC portfolios is higher than that of MVN portfolios, particularly when return target of an international portfolio is medium or high. This shows that the merit of employing a regular-vine copula in generating scenarios is

demonstrated when portfolio holds a substantial proportion of risky assets such as equities whose returns do not follow a normal distribution.

Chapter 5 studies the concept of the second-order stochastic dominance (SSD) criterion in developing a portfolio rebalancing model. The proposed strategy rebalances a portfolio when its floated allocation is second-order stochastically dominated by a newly optimised allocation. The rebalancing model is expressed as a stochastic optimisation problem. The objective function is to find an allocation which attains the SSD condition (either a deviated or optimised allocation) where scenarios are regenerated periodically to update market information. The SSD criterion is formulated with a summation of CVaRs over confidence levels. The two-stage stochastic international portfolio optimisation model in Chapter 4 can thus be modified to represent the SSD rebalancing strategy. The resulting optimisation problem for portfolio rebalancing is consequently similar to the optimisation problem of Chapter 4 in terms of mathematical expression and decision variables.

The experiments are conducted to compare performance of portfolios that implement the new rebalancing strategy and conventional rebalancing methods which are buy-and-hold and periodic. The buy-and-hold strategy is that portfolio allocation is allowed to deviate over time by market fluctuation. Performing assets thereby increase their share in the portfolio and vice versa. The periodic rebalancing, in contrast, reallocate the portfolio back to its original positions or its optimal allocation in every pre-defined period. Our SSD rebalancing strategy adopts the buy-and-hold strategy as a baseline and compare if the deviated allocation still second-order stochastically dominate the newly-optimised allocation with updated information from each new period. If the new allocation is second-order stochastically dominant, then the portfolio is reallocated to the new allocation, otherwise we let the portfolio allocation float with market fluctuation.

Three efficient portfolios from Chapter 4 with low-, medium- and high-risk preferences are used as initial allocations. The empirical results show that SSD rebalancing strategy generates higher average returns, higher maximum returns and lower minimum returns in all risk levels compared to those of buy-and-hold and periodic rebalancing strategies. This shows that our rebalancing strategy is more adaptive to changes in financial market conditions than conventional approaches.

The results from Chapter 4 and Chapter 5 exhibit the importance of scenarios in producing a good portfolio. We show a method that efficiently extracts realistic information regarding risk-return characteristics of individual asset and interdependency among

multiple assets for generating realistic scenarios. Nevertheless, apart from the quality of model, data availability is also crucial. It should be noted that our in-sample data ranges from Jan-02 to Dec-11 which covers several major boom-and-bust economic cycles. The length of this in-sample data provides a countless number of valuable facts on extreme events and asymmetric dependence of assets. If our scenarios were created from data containing only mild events or incomplete economic cycles, the resulting portfolios should exhibit lower performances and experience severe losses when extreme events occur.

### 6.1.1 Answers to The Research Questions

Recalling the research questions made in Section 1.2 of Chapter 1, the answers are obtained from the experiment results conducted in the thesis as follows:

- How could the incorporation of a currency overlay affect the risk-return reward of a resulting portfolio?
  - It is shown in Figure 3.10 on page 64 that allowing an adjustment of currency exposure ($V_u > 0$) on a portfolio results in more efficient Pareto front (comparing to the case that $V_u = 0$). Modifying currency exposure can reduce overall portfolio risk (especially at the lower end of an efficient frontier) and also increase portfolio expected return (particularly at the upper end of an efficient frontier).
- Does exchange rate hedging on an international portfolio result in more efficient portfolio?
  - Yes, it is illustrated in Figure 3.5 on page 58 that exchange rate hedging has an impact on risk-return profile of a portfolio, and the degree of risk-return reward improvement depends on the extent of currency hedging on a portfolio. Specifically, when comparing between three types of hedging; foreign (only non-USD exposure), fully-hedged (only USD exposure) and unrestricted (increase or decrease exposure on any currency), the resulting efficient frontiers are cascading by the level of hedging flexibility. The unrestricted is the most efficient, followed by the foreign and the fully-hedged.

- How would the risk-return compensation of a portfolio be affected when constraints associated to forward contracts holding are altered?

  - There are three constraints relevant to forward contracts holding, i.e., margin requirement (Figure 3.7 on page 60), total currency overlay limit (Figure 3.10 on page 64) and the number of forward contract in a currency overlay (Figure 3.14 on page 67). The experiment results show that the tighter the restrictions, the less efficient of resulting portfolios.

- Whether the proposed scenario generation method produces portfolios with better risk-return compensation than those produced by standard scenario generation approach?

  - The quality of scenarios generated from the proposed method can be assessed at the first instant by comparing the descriptive statistics as exhibited in Table 4.14 on page 145. It is shown that the new approach produces return distributions that are more resemble to the raw data than those of a conventional scenario generation method. Moreover, when comparing the out-of-sample performance of efficient portfolios (as in Figure 4.20 on page 153), it can be seen that portfolios constructed with scenarios from the new method demonstrate higher return per risk than portfolios created with scenarios generated from a traditional method.

- Would genetic algorithms provide reliable solutions for our international portfolio optimisation model?

  - Yes, the reliability of optimisation results is ascertained by investigating the evolution of fitness values over generations. Figure 4.8 to Figure 4.11 from page 130 to page 132 show that the fitness value reaches the minimum (and there is no further improvement) at around $250^{th}$ generation, long before the maximum number of generations at 500.

- Would the new rebalancing strategy outperform traditional rebalancing strategies in terms of portfolio return?

  - Yes, portfolios following the SSD rebalancing strategy proposed in Chapter 5 have higher average monthly return and lower minimum return comparing to portfolios adopting traditional rebalancing strategies (see Table 5.5 on page 182).

## 6.2 Contributions

The contributions of this thesis to the literature are primarily on the improvement of realistic international portfolio optimisation model.

In Chapter 3, in creating a hedged international portfolio, there are previous works that incorporated a currency overlay in an optimisation model. Thus an integration of currency overlay to find optimal hedging positions and asset allocation is not a new idea. However, none includes realistic costs of holding hedging instruments and transaction costs associated. Inclusion of such hedging costs improves accuracy of optimisation solutions and reflects true operational burdens involved in managing an international portfolio. Additionally, existing international portfolio optimisation models are not designed to accommodate transaction costs related to foreign exchange hedging (e.g., the cost of carry), thus the model presented in the thesis needed to be built from scratch. The constructed model is thus able to provide us an opportunity to study the effects of different constraint settings on risk-return profiles of international portfolios.

In Chapter 4, the deterministic international portfolio optimisation problem in Chapter 3 is reformulated as a stochastic programming problem. This improvement is made to improve the capability to capture true characteristics of financial assets both individually and collectively. The development of a new scenario generation method, making use of a regular-vine copula, allows better extraction of information on asset dependency, which is also a contribution to data science and computer science communities. The realistic scenarios constructed, along with realistic costs and constraints constitute a richer model which is more appropriate to handle sophisticated uncertainty in financial markets than existing approaches.

The resulting complex model is more challenging to solve. We show an application of the genetic algorithm to find efficient portfolios which is a stepping stone for further improvement on solving this kind of problem more efficiently. Finally in Chapter 5, the stochastic international portfolio optimisation model in Chapter 4 is augmented with the second-order stochastic dominance criterion to create a new portfolio rebalancing model. The results of the new rebalancing strategy comparing to those of traditional methods are thus new and not presented anywhere before.

To summarise, the contributions of this thesis are:

- The development of deterministic (Chapter 3) and stochastic (Chapter 4) international portfolio optimisation models which are able to manage currency exposure effectively through a currency overlay.
- The estimation of return and risk of resulting portfolio is accurate with the inclusion of related hedging costs and transaction costs (Chapter 3).
- Application of a copula-based approach helps capture financial assets uncertainty which creates realistic scenarios for a stochastic portfolio optimisation model (Chapter 4). It is also shown in the study that a portfolio created from a regular-vine-copula-based scenarios tends to have lower allocation of risky assets (as it is more aware of extreme events) than a portfolio created from a multivariate-normal scenarios.
- An interesting finding regarding the incorporation of currency overlay in an international portfolio is discovered. It is found that a portfolio requires different amount of currency overlay at different levels of expected return (Chapter 3). An international portfolio tends to have large currency overlay positions when a target return is low (a currency overlay is employed to reduce currency risk) and when a target return is high (a currency overlay is used to substitute equities to earn comparably high return but with lower risk).
- It is shown in Chapter 4 that the proposed stochastic programming can be solved by genetic algorithms. Although future work may find better algorithms to solve the problem, what is shown in this thesis is believed to be a good starting point for future development.
- An application of the second-order stochastic dominance criterion in creating a portfolio rebalancing model (Chapter 5) proves a better monthly return performance than that of traditional rebalancing methods.

## 6.3 Future Research Directions

This thesis has explored several issues thoroughly, nonetheless, we see that there is still room for further extensions and improvements.

1. **Extending the number of assets in an international portfolio.** Instead of holding indices of, for example, bonds or equities, the portfolio can invest in individual bonds or individual stocks.

   Since, a weight of each constituent in an index is determined by constituents' market value, composition of an index does not reflect true risk-return characteristic of each constituent. This implies that stock or bond indices are sub-optimal. Investing in individual assets can thus enhance efficiency of the portfolio.

   Increasing number of assets accompanied by a number of scenarios to express their uncertainties lead to even more challenging optimisation problem. It is likely that there are other heuristic methods that can solve such optimisation problem more efficiently than the genetic algorithm. Hybrid algorithms are also promising candidates for solving this large-scale problem. Hence, we can propose this optimisation model as a challenge for future works.

2. **Expanding portfolio investment to China.** Bond and stock markets in China are attractive investment venues because of their rapid growths and market sizes. However, capital control measures are the main obstacle for investment in China. Restrictions on the minimum stay of funds and cross-border transferring of funds make the optimisation model more complicated as proceeds from selling Chinese securities can only be used to purchased assets in China.

   Moreover, the Chinese Yuan is even more difficult to model due to its continual switching between fixed and floated exchange rate regimes. The current scenario generation method does not have the attempt to capture such uncertainty. A more advanced model is required to include asset and currency investments in China in an international portfolio.

3. **SSD condition under non-uniform risk preference.** In Chapter 5, it is shown that SSD criterion can be represented by a summation of CVaRs over a range of confidence levels. Each confidence level is assigned with a uniform weight throughout the range. Usually, CVaR is measured at 5% level confidence level; that is, an average of losses of events that occur 5% of the time. Thus, the lower values of confidence level indicate an average loss of more extreme events. It is sensible, when establishing the SSD criterion, that more emphasis could be put at lower confidence levels rather than higher. One alternative to construct such SSD condition is by summing weighted CVaRs across all the specified confidence levels where lower confidence levels are assigned with larger weights than

higher confidence levels. The resulting SSD condition is still valid in the sense of stochastic dominance with more emphasis on rare events. An application of the non-uniform-risk-preference SSD criterion on portfolio rebalancing is expected to provide even more secure portfolio against volatile markets than the uniform case applied in this thesis.

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# Appendix A

# Financial Terminologies

Descriptions and definitions of financial terms frequently used throughout the thesis are provided as follows:

## A.1 Portfolio

A portfolio is a collection of investments which are different in terms of return and risk. The main purpose of constructing a portfolio is to gain a benefit from risk diversification, as opposed to having only one single investment. Risk in a portfolio can be defined in various ways, but is typically associated with how likely to have a return lower than expected. A risky portfolio thus has a higher chance for its return to deviate from the target, which can although be higher or lower. A portfolio that invests in more than one currencies is defined as an international portfolio in this thesis. The proportion of fund allocated to each investment in a portfolio is usually referred to as a "position" or portfolio allocation.

## A.2 Position

A position is an amount of money invested in financial securities (e.g., bonds or equities) or financial instruments (e.g., forwards or options). In the context of portfolio investment, a position is referred to a portfolio allocation. There are two types of positions, long and short.

### A.2.1 Long Position

When a trader buys a security or an instrument into a portfolio, he/she is said to be opening a "long position" (allocation $> 0$). Such a position is said to be "closed", "squared" or "covered" (allocation $= 0$) if the trader sells the security or the instrument he/she is already long. A long position is in profit when the price increases, and vice versa.

### A.2.2 Short Position

Basically, a short position works in an opposite way to a long position. A short position or "short selling" (allocation $< 0$) is a practice to sell a security or an instrument that are not currently owned (by borrowing), and subsequently repurchase it to close the position at later time. A short position is thus in profit if the price falls, as the buy-back price is lower than the price of a security or an instrument when it was borrowed.

## A.3 Exposure

Exposure is typically referred to a state or condition of being unprotected and open to risk of suffering a loss in a transaction, or uncertainty. An investment can be exposed to multiple risk factors; in that case, the investment is said to have multiple risk exposure. For instance, a USD-funded trader who invests in the UK stock market is basically exposed to risk from stock market uncertainty and exchange rate uncertainty. In the thesis, risk associated with fluctuation in security prices is defined as "asset exposure" while risk associated with fluctuation in exchange rates (which affects valuation of foreign investment when converted to a base currency) is defined as "currency exposure" or "exchange rate exposure". Since a portfolio can have multiple positions on one asset class or one country, exposure on the respective asset class/country is thus calculated from the net position.

## A.4 Exchange Rate Pair

An exchange rate pair or a currency pair consists of two currencies denoting the rate at which one currency will be exchanged for another. The exchange rate market

convention quotes a "fixed" or "based" currency first, followed by a "variable" or "counter" currency. For instance, the quotation EUR/USD or EURUSD 1.2345 means that selling 1 Euro earns 1.2345 US dollars. It is said that when a trader sells EUR for USD, he/she enters a short position on EURUSD, creating a short position on EUR and a long position on USD. The following example shows how a trader gains or loses from exchange rate changes subject to his position on EURUSD pair.

| Action | Position/Exposure | |
|---|---|---|
| | EUR | USD |
| (1) A USD-funded trader purchases (long) 10,000 Euros at the EURUSD exchange rate of 1.1800. | 10,000 | -11,800 |
| (2) Later, he/she exchanges the 10,000 euros back into U.S. dollars at the exchange rate of 1.2500. | -10,000 | 12,500 |
| (3) Finally, he/she earns a profit of USD 700. | 0 | 700 |

There are two different types of exchange rates, "spot" and "forward" exchange rates.

### A.4.1 Spot Exchange Rate

The term "exchange rate" is generally referred to a spot exchange rate which is a current rate of exchange between two currencies.

### A.4.2 Forward Exchange Rate

A forward exchange rate[1] essentially refers to an exchange rate agreed today to exchange two currencies on a set future date.

[1] A forward rate is usually different from a spot exchange rate as it takes into account the interest rate differential between two countries. For example, a forward rate for GBPUSD can be calculated using the following formula:

$$F_{GBPUSD} = S_{GBPUSD}\frac{(1+i_{GBP})}{(1+i_{USD})}$$

where $i_{GBP}$ and $i_{USD}$ are respectively the interest rates of UK and US, $S_{GBPUSD}$ is the spot exchange rate and $F_{GBPUSD}$ is the forward exchange rate of GBPUSD.

## A.5 Foreign Exchange Rate Forward Contract

Foreign exchange forward contracts or FX forwards are used as a foreign currency hedge when an investor has an obligation to either make or take a foreign currency payment at some point in the future. The advantage is that the investor can lock in an exchange rate of future transaction from today, preventing further exchange rate uncertainty.

The agreed exchange rate of a forward contract is referred to a forward exchange rate (see A.4.2). The length of a forward contract can be tailored between two parties to suit their requirements, unlike future contracts.

During the time of forward contract holding, a varying spot exchange rate could differ from the locked-in forward rate, gain and loss of outstanding forward position thus occur accordingly. This is the key feature of FX forwards that are used to modify (either to increase or decrease) currency exposure of an international portfolio.

## A.6 Currency Overlay

Typically, a currency overlay is referred to a strategy conducted to adjust currency exposure of a portfolio in attempt to hedge or speculate a specific currency on the portfolio. Normally, a currency overlay is implemented as an ex-post decision once an asset allocation is decided. This, however, is a sub-optimal decision as final positions on asset and currency are decided independently.

In the thesis, since the aim is to obtain a portfolio that is optimal both in terms of asset and currency exposure, a currency overlay is also a part of decision variables along with asset allocation and is hence considered as an ex-ante decision. A currency overlay in the thesis is constructed from one or multiple forward contracts (from different exchange rate pair). Thus, alternatively, a currency overlay can be viewed as a portfolio of forward contracts.

To exemplify, consider a portfolio manager with an endowment of USD 10,000 investing in US, UK and German stock markets. Thus, his/her asset exposure is due to uncertain stock prices while his currency exposure is from uncertain GBPUSD and EURUSD exchange rates. Suppose that he/she allocates 50% of his fund for US stocks

30% for UK stocks and 20% to German stocks, his/her portfolio then has the following exposure:

| | US | UK | Germany |
|---|---|---|---|
| asset exposure (%) | 50 | 30 | 20 |
| currency exposure (%) | 50 | 30 | 20 |

That is, changes of stock prices in the US, the UK and Germany account for 50%, 30% and 20% of a portfolio value, respectively. In addition, the portfolio value is also affected by exchange rate uncertainties of GBPUSD and EURUSD by 30% and 20%, respectively. Note that the currency exposure on USD does not affect the portfolio value as it is funded by US dollars.

Suppose that the portfolio manager prefers to reduce currency exposure in EUR to 0% and increase exposure in GBP to 45% as follows:

| | US | UK | Germany |
|---|---|---|---|
| asset exposure (%) | 50 | 30 | 20 |
| currency exposure (%) | 55 | 45 | 0 |
| currency overlay (%) | 5 | 15 | -20 |

The adjustment on currency exposure creates the difference between currency and asset exposure in the portfolio, and that results in a “currency overlay”. Such currency exposure adjustment can be created from a combination of forward contracts, which can have many possibilities as exemplified below.

| | US | UK | Germany |
|---|---|---|---|
| currency overlay (%) | 5 | 15 | -20 |
| choice 1 | | | |
| short GBPUSD forward | 5 | -5 | |
| short EURGBP forward | | 20 | -20 |
| choice 2 | | | |
| long GBPUSD forward | -15 | 15 | |
| short EURUSD forward | 20 | | -20 |

It can be seen that the overlay position above can be constructed from various combination of forward contracts (in fact, there are more than 2 choices possible). Factors that determines the right combination of forward contracts are, for instance, transaction cost and forward exchange rate of each forward contract (see thorough details in Chapter 3).

Based on the example given above, a currency overlay in this thesis is therefore defined as the summation of positions of forward contracts (net forward positions). Alternatively, it can be considered as a portfolio of forward contracts themselves, or it can be viewed as the difference between currency exposure and asset exposure in a portfolio. In sum, depending on the context, a currency overlay can be viewed as:

- Net forward positions.
- A portfolio of forward contracts
- The difference between currency and asset exposure in a portfolio.

# Appendix B

# Additional information for Chapter 4

## B.1 Data Description

Figures B.1 and B.2 demonstrate that government bond and corporate bond indices are the asset classes that demonstrate lowest return dispersion (low uncertainty), followed by currencies and gold. Stock indices show highest fluctuation of returns and also asymmetric distributions with more chances for extreme negative returns.

In addition to details of return distributions from the box plots, we conduct the Jarque-Bera Test to examine if each return time series is normally distributed. The test results are presented in Table B.1. It is found that, at 5 percent level of significance, only return distributions of France government bond index (govt.EUR.FR), Germany government bond index (govt.EUR.DE) and EU corporate bond index are normal.

**Figure B.1 The box plot of government bond indices and currency returns.** Raw data of monthly returns from Jan-02 to Dec-11. The box plots provide a visualisation of summary statistics for our data. The left and right ends of each box denote the $25^{th}$- and $75^{th}$-percentiles of the series, respectively. The line in the middle of each box locates the sample median. If the median is not centered in the box, the data distribution is skew. The whisker lengths are 1.5 times the interquartile range away from the left and right ends of the box. Observations beyond the whisker length are considered as outliers.

**Figure B.2 The box plot of stock indices, corporate bond indices and gold returns.** Raw data of monthly returns from Jan-02 to Dec-11. The box plots provide a visualisation of summary statistics for our data. The left and right ends of each box denote the $25^{th}$- and $75^{th}$-percentiles of the series, respectively. The line in the middle of each box locates the sample median. If the median is not centered in the box, the data distribution is skew. The whisker lengths are 1.5 times the interquartile range away from the left and right ends of the box. Observations beyond the whisker length are considered as outliers.

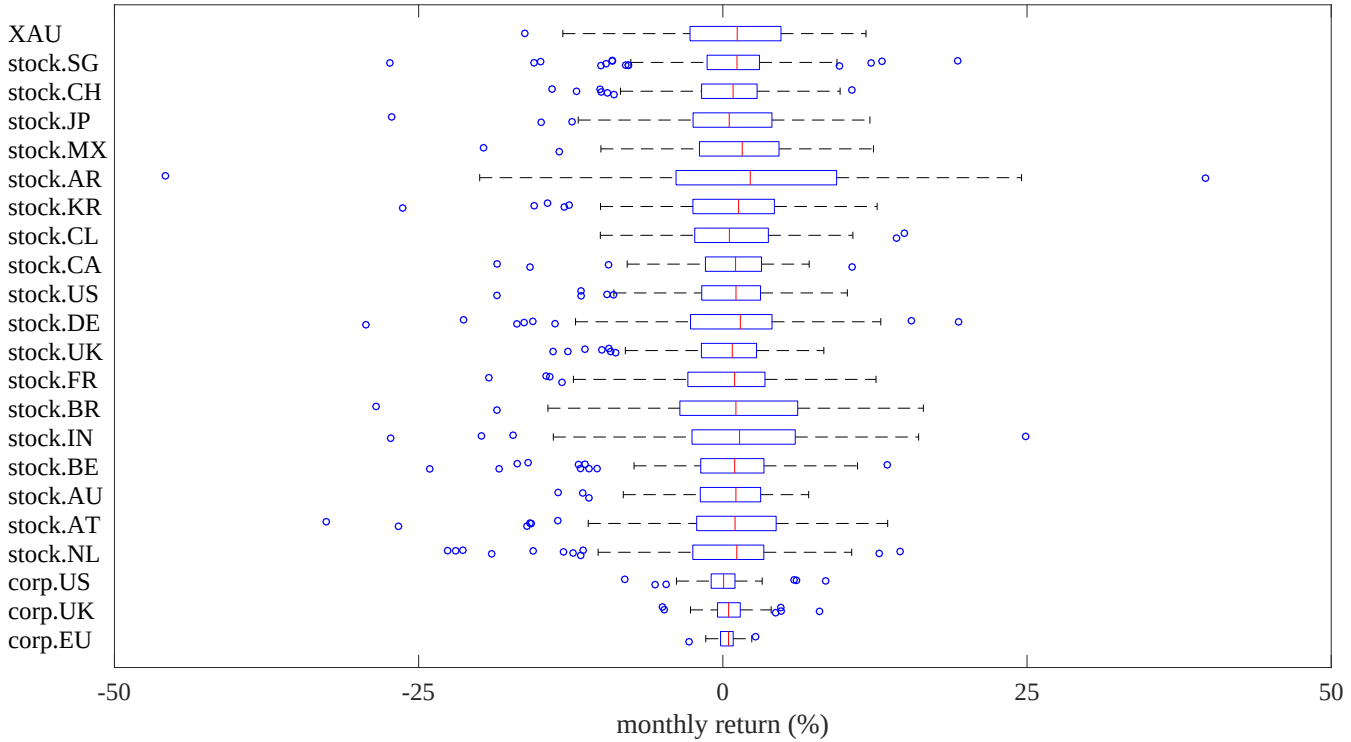


**Table B.1 p-values of the Jarque-Bera test for normal distribution.** The variable notations are as given in Table 4.7. At 5 percent level of significance, a variable follows a normal distribution if its p-value is greater than 0.05 (indicated in bold).

| variable | ARS | BRL | CAD | CLP | MXN | EUR | GBP | CHF |
|---|---|---|---|---|---|---|---|---|
| p-value | 0.001 | 0.001 | 0.001 | 0.001 | 0.001 | 0.006 | 0.007 | 0.001 |
| variable | AUD | INR | JPY | KRW | SGD | govt.AU | govt.AT | govt.BE |
| p-value | 0.001 | 0.002 | 0.230 | 0.001 | 0.001 | 0.032 | 0.040 | 0.001 |
| variable | govt.CA | govt.FR | govt.DE | govt.KR | govt.JP | govt.MX | govt.NL | govt.SG |
| p-value | 0.011 | 0.070 | **0.241** | 0.001 | 0.001 | 0.008 | 0.019 | 0.002 |
| variable | govt.UK | govt.US | corp.EU | corp.UK | corp.US | stock.NL | stock.AT | stock.AU |
| p-value | 0.034 | 0.005 | **0.146** | 0.001 | 0.001 | 0.001 | 0.001 | 0.001 |
| variable | stock.BE | stock.IN | stock.BR | stock.FR | stock.UK | stock.DE | stock.US | stock.CA |
| p-value | 0.001 | 0.001 | 0.004 | 0.003 | 0.002 | 0.001 | 0.001 | 0.001 |
| variable | stock.CL | stock.KR | stock.AR | stock.MX | stock.JP | stock.CH | stock.SG | XAU |
| p-value | 0.500 | 0.001 | 0.001 | 0.002 | 0.001 | 0.002 | 0.001 | **0.252** |

## B.2 Sequential Method to Select an R-Vine Model

Dissmann et al. [36] proposed that the algorithm involves searching for an appropriate R-vine tree structure, the pair-copula families, and the parameter values of the chosen pair-copula families which is summarised as follows.

1. Calculate the empirical Kendall's tau for all possible variable pairs.
2. Select the tree that maximises the sum of absolute values of Kendall's taus.
3. Select a copula for each pair and fit the corresponding parameters.
4. Transform the observations using the copula and parameters from Step 3. To obtain the transformed values.
5. Use transformed observations to calculate empirical Kendall's taus for all possible pairs.
6. Proceed with Step 2. Repeat until the R-Vine is fully specified.

## B.3 Scenario Generation using R-Vine Copula

1. *Modelling marginal distributions* - The example of fitting marginal distributions to monthly returns of US treasury index, S&P500 index and EURUSD is shown in Figure B.3. In our scenario generation method, the empirical distribution estimates are employed to fit the return distributions.
2. *Estimating a regular-vine copula* - The regular-vine copula specification is estimated by the sequential method of Dissmann et al. [36] as outlined in B.2. Let $X_1, X_2$ and $X_3$ denote the monthly returns of US treasury index, S&P500 index and EURUSD, respectively. The estimation results from the sequential method show that the joint density decomposition is as shown in Figure B.4.
3. *Sampling for a regular-vine density* - We simulate 1,000 scenarios for returns of each asset. Figure B.5 display histograms of return distributions from raw data (left panels) and histograms of return distributions from simulated data (right panels). The simulated marginal distributions exhibit similar means and variances to those from raw data. Figure B.6 shows the bivariate return distributions of raw data and simulated data. The joint distribution shows the

interdependence between asset returns which is characterised by the copula family and copula parameters estimated earlier.

**Figure B.3 Marginal distribution fitting of US treasury index, S&P500 index and EURUSD monthly returns over Jan-02 to Dec-11.** The histograms illustrate distributions of raw data, the blue curves show the assumed normal distributions while the red curves demonstrate empirical distributions estimated by Epanechnikov kernel and optimal bandwidth. It can be noticed that extreme returns can be better captured by empirical distributions that normal distributions.

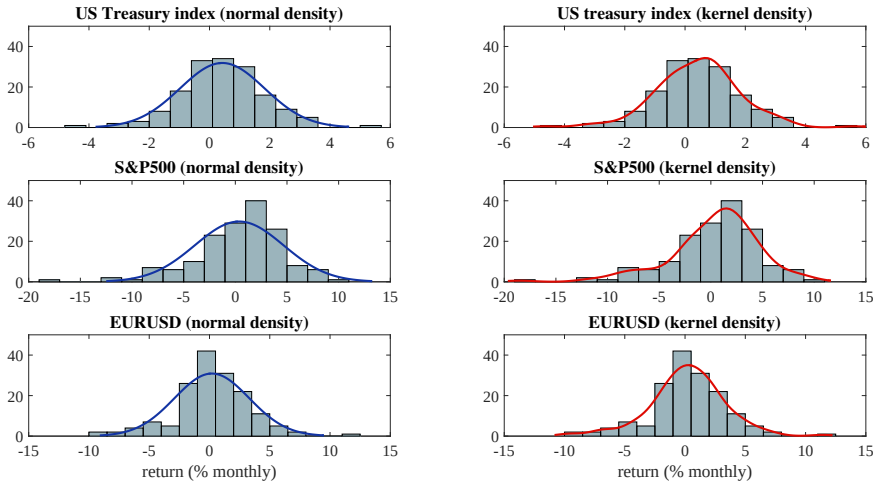



**Figure B.4 The regular vine copula of US government bond index, S&P500 index and EURUSD.** With three variables, there exists two trees composing the vine. The yellow nodes represent the return distributions while the edges linking the nodes represents the associated copulas. The combination of nodes and edges from all trees produces the joint density function.

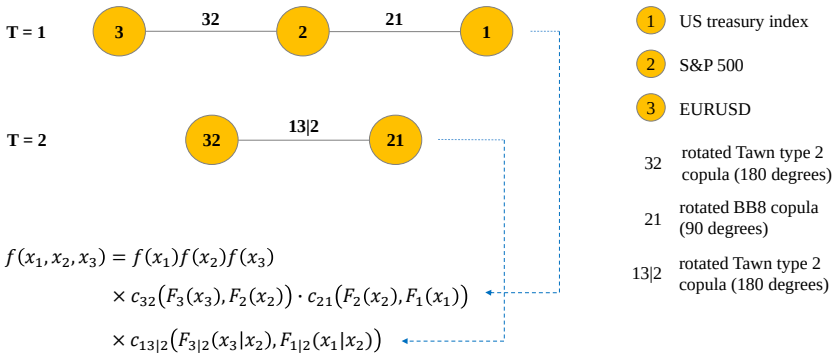


**Figure B.5 The return histograms of raw data (left panels) and simulated data (right panels).**

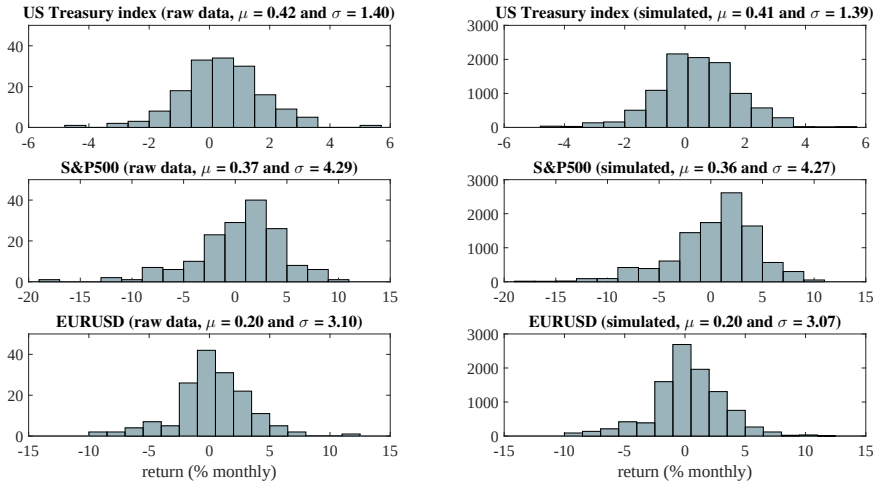
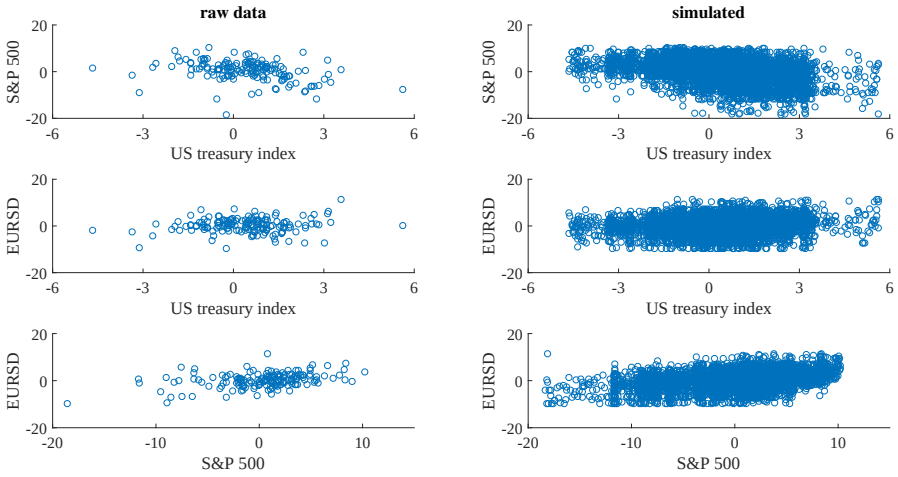



**Figure B.6 The joint return distributions by asset pairs.** The left panels show the bivariate distributions of raw data and the right panels show the bivariate distributions of simulated data.




# B.4 Optimal Portfolio Allocations

This sections present allocations obtained from optimising the stochastic portfolio optimisation on Regular-Vine-Copula base (RVC) and Multivariate Normal (MVN) scenarios. (See Chapter 4 for details).

**Table B.2 Optimal portfolio allocations under regular-vine-copula-based scenarios at return targets 0.55% to 0.75% monthly.**

| Return | CVaR | Allocation | USD | ARS | BRL | CAD | CLP | MXN | | | EUR | | | GBP | CHF | AUD | INR | JPY | KRW | SGD | XAU | Total |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | US | Argentina | Brazil | Canada | Chile | Mexico | Austria | Belgium | Netherlands | France | Germany | UK | Switzerland | Australia | India | Japan | South Korea | Singapore | | |
| | | Govt bond | 5.73 | | | 4.33 | | 4.94 | 6.37 | 4.80 | 6.68 | 5.79 | 6.05 | 4.85 | | 4.52 | | 3.67 | 3.32 | 3.95 | | 65.00 |
| | | Corp bond | 2.78 | | | | | | 5.73 | | | 3.55 | | | | | | | | | | 12.06 |
| | | Equity | 0.69 | 0.10 | 0.17 | 0.01 | 1.99 | 0.85 | 0.15 | 0.64 | 0.16 | 0.50 | 0.11 | 0.32 | 0.90 | 1.17 | 0.13 | 0.02 | 0.34 | 0.15 | | 8.38 |
| 0.55 | 3.32 | Cash | 9.84 | | | | | | | | | | | | | | | | | | | 9.84 |
| | | Total Asset | **19.03** | **0.10** | **0.17** | **4.33** | **1.99** | **5.79** | **12.25** | **5.44** | **6.84** | **9.84** | **6.16** | **5.16** | **0.90** | **5.69** | **0.13** | **3.69** | **3.66** | **4.10** | **4.73** | **100.00** |
| | | Total Currency | **30.26** | **2.18** | **2.38** | **4.86** | **0.96** | **3.98** | | | **10.65** | | | **6.57** | **4.50** | **1.53** | **1.79** | **11.52** | **4.41** | **9.70** | | **100.00** |
| | | Overlay | **11.23** | **2.08** | **2.20** | **0.52** | **-1.03** | **-1.82** | | | **-29.89** | | | **1.40** | **3.60** | **-4.16** | **1.66** | **7.83** | **0.75** | **5.60** | **0.00** | **36.90** |
| | | Govt bond | 5.82 | | | 4.41 | | 5.09 | 6.36 | 4.74 | 6.67 | 5.76 | 6.01 | 4.87 | | 4.65 | | 3.66 | 3.45 | 4.03 | | 65.53 |
| | | Corp bond | 2.71 | | | | | | 5.69 | | | 3.54 | | | | | | | | | | 11.94 |
| | | Equity | 0.46 | 0.12 | 0.11 | 0.04 | 2.25 | 1.24 | 0.12 | 0.39 | 0.31 | 0.40 | 0.11 | 0.19 | 1.06 | 0.82 | 0.23 | 0.01 | 0.21 | 0.12 | | 8.17 |
| 0.60 | 3.37 | Cash | 9.44 | | | | | | | | | | | | | | | | | | | 9.44 |
| | | Total Asset | **18.43** | **0.12** | **0.11** | **4.45** | **2.25** | **6.33** | **12.17** | **5.13** | **6.98** | **9.70** | **6.12** | **5.06** | **1.06** | **5.47** | **0.23** | **3.67** | **3.65** | **4.16** | **4.92** | **100.00** |
| | | Total Currency | **29.63** | **1.67** | **2.26** | **5.22** | **1.90** | **3.92** | | | **11.08** | | | **6.71** | **4.85** | **2.62** | **1.66** | **9.32** | **4.64** | **9.60** | | **100.00** |
| | | Overlay | **11.20** | **1.55** | **2.15** | **0.77** | **-0.35** | **-2.41** | | | **-29.02** | | | **1.65** | **3.79** | **-2.85** | **1.42** | **5.66** | **0.99** | **5.45** | **0.00** | **34.63** |
| | | Govt bond | 5.91 | | | 4.49 | | 5.24 | 6.35 | 4.67 | 6.67 | 5.72 | 5.97 | 4.90 | | 4.78 | | 3.64 | 3.58 | 4.12 | | 66.06 |
| | | Corp bond | 2.64 | | | | | | 5.64 | | | 3.53 | | | | | | | | | | 11.82 |
| | | Equity | 0.23 | 0.14 | 0.05 | 0.08 | 2.51 | 1.62 | 0.09 | 0.15 | 0.46 | 0.30 | 0.10 | 0.06 | 1.21 | 0.47 | 0.34 | 0.00 | 0.07 | 0.09 | | 7.96 |
| 0.65 | 3.49 | Cash | 9.04 | | | | | | | | | | | | | | | | | | | 9.04 |
| | | Total Asset | **17.82** | **0.14** | **0.05** | **4.56** | **2.51** | **6.86** | **12.09** | **4.82** | **7.13** | **9.56** | **6.07** | **4.96** | **1.21** | **5.25** | **0.34** | **3.65** | **3.65** | **4.22** | **5.12** | **100.00** |
| | | Total Currency | **28.99** | **1.15** | **2.14** | **5.58** | **2.09** | **3.86** | | | **11.52** | | | **6.85** | **5.20** | **2.95** | **1.52** | **8.63** | **4.88** | **9.51** | | **100.00** |
| | | Overlay | **11.17** | **1.02** | **2.09** | **1.02** | **-0.42** | **-3.00** | | | **-28.15** | | | **1.90** | **3.98** | **-2.30** | **1.18** | **4.98** | **1.23** | **5.30** | **0.00** | **33.87** |
| | | Govt bond | 5.93 | | | 4.52 | | 5.42 | 6.18 | 4.49 | 6.48 | 5.51 | 5.75 | 4.85 | | 4.97 | | 3.40 | 3.78 | 4.14 | | 65.43 |
| | | Corp bond | 2.29 | | | | | | 5.37 | | | 3.48 | | | | | | | | | | 11.14 |
| | | Equity | 0.12 | 0.72 | 0.11 | 0.09 | 3.01 | 2.51 | 0.11 | 0.03 | 0.46 | 0.00 | 0.26 | 0.21 | 1.31 | 0.15 | 0.06 | 0.17 | 0.02 | 0.08 | | 9.40 |
| 0.70 | 3.57 | Cash | 8.43 | | | | | | | | | | | | | | | | | | | 8.43 |
| | | Total Asset | **16.77** | **0.72** | **0.11** | **4.61** | **3.01** | **7.94** | **11.66** | **4.51** | **6.94** | **9.00** | **6.01** | **5.06** | **1.31** | **5.12** | **0.06** | **3.57** | **3.81** | **4.22** | **5.59** | **100.00** |
| | | Total Currency | **27.20** | **0.87** | **2.60** | **6.13** | **2.56** | **3.90** | | | **11.59** | | | **6.92** | **6.16** | **2.75** | **0.79** | **8.40** | **5.29** | **9.25** | | **100.00** |
| | | Overlay | **10.43** | **0.15** | **2.50** | **1.53** | **-0.45** | **-4.03** | | | **-26.52** | | | **1.87** | **4.85** | **-2.37** | **0.73** | **4.83** | **1.48** | **5.03** | **0.00** | **33.38** |
| | | Govt bond | 5.89 | | | 4.45 | | 5.47 | 6.13 | 4.43 | 6.41 | 5.44 | 5.67 | 4.75 | | 4.97 | | 3.10 | 3.77 | 4.02 | | 64.49 |
| | | Corp bond | 1.95 | | | | | | 5.24 | | | 3.37 | | | | | | | | | | 10.56 |
| | | Equity | 0.03 | 1.53 | 0.03 | 0.12 | 3.12 | 2.84 | 0.38 | 0.16 | 0.13 | 0.25 | 0.23 | 0.51 | 1.17 | 0.04 | 0.36 | 0.04 | 0.16 | 0.00 | | 11.11 |
| 0.75 | 3.73 | Cash | 8.13 | | | | | | | | | | | | | | | | | | | 8.13 |
| | | Total Asset | **15.99** | **1.53** | **0.03** | **4.57** | **3.12** | **8.31** | **11.75** | **4.59** | **6.54** | **9.06** | **5.90** | **5.26** | **1.17** | **5.01** | **0.36** | **3.15** | **3.94** | **4.02** | **5.72** | **100.00** |
| | | Total Currency | **26.24** | **0.50** | **2.29** | **6.39** | **3.08** | **4.31** | | | **11.68** | | | **7.09** | **6.58** | **3.38** | **1.15** | **6.85** | **5.54** | **9.20** | | **100.00** |
| | | Overlay | **10.25** | **-1.02** | **2.26** | **1.81** | **-0.03** | **-4.00** | | | **-26.16** | | | **1.83** | **5.41** | **-1.62** | **0.79** | **3.70** | **1.60** | **5.18** | **0.00** | **32.84** |

**Table B.3 Optimal portfolio allocations under regular-vine-copula-based scenarios at return targets 0.80% to 1.00% monthly.**

| Return | CVaR | Allocation | USD | ARS | BRL | CAD | CLP | MXN | EUR | | | | | GBP | CHF | AUD | INR | JPY | KRW | SGD | XAU | Total |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | US | Argentina | Brazil | Canada | Chile | Mexico | Austria | Belgium | Netherlands | France | Germany | UK | Switzerland | Australia | India | Japan | South Korea | Singapore | | |
| | | Govt bond | 5.68 | | | 4.41 | | 5.47 | 5.75 | 4.08 | 6.02 | 5.06 | 5.27 | 4.58 | | 5.01 | | 2.98 | 3.87 | 4.02 | | 62.18 |
| | | Corp bond | 1.81 | | | | | | 4.86 | | | 3.24 | | | | | | | | | | 9.92 |
| | | Equity | 0.01 | 2.96 | 0.19 | 0.03 | 3.51 | 3.49 | 0.16 | 0.46 | 0.29 | 0.01 | 0.62 | 0.08 | 1.46 | 0.02 | 0.46 | 0.50 | 0.25 | 0.21 | | 14.73 |
| 0.80 | 4.13 | Cash | 7.22 | | | | | | | | | | | | | | | | | | | 7.22 |
| | | Total Asset | **14.72** | **2.96** | **0.19** | **4.44** | **3.51** | **8.96** | **10.77** | **4.54** | **6.31** | **8.32** | **5.89** | **4.66** | **1.46** | **5.02** | **0.46** | **3.48** | **4.13** | **4.23** | **5.95** | **100.00** |
| | | Total Currency | **24.15** | **1.06** | **3.08** | **6.56** | **2.64** | **4.04** | | | **12.17** | | | **6.93** | **7.06** | **3.79** | **0.40** | **7.42** | **5.96** | **8.78** | | **100.00** |
| | | Overlay | **9.43** | **-1.90** | **2.88** | **2.12** | **-0.87** | **-4.92** | | | **-23.66** | | | **2.27** | **5.60** | **-1.23** | **-0.06** | **3.94** | **1.83** | **4.55** | **0.00** | **32.63** |
| | | Govt bond | 5.64 | | | 4.26 | | 5.54 | 5.84 | 4.17 | 6.06 | 5.09 | 5.29 | 4.48 | | 5.09 | | 2.28 | 3.84 | 3.68 | | 61.26 |
| | | Corp bond | 0.89 | | | | | | 4.71 | | | 3.12 | | | | | | | | | | 8.72 |
| | | Equity | 0.08 | 4.59 | 0.23 | 0.09 | 3.42 | 3.87 | 0.38 | 0.12 | 0.68 | 0.19 | 0.01 | 0.12 | 0.66 | 0.02 | 1.12 | 0.37 | 0.18 | 0.66 | | 16.77 |
| 0.85 | 4.41 | Cash | 6.89 | | | | | | | | | | | | | | | | | | | 6.89 |
| | | Total Asset | **13.49** | **4.59** | **0.23** | **4.35** | **3.42** | **9.40** | **10.93** | **4.29** | **6.74** | **8.40** | **5.30** | **4.60** | **0.66** | **5.11** | **1.12** | **2.65** | **4.02** | **4.34** | **6.37** | **100.00** |
| | | Total Currency | **22.73** | **1.78** | **2.09** | **6.83** | **3.40** | **3.65** | | | **12.10** | | | **6.86** | **7.57** | **4.71** | **0.73** | **6.33** | **5.83** | **9.04** | | **100.00** |
| | | Overlay | **9.23** | **-2.80** | **1.86** | **2.48** | **-0.03** | **-5.76** | | | **-23.56** | | | **2.25** | **6.91** | **-0.40** | **-0.39** | **3.68** | **1.82** | **4.70** | **0.00** | **32.94** |
| | | Govt bond | 5.55 | | | 4.32 | | 5.86 | 5.42 | 3.66 | 5.66 | 4.64 | 4.83 | 4.31 | | 5.05 | | 2.20 | 3.95 | 3.71 | | 59.16 |
| | | Corp bond | 0.85 | | | | | | 4.33 | | | 2.93 | | | | | | | | | | 8.10 |
| | | Equity | 0.36 | 6.68 | 0.08 | 0.25 | 4.19 | 5.02 | 0.16 | 0.02 | 0.01 | 0.03 | 0.26 | 0.48 | 1.22 | 0.31 | 0.27 | 0.19 | 0.54 | 0.32 | | 20.39 |
| 0.90 | 4.87 | Cash | 5.78 | | | | | | | | | | | | | | | | | | | 5.78 |
| | | Total Asset | **12.53** | **6.68** | **0.08** | **4.57** | **4.19** | **10.89** | **9.90** | **3.68** | **5.67** | **7.60** | **5.09** | **4.79** | **1.22** | **5.36** | **0.27** | **2.39** | **4.49** | **4.03** | **6.56** | **100.00** |
| | | Total Currency | **20.40** | **2.44** | **2.64** | **7.00** | **2.89** | **4.15** | | | **11.77** | | | **6.93** | **7.88** | **3.96** | **1.13** | **7.54** | **6.48** | **8.23** | | **100.00** |
| | | Overlay | **7.87** | **-4.23** | **2.55** | **2.44** | **-1.30** | **-6.74** | | | **-20.17** | | | **2.14** | **6.66** | **-1.40** | **0.86** | **5.15** | **1.98** | **4.19** | **0.00** | **33.85** |
| | | Govt bond | 5.20 | | | 3.91 | | 5.61 | 5.70 | 3.90 | 5.92 | 4.86 | 5.05 | 3.90 | | 5.03 | | 1.34 | 3.73 | 3.20 | | 57.35 |
| | | Corp bond | 0.77 | | | | | | 4.42 | | | 2.46 | | | | | | | | | | 7.66 |
| | | Equity | 0.18 | 7.69 | 0.01 | 0.18 | 3.88 | 5.17 | 0.16 | 0.05 | 0.01 | 0.17 | 0.55 | 0.41 | 0.70 | 0.05 | 2.57 | 0.29 | 0.05 | 0.21 | | 22.32 |
| 0.95 | 5.13 | Cash | 5.93 | | | | | | | | | | | | | | | | | | | 5.93 |
| | | Total Asset | **12.09** | **7.69** | **0.01** | **4.09** | **3.88** | **10.77** | **10.28** | **3.94** | **5.93** | **7.50** | **5.60** | **4.32** | **0.70** | **5.08** | **2.57** | **1.63** | **3.77** | **3.42** | **6.74** | **100.00** |
| | | Total Currency | **19.99** | **4.20** | **3.13** | **6.71** | **1.87** | **6.54** | | | **11.76** | | | **6.72** | **8.34** | **3.99** | **0.95** | **4.93** | **5.63** | **8.51** | | **100.00** |
| | | Overlay | **7.91** | **-3.50** | **3.13** | **2.62** | **-2.01** | **-4.24** | | | **-21.49** | | | **2.40** | **7.64** | **-1.09** | **-1.62** | **3.30** | **1.86** | **5.09** | **0.00** | **33.95** |
| | | Govt bond | 5.24 | | | 3.75 | | 5.59 | 5.51 | 3.58 | 5.75 | 4.62 | 4.80 | 3.63 | | 4.94 | | 1.12 | 3.70 | 3.19 | | 55.42 |
| | | Corp bond | 0.08 | | | | | | 4.21 | | | 2.11 | | | | | | | | | | 6.40 |
| | | Equity | 0.18 | 8.76 | 0.35 | 0.66 | 4.12 | 5.73 | 0.06 | 0.17 | 0.07 | 0.21 | 0.03 | 0.45 | 1.10 | 0.19 | 3.13 | 0.18 | 0.28 | 0.35 | | 25.99 |
| 1.00 | 5.56 | Cash | 5.55 | | | | | | | | | | | | | | | | | | | 5.55 |
| | | Total Asset | **11.05** | **8.76** | **0.35** | **4.41** | **4.12** | **11.31** | **9.77** | **3.74** | **5.82** | **6.94** | **4.83** | **4.08** | **1.10** | **5.12** | **3.13** | **1.31** | **3.97** | **3.54** | **6.64** | **100.00** |
| | | Total Currency | **18.53** | **2.94** | **2.24** | **7.51** | **2.63** | **4.75** | | | **11.91** | | | **6.57** | **8.55** | **5.05** | **1.61** | **6.80** | **6.24** | **8.03** | | **100.00** |
| | | Overlay | **7.49** | **-5.82** | **1.89** | **3.11** | **-1.49** | **-6.57** | | | **-19.20** | | | **2.49** | **7.46** | **-0.07** | **-1.52** | **5.49** | **2.26** | **4.48** | **0.00** | **34.67** |

**Table B.4 Optimal portfolio allocations under regular-vine-copula-based scenarios at return targets 1.05% to 1.25% monthly.**

| Return | CVaR | Allocation | USD | ARS | BRL | CAD | CLP | MXN | EUR | | | | | GBP | CHF | AUD | INR | JPY | KRW | SGD | XAU | Total |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | US | Argentina | Brazil | Canada | Chile | Mexico | Austria | Belgium | Netherlands | France | Germany | UK | Switzerland | Australia | India | Japan | South Korea | Singapore | | |
| | | Govt bond | 4.73 | | | 3.44 | | 5.31 | 5.11 | 3.31 | 5.26 | 4.24 | 4.40 | 3.36 | | 4.80 | | 0.88 | 3.59 | 2.69 | | 51.11 |
| | | Corp bond | 1.15 | | | | | | 3.65 | | | 1.99 | | | | | | | | | | 6.80 |
| | | Equity | 0.22 | 10.36 | 1.29 | 0.09 | 3.90 | 5.88 | 0.34 | 1.59 | 0.21 | 0.15 | 0.07 | 0.63 | 0.16 | 0.17 | 4.02 | 0.59 | 0.38 | 0.08 | | 30.14 |
| 1.05 | 6.17 | Cash | 5.08 | | | | | | | | | | | | | | | | | | | 5.08 |
| | | Total Asset | **11.18** | **10.36** | **1.29** | **3.53** | **3.90** | **11.19** | **9.10** | **4.91** | **5.46** | **6.39** | **4.47** | **3.99** | **0.16** | **4.98** | **4.02** | **1.47** | **3.97** | **2.76** | **6.87** | **100.00** |
| | | Total Currency | **17.00** | **2.86** | **2.53** | **7.39** | **2.52** | **4.46** | | | **12.50** | | | **6.44** | **9.03** | **5.68** | **2.02** | **6.86** | **6.09** | **7.74** | | **100.00** |
| | | Overlay | **5.82** | **-7.50** | **1.23** | **3.86** | **-1.39** | **-6.72** | | | **-17.83** | | | **2.45** | **8.87** | **0.71** | **-2.00** | **5.39** | **2.12** | **4.98** | **0.00** | **35.43** |
| | | Govt bond | 4.73 | | | 3.49 | | 5.73 | 5.08 | 3.48 | 5.21 | 4.17 | 4.25 | 3.14 | | 4.93 | | 0.29 | 3.51 | 2.57 | | 50.58 |
| | | Corp bond | 0.86 | | | | | | 3.42 | | | 1.74 | | | | | | | | | | 6.02 |
| | | Equity | 0.28 | 12.27 | 0.19 | 0.03 | 4.37 | 6.80 | 0.41 | 0.25 | 0.21 | 0.51 | 0.05 | 0.07 | 0.28 | 0.14 | 4.96 | 0.09 | 0.04 | 0.13 | | 31.07 |
| 1.10 | 6.33 | Cash | 5.00 | | | | | | | | | | | | | | | | | | | 5.00 |
| | | Total Asset | **10.87** | **12.27** | **0.19** | **3.51** | **4.37** | **12.52** | **8.91** | **3.73** | **5.42** | **6.41** | **4.30** | **3.21** | **0.28** | **5.07** | **4.96** | **0.38** | **3.55** | **2.70** | **7.33** | **100.00** |
| | | Total Currency | **16.30** | **3.29** | **1.67** | **7.02** | **2.21** | **7.32** | | | **11.31** | | | **6.03** | **9.13** | **5.78** | **2.68** | **5.85** | **6.32** | **7.76** | | **100.00** |
| | | Overlay | **5.43** | **-8.98** | **1.48** | **3.51** | **-2.16** | **-5.20** | | | **-17.46** | | | **2.82** | **8.85** | **0.71** | **-2.28** | **5.48** | **2.77** | **5.05** | **0.00** | **36.09** |
| | | Govt bond | 4.11 | | | 2.88 | | 5.47 | 5.12 | 3.62 | 5.27 | 4.11 | 4.62 | 2.53 | | 4.66 | | 0.45 | 2.98 | 2.73 | | 48.57 |
| | | Corp bond | 0.12 | | | | | | 3.86 | | | 1.28 | | | | | | | | | | 5.26 |
| | | Equity | 0.23 | 13.97 | 0.54 | 0.02 | 4.02 | 7.20 | 0.62 | 0.18 | 0.07 | 0.31 | 0.44 | 0.01 | 0.13 | 0.35 | 5.28 | 0.12 | 0.33 | 0.22 | | 34.03 |
| 1.15 | 6.80 | Cash | 4.81 | | | | | | | | | | | | | | | | | | | 4.81 |
| | | Total Asset | **9.28** | **13.97** | **0.54** | **2.90** | **4.02** | **12.68** | **9.60** | **3.80** | **5.34** | **5.69** | **5.06** | **2.55** | **0.13** | **5.01** | **5.28** | **0.57** | **3.32** | **2.95** | **7.34** | **100.00** |
| | | Total Currency | **15.60** | **4.56** | **2.68** | **6.81** | **3.12** | **6.92** | | | **11.32** | | | **5.75** | **9.31** | **5.66** | **2.75** | **5.86** | **5.81** | **6.52** | | **100.00** |
| | | Overlay | **6.32** | **-9.41** | **2.14** | **3.91** | **-0.89** | **-5.76** | | | **-18.16** | | | **3.21** | **9.18** | **0.65** | **-2.54** | **5.28** | **2.49** | **3.57** | **0.00** | **36.76** |
| | | Govt bond | 3.60 | | | 2.64 | | 5.24 | 4.48 | 3.63 | 4.59 | 3.45 | 4.23 | 1.91 | | 4.32 | | 0.09 | 2.97 | 3.03 | | 44.19 |
| | | Corp bond | 0.01 | | | | | | 4.07 | | | 0.57 | | | | | | | | | | 4.65 |
| | | Equity | 0.50 | 15.85 | 0.67 | 0.31 | 4.30 | 7.83 | 0.87 | 0.03 | 0.55 | 0.38 | 0.33 | 0.33 | 0.33 | 0.25 | 6.23 | 0.12 | 0.19 | 0.17 | | 39.26 |
| 1.20 | 7.43 | Cash | 4.61 | | | | | | | | | | | | | | | | | | | 4.61 |
| | | Total Asset | **8.73** | **15.85** | **0.67** | **2.95** | **4.30** | **13.07** | **9.42** | **3.66** | **5.14** | **4.41** | **4.56** | **2.24** | **0.33** | **4.57** | **6.23** | **0.21** | **3.16** | **3.20** | **7.29** | **100.00** |
| | | Total Currency | **13.10** | **6.67** | **1.24** | **6.92** | **2.90** | **5.52** | | | **10.72** | | | **5.90** | **8.28** | **6.70** | **3.51** | **7.22** | **5.99** | **8.04** | | **100.00** |
| | | Overlay | **4.37** | **-9.18** | **0.57** | **3.97** | **-1.40** | **-7.55** | | | **-16.47** | | | **3.66** | **7.95** | **2.13** | **-2.72** | **7.01** | **2.83** | **4.83** | **0.00** | **37.32** |
| | | Govt bond | 2.63 | | | 1.55 | | 4.78 | 4.59 | 3.67 | 4.69 | 3.31 | 5.09 | 1.21 | | 3.89 | | 0.06 | 3.24 | 3.29 | | 41.99 |
| | | Corp bond | 0.77 | | | | | | 4.33 | | | 0.09 | | | | | | | | | | 5.20 |
| | | Equity | 0.69 | 17.84 | 0.38 | 0.29 | 3.76 | 8.24 | 0.90 | 0.03 | 0.04 | 0.00 | 0.44 | 0.03 | 0.21 | 0.82 | 6.89 | 0.13 | 0.27 | 0.60 | | 41.55 |
| 1.25 | 7.80 | Cash | 4.12 | | | | | | | | | | | | | | | | | | | 4.12 |
| | | Total Asset | **8.20** | **17.84** | **0.38** | **1.85** | **3.76** | **13.02** | **9.82** | **3.70** | **4.73** | **3.41** | **5.53** | **1.23** | **0.21** | **4.70** | **6.89** | **0.19** | **3.51** | **3.89** | **7.15** | **100.00** |
| | | Total Currency | **14.14** | **5.03** | **1.97** | **6.60** | **4.35** | **7.20** | | | **11.12** | | | **5.47** | **8.12** | **5.81** | **3.40** | **5.88** | **6.19** | **7.59** | | **100.00** |
| | | Overlay | **5.93** | **-12.80** | **1.59** | **4.75** | **0.59** | **-5.82** | | | **-16.07** | | | **4.23** | **7.90** | **1.11** | **-3.49** | **5.68** | **2.69** | **3.70** | **0.00** | **38.18** |

**Table B.5 Optimal portfolio allocations under regular-vine-copula-based scenarios at return targets 1.30% to 1.50% monthly.**

| Return | CVaR | Allocation | USD | ARS | BRL | CAD | CLP | MXN | EUR | | | | | GBP | CHF | AUD | INR | JPY | KRW | SGD | XAU | Total |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | US | Argentina | Brazil | Canada | Chile | Mexico | Austria | Belgium | Netherlands | France | Germany | UK | Switzerland | Australia | India | Japan | South Korea | Singapore | | |
| | | Govt bond | 2.46 | | | 1.27 | | 4.31 | 3.50 | 2.19 | 3.54 | 3.18 | 4.20 | 0.86 | | 3.51 | | 0.32 | 3.24 | 3.77 | | 36.35 |
| | | Corp bond | 0.32 | | | | | | 3.60 | | | 1.51 | | | | | | | | | | 5.43 |
| | | Equity | 0.07 | 19.88 | 0.03 | 0.57 | 4.22 | 8.91 | 0.61 | 0.19 | 0.06 | 1.86 | 0.57 | 0.07 | 0.23 | 0.35 | 8.48 | 0.07 | 0.35 | 0.79 | | 47.33 |
| 1.30 | 8.75 | Cash | 3.87 | | | | | | | | | | | | | | | | | | | 3.87 |
| | | Total Asset | **6.72** | **19.88** | **0.03** | **1.84** | **4.22** | **13.22** | **7.71** | **2.38** | **3.60** | **6.55** | **4.77** | **0.93** | **0.23** | **3.86** | **8.48** | **0.39** | **3.60** | **4.56** | **7.02** | **100.00** |
| | | Total Currency | **9.04** | **5.39** | **0.75** | **7.48** | **3.70** | **6.05** | | | **12.20** | | | **6.15** | **11.68** | **7.95** | **4.44** | **3.27** | **6.35** | **8.53** | | **100.00** |
| | | Overlay | **2.33** | **-14.49** | **0.72** | **5.64** | **-0.52** | **-7.17** | | | **-12.82** | | | **5.22** | **11.45** | **4.09** | **-4.04** | **2.88** | **2.75** | **3.97** | **0.00** | **39.04** |
| | | Govt bond | 1.73 | | | 1.35 | | 3.10 | 4.13 | 3.26 | 4.02 | 2.98 | 3.67 | 2.45 | | 2.74 | | 0.43 | 2.13 | 1.46 | | 33.45 |
| | | Corp bond | 0.23 | | | | | | 2.45 | | | 0.98 | | | | | | | | | | 3.66 |
| | | Equity | 0.57 | 24.16 | 0.38 | 0.26 | 3.37 | 9.44 | 0.00 | 0.79 | 0.46 | 0.45 | 0.03 | 0.01 | 0.14 | 0.01 | 9.77 | 0.33 | 0.21 | 0.72 | | 51.09 |
| 1.35 | 9.44 | Cash | 4.83 | | | | | | | | | | | | | | | | | | | 4.83 |
| | | Total Asset | **7.35** | **24.16** | **0.38** | **1.60** | **3.37** | **12.54** | **6.58** | **4.05** | **4.48** | **4.41** | **3.70** | **2.47** | **0.14** | **2.74** | **9.77** | **0.76** | **2.34** | **2.18** | **6.97** | **100.00** |
| | | Total Currency | **8.41** | **7.28** | **-0.39** | **7.08** | **3.55** | **3.48** | | | **14.97** | | | **7.38** | **13.24** | **8.55** | **5.73** | **3.99** | **5.91** | **3.85** | | **100.00** |
| | | Overlay | **1.06** | **-16.88** | **-0.77** | **5.48** | **0.18** | **-9.07** | | | **-8.25** | | | **4.91** | **13.10** | **5.81** | **-4.04** | **3.23** | **3.57** | **1.67** | **0.00** | **39.00** |
| | | Govt bond | 1.17 | | | 1.72 | | 3.11 | 1.91 | 2.23 | 1.90 | 2.06 | 2.48 | 1.74 | | 2.63 | | 0.09 | 2.06 | 2.49 | | 25.58 |
| | | Corp bond | 0.63 | | | | | | 0.74 | | | 0.41 | | | | | | | | | | 1.78 |
| | | Equity | 0.32 | 28.00 | 0.97 | 0.02 | 2.74 | 10.11 | 1.43 | 0.53 | 0.25 | 0.53 | 0.25 | 0.77 | 0.35 | 0.76 | 11.93 | 0.54 | 0.18 | 0.59 | | 60.26 |
| 1.40 | 10.74 | Cash | 5.13 | | | | | | | | | | | | | | | | | | | 5.13 |
| | | Total Asset | **7.24** | **28.00** | **0.97** | **1.74** | **2.74** | **13.22** | **4.08** | **2.76** | **2.15** | **3.00** | **2.73** | **2.51** | **0.35** | **3.38** | **11.93** | **0.62** | **2.23** | **3.08** | **7.25** | **100.00** |
| | | Total Currency | **7.13** | **5.98** | **0.48** | **7.83** | **6.67** | **7.83** | | | **7.89** | | | **5.71** | **10.50** | **8.18** | **5.91** | **4.96** | **6.51** | **7.15** | | **100.00** |
| | | Overlay | **-0.12** | **-22.01** | **-0.49** | **6.09** | **3.93** | **-5.39** | | | **-6.83** | | | **3.20** | **10.15** | **4.80** | **-6.02** | **4.34** | **4.28** | **4.07** | **0.00** | **40.86** |
| | | Govt bond | 1.19 | | | 2.27 | | 2.13 | 1.22 | 2.25 | 2.15 | 0.61 | 2.18 | 1.04 | | 1.78 | | 1.02 | 1.80 | 1.40 | | 21.04 |
| | | Corp bond | 1.44 | | | | | | 2.02 | | | 2.05 | | | | | | | | | | 5.51 |
| | | Equity | 0.02 | 31.34 | 0.22 | 0.36 | 1.55 | 11.01 | 0.27 | 1.42 | 1.31 | 0.04 | 0.04 | 0.19 | 0.11 | 0.32 | 13.01 | 0.38 | 0.23 | 0.30 | | 62.12 |
| 1.45 | 11.10 | Cash | 5.13 | | | | | | | | | | | | | | | | | | | 5.13 |
| | | Total Asset | **7.77** | **31.34** | **0.22** | **2.62** | **1.55** | **13.14** | **3.52** | **3.67** | **3.46** | **2.70** | **2.23** | **1.23** | **0.11** | **2.10** | **13.01** | **1.40** | **2.04** | **1.70** | **6.21** | **100.00** |
| | | Total Currency | **10.78** | **6.64** | **2.62** | **7.04** | **1.48** | **7.41** | | | **10.54** | | | **5.57** | **9.35** | **7.43** | **6.83** | **5.57** | **5.70** | **6.82** | | **100.00** |
| | | Overlay | **3.01** | **-24.69** | **2.41** | **4.42** | **-0.07** | **-5.74** | | | **-5.03** | | | **4.34** | **9.24** | **5.33** | **-6.19** | **4.17** | **3.67** | **5.12** | **0.00** | **41.71** |
| | | Govt bond | 2.06 | | | 1.53 | | 2.52 | 1.08 | 1.87 | 1.17 | 0.62 | 2.11 | 0.26 | | 1.58 | | 0.04 | 2.28 | 2.06 | | 19.17 |
| | | Corp bond | 0.53 | | | | | | 2.13 | | | 0.43 | | | | | | | | | | 3.10 |
| | | Equity | 0.36 | 32.50 | 1.78 | 0.10 | 2.76 | 11.09 | 0.29 | 0.82 | 0.46 | 0.01 | 0.19 | 0.28 | 0.43 | 0.13 | 13.95 | 0.03 | 1.10 | 0.12 | | 66.38 |
| 1.50 | 11.67 | Cash | 5.80 | | | | | | | | | | | | | | | | | | | 5.80 |
| | | Total Asset | **8.75** | **32.50** | **1.78** | **1.62** | **2.76** | **13.61** | **3.49** | **2.69** | **1.63** | **1.06** | **2.30** | **0.53** | **0.43** | **1.70** | **13.95** | **0.06** | **3.39** | **2.18** | **5.55** | **100.00** |
| | | Total Currency | **12.14** | **7.87** | **3.82** | **6.92** | **2.06** | **5.51** | | | **9.83** | | | **5.19** | **10.27** | **9.04** | **4.97** | **4.99** | **5.74** | **6.10** | | **100.00** |
| | | Overlay | **3.39** | **-24.63** | **2.04** | **5.30** | **-0.70** | **-8.10** | | | **-1.35** | | | **4.66** | **9.84** | **7.34** | **-8.98** | **4.93** | **2.35** | **3.92** | **0.00** | **43.77** |

**Table B.6 Optimal portfolio allocations under regular-vine-copula-based scenarios at return targets 1.55% to 1.60% monthly.**

| Return | CVaR | Allocation | USD | ARS | BRL | CAD | CLP | MXN | EUR | | | | | GBP | CHF | AUD | INR | JPY | KRW | SGD | XAU | Total |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | US | Argentina | Brazil | Canada | Chile | Mexico | Austria | Belgium | Netherlands | France | Germany | UK | Switzerland | Australia | India | Japan | South Korea | Singapore | | |
| | | Govt bond | 2.07 | | | 0.29 | | 2.29 | 1.11 | 0.55 | 1.42 | 1.51 | 1.38 | 0.11 | | 1.00 | | 0.23 | 0.74 | 1.12 | | 13.84 |
| | | Corp bond | 0.27 | | | | | | 0.98 | | | 0.74 | | | | | | | | | | 1.99 |
| | | Equity | 1.40 | 35.33 | 2.11 | 0.07 | 3.03 | 12.58 | 0.11 | 0.07 | 0.04 | 0.24 | 0.07 | 0.05 | 0.33 | 0.55 | 16.20 | 0.11 | 0.13 | 0.86 | | 73.28 |
| 1.55 | 12.32 | Cash | 5.75 | | | | | | | | | | | | | | | | | | | 5.75 |
| | | Total Asset | **9.49** | **35.33** | **2.11** | **0.36** | **3.03** | **14.87** | **2.21** | **0.62** | **1.46** | **2.50** | **1.46** | **0.16** | **0.33** | **1.55** | **16.20** | **0.34** | **0.87** | **1.98** | **5.14** | **100.00** |
| | | Total Currency | **10.84** | **10.76** | **4.76** | **6.07** | **2.94** | **2.72** | | | **7.82** | | | **4.80** | **9.53** | **9.07** | **7.19** | **4.02** | **6.56** | **7.79** | | **100.00** |
| | | Overlay | **1.35** | **-24.57** | **2.65** | **5.71** | **-0.09** | **-12.14** | | | **-0.43** | | | **4.64** | **9.20** | **7.52** | **-9.01** | **3.67** | **5.69** | **5.81** | **0.00** | **46.24** |
| | | Govt bond | 1.13 | | | 0.79 | | 2.90 | 1.60 | 1.67 | 1.77 | 1.14 | 2.16 | 0.13 | | 2.03 | | 0.00 | 1.85 | 1.59 | | 18.75 |
| | | Corp bond | 0.00 | | | | | | 1.44 | | | 0.11 | | | | | | | | | | 1.54 |
| | | Equity | 0.55 | 33.71 | 1.16 | 0.26 | 3.41 | 12.49 | 0.61 | 0.61 | 0.39 | 0.43 | 0.21 | 0.22 | 0.00 | 0.33 | 14.32 | 0.25 | 0.40 | 0.59 | | 69.94 |
| 1.60 | 13.40 | Cash | 3.28 | | | | | | | | | | | | | | | | | | | 3.28 |
| | | Total Asset | **4.96** | **33.71** | **1.16** | **1.05** | **3.41** | **15.39** | **3.64** | **2.28** | **2.16** | **1.67** | **2.37** | **0.35** | **0.00** | **2.36** | **14.32** | **0.25** | **2.25** | **2.19** | **6.49** | **100.00** |
| | | Total Currency | **13.04** | **11.95** | **6.28** | **7.54** | **1.98** | **1.53** | | | **10.32** | | | **4.44** | **8.69** | **6.18** | **4.51** | **3.94** | **6.03** | **7.10** | | **100.00** |
| | | Overlay | **8.08** | **-21.76** | **5.12** | **6.49** | **-1.42** | **-13.86** | | | **-1.81** | | | **4.09** | **8.69** | **3.82** | **-9.81** | **3.68** | **3.78** | **4.91** | **0.00** | **48.67** |

**Table B.7 Optimal portfolio allocations under multivariate normal distribution scenarios at return targets 0.55% to 0.75% monthly.**

| Return | CVaR | Allocation | USD | ARS | BRL | CAD | CLP | MXN | | | EUR | | | GBP | CHF | AUD | INR | JPY | KRW | SGD | XAU | Total |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | US | Argentina | Brazil | Canada | Chile | Mexico | Austria | Belgium | Netherlands | France | Germany | UK | Switzerland | Australia | India | Japan | South Korea | Singapore | | |
| | | Govt bond | 6.04 | | | 6.22 | | 5.86 | 4.58 | 3.81 | 2.30 | 7.90 | 5.30 | 3.35 | | 8.56 | | 3.17 | 2.03 | 5.29 | | 64.40 |
| | | Corp bond | 1.29 | | | | | | 6.01 | | | 0.19 | | | | | | | | | | 7.50 |
| | | Equity | 1.10 | 0.59 | 0.37 | 0.57 | 0.25 | 0.36 | 0.06 | 0.07 | 0.17 | 0.00 | 0.39 | 0.15 | 0.46 | 0.27 | 0.00 | 0.06 | 0.61 | 5.59 | | 11.06 |
| 0.55 | 3.10 | Cash | 11.04 | | | | | | | | | | | | | | | | | | | 11.04 |
| | | Total Asset | **19.47** | **0.59** | **0.37** | **6.79** | **0.25** | **6.21** | **10.65** | **3.87** | **2.47** | **8.09** | **5.69** | **3.50** | **0.46** | **8.83** | **0.00** | **3.23** | **2.64** | **10.87** | **6.00** | **100.00** |
| | | Total Currency | **30.32** | **0.39** | **5.09** | **4.34** | **0.80** | **1.79** | | | **7.73** | | | **6.94** | **4.76** | **1.36** | **1.60** | **12.66** | **4.03** | **12.20** | | **100.00** |
| | | Overlay | **10.85** | **-0.20** | **4.72** | **-2.45** | **0.55** | **-4.43** | | | **-23.05** | | | **3.44** | **4.29** | **-7.47** | **1.60** | **9.43** | **1.39** | **1.33** | **0.00** | **37.60** |
| | | Govt bond | 5.61 | | | 5.79 | | 5.38 | 4.02 | 3.99 | 2.47 | 7.61 | 5.95 | 3.44 | | 8.46 | | 3.55 | 1.76 | 5.27 | | 63.29 |
| | | Corp bond | 0.02 | | | | | | 6.14 | | | 0.23 | | | | | | | | | | 6.39 |
| | | Equity | 0.42 | 2.14 | 0.15 | 0.13 | 0.53 | 1.21 | 0.08 | 0.11 | 0.64 | 0.08 | 0.17 | 0.35 | 0.51 | 0.04 | 0.03 | 0.36 | 0.47 | 6.45 | | 13.87 |
| 0.60 | 3.30 | Cash | 10.03 | | | | | | | | | | | | | | | | | | | 10.03 |
| | | Total Asset | **16.08** | **2.14** | **0.15** | **5.92** | **0.53** | **6.59** | **10.25** | **4.10** | **3.11** | **7.92** | **6.11** | **3.79** | **0.51** | **8.50** | **0.03** | **3.90** | **2.23** | **11.72** | **6.43** | **100.00** |
| | | Total Currency | **27.41** | **-1.33** | **4.18** | **3.59** | **2.59** | **3.04** | | | **8.30** | | | **8.39** | **6.31** | **1.77** | **3.25** | **12.18** | **5.27** | **8.63** | | **100.00** |
| | | Overlay | **11.33** | **-3.47** | **4.04** | **-2.33** | **2.06** | **-3.55** | | | **-23.18** | | | **4.60** | **5.79** | **-6.73** | **3.22** | **8.28** | **3.04** | **-3.09** | **0.00** | **42.35** |
| | | Govt bond | 5.53 | | | 5.80 | | 5.38 | 4.07 | 3.69 | 2.05 | 7.62 | 5.69 | 3.28 | | 8.52 | | 3.31 | 1.22 | 5.01 | | 61.17 |
| | | Corp bond | 0.02 | | | | | | 5.88 | | | 0.27 | | | | | | | | | | 6.17 |
| | | Equity | 0.58 | 2.28 | 0.49 | 0.23 | 1.08 | 0.92 | 0.53 | 0.27 | 0.90 | 0.04 | 0.46 | 0.07 | 0.66 | 0.34 | 0.09 | 0.13 | 1.29 | 6.39 | | 16.74 |
| 0.65 | 3.44 | Cash | 9.74 | | | | | | | | | | | | | | | | | | | 9.74 |
| | | Total Asset | **15.87** | **2.28** | **0.49** | **6.03** | **1.08** | **6.30** | **10.47** | **3.96** | **2.96** | **7.93** | **6.15** | **3.35** | **0.66** | **8.86** | **0.09** | **3.44** | **2.51** | **11.40** | **6.18** | **100.00** |
| | | Total Currency | **27.66** | **0.27** | **3.64** | **9.39** | **0.69** | **2.40** | | | **7.76** | | | **7.08** | **6.70** | **1.86** | **4.07** | **11.81** | **3.63** | **6.87** | | **100.00** |
| | | Overlay | **11.79** | **-2.01** | **3.15** | **3.36** | **-0.39** | **-3.90** | | | **-23.71** | | | **3.73** | **6.04** | **-7.00** | **3.98** | **8.37** | **1.12** | **-4.54** | **0.00** | **41.54** |
| | | Govt bond | 5.50 | | | 5.85 | | 5.41 | 4.08 | 3.72 | 1.94 | 7.72 | 5.96 | 3.32 | | 8.70 | | 3.52 | 0.91 | 5.07 | | 61.69 |
| | | Corp bond | 0.08 | | | | | | 5.96 | | | 0.23 | | | | | | | | | | 6.28 |
| | | Equity | 0.22 | 2.91 | 0.05 | 0.43 | 0.20 | 0.80 | 0.05 | 0.45 | 0.37 | 0.17 | 0.14 | 0.12 | 0.19 | 0.02 | 0.35 | 0.09 | 2.79 | 7.01 | | 16.37 |
| 0.70 | 3.58 | Cash | 9.24 | | | | | | | | | | | | | | | | | | | 9.24 |
| | | Total Asset | **15.05** | **2.91** | **0.05** | **6.28** | **0.20** | **6.21** | **10.08** | **4.17** | **2.31** | **8.13** | **6.10** | **3.44** | **0.19** | **8.73** | **0.35** | **3.61** | **3.70** | **12.08** | **6.42** | **100.00** |
| | | Total Currency | **25.56** | **0.97** | **2.91** | **8.61** | **3.38** | **2.23** | | | **7.49** | | | **7.28** | **7.55** | **2.02** | **2.51** | **11.88** | **1.59** | **9.60** | | **100.00** |
| | | Overlay | **10.51** | **-1.94** | **2.86** | **2.33** | **3.18** | **-3.99** | | | **-23.30** | | | **3.85** | **7.36** | **-6.70** | **2.16** | **8.28** | **-2.11** | **-2.48** | **0.00** | **40.53** |
| | | Govt bond | 5.12 | | | 5.37 | | 4.92 | 3.57 | 3.72 | 2.12 | 7.23 | 6.08 | 3.19 | | 8.25 | | 3.53 | 0.97 | 4.87 | | 58.93 |
| | | Corp bond | 0.01 | | | | | | 5.83 | | | 0.02 | | | | | | | | | | 5.85 |
| | | Equity | 0.32 | 3.51 | 0.05 | 1.10 | 0.14 | 1.45 | 0.05 | 1.57 | 0.24 | 0.02 | 0.51 | 0.05 | 0.21 | 0.07 | 0.65 | 0.07 | 3.59 | 7.15 | | 20.74 |
| 0.75 | 3.94 | Cash | 8.11 | | | | | | | | | | | | | | | | | | | 8.11 |
| | | Total Asset | **13.55** | **3.51** | **0.05** | **6.47** | **0.14** | **6.37** | **9.45** | **5.29** | **2.36** | **7.26** | **6.59** | **3.24** | **0.21** | **8.33** | **0.65** | **3.59** | **4.56** | **12.01** | **6.37** | **100.00** |
| | | Total Currency | **22.78** | **1.48** | **4.39** | **4.99** | **4.55** | **3.44** | | | **7.47** | | | **6.93** | **9.96** | **2.82** | **0.55** | **11.70** | **3.77** | **8.79** | | **100.00** |
| | | Overlay | **9.23** | **-2.03** | **4.34** | **-1.48** | **4.42** | **-2.93** | | | **-23.48** | | | **3.69** | **9.75** | **-5.51** | **-0.10** | **8.11** | **-0.79** | **-3.22** | **0.00** | **39.54** |

**Table B.8 Optimal portfolio allocations under multivariate normal distribution scenarios at return targets 0.80% to 1.00% monthly.**

| Return | CVaR | Allocation | USD | ARS | BRL | CAD | CLP | MXN | | | EUR | | | GBP | CHF | AUD | INR | JPY | KRW | SGD | XAU | Total |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | US | Argentina | Brazil | Canada | Chile | Mexico | Austria | Belgium | Netherlands | France | Germany | UK | Switzerland | Australia | India | Japan | South Korea | Singapore | | |
| | | Govt bond | 4.97 | | | 5.27 | | 4.80 | 3.42 | 3.64 | 1.87 | 7.20 | 6.11 | 3.21 | | 8.32 | | 3.55 | 0.58 | 4.69 | | 57.63 |
| | | Corp bond | 0.04 | | | | | | 5.71 | | | 0.04 | | | | | | | | | | 5.79 |
| | | Equity | 0.21 | 4.09 | 0.05 | 1.28 | 0.27 | 1.50 | 0.07 | 2.20 | 0.02 | 0.01 | 0.33 | 0.03 | 0.13 | 0.05 | 0.07 | 0.09 | 5.04 | 7.39 | | 22.83 |
| 0.80 | 4.16 | Cash | 7.40 | | | | | | | | | | | | | | | | | | | 7.40 |
| | | Total Asset | **12.62** | **4.09** | **0.05** | **6.55** | **0.27** | **6.30** | **9.20** | **5.85** | **1.89** | **7.25** | **6.44** | **3.24** | **0.13** | **8.36** | **0.07** | **3.65** | **5.63** | **12.07** | **6.34** | **100.00** |
| | | Total Currency | **21.35** | **0.95** | **2.93** | **9.56** | **2.41** | **3.81** | | | **7.23** | | | **7.00** | **8.69** | **2.63** | **2.91** | **10.79** | **3.56** | **9.83** | | **100.00** |
| | | Overlay | **8.73** | **-3.14** | **2.88** | **3.01** | **2.14** | **-2.49** | | | **-23.40** | | | **3.76** | **8.57** | **-5.73** | **2.84** | **7.14** | **-2.07** | **-2.24** | **0.00** | **39.07** |
| | | Govt bond | 4.56 | | | 4.83 | | 4.36 | 3.02 | 3.35 | 1.73 | 6.74 | 5.98 | 2.83 | | 7.90 | | 3.24 | 0.28 | 4.34 | | 53.16 |
| | | Corp bond | 0.68 | | | | | | 5.40 | | | 0.05 | | | | | | | | | | 6.14 |
| | | Equity | 0.07 | 4.49 | 0.02 | 1.43 | 0.56 | 1.88 | 0.57 | 3.18 | 0.98 | 0.48 | 0.23 | 0.20 | 0.07 | 0.10 | 0.05 | 0.00 | 6.13 | 7.30 | | 27.75 |
| 0.85 | 4.68 | Cash | 6.76 | | | | | | | | | | | | | | | | | | | 6.76 |
| | | Total Asset | **12.07** | **4.49** | **0.02** | **6.26** | **0.56** | **6.24** | **8.99** | **6.53** | **2.70** | **7.27** | **6.22** | **3.03** | **0.07** | **8.01** | **0.05** | **3.24** | **6.40** | **11.65** | **6.19** | **100.00** |
| | | Total Currency | **19.84** | **1.16** | **1.70** | **9.22** | **3.01** | **2.78** | | | **7.37** | | | **7.28** | **9.24** | **4.98** | **3.50** | **10.23** | **4.15** | **9.34** | | **100.00** |
| | | Overlay | **7.77** | **-3.33** | **1.69** | **2.96** | **2.45** | **-3.46** | | | **-24.35** | | | **4.25** | **9.17** | **-3.03** | **3.45** | **6.99** | **-2.25** | **-2.31** | **0.00** | **38.74** |
| | | Govt bond | 4.55 | | | 5.01 | | 4.53 | 3.26 | 3.08 | 1.19 | 6.91 | 5.75 | 2.23 | | 8.15 | | 3.15 | 0.18 | 4.09 | | 52.07 |
| | | Corp bond | 0.54 | | | | | | 5.12 | | | 0.04 | | | | | | | | | | 5.71 |
| | | Equity | 0.49 | 5.07 | 0.08 | 1.08 | 0.54 | 0.94 | 0.14 | 3.74 | 0.28 | 0.16 | 0.25 | 0.05 | 0.48 | 0.15 | 0.28 | 0.11 | 7.71 | 7.76 | | 29.31 |
| 0.90 | 4.90 | Cash | 6.68 | | | | | | | | | | | | | | | | | | | 6.68 |
| | | Total Asset | **12.26** | **5.07** | **0.08** | **6.09** | **0.54** | **5.48** | **8.52** | **6.82** | **1.47** | **7.11** | **6.00** | **2.28** | **0.48** | **8.30** | **0.28** | **3.26** | **7.89** | **11.85** | **6.23** | **100.00** |
| | | Total Currency | **19.48** | **1.60** | **2.40** | **9.21** | **3.70** | **2.70** | | | **7.84** | | | **6.68** | **9.05** | **4.94** | **2.99** | **9.92** | **4.05** | **9.22** | | **100.00** |
| | | Overlay | **7.22** | **-3.47** | **2.32** | **3.12** | **3.16** | **-2.77** | | | **-22.08** | | | **4.40** | **8.58** | **-3.36** | **2.70** | **6.66** | **-3.84** | **-2.63** | **0.00** | **38.16** |
| | | Govt bond | 4.29 | | | 4.75 | | 4.26 | 2.96 | 2.88 | 1.25 | 6.72 | 5.78 | 2.10 | | 8.04 | | 3.31 | 0.14 | 3.86 | | 50.34 |
| | | Corp bond | 0.03 | | | | | | 4.97 | | | 0.06 | | | | | | | | | | 5.05 |
| | | Equity | 0.55 | 5.73 | 0.50 | 1.37 | 0.16 | 1.17 | 0.35 | 4.43 | 0.02 | 0.09 | 0.35 | 0.09 | 0.17 | 0.30 | 0.01 | 0.24 | 9.06 | 7.93 | | 32.53 |
| 0.95 | 5.20 | Cash | 5.91 | | | | | | | | | | | | | | | | | | | 5.91 |
| | | Total Asset | **10.79** | **5.73** | **0.50** | **6.12** | **0.16** | **5.43** | **8.27** | **7.31** | **1.27** | **6.87** | **6.12** | **2.20** | **0.17** | **8.34** | **0.01** | **3.55** | **9.20** | **11.79** | **6.17** | **100.00** |
| | | Total Currency | **17.79** | **1.97** | **3.60** | **9.24** | **3.31** | **1.67** | | | **10.37** | | | **6.79** | **9.35** | **5.47** | **1.47** | **9.57** | **4.25** | **8.99** | | **100.00** |
| | | Overlay | **7.00** | **-3.76** | **3.10** | **3.12** | **3.15** | **-3.76** | | | **-19.47** | | | **4.59** | **9.18** | **-2.86** | **1.45** | **6.02** | **-4.95** | **-2.81** | **0.00** | **37.63** |
| | | Govt bond | 3.93 | | | 4.43 | | 3.92 | 2.65 | 2.79 | 1.87 | 6.39 | 5.73 | 2.06 | | 7.77 | | 3.28 | 0.10 | 3.52 | | 48.43 |
| | | Corp bond | 0.32 | | | | | | 4.65 | | | 0.04 | | | | | | | | | | 5.01 |
| | | Equity | 0.22 | 6.39 | 0.02 | 1.61 | 0.17 | 1.36 | 0.11 | 5.46 | 0.19 | 0.12 | 0.02 | 0.01 | 1.08 | 0.03 | 0.22 | 0.09 | 10.77 | 8.00 | | 35.88 |
| 1.00 | 5.75 | Cash | 4.55 | | | | | | | | | | | | | | | | | | | 4.55 |
| | | Total Asset | **9.02** | **6.39** | **0.02** | **6.04** | **0.17** | **5.28** | **7.41** | **8.25** | **2.07** | **6.55** | **5.75** | **2.07** | **1.08** | **7.80** | **0.22** | **3.38** | **10.88** | **11.52** | **6.13** | **100.00** |
| | | Total Currency | **16.76** | **2.35** | **2.09** | **8.10** | **2.81** | **3.03** | | | **11.98** | | | **6.73** | **9.38** | **4.64** | **3.02** | **10.03** | **4.22** | **8.74** | | **100.00** |
| | | Overlay | **7.74** | **-4.04** | **2.06** | **2.06** | **2.64** | **-2.24** | | | **-18.04** | | | **4.66** | **8.31** | **-3.16** | **2.80** | **6.65** | **-6.66** | **-2.78** | **0.00** | **36.92** |

**Table B.9 Optimal portfolio allocations under multivariate normal distribution scenarios at return targets 1.05% to 1.25% monthly.**

| Return | CVaR | Allocation | USD | ARS | BRL | CAD | CLP | MXN | | | EUR | | | GBP | CHF | AUD | INR | JPY | KRW | SGD | XAU | Total |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | US | Argentina | Brazil | Canada | Chile | Mexico | Austria | Belgium | Netherlands | France | Germany | UK | Switzerland | Australia | India | Japan | South Korea | Singapore | | |
| | | Govt bond | 3.84 | | | 4.41 | | 3.93 | 2.62 | 2.43 | 1.54 | 6.36 | 5.53 | 1.65 | | 7.83 | | 3.21 | 0.36 | 3.13 | | 46.83 |
| | | Corp bond | 0.01 | | | | | | 4.29 | | | 0.07 | | | | | | | | | | 4.37 |
| | | Equity | 0.05 | 7.09 | 0.09 | 1.56 | 0.09 | 1.02 | 0.01 | 6.19 | 0.03 | 0.36 | 0.03 | 0.36 | 0.29 | 0.23 | 0.05 | 0.11 | 12.39 | 8.27 | | 38.21 |
| 1.05 | 6.07 | Cash | 4.60 | | | | | | | | | | | | | | | | | | | 4.60 |
| | | Total Asset | **8.50** | **7.09** | **0.09** | **5.96** | **0.09** | **4.95** | **6.93** | **8.62** | **1.57** | **6.79** | **5.55** | **2.01** | **0.29** | **8.06** | **0.05** | **3.32** | **12.75** | **11.40** | **5.99** | **100.00** |
| | | Total Currency | **14.61** | **2.42** | **2.13** | **9.48** | **3.97** | **4.46** | | | **10.10** | | | **6.80** | **9.86** | **6.81** | **1.77** | **8.65** | **4.53** | **8.42** | | **100.00** |
| | | Overlay | **6.11** | **-4.66** | **2.04** | **3.51** | **3.89** | **-0.49** | | | **-19.36** | | | **4.79** | **9.56** | **-1.25** | **1.72** | **5.32** | **-8.22** | **-2.98** | **0.00** | **36.95** |
| | | Govt bond | 4.15 | | | 4.64 | | 4.12 | 2.09 | 1.54 | 1.18 | 6.00 | 4.97 | 0.79 | | 7.56 | | 2.58 | 0.95 | 2.45 | | 43.02 |
| | | Corp bond | 0.63 | | | | | | 3.63 | | | 0.00 | | | | | | | | | | 4.27 |
| | | Equity | 0.34 | 7.42 | 0.20 | 1.12 | 0.02 | 0.80 | 0.02 | 7.30 | 0.11 | 0.04 | 0.19 | 0.15 | 0.43 | 0.35 | 0.12 | 0.89 | 14.25 | 8.30 | | 42.06 |
| 1.10 | 6.52 | Cash | 4.92 | | | | | | | | | | | | | | | | | | | 4.92 |
| | | Total Asset | **10.05** | **7.42** | **0.20** | **5.75** | **0.02** | **4.92** | **5.75** | **8.84** | **1.30** | **6.04** | **5.16** | **0.93** | **0.43** | **7.91** | **0.12** | **3.47** | **15.19** | **10.75** | **5.73** | **100.00** |
| | | Total Currency | **15.29** | **2.33** | **3.15** | **9.46** | **4.50** | **4.06** | | | **8.10** | | | **5.82** | **9.47** | **6.19** | **4.56** | **8.57** | **4.34** | **8.41** | | **100.00** |
| | | Overlay | **5.24** | **-5.09** | **2.95** | **3.71** | **4.48** | **-0.86** | | | **-18.99** | | | **4.88** | **9.04** | **-1.72** | **4.44** | **5.10** | **-10.86** | **-2.34** | **0.00** | **39.85** |
| | | Govt bond | 4.26 | | | 4.63 | | 4.14 | 1.21 | 0.60 | 0.79 | 5.17 | 4.57 | 3.07 | | 6.71 | | 2.46 | 0.34 | 1.90 | | 39.84 |
| | | Corp bond | 0.41 | | | | | | 2.92 | | | 0.57 | | | | | | | | | | 3.90 |
| | | Equity | 0.05 | 7.84 | 0.16 | 1.31 | 0.09 | 0.69 | 0.01 | 8.06 | 0.60 | 0.47 | 0.28 | 0.32 | 1.59 | 0.82 | 0.06 | 0.60 | 15.99 | 8.27 | | 47.22 |
| 1.15 | 7.23 | Cash | 3.65 | | | | | | | | | | | | | | | | | | | 3.65 |
| | | Total Asset | **8.37** | **7.84** | **0.16** | **5.94** | **0.09** | **4.84** | **4.14** | **8.66** | **1.39** | **6.20** | **4.85** | **3.39** | **1.59** | **7.52** | **0.06** | **3.06** | **16.32** | **10.17** | **5.39** | **100.00** |
| | | Total Currency | **12.36** | **2.83** | **5.61** | **11.69** | **4.22** | **8.27** | | | **8.16** | | | **FALSE** | **10.06** | **6.25** | **5.25** | **8.08** | **3.61** | **8.23** | | **100.00** |
| | | Overlay | **3.98** | **-5.01** | **5.44** | **5.75** | **4.12** | **3.43** | | | **-17.08** | | | **-3.39** | **8.47** | **-1.27** | **5.19** | **5.02** | **-12.72** | **-1.94** | **0.00** | **41.41** |
| | | Govt bond | 4.83 | | | 5.29 | | 4.82 | 1.08 | 0.65 | 0.20 | 4.90 | 5.26 | 0.02 | | 6.66 | | 1.59 | 0.15 | 1.20 | | 36.65 |
| | | Corp bond | 0.26 | | | | | | 6.97 | | | 0.86 | | | | | | | | | | 8.09 |
| | | Equity | 0.10 | 8.04 | 0.04 | 1.78 | 0.05 | 0.23 | 0.01 | 8.53 | 0.09 | 0.16 | 0.01 | 0.02 | 0.78 | 0.06 | 0.27 | 0.05 | 17.63 | 8.48 | | 46.33 |
| 1.20 | 7.32 | Cash | 4.16 | | | | | | | | | | | | | | | | | | | 4.16 |
| | | Total Asset | **9.35** | **8.04** | **0.04** | **7.08** | **0.05** | **5.05** | **8.07** | **9.18** | **0.29** | **5.92** | **5.27** | **0.04** | **0.78** | **6.71** | **0.27** | **1.65** | **17.78** | **9.67** | **4.76** | **100.00** |
| | | Total Currency | **12.92** | **2.87** | **2.19** | **11.62** | **3.63** | **3.27** | | | **8.92** | | | **6.56** | **10.29** | **7.69** | **4.00** | **9.34** | **3.78** | **8.16** | | **100.00** |
| | | Overlay | **3.57** | **-5.18** | **2.14** | **4.54** | **3.58** | **-1.78** | | | **-19.81** | | | **6.52** | **9.52** | **0.98** | **3.73** | **7.69** | **-14.00** | **-1.51** | **0.00** | **42.28** |
| | | Govt bond | 4.08 | | | 4.60 | | 3.55 | 1.01 | 0.32 | 1.40 | 5.78 | 3.58 | 0.85 | | 6.75 | | 3.06 | 0.81 | 0.94 | | 36.73 |
| | | Corp bond | 0.00 | | | | | | 4.26 | | | 0.22 | | | | | | | | | | 4.48 |
| | | Equity | 0.01 | 8.83 | 0.51 | 0.99 | 0.18 | 0.20 | 0.51 | 9.37 | 0.03 | 0.13 | 0.10 | 0.14 | 1.53 | 0.15 | 0.05 | 0.14 | 19.40 | 8.54 | | 50.81 |
| 1.25 | 7.84 | Cash | 3.59 | | | | | | | | | | | | | | | | | | | 3.59 |
| | | Total Asset | **7.68** | **8.83** | **0.51** | **5.59** | **0.18** | **3.74** | **5.77** | **9.69** | **1.43** | **6.13** | **3.69** | **0.99** | **1.53** | **6.91** | **0.05** | **3.20** | **20.21** | **9.48** | **4.40** | **100.00** |
| | | Total Currency | **11.68** | **3.71** | **4.35** | **7.74** | **4.43** | **5.33** | | | **6.99** | | | **6.31** | **10.38** | **10.02** | **4.19** | **9.31** | **4.23** | **6.92** | | **100.00** |
| | | Overlay | **4.00** | **-5.11** | **3.84** | **2.16** | **4.25** | **1.59** | | | **-19.71** | | | **5.32** | **8.85** | **3.11** | **4.14** | **6.11** | **-15.99** | **-2.56** | **0.00** | **43.37** |

**Table B.10 Optimal portfolio allocations under multivariate normal distribution scenarios at return targets 1.30% to 1.50% monthly.**

| Return | CVaR | Allocation | USD | ARS | BRL | CAD | CLP | MXN | | | EUR | | | GBP | CHF | AUD | INR | JPY | KRW | SGD | XAU | Total |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | US | Argentina | Brazil | Canada | Chile | Mexico | Austria | Belgium | Netherlands | France | Germany | UK | Switzerland | Australia | India | Japan | South Korea | Singapore | | |
| | | Govt bond | 3.92 | | | 4.60 | | 3.95 | 1.63 | 0.23 | 0.02 | 5.58 | 2.96 | 0.77 | | 6.29 | | 2.34 | 0.31 | 0.45 | | 33.04 |
| | | Corp bond | 0.57 | | | | | | 2.97 | | | 0.15 | | | | | | | | | | 3.69 |
| | | Equity | 0.02 | 9.53 | 0.81 | 0.91 | 0.07 | 0.92 | 0.34 | 9.72 | 0.49 | 0.21 | 0.31 | 0.06 | 0.01 | 0.52 | 0.35 | 0.11 | 20.92 | 8.96 | | 54.25 |
| 1.30 | 8.07 | Cash | 4.79 | | | | | | | | | | | | | | | | | | | 4.79 |
| | | Total Asset | **9.30** | **9.53** | **0.81** | **5.52** | **0.07** | **4.87** | **4.93** | **9.94** | **0.51** | **5.94** | **3.27** | **0.83** | **0.01** | **6.81** | **0.35** | **2.46** | **21.22** | **9.40** | **4.24** | **100.00** |
| | | Total Currency | **12.25** | **3.63** | **-0.69** | **10.23** | **4.22** | **3.74** | | | **6.61** | | | **5.88** | **11.17** | **10.86** | **5.79** | **9.33** | **4.45** | **8.29** | | **100.00** |
| | | Overlay | **2.95** | **-5.90** | **-1.50** | **4.71** | **4.16** | **-1.13** | | | **-17.97** | | | **5.05** | **11.16** | **4.05** | **5.44** | **6.88** | **-16.77** | **-1.11** | **0.00** | **44.38** |
| | | Govt bond | 3.54 | | | 4.46 | | 3.29 | 1.34 | 0.10 | 0.11 | 5.02 | 2.33 | 1.43 | | 4.59 | | 1.99 | 0.13 | 1.08 | | 29.40 |
| | | Corp bond | 1.94 | | | | | | 3.21 | | | 2.13 | | | | | | | | | | 7.29 |
| | | Equity | 0.56 | 8.75 | 0.22 | 0.61 | 0.21 | 0.09 | 0.65 | 10.93 | 0.06 | 0.23 | 0.56 | 0.13 | 0.76 | 0.27 | 0.03 | 0.35 | 24.48 | 7.83 | | 56.73 |
| 1.35 | 9.02 | Cash | 4.62 | | | | | | | | | | | | | | | | | | | 4.62 |
| | | Total Asset | **10.66** | **8.75** | **0.22** | **5.07** | **0.21** | **3.38** | **5.21** | **11.03** | **0.17** | **7.38** | **2.88** | **1.56** | **0.76** | **4.86** | **0.03** | **2.34** | **24.61** | **8.91** | **1.97** | **100.00** |
| | | Total Currency | **8.07** | **4.57** | **3.81** | **8.22** | **4.14** | **2.83** | | | **8.48** | | | **6.52** | **11.16** | **10.79** | **6.13** | **6.06** | **5.12** | **12.12** | | **100.00** |
| | | Overlay | **-2.59** | **-4.18** | **3.59** | **3.15** | **3.93** | **-0.55** | | | **-18.19** | | | **4.97** | **10.41** | **5.92** | **6.11** | **3.72** | **-19.49** | **3.21** | **0.00** | **45.01** |
| | | Govt bond | 3.09 | | | 3.13 | | 2.72 | 1.88 | 1.37 | 2.40 | 3.56 | 2.54 | 1.63 | | 3.90 | | 1.52 | 1.48 | 1.18 | | 30.40 |
| | | Corp bond | 0.01 | | | | | | 1.30 | | | 0.98 | | | | | | | | | | 2.28 |
| | | Equity | 1.18 | 9.82 | 0.02 | 0.93 | 0.32 | 0.07 | 0.01 | 12.07 | 0.10 | 0.30 | 0.01 | 0.08 | 0.12 | 0.08 | 1.51 | 1.32 | 26.37 | 6.92 | | 61.25 |
| 1.40 | 9.65 | Cash | 4.51 | | | | | | | | | | | | | | | | | | | 4.51 |
| | | Total Asset | **8.78** | **9.82** | **0.02** | **4.06** | **0.32** | **2.79** | **3.19** | **13.44** | **2.50** | **4.84** | **2.55** | **1.71** | **0.12** | **3.98** | **1.51** | **2.84** | **27.86** | **8.11** | **1.56** | **100.00** |
| | | Total Currency | **8.04** | **5.84** | **4.03** | **8.60** | **5.11** | **4.67** | | | **8.31** | | | **4.83** | **9.71** | **9.88** | **5.43** | **7.71** | **5.24** | **11.04** | | **100.00** |
| | | Overlay | **-0.74** | **-3.98** | **4.00** | **4.54** | **4.79** | **1.88** | | | **-18.20** | | | **3.12** | **9.59** | **5.90** | **3.92** | **4.87** | **-22.61** | **2.94** | **0.00** | **45.54** |
| | | Govt bond | 2.61 | | | 2.92 | | 2.43 | 1.90 | 2.12 | 0.98 | 3.57 | 0.25 | 1.55 | | 4.04 | | 1.18 | 0.26 | 1.86 | | 25.67 |
| | | Corp bond | 0.01 | | | | | | 1.13 | | | 1.04 | | | | | | | | | | 2.18 |
| | | Equity | 0.48 | 9.61 | 0.05 | 1.51 | 0.89 | 0.00 | 0.47 | 12.68 | 2.95 | 1.01 | 0.05 | 1.69 | 0.02 | 0.05 | 2.59 | 0.04 | 27.05 | 5.71 | | 66.84 |
| 1.45 | 10.51 | Cash | 4.72 | | | | | | | | | | | | | | | | | | | 4.72 |
| | | Total Asset | **7.83** | **9.61** | **0.05** | **4.42** | **0.89** | **2.43** | **3.49** | **14.80** | **3.92** | **5.62** | **0.30** | **3.24** | **0.02** | **4.08** | **2.59** | **1.22** | **27.32** | **7.57** | **0.58** | **100.00** |
| | | Total Currency | **10.89** | **3.50** | **6.88** | **8.38** | **5.17** | **6.23** | | | **10.41** | | | **6.08** | **8.23** | **8.79** | **6.24** | **5.80** | **5.75** | **7.07** | | **100.00** |
| | | Overlay | **3.06** | **-6.11** | **6.83** | **3.96** | **4.29** | **3.80** | | | **-17.74** | | | **2.83** | **8.21** | **4.71** | **3.66** | **4.58** | **-21.57** | **-0.50** | **0.00** | **45.92** |
| | | Govt bond | 2.59 | | | 2.24 | | 1.76 | 1.45 | 0.92 | 1.43 | 2.88 | 1.34 | 0.20 | | 3.16 | | 1.28 | 2.25 | 0.58 | | 22.08 |
| | | Corp bond | 0.08 | | | | | | 2.10 | | | 1.93 | | | | | | | | | | 4.11 |
| | | Equity | 0.88 | 10.53 | 0.38 | 0.78 | 0.24 | 0.04 | 0.50 | 13.50 | 0.40 | 0.00 | 0.15 | 0.03 | 1.60 | 0.29 | 0.79 | 0.14 | 31.17 | 6.42 | | 67.88 |
| 1.50 | 10.71 | Cash | 5.36 | | | | | | | | | | | | | | | | | | | 5.36 |
| | | Total Asset | **8.91** | **10.53** | **0.38** | **3.02** | **0.24** | **1.80** | **4.05** | **14.42** | **1.83** | **4.82** | **1.49** | **0.23** | **1.60** | **3.46** | **0.79** | **1.42** | **33.42** | **7.00** | **0.58** | **100.00** |
| | | Total Currency | **7.08** | **6.12** | **2.26** | **8.12** | **3.76** | **6.01** | | | **12.32** | | | **4.97** | **9.42** | **7.70** | **6.26** | **8.44** | **7.13** | **9.85** | | **100.00** |
| | | Overlay | **-1.83** | **-4.41** | **1.88** | **5.10** | **3.51** | **4.21** | | | **-14.29** | | | **4.74** | **7.81** | **4.24** | **5.47** | **7.02** | **-26.30** | **2.85** | **0.00** | **46.83** |

**Table B.11 Optimal portfolio allocations under multivariate normal distribution scenarios at return targets 1.55% to 1.60% monthly.**

| Return | CVaR | Allocation | USD | ARS | BRL | CAD | CLP | MXN | EUR | | | | | GBP | CHF | AUD | INR | JPY | KRW | SGD | XAU | Total |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | US | Argentina | Brazil | Canada | Chile | Mexico | Austria | Belgium | Netherlands | France | Germany | UK | Switzerland | Australia | India | Japan | South Korea | Singapore | | |
| | | Govt bond | 2.27 | | | 2.65 | | 2.02 | 1.93 | 0.36 | 1.72 | 3.15 | 1.08 | 0.38 | | 3.14 | | 1.98 | 0.87 | 0.30 | | 21.85 |
| | | Corp bond | 0.79 | | | | | | 1.80 | | | 1.67 | | | | | | | | | | 4.26 |
| | | Equity | 0.07 | 11.06 | 0.11 | 0.55 | 0.28 | 0.02 | 0.09 | 14.12 | 0.24 | 0.42 | 0.60 | 0.34 | 0.27 | 0.02 | 0.69 | 0.37 | 32.27 | 6.62 | | 68.14 |
| 1.55 | 10.81 | Cash | 5.19 | | | | | | | | | | | | | | | | | | | 5.19 |
| | | Total Asset | **8.33** | **11.06** | **0.11** | **3.20** | **0.28** | **2.04** | **3.82** | **14.48** | **1.95** | **5.24** | **1.68** | **0.72** | **0.27** | **3.16** | **0.69** | **2.35** | **33.14** | **6.92** | **0.56** | **100.00** |
| | | Total Currency | **10.20** | **4.61** | **1.64** | **7.93** | **3.94** | **6.64** | | | **12.57** | | | **5.32** | **9.61** | **6.62** | **5.46** | **8.43** | **6.48** | **10.00** | | **100.00** |
| | | Overlay | **1.88** | **-6.45** | **1.53** | **4.74** | **3.66** | **4.60** | | | **-14.61** | | | **4.60** | **9.34** | **3.47** | **4.77** | **6.08** | **-26.66** | **3.08** | **0.00** | **47.72** |
| | | Govt bond | 2.79 | | | 2.94 | | 1.95 | 0.98 | 1.12 | 0.19 | 3.62 | 0.76 | 1.03 | | 3.64 | | 1.54 | 0.66 | 0.09 | | 21.30 |
| | | Corp bond | 0.24 | | | | | | 1.15 | | | 0.09 | | | | | | | | | | 1.48 |
| | | Equity | 0.39 | 10.34 | 0.79 | 1.09 | 0.96 | 0.06 | 0.31 | 14.05 | 0.15 | 0.48 | 0.03 | 0.15 | 0.95 | 0.05 | 0.20 | 0.36 | 33.71 | 5.51 | | 69.57 |
| 1.60 | 11.24 | Cash | 5.58 | | | | | | | | | | | | | | | | | | | 5.58 |
| | | Total Asset | **8.99** | **10.34** | **0.79** | **4.03** | **0.96** | **2.00** | **2.44** | **15.17** | **0.35** | **4.18** | **0.79** | **1.18** | **0.95** | **3.69** | **0.20** | **1.90** | **34.37** | **5.60** | **2.07** | **100.00** |
| | | Total Currency | **8.68** | **5.30** | **2.44** | **8.24** | **5.33** | **5.86** | | | **8.71** | | | **6.27** | **10.51** | **9.18** | **7.34** | **5.22** | **5.79** | **9.06** | | **100.00** |
| | | Overlay | **-0.32** | **-5.04** | **1.65** | **4.21** | **4.37** | **3.86** | | | **-14.22** | | | **5.09** | **9.56** | **5.50** | **7.13** | **3.32** | **-28.58** | **3.45** | **0.00** | **48.16** |

# Appendix C

# Additional information for Chapter 5

## C.1 Rebalancing Allocation Deviation

This section shows the deviations of allocation of portfolios applying a Second-Order Stochastic Dominance Criterion (SSD) rebalancing strategy from portfolios applying a buy-and-hold rebalancing strategy. (See Section 5.4 of Chapter 5 for more details).

**Figure C.1 Currency allocation deviation between the SSD and BHD rebalancing strategies implemented on the low-risk portfolio.** A positive value in each cell indicates that the SSD rebalancing strategy overweighs an investment over the BHD rebalancing strategy and vice versa. Each cell is coloured upto its degree of deviation as given in the scale on the right of the table. Non-coloured rows indicate months SSD rebalancing is discarded as BHD allocations in those periods are already second-order stochastic dominant and hence there are no deviation.

| | USD | ARS | BRL | CAD | CLP | MXN | EUR | GBP | CHF | AUD | INR | JPY | KRW | SGD | XAU |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Jan-12 | 0.05 | -0.02 | -0.02 | 0.05 | -0.01 | 0.03 | -0.05 | -0.05 | -0.04 | -0.02 | 0.05 | -0.03 | -0.04 | 0.05 | 0.05 |
| Feb-12 | 0.05 | -0.02 | -0.02 | 0.06 | -0.01 | 0.03 | -0.05 | -0.05 | -0.04 | -0.02 | 0.04 | -0.01 | -0.04 | 0.05 | 0.05 |
| Mar-12 | 0.05 | -0.02 | -0.02 | 0.05 | -0.01 | 0.03 | -0.05 | -0.05 | -0.05 | -0.02 | 0.04 | 0.00 | -0.04 | 0.05 | 0.05 |
| Apr-12 | 0.05 | -0.02 | -0.02 | 0.05 | -0.01 | 0.02 | -0.05 | -0.05 | -0.05 | -0.02 | 0.03 | 0.00 | -0.04 | 0.05 | 0.05 |
| May-12 | 0.05 | -0.02 | -0.02 | -0.02 | -0.01 | -0.04 | -0.05 | -0.05 | -0.04 | -0.02 | -0.02 | -0.05 | -0.04 | 0.29 | 0.05 |
| Jun-12 | 0.05 | -0.02 | -0.02 | -0.02 | -0.01 | -0.04 | -0.05 | -0.05 | -0.04 | -0.01 | -0.02 | -0.05 | -0.04 | 0.28 | 0.05 |
| Jul-12 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| Aug-12 | 0.05 | -0.02 | -0.02 | -0.02 | -0.01 | -0.04 | -0.05 | -0.05 | -0.04 | -0.02 | -0.02 | -0.05 | -0.04 | 0.28 | 0.05 |
| Sep-12 | 0.05 | -0.02 | -0.02 | -0.02 | -0.01 | -0.04 | -0.05 | -0.05 | -0.04 | -0.02 | -0.02 | -0.05 | -0.04 | 0.29 | 0.05 |
| Oct-12 | 0.05 | -0.02 | -0.02 | -0.02 | -0.01 | -0.04 | -0.05 | -0.05 | -0.04 | -0.02 | -0.02 | -0.05 | -0.05 | 0.29 | 0.05 |
| Nov-12 | 0.05 | -0.02 | -0.02 | -0.02 | -0.01 | -0.04 | -0.05 | -0.05 | -0.04 | -0.02 | -0.02 | -0.05 | -0.05 | 0.29 | 0.05 |
| Dec-12 | 0.05 | -0.02 | -0.02 | -0.02 | -0.01 | -0.04 | -0.05 | -0.05 | -0.04 | -0.02 | -0.02 | -0.05 | -0.05 | 0.29 | 0.05 |
| Jan-13 | 0.05 | -0.02 | -0.02 | -0.02 | -0.01 | -0.04 | -0.05 | -0.05 | -0.04 | -0.02 | -0.02 | -0.05 | -0.05 | 0.29 | 0.05 |
| Feb-13 | 0.05 | -0.02 | -0.02 | -0.02 | -0.01 | -0.04 | -0.05 | -0.05 | -0.05 | -0.02 | -0.02 | -0.05 | -0.05 | 0.29 | 0.05 |
| Mar-13 | 0.05 | -0.02 | -0.02 | -0.02 | -0.01 | -0.04 | -0.05 | -0.05 | -0.04 | -0.02 | -0.02 | -0.05 | -0.05 | 0.29 | 0.05 |
| Apr-13 | 0.05 | -0.02 | -0.02 | -0.02 | -0.01 | -0.05 | -0.05 | -0.05 | -0.04 | -0.02 | -0.02 | -0.05 | -0.05 | 0.29 | 0.05 |
| May-13 | 0.05 | -0.02 | -0.02 | -0.02 | -0.01 | -0.05 | -0.05 | -0.05 | -0.04 | -0.02 | -0.02 | -0.05 | -0.05 | 0.29 | 0.05 |
| Jun-13 | 0.05 | -0.02 | -0.02 | -0.02 | -0.01 | -0.04 | -0.05 | -0.05 | -0.04 | -0.01 | -0.02 | -0.05 | -0.05 | 0.29 | 0.05 |
| Jul-13 | 0.05 | -0.02 | -0.02 | -0.02 | -0.01 | -0.04 | -0.05 | -0.05 | -0.05 | -0.01 | -0.02 | -0.05 | -0.05 | 0.29 | 0.05 |
| Aug-13 | 0.05 | -0.02 | -0.02 | -0.02 | -0.01 | -0.05 | -0.05 | -0.05 | -0.05 | -0.01 | -0.02 | -0.05 | -0.05 | 0.29 | 0.05 |
| Sep-13 | 0.05 | -0.02 | -0.02 | -0.02 | -0.01 | -0.04 | -0.05 | -0.05 | -0.05 | -0.01 | -0.01 | -0.05 | -0.05 | 0.29 | 0.05 |
| Oct-13 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| Nov-13 | 0.05 | -0.02 | -0.02 | -0.02 | -0.01 | -0.04 | -0.05 | -0.05 | -0.05 | -0.01 | -0.02 | -0.05 | -0.05 | 0.29 | 0.05 |
| Dec-13 | 0.05 | -0.02 | -0.02 | -0.02 | -0.01 | -0.04 | -0.05 | -0.05 | -0.05 | -0.01 | -0.02 | -0.05 | -0.05 | 0.29 | 0.05 |
| Jan-14 | 0.05 | -0.01 | -0.02 | -0.02 | -0.01 | -0.04 | -0.05 | -0.05 | -0.05 | -0.01 | -0.02 | -0.05 | -0.05 | 0.29 | 0.05 |
| Feb-14 | 0.05 | -0.01 | -0.02 | -0.02 | -0.01 | -0.04 | -0.05 | -0.05 | -0.05 | -0.01 | -0.02 | -0.05 | -0.05 | 0.28 | 0.05 |
| Mar-14 | 0.05 | -0.01 | -0.02 | -0.02 | -0.01 | -0.04 | -0.05 | -0.05 | -0.05 | -0.01 | -0.02 | -0.05 | -0.05 | 0.28 | 0.05 |
| Apr-14 | 0.05 | -0.01 | -0.02 | -0.02 | -0.01 | -0.04 | -0.05 | -0.05 | -0.05 | -0.01 | -0.02 | -0.05 | -0.05 | 0.29 | 0.05 |
| May-14 | 0.05 | -0.01 | -0.02 | -0.02 | -0.01 | -0.04 | -0.05 | -0.05 | -0.05 | -0.01 | -0.02 | -0.05 | -0.05 | 0.29 | 0.05 |
| Jun-14 | 0.05 | -0.01 | -0.02 | -0.02 | -0.01 | -0.04 | -0.05 | -0.05 | -0.05 | -0.01 | -0.02 | -0.05 | -0.05 | 0.29 | 0.05 |
| Jul-14 | 0.05 | -0.01 | -0.02 | -0.02 | -0.01 | -0.04 | -0.05 | -0.05 | -0.05 | -0.01 | -0.02 | -0.05 | -0.05 | 0.29 | 0.05 |
| Aug-14 | 0.05 | -0.01 | -0.02 | -0.02 | -0.01 | -0.04 | -0.05 | -0.05 | -0.05 | -0.01 | -0.02 | -0.05 | -0.05 | 0.28 | 0.05 |
| Sep-14 | 0.05 | -0.01 | -0.02 | -0.02 | -0.01 | -0.04 | -0.05 | -0.05 | -0.05 | -0.01 | -0.02 | -0.05 | -0.05 | 0.28 | 0.05 |
| Oct-14 | 0.05 | -0.01 | -0.02 | -0.02 | -0.01 | -0.04 | -0.05 | -0.05 | -0.05 | -0.01 | -0.02 | -0.05 | -0.05 | 0.28 | 0.05 |
| Nov-14 | 0.05 | -0.01 | -0.02 | -0.02 | -0.01 | -0.04 | -0.05 | -0.05 | -0.05 | -0.01 | -0.02 | -0.05 | -0.05 | 0.28 | 0.05 |
| Dec-14 | 0.05 | -0.01 | -0.02 | -0.02 | -0.01 | -0.04 | -0.05 | -0.05 | -0.05 | -0.01 | -0.02 | -0.05 | -0.05 | 0.28 | 0.05 |
| Jan-15 | 0.05 | -0.01 | -0.02 | -0.03 | -0.01 | -0.04 | -0.05 | -0.05 | -0.05 | -0.01 | -0.02 | -0.05 | -0.05 | 0.28 | 0.05 |
| Feb-15 | 0.05 | -0.01 | -0.02 | -0.02 | -0.01 | -0.04 | -0.05 | -0.05 | -0.05 | -0.01 | -0.02 | -0.05 | -0.05 | 0.28 | 0.05 |
| Mar-15 | 0.05 | -0.01 | -0.02 | -0.02 | -0.01 | -0.04 | -0.05 | -0.05 | -0.05 | -0.01 | -0.02 | -0.05 | -0.05 | 0.28 | 0.05 |
| Apr-15 | 0.05 | -0.01 | -0.02 | -0.02 | -0.01 | -0.04 | -0.05 | -0.05 | -0.05 | -0.01 | -0.02 | -0.05 | -0.05 | 0.28 | 0.05 |
| May-15 | 0.05 | -0.01 | -0.02 | -0.02 | -0.01 | -0.04 | -0.05 | -0.05 | -0.05 | -0.01 | -0.02 | -0.05 | -0.05 | 0.28 | 0.05 |

Deviation scale

| | | | |
|---|---|---|---|
| -0.30 | to | -0.25 |
| -0.25 | to | -0.20 |
| -0.20 | to | -0.15 |
| -0.15 | to | -0.10 |
| -0.10 | to | -0.05 |
| -0.05 | to | 0.00 |
| 0.00 | to | 0.05 |
| 0.05 | to | 0.10 |
| 0.10 | to | 0.15 |
| 0.15 | to | 0.20 |
| 0.20 | to | 0.25 |
| 0.25 | to | 0.30 |
| 0.30 | to | 0.35 |
| 0.35 | to | 0.40 |
| 0.40 | to | 0.45 |
| 0.45 | to | 0.50 |
| 0.50 | to | 0.55 |

**Figure C.2 Bond allocation deviation between the SSD and BHD rebalancing strategies implemented on the low-risk portfolio.** A positive value in each cell indicates that the SSD rebalancing strategy overweighs an investment over the BHD rebalancing strategy and vice versa. Each cell is coloured upto its degree of deviation as given in the scale on the right of the table. Non-coloured rows indicate months SSD rebalancing is discarded as BHD allocations in those periods are already second-order stochastic dominant and hence there are no deviation.

| | govt.AUD.AU | govt.EUR.AT | govt.EUR.BE | govt.CAD.CA | govt.EUR.FR | govt.EUR.DE | govt.KRW.KR | govt.JPY.JP | govt.MXN.MX | govt.EUR.NL | govt.SGD.SG | govt.GBP.UK | govt.USD.US | corp.EUR.EU | corp.GBP.UK | corp.USD.US |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Jan-12 | 0.02 | -0.01 | 0.02 | -0.02 | 0.05 | 0.05 | -0.04 | -0.02 | 0.03 | -0.04 | -0.05 | -0.05 | 0.01 | -0.04 | -0.03 | 0.00 |
| Feb-12 | 0.02 | -0.01 | 0.02 | -0.01 | 0.05 | 0.05 | -0.04 | 0.00 | 0.03 | -0.04 | -0.05 | -0.05 | 0.01 | -0.04 | -0.03 | 0.00 |
| Mar-12 | 0.02 | -0.01 | 0.02 | -0.01 | 0.05 | 0.05 | -0.04 | 0.00 | 0.03 | -0.04 | -0.05 | -0.05 | 0.00 | -0.04 | -0.03 | 0.00 |
| Apr-12 | 0.02 | -0.01 | 0.02 | -0.01 | 0.05 | 0.05 | -0.04 | 0.00 | 0.03 | -0.04 | -0.05 | -0.05 | 0.00 | -0.04 | -0.03 | 0.00 |
| May-12 | 0.01 | -0.02 | 0.00 | -0.02 | 0.03 | 0.05 | -0.04 | 0.02 | 0.02 | -0.04 | -0.05 | -0.05 | 0.00 | -0.04 | -0.03 | 0.00 |
| Jun-12 | 0.00 | -0.03 | -0.01 | -0.03 | 0.03 | 0.05 | -0.04 | 0.02 | 0.01 | -0.04 | -0.05 | -0.05 | 0.00 | -0.04 | -0.03 | 0.00 |
| Jul-12 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| Aug-12 | 0.00 | -0.03 | -0.01 | -0.03 | 0.02 | 0.05 | -0.04 | 0.02 | 0.01 | -0.04 | -0.05 | -0.05 | 0.00 | -0.04 | -0.03 | 0.00 |
| Sep-12 | 0.00 | -0.03 | -0.01 | -0.03 | 0.02 | 0.05 | -0.04 | 0.02 | 0.01 | -0.04 | -0.05 | -0.05 | 0.00 | -0.04 | -0.03 | 0.00 |
| Oct-12 | 0.00 | -0.03 | -0.01 | -0.03 | 0.02 | 0.05 | -0.04 | 0.02 | 0.01 | -0.04 | -0.05 | -0.05 | 0.00 | -0.04 | -0.03 | 0.00 |
| Nov-12 | 0.00 | -0.03 | -0.01 | -0.03 | 0.03 | 0.05 | -0.04 | 0.02 | 0.01 | -0.04 | -0.05 | -0.05 | 0.00 | -0.04 | -0.03 | 0.00 |
| Dec-12 | 0.00 | -0.03 | -0.01 | -0.03 | 0.02 | 0.05 | -0.04 | 0.02 | 0.01 | -0.04 | -0.05 | -0.05 | 0.00 | -0.04 | -0.03 | 0.00 |
| Jan-13 | 0.00 | -0.03 | -0.01 | -0.03 | 0.03 | 0.05 | -0.04 | 0.02 | 0.01 | -0.04 | -0.05 | -0.05 | 0.00 | -0.04 | -0.03 | 0.00 |
| Feb-13 | 0.00 | -0.03 | -0.01 | -0.03 | 0.03 | 0.05 | -0.04 | 0.02 | 0.01 | -0.04 | -0.05 | -0.05 | 0.00 | -0.04 | -0.03 | 0.00 |
| Mar-13 | 0.00 | -0.03 | -0.01 | -0.03 | 0.03 | 0.05 | -0.04 | 0.01 | 0.01 | -0.04 | -0.05 | -0.05 | 0.00 | -0.04 | -0.03 | 0.00 |
| Apr-13 | 0.00 | -0.03 | -0.01 | -0.03 | 0.03 | 0.05 | -0.04 | 0.01 | 0.01 | -0.04 | -0.05 | -0.05 | 0.00 | -0.04 | -0.03 | 0.00 |
| May-13 | 0.00 | -0.03 | -0.01 | -0.03 | 0.03 | 0.05 | -0.04 | 0.01 | 0.01 | -0.04 | -0.05 | -0.05 | 0.00 | -0.04 | -0.03 | 0.00 |
| Jun-13 | 0.00 | -0.03 | -0.01 | -0.03 | 0.03 | 0.05 | -0.04 | 0.01 | 0.01 | -0.04 | -0.05 | -0.05 | 0.00 | -0.04 | -0.03 | 0.00 |
| Jul-13 | 0.00 | -0.03 | 0.00 | -0.03 | 0.03 | 0.05 | -0.04 | 0.02 | 0.01 | -0.04 | -0.05 | -0.05 | 0.00 | -0.04 | -0.03 | 0.00 |
| Aug-13 | 0.00 | -0.03 | 0.00 | -0.03 | 0.03 | 0.05 | -0.04 | 0.02 | 0.01 | -0.04 | -0.05 | -0.05 | 0.00 | -0.04 | -0.03 | 0.00 |
| Sep-13 | 0.00 | -0.03 | 0.00 | -0.03 | 0.03 | 0.05 | -0.04 | 0.02 | 0.01 | -0.04 | -0.05 | -0.05 | 0.00 | -0.04 | -0.03 | 0.00 |
| Oct-13 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| Nov-13 | 0.00 | -0.03 | 0.00 | -0.03 | 0.03 | 0.05 | -0.04 | 0.02 | 0.01 | -0.04 | -0.05 | -0.05 | 0.00 | -0.04 | -0.03 | 0.00 |
| Dec-13 | 0.00 | -0.03 | 0.00 | -0.03 | 0.03 | 0.05 | -0.04 | 0.02 | 0.01 | -0.04 | -0.05 | -0.05 | 0.00 | -0.04 | -0.03 | 0.00 |
| Jan-14 | 0.00 | -0.03 | 0.00 | -0.03 | 0.03 | 0.05 | -0.04 | 0.02 | 0.01 | -0.04 | -0.05 | -0.05 | 0.00 | -0.04 | -0.03 | 0.00 |
| Feb-14 | 0.00 | -0.03 | -0.01 | -0.03 | 0.03 | 0.05 | -0.04 | 0.02 | 0.01 | -0.04 | -0.05 | -0.05 | -0.01 | -0.04 | -0.03 | 0.00 |
| Mar-14 | 0.00 | -0.03 | -0.01 | -0.03 | 0.03 | 0.05 | -0.04 | 0.01 | 0.01 | -0.04 | -0.05 | -0.02 | 0.03 | -0.04 | -0.03 | 0.00 |
| Apr-14 | 0.00 | -0.03 | -0.01 | -0.03 | 0.03 | 0.05 | -0.04 | 0.01 | 0.01 | -0.04 | -0.05 | -0.02 | 0.03 | -0.04 | -0.03 | 0.00 |
| May-14 | 0.00 | -0.03 | -0.01 | -0.03 | 0.03 | 0.05 | -0.04 | 0.01 | 0.01 | -0.04 | -0.05 | -0.02 | 0.03 | -0.04 | -0.03 | 0.00 |
| Jun-14 | 0.00 | -0.03 | -0.01 | -0.03 | 0.03 | 0.05 | -0.04 | 0.01 | 0.01 | -0.04 | -0.05 | -0.02 | 0.03 | -0.04 | -0.03 | 0.00 |
| Jul-14 | -0.01 | -0.03 | -0.01 | -0.04 | 0.03 | 0.05 | -0.04 | 0.01 | 0.01 | -0.04 | -0.05 | -0.02 | 0.03 | -0.04 | -0.03 | 0.00 |
| Aug-14 | -0.01 | -0.04 | -0.01 | -0.04 | 0.03 | 0.05 | -0.04 | 0.01 | 0.01 | -0.04 | -0.05 | -0.02 | 0.02 | -0.04 | -0.03 | 0.00 |
| Sep-14 | -0.01 | -0.04 | -0.01 | -0.04 | 0.02 | 0.05 | -0.04 | 0.01 | 0.00 | -0.04 | -0.05 | -0.02 | 0.02 | -0.04 | -0.03 | 0.00 |
| Oct-14 | -0.01 | -0.04 | -0.01 | -0.04 | 0.02 | 0.05 | -0.04 | 0.01 | 0.00 | -0.04 | -0.05 | -0.02 | 0.02 | -0.04 | -0.03 | 0.00 |
| Nov-14 | 0.03 | -0.04 | -0.01 | -0.04 | 0.02 | 0.05 | -0.04 | 0.01 | 0.00 | -0.04 | -0.05 | -0.02 | 0.02 | -0.04 | -0.03 | 0.00 |
| Dec-14 | 0.03 | -0.04 | -0.01 | -0.04 | 0.02 | 0.05 | -0.04 | 0.01 | 0.00 | -0.04 | -0.05 | -0.02 | 0.02 | -0.04 | -0.03 | 0.00 |
| Jan-15 | 0.03 | -0.04 | -0.01 | -0.04 | 0.02 | 0.05 | -0.04 | 0.01 | 0.00 | -0.04 | -0.05 | -0.02 | 0.02 | -0.05 | -0.03 | 0.00 |
| Feb-15 | 0.02 | -0.05 | -0.02 | -0.05 | 0.01 | 0.05 | -0.04 | 0.01 | -0.01 | -0.04 | -0.05 | -0.02 | 0.02 | -0.05 | -0.03 | 0.00 |
| Mar-15 | 0.02 | -0.05 | -0.02 | -0.05 | 0.01 | 0.05 | -0.04 | 0.01 | -0.01 | -0.04 | -0.05 | -0.02 | 0.02 | -0.05 | -0.03 | 0.00 |
| Apr-15 | 0.02 | -0.05 | -0.02 | -0.05 | 0.01 | 0.05 | -0.04 | 0.01 | -0.01 | -0.04 | -0.05 | -0.02 | 0.02 | -0.05 | -0.03 | 0.00 |
| May-15 | 0.02 | -0.05 | -0.02 | -0.05 | 0.01 | 0.05 | -0.04 | 0.01 | -0.01 | -0.04 | -0.05 | -0.02 | 0.02 | -0.05 | -0.03 | 0.00 |

Deviation scale

| | | | |
|---|---|---|---|
| -0.30 | to | -0.25 |
| -0.25 | to | -0.20 |
| -0.20 | to | -0.15 |
| -0.15 | to | -0.10 |
| -0.10 | to | -0.05 |
| -0.05 | to | 0.00 |
| 0.00 | to | 0.05 |
| 0.05 | to | 0.10 |
| 0.10 | to | 0.15 |
| 0.15 | to | 0.20 |
| 0.20 | to | 0.25 |
| 0.25 | to | 0.30 |
| 0.30 | to | 0.35 |
| 0.35 | to | 0.40 |
| 0.40 | to | 0.45 |
| 0.45 | to | 0.50 |
| 0.50 | to | 0.55 |

**Figure C.3 Equity allocation deviation between the SSD and BHD rebalancing strategies implemented on the low-risk portfolio.** A positive value in each cell indicates that the SSD rebalancing strategy overweighs an investment over the BHD rebalancing strategy and vice versa. Each cell is coloured upto its degree of deviation as given in the scale on the right of the table. Non-coloured rows indicate months SSD rebalancing is discarded as BHD allocations in those periods are already second-order stochastic dominant and hence there are no deviation.

| | stock.EUR.NL | stock.EUR.AT | stock.AUD.AU | stock.EUR.BE | stock.INR.IN | stock.BRL.BR | stock.EUR.FR | stock.GBP.UK | stock.EUR.DE | stock.USD.US | stock.CAD.CA | stock.CLP.CL | stock.KRW.KR | stock.ARS.AR | stock.MXN.MX | stock.JPY.JP | stock.CHF.CH | stock.SGD.SG | CASH |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Jan-12 | 0.00 | -0.01 | -0.01 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | -0.01 | 0.00 | 0.06 | 0.00 | 0.00 | -0.01 | 0.01 | 0.06 | 0.00 | -0.04 | 0.01 |
| Feb-12 | 0.00 | -0.01 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.05 | 0.00 | 0.00 | -0.01 | 0.00 | 0.06 | 0.00 | -0.05 | 0.03 |
| Mar-12 | 0.00 | -0.01 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.05 | 0.00 | 0.00 | -0.01 | 0.00 | 0.06 | 0.00 | -0.05 | 0.03 |
| Apr-12 | 0.00 | -0.01 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.05 | 0.00 | 0.00 | -0.01 | 0.00 | 0.06 | 0.00 | -0.05 | 0.03 |
| May-12 | 0.00 | -0.01 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.08 | 0.00 | 0.00 | 0.00 | 0.01 | 0.08 | 0.00 | -0.04 | 0.04 |
| Jun-12 | 0.00 | -0.01 | -0.01 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | -0.01 | 0.00 | 0.09 | 0.00 | 0.00 | 0.01 | 0.02 | 0.07 | 0.00 | -0.04 | 0.05 |
| Jul-12 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| Aug-12 | 0.00 | -0.01 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.09 | 0.00 | 0.00 | 0.01 | 0.02 | 0.07 | 0.00 | -0.04 | 0.07 |
| Sep-12 | 0.00 | -0.01 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.09 | 0.00 | 0.00 | 0.01 | 0.02 | 0.07 | 0.00 | -0.04 | 0.07 |
| Oct-12 | 0.00 | -0.01 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.09 | 0.00 | 0.00 | 0.01 | 0.02 | 0.07 | 0.00 | -0.05 | 0.08 |
| Nov-12 | 0.00 | -0.01 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.09 | 0.00 | 0.00 | 0.01 | 0.02 | 0.07 | 0.00 | -0.05 | 0.08 |
| Dec-12 | 0.00 | -0.01 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.09 | 0.00 | 0.00 | 0.01 | 0.02 | 0.07 | 0.00 | -0.05 | 0.08 |
| Jan-13 | 0.00 | -0.01 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.08 | 0.00 | 0.00 | -0.01 | 0.01 | 0.08 | 0.00 | -0.04 | 0.09 |
| Feb-13 | 0.00 | -0.01 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.08 | 0.00 | 0.00 | -0.01 | 0.01 | 0.08 | 0.00 | -0.04 | 0.08 |
| Mar-13 | 0.00 | -0.01 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.09 | 0.00 | 0.00 | -0.01 | 0.01 | 0.08 | 0.00 | -0.04 | 0.09 |
| Apr-13 | 0.00 | -0.01 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.09 | 0.00 | 0.00 | -0.01 | 0.01 | 0.08 | 0.00 | -0.04 | 0.09 |
| May-13 | 0.00 | -0.01 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.09 | 0.00 | 0.00 | -0.01 | 0.01 | 0.08 | 0.00 | -0.04 | 0.10 |
| Jun-13 | 0.00 | -0.01 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.09 | 0.00 | 0.00 | 0.00 | 0.01 | 0.08 | 0.00 | -0.04 | 0.08 |
| Jul-13 | 0.00 | -0.01 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.09 | 0.00 | 0.00 | 0.00 | 0.01 | 0.08 | 0.00 | -0.03 | 0.06 |
| Aug-13 | 0.00 | -0.01 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.09 | 0.00 | 0.00 | 0.00 | 0.01 | 0.08 | 0.00 | -0.04 | 0.07 |
| Sep-13 | 0.00 | -0.01 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.09 | 0.00 | 0.00 | 0.00 | 0.01 | 0.08 | 0.00 | -0.04 | 0.06 |
| Oct-13 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| Nov-13 | 0.00 | -0.02 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.09 | 0.00 | 0.00 | -0.01 | 0.01 | 0.08 | 0.00 | -0.04 | 0.08 |
| Dec-13 | 0.00 | -0.01 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.09 | 0.00 | 0.00 | -0.01 | 0.01 | 0.08 | 0.00 | -0.03 | 0.08 |
| Jan-14 | 0.00 | -0.02 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.09 | 0.00 | 0.00 | -0.01 | 0.01 | 0.08 | 0.00 | -0.03 | 0.07 |
| Feb-14 | 0.00 | -0.01 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.09 | 0.00 | 0.00 | -0.01 | 0.01 | 0.08 | 0.00 | -0.03 | 0.08 |
| Mar-14 | 0.00 | -0.02 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.09 | 0.00 | 0.00 | -0.01 | 0.01 | 0.08 | 0.00 | -0.03 | 0.03 |
| Apr-14 | 0.00 | -0.02 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.09 | 0.00 | 0.00 | -0.01 | 0.01 | 0.08 | 0.00 | -0.03 | 0.03 |
| May-14 | 0.00 | -0.02 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.09 | 0.00 | 0.00 | -0.01 | 0.01 | 0.08 | 0.00 | -0.03 | 0.04 |
| Jun-14 | 0.00 | -0.02 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.09 | 0.00 | 0.00 | -0.01 | 0.01 | 0.08 | 0.00 | -0.03 | 0.05 |
| Jul-14 | 0.00 | -0.02 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.09 | 0.00 | 0.00 | -0.01 | 0.01 | 0.08 | 0.00 | -0.03 | 0.06 |
| Aug-14 | 0.00 | -0.02 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.09 | 0.00 | 0.00 | -0.01 | 0.01 | 0.08 | 0.00 | -0.03 | 0.06 |
| Sep-14 | 0.00 | -0.02 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.09 | 0.00 | 0.00 | -0.01 | 0.01 | 0.08 | 0.00 | -0.03 | 0.08 |
| Oct-14 | 0.00 | -0.01 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.09 | 0.00 | 0.00 | 0.00 | 0.01 | 0.08 | 0.00 | -0.03 | 0.07 |
| Nov-14 | 0.00 | -0.02 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.09 | 0.00 | 0.00 | 0.00 | 0.01 | 0.08 | 0.00 | -0.03 | 0.04 |
| Dec-14 | 0.00 | -0.01 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.09 | 0.00 | 0.00 | 0.00 | 0.01 | 0.08 | 0.00 | -0.03 | 0.05 |
| Jan-15 | 0.00 | -0.02 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.09 | 0.00 | 0.00 | 0.00 | 0.01 | 0.08 | 0.00 | -0.03 | 0.06 |
| Feb-15 | 0.00 | -0.02 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.10 | 0.00 | 0.00 | 0.01 | 0.02 | 0.07 | 0.00 | -0.03 | 0.09 |
| Mar-15 | 0.00 | -0.02 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.10 | 0.00 | 0.00 | 0.01 | 0.02 | 0.07 | 0.00 | -0.03 | 0.09 |
| Apr-15 | 0.00 | -0.02 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.10 | 0.00 | 0.00 | 0.01 | 0.02 | 0.07 | 0.00 | -0.03 | 0.10 |
| May-15 | 0.00 | -0.02 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.00 | -0.01 | 0.00 | 0.10 | 0.00 | 0.00 | 0.01 | 0.02 | 0.07 | 0.00 | -0.03 | 0.09 |

Deviation scale

| | | | |
|---|---|---|---|
| -0.30 | to | -0.25 |
| -0.25 | to | -0.20 |
| -0.20 | to | -0.15 |
| -0.15 | to | -0.10 |
| -0.10 | to | -0.05 |
| -0.05 | to | 0.00 |
| 0.00 | to | 0.05 |
| 0.05 | to | 0.10 |
| 0.10 | to | 0.15 |
| 0.15 | to | 0.20 |
| 0.20 | to | 0.25 |
| 0.25 | to | 0.30 |
| 0.30 | to | 0.35 |
| 0.35 | to | 0.40 |
| 0.40 | to | 0.45 |
| 0.45 | to | 0.50 |
| 0.50 | to | 0.55 |

**Figure C.4 Currency allocation deviation between the SSD and BHD rebalancing strategies implemented on the medium-risk portfolio.** A positive value in each cell indicates that the SSD rebalancing strategy overweighs an investment over the BHD rebalancing strategy and vice versa. Each cell is coloured upto its degree of deviation as given in the scale on the right of the table. Non-coloured rows indicate months SSD rebalancing is discarded as BHD allocations in those periods are already second-order stochastic dominant and hence there are no deviation.

| | USD | ARS | BRL | CAD | CLP | MXN | EUR | GBP | CHF | AUD | INR | JPY | KRW | SGD | XAU |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Jan-12 | 0.05 | -0.03 | -0.03 | -0.05 | -0.03 | -0.04 | -0.05 | -0.05 | -0.05 | -0.05 | -0.02 | 0.00 | -0.05 | 0.34 | 0.05 |
| Feb-12 | 0.05 | -0.03 | -0.03 | -0.05 | -0.03 | -0.04 | -0.05 | -0.05 | -0.05 | -0.05 | -0.02 | 0.00 | -0.05 | 0.34 | 0.05 |
| Mar-12 | 0.05 | -0.03 | -0.03 | -0.05 | -0.03 | -0.04 | -0.05 | -0.05 | -0.05 | -0.05 | -0.02 | 0.00 | -0.05 | 0.34 | 0.05 |
| Apr-12 | 0.05 | -0.03 | -0.03 | -0.05 | -0.03 | -0.04 | -0.05 | -0.05 | -0.05 | -0.05 | -0.02 | 0.00 | -0.05 | 0.34 | 0.05 |
| May-12 | 0.05 | -0.03 | -0.02 | -0.05 | -0.03 | -0.04 | -0.05 | -0.05 | -0.05 | -0.05 | -0.02 | 0.00 | -0.05 | 0.34 | 0.05 |
| Jun-12 | 0.05 | -0.03 | -0.02 | -0.05 | -0.03 | -0.04 | -0.05 | -0.05 | -0.05 | -0.05 | -0.02 | -0.01 | -0.05 | 0.34 | 0.05 |
| Jul-12 | 0.05 | -0.03 | -0.02 | -0.05 | -0.03 | -0.04 | -0.05 | -0.05 | -0.05 | -0.05 | -0.02 | 0.00 | -0.05 | 0.34 | 0.05 |
| Aug-12 | 0.05 | -0.03 | -0.02 | -0.05 | -0.03 | -0.04 | -0.05 | -0.05 | -0.05 | -0.05 | -0.02 | 0.00 | -0.05 | 0.34 | 0.05 |
| Sep-12 | 0.05 | -0.03 | -0.02 | -0.05 | -0.03 | -0.04 | -0.05 | -0.05 | -0.05 | -0.05 | -0.02 | 0.00 | -0.05 | 0.34 | 0.05 |
| Oct-12 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| Nov-12 | 0.05 | -0.03 | -0.02 | -0.05 | -0.03 | -0.04 | -0.05 | -0.05 | -0.05 | -0.05 | -0.02 | 0.00 | -0.05 | 0.34 | 0.05 |
| Dec-12 | 0.05 | -0.03 | -0.02 | -0.05 | -0.03 | -0.04 | -0.05 | -0.05 | -0.05 | -0.05 | -0.02 | 0.00 | -0.05 | 0.34 | 0.05 |
| Jan-13 | 0.05 | -0.03 | -0.02 | -0.05 | -0.03 | -0.04 | -0.05 | -0.05 | -0.05 | -0.05 | -0.02 | 0.01 | -0.05 | 0.33 | 0.05 |
| Feb-13 | 0.05 | -0.03 | -0.02 | -0.05 | -0.03 | -0.04 | -0.05 | -0.05 | -0.05 | -0.05 | -0.02 | 0.03 | -0.05 | 0.31 | 0.05 |
| Mar-13 | 0.05 | -0.03 | -0.02 | -0.05 | -0.03 | -0.04 | -0.05 | -0.05 | -0.05 | -0.05 | -0.02 | 0.01 | -0.05 | 0.33 | 0.05 |
| Apr-13 | 0.05 | -0.02 | -0.02 | -0.05 | -0.03 | -0.04 | -0.05 | -0.05 | -0.05 | -0.05 | -0.02 | 0.02 | -0.05 | 0.32 | 0.05 |
| May-13 | 0.05 | -0.02 | -0.02 | -0.05 | -0.03 | -0.04 | -0.05 | -0.05 | -0.05 | -0.05 | -0.02 | 0.04 | -0.05 | 0.30 | 0.05 |
| Jun-13 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| Jul-13 | 0.05 | -0.02 | -0.02 | -0.05 | -0.03 | -0.04 | -0.05 | -0.05 | -0.05 | -0.05 | -0.02 | 0.01 | -0.05 | 0.33 | 0.05 |
| Aug-13 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| Sep-13 | 0.05 | -0.02 | -0.02 | -0.05 | -0.03 | -0.04 | -0.05 | -0.05 | -0.05 | -0.05 | -0.02 | 0.03 | -0.05 | 0.30 | 0.05 |
| Oct-13 | 0.05 | -0.02 | -0.02 | -0.05 | -0.03 | -0.04 | -0.05 | -0.05 | -0.05 | -0.05 | -0.02 | 0.07 | -0.05 | 0.26 | 0.05 |
| Nov-13 | 0.05 | -0.02 | -0.02 | -0.05 | -0.03 | -0.04 | -0.05 | -0.05 | -0.05 | -0.05 | -0.02 | 0.08 | -0.05 | 0.25 | 0.05 |
| Dec-13 | 0.05 | -0.02 | -0.02 | -0.05 | -0.03 | -0.04 | -0.05 | -0.05 | -0.05 | -0.05 | -0.02 | 0.11 | -0.05 | 0.21 | 0.05 |
| Jan-14 | 0.05 | -0.02 | -0.02 | -0.05 | -0.03 | -0.04 | -0.05 | -0.05 | -0.05 | -0.05 | -0.02 | 0.10 | -0.05 | 0.23 | 0.05 |
| Feb-14 | 0.05 | -0.02 | -0.02 | -0.05 | -0.02 | -0.04 | -0.05 | -0.05 | -0.05 | -0.05 | -0.02 | 0.12 | -0.05 | 0.20 | 0.05 |
| Mar-14 | 0.05 | -0.02 | -0.02 | -0.05 | -0.02 | -0.04 | -0.05 | -0.05 | -0.05 | -0.05 | -0.02 | 0.11 | -0.05 | 0.21 | 0.05 |
| Apr-14 | 0.05 | -0.02 | -0.02 | -0.05 | -0.02 | -0.04 | -0.05 | -0.05 | -0.05 | -0.05 | -0.02 | 0.14 | -0.05 | 0.19 | 0.05 |
| May-14 | 0.05 | -0.02 | -0.02 | -0.05 | -0.02 | -0.04 | -0.05 | -0.05 | -0.05 | -0.05 | -0.02 | 0.15 | -0.05 | 0.17 | 0.05 |
| Jun-14 | 0.05 | -0.02 | -0.02 | -0.05 | -0.02 | -0.04 | -0.05 | -0.05 | -0.05 | -0.05 | -0.02 | 0.19 | -0.05 | 0.14 | 0.05 |
| Jul-14 | 0.05 | -0.02 | -0.02 | -0.05 | -0.02 | -0.04 | -0.05 | -0.05 | -0.05 | -0.05 | -0.02 | 0.20 | -0.05 | 0.12 | 0.05 |
| Aug-14 | 0.05 | -0.02 | -0.02 | -0.05 | -0.02 | -0.04 | -0.05 | -0.05 | -0.05 | -0.05 | -0.02 | 0.21 | -0.05 | 0.11 | 0.05 |
| Sep-14 | 0.05 | -0.02 | -0.02 | -0.05 | -0.02 | -0.04 | -0.05 | -0.05 | -0.05 | -0.05 | -0.02 | 0.26 | -0.05 | 0.07 | 0.05 |
| Oct-14 | 0.05 | -0.02 | -0.02 | -0.05 | -0.02 | -0.04 | -0.05 | -0.05 | -0.05 | -0.05 | -0.02 | 0.21 | -0.05 | 0.11 | 0.05 |
| Nov-14 | 0.05 | -0.02 | -0.02 | -0.05 | -0.02 | -0.04 | -0.05 | -0.05 | -0.05 | -0.05 | -0.02 | 0.05 | -0.05 | 0.27 | 0.05 |
| Dec-14 | 0.05 | -0.02 | -0.02 | -0.05 | -0.02 | -0.04 | -0.05 | -0.05 | -0.05 | -0.05 | -0.02 | 0.05 | -0.05 | 0.27 | 0.05 |
| Jan-15 | 0.05 | -0.02 | -0.02 | -0.05 | -0.02 | -0.04 | -0.05 | -0.05 | -0.05 | -0.05 | -0.02 | 0.05 | -0.05 | 0.27 | 0.05 |
| Feb-15 | 0.05 | -0.02 | -0.02 | -0.05 | -0.02 | -0.04 | -0.05 | -0.05 | -0.05 | -0.05 | -0.02 | 0.05 | -0.05 | 0.27 | 0.05 |
| Mar-15 | 0.05 | -0.02 | -0.02 | -0.05 | -0.02 | -0.04 | -0.05 | -0.05 | -0.05 | -0.05 | -0.02 | 0.05 | -0.05 | 0.27 | 0.05 |
| Apr-15 | 0.05 | -0.02 | -0.02 | -0.05 | -0.02 | -0.04 | -0.05 | -0.05 | -0.05 | -0.05 | -0.02 | 0.05 | -0.05 | 0.27 | 0.05 |
| May-15 | 0.05 | -0.02 | -0.02 | -0.05 | -0.02 | -0.04 | -0.05 | -0.05 | -0.05 | -0.05 | -0.02 | 0.05 | -0.05 | 0.26 | 0.05 |

Deviation scale

| | | | |
|---|---|---|---|
| -0.30 | to | -0.25 | |
| -0.25 | to | -0.20 | |
| -0.20 | to | -0.15 | |
| -0.15 | to | -0.10 | |
| -0.10 | to | -0.05 | |
| -0.05 | to | 0.00 | |
| 0.00 | to | 0.05 | |
| 0.05 | to | 0.10 | |
| 0.10 | to | 0.15 | |
| 0.15 | to | 0.20 | |
| 0.20 | to | 0.25 | |
| 0.25 | to | 0.30 | |
| 0.30 | to | 0.35 | |
| 0.35 | to | 0.40 | |
| 0.40 | to | 0.45 | |
| 0.45 | to | 0.50 | |
| 0.50 | to | 0.55 | |

**Figure C.5 Bond allocation deviation between the SSD and BHD rebalancing strategies implemented on the medium-risk portfolio.** A positive value in each cell indicates that the SSD rebalancing strategy overweighs an investment over the BHD rebalancing strategy and vice versa. Each cell is coloured according to its degree of deviation as given in the scale on the right of the table. Non-coloured rows indicate months SSD rebalancing is discarded as BHD allocations in those periods are already second-order stochastic dominant and hence there are no deviation.

| | govt.AUD.AU | govt.EUR.AT | govt.EUR.BE | govt.CAD.CA | govt.EUR.FR | govt.EUR.DE | govt.KRW.KR | govt.JPY.JP | govt.MXN.MX | govt.EUR.NL | govt.SGD.SG | govt.GBP.UK | govt.USD.US | corp.EUR.EU | corp.GBP.UK | corp.USD.US |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Jan-12 | 0.03 | 0.01 | 0.01 | 0.01 | 0.05 | 0.05 | -0.01 | 0.02 | 0.04 | -0.03 | -0.03 | -0.05 | 0.03 | -0.02 | -0.01 | 0.00 |
| Feb-12 | 0.03 | 0.01 | 0.01 | 0.01 | 0.05 | 0.05 | -0.01 | 0.02 | 0.04 | -0.03 | -0.03 | -0.05 | 0.03 | -0.02 | -0.01 | 0.00 |
| Mar-12 | 0.03 | 0.01 | 0.01 | 0.01 | 0.05 | 0.05 | -0.01 | 0.02 | 0.04 | -0.03 | -0.03 | -0.05 | 0.03 | -0.02 | -0.01 | 0.00 |
| Apr-12 | 0.03 | 0.01 | 0.01 | 0.01 | 0.05 | 0.05 | -0.01 | 0.02 | 0.04 | -0.03 | -0.03 | -0.05 | 0.03 | -0.02 | -0.01 | 0.00 |
| May-12 | 0.04 | 0.01 | 0.02 | 0.01 | 0.05 | 0.05 | -0.01 | 0.01 | 0.05 | -0.03 | -0.03 | -0.05 | 0.03 | -0.02 | -0.01 | 0.00 |
| Jun-12 | 0.03 | 0.01 | 0.01 | 0.01 | 0.05 | 0.05 | -0.01 | 0.01 | 0.04 | -0.03 | -0.04 | -0.05 | 0.03 | -0.02 | -0.01 | 0.00 |
| Jul-12 | 0.03 | 0.01 | 0.01 | 0.01 | 0.05 | 0.05 | -0.01 | 0.01 | 0.04 | -0.03 | -0.04 | -0.05 | 0.03 | -0.02 | -0.01 | 0.00 |
| Aug-12 | 0.03 | 0.01 | 0.01 | 0.01 | 0.05 | 0.05 | -0.01 | 0.01 | 0.04 | -0.03 | -0.04 | -0.05 | 0.03 | -0.02 | -0.01 | 0.00 |
| Sep-12 | 0.03 | 0.01 | 0.01 | 0.01 | 0.05 | 0.05 | -0.01 | 0.01 | 0.04 | -0.03 | -0.04 | -0.05 | 0.03 | -0.02 | -0.01 | 0.00 |
| Oct-12 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| Nov-12 | 0.03 | 0.01 | 0.01 | 0.01 | 0.05 | 0.05 | -0.01 | 0.01 | 0.04 | -0.03 | -0.04 | -0.05 | 0.03 | -0.02 | -0.01 | 0.00 |
| Dec-12 | 0.03 | 0.01 | 0.01 | 0.01 | 0.05 | 0.05 | -0.01 | 0.01 | 0.04 | -0.03 | -0.04 | -0.05 | 0.03 | -0.02 | -0.01 | 0.00 |
| Jan-13 | 0.03 | 0.00 | 0.01 | 0.01 | 0.05 | 0.05 | -0.01 | 0.02 | 0.04 | -0.03 | -0.04 | -0.05 | 0.03 | -0.02 | -0.01 | 0.00 |
| Feb-13 | 0.04 | 0.01 | 0.02 | 0.01 | 0.05 | 0.05 | -0.01 | 0.01 | 0.05 | -0.03 | -0.04 | -0.05 | 0.02 | -0.02 | -0.01 | 0.00 |
| Mar-13 | 0.03 | 0.01 | 0.01 | 0.01 | 0.05 | 0.05 | -0.01 | 0.02 | 0.04 | -0.03 | -0.04 | -0.05 | 0.03 | -0.02 | -0.01 | 0.00 |
| Apr-13 | 0.03 | 0.00 | 0.01 | 0.01 | 0.07 | 0.05 | -0.01 | 0.01 | 0.04 | -0.03 | -0.04 | -0.05 | 0.02 | -0.02 | -0.01 | 0.00 |
| May-13 | 0.03 | 0.00 | 0.01 | 0.01 | 0.07 | 0.05 | -0.01 | 0.01 | 0.04 | -0.03 | -0.04 | -0.05 | 0.01 | -0.02 | -0.01 | 0.00 |
| Jun-13 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| Jul-13 | 0.03 | 0.00 | 0.01 | 0.00 | 0.05 | 0.05 | -0.01 | 0.02 | 0.04 | -0.03 | -0.03 | -0.05 | 0.03 | -0.02 | -0.01 | 0.00 |
| Aug-13 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| Sep-13 | 0.04 | 0.00 | 0.02 | 0.01 | 0.05 | 0.05 | -0.01 | 0.03 | 0.05 | -0.03 | -0.03 | -0.05 | 0.02 | -0.02 | -0.01 | 0.00 |
| Oct-13 | 0.04 | 0.00 | 0.02 | 0.01 | 0.05 | 0.05 | -0.01 | 0.01 | 0.05 | -0.03 | -0.02 | -0.05 | 0.00 | -0.02 | -0.01 | 0.00 |
| Nov-13 | 0.03 | 0.00 | 0.02 | 0.01 | 0.07 | 0.05 | -0.01 | -0.01 | 0.04 | -0.03 | -0.02 | -0.05 | -0.01 | -0.02 | -0.01 | 0.00 |
| Dec-13 | 0.04 | 0.00 | 0.02 | 0.01 | 0.05 | 0.05 | -0.01 | -0.05 | 0.05 | -0.03 | 0.01 | -0.05 | -0.02 | -0.02 | -0.01 | 0.00 |
| Jan-14 | 0.04 | 0.00 | 0.02 | 0.01 | 0.05 | 0.05 | -0.01 | -0.04 | 0.05 | -0.03 | 0.00 | -0.05 | -0.01 | -0.02 | -0.01 | 0.00 |
| Feb-14 | 0.04 | 0.00 | 0.02 | 0.01 | 0.05 | 0.05 | -0.01 | -0.05 | 0.05 | -0.03 | 0.02 | -0.05 | -0.03 | -0.02 | -0.01 | 0.00 |
| Mar-14 | 0.04 | 0.00 | 0.02 | 0.01 | 0.05 | 0.05 | -0.01 | -0.05 | 0.05 | -0.03 | 0.01 | -0.05 | -0.02 | -0.02 | -0.01 | 0.00 |
| Apr-14 | 0.04 | 0.00 | 0.02 | 0.01 | 0.05 | 0.04 | -0.01 | -0.05 | 0.04 | -0.03 | 0.03 | -0.05 | -0.04 | -0.02 | -0.01 | 0.00 |
| May-14 | 0.03 | -0.01 | 0.01 | 0.00 | 0.08 | 0.02 | -0.01 | -0.05 | 0.04 | -0.03 | 0.03 | -0.05 | -0.04 | -0.02 | -0.01 | 0.00 |
| Jun-14 | 0.03 | -0.01 | 0.00 | 0.00 | 0.07 | -0.01 | -0.01 | -0.05 | 0.03 | -0.03 | 0.06 | -0.05 | -0.04 | -0.02 | -0.01 | 0.00 |
| Jul-14 | 0.02 | -0.02 | 0.00 | -0.01 | 0.07 | -0.01 | -0.01 | -0.05 | 0.02 | -0.03 | 0.06 | -0.05 | -0.04 | -0.02 | -0.01 | 0.00 |
| Aug-14 | 0.02 | -0.02 | 0.00 | -0.01 | 0.07 | -0.02 | -0.01 | -0.05 | 0.02 | -0.03 | 0.07 | -0.05 | -0.04 | -0.02 | -0.01 | 0.00 |
| Sep-14 | 0.01 | -0.03 | -0.01 | -0.02 | 0.05 | -0.04 | -0.01 | -0.05 | 0.00 | -0.03 | 0.09 | -0.05 | -0.04 | -0.02 | -0.01 | 0.00 |
| Oct-14 | 0.00 | -0.02 | -0.01 | -0.03 | 0.03 | -0.04 | -0.01 | -0.05 | -0.01 | -0.03 | 0.07 | -0.05 | -0.04 | -0.02 | -0.01 | 0.00 |
| Nov-14 | 0.00 | -0.04 | -0.02 | -0.03 | 0.04 | -0.04 | -0.01 | -0.05 | -0.01 | -0.03 | 0.12 | -0.05 | -0.04 | -0.01 | -0.01 | 0.00 |
| Dec-14 | 0.01 | -0.03 | -0.01 | -0.02 | 0.05 | -0.04 | -0.01 | -0.05 | 0.00 | -0.03 | 0.09 | -0.05 | -0.04 | -0.02 | -0.01 | 0.00 |
| Jan-15 | 0.02 | -0.02 | 0.00 | -0.01 | 0.07 | -0.02 | -0.01 | -0.05 | 0.02 | -0.03 | 0.07 | -0.05 | -0.04 | -0.03 | -0.01 | 0.00 |
| Feb-15 | 0.02 | -0.02 | 0.00 | -0.01 | 0.06 | -0.02 | -0.01 | -0.05 | 0.01 | -0.03 | 0.06 | -0.05 | -0.04 | -0.03 | -0.01 | 0.00 |
| Mar-15 | 0.01 | -0.03 | -0.01 | -0.02 | 0.05 | -0.04 | -0.01 | -0.05 | 0.00 | -0.03 | 0.09 | -0.05 | -0.04 | -0.03 | -0.01 | 0.00 |
| Apr-15 | 0.00 | -0.04 | -0.02 | -0.03 | 0.04 | -0.04 | -0.01 | -0.05 | -0.01 | -0.03 | 0.10 | -0.03 | -0.04 | -0.02 | -0.01 | 0.00 |
| May-15 | -0.01 | -0.05 | -0.02 | -0.05 | 0.03 | -0.04 | -0.01 | -0.05 | -0.02 | -0.03 | 0.12 | -0.03 | -0.04 | 0.00 | -0.01 | 0.00 |

Deviation scale

| | | | |
|---|---|---|---|
| -0.30 | to | -0.25 |
| -0.25 | to | -0.20 |
| -0.20 | to | -0.15 |
| -0.15 | to | -0.10 |
| -0.10 | to | -0.05 |
| -0.05 | to | 0.00 |
| 0.00 | to | 0.05 |
| 0.05 | to | 0.10 |
| 0.10 | to | 0.15 |
| 0.15 | to | 0.20 |
| 0.20 | to | 0.25 |
| 0.25 | to | 0.30 |
| 0.30 | to | 0.35 |
| 0.35 | to | 0.40 |
| 0.40 | to | 0.45 |
| 0.45 | to | 0.50 |
| 0.50 | to | 0.55 |

**Figure C.6 Equity allocation deviation between the SSD and BHD rebalancing strategies implemented on the medium-risk portfolio.** A positive value in each cell indicates that the SSD rebalancing strategy overweighs an investment over the BHD rebalancing strategy and vice versa. Each cell is coloured according to its degree of deviation as given in the scale on the right of the table. Non-coloured rows indicate months SSD rebalancing is discarded as BHD allocations in those periods are already second-order stochastic dominant and hence there are no deviation.

| | stock.EUR.NL | stock.EUR.AT | stock.AUD.AU | stock.EUR.BE | stock.INR.IN | stock.BRL.BR | stock.EUR.FR | stock.GBP.UK | stock.EUR.DE | stock.USD.US | stock.CAD.CA | stock.CLP.CL | stock.KRW.KR | stock.ARS.AR | stock.MXN.MX | stock.JPY.JP | stock.CHF.CH | stock.SGD.SG | CASH |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Jan-12 | 0.00 | 0.00 | -0.02 | -0.04 | -0.01 | 0.00 | -0.01 | 0.00 | 0.00 | 0.00 | -0.01 | 0.00 | -0.05 | -0.05 | -0.01 | 0.04 | 0.00 | -0.05 | 0.05 |
| Feb-12 | 0.00 | 0.00 | -0.02 | -0.04 | -0.01 | 0.00 | -0.01 | 0.00 | 0.00 | 0.00 | -0.02 | 0.00 | -0.05 | -0.05 | -0.01 | 0.04 | 0.00 | -0.05 | 0.08 |
| Mar-12 | 0.00 | 0.00 | -0.02 | -0.04 | -0.01 | 0.00 | -0.01 | 0.00 | 0.00 | 0.00 | -0.02 | 0.00 | -0.05 | -0.05 | -0.01 | 0.04 | 0.00 | -0.05 | 0.09 |
| Apr-12 | 0.00 | 0.00 | -0.02 | -0.04 | -0.01 | 0.00 | -0.01 | 0.00 | 0.00 | 0.00 | -0.02 | 0.00 | -0.05 | -0.05 | -0.01 | 0.04 | 0.00 | -0.05 | 0.09 |
| May-12 | 0.00 | 0.00 | -0.02 | -0.04 | -0.01 | 0.00 | -0.01 | 0.00 | 0.00 | 0.00 | -0.02 | 0.00 | -0.05 | -0.05 | -0.01 | 0.04 | 0.00 | -0.05 | 0.07 |
| Jun-12 | 0.00 | 0.00 | -0.02 | -0.04 | -0.01 | 0.00 | -0.01 | 0.00 | 0.00 | 0.00 | -0.01 | 0.00 | -0.05 | -0.05 | -0.01 | 0.04 | 0.00 | -0.05 | 0.07 |
| Jul-12 | 0.00 | 0.00 | -0.02 | -0.04 | -0.01 | 0.00 | -0.01 | 0.00 | 0.00 | 0.00 | -0.01 | 0.00 | -0.05 | -0.05 | -0.01 | 0.04 | 0.00 | -0.05 | 0.08 |
| Aug-12 | 0.00 | 0.00 | -0.02 | -0.04 | -0.01 | 0.00 | -0.01 | 0.00 | 0.00 | 0.00 | -0.01 | 0.00 | -0.05 | -0.05 | -0.01 | 0.03 | 0.00 | -0.05 | 0.10 |
| Sep-12 | 0.00 | 0.00 | -0.02 | -0.04 | -0.01 | 0.00 | -0.01 | 0.00 | 0.00 | 0.00 | -0.01 | 0.00 | 0.00 | -0.05 | -0.01 | 0.04 | 0.00 | -0.05 | 0.05 |
| Oct-12 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| Nov-12 | 0.00 | 0.00 | -0.02 | -0.05 | -0.01 | 0.00 | -0.01 | 0.00 | 0.00 | 0.00 | -0.02 | 0.00 | 0.00 | -0.05 | -0.01 | 0.03 | 0.00 | -0.05 | 0.06 |
| Dec-12 | 0.00 | 0.00 | -0.02 | -0.05 | -0.01 | 0.00 | -0.01 | 0.00 | 0.00 | 0.00 | -0.02 | 0.00 | 0.00 | -0.05 | -0.01 | 0.03 | 0.00 | -0.05 | 0.07 |
| Jan-13 | 0.00 | 0.00 | -0.02 | -0.05 | -0.01 | 0.00 | -0.01 | 0.00 | 0.00 | 0.00 | -0.03 | 0.00 | 0.01 | -0.05 | -0.01 | 0.02 | 0.00 | -0.05 | 0.08 |
| Feb-13 | 0.00 | 0.00 | -0.02 | -0.05 | -0.01 | 0.00 | -0.01 | 0.00 | 0.00 | 0.00 | -0.04 | 0.00 | 0.02 | -0.05 | -0.01 | 0.00 | 0.00 | -0.05 | 0.10 |
| Mar-13 | 0.00 | 0.00 | -0.02 | -0.05 | -0.01 | 0.00 | -0.01 | 0.00 | 0.00 | 0.00 | -0.04 | 0.00 | 0.01 | -0.05 | -0.01 | 0.00 | 0.00 | -0.05 | 0.09 |
| Apr-13 | 0.00 | 0.00 | -0.02 | -0.05 | -0.01 | 0.00 | -0.01 | 0.00 | 0.00 | 0.00 | -0.04 | 0.00 | 0.01 | -0.05 | -0.01 | 0.00 | 0.00 | -0.05 | 0.10 |
| May-13 | 0.00 | 0.00 | -0.02 | -0.05 | -0.01 | 0.00 | -0.01 | 0.00 | 0.00 | 0.00 | -0.04 | 0.00 | 0.06 | -0.05 | -0.01 | 0.00 | 0.00 | -0.05 | 0.07 |
| Jun-13 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| Jul-13 | 0.00 | 0.00 | -0.02 | -0.05 | -0.01 | 0.00 | -0.01 | 0.00 | 0.00 | 0.00 | -0.01 | 0.00 | 0.03 | -0.05 | -0.01 | 0.04 | 0.00 | -0.05 | 0.02 |
| Aug-13 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| Sep-13 | 0.00 | 0.00 | -0.02 | -0.05 | -0.01 | 0.00 | -0.01 | 0.00 | 0.00 | 0.00 | -0.03 | 0.00 | 0.06 | -0.05 | -0.01 | 0.00 | 0.00 | -0.05 | 0.03 |
| Oct-13 | 0.00 | 0.00 | -0.02 | -0.05 | -0.01 | 0.00 | -0.01 | 0.00 | 0.00 | 0.00 | -0.04 | 0.00 | 0.08 | -0.05 | -0.01 | 0.00 | 0.00 | -0.05 | 0.05 |
| Nov-13 | 0.00 | 0.00 | -0.02 | -0.05 | -0.01 | 0.00 | -0.01 | 0.00 | 0.00 | 0.00 | -0.04 | 0.00 | 0.09 | -0.05 | -0.01 | 0.00 | 0.00 | -0.05 | 0.06 |
| Dec-13 | 0.00 | 0.00 | -0.02 | -0.05 | -0.01 | 0.00 | -0.01 | 0.00 | 0.00 | 0.00 | -0.03 | 0.00 | 0.10 | -0.05 | -0.01 | 0.00 | 0.00 | -0.05 | 0.06 |
| Jan-14 | 0.00 | 0.00 | -0.02 | -0.05 | -0.01 | 0.00 | -0.01 | 0.00 | 0.00 | 0.00 | -0.03 | 0.00 | 0.09 | -0.05 | -0.01 | 0.00 | 0.00 | -0.05 | 0.05 |
| Feb-14 | 0.00 | 0.00 | -0.02 | -0.05 | -0.01 | 0.00 | -0.01 | 0.00 | 0.00 | 0.00 | -0.03 | 0.00 | 0.11 | -0.05 | -0.01 | 0.00 | 0.00 | -0.05 | 0.06 |
| Mar-14 | 0.00 | 0.00 | -0.02 | -0.05 | -0.01 | 0.00 | -0.01 | 0.00 | 0.00 | 0.00 | -0.03 | 0.00 | 0.10 | -0.01 | -0.01 | 0.00 | 0.00 | -0.05 | 0.03 |
| Apr-14 | 0.00 | 0.00 | -0.02 | -0.05 | -0.01 | 0.00 | -0.01 | 0.00 | 0.00 | 0.00 | -0.03 | 0.00 | 0.12 | -0.01 | -0.01 | 0.00 | 0.00 | -0.05 | 0.05 |
| May-14 | 0.00 | 0.00 | -0.02 | -0.05 | -0.01 | 0.00 | -0.01 | 0.00 | 0.00 | 0.00 | -0.04 | 0.00 | 0.13 | -0.02 | -0.01 | 0.00 | 0.00 | -0.05 | 0.06 |
| Jun-14 | 0.00 | 0.00 | -0.02 | -0.05 | -0.01 | 0.00 | -0.01 | 0.00 | 0.00 | 0.00 | -0.04 | 0.00 | 0.15 | -0.02 | -0.01 | 0.00 | 0.00 | -0.05 | 0.08 |
| Jul-14 | 0.00 | 0.00 | -0.02 | -0.05 | -0.01 | 0.00 | -0.01 | 0.00 | 0.00 | 0.00 | -0.04 | 0.00 | 0.15 | -0.02 | -0.01 | 0.00 | 0.00 | -0.05 | 0.09 |
| Aug-14 | 0.00 | 0.00 | -0.02 | -0.05 | -0.01 | 0.00 | -0.01 | 0.00 | 0.00 | 0.00 | -0.04 | 0.00 | 0.20 | -0.02 | -0.01 | 0.00 | 0.00 | -0.05 | 0.07 |
| Sep-14 | 0.00 | 0.00 | -0.02 | -0.01 | -0.01 | 0.00 | -0.01 | 0.00 | 0.00 | 0.00 | -0.04 | 0.00 | 0.24 | -0.02 | -0.01 | 0.00 | 0.00 | -0.05 | 0.06 |
| Oct-14 | 0.00 | 0.00 | -0.02 | -0.01 | -0.01 | 0.00 | -0.01 | 0.00 | 0.00 | 0.00 | -0.04 | 0.00 | 0.31 | -0.03 | -0.01 | 0.00 | 0.00 | -0.05 | 0.07 |
| Nov-14 | 0.00 | 0.00 | -0.02 | -0.01 | -0.01 | 0.00 | -0.01 | 0.00 | 0.00 | 0.00 | -0.04 | 0.00 | 0.26 | -0.02 | -0.01 | 0.00 | 0.00 | -0.04 | 0.06 |
| Dec-14 | 0.00 | 0.00 | -0.02 | -0.01 | -0.01 | 0.00 | -0.01 | 0.00 | 0.00 | 0.00 | -0.04 | 0.00 | 0.23 | -0.02 | -0.01 | 0.00 | 0.00 | -0.04 | 0.06 |
| Jan-15 | 0.00 | 0.00 | -0.02 | -0.01 | -0.01 | 0.00 | -0.01 | 0.00 | 0.00 | 0.00 | -0.04 | 0.00 | 0.19 | -0.02 | -0.01 | 0.00 | 0.00 | -0.05 | 0.06 |
| Feb-15 | 0.00 | 0.00 | -0.03 | -0.01 | -0.01 | 0.00 | -0.01 | 0.00 | 0.00 | 0.00 | -0.04 | 0.00 | 0.19 | -0.02 | -0.01 | 0.00 | 0.00 | -0.05 | 0.09 |
| Mar-15 | 0.00 | 0.00 | -0.03 | -0.01 | -0.01 | 0.00 | -0.01 | 0.00 | 0.00 | 0.00 | -0.04 | 0.00 | 0.22 | -0.02 | -0.01 | 0.00 | 0.00 | -0.05 | 0.10 |
| Apr-15 | 0.00 | 0.00 | -0.03 | -0.01 | -0.01 | 0.00 | -0.01 | 0.00 | 0.00 | 0.00 | -0.04 | 0.00 | 0.25 | -0.02 | -0.01 | 0.00 | 0.00 | -0.04 | 0.09 |
| May-15 | 0.00 | 0.00 | -0.03 | -0.01 | -0.01 | 0.00 | -0.01 | 0.00 | 0.00 | 0.00 | -0.04 | 0.00 | 0.28 | -0.03 | -0.01 | 0.00 | 0.00 | -0.05 | 0.10 |

Deviation scale

| | | | |
|---|---|---|---|
| -0.30 | to | -0.25 |
| -0.25 | to | -0.20 |
| -0.20 | to | -0.15 |
| -0.15 | to | -0.10 |
| -0.10 | to | -0.05 |
| -0.05 | to | 0.00 |
| 0.00 | to | 0.05 |
| 0.05 | to | 0.10 |
| 0.10 | to | 0.15 |
| 0.15 | to | 0.20 |
| 0.20 | to | 0.25 |
| 0.25 | to | 0.30 |
| 0.30 | to | 0.35 |
| 0.35 | to | 0.40 |
| 0.40 | to | 0.45 |
| 0.45 | to | 0.50 |
| 0.50 | to | 0.55 |

**Figure C.7 Currency allocation deviation between the SSD and BHD rebalancing strategies implemented on the high-risk portfolio.** A positive value in each cell indicates that the SSD rebalancing strategy overweighs an investment over the BHD rebalancing strategy and vice versa. Each cell is coloured according to its degree of deviation as given in the scale on the right of the table. Non-coloured rows indicate months SSD rebalancing is discarded as BHD allocations in those periods are already second-order stochastic dominant and hence there are no deviation.

| | USD | ARS | BRL | CAD | CLP | MXN | EUR | GBP | CHF | AUD | INR | JPY | KRW | SGD | XAU |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Jan-12 | 0.05 | -0.05 | -0.05 | -0.09 | -0.03 | -0.03 | -0.05 | 0.05 | 0.20 | -0.05 | -0.05 | 0.05 | -0.05 | 0.05 | 0.05 |
| Feb-12 | 0.05 | -0.05 | -0.05 | -0.09 | -0.03 | -0.03 | -0.05 | 0.05 | 0.20 | -0.05 | -0.06 | 0.05 | -0.05 | 0.05 | 0.05 |
| Mar-12 | 0.05 | -0.05 | -0.05 | -0.09 | -0.03 | -0.03 | -0.05 | 0.05 | 0.20 | -0.05 | -0.06 | 0.05 | -0.05 | 0.05 | 0.05 |
| Apr-12 | 0.05 | -0.05 | -0.05 | -0.09 | -0.03 | -0.03 | -0.05 | 0.05 | 0.20 | -0.05 | -0.05 | 0.05 | -0.05 | 0.05 | 0.05 |
| May-12 | 0.05 | -0.05 | -0.05 | -0.09 | -0.03 | -0.03 | -0.05 | 0.05 | 0.20 | -0.05 | -0.05 | 0.05 | -0.05 | 0.05 | 0.05 |
| Jun-12 | 0.05 | -0.05 | -0.05 | -0.09 | -0.03 | -0.03 | -0.05 | 0.05 | 0.19 | -0.05 | -0.05 | 0.05 | -0.05 | 0.05 | 0.05 |
| Jul-12 | 0.05 | -0.05 | -0.05 | -0.09 | -0.03 | -0.03 | -0.05 | 0.05 | 0.20 | -0.05 | -0.05 | 0.05 | -0.05 | 0.05 | 0.05 |
| Aug-12 | 0.05 | -0.05 | -0.04 | -0.09 | -0.03 | -0.03 | -0.05 | 0.05 | 0.20 | -0.05 | -0.05 | 0.05 | -0.05 | 0.05 | 0.05 |
| Sep-12 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| Oct-12 | 0.05 | -0.05 | -0.04 | -0.09 | -0.03 | -0.03 | -0.05 | 0.05 | 0.20 | -0.05 | -0.05 | 0.05 | -0.05 | 0.05 | 0.05 |
| Nov-12 | 0.05 | -0.05 | -0.04 | -0.09 | -0.03 | -0.03 | -0.05 | 0.05 | 0.20 | -0.05 | -0.05 | 0.05 | -0.05 | 0.05 | 0.05 |
| Dec-12 | 0.05 | -0.05 | -0.04 | -0.09 | -0.03 | -0.03 | -0.05 | 0.05 | 0.20 | -0.05 | -0.05 | 0.05 | -0.05 | 0.05 | 0.05 |
| Jan-13 | 0.05 | -0.05 | -0.04 | -0.09 | -0.03 | -0.03 | -0.05 | 0.05 | 0.20 | -0.05 | -0.05 | 0.05 | -0.05 | 0.05 | 0.05 |
| Feb-13 | 0.05 | -0.05 | -0.05 | -0.09 | -0.03 | -0.03 | -0.05 | 0.05 | 0.20 | -0.05 | -0.05 | 0.05 | -0.05 | 0.05 | 0.05 |
| Mar-13 | 0.05 | -0.05 | -0.05 | -0.09 | -0.03 | -0.03 | -0.05 | 0.05 | 0.20 | -0.05 | -0.05 | 0.05 | -0.05 | 0.05 | 0.05 |
| Apr-13 | 0.05 | -0.05 | -0.05 | -0.09 | -0.03 | -0.03 | -0.05 | 0.05 | 0.20 | -0.05 | -0.05 | 0.05 | -0.05 | 0.05 | 0.05 |
| May-13 | 0.05 | -0.05 | -0.05 | -0.09 | -0.03 | -0.03 | -0.05 | 0.05 | 0.20 | -0.05 | -0.05 | 0.05 | -0.05 | 0.05 | 0.05 |
| Jun-13 | 0.05 | -0.05 | -0.04 | -0.09 | -0.03 | -0.03 | -0.05 | 0.05 | 0.20 | -0.05 | -0.05 | 0.05 | -0.05 | 0.05 | 0.05 |
| Jul-13 | 0.05 | -0.05 | -0.04 | -0.09 | -0.03 | -0.03 | -0.05 | 0.05 | 0.20 | -0.05 | -0.05 | 0.05 | -0.05 | 0.05 | 0.05 |
| Aug-13 | 0.05 | -0.05 | -0.04 | -0.09 | -0.03 | -0.03 | -0.05 | 0.05 | 0.20 | -0.05 | -0.05 | 0.05 | -0.05 | 0.05 | 0.05 |
| Sep-13 | 0.05 | -0.05 | -0.04 | -0.09 | -0.03 | -0.03 | -0.05 | 0.05 | 0.19 | -0.05 | -0.05 | 0.05 | -0.05 | 0.05 | 0.05 |
| Oct-13 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| Nov-13 | 0.05 | -0.05 | -0.04 | -0.09 | -0.03 | -0.03 | -0.05 | 0.05 | 0.19 | -0.05 | -0.05 | 0.05 | -0.05 | 0.05 | 0.05 |
| Dec-13 | 0.05 | -0.05 | -0.04 | -0.09 | -0.03 | -0.03 | -0.05 | 0.05 | 0.23 | -0.05 | -0.05 | 0.05 | -0.05 | 0.05 | 0.01 |
| Jan-14 | 0.05 | -0.05 | -0.04 | -0.09 | -0.03 | -0.03 | -0.05 | 0.05 | 0.20 | -0.05 | -0.05 | 0.05 | -0.05 | 0.05 | 0.04 |
| Feb-14 | 0.05 | -0.02 | 0.00 | -0.02 | -0.01 | 0.02 | 0.03 | 0.02 | -0.05 | -0.05 | -0.02 | 0.05 | -0.02 | 0.02 | -0.01 |
| Mar-14 | 0.05 | -0.05 | -0.04 | -0.08 | -0.03 | -0.03 | -0.05 | 0.05 | 0.22 | -0.05 | -0.05 | 0.05 | -0.05 | 0.05 | 0.01 |
| Apr-14 | 0.05 | -0.02 | 0.00 | -0.02 | -0.01 | 0.02 | 0.03 | 0.02 | -0.05 | -0.05 | -0.02 | 0.05 | -0.02 | 0.02 | -0.01 |
| May-14 | 0.05 | -0.02 | -0.01 | -0.02 | -0.01 | 0.02 | 0.03 | 0.02 | -0.05 | -0.05 | -0.02 | 0.05 | -0.02 | 0.02 | -0.01 |
| Jun-14 | 0.06 | -0.02 | -0.01 | -0.02 | -0.01 | 0.02 | 0.03 | 0.02 | -0.06 | -0.06 | -0.02 | 0.06 | -0.02 | 0.02 | -0.01 |
| Jul-14 | 0.06 | -0.02 | -0.01 | -0.02 | -0.01 | 0.02 | 0.03 | 0.02 | -0.06 | -0.06 | -0.02 | 0.06 | -0.02 | 0.02 | -0.01 |
| Aug-14 | 0.06 | -0.02 | -0.01 | -0.02 | -0.01 | 0.02 | 0.03 | 0.02 | -0.06 | -0.06 | -0.02 | 0.06 | -0.02 | 0.02 | -0.01 |
| Sep-14 | 0.09 | -0.03 | -0.02 | -0.04 | -0.02 | 0.01 | 0.02 | 0.01 | -0.09 | -0.08 | -0.03 | 0.08 | -0.04 | 0.01 | 0.00 |
| Oct-14 | 0.18 | -0.03 | -0.02 | -0.04 | -0.02 | 0.01 | 0.02 | 0.01 | -0.08 | -0.07 | -0.03 | 0.08 | -0.04 | 0.01 | 0.03 |
| Nov-14 | 0.14 | -0.03 | -0.02 | -0.04 | -0.02 | 0.01 | 0.02 | 0.01 | -0.08 | -0.08 | -0.03 | 0.08 | -0.04 | 0.01 | 0.01 |
| Dec-14 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| Jan-15 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| Feb-15 | 0.06 | -0.02 | 0.00 | -0.02 | -0.01 | 0.02 | 0.04 | 0.02 | -0.06 | -0.06 | -0.02 | 0.06 | -0.03 | 0.02 | -0.02 |
| Mar-15 | 0.06 | -0.02 | 0.00 | -0.02 | -0.01 | 0.02 | 0.03 | 0.02 | -0.06 | -0.06 | -0.02 | 0.06 | -0.03 | 0.02 | -0.02 |
| Apr-15 | 0.11 | -0.03 | -0.01 | -0.03 | -0.02 | 0.01 | 0.03 | 0.01 | -0.09 | -0.07 | -0.04 | 0.08 | -0.04 | 0.01 | 0.00 |
| May-15 | 0.17 | -0.03 | -0.01 | -0.03 | -0.02 | 0.01 | 0.02 | 0.01 | -0.09 | -0.07 | -0.03 | 0.08 | -0.04 | 0.00 | 0.02 |

Deviation scale

| | | | |
|---|---|---|---|
| -0.30 | to | -0.25 |
| -0.25 | to | -0.20 |
| -0.20 | to | -0.15 |
| -0.15 | to | -0.10 |
| -0.10 | to | -0.05 |
| -0.05 | to | 0.00 |
| 0.00 | to | 0.05 |
| 0.05 | to | 0.10 |
| 0.10 | to | 0.15 |
| 0.15 | to | 0.20 |
| 0.20 | to | 0.25 |
| 0.25 | to | 0.30 |
| 0.30 | to | 0.35 |
| 0.35 | to | 0.40 |
| 0.40 | to | 0.45 |
| 0.45 | to | 0.50 |
| 0.50 | to | 0.55 |

**Figure C.8 Bond allocation deviation between the SSD and BHD rebalancing strategies implemented on the high-risk portfolio.** A positive value in each cell indicates that the SSD rebalancing strategy overweighs an investment over the BHD rebalancing strategy and vice versa. Each cell is coloured according to its degree of deviation as given in the scale on the right of the table. Non-coloured rows indicate months SSD rebalancing is discarded as BHD allocations in those periods are already second-order stochastic dominant and hence there are no deviation.

| | govt.AUD.AU | govt.EUR.AT | govt.EUR.BE | govt.CAD.CA | govt.EUR.FR | govt.EUR.DE | govt.KRW.KR | govt.JPY.JP | govt.MXN.MX | govt.EUR.NL | govt.SGD.SG | govt.GBP.UK | govt.USD.US | corp.EUR.EU | corp.GBP.UK | corp.USD.US |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Jan-12 | 0.05 | 0.02 | 0.04 | 0.01 | 0.05 | -0.01 | 0.00 | -0.02 | 0.05 | -0.01 | 0.08 | -0.02 | -0.01 | -0.01 | 0.00 | 0.00 |
| Feb-12 | 0.05 | 0.00 | 0.02 | -0.01 | 0.05 | -0.01 | 0.00 | -0.02 | 0.05 | -0.01 | 0.08 | -0.02 | -0.01 | -0.01 | 0.00 | 0.00 |
| Mar-12 | 0.05 | 0.01 | 0.02 | 0.00 | 0.05 | -0.01 | 0.00 | -0.02 | 0.05 | -0.01 | 0.08 | -0.02 | -0.01 | -0.01 | 0.00 | 0.00 |
| Apr-12 | 0.05 | 0.01 | 0.02 | -0.01 | 0.05 | -0.01 | 0.00 | -0.02 | 0.05 | -0.01 | 0.08 | -0.02 | -0.01 | -0.01 | 0.00 | 0.00 |
| May-12 | 0.05 | 0.03 | 0.04 | 0.02 | 0.05 | -0.01 | 0.00 | -0.02 | 0.05 | -0.01 | 0.07 | -0.02 | -0.01 | -0.01 | 0.00 | 0.00 |
| Jun-12 | 0.05 | 0.03 | 0.05 | 0.02 | 0.05 | -0.01 | 0.00 | -0.02 | 0.05 | -0.01 | 0.07 | -0.02 | -0.01 | -0.01 | 0.00 | 0.00 |
| Jul-12 | 0.05 | 0.02 | 0.04 | 0.01 | 0.05 | -0.01 | 0.00 | -0.02 | 0.05 | -0.01 | 0.07 | -0.02 | -0.01 | -0.01 | 0.00 | 0.00 |
| Aug-12 | 0.05 | 0.02 | 0.03 | 0.01 | 0.05 | -0.01 | 0.00 | -0.03 | 0.05 | -0.01 | 0.07 | -0.02 | -0.01 | -0.01 | 0.00 | 0.00 |
| Sep-12 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| Oct-12 | 0.05 | 0.01 | 0.03 | 0.00 | 0.05 | -0.01 | 0.00 | -0.03 | 0.05 | -0.01 | 0.07 | -0.02 | -0.01 | -0.01 | 0.00 | 0.00 |
| Nov-12 | 0.05 | 0.02 | 0.03 | 0.01 | 0.05 | -0.01 | 0.00 | -0.03 | 0.05 | -0.01 | 0.07 | -0.02 | -0.01 | -0.01 | 0.00 | 0.00 |
| Dec-12 | 0.05 | 0.01 | 0.03 | 0.00 | 0.05 | -0.01 | 0.00 | -0.03 | 0.05 | -0.01 | 0.07 | -0.02 | -0.01 | -0.01 | 0.00 | 0.00 |
| Jan-13 | 0.05 | -0.01 | 0.01 | -0.02 | 0.05 | -0.01 | 0.00 | -0.03 | 0.04 | -0.01 | 0.07 | -0.02 | -0.01 | -0.01 | 0.00 | 0.00 |
| Feb-13 | 0.02 | -0.01 | 0.00 | -0.02 | 0.05 | -0.01 | 0.00 | -0.03 | 0.01 | -0.01 | 0.07 | -0.02 | -0.01 | -0.01 | 0.00 | 0.00 |
| Mar-13 | 0.04 | -0.01 | 0.00 | -0.02 | 0.05 | -0.01 | 0.00 | -0.03 | 0.04 | -0.01 | 0.07 | -0.02 | -0.01 | -0.01 | 0.00 | 0.00 |
| Apr-13 | 0.03 | -0.01 | 0.00 | -0.02 | 0.05 | -0.01 | 0.00 | -0.03 | 0.02 | -0.01 | 0.07 | -0.02 | -0.01 | -0.01 | 0.00 | 0.00 |
| May-13 | 0.01 | -0.01 | 0.00 | -0.02 | 0.04 | -0.01 | 0.00 | -0.03 | 0.00 | -0.01 | 0.07 | -0.02 | -0.01 | -0.01 | 0.00 | 0.00 |
| Jun-13 | 0.03 | -0.01 | 0.00 | -0.02 | 0.05 | -0.01 | 0.00 | -0.03 | 0.02 | -0.01 | 0.07 | -0.02 | -0.01 | -0.01 | 0.00 | 0.00 |
| Jul-13 | 0.05 | 0.00 | 0.01 | -0.01 | 0.05 | -0.01 | 0.00 | -0.03 | 0.05 | -0.01 | 0.07 | -0.02 | -0.01 | -0.01 | 0.00 | 0.00 |
| Aug-13 | 0.04 | -0.01 | 0.00 | -0.02 | 0.05 | -0.01 | 0.00 | -0.03 | 0.03 | -0.01 | 0.07 | -0.02 | -0.01 | -0.01 | 0.00 | 0.00 |
| Sep-13 | 0.02 | -0.01 | 0.00 | -0.02 | 0.05 | -0.01 | 0.00 | -0.03 | 0.01 | -0.01 | 0.07 | -0.02 | -0.01 | -0.01 | 0.00 | 0.00 |
| Oct-13 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| Nov-13 | -0.01 | -0.01 | 0.00 | -0.02 | -0.02 | -0.01 | 0.00 | -0.03 | -0.02 | -0.01 | 0.02 | -0.02 | -0.01 | -0.01 | 0.00 | 0.00 |
| Dec-13 | -0.01 | -0.01 | 0.00 | -0.02 | -0.02 | -0.01 | 0.00 | -0.03 | -0.02 | -0.01 | 0.00 | -0.02 | -0.01 | -0.01 | 0.00 | 0.00 |
| Jan-14 | -0.01 | -0.01 | 0.00 | -0.02 | -0.01 | -0.01 | 0.00 | -0.03 | -0.02 | -0.01 | 0.00 | -0.02 | -0.01 | -0.01 | 0.00 | 0.00 |
| Feb-14 | -0.01 | -0.01 | 0.00 | -0.02 | -0.02 | -0.01 | 0.00 | -0.03 | -0.02 | -0.01 | 0.00 | -0.02 | -0.01 | -0.01 | 0.00 | 0.00 |
| Mar-14 | -0.01 | -0.01 | 0.00 | -0.02 | -0.02 | -0.01 | 0.00 | -0.03 | -0.02 | -0.01 | 0.00 | -0.02 | -0.01 | -0.01 | 0.00 | 0.00 |
| Apr-14 | -0.01 | -0.01 | 0.00 | -0.02 | -0.02 | -0.01 | 0.00 | -0.03 | -0.02 | -0.01 | 0.00 | -0.02 | -0.01 | -0.01 | 0.00 | 0.00 |
| May-14 | -0.01 | -0.01 | 0.00 | -0.02 | -0.02 | -0.01 | 0.00 | -0.03 | -0.02 | -0.01 | 0.00 | -0.02 | -0.01 | -0.01 | 0.00 | 0.00 |
| Jun-14 | -0.01 | -0.01 | 0.00 | -0.02 | -0.02 | -0.01 | 0.00 | -0.03 | -0.02 | -0.01 | 0.00 | -0.02 | -0.01 | -0.01 | 0.00 | 0.00 |
| Jul-14 | -0.01 | -0.01 | 0.00 | -0.02 | -0.02 | -0.01 | 0.00 | -0.03 | -0.02 | -0.01 | 0.00 | -0.02 | -0.01 | -0.01 | 0.00 | 0.00 |
| Aug-14 | -0.01 | -0.01 | 0.00 | -0.02 | -0.02 | -0.01 | 0.00 | -0.03 | -0.02 | -0.01 | 0.00 | -0.02 | -0.01 | -0.01 | 0.00 | 0.00 |
| Sep-14 | 0.02 | 0.01 | 0.01 | 0.01 | 0.01 | -0.01 | 0.00 | -0.01 | 0.02 | 0.00 | 0.02 | 0.00 | 0.01 | -0.01 | 0.00 | 0.00 |
| Oct-14 | 0.05 | 0.04 | 0.04 | 0.04 | 0.04 | 0.02 | 0.03 | 0.02 | 0.05 | 0.02 | 0.04 | 0.03 | 0.04 | 0.02 | 0.02 | 0.00 |
| Nov-14 | 0.03 | 0.02 | 0.02 | 0.02 | 0.02 | 0.00 | 0.01 | 0.00 | 0.03 | 0.01 | 0.03 | 0.01 | 0.02 | 0.00 | 0.00 | 0.00 |
| Dec-14 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| Jan-15 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| Feb-15 | -0.02 | -0.01 | 0.00 | -0.02 | -0.02 | -0.01 | 0.00 | -0.03 | -0.02 | -0.01 | 0.00 | -0.02 | -0.01 | -0.01 | 0.00 | 0.00 |
| Mar-15 | -0.02 | -0.01 | 0.00 | -0.02 | -0.02 | -0.01 | 0.00 | -0.03 | -0.02 | -0.01 | 0.00 | -0.02 | -0.01 | -0.01 | 0.00 | 0.00 |
| Apr-15 | 0.02 | 0.01 | 0.01 | 0.01 | 0.02 | 0.00 | 0.01 | -0.01 | 0.02 | 0.00 | 0.02 | 0.01 | 0.02 | 0.00 | 0.00 | 0.00 |
| May-15 | 0.04 | 0.03 | 0.03 | 0.02 | 0.03 | 0.01 | 0.02 | 0.01 | 0.04 | 0.01 | 0.03 | 0.02 | 0.03 | 0.01 | 0.01 | 0.00 |

Deviation scale

| | | | |
|---|---|---|---|
| -0.30 | to | -0.25 | |
| -0.25 | to | -0.20 | |
| -0.20 | to | -0.15 | |
| -0.15 | to | -0.10 | |
| -0.10 | to | -0.05 | |
| -0.05 | to | 0.00 | |
| 0.00 | to | 0.05 | |
| 0.05 | to | 0.10 | |
| 0.10 | to | 0.15 | |
| 0.15 | to | 0.20 | |
| 0.20 | to | 0.25 | |
| 0.25 | to | 0.30 | |
| 0.30 | to | 0.35 | |
| 0.35 | to | 0.40 | |
| 0.40 | to | 0.45 | |
| 0.45 | to | 0.50 | |
| 0.50 | to | 0.55 | |

**Figure C.9 Equity allocation deviation between the SSD and BHD rebalancing strategies implemented on the high-risk portfolio.** A positive value in each cell indicates that the SSD rebalancing strategy overweighs an investment over the BHD rebalancing strategy and vice versa. Each cell is coloured according to its degree of deviation as given in the scale on the right of the table. Non-coloured rows indicate months SSD rebalancing is discarded as BHD allocations in those periods are already second-order stochastic dominant and hence there are no deviation.

| | stock.EUR.NL | stock.EUR.AT | stock.AUD.AU | stock.EUR.BE | stock.INR.IN | stock.BRL.BR | stock.EUR.FR | stock.GBP.UK | stock.EUR.DE | stock.USD.US | stock.CAD.CA | stock.CLP.CL | stock.KRW.KR | stock.ARS.AR | stock.MXN.MX | stock.JPY.JP | stock.CHF.CH | stock.SGD.SG | CASH |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Jan-12 | 0.00 | -0.01 | 0.00 | -0.05 | -0.02 | 0.00 | 0.00 | 0.00 | -0.01 | 0.00 | -0.03 | 0.00 | -0.05 | -0.05 | 0.00 | 0.00 | -0.01 | -0.05 | 0.02 |
| Feb-12 | 0.00 | -0.01 | 0.00 | -0.05 | -0.02 | 0.00 | 0.00 | 0.00 | -0.01 | 0.00 | -0.03 | 0.00 | -0.05 | -0.05 | 0.00 | 0.00 | -0.01 | -0.05 | 0.09 |
| Mar-12 | 0.00 | -0.01 | 0.00 | -0.05 | -0.02 | 0.00 | 0.00 | 0.00 | -0.02 | 0.00 | -0.03 | 0.00 | -0.05 | -0.05 | 0.00 | 0.00 | -0.01 | -0.05 | 0.08 |
| Apr-12 | 0.00 | -0.01 | 0.00 | -0.05 | -0.02 | 0.00 | 0.00 | 0.00 | -0.02 | 0.00 | -0.03 | 0.00 | -0.05 | -0.05 | 0.00 | 0.00 | -0.01 | -0.05 | 0.09 |
| May-12 | 0.00 | -0.01 | 0.00 | -0.05 | -0.02 | 0.00 | 0.00 | 0.00 | -0.02 | 0.00 | -0.03 | 0.00 | -0.05 | -0.05 | 0.00 | 0.00 | -0.01 | -0.05 | 0.03 |
| Jun-12 | 0.00 | -0.01 | 0.00 | -0.05 | -0.02 | 0.00 | 0.00 | 0.00 | -0.01 | 0.00 | -0.03 | 0.00 | -0.05 | -0.05 | 0.00 | 0.00 | -0.01 | -0.05 | 0.00 |
| Jul-12 | 0.00 | -0.01 | 0.00 | -0.05 | -0.02 | 0.00 | 0.00 | 0.00 | -0.02 | 0.00 | -0.03 | 0.00 | -0.05 | -0.05 | 0.00 | 0.00 | -0.01 | -0.05 | 0.04 |
| Aug-12 | 0.00 | -0.01 | 0.00 | -0.05 | -0.02 | 0.00 | 0.00 | 0.00 | -0.02 | 0.00 | -0.03 | 0.00 | -0.05 | -0.05 | 0.00 | 0.00 | -0.01 | -0.05 | 0.05 |
| Sep-12 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| Oct-12 | 0.00 | -0.01 | 0.00 | -0.05 | -0.02 | 0.00 | 0.00 | 0.00 | -0.02 | 0.00 | -0.03 | 0.00 | -0.05 | -0.05 | 0.00 | 0.00 | -0.01 | -0.05 | 0.08 |
| Nov-12 | 0.00 | -0.01 | 0.00 | -0.05 | -0.02 | 0.00 | 0.00 | 0.00 | -0.02 | 0.00 | -0.03 | 0.00 | -0.05 | -0.05 | 0.00 | 0.00 | -0.01 | -0.05 | 0.06 |
| Dec-12 | 0.00 | -0.01 | 0.00 | -0.05 | -0.02 | 0.00 | 0.00 | 0.00 | -0.02 | 0.00 | -0.03 | 0.00 | 0.11 | -0.05 | 0.00 | 0.00 | -0.01 | -0.05 | -0.08 |
| Jan-13 | 0.00 | -0.01 | 0.00 | -0.05 | -0.02 | 0.00 | 0.00 | 0.00 | -0.02 | 0.00 | -0.03 | 0.00 | 0.14 | -0.05 | 0.00 | 0.00 | -0.01 | -0.05 | -0.05 |
| Feb-13 | 0.00 | -0.01 | 0.00 | -0.05 | -0.02 | 0.00 | 0.00 | 0.00 | -0.02 | 0.00 | -0.03 | 0.00 | 0.18 | -0.05 | 0.00 | 0.00 | -0.01 | -0.05 | 0.00 |
| Mar-13 | 0.00 | -0.01 | 0.00 | -0.05 | -0.02 | 0.00 | 0.00 | 0.00 | -0.02 | 0.00 | -0.03 | 0.00 | 0.15 | -0.05 | 0.00 | 0.00 | -0.01 | -0.05 | -0.04 |
| Apr-13 | 0.00 | -0.01 | 0.00 | -0.05 | -0.02 | 0.00 | 0.00 | 0.00 | -0.02 | 0.00 | -0.03 | 0.00 | 0.17 | -0.05 | 0.00 | 0.00 | -0.01 | -0.05 | -0.02 |
| May-13 | 0.00 | -0.01 | 0.00 | -0.05 | -0.02 | 0.00 | 0.00 | 0.00 | -0.02 | 0.00 | -0.03 | 0.00 | 0.19 | -0.05 | 0.00 | 0.00 | -0.01 | -0.04 | 0.00 |
| Jun-13 | 0.00 | -0.01 | 0.00 | -0.05 | -0.02 | 0.00 | 0.00 | 0.00 | -0.02 | 0.00 | -0.03 | 0.00 | 0.17 | -0.05 | 0.00 | 0.00 | -0.01 | -0.04 | -0.03 |
| Jul-13 | 0.00 | -0.01 | 0.00 | -0.05 | -0.02 | 0.00 | 0.00 | 0.00 | -0.02 | 0.00 | -0.03 | 0.00 | 0.14 | -0.05 | 0.00 | 0.00 | -0.01 | -0.04 | -0.08 |
| Aug-13 | 0.00 | -0.01 | 0.00 | -0.05 | -0.02 | 0.00 | 0.00 | 0.00 | -0.02 | 0.00 | -0.03 | 0.00 | 0.16 | -0.05 | 0.00 | 0.00 | -0.01 | -0.04 | -0.06 |
| Sep-13 | 0.00 | -0.01 | 0.00 | -0.05 | -0.02 | 0.00 | 0.00 | 0.00 | -0.02 | 0.00 | -0.03 | 0.00 | 0.19 | -0.05 | 0.00 | 0.00 | -0.01 | -0.04 | -0.04 |
| Oct-13 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| Nov-13 | 0.00 | -0.01 | 0.00 | -0.05 | -0.02 | 0.00 | 0.00 | 0.00 | -0.02 | 0.00 | -0.03 | 0.00 | 0.26 | -0.05 | 0.00 | 0.00 | -0.01 | -0.04 | 0.07 |
| Dec-13 | 0.00 | -0.01 | 0.00 | -0.05 | -0.02 | 0.00 | 0.00 | 0.00 | -0.02 | 0.00 | -0.03 | 0.00 | 0.29 | -0.05 | 0.00 | 0.00 | -0.01 | -0.04 | 0.10 |
| Jan-14 | 0.00 | -0.01 | 0.00 | -0.05 | -0.02 | 0.00 | 0.00 | 0.00 | -0.02 | 0.00 | -0.03 | 0.00 | 0.27 | -0.05 | 0.00 | 0.00 | -0.01 | -0.03 | 0.08 |
| Feb-14 | 0.00 | -0.01 | 0.00 | -0.05 | -0.02 | 0.00 | 0.00 | 0.00 | -0.02 | 0.00 | -0.02 | 0.00 | 0.49 | -0.05 | 0.00 | 0.00 | -0.01 | -0.04 | -0.08 |
| Mar-14 | 0.00 | -0.01 | 0.00 | -0.05 | -0.02 | 0.00 | 0.00 | 0.00 | -0.02 | 0.00 | -0.03 | 0.00 | 0.47 | -0.05 | 0.00 | 0.00 | -0.01 | -0.04 | -0.08 |
| Apr-14 | 0.00 | -0.01 | 0.00 | -0.05 | -0.02 | 0.00 | 0.00 | 0.00 | -0.02 | 0.00 | -0.03 | 0.00 | 0.51 | -0.05 | 0.00 | 0.00 | -0.01 | -0.04 | -0.10 |
| May-14 | 0.00 | -0.01 | 0.00 | -0.05 | -0.02 | 0.00 | 0.00 | 0.00 | -0.02 | 0.00 | -0.03 | 0.00 | 0.55 | -0.05 | 0.00 | 0.00 | -0.01 | -0.04 | -0.13 |
| Jun-14 | 0.00 | -0.01 | 0.00 | -0.06 | -0.02 | 0.00 | 0.00 | 0.00 | -0.02 | 0.00 | -0.03 | 0.00 | 0.42 | -0.06 | 0.00 | -0.01 | -0.01 | -0.04 | 0.01 |
| Jul-14 | 0.00 | -0.01 | 0.00 | -0.06 | -0.02 | 0.00 | 0.00 | 0.00 | -0.02 | 0.00 | -0.03 | 0.00 | 0.44 | -0.06 | 0.00 | 0.00 | -0.01 | -0.04 | 0.00 |
| Aug-14 | 0.00 | -0.01 | 0.00 | -0.06 | -0.02 | 0.00 | 0.00 | 0.00 | -0.02 | 0.00 | -0.03 | 0.00 | 0.46 | -0.06 | 0.00 | 0.00 | -0.01 | -0.04 | -0.02 |
| Sep-14 | 0.00 | -0.01 | 0.00 | 0.04 | -0.02 | 0.00 | 0.00 | 0.00 | -0.02 | 0.00 | -0.03 | 0.00 | 0.12 | -0.09 | 0.00 | -0.01 | -0.01 | -0.03 | -0.02 |
| Oct-14 | 0.00 | 0.00 | 0.00 | -0.10 | -0.02 | 0.00 | 0.00 | 0.00 | -0.02 | 0.00 | -0.01 | 0.00 | -0.27 | -0.15 | 0.00 | 0.00 | -0.01 | 0.00 | 0.06 |
| Nov-14 | 0.00 | -0.01 | 0.00 | 0.00 | -0.02 | 0.00 | 0.00 | 0.00 | -0.02 | 0.00 | -0.03 | 0.00 | 0.02 | -0.14 | 0.00 | -0.01 | -0.01 | -0.01 | 0.00 |
| Dec-14 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| Jan-15 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| Feb-15 | 0.00 | -0.01 | 0.00 | 0.08 | -0.02 | 0.00 | 0.00 | 0.00 | -0.02 | 0.00 | -0.03 | 0.00 | 0.34 | -0.06 | 0.00 | 0.00 | -0.01 | -0.04 | -0.03 |
| Mar-15 | 0.00 | -0.01 | 0.00 | 0.08 | -0.02 | 0.00 | 0.00 | 0.00 | -0.02 | 0.00 | -0.03 | 0.00 | 0.25 | -0.06 | 0.00 | -0.01 | -0.01 | -0.03 | 0.07 |
| Apr-15 | 0.00 | -0.01 | 0.00 | 0.03 | -0.02 | 0.00 | 0.00 | 0.00 | -0.02 | 0.00 | -0.03 | 0.00 | 0.10 | -0.11 | 0.00 | -0.01 | -0.01 | -0.02 | -0.03 |
| May-15 | 0.00 | -0.01 | 0.00 | -0.04 | -0.02 | 0.00 | 0.00 | 0.00 | -0.02 | 0.00 | -0.02 | 0.00 | -0.10 | -0.15 | 0.00 | -0.01 | -0.01 | -0.01 | 0.06 |

Deviation scale

| | | | |
|---|---|---|---|
| -0.30 | to | -0.25 |
| -0.25 | to | -0.20 |
| -0.20 | to | -0.15 |
| -0.15 | to | -0.10 |
| -0.10 | to | -0.05 |
| -0.05 | to | 0.00 |
| 0.00 | to | 0.05 |
| 0.05 | to | 0.10 |
| 0.10 | to | 0.15 |
| 0.15 | to | 0.20 |
| 0.20 | to | 0.25 |
| 0.25 | to | 0.30 |
| 0.30 | to | 0.35 |
| 0.35 | to | 0.40 |
| 0.40 | to | 0.45 |
| 0.45 | to | 0.50 |
| 0.50 | to | 0.55 |

# Index

Asset exposure, 233
Asymmetric Dependence, 6

Buy-and-Hold Rebalancing, 188

Clayton Copula, 96
Conditional Value-at-Risk (CVaR), 127
Copulas, 85
Correlation, 88
Cost of Carry, 26
Currency Overlay, 17

Efficient Frontier, 43
Exchange rate pair, 233
Exposure, 232

First-Order Stochastic Dominance, 181
Foreign Exchange Forward Contracts, 25
Forward Exchange rate, 234
Frank Copula, 97

Genetic Algorithm, 147
Gumbel Copula, 96

Kendall's Tau, 90
Kernel Density Estimation, 131

Long position, 232

Margin Requirement, 48

Normal Copula, 94

Pair-Copula Construction, 100
Periodic Rebalancing, 188
Portfolio, 231
Portfolio Rebalancing, 187
Position, 232

Regular-Vine Copula, 102

Scenario Generation, 130
Second-Order Stochastic Dominance, 183
Short position, 232
Spot Exchange rate, 234
SSD Rebalancing, 190
Stochastic Dominance, 180
Student's t Copula, 95

Tail Dependence, 92
The Constant Expected Return Model, 122
Two-Stage Stochastic Programming with Recourse, 81