

Particle detectors in fermionic and bosonic quantum field theory in flat and curved spacetimes

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Abstract

This thesis is concerned with aspects of quantum theory of fields in flat and curved spacetimes of arbitrary dimensions along with detecting bosons and fermions on these spacetimes. The thesis is divided into two main parts. In the first part, we analyse an Unruh-DeWitt particle detector that is coupled linearly to the scalar density of a massless Dirac field (neutrino field) in Minkowski spacetimes of dimension $d \geq 2$ and on the two-dimensional static Minkowski cylinder, allowing the detector's motion to remain arbitrary and working to leading order in perturbation theory. In d -dimensional Minkowski spacetime, with the field in the usual Fock vacuum, we show that the detector's response is identical to that of a detector coupled linearly to a massless scalar field in $2d$ -dimensional Minkowski. In the special case of uniform linear acceleration, the detector's response hence exhibits the Unruh effect with a Planckian factor in both even and odd dimensions, in contrast to the Rindler power spectrum of the Dirac field, which has a Planckian factor for odd d but a Fermi-Dirac factor for even d . On the two-dimensional cylinder, we set the oscillator modes in the usual Fock vacuum but allow an arbitrary state for the zero mode of the periodic spinor. We show that the detector's response distinguishes the periodic and antiperiodic spin structures, and the zero mode of the periodic spinor contributes to the response by a state-dependent but well defined amount. Explicit analytic and numerical results on the cylinder are obtained for inertial and uniformly accelerated trajectories, recovering the $d = 2$ Minkowski results in the limit of large circumference. The detector's response has no infrared ambiguity for $d = 2$, neither in Minkowski nor on the cylinder.

In the second part, firstly, we give a thorough discussion for the Bogolubov transformation for Dirac field, and discuss pair creation in a non-stationary spacetime. Secondly, we derive the *in* and *out* vacua Wightman two-point functions for the Dirac field and the Klein-Gordon field for certain class of spatially flat Friedmann-Robertson-Walker (FRW) cosmological spacetimes wherein the two-point functions

have the Hadamard form. We then establish the equivalence between the adiabatic vacuum of infinite order and the conformal vacuum in the massless limit. With the field in the conformal Fock vacuum, we then show that the detector's response to an UDW particle detector coupled linearly to the scalar density of a massless Dirac field in the spatially flat FRW spacetimes in d -dimensions is identical to the response of a detector coupled to the massless scalar field in the spatially flat FRW spacetimes in $2d$ -dimensions.

Lastly, we discuss a massive scalar field in the spatially compactified $(1 + 1)$ -dimensional FRW spacetime. There, the issue of the conformal zero momentum mode arises. To resolve this issue, we develop a new scheme for quantizing the conformal zero-mode. This new quantization scheme introduces a family of two real parameters for every zero-momentum mode with an associated two-real-parameter set of *in/out* vacua. We then show that the zero momentum initial state's wave functional corresponds to a two-real parameter set of Gaussian wave packets. For applications, we examine the finite-time detector's response to a massive scalar field in the $(1 + 1)$ -dimensional, spatially compactified Milne spacetime. Explicit analytic results are obtained for the comoving and inertially non-comoving trajectories. Numerical results are provided for the comoving trajectory. The numerical results suggest that when the in-vacuum is chosen to be very far from the conventional Minkowski vacuum state, then it contains particles. As result, spontaneous excitation of the comoving detector occurs.

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Notation

In this thesis our notation follows that of Wald [1] in which we use the mostly minus sign convention $(+, -, \dots, -)$ for dealing with spinor field and the mostly plus sign convention $(-, +, \dots, +)$ for dealing with the scalar field. For the majority of the thesis, we use units in which $\hbar = c = G = k_B = 1$. Furthermore, spacetime points are denoted by math italic letters. We denote the set of n-bein fields by $e^\cdot_\cdot$, $e^\cdot_\cdot$, $e_{\cdot\cdot}$, $e^{\cdot\cdot}$, where the first indices are the n-bein indices of the local frame, which we shall label with Latin letters from the first part of the Roman alphabet ($a - h$). The second indices are the spacetime or tensor indices which we shall label with Greek letters ($\alpha, \beta, \mu, \sigma, \dots$). Therefore the covariant derivative of the n-bein field operates on the second (tensor) indices only. To raise or lower the first indices we use the Minkowski metric $\eta_{ab}, \dots$, while to raise or lower the second indices we use the curved spacetime metric $g_{\mu\nu}(x)$. Moreover, we use the latter part of the Roman alphabet ($i, j, k, l, m, n, \dots$) to label spatial indices. We underline the spacetime-dependent γ -matrices to differentiate them from the constant γ -matrices.

The Christoffel symbols, Riemann tensor, Ricci Tensor and Einstein Equation in a coordinate basis, are given by

$$\begin{aligned}\Gamma_{\mu\nu}^\alpha &= \frac{1}{2}g^{\alpha\rho}(g_{\mu\rho,\nu} + g_{\nu\rho,\mu} - g_{\mu\nu,\rho}) , \\ R^\mu{}_{\nu\rho\sigma} &= \partial_\rho\Gamma_{\nu\sigma}^\mu - \partial_\sigma\Gamma_{\nu\rho}^\mu + \Gamma_{\nu\sigma}^\lambda\Gamma_{\lambda\rho}^\mu - \Gamma_{\nu\rho}^\lambda\Gamma_{\lambda\sigma}^\mu , \\ R^\alpha{}_{\mu\nu\alpha\nu} &= R_{\mu\nu} , \\ R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} &= 8\pi GT_{\mu\nu} ,\end{aligned}$$

where the Ricci curvature is defined as $R = R^\mu{}_\mu$, Λ and G are the cosmological and gravitational constants. $T_{\mu\nu}$ is the stress-energy momentum tensor representing the distribution of matter in the spacetime.

The complex conjugate of a complex number A is denoted by A^* . The adjoint of an operator B acting on a Hilbert space is denoted by $B^\dagger$.

Part I

Field Theory and Fermionic Detectors in Flat Spacetimes

Chapter 1

Introduction

Quantum theory of matter has passed a variety of experimental tests and is considered a well-established theory, except for ongoing discussion about its interpretational aspects [2, 3]. Quantum theory together with Einstein's gravitational theory, which revolutionised our understanding of the geometry of spacetime, are considered two pillars of modern science. In quantum theory the states of a system are represented by vectors in an abstract Hilbert space, $\mathcal{H}$, and observables are represented by Hermitian, linear operators on $\mathcal{H}$. In general, the observables will not have definite values unless the state of system happens to be in an eigenstate of the observable under consideration. In an experiment, quantum theory allows us, in principle, to predict the various possible outcomes of a long series of identical experiments along with the probability of each possible outcome. This is in contrast with Einstein's gravitational theory where observable quantities always have definite value [1]. Thus, if we regard quantum theory as a more fundamental mathematical framework to describe natural phenomena, then Einstein's gravitational theory must at best approximate a more fundamental theory of gravity valid when quantum effects can be ignored in perhaps a similar fashion as Maxwell's electromagnetic theory can be regarded as an approximation to quantum electrodynamics in that range where it is safe to ignore quantum effects. Although such a fundamental theory of gravity remains elusive, there exists a regime where a semi-classical approximation

of such a theory readily predicts the quantum effects of the gravitational field. This approach is analogous to the semi-classical description of quantum electrodynamics, in which the electromagnetic fields is treated as a fixed classical background interacting with the quantized matter.

Dimensional analysis suggests that the length scale at which we suspect a classical description of spacetime structure (the left side of Einstein's equation (see Notation)) to break down should occur at the order of the *Planck length*, $l_p := (G\hbar/c^3)^{1/2}$, and smaller, where $\hbar$, c and G are respectively Planck's constant, the speed of light and the gravitational constant. And similarly, the time scale at which quantum gravitational effects are suspected to be dominant should occur at the order of *Planck time*, $t_p := (G\hbar/c^5)^{1/2}$, or smaller. Consequently, in the regime above Planck scale, we expect that quantum effects of the gravitational field are likely to be negligible and one can work within a framework of a semi-classical approximation of the Einstein's gravitational theory.

The semiclassical theory which consists of quantum field theory in curved spacetime (QFTCS) and the semiclassical Einstein equation provides us with a first step in the direction of formulating a theory of quantum gravity. QFTCS describes the behaviour of quantum fields in a fixed classical curved spacetime background [5–7], while the semiclassical Einstein equation describes the effects induced by the quantized matter fields on the gravitational background. The semiclassical Einstein equation is given by

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G \langle \Psi | \hat{T}_{\mu\nu} | \Psi \rangle_{ren}, \quad (1.1)$$

where $\hat{T}_{\mu\nu}$ is the stress-energy tensor operator, $|\Psi\rangle$ is the given quantum state of the matter field and the subscript “ren” means that all infinities are removed from the matrix elements [4]. The semiclassical theory is phenomenologically relevant in the Big-Bang cosmological model and in the vicinity of black hole singularity.

One of the remarkable predictions of QFTCS is that black holes emit Hawking radiation [8]. Given a Schwarzschild spacetime of mass M , and a quantum field

prepared in the Hartle-Hawking vacuum state, a static particle detector outside the horizon, at Schwarzschild radial coordinate $r > 2M$, will perceive in the long time limit the temperature $T = \kappa/(2\pi V)$, where $V = \sqrt{1 - \frac{2M}{r}}$ is the redshift factor appropriate to this radius and $\kappa = (4M)^{-1}$ is the surface gravity of the black hole [1]. Far from the black hole ($r \rightarrow \infty$), we have $V = 1$, and the effect may be interpreted as the response of the detector to the flux of radiation produced by the black hole and radiation falling in from infinity [1, 9]. In other words, from the perspective of asymptotic inertial observers, the Schwarzschild black hole emits a thermal spectrum of particles with the Hawking Temperature $T_H = \frac{1}{8\pi M}$. In cgs units, $T_H \approx 6 \times 10^{-8} K \times \frac{M_\odot}{M}$, where $M_\odot$ is the solar mass. For macroscopic black holes $M \gtrsim M_\odot$, so $T_H \lesssim 6 \times 10^{-8} K$, which is considerably smaller than the cosmic microwave background temperature, $T_{CMB} \approx 2.72 K$ [10, 11]. As such, for macroscopic black holes, this effect is overshadowed by the CMB and is not within our contemporary observational reach. At $r \rightarrow 2M$, T diverges, being asymptotically equal to $a/(2\pi)$, where $a = \frac{M}{r^2 \sqrt{1 - \frac{2M}{r}}}$ is the static detector's acceleration [12]. This shows that near the horizon the Hawking effect is closely related to the Unruh effect [9], to which we now turn.

The Unruh Effect in Minkowski spacetime occurs when a particle detector, coupled to a quantum field initially in its vacuum state, is uniformly accelerated through Minkowski spacetime with acceleration a : in the long time limit, the detector perceives a thermal distribution of radiation with the temperature $T = a/(2\pi)$ [1, 9]. In other words, in the long detection time limit, the uniformly accelerated detector perceives the ordinary Minkowski Fock vacuum state as a thermal bath. In cgs units, the temperature of the thermal distribution of radiation is given by $T \approx 4 \times 10^{-23} a K$. While for terrestrial free fall acceleration $a = g$ (Earth's surface gravity), the Unruh Effect is far below any experimentally accessible regime, novel short-pulse, laser-driven acceleration is anticipated to push the Unruh Effect into an experimentally testable regime by providing an acceleration gradient as high as $a = 10^{23}g$ or $a = 10^{25}g$ [13, 14].

In 1981, Unruh raised the possibility that the aforementioned effects in QFTCS could be verified in analogue acoustic counterparts (toy models) [15] (and references cited there). He noticed that the motion of sound waves propagating in a non-rotational fluid simulates the behaviour of a massless scalar field in Schwarzschild spacetime near the horizon. Despite that his fluid method broke Lorentz invariance, the sonic black hole nevertheless produced a sonic spectrum of sound that is approximately a thermal spectrum [16]. This is presently an active area of research that can demonstrate the universality and robustness of several effects predicted within the framework of QFTCS.

In this thesis, we focus on aspects of quantum field theory in flat and curved spacetimes along with studying fermionic and bosonic particle detectors' responses on these spacetimes. We present a collection of new results in the following related areas of research:

1. Fermionic detector's response in Minkowski spacetime of arbitrary dimensions and connection with bosonic detector's response
2. Twisted and untwisted Dirac fields in the $(1+1)$ -dimensional, static cylindrical spacetime $(S^1(\text{space}) \times R^1(\text{time}))$ and quantization of the fermionic zero-mode
3. Wightman two-point functions for the Dirac field and the scalar field for certain classes of spatially flat Friedman-Robertson-Walker (FRW) cosmological spacetimes of arbitrary dimensions wherein the two-point functions have the Hadamard form
4. Bosonic quantum field theory in the $(1+1)$ -dimensional, spatially compactified FRW cosmological models and quantization of the conformal-zero momentum mode
5. Finite-time detection of massive bosonic particles in the $(1+1)$ -dimensional, spatially compactified Milne spacetime

Research on topics 1 and 2 was published in [17] and Research on topics 3, 4 and 5 (Chapters 4-5) is currently being prepared for publications.

This thesis is divided in two main parts. Part I comprises of Chapters 1-3 and part II of Chapters 3-6. Part I starts with Chapter 1 beginning with the above brief overview of QFTCS. In Chapter 2, We examine an Unruh-DeWitt particle detector that is coupled linearly to the scalar density $\bar{\psi}\psi$ of a massless Dirac field in Minkowski spacetimes of dimension $d \geq 2$, allowing the detector's motion to remain arbitrary and working to leading order in perturbation theory. The product of $\bar{\psi}$ and ψ at the same spacetime point makes this interaction more challenging than the conventional linear coupling to a scalar field [9, 18]. In d -dimensional Minkowski spacetime, with the field in the usual Fock vacuum, we show that the detector's response is identical to that of a detector coupled linearly to a massless scalar field in $2d$ -dimensional Minkowski spacetime. In the special case of uniform linear acceleration, the detector's response hence exhibits the Unruh effect with a Planckian factor in both even and odd dimensions, in contrast to the Rindler power spectrum of the Dirac field, which has a Planckian factor for odd d but a Fermi-Dirac factor for even d .

In Chapter 3, we examine an UDW detector model coupled to a massless Dirac field on the static two-dimensional cylinder. Our main objective is to consider a detector on an arbitrary worldline on a $(1 + 1)$ -dimensional flat static cylinder ($S^1(\text{space}) \times R^1(\text{time})$). We consider both twisted and untwisted spinor fields, defined by applying periodic and antiperiodic boundary conditions to the field; this is identified in the literature as the two inequivalent spin structures. The main issue is that while the twisted field has a Minkowski-like Fock vacuum, the untwisted spinor field contains a vanishing frequency mode called the zero mode which renders the untwisted spinor field without a well-defined Fock vacuum state. We develop a new scheme to quantize the zero-mode. We evaluate the detector's response, showing that the response distinguishes the periodic and antiperiodic spin structures, and the zero mode of the periodic spinor contributes to the response by a state-dependent

but well defined amount. We also give a selection of analytic and numerical results for inertial and uniformly accelerated trajectories, recovering the $d = 2$ Minkowski results in the limit of large circumference.

Part II starts with Chapter 4 where we present a detailed overview on quantum theory of fields in a curved spacetime. We begin the chapter by discussing the quantization of the Dirac field on a general curved, globally hyperbolic spacetime. In Section 4.1.5, we discuss thoroughly the Bogolubov transformation of the Dirac field. This transformation shows that the choice of a basis of solutions to the Dirac field equation is not unique, and different choices may involve different splits into positive and negative frequencies. Consequently, there is no preferred Fock space and associated Fock vacuum state in a curved spacetime. Specifically, the definition of the n -particle Fock space is non-unique. Accordingly, the Fock vacuum associated with a set of basis will contain particles in another set of basis, thus the two sets do not share a common vacuum and will lead to a different number of particles. Physically this means that different observers will register different number of particles in a given state of the field. This is the mathematical reason for the Unruh-Hawking effect. In short, particle states (defined as eigenstates of the number operator) for a quantized field in a curved background is a basis-dependent notion.

In Minkowski spacetime there is a preferred set of basis of solutions to the Dirac equation chosen as the eigenfunctions of the global timelike Killing vector field. This allows us to define a natural notion of positive and negative frequency modes. And under the Lorentz transformation, positive frequency modes transform to positive frequency modes. Therefore, regardless of the Lorentz frame where t is the global time coordinate, this procedure defines the same, unique Minkowski vacuum state. Hence, there is a preferred Fock space for the free quantized Dirac field, with a Poincaré invariant Fock vacuum state. This procedure of defining positive and negative frequency mode functions generalises straightforwardly to stationary, curved backgrounds (see Section 4.1.5). However, for a non-stationary spacetime, there is no global timelike Killing vector field. There the states of the Dirac field will

not contain a meaningful particle interpretation, except, in the special case where the spacetime is asymptotically stationary (see Section 4.1.5), in which case we can define asymptotic particle states.

In Section 4.2.1 and Section 4.2.4 we discuss the quantization of the free Dirac field and the free scalar field for certain classes of Robertson-Walker (FRW) spacetimes wherein the two-point functions have the Hadamard form. In Section 4.2.5, we quantize the free, conformally coupled scalar field in the $(1 + 1)$ -dimensional spatially flat FRW spacetimes with compactified spatial sections. There, the issue of the conformal zero mode arises whenever the effective mass vanishes at early or late times. We develop a quantization scheme for the zero-momentum and obtain explicit expression for the Wightman two-point functions in the *in* and *out* vacua. In Section 4.3 we give a brief review of Unruh-DeWitt particle detector coupled to a linear quantized scalar field. We conclude this chapter in Section 4.3.1 by first establishing the equivalence between the adiabatic vacuum of infinite order and the conformal vacuum for massless field. Then, with the field prepared in the conformal Fock vacuum, we show that the response of an UDW particle detector coupled linearly to the scalar density of a massless Dirac field in the spatially flat FRW spacetimes in d dimensions is identical to the response of a detector coupled linearly to the massless scalar field in $2d$ dimensions.

In Chapter 5, we examine a conformally massive scalar field propagating in the $(1 + 1)$ -dimensional compactified Milne cosmological spacetime. Our main objective is to consider a detector on an arbitrary worldline on this $(1 + 1)$ -dimensional FRW cosmological spacetime. We consider both twisted and untwisted scalar fields, defined respectively, by imposing periodic and the antiperiodic boundary conditions to the scalar field. While adiabatic Fock vacua of infinite order can be defined at early and late times for the twisted field, the untwisted scalar field does not have a well defined infinite adiabatic order Fock vacuum state at early times due to the presence of a conformal zero mode resulting from the vanishing of the effective frequency of the zero-momentum mode at early times. We quantize the zero-momentum mode ac-

cording to the general formalisms we developed in Section 4.2.4 and Section 4.2.5 and show that its contribution to the *in*-vacuum Wightman two-point function is well-defined. Explicit numerical results are obtained for a comoving and non-comoving inertial trajectories, recovering the Minkowski spacetime results in the appropriate limits.

The thesis concludes with Chapter 6, where we present final remarks about the research described above and possible areas of extension. This is followed by eight appendices, which show explicit details of several calculations from the various chapters.

Chapter 2

Fermionic detector in Minkowski spacetime

In quantum field theory, the interaction between a scalar field and an observer is often studied by modelling the observer by a spatially pointlike system with discrete energy levels, an Unruh-DeWitt detector [9, 18]. Despite its mathematical simplicity, this physical model captures the core features of the dipole interaction whereby atomic orbitals couple to the electromagnetic field [19, 20]. Of special importance is the case of a uniformly linearly accelerated observer coupled to a quantized field in its Minkowski vacuum. There, the detector analyses have provided significant evidence that the Unruh effect [9], occurs whenever the interaction time is long, the interaction switch-on and switch-off are sufficiently slow and the back-reaction of the observer on the quantum field remains small [5, 6, 9, 18, 21–33]. In this Chapter we consider an Unruh-DeWitt detector coupled to a Dirac field, taking the interaction Hamiltonian to be linear in the Dirac field’s scalar density, $\bar{\psi}\psi$ [34–38]. The product of $\bar{\psi}$ and ψ at the same spacetime point makes this interaction more singular than the conventional linear coupling to a scalar field [9, 18]. Working in linear perturbation theory for a massive Dirac field, the detector’s response has a divergent additive term. In stationary situations this term has been viewed formally as a divergent

constant that should be dropped in the dual limit of long interaction and small coupling [35]. However, in nonstationary situations the response would need an additional regularisation, perhaps by a spatial profile or by an appropriate normal ordering [36, 39]. In the special case of Minkowski vacuum, the divergent term is however proportional to the mass of the field, and for a massless field a consistent regularisation is accomplished by simply dropping the additive term [37, 38]. In this chapter we therefore focus on the massless field.

Our first objective is to evaluate the detector's response on an arbitrary trajectory in Minkowski spacetime of dimension $d \geq 2$ when the field is initially prepared in Minkowski vacuum, working in linear perturbation theory and allowing the detector to be switched on and off in an arbitrary smooth way. We show that the response is identical to that of a detector coupled linearly to a massless scalar field in $2d$ spacetime dimensions. In the special case of uniform linear acceleration, the long time limit of the detector's response hence exhibits the Unruh effect with a Planckian factor for all d . By contrast, *the Rindler power spectrum*¹ of the Dirac field is known to have a Planckian factor for odd d but a Fermi-Dirac factor for even d [35]. These observations are compatible since the detector's response is not equal to the Rindler power spectrum but is given by *the convolution of the Rindler power spectrum*² with itself [35].

We begin by recalling in Section 2.1 the definition of the field-detector model with an interaction Hamiltonian that is linear in the Dirac field's scalar density.

¹ *The Rindler power spectrum of the Dirac field* is defined as the Fourier transform of the Wightman function S^+ (2.17) over the uniformly accelerated trajectory $x(\tau)$ (where τ is the proper time): $\tilde{\mathcal{P}}(\Omega) := N_d^{-1} \text{tr} \left[\int_{-\infty}^{\infty} d(\Delta\tau) e^{i\Omega\Delta\tau} \gamma^0 \tilde{g}(\Delta\tau) \right]$, where N_d is the dimension of the γ matrices (see Appendix B), tr stands for the trace, $\Delta\tau := \tau - \tau'$, the *Rindler noise of the Dirac field* is defined as the Fourier amplitude, $\tilde{g}(\Delta\tau) := S_{\tau} S^+(x(\tau), x(\tau')) S_{\tau'}^{-1}$ (which is assumed to be stationary and so depends on $\Delta\tau$ alone), and the Lorentz boost factor for Dirac spinor $S_{\tau} := \exp(-\frac{1}{2}a\tau\gamma^0\gamma^1)$, where a is the magnitude of the uniformly accelerated trajectory.

²For the massless Dirac field, the convolution of the Rindler power spectra is given by: $\mathcal{F}(\Omega) := N_d \int_{-\infty}^{\infty} \frac{d\Omega'}{2\pi} \tilde{\mathcal{P}}(\Omega - \Omega') \tilde{\mathcal{P}}(\Omega')$.

The spinorial conventions and notation are collected in Appendix A, and a selection of technical calculations are deferred to Appendices B–D.

The contents of this chapter were published in [17].

2.1 Unruh-DeWitt detector coupled to the Dirac field

In this section we briefly recall relevant properties of an Unruh-DeWitt detector that is coupled linearly to the scalar density of a Dirac field.

We consider a pointlike detector that moves in a (possibly) curved spacetime on the worldline $x(\tau)$, where τ is the proper time. The detector is a two-level system, with the Hilbert space $\mathbb{C}^2$, spanned by the orthonormal basis $\{|E_0\rangle, |E_1\rangle\}$ of eigenstates of the Hamiltonian H_D : $H_D|E_i\rangle = E_i|E_i\rangle$, where the eigenenergies E_0 and E_1 are real-valued constants.

The detector is coupled to a Dirac field ψ by the interaction Hamiltonian ³

$$H_{\text{int}} := c\chi(\tau)m(\tau)\bar{\psi}(x(\tau))\psi(x(\tau)) , \quad (2.1)$$

where $m(\tau)$ is the detector's monopole moment operator, evolving in the interaction picture by

$$m(\tau) = e^{iH_D\tau}m(0)e^{-iH_D\tau} , \quad (2.2)$$

c is a coupling constant, and the switching function χ is a smooth real-valued function that specifies how the interaction is turned on and off. We assume χ either to

³Bear in mind that it is difficult to model an interaction between the detector and a single fermion field $\psi(x)$ as this would lead to a violation in fermion number conservation by creating and annihilating single fermion quanta. By coupling the UDW detector quadratically to the field, $\bar{\psi}(x)\psi(x)$ (instead of a single fermion field), we ensure that the detector can only create or annihilate fermion and antifermion pairs [36]. This is the motivation behind the coupling term in (2.1). An example of this type of coupling was first considered by Takagi [35].

have compact support or to have a falloff so rapid that the system can be treated as uncoupled in the asymptotic past and future.

Before the interaction begins, the detector occupies the eigenstate $|E_0\rangle$ and the field occupies some Hadamard state ⁴ $|\Psi_0\rangle$. Working to linear order in c , the probability for the detector to be found in the state $|E_1\rangle$ after the interaction has ceased, regardless the final state of the field, is

$$P(\Omega) = |c|^2 |\langle E_1 | m(0) | E_0 \rangle|^2 \mathcal{F}(\Omega) , \quad (2.3)$$

where $\Omega = E_1 - E_0$, the detector's response function $\mathcal{F}(\Omega)$ is given by

$$\mathcal{F}(\Omega) := \int d\tau d\tau' \chi(\tau) \chi(\tau') e^{-i\Omega(\tau-\tau')} W^{(2,\bar{2})}(x(\tau), x(\tau')) , \quad (2.4)$$

and

$$W^{(2,\bar{2})}(x, y) := \langle \Psi_0 | \bar{\psi}(x) \psi(x) \bar{\psi}(y) \psi(y) | \Psi_0 \rangle . \quad (2.5)$$

The factor $|c|^2 \langle E_0 | m(0) | E_1 \rangle^2$ depends only on the inner working of the detector, and we drop it from now on, referring to the response function as the probability. Note that $\mathcal{F}(\Omega)$ gives the probability of an excitation for $\Omega > 0$ and the probability of a de-excitation for $\Omega < 0$.

Although $|\Psi_0\rangle$ is by assumption Hadamard, formula (2.5) as it stands does not define $W^{(2,\bar{2})}(x(\tau), x(\tau'))$ as a distribution on the detector's worldline because of the partial coincidence limit in (2.5) [35–38]. To make the response function (2.4) well defined, it will be necessary to give formula (2.5) an appropriate distributional interpretation. We shall address this in Sections 2.2 and 3.0.1 below.

⁴It turns out there is a unique vacuum singularity structure for the conventional Minkowski vacuum such as the short distance behaviour of the two-point function contains a leading singularity of the form $1/(4\pi^2\sigma_\epsilon)$ and a subleading singularity proportional to $\ln(\sigma_\epsilon)$ with all other term being regular, and where σ_ϵ is one-half of the geodesic distance between two spacetime points (for more details see (4.93)). A state with this property is said to be a *Hadamard state*.

2.2 Response in Minkowski vacuum

In this section we evaluate the detector's response to a massless Dirac field in Minkowski spacetime of dimension $d \geq 2$, with the field in the usual Minkowski vacuum. We first recall relevant properties of the massive field, and we then show that the massless limit of the correlation function $W^{(2,\bar{2})}(x, y)$ (2.5) can be interpreted as a distribution for which the response function (2.4) is well defined.

2.2.1 Quantum Dirac field

We first recall some basic facts and notation about a massive Dirac field on Minkowski spacetime.

We denote the spacetime points by $x = (x^0, \mathbf{x}) = (t, \mathbf{x})$, and the Minkowski metric is $\eta_{\mu\nu} = \text{diag}(1, -1, -1, \dots)$. The action of the Dirac field ψ is

$$S = \int d^d x \bar{\psi}(x) (i\gamma^\mu \partial_\mu - m) \psi(x) , \quad (2.6)$$

where $m > 0$ is the mass and the conventions for the gamma matrices γ^μ , $\mu = 0, 1, 2, \dots, d-1$, are summarised in Appendix A. The field equations for ψ and its Dirac conjugate $\bar{\psi} = \psi^\dagger \gamma^0$ are

$$0 = (i\gamma^\mu \partial_\mu - m)\psi(x) , \quad (2.7a)$$

$$0 = i\partial_\mu \bar{\psi}(x) \gamma^\mu + m\bar{\psi}(x) . \quad (2.7b)$$

A complete set of mode solutions to (2.7a) is

$$u_{\mathbf{k}}^{(s)}(x) = u^{(s)}(\mathbf{k}) e^{-ikx} , \quad (2.8a)$$

$$v_{\mathbf{k}}^{(s)}(x) = v^{(s)}(\mathbf{k}) e^{ikx} , \quad (2.8b)$$

where $k = (k^0, \mathbf{k})$, $k^0 := \omega_{\mathbf{k}} = (\mathbf{k}^2 + m^2)^{1/2}$, $kx = k^0 x^0 - \mathbf{k} \cdot \mathbf{x}$, and the spinors $u^{(s)}(\mathbf{k})$ and $v^{(s)}(\mathbf{k})$ are as given in Appendix A, with s being the helicity index. In the Dirac inner product, given by

$$\langle \psi, \phi \rangle = \int d^{d-1} x \bar{\psi}(t, \mathbf{x}) \gamma_0 \phi(t, \mathbf{x}) , \quad (2.9)$$

these mode solutions are normalised to

$$\langle u_{\mathbf{k}}^{(s)}, u_{\mathbf{k}'}^{(s')} \rangle = \langle v_{\mathbf{k}}^{(s)}, v_{\mathbf{k}'}^{(s')} \rangle = 2\omega_{\mathbf{k}}(2\pi)^{d-1} \delta^{ss'} \delta^{d-1}(\mathbf{k} - \mathbf{k}') , \quad (2.10a)$$

$$\langle u_{\mathbf{k}}^{(s)}, v_{\mathbf{k}'}^{(s')} \rangle = 0 . \quad (2.10b)$$

The quantised field is expanded as

$$\psi(x) = \int \widetilde{dk} \sum_s \left(b_s(\mathbf{k}) u_{\mathbf{k}}^{(s)}(x) + d_s^\dagger(\mathbf{k}) v_{\mathbf{k}}^{(s)}(x) \right) , \quad (2.11)$$

where

$$\widetilde{dk} := \frac{d^{d-1}k}{2\omega_{\mathbf{k}}(2\pi)^{d-1}} , \quad (2.12)$$

and the only nonvanishing anticommutators of the operator coefficients are

$$\{b_s(\mathbf{k}), b_{s'}^\dagger(\mathbf{k}')\} = \{d_s(\mathbf{k}), d_{s'}^\dagger(\mathbf{k}')\} = 2\omega_{\mathbf{k}}(2\pi)^{d-1} \delta_{ss'} \delta^{d-1}(\mathbf{k} - \mathbf{k}') . \quad (2.13)$$

The field's equal-time anticommutators are

$$\{\psi_a(t, \mathbf{x}), \psi_b(t, \mathbf{x}')\} = \{\psi_a^\dagger(t, \mathbf{x}), \psi_b^\dagger(t, \mathbf{x}')\} = 0 , \quad (2.14a)$$

$$\{\psi_a(t, \mathbf{x}), \psi_b^\dagger(t, \mathbf{x}')\} = \delta_{ab} \delta^{d-1}(\mathbf{x} - \mathbf{x}') , \quad (2.14b)$$

where we have explicitly written out the spinor indices. The fermionic Fock space is built on the Minkowski vacuum state $|0\rangle$ which satisfies $b_s(\mathbf{k})|0\rangle = d_s(\mathbf{k})|0\rangle = 0$.

The fields ψ and $\bar{\psi}$ may be decomposed into their positive and negative frequency components as

$$\psi(x) = \psi^+(x) + \psi^-(x) , \quad (2.15a)$$

$$\bar{\psi}(x) = \bar{\psi}^+(x) + \bar{\psi}^-(x) , \quad (2.15b)$$

where

$$\psi^+(x) = \int \widetilde{dk} \sum_s b_s(\mathbf{k}) u_{\mathbf{k}}^{(s)}(x) , \quad (2.16a)$$

$$\psi^-(x) = \int \widetilde{dk} \sum_s d_s^\dagger(\mathbf{k}) v_{\mathbf{k}}^{(s)}(x) , \quad (2.16b)$$

$$\bar{\psi}^+(x) = \int \widetilde{dk} \sum_s d_s(\mathbf{k}) \bar{v}_{\mathbf{k}}^{(s)}(x) , \quad (2.16c)$$

$$\bar{\psi}^-(x) = \int \widetilde{dk} \sum_s b_s^\dagger(\mathbf{k}) \bar{u}_{\mathbf{k}}^{(s)}(x) . \quad (2.16d)$$

In the conventions of [35], the Dirac field Wightman functions $S^\pm(x, y)$ are

$$\begin{aligned} S_{ab}^+(x, y) &:= \langle 0 | \psi_a(x) \bar{\psi}_b(y) | 0 \rangle = \{ \psi_a^+(x), \bar{\psi}_b^-(y) \} \\ &= (i\gamma^\mu \partial_{x^\mu} + m)_{ab} G^+(x, y) , \end{aligned} \quad (2.17a)$$

$$\begin{aligned} S_{ab}^-(x, y) &:= \langle 0 | \bar{\psi}_b(y) \psi_a(x) | 0 \rangle = \{ \psi_a^-(x), \bar{\psi}_b^+(y) \} \\ &= -(i\gamma^\mu \partial_{x^\mu} + m)_{ab} G^+(y, x) , \end{aligned} \quad (2.17b)$$

where G^+ is the Wightman function of a real scalar field of mass m ,

$$G^+(x, y) = \int \widetilde{dk} e^{-ik(x-y)} , \quad (2.18)$$

and the distributional sense in (2.18) is that of $x^0 - y^0 \rightarrow x^0 - y^0 - i\epsilon$ and the limit $\epsilon \rightarrow 0_+$. The explicit expression for $G^+(x, y)$ is [35]

$$G^+(x, y) = \frac{1}{2\pi} \left(\frac{m}{2\pi z(x, y)} \right)^{d/2-1} K_{d/2-1}(mz(x, y)) , \quad (2.19)$$

where $K_{d/2-1}$ is the modified Bessel function of the second kind [41] and

$$z(x, y) := \sqrt{(\mathbf{x} - \mathbf{y})^2 - (x^0 - y^0 - i\epsilon)^2} . \quad (2.20)$$

2.2.2 $W^{(2, \bar{2})}(x, y)$

We wish to examine the correlation function $W^{(2, \bar{2})}(x, y)$ (2.5).

We show in Appendix B that

$$W^{(2, \bar{2})}(x, y) = \text{Tr}[S^+(x, y)S^-(y, x)] + \text{Tr}[S^-(0, 0)] \text{Tr}[S^-(0, 0)] . \quad (2.21)$$

Using $\text{Tr}(\gamma^\mu) = 0$ and $\text{Tr}(\gamma^\mu \gamma^\nu) = N_d \eta^{\mu\nu}$ (see Appendix B for definition of N_d), (2.17) and (2.19) show that the first term in (2.21) can be written as

$$\begin{aligned} \text{Tr}[S^+(x, y)S^-(y, x)] &= \frac{N_d m^2}{(2\pi)^2} \left(\frac{m}{2\pi z(x, y)} \right)^{d-2} \times \\ &\quad \times \left\{ [K_{d/2}(mz(x, y))]^2 - [K_{d/2-1}(mz(x, y))]^2 \right\} , \end{aligned} \quad (2.22)$$

which is a well-defined distribution. To obtain this result, we make use of the chain rule:

$$\begin{aligned}
 \partial_\mu (z^{-\alpha} K_\alpha(mz)) &= \frac{d}{dz} (z^{-\alpha} K_\alpha(mz)) \partial_\mu z \\
 &= m^{\alpha+1} \frac{d}{d(mz)} ((mz)^{-\alpha} K_\alpha(mz)) \partial_\mu z \\
 &= -mz^{-\alpha} K_{\alpha+1}(mz) \partial_\mu z,
 \end{aligned} \tag{2.23}$$

together with the following recursive relation for the modified Bessel function:

$$\frac{d}{dx} (x^{-\alpha} K_\alpha(x)) = -x^{-\alpha} K_{\alpha+1}(x). \tag{2.24}$$

In the second term in (2.21), by contrast, we have, using (2.17b) and $\text{Tr}(\gamma^\mu) = 0$,

$$\text{Tr}[S^-(x, y)] = -N_d m G^+(y, x), \tag{2.25}$$

which diverges as $(x, y) \rightarrow (0, 0)$ by (2.19). $W^{(2, \bar{2})}(x, y)$ is hence not well defined, due to a divergent additive constant in the second term in (2.21) [35–38].

Consider however now the limit $m \rightarrow 0$. If the second term in (2.21) is dropped in this limit, we obtain

$$W^{(2, \bar{2})}(x, y) = \frac{N_d (\Gamma(d/2))^2}{4\pi^d (z(x, y))^{2d-2}}, \tag{2.26}$$

using (2.22) and the small argument form of the modified Bessel function [41]. We adopt (2.26) as the definition of $W^{(2, \bar{2})}$ for the massless field.

We shall not attempt to justify dropping the second term in (2.21) as $m \rightarrow 0$ from some underlying framework that would provide a definition for the coincidence limit of a squared distribution, but we can make two consistency observations.

First, from (2.19), (2.25) and the small argument form of the modified Bessel function [41] we see that $\text{Tr}[S^-(x, y)]$ has a well defined distributional limit as $m \rightarrow 0$, and this limit is the zero distribution.

Second, recall that the Wightman function of a massless scalar field is given

by [40]

$$G^+(x, y) = \begin{cases} \frac{\Gamma(\frac{d}{2} - 1)}{4\pi^{d/2}(z(x, y))^{d-2}} & \text{for } d > 2, \\ -(2\pi)^{-1} \ln(\mu z(x, y)) & \text{for } d = 2, \end{cases} \quad (2.27)$$

where μ is an undetermined positive constant of the dimension of inverse length. For $d > 2$, (2.27) is obtained as the $m \rightarrow 0$ limit of (2.19). For $d = 2$, (2.27) is obtained as the $m \rightarrow 0$ limit of (2.19) after subtracting an m -dependent constant that diverges as $m \rightarrow 0$, and the arbitrariness (“infrared ambiguity”) in this subtraction is encoded in the positive constant μ in (2.27). Substituting (2.27) in (2.17) with $m = 0$ gives $S^\pm(x, y)$ such that $\text{Tr}[S^-(x, y)]$ vanishes as a distribution, and substituting these $S^\pm(x, y)$ in the first term in (2.21) and $\text{Tr}[S^-(0, 0)]$ in the second term in (2.21) gives (2.26).

Note that $W^{(2, \bar{2})}(x, y)$ (2.26) tends to zero in the limit of large spacelike separation for all $d \geq 2$, including $d = 2$. For $d = 2$, $W^{(2, \bar{2})}$ has no infrared ambiguity, in contrast to the infrared ambiguity of G^+ (2.27).

2.2.3 Detector’s response to a massless field

Collecting (2.4) and (2.26), we see that the detector’s response to a massless Dirac field is given by

$$\mathcal{F}(\Omega) = \frac{N_d(\Gamma(d/2))^2}{4\pi^d} \int d\tau d\tau' \frac{\chi(\tau)\chi(\tau') e^{-i\Omega(\tau-\tau')}}{[z(x(\tau), x(\tau'))]^{2d-2}}, \quad (2.28)$$

where we recall from (2.20) that

$$z(x, y) = \sqrt{(\mathbf{x} - \mathbf{y})^2 - (x^0 - y^0 - i\epsilon)^2} \quad (2.29)$$

with $\epsilon \rightarrow 0_+$. This result is valid for general trajectory and arbitrary switching functions in the massless limit and agrees with the special cases for the inertial and uniformly accelerated detectors considered in [35–38]. To set this result in context,

recall that the response of an Unruh-DeWitt detector that is linearly coupled to a scalar field in its Minkowski vacuum is [5, 6, 42]

$$\mathcal{F}_{\text{sc}}(\Omega) = \int d\tau d\tau' \chi(\tau) \chi(\tau') e^{-i\Omega(\tau-\tau')} G^+(x(\tau), x(\tau')) , \quad (2.30)$$

where G^+ is the scalar field's Wightman function. By (2.27), (2.28) and (2.30), we may hence formalise our observations as the following theorem:

Theorem 2.1. *The response function of an Unruh-DeWitt detector coupled quadratically to a massless Dirac field in Minkowski vacuum in $d \geq 2$ spacetime dimensions equals*

$$\frac{N_d(\Gamma(d/2))^2}{\Gamma(d-1)} \quad (2.31)$$

times the response function of an Unruh-DeWitt detector coupled linearly to a massless scalar field in Minkowski vacuum in $2d$ spacetime dimensions.

A direct consequence of Theorem 2.1 is that the Dirac field detector's response is well defined whenever the detector's worldline is smooth. This follows from the corresponding result for the scalar field detector [43, 44].

The special case of a uniformly linearly accelerated detector deserves a comment. In the limit in which the detector operates for a long time and the switching effects are negligible, it is well known [35] that both the response function of the scalar field detector and the response function of the Dirac field detector satisfy the detailed balance condition,

$$\mathcal{F}(-\Omega) = e^{\Omega/T_U} \mathcal{F}(\Omega) , \quad (2.32)$$

where $T_U := a/(2\pi)$ and a is the magnitude of the detector's proper acceleration. This is the celebrated Unruh effect, and T_U is the Unruh temperature [9]. It was observed in [35] that the response function of the scalar field detector involves a Planck factor in even spacetime dimensions but a Fermi-Dirac factor in odd spacetime dimensions. Theorem 2.1 hence implies that the response function of the Dirac field detector involves a Planck factor in *all* spacetime dimensions.

By contrast, the *Rindler power spectrum* of the Dirac field (See footnote 1 on page 11) involves a Fermi-Dirac factor⁵ in even spacetime dimensions and a Planck factor in odd spacetime dimensions [35]. This is fully compatible with our observation that the detector's response involves a Planck factor in all spacetime dimensions: the response function is not directly the Rindler power spectrum but rather the self-convolution of the Rindler power spectrum, as defined on footnote 1 on page 10. Consequently, a Fermi-Dirac factor in the Rindler power spectrum of the Dirac field does not imply a Fermi-Dirac factor in the response function. We have explicitly checked that our Theorem 2.1 agrees with (8.5.13) in [35] for a Dirac field in spacetime dimensions 2, 3 and 4. The verbal description that the Dirac field detector's response is characterized by the Fermi-Dirac distribution for even-dimensional spacetimes and by the Bose-Einstein distribution for odd dimensional space-times given in [35], in the full paragraph between (8.5.14) and (8.5.15), is hence not accurate.

2.3 Conclusions

We have analysed the response of a spatially pointlike Unruh-DeWitt detector coupled linearly to the scalar density of a massless Dirac field in Minkowski spacetimes in dimension $d \geq 2$, allowing the detector's motion to remain arbitrary and allowing the detector to be switched on and off in an arbitrary smooth way. Working within first-order perturbation theory, we regularised the interaction by dropping an additive term that is technically ill-defined but formally proportional to the field's mass [37, 38].

In d -dimensional Minkowski, with the field in its Fock vacuum, we found that the response is identical to that of a detector coupled linearly to a massless scalar field in

⁵Takagi [35] shows that $\tilde{\mathcal{P}}_d(\Omega) \sim f_d(\Omega) \left[\exp\left(\frac{\Omega}{T_U}\right) + (-1)^d \right]^{-1}$, where the subscript d is included to show the dimension of the spacetime. Note: for even dimensions $\tilde{\mathcal{P}}_d(\Omega)$ is proportional to the Fermi-Dirac distribution $\left[\exp\left(\frac{\Omega}{T_U}\right) + 1 \right]^{-1}$ while for even dimension $\tilde{\mathcal{P}}_d(\Omega)$ is proportional to the Bose-Einstein distribution $\left[\exp\left(\frac{\Omega}{T_U}\right) - 1 \right]^{-1}$.

$2d$ spacetime dimensions. For a uniformly linearly accelerated detector, this implies that the long time limit of the response exhibits the Unruh effect with a Planckian frequency dependence factor, for all d . While the Rindler power spectrum of the Dirac field is known to have a Planckian factor for odd d but a Fermi-Dirac factor for even d [35], the detector's response is Planckian for all d because the response is not proportional to the Rindler power spectrum but to the convolution of the Rindler power spectrum with itself.

In the special case of two-dimensional Minkowski spacetime, we saw that the detector's response has no infrared ambiguity. In this respect our detector differs from the detector coupled linearly to a massless scalar field, where in two dimensions the response is ambiguous due to the infrared ambiguity of the Wightman function [40].

Chapter 3

Cylindrical $(1+1)$ -dimensional spacetime

In this chapter, we consider a detector coupled to a massless Dirac field in a flat static cylindrical spacetime in $1 + 1$ dimensions. Our main objective is to consider a detector on an arbitrary worldline on a $(1 + 1)$ -dimensional flat static cylinder ($S^1(\text{space}) \times R^1(\text{time})$). The main issue here is that the field has two spin structures, often referred to as the periodic field and the antiperiodic field, and while the antiperiodic field has a Minkowski-like Fock vacuum, the zero mode of the periodic field does not have a Fock vacuum. We evaluate the detector's response, showing that the response distinguishes the periodic and antiperiodic spin structures, and the zero mode of the periodic spinor contributes to the response by a state-dependent but well defined way. We also give a selection of analytic and numerical results for inertial and uniformly accelerated trajectories, recovering the $d = 2$ Minkowski results in the limit of large circumference.

Our results show that the detector's response has no infrared ambiguity, on the cylinder. In this respect the massless Dirac field differs from the massless scalar field, whose response in two-dimensional Minkowski vacuum is ambiguous due to the additive ambiguity in the Wightman function [40].

The response on the $(1 + 1)$ -dimensional flat static cylinder is evaluated in Section 3.0.1. Inertial and uniformly accelerated trajectories on the cylinder are analysed in Section 3.1. Section 3.2 gives a summary and brief concluding remarks.

The contents of this chapter were published in [17].

3.0.1 Massive Dirac field on the cylindrical spacetime

The spacetime is a flat static $(1 + 1)$ -dimensional cylinder with spatial circumference $L > 0$. We work in standard Minkowski coordinates (t, x) in which the metric reads

$$ds^2 = dt^2 - dx^2, \quad (3.1)$$

with the periodic identification $(t, x) \sim (t, x + L)$.

We consider a Dirac field ψ with mass $m > 0$. We use the Minkowski spacetime notation of Section 2.2 with the exception that ψ is now either periodic or antiperiodic as $(t, x) \mapsto (t, x + L)$. The choice of periodicity versus antiperiodicity implements the choice between the two inequivalent spin structures of the field [5]: we refer to these spin structures as respectively the *periodic* or *untwisted* spin structure and the *antiperiodic* or *twisted* spin structure. We suppress explicit references to the spin structure in the formulas until the final expressions for the detector's response in (3.12) and (3.28).

A complete set of mode solutions for each spin structure is

$$u_n(t, x) = (2L\omega_n)^{-1/2} u(k_n) e^{-i(\omega_n t - k_n x)}, \quad (3.2a)$$

$$v_n(t, x) = (2L\omega_n)^{-1/2} v(k_n) e^{i(\omega_n t - k_n x)}, \quad (3.2b)$$

where $n \in \mathbb{Z}$ and

$$k_n := \begin{cases} 2\pi n/L & \text{for untwisted spinors,} \\ 2\pi(n + \frac{1}{2})/L & \text{for twisted spinors,} \end{cases} \quad (3.3a)$$

$$\omega_n := (m^2 + k_n^2)^{1/2}, \quad (3.3b)$$

and the spinors $u(k_n)$ and $v(k_n)$ are the (1+1)-dimensional special case of the spinors introduced in Appendix A. Note that the spinors carry no spin index. The Dirac inner product (2.9) is modified to

$$\langle \psi, \phi \rangle = \int_0^L dx \bar{\psi}(t, \mathbf{x}) \gamma_0 \phi(t, \mathbf{x}) , \quad (3.4)$$

in which the mode solutions (3.2) are normalised to

$$\langle u_n, u_{n'} \rangle = \langle v_n, v_{n'} \rangle = \delta_{nn'} , \quad (3.5a)$$

$$\langle u_n, v_{n'} \rangle = 0 . \quad (3.5b)$$

The quantised field is expanded as

$$\psi(t, x) = \sum_n (b_n u_n(t, x) + d_n^\dagger v_n(t, x)) , \quad (3.6)$$

where the only nonvanishing anticommutators of the coefficients are

$$\{b_n, b_{n'}^\dagger\} = \{d_n, d_{n'}^\dagger\} = \delta_{nn'} . \quad (3.7)$$

The field's equal-time anticommutators are

$$\{\psi_a(t, x), \psi_b(t, x')\} = \{\psi_a^\dagger(t, x), \psi_b^\dagger(t, x')\} = 0 , \quad (3.8a)$$

$$\{\psi_a(t, x), \psi_b^\dagger(t, x')\} = \delta_{ab} \delta(x, x') , \quad (3.8b)$$

where $\delta(x, x')$ denotes Dirac's delta-function on the circle. The fermionic Fock space is built on the vacuum state $|0\rangle$ which satisfies $b_n|0\rangle = d_n|0\rangle = 0$.

Proceeding as in Section 2.2, we have

$$S_{ab}^+(t, x; t', x') := \langle 0 | \psi_a(t, x) \bar{\psi}_b(t', x') | 0 \rangle = (i\gamma^\mu \partial_\mu + m)_{ab} G^+(t, x; t', x') , \quad (3.9a)$$

$$S_{ab}^-(t, x; t', x') := \langle 0 | \bar{\psi}_b(t', x') \psi_a(t, x) | 0 \rangle = -(i\gamma^\mu \partial_\mu + m)_{ab} G^+(t', x'; t, x) , \quad (3.9b)$$

where

$$G^+(t, x; t', x') = \sum_n \frac{1}{2\omega_n L} \exp[-i\omega_n(t - t' - i\epsilon) + ik_n(x - x')] , \quad (3.10)$$

understood in the sense $\epsilon \rightarrow 0_+$, and the differentiation in (3.9) is with respect to the unprimed argument. For the untwisted spinor, G^+ is the Wightman function of a real scalar field of mass m . For the twisted spinor, G^+ is the Wightman function of a scalar field that takes values on a twisted $\mathbb{R}/\mathbb{Z}_2$ bundle [5].

The correlation function $W^{(2,\bar{2})}$ (2.5) is again given by the ill-defined expression (2.21). We shall give a well-defined interpretation for this expression in the massless limit for each of the two spin structures in turn.

3.0.2 Twisted massless field

Consider the twisted massless Dirac field. Proceeding as in subsection 2.2.2, we drop the second term in (2.21) and use in the first term (3.9) with $m = 0$, obtaining

$$\begin{aligned}
W_t^{(2,\bar{2})}(t, x; t', x') &= \text{Tr} \left\{ [S^+(t, x; t', x')]^2 \right\} \\
&= -2 [(\partial_t - \partial_x)G^+(t, x; t', x')] [(\partial_t + \partial_x)G^+(t, x; t', x')] \\
&= \frac{2}{L^2} e^{-2\pi i(\Delta t - i\epsilon)/L} \sum_{n,m=0}^{\infty} e^{-2\pi i n(\Delta t + \Delta x - i\epsilon)/L} e^{-2\pi i m(\Delta t - \Delta x - i\epsilon)/L} \\
&= -\frac{1}{2L^2 \sin[\pi(\Delta t + \Delta x - i\epsilon)/L] \sin[\pi(\Delta t - \Delta x - i\epsilon)/L]} ,
\end{aligned} \tag{3.11}$$

where $\Delta t := t - t'$ and $\Delta x := x - x'$, and the subscript in $W_t^{(2,\bar{2})}$ refers to the twisted spin structure.

The response of the detector is obtained from (2.4) with (3.11). Note that the response contains no infrared ambiguities. A formula that is well suited for numerical evaluation is obtained by using the sum form in (3.11), yielding

$$\begin{aligned}
\mathcal{F}_t(\Omega) &= \frac{2}{L^2} \sum_{n,m=0}^{\infty} \int d\tau d\tau' \chi(\tau) \chi(\tau') e^{-i\Omega(\tau - \tau')} \\
&\quad \times \exp \left(-\frac{2\pi i \{ (n+m+1)[t(\tau) - t(\tau') - i\epsilon] + (n-m)[x(\tau) - x(\tau')] \}}{L} \right) .
\end{aligned} \tag{3.12}$$

3.0.3 Untwisted massless field

Consider the untwisted massless Dirac field. As the $n = 0$ term in (3.6) has vanishing frequency, this mode does not have a Fock vacuum. We hence split ψ as

$$\psi(t, x) = \psi^{\text{osc}}(t, x) + \psi^{\text{zm}}(t) , \quad (3.13)$$

where

$$\psi^{\text{osc}}(t, x) = \sum_{n \neq 0} (b_n u_n(t, x) + d_n^\dagger v_n(t, x)) , \quad (3.14)$$

and $\psi^{\text{zm}}(t)$ is spatially constant. We treat ψ^{osc} and ψ^{zm} in turn and then combine the two.

Oscillator modes

We quantise the oscillator modes ψ^{osc} with the usual anticommutators (3.7). It follows that the equal-time anticommutators of ψ^{osc} are

$$\{\psi_a^{\text{osc}}(t, x), \psi_b^{\text{osc}}(t, x')\} = \{\psi_a^{\text{osc} \dagger}(t, x), \psi_b^{\text{osc} \dagger}(t, x')\} = 0 , \quad (3.15a)$$

$$\{\psi_a^{\text{osc}}(t, x), \psi_b^{\text{osc} \dagger}(t, x')\} = \delta_{ab} \delta(x, x') - \delta_{ab}/L . \quad (3.15b)$$

Let $|0^{\text{osc}}\rangle$ denote the oscillator mode Fock vacuum, satisfying $b_n|0^{\text{osc}}\rangle = d_n|0^{\text{osc}}\rangle = 0$ for $n \neq 0$. Proceeding as in (3.9), we find

$$S_{ab}^{\text{osc}+}(t, x; t', x') := \langle 0^{\text{osc}} | \psi_a^{\text{osc}}(t, x) \overline{\psi_b^{\text{osc}}}(t', x') | 0^{\text{osc}} \rangle = i (\gamma^\mu)_{ab} \partial_\mu G^{\text{osc}+}(t, x; t', x') , \quad (3.16a)$$

$$S_{ab}^{\text{osc}-}(t, x; t', x') := \langle 0^{\text{osc}} | \overline{\psi_b^{\text{osc}}}(t', x') \psi_a^{\text{osc}}(t, x) | 0^{\text{osc}} \rangle = -i (\gamma^\mu)_{ab} \partial_\mu G^{\text{osc}+}(t', x'; t, x) , \quad (3.16b)$$

where the differentiation is with respect to the unprimed argument and

$$G^{\text{osc}+}(t, x; t', x') = \sum_{n \neq 0} \frac{1}{2\omega_n L} \exp[-i\omega_n(t - t' - i\epsilon) + ik_n(x - x')] . \quad (3.17)$$

Hence

$$\begin{aligned}
W_{\text{osc}}^{(2,\bar{2})}(t, x; t', x') &:= \text{Tr} [S^{\text{osc}+}(t, x; t', x') S^{\text{osc}-}(t', x'; t, x)] \\
&= \text{Tr} \left\{ [S^{\text{osc}+}(t, x; t', x')]^2 \right\} \\
&= -2 [(\partial_t - \partial_x) G^{\text{osc}+}(t, x; t', x')] [(\partial_t + \partial_x) G^{\text{osc}+}(t, x; t', x')] \\
&= \frac{2}{L^2} \sum_{n,m=1}^{\infty} e^{-2\pi i n(\Delta t + \Delta x - i\epsilon)/L} e^{-2\pi i m(\Delta t - \Delta x - i\epsilon)/L} \\
&= -\frac{\exp[-2\pi i(\Delta t - i\epsilon)/L]}{2L^2 \sin[\pi(\Delta t + \Delta x - i\epsilon)/L] \sin[\pi(\Delta t - \Delta x - i\epsilon)/L]} .
\end{aligned} \tag{3.18}$$

Zero mode

We quantise the zero mode ψ^{zm} so that $\psi^{\text{zm}}(t)$ and $\psi^{\text{zm}\dagger}(t)$ anticommute with $\psi^{\text{osc}}(t, x)$ and $\psi^{\text{osc}\dagger}(t, x')$ and satisfy

$$\{\psi_a^{\text{zm}}(t), \psi_b^{\text{zm}}(t)\} = \{\psi_a^{\text{zm}\dagger}(t), \psi_b^{\text{zm}\dagger}(t)\} = 0 , \tag{3.19a}$$

$$\{\psi_a^{\text{zm}}(t), \psi_b^{\text{zm}\dagger}(t)\} = \delta_{ab}/L . \tag{3.19b}$$

Together with (3.15), this ensures that the full Dirac field (3.13) satisfies the equal-time anticommutators (3.8).

Inserting (3.13) in the action shows that ψ^{zm} is independent of t . To satisfy (3.19), we write (cf. Chapter 20 of [45])

$$\psi^{\text{zm}} = \frac{1}{\sqrt{L}} \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} , \tag{3.20}$$

where Q_a are independent of t and satisfy

$$\{Q_a, Q_b\} = \{Q_a^\dagger, Q_b^\dagger\} = 0 , \tag{3.21a}$$

$$\{Q_a, Q_b^\dagger\} = \delta_{ab} . \tag{3.21b}$$

The Hilbert space is built on the normalised state $|0^{\text{zm}}\rangle$ that satisfies $Q_a|0^{\text{zm}}\rangle = 0$. The Hilbert space has dimension four, and an orthonormal basis is $\{|0^{\text{zm}}\rangle, Q_1^\dagger|0^{\text{zm}}\rangle, Q_2^\dagger|0^{\text{zm}}\rangle, Q_1^\dagger Q_2^\dagger|0^{\text{zm}}\rangle\}$.

For concreteness, we may work in a representation in which $\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. We then have

$$\overline{\psi^{\text{zm}}} \psi^{\text{zm}} = L^{-1} (Q_2 Q_1 + Q_1^\dagger Q_2^\dagger) . \quad (3.22)$$

If the zero mode is in the normalised state

$$|\text{ZM}\rangle := \frac{(a_0 + a_1 Q_1^\dagger + a_2 Q_2^\dagger + a_3 Q_1^\dagger Q_2^\dagger) |0^{\text{zm}}\rangle}{\sqrt{|a_0|^2 + |a_1|^2 + |a_2|^2 + |a_3|^2}} , \quad (3.23)$$

where the four a_i are complex numbers, not all of them vanishing, we find

$$W_{\text{zm}}^{(2,\bar{2})} := \langle \text{ZM} | \overline{\psi^{\text{zm}}} \psi^{\text{zm}} \overline{\psi^{\text{zm}}} \psi^{\text{zm}} | \text{ZM} \rangle = \theta L^{-2} , \quad (3.24)$$

where $\theta = (|a_0|^2 + |a_3|^2)(|a_0|^2 + |a_1|^2 + |a_2|^2 + |a_3|^2)^{-1}$. Note that $\theta \in [0, 1]$, and when $|\text{ZM}\rangle = |0^{\text{zm}}\rangle$, we have $\theta = 1$.

Full field

Consider now the full field (3.13), consisting of both the oscillator modes and the zero mode. We put the field in the state

$$\begin{aligned} |\widetilde{\text{ZM}}\rangle &:= |\text{ZM}\rangle \otimes |0^{\text{osc}}\rangle \\ &= \frac{(a_0 + a_1 Q_1^\dagger + a_2 Q_2^\dagger + a_3 Q_1^\dagger Q_2^\dagger) (|0^{\text{zm}}\rangle \otimes |0^{\text{osc}}\rangle)}{\sqrt{|a_0|^2 + |a_1|^2 + |a_2|^2 + |a_3|^2}} , \end{aligned} \quad (3.25)$$

We show in Appendix C that

$$\begin{aligned} W_u^{(2,\bar{2})}(t, x; t', x') &:= \langle \widetilde{\text{ZM}} | \bar{\psi}(t, x) \psi(t, x) \bar{\psi}(t', x') \psi(t', x') | \widetilde{\text{ZM}} \rangle \\ &= W_{\text{osc}}^{(2,\bar{2})}(t, x; t', x') + W_{\text{mix}}^{(2,\bar{2})}(t, x; t', x') + W_{\text{zm}}^{(2,\bar{2})} , \end{aligned} \quad (3.26)$$

where $W_{\text{osc}}^{(2,\bar{2})}$ is given by (3.18), $W_{\text{zm}}^{(2,\bar{2})}$ is given by (3.24),

$$\begin{aligned} W_{\text{mix}}^{(2,\bar{2})} &= L^{-2} \sum_{n=1}^{\infty} (e^{-2\pi i n(\Delta t + \Delta x - i\epsilon)/L} + e^{-2\pi i n(\Delta t - \Delta x - i\epsilon)/L}) \\ &= L^{-2} \left(\frac{1}{e^{2\pi i(\Delta t + \Delta x - i\epsilon)/L} - 1} + \frac{1}{e^{2\pi i(\Delta t - \Delta x - i\epsilon)/L} - 1} \right) , \end{aligned} \quad (3.27)$$

and the subscript in $W_u^{(2,\bar{2})}$ refers to the untwisted spin structure.

The response of the detector is obtained from (2.4) with (3.26). Note that the response again contains no infrared ambiguities. We may break the response as

$$\mathcal{F}_u(\Omega) = \mathcal{F}_u^{\text{osc}}(\Omega) + \mathcal{F}_u^{\text{mix}}(\Omega) + \mathcal{F}_u^{\text{zm}}(\Omega) , \quad (3.28)$$

where numerically efficient formulas for $\mathcal{F}_u^{\text{osc}}$ and $\mathcal{F}_u^{\text{mix}}$ are obtained from the sums in (3.18) and (3.27),

$$\begin{aligned} \mathcal{F}_u^{\text{osc}}(\Omega) &= \frac{2}{L^2} \sum_{n,m=1}^{\infty} \int d\tau d\tau' \chi(\tau) \chi(\tau') e^{-i\Omega(\tau-\tau')} \\ &\times \exp\left(-\frac{2\pi i \{(n+m)[t(\tau)-t(\tau')-i\epsilon] + (n-m)[(x(\tau)-x(\tau'))]\}}{L}\right) , \end{aligned} \quad (3.29a)$$

$$\begin{aligned} \mathcal{F}_u^{\text{mix}}(\Omega) &= \frac{2}{L^2} \sum_{n=1}^{\infty} \int d\tau d\tau' \chi(\tau) \chi(\tau') e^{-i\Omega(\tau-\tau')} \\ &\times \exp\left(-\frac{2\pi i n[t(\tau)-t(\tau')-i\epsilon]}{L}\right) \cos\left(\frac{2\pi n[x(\tau)-x(\tau')]}{L}\right) , \end{aligned} \quad (3.29b)$$

while

$$\mathcal{F}_u^{\text{zm}}(\Omega) = \frac{\theta}{L^2} \int d\tau d\tau' \chi(\tau) \chi(\tau') e^{-i\Omega(\tau-\tau')} = \frac{\theta}{L^2} |\hat{\chi}(\Omega)|^2 , \quad (3.30)$$

where the hat denotes the Fourier transform, $\hat{\chi}(\Omega) := \int_{-\infty}^{\infty} d\tau \chi(\tau) e^{-i\Omega\tau}$.

3.0.4 $L \rightarrow \infty$ limit

In the limit $L \rightarrow \infty$, the final expressions in (3.11), (3.18), (3.24) and (3.27) show that both $W_t^{(2,\bar{2})}(t, x; t', x')$ and $W_u^{(2,\bar{2})}(t, x; t', x')$ approach the same limit,

$$W_{t,u}^{(2,\bar{2})}(t, x; t', x') \xrightarrow{L \rightarrow \infty} \frac{1}{2\pi^2 [(x-x')^2 - (t-t'-i\epsilon)^2]} , \quad (3.31)$$

which by (2.26) is equal to $W^{(2,\bar{2})}(t, x; t', x')$ in the Minkowski vacuum in two-dimensional Minkowski spacetime. This is as expected: in the limit of large spatial circumference, the detector's response for either spin structure reduces to that in the Minkowski vacuum in Minkowski spacetime.

3.1 Inertial and uniformly accelerated trajectories on the cylindrical spacetime

In this section we consider inertial and uniformly accelerated detectors on the cylindrical spacetime of Section 3.0.1.

3.1.1 Inertial detector

Consider a detector on the inertial worldline

$$t = \tau \cosh \beta, \quad x = \tau \sinh \beta, \quad (3.32)$$

where $\beta \in \mathbb{R}$ is the rapidity with respect to the worldlines of constant x . We take the switching function to be Gaussian,

$$\chi(\tau) = \frac{1}{\pi^{1/4} \sigma^{1/2}} e^{-\tau^2/(2\sigma^2)}, \quad (3.33)$$

where the positive parameter σ is the effective duration of the interaction. The normalisation is such that $\int_{-\infty}^{\infty} \chi^2(\tau) d\tau = 1$, and

$$\widehat{\chi}(\Omega) = \pi^{1/4} (2\sigma)^{1/2} e^{-\sigma^2 \Omega^2/2}. \quad (3.34)$$

For the twisted field, (3.12) gives

$$\mathcal{F}_t(\Omega) = \frac{4\pi^{1/2}\sigma}{L^2} \sum_{n,m=0}^{\infty} \exp \left[-\sigma^2 \left(\Omega + \frac{2\pi((n + \frac{1}{2})e^\beta + (m + \frac{1}{2})e^{-\beta})}{L} \right)^2 \right]. \quad (3.35)$$

For the untwisted field, (3.28), (3.29) and (3.30) give

$$\begin{aligned} \mathcal{F}_u(\Omega) = & \frac{4\pi^{1/2}\sigma}{L^2} \sum_{n,m=1}^{\infty} \exp \left[-\sigma^2 \left(\Omega + \frac{2\pi(ne^\beta + me^{-\beta})}{L} \right)^2 \right] \\ & + \frac{2\pi^{1/2}\sigma}{L^2} \sum_{n=1}^{\infty} \left\{ \exp \left[-\sigma^2 \left(\Omega + \frac{2\pi ne^\beta}{L} \right)^2 \right] + \exp \left[-\sigma^2 \left(\Omega + \frac{2\pi ne^{-\beta}}{L} \right)^2 \right] \right\} \\ & + \frac{2\pi^{1/2}\sigma\theta}{L^2} e^{-\sigma^2 \Omega^2}. \end{aligned} \quad (3.36)$$

Three comments are in order.

First, consider the limit of long detection, $\sigma \rightarrow \infty$, with the other parameters fixed. In this limit, $\mathcal{F}_t$ and $\mathcal{F}_u$ each reduce to a series of delta-peaks,

$$\mathcal{F}_t(\Omega) = \frac{4\pi}{L^2} \sum_{n,m=0}^{\infty} \delta\left(\Omega + \frac{2\pi((n + \frac{1}{2})e^\beta + (m + \frac{1}{2})e^{-\beta})}{L}\right), \quad (3.37a)$$

$$\begin{aligned} \mathcal{F}_u(\Omega) &= \frac{4\pi}{L^2} \sum_{n,m=1}^{\infty} \delta\left(\Omega + \frac{2\pi(ne^\beta + me^{-\beta})}{L}\right) \\ &\quad + \frac{2\pi}{L^2} \sum_{n=1}^{\infty} \left[\delta\left(\Omega + \frac{2\pi ne^\beta}{L}\right) + \delta\left(\Omega + \frac{2\pi ne^{-\beta}}{L}\right) \right] \\ &\quad + \frac{2\pi\theta}{L^2} \delta(\Omega), \end{aligned} \quad (3.37b)$$

where δ is Dirac's delta function. The Doppler shift factors $e^{\pm\beta}$ show that the peaks in $\mathcal{F}_t$ correspond to the creation of a pair of field excitations, one left-moving and the other right-moving. The peaks in $\mathcal{F}_u$ are similar but also contain the special cases where one or both of the field excitations are in the zero mode, with vanishing energy. That the excitations occur in pairs is a consequence of the quadratic interaction Hamiltonian H_{int} (4.139). By contrast, the peaks for a detector coupled linearly to a scalar field [46] correspond to emission of just single field quanta.

Second, consider the ultrarelativistic velocity limit, $|\beta| \rightarrow \infty$, with the other parameters fixed. $\mathcal{F}_t$ vanishes in this limit, exponentially in $e^{|\beta|}$: the physical reason is that the detector would need to excite field quanta in pairs and one member of each pair is necessarily highly blueshifted in the detector's local rest frame. For $\mathcal{F}_u$ (3.36), however, the first of the terms gives an exponentially small contribution while the last term is independent of β . The part that is (potentially) large is

$$\begin{aligned} \mathcal{F}_u(\Omega) &= \frac{2\sqrt{\pi}\sigma}{L^2} \sum_{n=1}^{\infty} \left\{ \exp\left[-\sigma^2 \left(\Omega + \frac{2\pi ne^{-|\beta|}}{L}\right)^2\right] \right\} \\ &= \frac{\sigma}{\sqrt{\pi}L} e^{|\beta|} \frac{2\pi}{L} e^{-|\beta|} \sum_{n=1}^{\infty} \left\{ \exp\left[-\sigma^2 \left(\Omega + \frac{2\pi ne^{-|\beta|}}{L}\right)^2\right] \right\} \\ &= \frac{\sigma}{\sqrt{\pi}L} e^{|\beta|} \left\{ \int_0^\infty dx \exp[-\sigma^2 (\Omega + x)^2] + o(1) \right\}. \end{aligned} \quad (3.38)$$

Using the change of integration variable, $x = -\Omega + y/\sigma$, where $dx = dy/\sigma$, we obtain

$$\begin{aligned}\mathcal{F}_u(\Omega) &= \frac{\sigma}{\sqrt{\pi}L} e^{|\beta|} \left\{ \frac{1}{\sigma} \int_{\Omega\sigma}^{\infty} dy \exp(-y^2) + o(1) \right\} \\ &= \frac{e^{|\beta|}}{2L} [\operatorname{erfc}(\sigma\Omega) + o(1)] \quad \text{as } |\beta| \rightarrow \infty,\end{aligned}\tag{3.39}$$

where erfc is the error complement function [41]. The physical interpretation is that at ultrarelativistic velocities the detector has an exponentially large probability to generate field excitation pairs in which one excitation is highly redshifted with respect to the detector's local rest frame and the other excitation is a zero mode. This phenomenon has no counterpart for a detector coupled linearly to a scalar field [46].

Third, consider the large circumference limit, $L \rightarrow \infty$, with the other parameters fixed. As noted in subsection 3.0.4, in this limit both $\mathcal{F}_t$ and $\mathcal{F}_u$ approach the response of an inertial detector in Minkowski vacuum in (1 + 1)-dimensional Minkowski spacetime, evaluated in Appendix D, with the result

$$\mathcal{F}_{t,u}(\Omega) \xrightarrow{L \rightarrow \infty} \frac{1}{2\pi\sigma} \left(\frac{e^{-\sigma^2\Omega^2}}{\pi^{1/2}} - \sigma\Omega \operatorname{erfc}(\sigma\Omega) \right). \tag{3.40}$$

In the limit $\sigma \rightarrow \infty$, (3.40) reduces to

$$\mathcal{F}(\Omega) = -\frac{\Omega\Theta(-\Omega)}{\pi}, \tag{3.41}$$

where Θ is the Heaviside function. Formula (3.41) equals twice the response of an inertial Unruh-DeWitt detector coupled linearly to a scalar field in four-dimensional Minkowski space in the long interaction limit [5], as must be the case by Theorem 2.1.

Plots of $L\mathcal{F}_t$ and $L\mathcal{F}_u$ as a function of the dimensionless variables $\sigma\Omega$ and σ/L are shown in Figures 3.1 and 3.2.

3.1.2 Uniformly accelerated detector

Consider a detector on the uniformly accelerated worldline

$$t = a^{-1} \sinh(a\tau), \quad x = a^{-1} \cosh(a\tau), \tag{3.42}$$

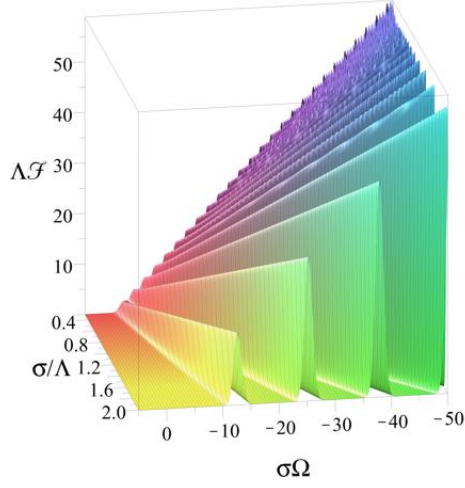
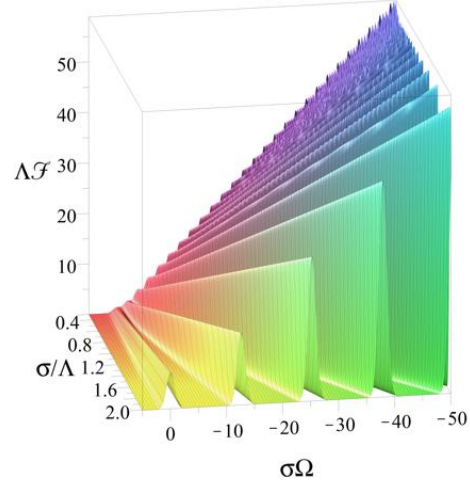
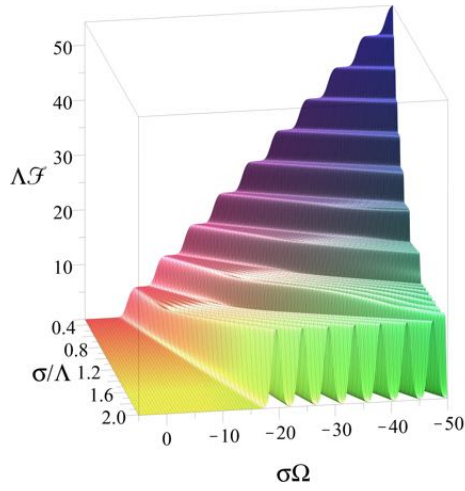
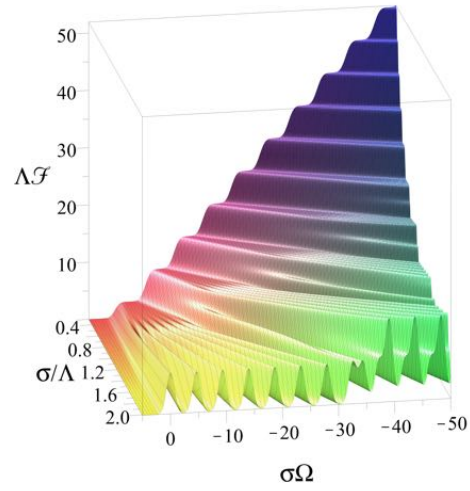
(a) $L\mathcal{F}_t$ for $\beta = 0$ (b) $L\mathcal{F}_u$ for $\beta = 0$ (c) $L\mathcal{F}_t$ for $\beta = 1$ (d) $L\mathcal{F}_u$ for $\beta = 1$

Figure 3.1: Perspective plots of $L\mathcal{F}_t$ and $L\mathcal{F}_u$ for the inertial detector in terms of the dimensionless variables $\sigma\Omega$ and σ/Λ , for $\beta = 0$ and $\beta = 1$. In $L\mathcal{F}_u$ we have set $\theta = 1$. Λ in the axis labels stands for L .

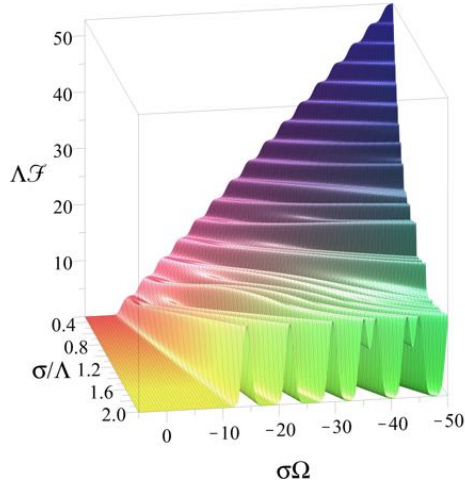
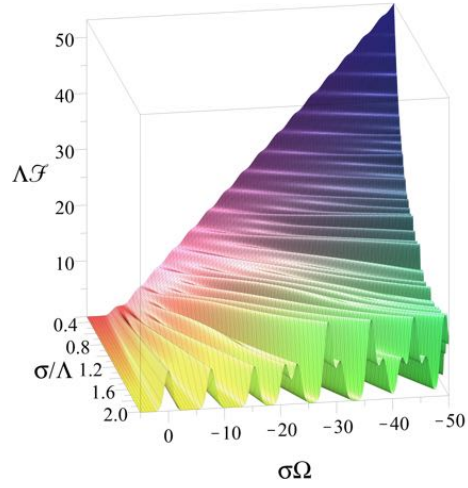
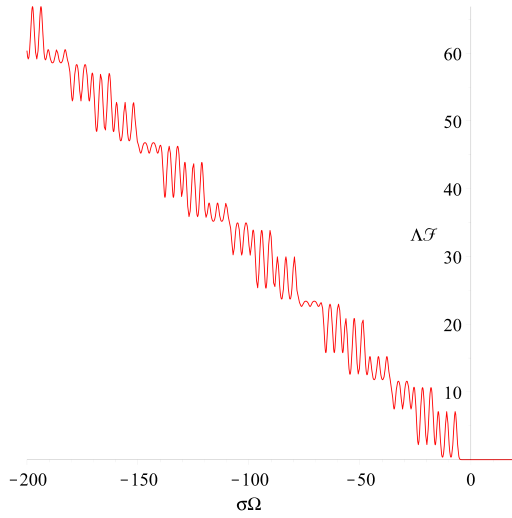
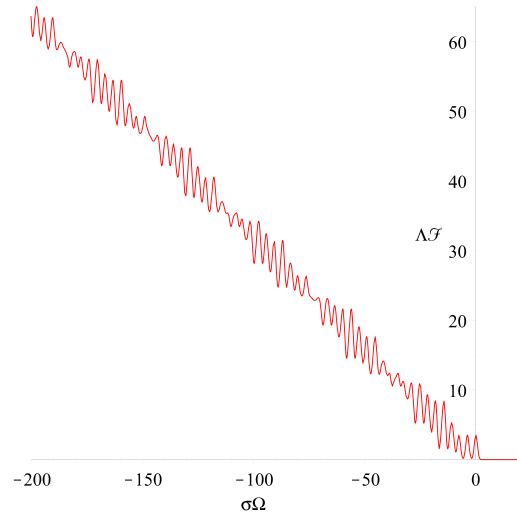

 (a) $L\mathcal{F}_t$ for $\beta = 0.5$

 (b) $L\mathcal{F}_u$ for $\beta = 0.5$

 (c) Cross-section of (a) at $\sigma/\Lambda = 1$.

 (d) Cross-section of (b) at $\sigma/\Lambda = 1$.

Figure 3.2: As in Figure 3.1 but with $\beta = 0.5$, showing also cross-sections at $\sigma/\Lambda = 1$ which reveal finer detail.

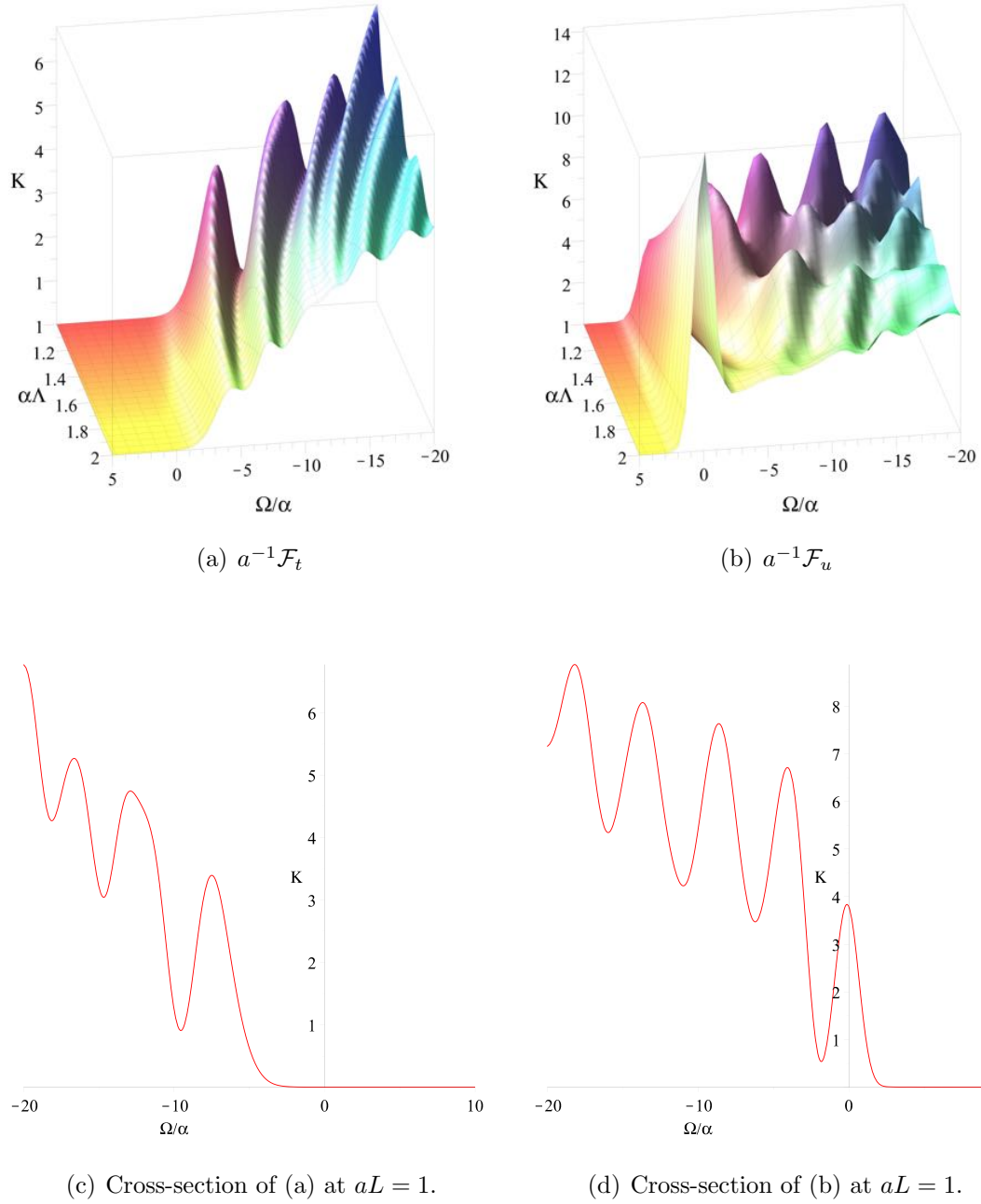


Figure 3.3: Perspective plots of $a^{-1}\mathcal{F}_t$ and $a^{-1}\mathcal{F}_u$ for the uniformly accelerated detector in terms of the dimensionless variables aL and Ω/a , for $\tau_0 = 0$, and cross-sections at $aL = 1$. In $a^{-1}\mathcal{F}_u$ we have set $\theta = 1$. K , Λ and α in the axis labels stand respectively for $a^{-1}\mathcal{F}$, L and a .

where the positive parameter a is the proper acceleration. As this trajectory is not stationary on the cylinder, we now consider the Gaussian switching function

$$\chi_{\tau_0}(\tau) = \frac{1}{\pi^{1/4}\sigma^{1/2}} e^{-(\tau-\tau_0)^2/(2\sigma^2)} , \quad (3.43)$$

where the new real-valued parameter τ_0 specifies the moment about which χ_{τ_0} is peaked.

For the twisted field, by using the change of integration $e^{a\tau} = y$, $\tau = \frac{1}{a} \ln y$, and $d\tau = \frac{1}{a} \frac{dy}{y}$, (3.12) gives

$$\mathcal{F}_t(\Omega) = \frac{2}{\pi^{1/2}(aL)^2\sigma} \sum_{n,m=0}^{\infty} |I_{nm}^t|^2 , \quad (3.44)$$

where

$$I_{nm}^t = \int_0^\infty \frac{dy}{y} \exp \left[-\frac{(\ln y - a\tau_0)^2}{2\sigma^2 a^2} - \frac{i\Omega \ln y}{a} - \frac{2\pi i}{aL} \left[(n + \frac{1}{2})y - (m + \frac{1}{2})y^{-1} \right] \right] . \quad (3.45)$$

For the untwisted field, (3.28), (3.29) and (3.30) give

$$\begin{aligned} \mathcal{F}_u(\Omega) &= \frac{2}{\pi^{1/2}(aL)^2\sigma} \sum_{n,m=1}^{\infty} |I_{nm}^u|^2 \\ &\quad + \frac{1}{\pi^{1/2}(aL)^2\sigma} \sum_{n=1}^{\infty} (|J_n^+|^2 + |J_n^-|^2) \\ &\quad + \frac{2\pi^{1/2}\sigma\theta}{L^2} e^{-\sigma^2\Omega^2} , \end{aligned} \quad (3.46)$$

where

$$I_{nm}^u = \int_0^\infty \frac{dy}{y} \exp \left[-\frac{(\ln y - a\tau_0)^2}{2\sigma^2 a^2} - \frac{i\Omega \ln y}{a} - \frac{2\pi i}{aL} (ny - my^{-1}) \right] , \quad (3.47a)$$

$$J_n^+ = \int_0^\infty \frac{dy}{y} \exp \left[-\frac{(\ln y - a\tau_0)^2}{2\sigma^2 a^2} - \frac{i\Omega \ln y}{a} - \frac{2\pi ni}{aL} y \right] , \quad (3.47b)$$

$$J_n^- = \int_0^\infty \frac{dy}{y} \exp \left[-\frac{(\ln y - a\tau_0)^2}{2\sigma^2 a^2} - \frac{i\Omega \ln y}{a} + \frac{2\pi ni}{aL} y^{-1} \right] . \quad (3.47c)$$

As the detector's worldline is not stationary, analytic investigation of $\mathcal{F}_t$ and $\mathcal{F}_u$ in the limit of large σ and in the limit of large τ_0 is not straightforward. In the limit

of large circumference, however, we recall from subsection 3.0.4 that both $\mathcal{F}_t$ and $\mathcal{F}_u$ approach the response in Minkowski vacuum, evaluated in Appendix D, with the result

$$\mathcal{F}_{t,u}(\Omega) \xrightarrow{L \rightarrow \infty} \frac{ae^{-\pi\Omega/a}}{4\pi^2} \int_{-\infty}^{\infty} \frac{dr}{\cosh^2 r} \exp\left(-\frac{(r - i\pi/2)^2}{\sigma^2 a^2} - \frac{2i\Omega r}{a}\right). \quad (3.48)$$

In the limit $\sigma \rightarrow \infty$, (3.48) reduces by formula 3.982.1 in [47] to the Planckian distribution in the Unruh temperature $a/(2\pi)$,

$$\mathcal{F}(\Omega) = \frac{\Omega}{\pi(e^{2\pi\Omega/a} - 1)}. \quad (3.49)$$

Formula (3.49) equals twice the response of a uniformly accelerated Unruh-DeWitt detector coupled linearly to a scalar field in four-dimensional Minkowski space in the long interaction limit [5], as must be the case by Theorem 2.1.

Plots of $a^{-1}\mathcal{F}_t(\Omega)$ and $a^{-1}\mathcal{F}_u(\Omega)$ as a function of the dimensionless variables aL and Ω/a are shown in Figure 3.3.

3.2 Conclusions

We have analysed the response of a spatially pointlike Unruh-DeWitt detector coupled linearly to the scalar density on the $(1+1)$ -dimensional flat static cylinder, allowing the detector's motion to remain arbitrary and allowing the detector to be switched on and off in an arbitrary smooth way.

We found that the response distinguishes the Fock vacua of the field's oscillator modes for periodic and antiperiodic spin structures, and the zero mode that occurs for the periodic spin structure contributes to the response in a way that depends on zero mode's initial state. We also provided a selection of analytic and numerical results for inertial and uniformly accelerated trajectories on the cylinder, recovering the $d = 2$ Minkowski results in the limit of large circumference.

Part II

Field Theory and Particle detectors in curved spacetimes

Chapter 4

Quantum field theory in curved spacetimes

In this chapter the formalism of quantum Dirac field theory in flat spacetime is generalised to a curved spacetime in a pedagogical way. The treatment is not intended to be very rigorous or complete (for more detailed discussion see [48–51]). Our generalisation of quantum field theory to a curved spacetime background is based on the *minimal coupling prescription*, and is consistent with Einstein Equivalence Principle. According to this principal, local gravitational effects are not present in a neighbourhood of the space time origin of a locally inertial frame of reference. Thus, our curved spacetime field theory reduces to the flat-spacetime theory locally. Mathematically, this means we construct the field theory so that it is invariant under general coordinate transformations. Furthermore, we will ignore the back reaction of the quantized field on the geometry by assuming that the metric is a fixed classical background that is not affected by the quantum field. This procedure will lead to the field equation. However, the classical existence and uniqueness of the solutions to the field equation along with the fact that we require a globally defined time coordinate in the canonical quantization approach will demand that we impose additional restrictions on the choice of our spacetime.

We give what is intended to be a complete treatment for the Bogolubov transformation of Dirac field in Section 4.1.5 since this is not as widely discussed as the Bogolubov transformation for the scalar field. This transformation reveals that the choice of a complete set of solutions to the Dirac equation is generally not unique. Neither is the constructed Fock space over the Hilbert space generated by a complete set of mode solutions. Therefore there is no preferred Fock space and associated Fock vacuum state. Specifically, the definition of Fock particle states (as eigenstates of the number operator in the Fock space) is non-unique. The vacuum associated with a set of modes will contain particles in another set of modes, thus the two sets do not share a common vacuum and will lead to a different number of particles. Physically this means that different observers will register different number of particles in a given state of the field. This is the mathematical reason for the Unruh-Hawking effect. Hence, we conclude that the notion of particle states for a quantized field in a curved background is a basis-dependent notion.

As is well known, in Minkowski spacetime there is a preferred set of mode solutions to Dirac equation chosen as the eigenfunctions of the global timelike Killing vector field. This approach allows us to define a natural basis of solutions in terms of positive and negative frequency modes. And under the Lorentz transformation, positive frequency modes transform to positive frequency modes. Therefore, regardless of the Lorentz frame where t is the global time coordinate, this procedure defines the same, unique Minkowski vacuum state. Hence, there is a preferred Fock space for the free quantized Dirac field, with a Poincaré invariant vacuum. This procedure of defining positive and negative frequency mode functions generalises straightforwardly to stationary, curved backgrounds because of the presence of the global timelike killing vector field (see Section 4.1.5). Therefore we have a preferred set of mode solutions to the field equation and a preferred Fock state and a natural, unique notion of particles. However, in a generic non-stationary, curved background there is no global timelike Killing vector field. There the states of the field will not contain a meaningful particle interpretation, except in the special case where

the spacetime is asymptotically stationary (see Section 4.1.5) where we can define asymptotic particle states. For non-stationary spacetimes with asymptotically stationary regions, this leads to particle creation due to the changing geometry.

In Section 4.2.1 and Section 4.2.4 we discuss the quantization of the free Dirac field and the free scalar field for certain classes of Robertson-Walker (FRW) spacetimes wherein the two-point functions have the Hadamard form. In Section 4.2.5, we quantize the free scalar field in $(1 + 1)$ -dimensional spatially flat FRW spacetimes with compactified spatial sections. There, the issue of the conformal zero mode arises and we develop a quantization scheme for this mode. In Section 4.3 we give a brief review of Unruh-DeWitt particle detector coupled to a linear quantized scalar field. We conclude this chapter in Section 4.3.1 by first establishing the equivalence between the adiabatic vacuum of infinite order and the conformal vacuum for massless field. Then, with the field prepared in the conformal Fock vacuum, we show that the response of an UDW particle detector coupled linearly to the scalar density of a massless Dirac field in the spatially flat FRW spacetimes in d dimensions is identical to the response of a detector coupled linearly to the massless scalar field in $2d$ dimensions.

4.1 Quantum Dirac field in a curved spacetime

4.1.1 Spacetime Structure: Global Hyperbolicity

We consider a spacetime $(\mathcal{M}, \mathbf{g})$, where $\mathcal{M}$ is a real smooth (i.e. C^∞) manifold of dimension $d \geq 2$ and $\mathbf{g}$ is a Lorentzian metric on $\mathcal{M}$ of signature $(+, -, -, \dots, -)$. To have a global notion of causality, i.e. a unique notion of past and future continuously over the whole spacetime, we require the spacetime to be time oriented. We do this by recalling the following definitions from [52].

Definition 4.1.1.

*A Lorentzian manifold $(\mathcal{M}, g)$ is called **time orientable** if there exists a globally*

defined smooth time-like vector field t^μ , i.e. $\forall p \in \mathcal{M} : g(p)(t, t) = g_{\mu\nu}(p)t^\mu(p)t^\nu(p) > 0$. A time-like vector $v \in T_p\mathcal{M}$ is called future oriented if $g(p)(t, v) > 0$.

Furthermore, to guarantee that the field equation has a well-posed initial value formulation analogous to that of Minkowski spacetime, we require that the spacetime be globally hyperbolic. We do this by recalling the following definitions from [52, 53].

Definition 4.1.2.

- a smooth curve $c : \mathbb{R} \supset I \rightarrow \mathcal{M}$ is called a **causal curve** if $\forall s \in I : g(c(s))(\dot{c}(s), \dot{c}(s)) \geq 0$.
- A point $p \in \mathcal{M}$ is called an **endpoint** of the curve c if for every neighbourhood U of p there exists an s_0 such that $c(s) \in U$ either for all $s > s_0$ or for all $s < s_0$.
- A causal curve is called **inextendible** if it has no endpoint.
- A hypersurface $\Sigma \subset \mathcal{M}$ is called a **Cauchy surface** if it is intersected exactly once by every inextendible causal curve.
- A spacetime $(\mathcal{M}, g)$ is called **globally hyperbolic** if it contains a Cauchy surface. (It is then time orientable and diffeomorphic to $\Sigma \times \mathbb{R}$ as shown in [53]).

Any globally hyperbolic spacetime can be foliated by a one-parameter family of Cauchy surfaces $(\Sigma_t)_{t \in \mathbb{R}}$ [54]. By foliation we mean there exists a smooth scalar field f on $\mathcal{M}$ which is regular (in the sense that its gradient vanishes nowhere) such that each level hypersurface of this scalar field is a Cauchy surface:

$$\forall t \in \mathbb{R}, \quad \Sigma_t := \{p \in \mathcal{M}, f(p) = t\} \quad (4.1)$$

Since f is regular, the Cauchy surfaces Σ_t are non-intersecting:

$$\Sigma_t \cap \Sigma_{t'} = \emptyset \quad \text{for } t \neq t'. \quad (4.2)$$

We assume that all the Cauchy surfaces Σ_t are spacelike and that the foliation covers $\mathcal{M}$ [54]:

$$\mathcal{M} = \bigcup_{t \in \mathbb{R}} \Sigma_t. \quad (4.3)$$

The normal of a foliation is a normalised timelike vector field n^μ orthogonal to each Cauchy surface Σ_t . In general, we can write

$$n_\mu := N \nabla_\mu f = N \partial_\mu f, \quad \text{for } N := (\partial_\mu f \partial^\mu f)^{-1/2}, \quad n^\mu n_\mu = 1, \quad (4.4)$$

where the proportionality factor is called the lapse of the foliation and we choose the normal n^μ to be future-directed. The foliation of the spacetime into a family of Cauchy surfaces Σ_t , labeled by their coordinate t is the basis of the ADM decomposition [55], which we demonstrate below.

Each time-slice Σ_t is equipped with its own Riemannian structure. The induced metric $h_{\mu\nu}$ on each Σ_t can be uniquely determined by the two conditions [56]

$$h_{\mu\nu} n^\mu = 0, \quad (4.5a)$$

$$h_{\mu\nu} s^\mu = -g_{\mu\nu} s^\mu, \quad \forall s^\mu \in \mathcal{T}_p(\Sigma_t). \quad (4.5b)$$

The first condition (4.5a) implies that $h_{\mu\nu}$ is purely spatial, thus has no component along n^μ . The minus sign is included in the second condition (4.5b) as to agree with our sign convention. Therefore, the induced $(d-1)$ -dimensional Riemannian metric on each Cauchy surface Σ_t is given according to the formula

$$h_{\mu\nu} = n_\mu n_\nu - g_{\mu\nu}. \quad (4.6)$$

Coordinates adapted to the foliation are set up in the following way. On each Cauchy surface Σ_t , we introduce some coordinates $(x^i) = (x^1, x^2, \dots, x^{d-1})$. If these coordinates vary smoothly between neighbouring Cauchy surfaces, then $(x^\alpha) = (t, x^1, x^2, \dots, x^{d-1})$ constitutes a smooth coordinate system on $\mathcal{M}$, where t is the global time coordinate and (x^i) are the spatial coordinates [54].

The idea behind the foliation is that the physics on any time-slice Σ_t completely determines the physics in the future and the past of Σ_t . In particular, initial data given on any time-slice (Σ_t) is sufficient to determine the solution to the field equation throughout the spacetime. In short, for globally hyperbolic spacetimes, the Klein-Gordon and Dirac equations have unique solutions that are globally defined in the spacetime [57] and there is a globally defined time coordinate which can be used as the distinguished time in the canonical quantization [58].

Let t^μ be a “time evolution” vector field in $\mathcal{M}$ satisfying $t^\mu \nabla_\mu f = 1$. Then the small displacement $\delta t t^\mu$ evolves a point $p \in \Sigma_t$ to a point $p' = p + \delta t t^\mu \in \Sigma_{t+\delta t}$ as is evident by writing

$$f(p') = f(p + \delta t t^\mu) = f(p) + \delta t t^\mu \nabla_\mu f = f(p) + \delta t. \quad (4.7)$$

We may see this as the effect of moving forward in time from the time-slice Σ_t with induced metric $h_{ij}(t)$ to the time slice $\Sigma_{t+\delta t, \delta t > 0}$ with induced metric $h_{ij}(t + \delta t)$. For this reason the vector field t^μ is seen as representing the “flow of time”. Hence, we can view a hyperbolic spacetime $(\mathcal{M}, g_{\mu\nu})$ as representing the time evolution of a Riemannian metric on a fixed $(d-1)$ -dimensional Cauchy surface [1]. Furthermore, we can decompose t^μ into its components normal and tangential to Σ_t

$$t^\mu = N n^\mu + N^\mu, \quad \rightarrow \quad n^\mu = \frac{t^\mu - N^\mu}{N} \quad (4.8)$$

where N is the lapse function defined in (4.4), quantifying the temporal distance between neighbouring time-slices and N^μ is called the shift vector and is defined with respect to t^μ by

$$N_\mu := h_{\mu\nu} t^\nu, \quad n^\mu N_\mu = 0 \quad \rightarrow \quad N^\mu = (0, N^i). \quad (4.9)$$

Contracting both sides of (4.4) with n^μ , again contracting (4.4) with t^μ yield the following formula for the lapse function

$$N = \frac{1}{n^\mu \nabla_\mu f} = t^\mu n_\mu. \quad (4.10)$$

In the adapted coordinates (x^μ) system, the vector ∂_t is tangent to the lines of constant spatial coordinates, so it is standard [1, 54] to choose the time evolution vector field

$$t^\mu =: \partial_t x^\mu = (1, 0, \dots, 0). \quad (4.11)$$

Now collecting (4.6) and (4.8), we obtain

$$g^{\mu\nu} = \frac{(t^\mu - N^\mu)(t^\nu - N^\nu)}{N^2} - h_{\mu\nu}. \quad (4.12)$$

Using (4.9) and (4.11), the spacetime metric (4.12) and its inverse can then be decomposed into

$$g_{\cdot\cdot} = \begin{pmatrix} \frac{1}{N^2} & -\frac{N^i}{N^2} \\ -\frac{N^i}{N^2} & \frac{N^i N^j}{N^2} - h_{ij} \end{pmatrix}, \quad g_{\cdot\cdot} = \begin{pmatrix} N^2 - N_i N^i & -N_j \\ -N_j & -h_{ij} \end{pmatrix}, \quad (4.13)$$

where $N_i := h_{ij} N^j$. The components of the normal vector can be found from the one-form $n_\mu dx^\mu = -N dt$ and the metric $g_{\cdot\cdot}$ in (4.13) to read

$$n_\mu = (N, 0), \quad n^\mu = g^{\mu\nu} n_\nu = \left(\frac{1}{N}, -\frac{N^i}{N} \right). \quad (4.14)$$

Using the relation $\sqrt{|g|} = N\sqrt{|h|}$ [54], the normal component to Σ_t can be cast in the more convenient form

$$n^\mu = g^{0\mu} \sqrt{\frac{|g|}{h}}. \quad (4.15)$$

Thus, the spacetime line element and the Cauchy surface element read

$$ds^2 = N^2 dt^2 - h_{ij} (dx^i + N^i dt)(dx^j + N^j dt), \quad (4.16a)$$

$$d\Sigma = \sqrt{h} dx^1 \dots dx^{d-1}. \quad (4.16b)$$

(4.16a) is the ADM form of the metric [59, 60].

4.1.2 Construction of Dirac field theory in a curved space-time

The generalization of the action of the Dirac field (2.6) to curved spacetime, based on the *minimal coupling prescription*, reads

$$S[\psi] = \int_{\Sigma_t \times \mathbb{R}} d^d x |g|^{1/2} \bar{\psi}(x) (i \underline{\gamma}^\mu(x) \nabla_\mu - m) \psi(x) , \quad (4.17)$$

where the constant γ -matrices are superseded by spacetime-dependent γ -matrices $\underline{\gamma}^\mu$, $\mu = 0, 1, \dots, d-1$, satisfying

$$\{\underline{\gamma}^\mu, \underline{\gamma}^\nu\} = 2g^{\mu\nu}(x) , \quad (4.18)$$

where on the right hand side we have suppressed the identity matrix. The covariant derivative of the spinor field is defined as

$$\nabla_\mu \psi(x) := (\partial_\mu - \Gamma_\mu(x)) \psi(x) , \quad \nabla_\mu \psi^\dagger(x) = \partial_\mu \psi^\dagger(x) - \psi^\dagger(x) \Gamma_\mu^\dagger(x) , \quad (4.19)$$

where $\Gamma_\mu(x)$ is the spinorial affine connection which is required for covariantisation. The field $\bar{\psi}$ is the Dirac conjugate spinor defined as $\bar{\psi}(x) := \psi^\dagger(x) \gamma^0$, where γ^0 is the constant γ -matrix of a local orthonormal frame. To obtain an explicit expression for the spinorial affine connection Γ_μ , we have to impose additional conditions. As in flat spacetime, we require that $\bar{\psi}\psi$ and $\bar{\psi}\underline{\gamma}^\nu\psi$ transform as a scalar and a vector respectively. The transformations properties

$$\nabla_\mu (\bar{\psi}\psi) = \partial_\mu (\bar{\psi}\psi) , \quad (4.20a)$$

$$\nabla_\mu (\bar{\psi}\underline{\gamma}^\nu\psi) = \partial_\mu (\bar{\psi}\underline{\gamma}^\nu\psi) + \Gamma_{\lambda\mu}^\nu (\bar{\psi}\underline{\gamma}^\lambda\psi) , \quad (4.20b)$$

lead to the following additional conditions on the spinorial affine connection

$$\gamma^0 \Gamma_\mu \gamma^0 = -\Gamma_\mu^\dagger , \quad (4.21a)$$

$$\nabla_\mu \underline{\gamma}^\nu(x) = \partial_\mu \underline{\gamma}^\nu(x) + \Gamma_{\mu\lambda}^\nu(x) \underline{\gamma}^\lambda(x) - [\Gamma_\mu(x), \underline{\gamma}^\nu(x)] , \quad (4.21b)$$

where $\Gamma_{\mu\lambda}^\nu(x)$ are the Riemann-Christoffel connection. Moreover, it is standard to choose the spinorial affine connection by requiring that $\underline{\gamma}$ -matrix be covariantly

conserved $\nabla_\mu \underline{\gamma}^\nu = 0$. An explicit expression for the spinorial affine connection is obtained by first multiplying (4.21b) by $\underline{\gamma}_\nu(x) := g_{\nu\mu}(x)\underline{\gamma}^\mu(x)$ and using the identity $\underline{\gamma}_\nu \Gamma_\mu \underline{\gamma}^\nu \equiv (d-4)\Gamma_\mu$ [61]

$$\Gamma_\mu(x) = -\frac{1}{4}\underline{\gamma}_\nu(x) \left(\partial_\mu \underline{\gamma}^\nu(x) + \Gamma_{\mu\lambda}^\nu(x) \underline{\gamma}^\lambda(x) \right). \quad (4.22)$$

The spacetime-dependent $\underline{\gamma}$ -matrices can be expressed as a linear combination of the constant γ -matrices of a local orthonormal frame

$$\underline{\gamma}^\mu(x) = e_a^\mu(x) \gamma^a, \quad \underline{\gamma}_\mu(x) = e_\mu^b(x) \gamma_b, \quad \{a, b, \dots = 0, \dots, d-1\}. \quad (4.23)$$

The constraints on the coefficients $e_a^\mu(x)$ and $e_\mu^b(x)$ follow from the anticommutator algebra (A.2)

$$g^{\mu\nu}(x) = \eta^{ab} e_a^\mu(x) e_b^\nu(x), \quad g_{\mu\nu}(x) = \eta_{ab} e_\mu^a(x) e_\nu^b(x), \quad (4.24)$$

which implies that $e_a^\mu(x) e_\mu^\nu(x) = \delta_a^\nu$. Therefore, the coefficient fields $e_a^\mu(x)$ are the n-bein fields describing the local orthonormal frame. Using the relations (4.23) and (4.24), we can express the spinorial affine connection (4.22) as

$$\Gamma_\mu(x) = -\frac{1}{8} [\gamma^a, \gamma^b] e_{a\nu}(x) \nabla_\mu e_b^\nu(x) := -\frac{1}{4} \sigma^{ab} \omega_{ab\mu}, \quad (4.25)$$

where $\sigma^{ab} := (1/2) [\gamma^a, \gamma^b]$ and the spin connection $\omega_{ab\mu} := e_{a\nu}(x) \nabla_\mu e_b^\nu(x) = -\omega_{ba\mu}$, with the covariant derivative of the n-bein field given by $\nabla_\mu e_b^\nu(x) = \partial_\mu e_b^\nu(x) + \Gamma_{\mu\alpha}^\nu e_b^\alpha(x)$. The field equations for ψ and its Dirac conjugate $\bar{\psi}$ follows from the action functional (4.17)

$$(i e_a^\mu(x) \gamma^a \nabla_\mu - m) \psi(x) = 0, \quad (4.26a)$$

$$i (\nabla_\mu \bar{\psi}(x)) e_a^\mu(x) \gamma^a + m \bar{\psi}(x) = 0, \quad (4.26b)$$

where

$$\nabla_\mu \bar{\psi}(x) := \partial_\mu \bar{\psi}(x) + \bar{\psi}(x) \Gamma_\mu(x). \quad (4.27)$$

4.1.3 Normalization, Quantization, Wightman functions and Two-point function: $W^{(2,\bar{2})}$

Normalization

For any pair of solutions ψ and ϕ to Dirac equation (4.26a) the following current is conserved

$$J^\mu(\psi, \phi) := \bar{\psi} \underline{\gamma}^\mu \phi, \quad \nabla_\mu J^\mu = 0. \quad (4.28)$$

This conserved current allows us to define a positive definite Hermitian product (ψ, ϕ) of two-well behaved solutions across any smooth spacelike Cauchy surface Σ_t

$$(\psi, \phi) := \int_{\Sigma_t} d\Sigma^\mu J_\mu(\psi, \phi) = \int_{\Sigma_t} d\Sigma n^\mu \bar{\psi}(x) \underline{\gamma}_\mu \phi(x), \quad (4.29a)$$

$$= \int_{\Sigma_t} d^{d-1}x |g|^{1/2} g^{0\mu} e^a{}_\mu(x) \bar{\psi}(x) \gamma_a \phi(x), \quad (4.29b)$$

where $d\Sigma$ is given in (4.16b), n^μ in (4.15) and $\underline{\gamma}_\mu$ in (4.23). When the metric of the spacetime $g_{\mu\nu}(x)$ is diagonal, (4.29) reduces to

$$(\psi, \phi) = \int_{\Sigma_t} d^{d-1}x |g|^{1/2} g^{00} e^0{}_0(x) \psi^\dagger(x) \phi(x). \quad (4.30)$$

This Hermitian inner product satisfy the following properties

- (i) $\frac{d}{dt}(\psi, \phi) = 0$,
- (ii) $(\psi, \phi)^* = (\phi, \psi)$,
- (iii) $(\psi, \psi) \geq 0$.

Thus, the above properties imply that the space of solutions to the Dirac equation with (ψ, ϕ) finite forms an inner product space. The Hilbert space completion of this space is taken as the Hilbert space $\mathcal{H}$ of the state of the classical spinor field system.

4.1.4 Quantization

On a globally hyperbolic spacetime, which is equipped with a globally defined time coordinate, we can perform canonical quantization of the classical free spinor field in a straightforward way. Let Σ_{t_0} be a particular Cauchy surface corresponding to the initial time-slice. The momentum canonically conjugated to ψ on Σ_{t_0} is defined by

$$\pi := \frac{\delta S}{\delta(\partial_t \psi)} \Big|_{\Sigma_{t_0}} = i|g(x)|^{1/2} \bar{\psi}(x) e_a^0(x) \gamma^a \Big|_{\Sigma_{t_0}} . \quad (4.31)$$

We construct a quantum field theory for Dirac spinor field by promoting the classical fields $[\psi, \pi]_{\Sigma_{t_0}}$ to operators $[\hat{\psi}, \hat{\pi}]_{\Sigma_{t_0}}$, satisfying the equal-time canonical anticommutation relations (CARs) on the time-slice Σ_{t_0} ¹

$$\{\psi_a(t, \mathbf{x}), \psi_b(t, \mathbf{x}')\} \Big|_{\Sigma_{t_0}} = \{\pi_a(t, \mathbf{x}), \pi_b(t, \mathbf{x}')\} \Big|_{\Sigma_{t_0}} = 0 , \quad (4.32a)$$

$$\{\psi_a(t, \mathbf{x}), \pi_b(t, \mathbf{x}')\} \Big|_{\Sigma_{t_0}} = i\delta_{ab} \delta^{d-1}(\mathbf{x}, \mathbf{x}') \Big|_{\Sigma_{t_0}} \mathbb{1} . \quad (4.32b)$$

and all other CARs vanish. Here $\delta^{d-1}(\mathbf{x} - \mathbf{x}')$ is the delta function in the time-slice Σ_{t_0} which satisfies

$$\int_{\Sigma_{t_0}} d^{d-1}x \delta^{d-1}(\mathbf{x}, \mathbf{x}') = 1 . \quad (4.33)$$

Furthermore, we may use the fact that Dirac norm $(,)_{\Sigma}$ is slice independent to show that if the CAR's are imposed on any time-slice, they hold on all time-slices [7]. Hence, the quantum field theory does not depend on how we slice the spacetime. Collecting (4.31) and (4.32), we obtain

$$e^0_c(t, \mathbf{x}) e^0_d(t, \mathbf{x}') \{ (\bar{\psi}(t, \mathbf{x}) \gamma^c)_a, (\bar{\psi}(t, \mathbf{x}') \gamma^d)_b \} = 0 , \quad (4.34a)$$

$$\{ \psi_a(t, \mathbf{x}), (\bar{\psi}(t, \mathbf{x}') \gamma^c)_b \} = e^c_0(t, \mathbf{x}) |g(t, \mathbf{x})|^{-1/2} \delta_{ab} \delta^{d-1}(\mathbf{x}, \mathbf{x}') . \quad (4.34b)$$

¹Note we are using the subscripts $\{a, b, \dots\}$ for the local Lorentz indices as well as for the spinor indices.

Note (4.34a) is obtained from (4.32a) by substituting (4.31), then multiplying the result by $|g(t, \mathbf{x})g(t, \mathbf{x}')|^{-1/2}$ which cancels the metric factor out and yields the above result. Similarly, (4.34b) is obtained from (4.32b) by substituting (4.31), then multiplying the result by $e^c_0(t, \mathbf{x})|g(t, \mathbf{x}')|^{-1/2}$, and using the Vielbein identity $e^c_0(t, \mathbf{x})e^0_c(t, \mathbf{x}) = 1$ derived from the paragraph following (4.24). Hence, the results given in (4.34) are consistent with (4.32).

Consequently, in a general curved background of dimension $d \geq 2$, with diagonal metric (where $e^c_0(t, \mathbf{x}) = e^0_0(t, \mathbf{x})$), the spinor field is quantized by imposing the following equal-time anticommutator relations

$$\{\psi_a(\mathbf{x}, t), \psi_b(\mathbf{x}', t)\} = \{\psi^\dagger_a(\mathbf{x}, t), \psi^\dagger_b(\mathbf{x}', t)\} = 0, \quad (4.35a)$$

$$\{\psi_a(\mathbf{x}, t), \psi^\dagger_b(\mathbf{x}', t)\} = e^0_0(\mathbf{x}, t)|g(\mathbf{x}, t)|^{-1/2}\delta_{ab}\delta^{d-1}(\mathbf{x}, \mathbf{x}'). \quad (4.35b)$$

We recovered (4.35) as a special case of the more general quantization procedure given in (4.34) upon substitution of $\bar{\psi}(x) = \psi^\dagger(x)\gamma^0$. This field theory is causal in the sense that (4.32) holds whenever there is a spacelike slice that contains both $(t, \mathbf{x})$ and $(t, \mathbf{x}')$ [57].

4.1.5 Quantized spinor field representation and the Bogolubov transformation for the spinor field

We can choose a complete set of mode solutions $\{u_i(x), v_i(x)\}$ to the Dirac equation which are orthonormal in Dirac norm (4.29), i.e, satisfying

$$(u_i(x), u_j(x)) = (v_i(x), v_j(x)) = \delta_{ij} \quad (4.36a)$$

$$(u_i(x), v_j(x)) = 0, \quad (4.36b)$$

where i, j denote a collective index for the momentum and the independent spin state. The quantized field (in unsmeared form) may be represented as

$$\hat{\psi}(x) = \sum_j \hat{b}_j u_j(x) + \hat{d}^\dagger_j v_j(x), \quad (4.37)$$

on a Hilbert space carrying a representation of the CAR's (4.34a), and Σ_j is shorthand notation to indicate an integral over momentum and a sum over spin states. The CAR's (4.34a) then simplify into an algebra in the expansion coefficients given by

$$\{\hat{b}_i, \hat{b}_j^\dagger\} = \{\hat{d}_i, \hat{d}_j^\dagger\} = \delta_{ij} \mathbb{1} . \quad (4.38)$$

These coefficient operators representing the CAR's act on a Hilbert space of all possible states of the Dirac field defined to be the antisymmetric Fock space $\mathcal{F}_a(\mathcal{H})$ constructed by first postulating the existence of a Fermionic Fock vacuum state $|0\rangle$ satisfying

$$\hat{b}_i|0\rangle = \hat{d}_i|0\rangle = 0, \quad \forall i, \quad \langle 0|0\rangle = 1 . \quad (4.39)$$

From this Fock vacuum we can then construct an entire Fock basis for the Hilbert space $\mathcal{F}_a(\mathcal{H})$, i.e., $\{|0\rangle, \hat{b}_s^\dagger(i)|0\rangle, \hat{b}_i^\dagger\hat{b}_i^\dagger|0\rangle\ldots\hat{d}_i^\dagger|0\rangle, \hat{d}_i^\dagger\hat{d}_i^\dagger|0\rangle\ldots\hat{d}_i^\dagger\hat{b}_i^\dagger|0\rangle\ldots\}$.

A second complete set of modes solutions $\{u'_i(x), v'_j(x)\}$ to Dirac equation is related to the first by a Bogolubov transformation

$$u'_i(x) = \sum_j \alpha_{ij} u_j(x) + \beta_{ij} v_j(x) , \quad (4.40a)$$

$$v'_i(x) = \sum_j \beta_{ij}^* u_j(x) + \alpha_{ij}^* v_j(x) , \quad (4.40b)$$

where (4.40b) is obtained by applying the charge conjugate operator $Cu_j(v_j) = v_j(u_j)$ to (4.40a), which is an antilinear operator that swaps the u_j -modes with the v_j -modes and vice versa [62]. Furthermore, these modes are orthonormal in Dirac norm (4.29) provided the coefficients are constrained to

$$\sum_k (\alpha_{ik} \alpha_{jk}^* + \beta_{ik} \beta_{jk}^*) = \delta_{ij} , \quad (4.41a)$$

$$\sum_k (\alpha_{ik} \beta_{jk} + \beta_{ik} \alpha_{jk}) = 0 . \quad (4.41b)$$

In matrix form the relations (4.41) read

$$\alpha \alpha^\dagger + \beta \beta^\dagger = \mathbb{1} , \quad \alpha \beta^T + \beta \alpha^T = 0 . \quad (4.42)$$

Consequently it follows from (4.40) and (4.42) that the Bogolubov transformation for the spinor field is effected by the unitary matrix

$$U = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix}, \quad U^\dagger = \begin{pmatrix} \alpha^\dagger & \beta^T \\ \beta^\dagger & \alpha^T \end{pmatrix}, \quad UU^\dagger = I. \quad (4.43)$$

The matrices α_{ij} and β_{ij} are called Bogolubov coefficients. Using (4.36) and (4.40), they can be evaluated as

$$\alpha_{ij} = (u_i(x), u'_j(x)), \quad \beta_{ij} = (u_i(x), v'_j(x)). \quad (4.44)$$

Inverting (4.40), we have

$$u_i(x) = \sum_j \alpha'_{ij} u'_j(x) + \beta'_{ij} v'_j(x), \quad (4.45a)$$

$$v_i(x) = \sum_j \beta'^*_{ij} u'_j(x) + \alpha'^*_{ij} v'_j(x), \quad (4.45b)$$

where from (4.40), (4.41) and (4.45), we have $\alpha' = \alpha^\dagger$ ($\alpha'_{ij} = \alpha^*_{ji}$) and $\beta' = \beta^T$ ($\beta'_{ij} = \beta_{ji}$). With respect to this new basis the quantized Dirac field can be represented as

$$\hat{\psi}(x) = \sum_j \hat{b}'_j u'_j(x) + \hat{d}'^\dagger_j v'_j(x), \quad (4.46)$$

where collecting from (4.37), (4.45) and (4.46) we have

$$\hat{b}'_j = \sum_i \alpha^*_{ji} \hat{b}_i + \beta^*_{ji} \hat{d}_i^\dagger, \quad (4.47a)$$

$$\hat{d}'^\dagger_j = \sum_i \beta_{ji} \hat{b}_i + \alpha_{ji} \hat{d}_i^\dagger. \quad (4.47b)$$

Equivalently we have from (4.37), (4.40) and (4.46)

$$\hat{b}_j = \sum_k \alpha_{kj} \hat{b}'_k + \beta^*_{kj} \hat{d}'^\dagger_k, \quad (4.48a)$$

$$\hat{d}_j^\dagger = \sum_k \beta_{kj} \hat{b}'_k + \alpha^*_{kj} \hat{d}'^\dagger_k. \quad (4.48b)$$

Substituting (4.47) into (4.48), we obtained the constraints

$$\sum_i \alpha_{ij} \alpha^*_{ik} + \beta^*_{ij} \beta_{ik} = \delta_{jk}, \quad (4.49a)$$

$$\sum_i \alpha_{ij} \beta^*_{ik} + \beta^*_{ij} \alpha_{ik} = 0. \quad (4.49b)$$

In matrix notation, they read

$$\alpha^T \alpha^* + \beta^\dagger \beta = 1, \quad \beta^T \alpha^* + \alpha^\dagger \beta = 0 = \alpha^T \beta^* + \beta^\dagger \alpha. \quad (4.50)$$

These relations (4.50) are equivalent to $U^\dagger U = I$, where U is given in (4.43). Although it may not be apparent, the relations given in (4.42) and (4.50) are equivalent. This follows from the Inverse Matrix Theorem [63, 64], which says that the inverse of a matrix is unique.

In short, the Bogolubov transformation for the Dirac field is effected by the unitary matrix (4.43) because Dirac norm (4.36) is positive definite for all modes. However, this is not true for the scalar field as Klein Gordon norm is not positive definite for all modes. Moreover, since the bases $\{u'_i, v'_i\}$ are normalised via Dirac norm (4.36), the creation and annihilation operators (4.47) equally satisfy the canonical anticommutator relations (4.38). Thus, the Bogolubov transformation can equally be seen as a linear transformation of the creation and annihilation operators that preserves the algebraic structure of the CAR's.

Consequently, it follows from (4.47) that the Fermionic Fock vacuum state $|0'\rangle$ defined by $\hat{b}'_j|0'\rangle = \hat{d}'_j|0'\rangle = 0, \forall j$ is not the same as the vacuum $|0\rangle$ defined in (4.39) so long as $\beta_{ji} \neq 0$. As a result, the Fock spaces based on the two choices of basis $\{u_i(x), v_i(x)\}$ and $\{u'_i(x), v'_i(x)\}$ of solutions to the Dirac equation (4.26a) are different. Therefore, the notion of particles in quantum field theory as excitations above the Fock vacuum of a free quantized field is a basis-dependent notion.

Remark 4.1. *The definition of a Fermionic Fock vacuum state depends on the choice of a basis of solutions $\{u_i(x), v_i(x)\}$ to the field equation, which itself is sensitive to both the local and global properties of the background spacetime. For instance the basis of solutions to the Dirac equation for the background geometry having the topologies $R \times R^{d-1}$ and $R \times R^{d-2} \times S^1$ are different, though both of these spacetimes are flat. Hence, the notion of the Fock vacuum state and the Fock particle states (as eigenstates of the number operator in the Fock space) are globally defined [65].*

Stationary Spacetime

In a stationary, globally hyperbolic spacetime (M, g) , there exists a global timelike Killing vector field $\xi = \xi^\mu \partial_\mu$ for the metric g [66],

$$\mathcal{L}_\xi g_{\mu\nu}(x) := \xi^\alpha(x) \partial_\alpha g_{\mu\nu}(x) + g_{\alpha\nu}(x) \partial_\mu \xi^\alpha(x) + g_{\mu\alpha}(x) \partial_\nu \xi^\alpha(x) = 0. \quad (4.51)$$

This implies the existence of a coordinate system $\{x^\mu\}$ where the metric is time independent in which $\xi^\mu = (1, 0, 0, 0)$, i.e., $\mathcal{L}_\xi g_{\mu\nu}(x) = \partial_t g_{\mu\nu}(x) = 0$. The Killing vector field satisfies the following properties

- The global timelike Killing vector field $\mathcal{L}_\xi$ commutes with Dirac operator $\nabla_\mu - m = \partial_\mu - \Gamma_\mu - m$, i.e., $[\partial_t, \nabla_\mu - m]\psi(x) = 0$. This follows since the metric (and hence the Riemann-Christoffel connection) are time-independent in a stationary setting.
- The timelike Killing vector field $\mathcal{L}_\xi = \partial_t$ is anti-Hermitian $(f, \xi g) = -(\xi f, g)$, where f and g are pair of solutions to Dirac equation. Therefore, $\mathcal{L}_\xi$ can be diagonalised with imaginary eigenvalues, i.e. $\xi f_j = -i\omega f_j$, for some real ω . Note that the Lie derivative of a spinor field reads [67] $\mathcal{L}_\xi \psi := \xi^a \nabla_a \psi - \frac{1}{8} (\nabla_a \xi_b - \nabla_b \xi_a) \gamma^a \gamma^b \psi = \partial_t \psi$ in the coordinate system in which $\xi^\mu = (1, 0, 0, 0)$.
- Eigenfunctions of the Killing field $\mathcal{L}_\xi = \partial_t$ with distinct eigenvalues will be orthogonal, so $(u_i, u'_j) = 0 = (v_i, v'_j)$. Upon normalisation, the mode functions satisfy (4.36).
- The basis of solutions to Dirac equation is unique up to relabelling the mode solutions $u_i^{(s)}$ and $v_j^{(s)}$, which corresponds to a Bogolubov transformation with $\beta_{ji} = 0$. In that case the vacuum state $|0'\rangle$ is the same as $|0\rangle$.

In short, for a stationary spacetime background, there exists a preferred set of basis vector $\{u_j(x), v_j(x)\}$ such that $\mathcal{L}_\xi u_j(x) = -i\omega u_j(x)$ and $\mathcal{L}_\xi v_j(x) = i\omega v_j(x)$ where $\omega > 0$. We called $u_j(x)$ and $v_j(x)$, respectively, the positive and negative

frequency modes with respect to the global timelike Killing vector field. In the context of QFT mode with $\omega = 0$ are normally excluded on the grounds that they are not normalizable [86]. In the context of compactified spacetimes, these modes are identified as zero-frequency modes, which we discuss later.

Particle Creation in a Non-Stationary Spacetime

In a generic, non-stationary, globally hyperbolic spacetime, there is no global timelike Killing vector field. Consequently, we have no natural choice for a positive and a negative frequency modes and no natural Fock vacuum state. In that case, the choice of a basis of solutions to Dirac equation will be motivated by the particular questions we seek to answer.

A situation that arises especially in collapsing star geometry and cosmological settings in which the spacetimes in non-stationary but where there are two natural choices of a Fock vacuum is the case of spacetime (M, g) composed of three regions $M = M_- \cup M_0 \cup M_+$. The spacetime is assumed to be stationary in the past region M_- for $t < t_1$ and future region M_+ for $t > t_2$ but nonstationary in M_0 for $t_1 < t < t_2$. By hypothesis, in the past region, we have a Killing vector field ξ^p which allows us to decompose the basis of solutions to Dirac equation in terms of positive and negative frequency modes $\{u_i(x), v_i(x)\}$. Then we may choose a complete set of exact mode solutions to Dirac equation $\{U_i^{in}(x), V_i^{in}(x)\}$ such that

$$U_i^{in}(x) \xrightarrow[t < t_1]{} u_i(x), \quad V_i^{in}(x) \xrightarrow[t < t_1]{} v_i(x). \quad (4.52)$$

Subsequently the basis of solution $\{U_i^{in}(x), V_i^{in}(x)\}$ is normalised according to (4.36) on any time slice (Cauchy surface) since this is true in the asymptotic past ($t < t_1$), and Dirac norm (4.29) is independent of the time slice. Similarly, we may choose an exact basis of solution $\{U_i^{out}(x), V_i^{out}(x)\}$ to Dirac equation that becomes a pair of positive and negative frequency modes in the future region M_+ for $t > t_2$

$$U_i^{out}(x) \xrightarrow[t > t_2]{} f_i(x), \quad V_i^{out}(x) \xrightarrow[t > t_2]{} g_i(x), \quad (4.53)$$

where $\{f_i(x), g_i(x)\}$ is the standard basis to Dirac equation in the stationary region M_+ . The bases (4.52) and (4.53) do not have to coincide. An exact basis of solution to Dirac equation which reduces to a pair of positive and negative frequency modes in the past region M_- will not necessarily be a pair of positive and negative frequency modes in the future region M_+ . To illustrate this point we make use of the Bogolubov transformation discussed previously. The bases of solutions (4.52) and (4.53) are related by the Bogolubov transformation (4.40)

$$U_i^{out}(x) = \sum_j \alpha_{ij} U_j^{in}(x) + \beta_{ij} V_j^{in}(x), \quad (4.54a)$$

$$V_i^{out}(x) = \sum_j \beta_{ij}^* U_j^{in}(x) + \alpha_{ij}^* V_j^{in}(x). \quad (4.54b)$$

Equivalently, we have from (4.45)

$$U_i^{in}(x) = \sum_j \alpha_{ji}^* U_j^{out}(x) + \beta_{ji} V_j^{out}(x), \quad (4.55a)$$

$$V_i^{in}(x) = \sum_j \beta_{ji}^* U_j^{out}(x) + \alpha_{ji} V_j^{out}(x). \quad (4.55b)$$

Therefore if

$$\hat{\psi}(x) = \sum_j \hat{b}_j^{in} U_j^{in}(x) + \hat{d}_j^{in\dagger} V_j^{in}(x), \quad (4.56a)$$

$$= \sum_j \hat{b}_j^{out} U_j^{out}(x) + \hat{d}_j^{out\dagger} V_j^{out}(x), \quad (4.56b)$$

where from (4.47) we have

$$\hat{b}_j^{out} = \sum_i \alpha_{ji}^* \hat{b}_i^{in} + \beta_{ji}^* \hat{d}_i^{in\dagger}, \quad (4.57a)$$

$$\hat{d}_j^{out\dagger} = \sum_i \beta_{ji} \hat{b}_i^{in} + \alpha_{ji} \hat{d}_i^{in\dagger}, \quad (4.57b)$$

the vacua in the past and future regions $M_{\pm}$ are $|0_{in/out}\rangle$ obeying $\hat{b}_j^{in/out}|0_{in/out}\rangle = \hat{d}_j^{in/out}|0_{in/out}\rangle = 0, \forall j$. The particle and antiparticle operators for the number of j th-modes in the future region M_+ are respectively $N_j^{out(+)} := \hat{b}_j^{out\dagger} \hat{b}_j^{out}$ and $N_j^{out(-)} := \hat{d}_j^{out\dagger} \hat{d}_j^{out}$. Consequently, the expectation values for the number of U_j^{out} -mode particles

and V_j^{out} -mode particles present in the future region M_+ for $t > t_2$ when the incoming state is the in-vacuum are given by

$$\langle_{in} 0 | N_j^{out(+)} | 0_{in} \rangle = \langle_{in} 0 | \hat{b}_j^{out\dagger} \hat{b}_j^{out} | 0_{in} \rangle = \sum_{i,k} \beta_{ki} \beta_{ji}^* \delta_{ik} = \sum_i |\beta_{ji}|^2, \quad (4.58a)$$

$$\langle_{in} 0 | N_j^{out(-)} | 0_{in} \rangle = \langle_{in} 0 | \hat{d}_j^{out\dagger} \hat{d}_j^{out} | 0_{in} \rangle = \sum_i \beta_{ki} \beta_{ji}^* \delta_{ik} = \sum_i |\beta_{ji}|^2, \quad (4.58b)$$

which is to say that the in-vacuum associated with the modes $\{U_j^{in}, V_j^{in}\}$ contains $\sum_i |\beta_{ji}|^2$ particles in the modes $\{U_j^{out}, V_j^{out}\}$ at late times. Hence, the expected total number of spin- $\frac{1}{2}$ particles or antiparticles produced in the future region M_+ is

$$\sum_j \langle_{in} 0 | N_j^{out(+)} | 0_{in} \rangle = \sum_j \langle_{in} 0 | N_j^{out(-)} | 0_{in} \rangle = \text{Tr}(\beta\beta^\dagger). \quad (4.59)$$

In short, the incoming Fermionic Fock vacuum state in the past region M_- evolves into a state that is not a Fermionic Fock vacuum in the future region M_+ . Henceforth, the time-dependent geometry leads to creation of spin- $\frac{1}{2}$ particles according to the Bogolubov approach.

For a general curved spacetime with no asymptotically stationary regions, a definition of Fock particle states cannot be based on the above construction. Instead a method consisting of selecting exact mode solutions to the field equation that comes “closest” to the stationary background limit was first introduced by Parker [68–71]. It was designed to give an optimal definition of “physical particles” based on the so-called adiabatic vacuum states in an expanding cosmology, optimal in the sense that there is minimal particle creation by the expansion of the cosmological background model. Furthermore, in the limit of vanishing expansion rate, the notion of particle should reduce to the usual notion of particle in the stationary background spacetime [72]. However, the so-called adiabatic particle states are generally not of the Hadamard form [73]. Only the subclass of adiabatic states of infinite adiabatic order are Hadamard states [74]. In the following section, we will give a treatment for the conformally coupled spinor field and the conformally coupled scalar field in the spatially flat Robertson-Walker spacetimes. Our work will consider certain

classes of cosmological backgrounds wherein the mode functions are chosen to be of infinite adiabatic order and the associated two-point functions for the field possess the Hadamard form.

4.2 Quantum field theory in the spatially flat FRW spacetimes

In this section we build up on the work of [5, 70, 74–77] to derive a general formula for the free Dirac field Wightman two-point functions in the “in” and “out” vacua for certain classes of conformally flat cosmological spacetimes wherein the two-point functions have the Hadamard short-distance singularity structure [6]. In addition, we show that the formula for the two-point functions in the conformally spatially flat FRW spacetimes reduce to Minkowski spacetime results in the limit where the conformal scaling factor goes to unity.

4.2.1 Quantum Dirac field theory in the spatially flat FRW spacetimes

Consider a conformally coupled massive spinor field propagating in the spatially flat Robertson-Walker spacetime (FRW) in $d \geq 2$ -dimensions with metric

$$ds^2 = C(\eta) (d\eta^2 - \delta_{ij} dx^i dx^j) , \quad \eta_0 < \eta < \eta_1 . \quad (4.60)$$

For this metric, the nonzero Christoffel symbols are $\Gamma_{00}^0 = \frac{\dot{C}}{2C}$, $\Gamma_{ij}^0 = \frac{\dot{C}}{2C} \delta_{ij}$, $\Gamma_{0j}^i = \frac{\dot{C}}{2C} \delta_j^i$. A suitable choice of n-bein fields satisfying the constraints (4.24) are $e^b{}_\nu(\eta) = C^{1/2}(\eta) \delta_\nu^b$ and $e_a{}^\nu(\eta) = C^{-1/2}(\eta) \delta_a^\nu$. From $e_{b\nu}(\eta) = \eta_{bc} e^c{}_\nu(\eta) = C^{1/2}(\eta) \eta_{b\nu}$, we use expression (4.25) to obtain the spinorial connection

$$\Gamma_0(x) = 0 , \quad \Gamma_i(x) = -(1/4)(\dot{C}/C) \gamma^0 \gamma^i \delta_{ki} . \quad (4.61)$$

Dirac equation (4.26a) in the spatially flat FRW spacetimes reads,

$$0 = \left[i\gamma^0 \left(\partial_\eta + \frac{d-1}{4} \frac{\dot{C}}{C} \right) + i\gamma^k \partial_k - mC^{1/2}(\eta) \right] \psi(\eta, \mathbf{x}) := (i\mathcal{D} - \mu(\eta)) \psi(\eta, \mathbf{x}), \quad (4.62)$$

where the effective mass, $\mu(\eta) := mC^{1/2}(\eta)$, and the FRW Dirac slashed operator $\mathcal{D} := \gamma^0 \left(\partial_\eta + \frac{d-1}{4} \frac{\dot{C}}{C} \right) + \gamma^k \partial_k$. The FRW Dirac slashed operator satisfies the following properties

$$\mathcal{D}C^{(1-d)/4}(\eta) = 0, \quad (4.63a)$$

$$\mathcal{D} [C^{(1-d)/4}(\eta) F(\phi, \partial_\mu \phi, \dots)] = C^{(1-d)/4}(\eta) \gamma^\mu \partial_\mu F(\phi, \partial_\mu \phi, \dots), \quad (4.63b)$$

where ϕ is an arbitrary spinor field and F is a smooth function of the spinor field ϕ and its partial derivatives.

It is convenient to write $\psi(\eta, \mathbf{x})$ in the form of $\psi(\eta, \mathbf{x}) = (i\mathcal{D} + \mu(\eta)) \phi(\eta, \mathbf{x})$. Moreover by letting $\phi \rightarrow C^{(1-d)/4} \phi$ and using properties (4.63), we obtained the following ansatz for Dirac equation (4.62)

$$\psi(\eta, \mathbf{x}) = (i\mathcal{D} + \mu(\eta)) C^{(1-d)/4}(\eta) \phi(\eta, \mathbf{x}) = C^{(1-d)/4}(\eta) (i\gamma^\mu \partial_\mu + \mu(\eta)) \phi(\eta, \mathbf{x}). \quad (4.64)$$

The spinor field ϕ obeys the spinorial Klein Gordon equation, obtained by plugging (4.64) into Dirac equation (4.62),

$$(\square_\eta + \mu^2(\eta) - i\gamma^0 \mu'(\eta)) \phi(\eta, \mathbf{x}) = 0, \quad (4.65)$$

where $\square_\eta := \eta^{\mu\nu} \partial_\mu \partial_\nu$ and the prime stands for $\partial/\partial\eta$.

We seek solutions of the form $\phi_a(\eta, \mathbf{x}) = \psi_a(\mathbf{k}) \chi_k(\eta) \exp(\pm i\mathbf{k} \cdot \mathbf{x})$, $k = |\mathbf{k}|$, where $\psi_a(\mathbf{k})$ is the momentum space spinor field and the scalar quantity $\chi_k(\eta)$ are the mode functions. For solutions of this form Dirac ansatz (4.64) becomes

$$\psi_{a,\mathbf{k}}(\eta, \mathbf{x}) = C^{\frac{1-d}{4}}(\eta) (2\pi)^{\frac{1-d}{2}} (i\gamma^0 \partial_\eta \mp \mathbf{k} \cdot \boldsymbol{\gamma} + \mu(\eta)) \chi_k(\eta) \psi_a(\mathbf{k}) e^{\pm i\mathbf{k} \cdot \mathbf{x}}, \quad (4.66)$$

where $\mathbf{k} \cdot \boldsymbol{\gamma} := \delta_{ij} k^i \gamma^j$. Additionally, the spinorial Klein Gordon equation (4.65) reduces to

$$[\chi_k''(\eta) + (\omega_k^2(\eta) - i\gamma^0 \mu'(\eta)) \chi_k(\eta)] \psi_a(\mathbf{k}) = 0, \quad \omega_k^2(\eta) := |\mathbf{k}|^2 + \mu^2(\eta), \quad (4.67)$$

Without any loss of generality, we choose $\psi_a(\mathbf{k})$ to satisfy the condition $\gamma^0 \psi_a(\mathbf{k}) = \pm \psi_a(\mathbf{k})$. Consequently (4.67) reduces to the following second order differential equation for the mode function

$$\chi_k^{(\pm)''}(\eta) + (\omega_k^2(\eta) \mp i\mu'(\eta)) \chi_k^{(\pm)}(\eta) = 0, \quad (4.68)$$

where $\chi_k^\pm(\eta)$ are the solutions on ± 1 eigenspace of γ^0 . The zero-momentum basis spinors $u_a^s(0)$ and $v_a^s(0)$ in Minkowski spacetime satisfy the aforementioned choice of $\psi_a(\mathbf{k})$,

$$\gamma^0 u_a^s(0) = u_a^s(0), \quad \gamma^0 v_a^s(0) = -v_a^s(0), \quad (4.69)$$

where the helicity index s takes the values $s = 1, \dots, N_d/2$. We should note that $u_a^s(0)$ is associated with the set of linearly independent solutions of $\chi_k^{(+)}$ and v_a^s with the set of linearly independent solutions of $\chi_k^{(-)}$. Consequently, we can construct solutions to the squared Dirac equation (4.67) compatible with the ansatz (4.66) by choosing

$$\chi_k(\eta) \psi_a(\mathbf{k}) = \begin{cases} \chi_k^{(+)}(\eta) u_a^s(0), \\ \chi_k^{(-)}(\eta) v_a^s(0). \end{cases} \quad (4.70)$$

A complete set of mode solutions to Dirac equation (4.62) constructed from the ansatz (4.66) are given by

$$U_{a,\mathbf{k}}^{(s)}(\eta, \mathbf{x}) = C^{\frac{1-d}{4}}(\eta) (2\pi)^{\frac{1-d}{2}} (i\partial_\eta - \mathbf{k} \cdot \boldsymbol{\gamma} + \mu(\eta)) \chi_k^{(+)}(\eta) u_a^s(0) e^{i\mathbf{k} \cdot \mathbf{x}}, \quad (4.71a)$$

$$V_{a,\mathbf{k}}^{(s)}(\eta, \mathbf{x}) = C^{\frac{1-d}{4}}(\eta) (2\pi)^{\frac{1-d}{2}} (-i\partial_\eta + \mathbf{k} \cdot \boldsymbol{\gamma} + \mu(\eta)) \chi_k^{(-)}(\eta) v_a^s(0) e^{-i\mathbf{k} \cdot \mathbf{x}}. \quad (4.71b)$$

The factor of $(2\pi)^{\frac{1-d}{2}}$ in (4.71) is included so the mode solutions are orthonormal (see Appendix E for details)

$$\left(U_{\mathbf{k}}^{(s)}(x), U_{\mathbf{k}'}^{(s')}(x) \right) = \left(V_{\mathbf{k}}^{(s)}(x), V_{\mathbf{k}'}^{(s')}(x) \right) = \delta^{ss'} \delta^{d-1}(\mathbf{k} - \mathbf{k}'), \quad (4.72a)$$

$$\left(U_{\mathbf{k}}^{(s)}(x), V_{\mathbf{k}'}^{(s')}(x) \right) = 0, \quad (4.72b)$$

in Dirac norm (4.29)

$$\begin{aligned} \langle \psi, \phi \rangle &:= \int_{\Sigma} d^{d-1}x |g|^{1/2} g^{0\mu} \bar{\psi}(x) e^a_{\mu}(x) \gamma_a \phi(x), \\ &= C^{(d-1)/2}(\eta) \int_{\Sigma} d^{d-1}x \psi^{\dagger}(x) \phi(x), \end{aligned} \quad (4.73)$$

provided we choose to normalise the mode function via

$$W[\chi_k^{(-)}, \chi_k^{(+)*}] := \chi_k^{(-)} \partial_{\eta} \chi_k^{(+)*} - \chi_k^{(+)*} \partial_{\eta} \chi_k^{(-)} = ik^{-1}. \quad (4.74)$$

Moreover, collecting (4.74) and (E.8d), we obtain the following identity

$$\left| \chi_k^{(+)}(\eta) \right|^2 + \left| \chi_k^{(-)}(\eta) \right|^2 = 1. \quad (4.75)$$

For completeness and since they have not been discussed in the literature [78], we wish to mention the following set of formal WKB-type solutions (WKB-type ansatz) for the mode functions (4.68) which can be used to define adiabatic mode functions

$$\chi_k^{(+)} = C_k^{+} \left(W_k^{(+)} \right)^{-\frac{1}{2}} \exp \left[-i \int^{\eta} \left(W_k^{(+)} - i \frac{\mu'}{2W_k^{(+)}} \right) \right], \quad (4.76a)$$

$$\chi_k^{(-)} = C_k^{-} \left(W_k^{(-)} \right)^{-\frac{1}{2}} \exp \left[-i \int^{\eta} \left(W_k^{(-)} + i \frac{\mu'}{2W_k^{(-)}} \right) \right], \quad (4.76b)$$

where $W_k^{(+)}$ and $W_k^{(-)}$ satisfy the nonlinear differential equations

$$W_k^{(+2)} = \omega_k^2 - \frac{1}{2} \left(\frac{W_k^{(+)''}}{W_k^{(+)}} - \frac{3}{2} \frac{W_k^{(+)'^2}}{W_k^{(+)2}} \right) - \frac{1}{2} \left(\frac{\mu''}{W_k^{(+)}} - \frac{1}{2} \frac{\mu'^2}{W_k^{(+)2}} - 2\mu' \frac{W_k^{(+)'}}{W_k^{(+)2}} \right), \quad (4.76c)$$

$$W_k^{(-2)} = W_k^{(+2)}(\mu \rightarrow -\mu), \quad (4.76d)$$

where $C_k^\pm$ are constants that must be chosen so that $\chi_k^\pm(\eta) = O(k^{-1})$, $\forall k$ and $\forall \eta$ [79]. In addition, (4.76) is defined to an arbitrary phase factor resulting from specifying an arbitrary lower limit to the integral.

Our choice of mode functions (χ_k) will be adapted to the following results so the vacuum two-point function satisfies the Hadamard condition. Hollands [74] has demonstrated that in the spatially flat FRW spacetime the adiabatic states of infinite order for Dirac field are of the Hadamard type. Therefore, a natural choice of mode functions are those associated with the infinite adiabatic order vacuum. Moreover, as we will demonstrate in Section 4.3.1 the adiabatic vacuum of infinite order is also identical to the conformal vacuum in the massless limit of Dirac field.

In order to properly define this choice of mode functions, it is useful to introduce the following definitions

$$\begin{cases} \omega_{in/out} := \omega_k(\eta \rightarrow \eta_0/\eta_1) & , \\ \mu_{in/out} := \mu(\eta \rightarrow \eta_0/\eta_1) & , \end{cases} \quad (4.77)$$

where η_0 and η_1 were introduced in (4.60). Note that $\omega_{in/out}$ depends on the momenta and can be time-dependent. For example, in Milne spacetime specified by the conformal factor $C(\eta) = e^{2a\eta}$, where a is a positive constant, $(\eta_0, \eta_1) = (-\infty, \infty)$, thus $\omega_{in} = |\mathbf{k}|$ and $\omega_{out} \approx me^{a\eta}$. In half of de Sitter manifold specified by $C(\eta) = (H\eta)^{-2}$, where H is the Hubble constant, $(\eta_0, \eta_1) = (-\infty, 0)$, thus $\omega_{in} = |\mathbf{k}|$ and $\omega_{out} \approx m(-H\eta)^{-1}$.

Our analysis will be restricted to the following class of spatially flat FRW models where there exist exact mode functions:

- (i) FRW models that are asymptotically stationary.
- (ii) FRW models that are asymptotically non-stationary but which nevertheless is undergoing a slow expansion as $\eta \rightarrow \pm\infty$ with the requirement

$$\frac{d^n}{d\eta^n} \frac{C'}{C} \longrightarrow 0, \quad n > 0. \quad (4.78)$$

For this class of spacetimes we may choose as positive frequency (of infinite adiabatic order) the exact solutions to the mode functions (4.68) with the adiabatic asymptotic behaviour which follows from (4.76)

$$\chi_k^{(\pm)in/out}(\eta) \xrightarrow{\eta \rightarrow \eta_0/\eta_1} \frac{1}{\sqrt{2k}} \sqrt{\frac{\omega_{in/out} \mp \mu_{in/out}}{\omega_{in/out}}} \exp\left(-i \int_{\eta_0}^{\eta} \omega_{in/out} d\eta'\right), \quad (4.79)$$

where we have chosen the constant $C_k^{\pm} = (\sqrt{2k})^{-1}$ in (4.76).

Note that using (4.68), the $\chi_k^{(+)}$ -ODE is obtained by taking the complex conjugate of the $\chi_k^{(-)}$ -ODE. Therefore the mode functions $\chi_k^{(+)}$ and $\chi_k^{(-)}$ satisfying (4.68) are not all independent. We shall denote the mode solutions associated with the positive frequency mode functions, $\chi_k^{(\pm)in/out}$, by $U_k^{(s)in/out}(x)$ and those associated with the negative frequency mode functions, $\chi_k^{(\mp)in/out*}$, by $V_k^{(s)in/out}(x)$. Consequently, the relation between the *in* and *out* bases of solutions to (4.68) defines the following Bogolubov transformations

$$\chi_k^{(\pm)in}(\eta) = \alpha_k \chi_k^{(\pm)out}(\eta) \pm \beta_k \chi_k^{(\mp)out*}(\eta). \quad (4.80)$$

Using either the Wronskian (4.74) or the identity (4.75), we obtain the normalisation condition

$$|\alpha_k|^2 + |\beta_k|^2 = 1. \quad (4.81)$$

The interpolating field satisfying Dirac equation (4.62) can be expressed as an integral expansion of either the “*in-modes*” or “*out-modes*”

$$\psi(x) = \int d^{d-1}k \sum_s \left(b_s^{in/out}(\mathbf{k}) U_{\mathbf{k}}^{(s)in/out}(x) + d_s^{in/out\dagger}(\mathbf{k}) V_{\mathbf{k}}^{(s)in/out}(x) \right). \quad (4.82)$$

In compliance with the CCRs (4.35), we have (see Appendix E for details)

$$\{\psi_a(\eta, \mathbf{x}), \psi_b(\eta, \mathbf{x}')\} = \{\psi_a^\dagger(\eta, \mathbf{x}), \psi_b^\dagger(\eta, \mathbf{x}')\} = 0, \quad (4.83a)$$

$$\{\psi_a(\eta, \mathbf{x}), \psi_b^\dagger(\eta, \mathbf{x}')\} = C^{(1-d)/2}(\eta) \delta_{ab} \delta^{d-1}(\mathbf{x}, \mathbf{x}'), \quad (4.83b)$$

which simplify into an algebra in the expansion coefficients given by

$$\{b_s^{in/out}(\mathbf{k}), b_{s'}^{in/out\dagger}(\mathbf{k}')\} = \{d_s^{in/out}(\mathbf{k}), d_{s'}^{in/out\dagger}(\mathbf{k}')\} = \delta_{ss'} \delta^{d-1}(\mathbf{k} - \mathbf{k}'). \quad (4.84)$$

Working in the Heisenberg picture, the initial vacuum state defined by $b_s^{in}(\mathbf{k})|0_{in}\rangle = d_s^{in}(\mathbf{k})|0_{in}\rangle = 0$, $\forall \mathbf{k}, \forall s$ is the normalized infinite adiabatic order vacuum state at early times associated with the mode functions $\chi_k^{(\pm)in}(\eta)$, which describes the situation where no spin- $\frac{1}{2}$ particles or antiparticles are present at early times. Similarly, the vacuum state in the remote future defined by $b_s^{out}(\mathbf{k})|0_{out}\rangle = d_s^{out}(\mathbf{k})|0_{out}\rangle = 0$, $\forall \mathbf{k}, \forall s$ is the normalized infinite adiabatic order vacuum at late times associated with the mode function $\chi_k^{(\pm)out}(\eta)$, which describes the situation when no spin- $\frac{1}{2}$ particles or antiparticles are present at late times. And according to [74], these vacuum states have the Hadamard form.

4.2.2 Wightman functions

In this section we derive the formulas for the ‘in’ and ‘out’ vacua Wightman two-point functions for the spinor field wherein the two-point functions have the required Hadamard form.

The spinor field $\psi(x)$ in (4.82) along with Dirac adjoint $\bar{\psi}(x) \equiv \psi(x)^\dagger \gamma^0$ may be decomposed into their positive and negative frequency components as

$$\psi(x) = \psi^+(x) + \psi^-(x) , \quad (4.85a)$$

$$\bar{\psi}(x) = \bar{\psi}^+(x) + \bar{\psi}^-(x) , \quad (4.85b)$$

where

$$\psi^+(x) = \int d^{d-1}k \sum_s b_s^{in/out}(\mathbf{k}) U_{\mathbf{k}}^{(s)in/out}(x) , \quad (4.86a)$$

$$\psi^-(x) = \int d^{d-1}k \sum_s d_s^{in/out\dagger}(\mathbf{k}) V_{\mathbf{k}}^{(s)in/out}(x) , \quad (4.86b)$$

$$\bar{\psi}^+(x) = \int d^{d-1}k \sum_s d_s^{in/out}(\mathbf{k}) \bar{V}_{\mathbf{k}}^{(s)in/out}(x) , \quad (4.86c)$$

$$\bar{\psi}^-(x) = \int d^{d-1}k \sum_s b_s^{in/out\dagger}(\mathbf{k}) \bar{U}_{\mathbf{k}}^{(s)in/out}(x) . \quad (4.86d)$$

The fields ψ and $\bar{\psi}$ are decomposed so the following relations hold

$$\psi^+|0_{in/out}\rangle = 0 = \langle 0_{in/out}|\psi^- , \quad (4.87a)$$

$$\bar{\psi}^+|0_{in/out}\rangle = 0 = \langle 0_{in/out}|\bar{\psi}^- . \quad (4.87b)$$

The *in* and *out* Wightman two-point functions, $S^{in/out(\pm)}(x, x')$, in the conformal FRW spacetimes are given by

$$\begin{aligned} S_{ab}^{in/out(+)}(x, x') &:= \langle 0_{in/out}|\psi_a(x)\bar{\psi}_b(x')|0_{in/out}\rangle = \{\psi_a^+(x), \bar{\psi}_b^-(x')\} \\ &= \int d^{d-1}k \sum_s U_{a,\mathbf{k}}^{(s)in/out}(x) \bar{U}_{b,\mathbf{k}}^{(s)in/out}(x') , \end{aligned} \quad (4.88a)$$

$$\begin{aligned} S_{ab}^{in/out(-)}(x, x') &:= \langle 0_{in/out}|\bar{\psi}_b(x')\psi_a(x)|0_{in/out}\rangle = \{\psi_a^-(x), \bar{\psi}_b^+(x')\} \\ &= \int d^{d-1}k \sum_s V_{a,\mathbf{k}}^{(s)in/out}(x) \bar{V}_{b,\mathbf{k}}^{(s)in/out}(x') . \end{aligned} \quad (4.88b)$$

Collecting (4.71), (4.88), and using the aforementioned choice of positive and negative frequency modes along with suppressing the matrix subscripts, we obtain the following formula for the spinor field two-point functions

$$\begin{aligned} S^{in/out(+)}(\eta, \mathbf{x}, \eta', \mathbf{x}') &= -[C(\eta)C(\eta')]^{\frac{1-d}{4}} \int \frac{d^{d-1}k}{(2\pi)^{d-1}} (i\cancel{\partial} + \mu(\eta)) \frac{I + \gamma_0}{2} (i\cancel{\partial}' - \mu(\eta')) \\ &\quad \chi_k^{(+in/out)}(\eta) \chi_k^{(+in/out*)}(\eta') e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}, \end{aligned} \quad (4.89a)$$

$$\begin{aligned} S^{in/out(-)}(\eta, \mathbf{x}, \eta', \mathbf{x}') &= [C(\eta)C(\eta')]^{\frac{1-d}{4}} \int \frac{d^{d-1}k}{(2\pi)^{d-1}} (i\cancel{\partial} + \mu(\eta)) \frac{I - \gamma_0}{2} (i\cancel{\partial}' - \mu(\eta')) \\ &\quad \chi_k^{(+in/out*)}(\eta) \chi_k^{(+in/out)}(\eta') e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}, \end{aligned} \quad (4.89b)$$

where $\cancel{\partial} := \gamma^\mu \partial_\mu$ and $\cancel{\partial}' := \gamma^\mu \partial_{\mu'}$. Moreover, in $d > 3$ dimensions, the Wightman two-point functions (4.89) for the spinor field can be rewritten as

$$\begin{aligned} S^{in/out(+)}(\eta, \mathbf{x}, \eta', \mathbf{x}') &= -[C(\eta)C(\eta')]^{\frac{1-d}{4}} (i\cancel{\partial} + \mu(\eta)) \frac{I + \gamma_0}{2} (i\cancel{\partial}' - \mu(\eta')) \times \\ &\quad \tilde{G}^{in/out(+)}(\eta, \mathbf{x}, \eta', \mathbf{x}'), \end{aligned} \quad (4.90a)$$

$$\begin{aligned} S_{ab}^{in/out(-)}(\eta, \mathbf{x}, \eta', \mathbf{x}') &= [C(\eta)C(\eta')]^{\frac{1-d}{4}} (i\cancel{\partial} + \mu(\eta)) \frac{I - \gamma_0}{2} (i\cancel{\partial}' - \mu(\eta')) \times \\ &\quad \tilde{G}^{in/out(+)}(\eta', \mathbf{x}', \eta, \mathbf{x}), \end{aligned} \quad (4.90b)$$

where

$$\begin{aligned}\tilde{G}^{in/out(+)}(\eta, \mathbf{x}, \eta', \mathbf{x}') &:= \int \frac{d^{d-1}k}{(2\pi)^{d-1}} \chi_k^{(+in/out)}(\eta) \chi_k^{(+in/out*)}(\eta') e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')} , \\ &= (2\pi)^{\frac{1-d}{2}} r^{\frac{3-d}{2}} \int_0^\infty dk k^{(d-1)/2} J_{(d-3)/2}(kr) \chi_k^{(+in/out)}(\eta) \chi_k^{(+in/out*)} , \\ &\equiv \tilde{G}_{FRW}^{in/out(+)}(\eta, \eta', r) ,\end{aligned}\tag{4.91}$$

where $r \doteq |\mathbf{x} - \mathbf{x}'|$, and $J_{(d-3)/2}(kr)$ are the Bessel functions of the first kind. In $d = 4$ dimensions, $J_{1/2}(kr) = \sqrt{2/(\pi kr)} \sin(kr)$. Then formula (4.91) yields

$$\tilde{G}^{in/out(+)}(\eta, \mathbf{x}, \eta', \mathbf{x}') = \frac{1}{2\pi^2 r} \int_0^\infty dk k \sin(kr) \chi_k^{(+in/out)}(\eta) \chi_k^{(+in/out*)} . \tag{4.92}$$

Since our choice of *in* and *out* vacua of infinite adiabatic order are of Hadamard type, then the short-distance singularity-structure of (4.92) is of the Hadamard form [80]

$$\tilde{G}^{in/out(+)}(x, x') = \lim_{\epsilon \rightarrow 0} \frac{U(x, x')}{\sigma_\epsilon(x, x')} + V(x, x') \ln(\sigma_\epsilon(x, x')) + H^{in/out}(x, x') , \tag{4.93}$$

where $2\sigma_\epsilon(x, x')^2$ is the squared geodesic distance (assumed spacelike) between x and x' , U and V are geometrical quantities that are determined by the spacetime metric and the wave equation of the scalar field while $H_{in/out}$ carries information about the *in* and *out* vacua states. (For a rigorous definition of these functions, see [81]).

Finally, note that in the limit $C(\eta) = 1$, collecting (4.77), (4.79) and (4.89) we obtain

$$S^{in/out(+)}(\eta, \mathbf{x}, \eta', \mathbf{x}') = \int \frac{d^{d-1}k}{2\omega_{in/out}(2\pi)^{d-1}} (\not{k} + \mu_{in/out}) e^{-ik(x-x')} \tag{4.94a}$$

$$= (i\gamma^\mu \partial_{x^\mu} + \mu_{in/out}) \int \widetilde{dk} e^{-ik(x-x')} , \tag{4.94b}$$

$$S^{in/out(-)}(\eta, \mathbf{x}, \eta', \mathbf{x}') = \int \frac{d^{d-1}k}{2\omega_{in/out}(2\pi)^{d-1}} (\not{k} - \mu_{in/out}) e^{ik(x-x')} \tag{4.94c}$$

$$= -(i\gamma^\mu \partial_{x^\mu} + \mu_{in/out}) \int \widetilde{dk} e^{-ik(x'-x)} , \tag{4.94d}$$

²In a Lorentzian manifold, the geodesic distance will vanish not only when they are coincident but also when they are null separated. Therefore, we include a small imaginary part and take the limit as it goes to zero by defining $\sigma_\epsilon(x, x') := \sigma(x, x') + 2i\epsilon(t - t') + \epsilon^2$, where t is the timelike coordinate, and the limit as $\epsilon \rightarrow 0^+$ is assumed.

where $k(x - x') := \omega_{in/out}(\eta - \eta') - \mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')$. In order to obtain the results given in (4.94), we used the following relations

$$(\not{k} + \mu_{in/out}) \frac{I + \gamma_0}{2} (\not{k} + \mu_{in/out}) = (\omega_{in/out} + \mu_{in/out}) (\not{k} + \mu_{in/out}) , \quad (4.95a)$$

$$(\not{k} + \mu_{in/out}) \frac{I - \gamma_0}{2} (\not{k} + \mu_{in/out}) = (\omega_{in/out} + \mu_{in/out}) (\not{k} - \mu_{in/out}) , \quad (4.95b)$$

whose proof depends on the following identities

$$\not{k} \gamma^0 \not{k} = \gamma^0 \not{k}^\dagger \not{k} = 2k_0 \not{k} - \gamma^0 k_\mu k^\mu = 2k_0 \not{k} - \mu_{in/out}^2 \gamma^0 , \quad k^\mu := (\omega_{in/out}, \mathbf{k}) . \quad (4.96a)$$

The second equality given in identity (4.96a) follows from $\not{k}^\dagger + \not{k} = 2\gamma^0 k_0$. Note that (4.94) is what we expect to obtain in the flat spacetime as was shown in (2.17).

4.2.3 Four-point correlation function: $W^{(2,\bar{2})}$

Because we decompose the spinor field ψ and Dirac adjoint field $\bar{\psi}$ so that relations (4.87) still hold, we obtained the same formula for the two-point correlation function $W^{(2,\bar{2})}(x, x')$ as we computed in our early paper [17],

$$\begin{aligned} W_{in}^{(2,\bar{2})}(x, x') &:= \langle 0_{in} | \bar{\psi}(x) \psi(x) \bar{\psi}(x') \psi(x') | 0_{in} \rangle , \\ &= \text{Tr}[S^{in(+)}(x, x') S^{in(-)}(x', x)] + \text{Tr}[S^{in(-)}(x, x)] \text{Tr}[S^{in(-)}(x', x')] , \end{aligned} \quad (4.97)$$

and with a similar formula in terms of the *out* vacuum.

4.2.4 Quantum scalar field theory in the spatially flat FRW spacetimes

In this section we build up on the work of [5, 75, 82, 83] to derive a general formula for the scalar field Wightman two-point function in the “in” and “out” vacua for certain classes of conformally flat cosmological spacetimes wherein the two-point function has the Hadamard short-distance singularity structure [6]. To adapt Duncan’s results to this section, we use the mostly plus sign convention, $(-, +, +, \dots, +)$.

Consider a real, conformally coupled, massive scalar field propagating on a curved background of dimensions $d \geq 2$, specified by the spacetime metric $g_{\mu\nu}(x)$. The dynamics of this scalar field is governed by the variation of the action functional

$$S = -\frac{1}{2} \int d^d x |g|^{1/2} [g^{\mu\nu}(x) \partial_\mu \phi(x) \partial_\nu \phi(x) + [m^2 + \xi R(x)] \phi(x)] , \quad (4.98)$$

where $\xi \equiv (d-2)/4(1-d)$. This leads to the wave equation

$$(\square_g - m^2 - \xi R(x)) \phi(x) = 0 , \quad (4.99)$$

where the Laplace-Beltrami operator is given by

$$\square_g \phi := g^{\mu\nu} \nabla_\mu \nabla_\nu \phi = |g|^{-\frac{1}{2}} \partial_\mu [|g|^{\frac{1}{2}} g^{\mu\nu}(x) \partial_\nu \phi(x)] , \quad (4.100)$$

and where the positive parameter m is the rest mass of the field's quanta, $R(x)$ is the Ricci scalar curvature, and ξ is the coupling constant between the scalar field and the scalar curvature.

Specifically, let us consider a linear scalar field propagating in the conformally flat Robertson-Walker (FRW) cosmological spacetimes with metric $g_{\mu\nu} = C(\eta)\eta_{\mu\nu}$, where $\eta_0 < \eta < \eta_1$. For this class of spacetimes we have $|g|^{1/2} = C^{d/2}$, and Ricci curvature given by

$$R = (1-d) \frac{1}{C} \left[\frac{C''}{C} + \left(\frac{d}{4} - \frac{3}{2} \right) \frac{C'^2}{C^2} \right] , \quad (4.101)$$

where the prime stands for $d/d\eta$. The scalar field equation of motion (4.99) reads

$$\square_\eta \phi + \frac{2-d}{2} \frac{C'}{C} \phi' - \mu^2 \phi - \frac{d-2}{4} \left[\frac{C''}{C} + \left(\frac{d}{4} - \frac{3}{2} \right) \left(\frac{C'}{C} \right)^2 \right] \phi = 0 , \quad (4.102)$$

where $\square_\eta := \eta^{\mu\nu} \partial_\mu \partial_\nu$ and the time-dependent effective mass is defined by $\mu^2(\eta) := m^2 C(\eta)$. After rescaling the field via $\phi = C^{(2-d)/4} \psi$, we obtain

$$(\square_\eta - \mu^2(\eta)) \psi = 0 . \quad (4.103)$$

The mode solutions to (4.99), using the aforementioned rescaling, are of the form

$$U_{\mathbf{k}}(\eta, \mathbf{x}) = C^{(2-d)/4}(\eta) (2\pi)^{(1-d)/2} \chi_k(\eta) \exp(i\mathbf{k} \cdot \mathbf{x}) , \quad (4.104)$$

where the mode function $\chi_k(\eta)$ satisfies the following harmonic differential equation

$$\chi_k''(\eta) + \omega_k^2(\eta)\chi_k(\eta) = 0, \quad \omega_k^2(\eta) := |\mathbf{k}|^2 + \mu^2(\eta) \geq 0, \quad k = |\mathbf{k}|. \quad (4.105)$$

The mode solutions are orthonormal in the scalar product given by

$$(U_{\mathbf{k}}(x), U_{\mathbf{k}'}(x)) = -i \int d^{d-1}x |g|^{1/2} g^{0\nu} U_{\mathbf{k}}(x) \overleftrightarrow{\partial}_\nu U_{\mathbf{k}'}^*(x) = \delta^{d-1}(\mathbf{k} - \mathbf{k}'), \quad (4.106)$$

provided the set of mode functions χ_k are normalized via

$$W[\chi_k, \chi_k^*] := \chi_k \partial_\eta \chi_k^* - \chi_k^* \partial_\eta \chi_k = i. \quad (4.107)$$

Our choice of mode functions (χ_k) will be adapted to the following results so the vacuum two-point function satisfying the Hadamard condition. Pirk [82] has proven that in a spatially flat FRW spacetime the adiabatic vacuum of a quantum scalar field is Hadamard if and only if it is of infinite order. Moreover, in de Sitter spacetime, the adiabatic vacuum of infinite order is identical (up to the uncertainties inherent to the adiabatic vacuum) to the Bunch-Davies vacuum or Euclidean vacuum (see Chung et al. [84]). We will also show in Section 4.3.1 that in a spatially flat FRW spacetime that the adiabatic vacuum of infinite order is identical to the conformal vacuum in the massless limit. Therefore, a natural choice of mode functions are those associated with the infinite adiabatic order vacuum.

For the class of cosmological spacetimes discussed in Section 4.2.1 (right after (4.77)), we may choose as positive frequency (of infinite adiabatic order) the exact solutions to the mode functions (4.105) with the adiabatic asymptotic behaviour [85]

$$\chi_k^{in/out}(\eta) \xrightarrow{\eta \rightarrow \pm\infty} \frac{1}{\sqrt{2\omega_{in/out}}} \exp\left(-i \int_{\eta_0}^{\eta} \omega_{in/out} d\eta'\right), \quad (4.108)$$

defined to within an arbitrary phase factor. We shall denote the corresponding mode solutions by $U_k^{in/out}(x)$. The relation between the *in* and *out* bases of solutions to (4.105) defines a Bogolubov transformation

$$\chi_k^{in}(\eta) = \alpha_k \chi_k^{out}(\eta) + \beta_k \chi_k^{out*}(\eta). \quad (4.109)$$

From the Wronskian (4.107), we obtain the normalization condition

$$|\alpha_k|^2 - |\beta_k|^2 = 1. \quad (4.110)$$

The interpolating field satisfying the scalar equation (4.99) can be expressed as an integral expansion of either the “*in-modes*” or “*out-modes*”

$$\phi(x) = \int d^{d-1}k \left(a_{in/out}(\mathbf{k}) U_{\mathbf{k}}^{in/out}(x) + a_{in/out}^\dagger(\mathbf{k}) U_{\mathbf{k}}^{in/out*}(x) \right), \quad (4.111a)$$

$$\equiv \phi^{in/out+}(x) + \phi^{in/out-}(x). \quad (4.111b)$$

In compliance with the quantization procedure, the coefficients of the mode expansion are operators whose only nonvanishing commutator relations are

$$[a_{in/out}(\mathbf{k}), a_{in/out}^\dagger(\mathbf{k}')] = \delta^{d-1}(\mathbf{k} - \mathbf{k}'). \quad (4.112)$$

Working in the Heisenberg picture, the initial vacuum state defined by $a_{in}(\mathbf{k})|0_{in}\rangle$, $\forall \mathbf{k}$, is the normalized infinite adiabatic order vacuum at early times associated with the mode functions $\chi_k^{in}(\eta)$. Similarly, the vacuum state in the remote future defined by $a_{out}(\mathbf{k})|0_{out}\rangle$, $\forall \mathbf{k}$, is the normalized infinite adiabatic order vacuum at late times associated with the mode function $\chi_k^{out}(\eta)$. And according to the proof given in [82], these vacuum states are of Hadamard type. Moreover, the number of particles in the k th mode detected at late times in the infinite adiabatic *in* vacuum state is given by [85]

$$\langle 0_{in} | N^{out}(\mathbf{k}) | 0_{in} \rangle := \langle 0_{in} | a_{out}^\dagger(\mathbf{k}) a_{out}(\mathbf{k}) | 0_{in} \rangle = |\beta_k|^2 \delta^{d-1}(0) \neq 0, \quad (4.113)$$

which is infinite. Note the divergent factor $\delta^{d-1}(0)$ is relating to the infinite volume of space. The meaningful quantities are the mean number density in the $\mathbf{k}$ mode given by $n_{\mathbf{k}} = |\beta_k|^2$

If the quantized scalar field is initially in the infinite-order adiabatic *in* vacuum state at the instant η_0 , then the comoving number of particle per unit proper volume at late times, $\eta \rightarrow \eta_1$, (i.e. the mean density of all particles produced in the out

region) is obtained by integrating the mean number density over all k -modes

$$n_d := [4\pi^2 C(\eta)]^{\frac{1-d}{2}} \int d^{d-1}k \langle 0_{in} | N^{out}(\mathbf{k}) | 0_{in} \rangle \quad (4.114)$$

$$= \frac{(d-1) [4\pi C(\eta)]^{\frac{1-d}{2}}}{\Gamma\left(\frac{d+1}{2}\right)} \int_0^\infty dk k^{d-2} |\beta_k|^2. \quad (4.115)$$

thus the density of particle created is finite. And their energy per unit proper volume is given by [5]

$$\rho_d = \frac{(d-1) (4\pi)^{\frac{1-d}{2}} C^{\frac{-d}{2}}(\eta)}{\Gamma\left(\frac{d+1}{2}\right)} \int_0^\infty dk k^{d-2} \omega_{out} |\beta_k|^2. \quad (4.116)$$

In addition, using the formalism above, we obtain the following general formula for the Wightman two-point function for this class of cosmological backgrounds

$$\mathcal{W}_{in/out}(\eta, \mathbf{x}, \eta', \mathbf{x}') := \langle 0_{in/out} | \phi(\eta, \mathbf{x}) \phi(\eta', \mathbf{x}') | 0_{in/out} \rangle \quad (4.117a)$$

$$= \int d^{d-1}k U_{\mathbf{k}}^{in/out}(x) U_{\mathbf{k}}^{in/out*}(x') \quad (4.117b)$$

$$= (2\pi)^{(1-d)/2} [C(\eta)C(\eta')]^{(2-d)/4} r^{(3-d)/2} \times \int_0^\infty dk k^{(d-1)/2} J_{(d-3)/2}(kr) \chi_k^{in/out}(\eta) \chi_k^{in/out*}(\eta') \quad (4.117c)$$

$$:= G_{FRW}^+(\eta, \eta', r),$$

where $r \doteq |\mathbf{x} - \mathbf{x}'|$, and $J_{(d-3)/2}(kr)$ are the Bessel functions of the first kind. For $d = 4$ dimensions $J_{1/2}(kr) = \sqrt{2/(\pi kr)} \sin(kr)$. In this case, the formula for the Wightman two-point function (4.117a) reads

$$\mathcal{W}_{in/out}(\eta, \mathbf{x}, \eta', \mathbf{x}') = \frac{[C(\eta)C(\eta')]^{-1/2}}{2\pi^2 r} \int_0^\infty dk k \sin(kr) \chi_k^{in/out}(\eta) \chi_k^{in/out*}(\eta'). \quad (4.118)$$

Since our choice of *in* and *out* vacua of infinite adiabatic order are of Hadamard type, the short-distance singularity-structure of (4.118) is of the Hadamard form [80]

$$\mathcal{W}_{in/out}(x, x') = \lim_{\epsilon \rightarrow 0} \frac{U(x, x')}{\sigma_\epsilon(x, x')} + V(x, x') \ln(\sigma_\epsilon(x, x')) + H_{in/out}(x, x'), \quad (4.119)$$

where $2\sigma(x, x')$ is the squared geodesic distance between x and x' , U and V are geometrical quantities that are determined by the spacetime metric and the wave

equation of the scalar field while $H_{in/out}$ carries information about the *in* and *out* vacua states. (For rigorous definition of these functions, see [81]).

4.2.5 Quantum scalar field theory in the $(1+1)$ -dimensional compactified, spatially flat FRW spacetimes

Consider a conformally coupled massive scalar field propagating in the $(1+1)$ -dimensional compactified, spatially flat FRW spacetimes with closed spatial section, i.e. the spacetime point (η, x) and $(\eta, x + L)$ are identified, with $L > 0$. The field ϕ is periodic or antiperiodic as $(\eta, x) = (\eta, x + L)$. The mode functions $\chi_k(\eta)$ of (4.105) takes on discrete index $n \in \mathbb{Z}$ according to

$$k_n := \begin{cases} 2\pi n/L & \text{for periodic or untwisted scalar field,} \\ 2\pi(n + \frac{1}{2})/L & \text{for antiperiodic or twisted scalar field,} \end{cases} \quad (4.120a)$$

$$\omega_n(\eta) := (k_n^2 + \mu^2(\eta))^{1/2}. \quad (4.120b)$$

The mode expansion of the field operator (4.2.4) now reads

$$\hat{\phi}(\eta, x) = \sum_n \left(\hat{a}_{in/out}(n) U_n^{in/out}(\eta, x) + \hat{a}_{in/out}^\dagger(n) U_n^{in/out*}(\eta, x) \right), \quad (4.121)$$

where the normal modes (4.104) are

$$U_n(\eta, x) = (L)^{-1/2} \chi_n(\eta) \exp(ik_n x). \quad (4.122)$$

They are normalised via

$$(U_n, U_m) = -i \int_0^L dx U_n(x, \eta) \overleftrightarrow{\partial}_\eta U_m^*(x, \eta) = \delta_{n,m}, \quad (4.123)$$

provided the set of mode functions χ_n are normalized via

$$W[\chi_n, \chi_n^*] := \chi_n \partial_\eta \chi_n^* - \chi_n^* \partial_\eta \chi_n = i. \quad (4.124)$$

The mode functions $\chi_n(\eta)$ satisfies the harmonic differential equation (4.105) with discrete labels

$$\ddot{\chi}_n(\eta) + \omega_n^2(\eta) \chi_n(\eta) = 0. \quad (4.125)$$

The positive frequency mode functions are chosen according to the adiabatic asymptotic behaviour given in (4.108) (with discrete index) for the $(1 + 1)$ -dimensional compactified version of classes (i) and (ii) of cosmological backgrounds discussed in Section 4.2.4.

The annihilation and creation operators satisfy the commutator relations in (4.112) for $d = 2$ dimension with discrete labels

$$[\hat{a}_{in/out}(n), \hat{a}_{in/out}^\dagger(m)] = \delta_{n,m} \mathbb{1} , \quad (4.126)$$

which follow from the equal-time canonical commutation relations (CCR's) of the quantum scalar field operator and its momentum conjugate operator $\hat{\pi}(\eta, x) = \partial_\eta \hat{\phi}(\eta, x)$

$$[\hat{\phi}(\eta, x), \hat{\pi}(\eta, x')] = i\delta(x - x') \mathbb{1} , \quad (4.127)$$

where all other CCR's vanish. The adiabatic in-vacuum and out-vacuum states of infinite order are defined, respectively, as $a_{in/out}(n)|0_L, in/out\rangle = 0, \forall n$, and describe the situation where no quantum scalar particles are present at earlier and late times. The $(1 + 1)$ -dimensional Wightman two-point function is given by

$$\begin{aligned} G^{in/out+}(\eta, x; \eta', x') &:= \langle 0_L, in/out | \phi(\eta, x) \phi(\eta', x') | 0_L, in/out \rangle , \\ &= \frac{1}{L} \sum_n \chi_n^{in/out}(\eta) \chi_n^{in/out*}(\eta') e^{ik_n(x-x')} . \end{aligned} \quad (4.128)$$

The above formalism provides an adequate scheme for quantizing the twisted scalar field. However, the untwisted scalar field generally contains an effectively vanishing frequency mode that renders the Wightman two-point function (4.128) divergent as can be seen from the asymptotic form of the mode functions $\chi_n(\eta)$ in (4.108). This conformal zero mode is defined as follows.

Definition 4.2.1. *A conformal zero mode of the conformal massive scalar field in a compactified spacetime is the Fourier term with an asymptotically vanishing effective frequency that corresponds to the zero-momentum ($n = 0$) and zero effective mass ($\mu_{in/out}(\eta) = 0$) mode, but which does not need to be massless.*

Remark 4.2. *Note that our massive conformal zero mode is unlike the massless minimally coupled zero mode ($m = 0, \xi = 0$) (see Ford and Pathinayake (1989) [86]; Allen and Folacci (1987) [87]; and Kirsten [88]. As we will show later, a massive conformal zero mode emerges from the asymptotic behaviour of the zero-momentum mode in a compactified spacetime [86].*

To address the issue of conformal zero mode, we quantize the untwisted scalar field by a slightly modified scheme. We split the mode expansion (4.121)

$$\hat{\phi}(\eta, x) = \hat{\phi}_0(\eta) + \hat{\phi}_{osc}(\eta, x), \quad (4.129)$$

where the oscillator modes are given by

$$\hat{\phi}_{osc}(\eta, x) = \sum_{n \neq 0} \left(\hat{a}_{in/out}(n) U_n^{in/out}(\eta, x) + \hat{a}_{in/out}^\dagger(n) U_n^{in/out*}(\eta, x) \right), \quad (4.130a)$$

and the zero-momentum mode is given by

$$\hat{\phi}_0(\eta) = \hat{a}_0^{in/out} U_0^{in/out}(\eta) + \hat{a}_0^{in/out\dagger} U_0^{in/out*}(\eta), \quad (4.130b)$$

where $U_0^{in/out}(\eta) := U_{n=0}^{in/out}(\eta, x)$. It follows from the equal-time CCR's (4.127) and the mode expansions (4.130b) that the oscillator modes and zero-momentum mode are quantized via

$$[\hat{\phi}_0(\eta), \hat{\pi}_0(\eta)] = i \mathbb{1} \quad (4.131a)$$

$$[\hat{\phi}_{osc}(\eta, x), \hat{\pi}_{osc}(\eta, x')] = i \left(\delta(x - x') - \frac{1}{L} \right) \mathbb{1}, \quad (4.131b)$$

and all other CCR's consisting of mixed terms vanish. The nonvanishing commutation relations for the creation and annihilation operators read

$$[\hat{a}_0^{in/out}, \hat{a}_0^{in/out\dagger}] = \mathbb{1}, \quad [\hat{a}_{in/out}(n), \hat{a}_{in/out}^\dagger(m)]_{n,m \neq 0} = \delta_{n,m} \mathbb{1}. \quad (4.132)$$

The positive-frequency modes of the zero-momentum and oscillator sectors are chosen according to the adiabatic asymptotic behaviour given in (4.108) (with discrete indices). However, if the effective mass vanishes ($\mu_{in/out} = 0$) either at early or

late times, then the zero momentum mode reduces to a massive conformal zero mode in the *in* or *out* regions. In this case, we can no longer defined positive frequency according to (4.108). The resolution of this issue is discussed below.

For any conformally coupled massive scalar field propagating in a spatially compactified cosmological spacetime of classes (i) and (ii) discussed in Section 4.2.1 for which the scalar field's effective mass vanishes asymptotically, $\mu_{in/out}(\eta) = 0$, the zero-momentum mode solution of the scalar field's wave equation reduces to a conformal zero-mode in the corresponding asymptotic region(s). In this case, the zero-momentum mode function $\chi_0(\eta)$ has the following asymptotic form

$$\chi_0^{in/out}(\eta) \xrightarrow{\eta \rightarrow \eta_0/\eta_1} \chi_{czm}(\eta) = a_1 + a_2\eta, \quad a_1, a_2 \in \mathbb{C}. \quad (4.133a)$$

This mode can naturally be normalized according to the Wronskian given in 4.124 provided the integration constants satisfy the constraints

$$a_1 a_2^* - a_1^* a_2 = i. \quad (4.133b)$$

The most general solution to this zero-momentum constraint (4.133b) is given by

$$a_1 = \alpha + i\beta^{-1} \left(\frac{1}{2} + \alpha\gamma \right), \quad a_2 = \beta + i\gamma, \quad \alpha, \gamma \in \mathbb{R}, \quad \beta \in \mathbb{R} \setminus \{0\}, \quad (4.133c)$$

$$a_1 = -\frac{\gamma^{-1}}{2} + i\alpha, \quad a_2 = i\gamma, \quad \alpha \in \mathbb{R}, \quad \gamma \in \mathbb{R} \setminus \{0\}, \quad \beta = 0. \quad (4.133d)$$

However because the constraint (4.133b) is invariant under a change of phase $a_1, a_2 \rightarrow e^{i\theta} a_1, e^{i\theta} a_2$, therefore without loss of generality we are free to choose

$$a_1 = \alpha + \frac{i}{2\beta}, \quad a_2 = \beta, \quad \alpha \in \mathbb{R}, \quad \beta > 0, \quad (4.133e)$$

thereby introducing a family of two-real parameter for every zero-momentum mode. Furthermore, the zero momentum initial state's wave functional corresponds to a two-real parameter set of normalised Gaussian wave packets (see Appendix F for details and notations):

$$\Psi_{\alpha\beta,L}^{in/out}(q_s) := \langle Q_s | 0_L, in/out \rangle_{\alpha,\beta} = \left(\frac{2\beta^2 L}{\pi \sqrt{1 + 4\alpha^2 \beta^2}} \right)^{\frac{1}{4}} \exp \left(-\frac{L\beta^2 e^{i\delta_{\alpha,\beta}}}{\sqrt{1 + 4\alpha^2 \beta^2}} q_s^2 \right), \quad (4.134)$$

where $|0_L, in/out\rangle_{\alpha,\beta}$ is the zero momentum initial state in the Heisenberg picture defined in (4.135) and $\hat{Q}_s|q_s\rangle = q_s|q_s\rangle$, where we define the Hermitian operator $\hat{Q}_s$ in Appendix F.

Remark 4.3. *The zero-momentum mode does not need to reduce to a conformal zero mode in both the in and out regions; it can be a conformal zero mode in one asymptotic region and a positive frequency mode in the other. This choice will depend on the explicit form of the metric.*

We work in the Heisenberg picture where the state of the full field (4.129) is time independent. The Hilbert space of the full field is taken as the tensor product of the oscillator mode (4.130a) Fock space and the two-real parameter set of Fock spaces associated with the zero-momentum mode (4.131a). These Fock spaces are constructed as follows.

The oscillator mode Fock spaces at early and late times are built on the oscillator modes normalised *in* and *out* Fock vacua states which satisfy $a_{in/out}(n)|0_L^{osc}, in/out\rangle = 0, \forall n \neq 0$. These vacuum states are of infinite adiabatic order since they correspond to positive frequency modes of infinite adiabatic order.

For every zero-momentum mode obeying (4.133), there exists a two-real parameter set of Fock vacua satisfying

$$a_{0,a_1,a_2}^{in/out}|0_L, in/out\rangle_{a_1,a_2} \equiv a_{0,\alpha,\beta}^{in/out}|0_L, in/out\rangle_{\alpha,\beta} = 0, \quad (4.135a)$$

$${}_{\alpha,\beta}\langle 0_L, in/out|0_L, in/out\rangle_{\alpha,\beta} = 1. \quad (4.135b)$$

As such, there exists an infinite number of Fock spaces associated to the two-real parameter set of vacua, each constructed by repeated application of the creation operator $a_{0,\alpha,\beta}^{in/out\dagger}$ on the corresponding two-real parameter set of vacua. The full field (4.129), consisting of both the oscillator mode and the zero-momentum mode is then taken as the state $|0_L, in/out\rangle_{\alpha,\beta} \otimes |0_L^{osc}, in/out\rangle$. The Wightman two-point

function of the full field (4.129) in this state is given by

$$\begin{aligned}
G_{\alpha,\beta}^{in/out+}(\eta, x; \eta', x') &:= \langle 0_L^{\text{osc}}, in/out | \otimes_{\alpha,\beta} \langle 0_L, in/out | \hat{\phi}(\eta, x) \hat{\phi}(\eta', x') | 0_{in/out} \rangle_{\alpha,\beta} \\
&\quad \otimes | 0_L^{\text{osc}}, in/out \rangle \\
&= G_{0,\alpha,\beta}^{in/out+}(\eta, \eta') + G_{osc}^{in/out+}(\eta, x; \eta', x'), \tag{4.136}
\end{aligned}$$

where the oscillator mode Wightman two-point function in the oscillator mode vacuum is given by

$$\begin{aligned}
G_{osc}^{in/out+}(\eta, x; \eta', x') &:= \langle 0_L^{\text{osc}}, in/out | \hat{\phi}(\eta, x) \hat{\phi}(\eta', x') | 0_L^{\text{osc}}, in/out \rangle \tag{4.137} \\
&= \sum_{n \neq 0} U_n^{in/out}(\eta, x) U_n^{in/out*}(\eta', x') \\
&= \frac{1}{L} \sum_{n \neq 0} \chi_n^{in/out}(\eta) \chi_n^{in/out*}(\eta') e^{ik_n(x-x')}.
\end{aligned}$$

and the zero-momentum Wightman two-point function in the two-real parameter set of vacua is given by

$$\begin{aligned}
G_{0,\alpha,\beta}^{in/out+}(\eta, \eta') &:= {}_{\alpha,\beta} \langle 0_L, in/out | \hat{\phi}_0(\eta) \hat{\phi}_0(\eta') | 0_L, in/out \rangle_{\alpha,\beta} \tag{4.138} \\
&= U_0^{in/out}(\eta) U_0^{in/out*}(\eta') = \frac{1}{L} \chi_0^{in/out}(\eta) \chi_0^{in/out*}(\eta').
\end{aligned}$$

4.3 Unruh-DeWitt detector coupled to the scalar field

In this section, we briefly recall relevant properties of an Unruh-DeWitt (UDW) detector couples linearly to the free, real-valued scalar field propagating on a curved background of Lorentzian signature [5, 6, 9, 18, 89]. Here we review the finite-time detector's responses rather than the asymptotic values of the probability of excitation of the UDW detector. Asymptotic values of the detector's probability of excitation do not reflect accurately the intermediate values of the transition probability for short observation times, which is more suited for realistic detector (see e.g., B. F. Svaiter and N. F. Svaiter [90] or Higuchi *et al.* [91]).

To probe the quantum state of the field, we employ as a detector a point-like two-level atom linearly coupled to the real quantum scalar field ϕ , on an arbitrary curved background in $d \geq 2$ dimensions. This particle detector can absorb or emit field quanta, and it moves along the smooth, timelike trajectory, $x(\tau)$, parametrized by its proper time. The unperturbed detector has two orthonormal energy eigenstates $|0\rangle_D$ and $|\omega\rangle_D$ with energy 0 and $\omega \neq 0$, respectively. The eigenstate $|0\rangle_D$ is the ground state if $\omega > 0$ and the excited state if $\omega < 0$. The coupling of the detector to the field allows the detector to become excited by absorbing a field quantum, or de-excite by emitting a field quantum [57].

The Hilbert space of the coupled detector-field system is taken as the direct tensor product $\mathcal{H}_D \otimes \mathcal{H}_\phi$ with the associated total Hamiltonian operator $H = H_D + H_\phi + H_{int}$, where, H_D is the (unperturbed) detector Hamiltonian, H_ϕ is the Hamiltonian operator of the free field, which can be left unspecified. The detector is coupled to the scalar field ϕ via the monopole interaction Hamiltonian

$$H_{int} := c\chi(\tau)\mu(\tau)\phi(x(\tau)) , \quad (4.139)$$

where $\mu(\tau)$ is the detector's monopole moment operator, evolving in the interaction picture via

$$\mu(\tau) = e^{iH_D\tau}\mu(0)e^{-iH_D\tau} , \quad (4.140)$$

c is a coupling constant, and the switching function χ is a smooth real-valued function of compact support that specifies how the interaction is turned on and off.

Suppose before the interaction begins at time τ_0 , the coupled detector-field system occupies the uncorrelated state $|\Psi\rangle \otimes |0\rangle_D$, where the scalar field occupies an arbitrary Hadamard state $|\Psi\rangle$ [92]. Working in first-order perturbation theory in c , the transition probability of the detector to the state $|\omega\rangle_D$ induced by its coupling to the field at a later time $\tau > \tau_0$ is given by [90, 93]

$$P_\tau(\omega, \tau_0) = \sum_f \left| \langle \Psi_f |_D \langle 0 | \int_{\tau_0}^{\tau} (H_{int}(\tau))_I |0\rangle_D |\Psi_f\rangle_D \right|^2 = c^2 |{}_d\langle 0 | \mu(0) | 1 \rangle_d \mathcal{F}(\omega, \tau, \tau_0)|^2 , \quad (4.141)$$

where the response function is given by

$$\mathcal{F}(\omega, \tau, \tau_0) := \int_{\tau_0}^{\tau} d\tau \int_{\tau_0}^{\tau} d\tau' \chi(\tau) \chi(\tau') e^{-i\omega(\tau-\tau')} G^+(x(\tau), x(\tau')) , \quad (4.142)$$

and the distributional correlation function is the positive frequency Wightman two-point function

$$G^+(x(\tau), x(\tau')) := \langle \Psi | \phi(x(\tau)) \phi(x(\tau')) | \Psi \rangle . \quad (4.143)$$

The integrals (4.142) are understood in the distributional sense and are well defined whenever the initial state of the field $|\Psi\rangle$ is Hadamard [44, 94–96]. We will relax this assumption when considering compactified spacetime models as Hadamard states are not always well-defined on compactified spacetimes. Moreover, we shall drop the multiplicative constant $c^2 |{}_d\langle 0 | \mu(0) | 1 \rangle_d$ in (4.141) which depends only on the internal structure of the detector and the coupling constant. We will refer to the detector's response $\mathcal{F}(\omega, \tau, \tau_0)$ as the transition probability induced by the coupling to the field in the finite time interval $[\tau_0, \tau]$ while the interaction is still on. If $\mathcal{F}(\omega, \tau, \tau_0)$ is nonzero for $\omega > 0$, the detector has a nonzero probability to end up in the excited state after the finite time of interaction $[\tau_0, \tau]$. Similarly, if $\mathcal{F}(\omega, \tau, \tau_0)$ is nonzero for $\omega < 0$, the detector has a non zero probability to end up in the ground state after the finite time of interaction $[\tau_0, \tau]$.

When issues of regularisation of the short-distance singularity of the Wightman two-point function can be ignored, we can adopt the switching function [28]

$$\chi(\tau') = \Theta(\tau' - \tau_0) \Theta(\tau - \tau') , \quad (4.144)$$

where Θ is the Heaviside step function,

$$\Theta(x) \doteq \begin{cases} 0 & \text{for } x < 0 , \\ 1 & \text{for } x \geq 0 . \end{cases} \quad (4.145)$$

The transition probability of the detector (4.142) becomes

$$\mathcal{F}(\omega, \tau, \tau_0) := \int_{\tau_0}^{\tau} d\tau \int_{\tau_0}^{\tau} d\tau' e^{-i\omega(\tau-\tau')} G^+(x(\tau), x(\tau')) . \quad (4.146)$$

4.3.1 Detector's response to a massless field in the spatially flat FRW spacetimes in $d \geq 2$ dimensions

In this section, we first establish the equivalence between the adiabatic vacuum of infinite order and the conformal vacuum in the massless limit for the class of cosmological backgrounds defined in Section 4.2.4. With the field in the conformal Fock vacuum, we then show that the detector's response to an UDW particle detector coupled linearly to the scalar density of a massless Dirac field in the spatially flat FRW spacetimes in d -dimensions is identical to the response of a detector coupled to the massless scalar field in the spatially flat FRW spacetimes in $2d$ -dimensions.

For a real massless scalar field propagating in the spatially flat Robertson-Walker spacetime of the class discussed in Section 4.2.4, the normalized purely positive frequency modes of infinite adiabatic order (4.108) to (4.105) are $\chi_k^{in/out}(\eta) = (2\omega_{in/out})^{-1/2} \exp(-i\omega_{in/out}\eta)$, where $\omega_{in}(k) = \omega_{out}(k) = k$. This implies that the *in* and *out* vacua are equivalent, $|0_{in}\rangle \equiv |0_{out}\rangle$. Moreover, because the adiabatic positive frequency modes of infinite adiabatic order are also positive frequency with respect to the global timelike conformal killing vector field ∂_η , therefore the *in* and *out* vacua of infinite adiabatic order are identical to the conformal vacuum $|0_{in}\rangle \equiv |0_{out}\rangle \equiv |\bar{0}\rangle$. In this case, the positive frequency Wightman two-point function (4.117a) reduces to

$$G_{in/out}^+(\eta, \mathbf{x}, \eta', \mathbf{x}') = [C(\eta)C(\eta')]^{(2-d)/4} G^+(\eta, \mathbf{x}, \eta', \mathbf{x}'), \quad (4.147a)$$

$$\equiv \bar{G}^+(\eta, \mathbf{x}, \eta', \mathbf{x}'), \quad (4.147b)$$

where G^+ is the Wightman two-point function of a massless scalar field in Minkowski spacetime, which reads [17]

$$G^+(x, x') = \begin{cases} \frac{\Gamma(d/2)}{2\pi^{d/2}(d-2)(Z(x, x'))^{d-2}} & \text{for } d > 2, \\ -(2\pi)^{-1} \ln(\mu Z(x, x')) & \text{for } d = 2, \end{cases} \quad (4.148)$$

where the infrared ambiguity in $d = 2$ is encoded in the undetermined positive constant μ and where

$$Z(x, x') := \sqrt{(\mathbf{x} - \mathbf{x}')^2 - (\eta - \eta' - i\epsilon)^2}, \quad (4.149)$$

with $\epsilon \rightarrow 0_+$. Collecting (4.147a), (4.148) and (4.142), we see that the detector's response to a massless scalar field in $d > 2$ dimensions for a scalar field that initially occupies the conformal vacuum state is given by

$$\mathcal{F}(\omega) = \frac{\Gamma(d/2)}{2\pi^{d/2}(d-2)} \int d\tau d\tau' \frac{\chi(\tau)\chi(\tau') [C(\eta(\tau))C(\eta(\tau'))]^{(2-d)/4} e^{-i\omega(\tau-\tau')}}{[Z(x(\tau), x(\tau'))]^{d-2}}. \quad (4.150)$$

Spinor field

Similarly, for a massless Dirac field in the spatially flat FRW spacetimes, the normalized purely positive frequency modes of infinite adiabatic order (4.79) to (4.68) are $\chi_k^{(\pm)in/out}(\eta) = (\sqrt{2}k)^{-1} \exp(-ik\eta)$, where $\omega_{in}(k) = \omega_{out}(k) = k$. This implies that the *in* and *out* vacua are equivalent, $|0_{in}\rangle \equiv |0_{out}\rangle$. Moreover, because the adiabatic positive frequency modes of infinite adiabatic order are also positive frequency with respect to the global timelike conformal killing vector field ∂_η , therefore the *in* and *out* vacua of infinite adiabatic order are identical to the conformal vacuum $|0_{in}\rangle \equiv |0_{out}\rangle \equiv |\bar{0}\rangle$. In this case, the spinor field two-point functions (4.89) reduces to

$$S_{ab}^{in/out(\pm)}(\eta, \mathbf{x}, \eta', \mathbf{x}') = [C(\eta)C(\eta')]^{\frac{1-d}{4}} iS_{ab}^\pm((\eta, \mathbf{x}, \eta', \mathbf{x}'), \quad (4.151a)$$

$$\equiv \bar{S}_{ab}^{(\pm)}(\eta, \mathbf{x}, \eta', \mathbf{x}'), \quad (4.151b)$$

where $S_{ab}^\pm$ are Dirac field Wightman functions in Minkowski spacetime (2.17) with t replaced by η . Similarly, the four-point correlation function $W_{in/out}^{(2,\bar{2})}(x, x')$ defined

in (4.97) reduces to

$$W_{in/out}^{(2,\bar{2})}(\eta, \mathbf{x}, \eta', \mathbf{x}') = [C(\eta)C(\eta')]^{\frac{1-d}{2}} W^{(2,\bar{2})}(\eta, \mathbf{x}, \eta', \mathbf{x}') , \quad (4.152a)$$

$$\equiv \overline{W}^{(2,\bar{2})}(\eta, \mathbf{x}, \eta', \mathbf{x}') , \quad (4.152b)$$

where $W^{(2,\bar{2})}$ is the four-point correlation function of a massless Dirac field in Minkowski spacetime, which reads [17]

$$W^{(2,\bar{2})}(x, x') = \frac{N_d (\Gamma(d/2))^2}{4\pi^d (Z(x, x'))^{2d-2}} , \quad (4.153)$$

and where $Z(x, x')$ is defined in (2.20). Collecting (2.4), (4.152) and (4.153), we see that the detector's response to a massless Dirac field in $d \geq 2$ dimensions for a Dirac field that initially occupies the conformal vacuum state is given by

$$\mathcal{F}_{sp}(\omega) = \frac{N_d (\Gamma(d/2))^2}{4\pi^d} \int d\tau d\tau' \frac{\chi(\tau)\chi(\tau') [C(\eta(\tau))C(\eta(\tau'))]^{(1-d)/2} e^{-i\omega(\tau-\tau')}}{[Z(x(\tau), x(\tau'))]^{2d-2}} . \quad (4.154)$$

The results for the response functions given in (4.150) and (4.154) can be reformulated via the following theorem

Theorem 4.1. *The response function of an Unruh-DeWitt detector coupled quadratically to a massless Dirac field in the conformal cosmological vacuum in $d \geq 2$ spacetime dimensions equals*

$$\frac{N_d (\Gamma(d/2))^2}{\Gamma(d-1)} \quad (4.155)$$

times the response function of an Unruh-DeWitt detector coupled linearly to a massless scalar field in the conformal cosmological vacuum in $2d$ spacetime dimensions.

This theorem is the conformal vacuum counterpart of Theorem 2.1 for Minkowski vacuum and thus has similar consequences for the class of spatially flat cosmological backgrounds discussed in this chapter. Note however as stated in [5] even a

comoving detector - the nearest analogue in the spatially flat FRW spacetimes to an inertial observer in Minkowski spacetime - will register the presence of particles in the conformal vacuum. Hence, in general, both a comoving and a non-comoving detector can experience a thermal response for the classes of spatially flat FRW cosmological spacetimes we consider.

Chapter 5

Finite-time detector's response for a massive scalar field in the (1+1)-dimensional spatially compactified Milne spacetime

In this chapter, we consider a massive scalar field propagating in the $(1 + 1)$ -dimensional spatially compactified Milne cosmological spacetime. We quantize the field according to the general formalisms discussed in Section 4.2.4 and Section 4.2.5. We treat twisted and untwisted scalar fields in turn.

5.1 Milne spacetime

The two-dimensional Milne universe can be expressed in coordinatization where the metric element reads

$$ds^2 = dt^2 - a^2 t^2 dx^2, \quad t > 0, \quad (5.1a)$$

$$= e^{2a\eta}(d\eta^2 - dx^2), \quad -\infty < \eta < \infty, \quad (5.1b)$$

where $t = a^{-1}e^{a\eta}$ and a is a positive constant. This spacetime has a cosmological singularity at $t = 0$. View as an embedding manifold, Milne model represents the patch of the two-dimensional Minkowski spacetime which lies in the forward light cone. This can be seen under the coordinate transformation

$$y^0 = a^{-1}e^{a\eta} \cosh(ax), \quad y^1 = a^{-1}e^{a\eta} \sinh(ax), \quad (5.2)$$

in which the metric (5.1) reduces to

$$ds^2 = (dy^0)^2 - (dy^1)^2, \quad 0 < y^0 < \infty, \quad -\infty < y^1 < \infty. \quad (5.3)$$

Several authors [5,97–101] have studied quantum field theory in the full Milne spacetime. We shall, however, restrict our analysis to a massive scalar field propagating in the spatially compactified Milne spacetime, where $(\eta, x) = (\eta, x + L)$ as described in the general formalism given in Section 4.2.5.

Twisted scalar field

Consider the twisted massive scalar field as quantized in the general formalism given in Section 4.2.5. The time-dependent effective frequency (4.120b) reads $\omega_n(\eta) = \sqrt{k_n^2 + m^2 e^{2a\eta}}$, with asymptotic values $\omega_{in} := \omega_n(\eta \rightarrow -\infty) = |k_n|$ and $\omega_{out} := \omega_n(\eta \rightarrow +\infty) \approx me^{a\eta}$. Therefore, by performing the change of variable $\chi_n(\eta) := f_n(s) = f_n(me^{a\eta}/a)$, (4.125) reduces to the Bessel's equation

$$f_n''(s) + s^{-1}f_n'(s) + \left(\frac{\nu^2}{s^2} + 1\right)f_n(s) = 0, \quad (5.4)$$

where $\nu \equiv |k_n|/a$. The general solution to this equation can be expressed in this form

$$f_n(s) = c_1 J_{i\nu}(s) + c_2 J_{-i\nu}(s) \equiv a_1 H_{i\nu}^{(1)}(s) + a_2 H_{i\nu}^{(2)}(s), \quad (5.5)$$

where $J_{i\nu}(s)$ and $H_{i\nu}^{(1)}(s)$ ($H_{i\nu}^{(2)}(s)$) are the Bessel and Hankel functions, respectively. For this cosmological model we choose the exact mode functions (5.5) according to

the adiabatic asymptotic behaviour (4.108),

$$\begin{cases} \chi_n^{in}(\eta) \xrightarrow{\eta \rightarrow -\infty} (2|k_n|)^{-1/2} \exp(-i|k_n|\eta) , \\ \chi_n^{out}(\eta) \xrightarrow{\eta \rightarrow +\infty} (2m e^{a\eta})^{-1/2} \exp(-im e^{a\eta}/a) . \end{cases} \quad (5.6)$$

The normalized positive mode functions that satisfy this required asymptotic condition as $\eta \rightarrow \pm\infty$ ($s \rightarrow -\infty/0$) are

$$\chi_n^{in}(\eta) = [(2a/\pi) \sinh(\pi|k_n|/a)]^{-1/2} J_{-i|k_n|/a}(m e^{a\eta}/a) , \quad (5.7a)$$

$$\chi_n^{out}(\eta) = \frac{1}{2}(\pi/a)^{1/2} e^{\pi|k_n|/2a} H_{i|k_n|/a}^{(2)}(m e^{a\eta}/a) . \quad (5.7b)$$

In the formalism discussed in Section 4.2.5, the vacua associated with these modes are the *in* and *out* vacua states of infinite adiabatic order. Collecting (4.128) and (5.7), we obtain the *in* and *out* vacua Wightman two-point functions

$$G^{in+}(s, x, s', x') = \frac{\pi}{2aL} \sum_n \frac{J_{-i|k_n|/a}(s) J_{i|k_n|/a}(s') e^{ik_n(x-x')}}{\sinh(\pi|k_n|/a)} , \quad (5.8a)$$

$$G^{out+}(s, x, s', x') = \frac{\pi}{4aL} \sum_n e^{\pi|k_n|/a} H_{i|k_n|/a}^{(2)}(s) H_{-i|k_n|/a}^{(1)}(s') e^{ik_n(x-x')} , \quad (5.8b)$$

where $s := m e^{a\eta}/a$. Similarly, collecting (4.146) and (5.8a), we see that the detector's response to a massive scalar field in the finite time of interaction $[\tau_0, \tau]$ when the scalar field occupies the *in* vacuum of infinite adiabatic order, $|0_L, in\rangle$ is given by

$$\mathcal{F}_t^{in}(\omega, \tau, \tau_0) = \frac{\pi}{2aL} \sum_{n=-\infty}^{\infty} \frac{|A_n(\omega, \tau, \tau_0)|^2}{\sinh(\pi|k_n|/a)} , \quad k_n = \frac{2\pi}{L} \left(n + \frac{1}{2} \right) , \quad (5.9a)$$

where

$$A_n(\omega, \tau, \tau_0) := \int_{\tau_0}^{\tau} d\tau J_{-i|k_n|/a}(m t(\tau)) e^{-i(\omega\tau - k_n x(\tau))} , \quad t(\tau) = e^{a\eta(\tau)}/a , \quad (5.9b)$$

Likewise, collecting (4.146) and (5.8b), we see that the detector's response to a massive scalar field in the finite time of interaction $[\tau_0, \tau]$ when the scalar field

occupies the *out* vacuum of infinite order, $|0_L, out\rangle$, is given by

$$\mathcal{F}_t^{out}(\omega, \tau, \tau_0) = \frac{\pi}{4aL} \sum_{n=-\infty}^{\infty} e^{\pi|k_n|/a} |B_n(\omega, \tau, \tau_0)|^2, \quad (5.10a)$$

where

$$B_n(\omega, \tau, \tau_0) := \int_{\tau_0}^{\tau} d\tau H_{i|k_n|/a}^{(2)}(m\tau(\tau)) e^{-i(\omega\tau - k_n x(\tau))}. \quad (5.10b)$$

Untwisted scalar field

Consider the untwisted massive scalar field as quantized in the general formalism given in Section 4.2.5. The scalar field's effective mass reads $\mu(\eta) = me^{a\eta}$ with asymptotic values $\mu_{in} := \mu(\eta \rightarrow -\infty) = 0$ and $\mu_{out} := \mu(\eta \rightarrow \infty) \approx me^{a\eta}$. Therefore, collecting (4.133) and (4.108), the zero-momentum mode reduces to a massive conformal zero mode at early times and a positive frequency mode at late times.

The zero-momentum mode function, $\chi_0(\eta)$, satisfies the Bessel differential equation of order zero (5.4) with $\nu = 0$ ($k_0 = 0$). The general solution is of the form

$$f_0(s) = c_1 J_0(s) + c_2 Y_0(s) \equiv a_1 H_0^{(1)}(s) + a_2 H_0^{(2)}(s), \quad (5.11)$$

where $J_0(s)$ and $Y_0(s)$ are zero order Bessel functions of the first and second kind; and $H_0^{(1)}(s)$ and $H_0^{(2)}(s)$ are zero order Hankel functions of the first and second kind. Using the asymptotic form of the zero-momentum mode given in (4.133) at early times $\eta \rightarrow -\infty$ (or $s \rightarrow 0$) along with the asymptotic form of the positive frequency mode given in (4.108) at late times $\eta \rightarrow \infty$ ($s \rightarrow \infty$), we obtain the following asymptotic behaviour for the zero-momentum mode function for the untwisted field

$$\begin{cases} \chi_0^{in}(\eta) \xrightarrow{\eta \rightarrow -\infty} \left(c_1 + \frac{2c_2}{\pi} \ln\left(\frac{m}{a}\right)\right) + \frac{2c_2 a}{\pi} \eta, \\ \chi_0^{out}(\eta) \xrightarrow{\eta \rightarrow +\infty} (2m e^{a\eta})^{-1/2} \exp(-im e^{a\eta}/a), \end{cases} \quad (5.12)$$

where c_1 and c_2 are constrained by the normalisation condition given in (4.133b)

$$c_1 c_2^* - c_1^* c_2 = \frac{\pi}{2a} i. \quad (5.13)$$

In terms of two-real parameter family of states given in (4.133), the coefficients c_1 and c_2 can be expressed as

$$c_1 = \alpha + i \frac{\pi}{4a\beta}, \quad c_2 = \beta, \quad \alpha \in \mathbb{R}, \quad \beta > 0. \quad (5.14)$$

The normalized asymptotic zero-momentum mode function (5.11) which reduces to the asymptotic form (5.12) are

$$\chi_0^{in}(\eta) = c_1 J_0(me^{a\eta}/a) + c_2 Y_0(me^{a\eta}/a), \quad (5.15a)$$

$$\chi_0^{out}(\eta) = \frac{1}{2}(\pi/a)^{1/2} H_0^{(2)}(me^{a\eta}/a). \quad (5.15b)$$

Remark 5.1. Note when $c_1 = \frac{i}{2}\sqrt{\frac{\pi}{a}}$ and $c_2 = \frac{1}{2}\sqrt{\frac{\pi}{a}}$ (or equivalently when $\alpha = 0$ and $\beta = \frac{1}{2}\sqrt{\frac{\pi}{a}}$), the mode functions $\chi_0^{in}(\eta)$ is, up to a phase, equal to $\chi_0^{out}(\eta)$. In this case, the zero-momentum contributions to the massive scalar field in the in and out vacua are equal.

Collecting (4.138) and (5.15), we see that the in and out vacua Wightman two-point functions for the zero-momentum mode are given by

$$G_{\alpha,\beta}^{in+}(s, s') = \frac{1}{L} \left(|c_1|^2 J_0(s)J_0(s') + |c_2|^2 Y_0(s)Y_0(s') + c_1 c_2^* J_0(s)Y_0(s') + c_1^* c_2 Y_0(s)J_0(s') \right) \quad (5.16a)$$

$$G_0^{out+}(s, s') = \frac{\pi}{4aL} H_0^{(2)}(s)H_0^{(1)}(s'). \quad (5.16b)$$

Similarly, collecting (4.146) and (5.16), we see that detector's responses for the zero momentum mode contribution to the massive scalar field in the finite-time of interaction $[\tau_0, \tau]$, both when the zero-momentum mode occupies a vacuum in the two-real parameter family, and when it occupies the adiabatic out vacuum of infinite order, $|0_L, out\rangle$, are given by

$$\mathcal{F}_{\alpha,\beta}^{in}(\omega, \tau, \tau_0) = \frac{1}{L} \left\{ |c_1 M_{\tau_0}(\omega, \tau)|^2 + |c_2 N_{\tau_0}(\omega, \tau)|^2 + 2\Re[c_1 c_2^* M_{\tau_0}(\omega, \tau) N_{\tau_0}^*(\tau, (\omega, \tau))] \right\} \quad (5.17a)$$

$$\mathcal{F}_0^{out}(\omega, \tau, \tau_0) = \frac{\pi}{4aL} |B_0(\omega, \tau, \tau_0)|^2, \quad (5.17b)$$

where

$$M_{\tau_0}(\omega, \tau) := \int_{\tau_0}^{\tau} d\tau J_0(mt(\tau)) e^{-i\omega\tau}, \quad (5.18a)$$

$$N_{\tau_0}(\omega, \tau) := \int_{\tau_0}^{\tau} d\tau Y_0(mt(\tau)) e^{-i\omega\tau} \quad (5.18b)$$

$$B_0(\omega, \tau, \tau_0) = \int_{\tau_0}^{\tau} d\tau H_0^{(2)}(mt(\tau)) e^{-i\omega\tau}. \quad (5.18c)$$

For the oscillator modes, (4.137) and (5.7) give the following *in* and *out* vacua Wightman two-point functions

$$G_{osc}^{in+}(s, x, s', x') = \frac{\pi}{2aL} \sum_{n \neq 0} \frac{J_{-i|k_n|/a}(s) J_{i|k_n|/a}(s') e^{ik_n(x-x')}}{\sinh(\pi|k_n|/a)}, \quad (5.19a)$$

$$G_{osc}^{out+}(s, x, s', x') = \frac{\pi}{4aL} \sum_{n \neq 0} e^{\pi|k_n|/a} H_{i|k_n|/a}^{(2)}(s) H_{-i|k_n|/a}^{(1)}(s') e^{ik_n(x-x')}. \quad (5.19b)$$

The detector's response to the oscillator modes in the finite-time of interaction $[\tau_0, \tau]$ when the oscillator modes contribution to the scalar field occupies the *in*-vacuum, $|0_L^{osc}, in\rangle$, of infinite adiabatic order follows from (4.146) and (5.19a)

$$\mathcal{F}_{osc}^{in}(\omega, \tau, \tau_0) = \frac{\pi}{2aL} \sum_{n \neq 0} \frac{|A_n(\omega, \tau, \tau_0)|^2}{\sinh(\pi|k_n|/a)}, \quad k_n = \frac{2\pi}{L}n, \quad (5.20a)$$

where $A_n(\omega, \tau, \tau_0)$ is defined via (5.9b) with k_n in (5.20a). Similarly, the response of the detector to the oscillator modes in the finite-time of interaction $[\tau_0, \tau]$ when the oscillator modes contribution to the scalar field occupies the *out* vacuum, $|0_L^{osc}, out\rangle$, of infinite adiabatic order is obtained from (4.146) and (5.19b)

$$\mathcal{F}_{osc}^{out}(\omega, \tau, \tau_0) = \frac{\pi}{4aL} \sum_{n \neq 0} e^{\pi|k_n|/a} |B_n(\omega, \tau, \tau_0)|^2, \quad (5.20b)$$

where $B_n(\omega, \tau, \tau_0)$ is given in (5.10b) with k_n in (5.20a).

Collecting (5.17a) and (5.20a), we find that the response of the detector to the full untwisted field (4.129) in the finite-time of interaction $[\tau_0, \tau]$ when the oscillator modes occupy the adiabatic *in* vacuum of infinite order, $|0_L^{osc}, in\rangle$, and the zero-momentum mode occupies a vacuum in the two-real parameter set of vacua,

$$\mathcal{F}_{u,\alpha,\beta}^{in}(\omega, \tau, \tau_0) = \mathcal{F}_{osc}^{in}(\omega, \tau, \tau_0) + \mathcal{F}_{\alpha,\beta}^{in}(\omega, \tau, \tau_0). \quad (5.21a)$$

Similarly, collecting (5.17b) and (5.20b), we obtain that the detector's response to the full untwisted field (4.129) in the finite-time of interaction $[\tau_0, \tau]$ when the oscillator modes sector occupy the adiabatic *out* vacuum, $|0_L^{osc}, out\rangle$, of infinite order and zero-momentum sector occupies the adiabatic *out* vacuum, $|0_L, out\rangle$, of infinite order is given by

$$\mathcal{F}_u^{out}(\tau_0) = \mathcal{F}_{osc}^{out}(\omega, \tau, \tau_0) + \mathcal{F}_0^{out}(\omega, \tau, \tau_0). \quad (5.21b)$$

5.2 Comoving and non-comoving trajectories on the spatially compactified Milne spacetime

In this section we consider a comoving detector and an inertial non-comoving detector in the $(1+1)$ -dimensional compactified Milne spacetime of Section 5.1.

5.2.1 Comoving detector

Consider a comoving detector, that is a detector that is kept at rest in the compactified Milne spacetime, with worldlines

$$t(\tau) = \tau > 0, \quad x(\tau) = 0, \quad (5.22)$$

where $t(\tau)$ is the cosmic time.

Twisted Field

The detector's responses to the twisted field are obtained from (5.9a) and (5.10a) with (5.22). Note that the responses contain no infrared ambiguities. Using formulas

that are well suited for numerical evaluation, we have

$$\mathcal{F}_t^{in} \left(\frac{\omega}{m}, m\tau, m\tau_0 \right) = \frac{\pi}{aL} \left[\sum_{n=0}^{\infty} \frac{|A_n(\frac{\omega}{m}, m\tau, m\tau_0)|^2}{\sinh(\frac{2\pi^2}{aL} (n + \frac{1}{2}))} \right], \quad (5.23a)$$

$$\mathcal{F}_t^{out} \left(\frac{\omega}{m}, m\tau, m\tau_0 \right) = \frac{\pi}{2aL} \left[\sum_{n=0}^{\infty} \exp \left(\frac{2\pi^2}{aL} \left(n + \frac{1}{2} \right) \right) \left| B_n \left(\frac{\omega}{m}, m\tau, m\tau_0 \right) \right|^2 \right], \quad (5.23b)$$

where

$$A_n \left(\frac{\omega}{m}, m\tau, m\tau_0 \right) = \frac{1}{m} \int_{m\tau_0}^{m\tau} du J_{-i\frac{2\pi}{aL} (n + \frac{1}{2})}(u) \exp \left(-i \frac{\omega}{m} u \right), \quad (5.24a)$$

$$B_n \left(\frac{\omega}{m}, m\tau, m\tau_0 \right) = \frac{1}{m} \int_{m\tau_0}^{m\tau} du H_{i\frac{2\pi}{aL} (n + \frac{1}{2})}^{(2)}(u) \exp \left(-i \frac{\omega}{m} u \right). \quad (5.24b)$$

Untwisted Field

The response to the detector for the oscillator modes contribution to the untwisted field are obtained from (5.20) with (5.22). Note that the responses contain no infrared ambiguities. Using formulas that are well suited for numerical evaluation, we have

$$\mathcal{F}_{osc}^{in} \left(\frac{\omega}{m}, m\tau, m\tau_0 \right) = \frac{\pi}{aL} \sum_{n=1}^{\infty} \frac{|A_n(\frac{\omega}{m}, m\tau, m\tau_0)|^2}{\sinh(\frac{2\pi^2}{aL} n)}, \quad (5.25a)$$

$$\mathcal{F}_{osc}^{out} \left(\frac{\omega}{m}, m\tau, m\tau_0 \right) = \frac{\pi}{2aL} \sum_{n=1}^{\infty} \exp \left(\frac{2\pi^2}{aL} n \right) \left| B_n \left(\frac{\omega}{m}, m\tau, m\tau_0 \right) \right|^2, \quad (5.25b)$$

where

$$A_n \left(\frac{\omega}{m}, m\tau, m\tau_0 \right) = \frac{1}{m} \int_{m\tau_0}^{m\tau} du J_{-i\frac{2\pi}{aL} n}(u) \exp \left(-i \frac{\omega}{m} u \right), \quad (5.26a)$$

$$B_n \left(\frac{\omega}{m}, m\tau, m\tau_0 \right) = \frac{1}{m} \int_{m\tau_0}^{m\tau} du H_{i\frac{2\pi}{aL} n}^{(2)}(u) \exp \left(-i \frac{\omega}{m} u \right). \quad (5.26b)$$

Collecting (5.14), (5.17), (5.18) and (5.22), we see that the responses for the comoving detector to the the zero-momentum mode, using formulas that are well

suited for numerical evaluation, are given by

$$\begin{aligned} \mathcal{F}_{\alpha,\beta}^{in} \left(\frac{\omega}{m}, m\tau, m\tau_0 \right) &= \frac{1}{aL} \left\{ \left| \left(a^{\frac{1}{2}}\alpha + i\frac{\pi}{4a^{\frac{1}{2}}\beta} \right) M \left(\frac{\omega}{m}, m\tau, m\tau_0 \right) \right|^2 \right. \\ &\quad + \left| a^{\frac{1}{2}}\beta N \left(\frac{\omega}{m}, m\tau, m\tau_0 \right) \right|^2 + 2\Re \left[\left(a^{\frac{1}{2}}\alpha \right) \left(a^{\frac{1}{2}}\beta \right) + i\frac{\pi}{4} \right] \times \\ &\quad \left. M \left(\frac{\omega}{m}, m\tau, m\tau_0 \right) N^* \left(\frac{\omega}{m}, m\tau, m\tau_0 \right) \right] \right\}, \end{aligned} \quad (5.27a)$$

$$\mathcal{F}_0^{out} \left(\frac{\omega}{m}, m\tau, m\tau_0 \right) = \frac{\pi}{4aL} \left| B_0 \left(\frac{\omega}{m}, m\tau, m\tau_0 \right) \right|^2, \quad (5.27b)$$

where

$$M \left(\frac{\omega}{m}, m\tau, m\tau_0 \right) = \frac{1}{m} \int_{m\tau_0}^{m\tau} du J_0(u) \exp \left(-i\frac{\omega}{m}u \right), \quad (5.28a)$$

$$N \left(\frac{\omega}{m}, m\tau, m\tau_0 \right) = \frac{1}{m} \int_{m\tau_0}^{m\tau} du Y_0(u) \exp \left(-i\frac{\omega}{m}u \right), \quad (5.28b)$$

$$B_0 \left(\frac{\omega}{m}, m\tau, m\tau_0 \right) = \frac{1}{m} \int_{m\tau_0}^{m\tau} du H_0^{(2)}(u) \exp \left(-i\frac{\omega}{m}u \right). \quad (5.28c)$$

Similarly, collecting (5.21), (5.25) and (5.27), we find that the response for a comoving detector to the full field are given by

$$\mathcal{F}_{u,\alpha,\beta}^{in} \left(\frac{\omega}{m}, m\tau, m\tau_0 \right) = \mathcal{F}_{osc}^{in} \left(\frac{\omega}{m}, m\tau, m\tau_0 \right) + \mathcal{F}_{\alpha,\beta}^{in} \left(\frac{\omega}{m}, m\tau, m\tau_0 \right), \quad (5.29a)$$

$$\mathcal{F}_u^{out} \left(\frac{\omega}{m}, m\tau, m\tau_0 \right) = \mathcal{F}_{osc}^{out} \left(\frac{\omega}{m}, m\tau, m\tau_0 \right) + \mathcal{F}_0^{out} \left(\frac{\omega}{m}, m\tau, m\tau_0 \right). \quad (5.29b)$$

5.2.2 Non-comoving detector

Consider a non-comoving inertial trajectory in the spatially flat FRW cosmology prescribed by the worldlines

$$t(\tau) = \sqrt{(t_0 + (\tau - t_0) \cosh \theta)^2 - ((\tau - t_0) \sinh \theta)^2}, \quad (5.30a)$$

$$x(\tau) = \frac{1}{a} \operatorname{arctanh} \left(\frac{(\tau - t_0) \sinh \theta}{t_0 + (\tau - t_0) \cosh \theta} \right), \quad \theta \in \mathbb{R}, \quad (5.30b)$$

where $t_0 > 0$ is some fixed cosmic reference time and θ is the rapidity with respect to the compactified Milne spacetime's rest frame (forward light cone's rest frame). Therefore, the detector's velocity with respect to this rest frame is $\tanh \theta$.

Twisted Field

Collecting (5.9a) and (5.10a) with (5.30), we see that the responses for the non-comoving detector to the twisted field are given by (5.23) with A_n and B_n given by

$$A_n \left(\frac{\omega}{m}, m\tau, m\tau_0 \right) := \frac{1}{m} \int_{m\tau_0}^{m\tau} du J_{-i\frac{2\pi}{aL}(n+\frac{1}{2})} (f_{\theta,mt_0}(u)) \times \exp \left(-i\frac{\omega}{m}u + i\frac{2\pi}{aL} \left(n + \frac{1}{2} \right) g_{\theta,mt_0}(u) \right), \quad (5.31a)$$

$$B_n \left(\frac{\omega}{m}, m\tau, m\tau_0 \right) := \frac{1}{m} \int_{m\tau_0}^{m\tau} du H_{i\frac{2\pi}{aL}(n+\frac{1}{2})}^{(2)} (f_{\theta,mt_0}(u)) \times \exp \left(-i\frac{\omega}{m}u + i\frac{2\pi}{aL} \left(n + \frac{1}{2} \right) g_{\theta,mt_0}(u) \right), \quad (5.31b)$$

where

$$f_{\theta,mt_0}(u) := \sqrt{[mt_0 + (u - mt_0) \cosh \theta]^2 - [(u - mt_0) \sinh \theta]^2}, \quad (5.32a)$$

$$g_{\theta,mt_0}(u) := \operatorname{arctanh} \left[\frac{(u - mt_0) \sinh \theta}{mt_0 + (u - mt_0) \cosh \theta} \right]. \quad (5.32b)$$

Untwisted Field

Oscillator Modes

Similarly, collecting (5.20) and (5.30), we see that the responses for the non-comoving detector to the oscillator modes are given by formulas (5.25) with A_n and B_n given by

$$A_n \left(\frac{\omega}{m}, m\tau, m\tau_0 \right) := \frac{1}{m} \int_{m\tau_0}^{m\tau} du J_{-i\frac{2\pi}{aL}n} (f_{\theta,mt_0}(u)) \times \exp \left(-i\frac{\omega}{m}u + i\frac{2\pi n}{aL} g_{\theta,mt_0}(u) \right), \quad (5.33a)$$

$$B_n \left(\frac{\omega}{m}, m\tau, m\tau_0 \right) := \frac{1}{m} \int_{m\tau_0}^{m\tau} du H_{i\frac{2\pi}{aL}n}^{(2)} (f_{\theta,mt_0}(u)) \times \exp \left(-i\frac{\omega}{m}u + i\frac{2\pi n}{aL} g_{\theta,mt_0}(u) \right), \quad (5.33b)$$

where $f_{\theta,mt_0}(u)$ and $g_{\theta,mt_0}(u)$ are defined in (5.32).

Zero-momentum Mode

From (5.14), (5.17), (5.18) and (5.30), we see that the responses for the non-comoving detector to the zero-momentum mode are given by formulas (5.27) with M , N and B_0 given by

$$M\left(\frac{\omega}{m}, m\tau, m\tau_0\right) = \frac{1}{m} \int_{m\tau_0}^{m\tau} du J_0(f_{\theta, m\tau_0}(u)) \exp\left(-i\frac{\omega}{m}u\right), \quad (5.34a)$$

$$N\left(\frac{\omega}{m}, m\tau, m\tau_0\right) = \frac{1}{m} \int_{m\tau_0}^{m\tau} du Y_0(f_{\theta, m\tau_0}(u)) \exp\left(-i\frac{\omega}{m}u\right), \quad (5.34b)$$

$$B_0\left(\frac{\omega}{m}, m\tau, m\tau_0\right) = \frac{1}{m} \int_{m\tau_0}^{m\tau} du H_0^{(2)}(f_{\theta, m\tau_0}(u)) \exp\left(-i\frac{\omega}{m}u\right), \quad (5.34c)$$

where $f_{\theta, m\tau_0}(u)$ is defined in (5.32a).

Full field

The non-comoving detector's responses to the full field are given as in formulas (5.29) with corresponding auxiliary functions for the non-comoving trajectory given in (5.33) and (5.34).

Numerics

The expressions obtained for the responses of the comoving and non-comoving detectors are derived above in a manner suited for numerical evaluation. Here we illustrate the results in the limits where we can compare the finite-time detector's responses for both comoving and non-comoving trajectories in the spatially compactified Milne to the finite-time response of a massive scalar field moving along an inertial trajectory in the $(1+1)$ -dimensional Minkowski spacetime.

The Wightman functions for an inertial UDW detector coupled linearly to a massive scalar field in the Minkowski vacuum follows from (2.19)

$$G^+(\tau, \tau') = \begin{cases} \frac{1}{2\pi} K_0(im(\tau - \tau')), & \tau > \tau', \\ \frac{1}{2\pi} K_0(im(\tau' - \tau)), & \text{for } \tau < \tau'. \end{cases} \quad (5.35)$$

Using the analytic continuation for the modified Bessel function K_0 [41], (5.35) can be rewritten as

$$G^+(\tau, \tau') = \begin{cases} -\frac{i}{4} H_0^{(2)}(m(\tau - \tau')), & \tau > \tau', \\ \frac{i}{4} H_0^{(1)}(m(\tau' - \tau)), & \text{for } \tau < \tau'. \end{cases} \quad (5.36)$$

Collecting (2.30) and (5.36) and performing a change of integration variables ($u = m\tau/u' = m\tau'$), we obtain the detector's response

$$\begin{aligned} \mathcal{F}(\omega) = & \frac{-i}{4m^2} \int_{T_0}^{T_1} du \int_{T_0}^u du' H_0^{(2)}(u - u') \exp\left(-i\frac{\omega}{m}(u - u')\right) \\ & + \frac{i}{4m^2} \int_{T_0}^{T_1} du \int_u^{T_1} du' H_0^{(1)}(u' - u) \exp\left(-i\frac{\omega}{m}(u' - u)\right), \end{aligned} \quad (5.37)$$

where $T_0 = m\tau_0$ and $T_1 = m\tau_1$. In order to present the response function (5.37) in a form that is manifestly real, we can perform the additional change(s) of integral variables (see Appendix G for details) to obtain,

$$\mathcal{F}(\omega) = \frac{-1}{2m^2} \int_0^{\Delta T} dv \int_0^v dx \left[J_0(x) \sin\left(\frac{\omega}{m}x\right) + Y_0(x) \cos\left(\frac{\omega}{m}x\right) \right], \quad (5.38)$$

where $\Delta T = m\tau_1 - m\tau_0$.

A perspective plot of $m^2\mathcal{F}$ (5.38) and a cross section at $\Delta T = 10$ as a function of the dimensionless variables ω/m are shown in Figure 5.1.

Fulling *et al.* [102] showed using the integral representation of the Hankel function that the mode solution associated with the (continuous) mode function $\chi_\nu^{out}(\eta)$ ¹ (5.7b) can be written as a superposition of purely positive-frequency mode solutions with respect to the Minkowski time y^0 in (5.2). Thus, if we view the two-dimensional spatially compactified Milne spacetime as a manifold embedded in Minkowski spacetime, then in the large L limit the vacuum state at late times $|0_L, out\rangle$ corresponds to the conventional $(1 + 1)$ -dimensional Minkowski vacuum state $|0\rangle_{Mink}$,

¹As stated in equation (5.4), we have $\nu = |k_n|/a$ in the compactified Milne spacetime, where k_n takes on a discrete set of values. While for the non-compactified Milne spacetime, the case considered by Fulling *et al.*, we have $\nu = |k|/a$, where $k \in \mathbb{R}$ is a continuous variable.

i.e. $\lim_{L \rightarrow \infty} |0_L, out\rangle = |0\rangle_{Mink}$. Consequently, the particular limits of interest for detector's responses (5.29) in the $(1+1)$ -dimensional spatially compactified Milne spacetime are $aL \rightarrow \infty$ and $m\tau_0 \rightarrow \infty$ with $\Delta T = const$. In these limits, we expect the response function for the comoving detector in the compactified Milne spacetime in the *out*-vacuum state to approach that of an inertial detector in the $(1+1)$ -dimensional Minkowski spacetime, initially prepared in the conventional Minkowski vacuum state (5.38). As pointed out in Remark 5.1, when $\alpha = 0$ and $\sqrt{a}\beta = \frac{\sqrt{\pi}}{2}$, the zero-momentum contributions to the massive scalar field in the *in* and *out* vacua are equal. Therefore, by changing the values of α and β in (5.29a) slightly from the aforementioned values, the limits of interest can equally be used to give some insight into how the *in*-vacuum state in the spatially compactified Milne differs from the *out*-vacuum state for both a comoving and a non-comoving observers. We illustrate this for the response functions of the comoving detector (5.29) in the form of the numerical results presented in Figures 5.2, 5.3 and 5.4.

5.2.3 Discussion

In essence, Figures 5.1, 5.2, 5.3 show that in the large aL limit or when $m\tau_0 \rightarrow \infty$ with $\Delta T = const.$, the late time vacuum state, $|0_L, out\rangle$, and the early time vacuum state with $\alpha = 0$ and $\sqrt{a}\beta = \frac{\sqrt{\pi}}{2}$ ($|0_L, in\rangle_{\frac{1}{2}\sqrt{\frac{\pi}{a}}} \otimes |0_L^{osc}, in\rangle$) are as close to the conventional Minkowski vacuum as you would expect; so there is almost no spontaneous excitation of the detector, only de-excitation which probability of detection that peaks when the field's quanta rest energy $|\omega| = m$ ($\hbar|\omega| = mc^2$). Furthermore, these plots suggest that the limits where $aL = 1$ or $[m\tau_0, m\tau_1] = [50, 60]$ is suitable enough for this observation. However, when we change the in-vacuum state to a different set of values for the two-real parameters $\alpha \neq 0$ and $\beta \neq \frac{1}{2}\sqrt{\frac{\pi}{a}}$, we observe, in addition to the expected de-excitation (see Figure 5.4), spontaneous excitation of the comoving detector with probability of detection that peaks when $|\omega| = m$. This seems to imply that when the in-vacuum is chosen to be very far from the conven-

tional Minkowski vacuum state, then it contains particles. As result, spontaneous excitation of the comoving detector occurs.

In contrast, the green curves in Figures 5.2, 5.3 display the response function of the comoving detector when aL takes on small values and when $[m\tau_0, m\tau_1] = [10, 20]$. We plotted these values to show that the response function in the compactified Milne vacuum is generally different than the response function in the conventional Minkowski vacuum. The two vacua are identified only in the limit where aL is large. Although we can not compare the behaviour of the response function in the limit where $aL = 0.2$ (as displayed by the green curves) with known results, we can nonetheless conclude that the wiggles are not due to numerical noise as they also present in the limit where $[m\tau_0, m\tau_1] = [10, 20]$.

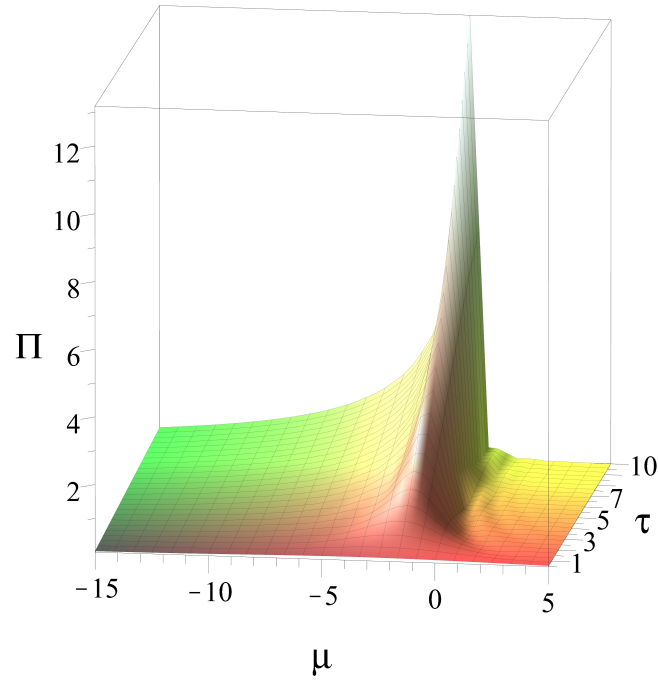
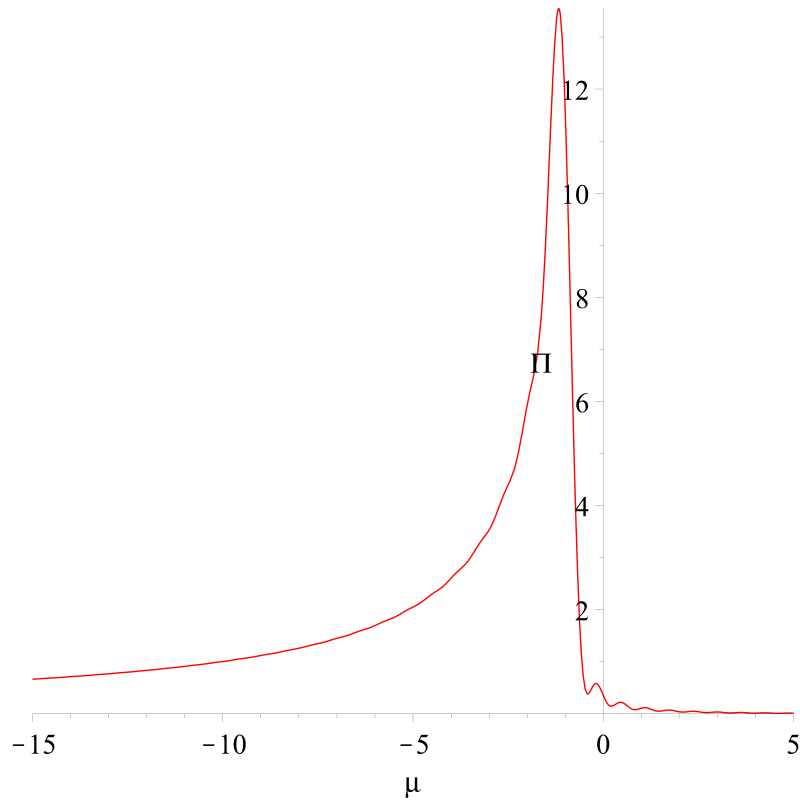

 (a) $m^2 \mathcal{F}$

 (b) $m^2 \mathcal{F}$ for $\Delta T = 10$

Figure 5.1: A perspective plot of $m^2 \mathcal{F}$ and a cross-section at $\Delta T = 10$ for the inertial detector (5.38) in terms of the dimensionless variable ω/m . Note: Π , τ and μ in the axis labels stand, respectively, for $m^2 \mathcal{F}$, ΔT and ω/m .

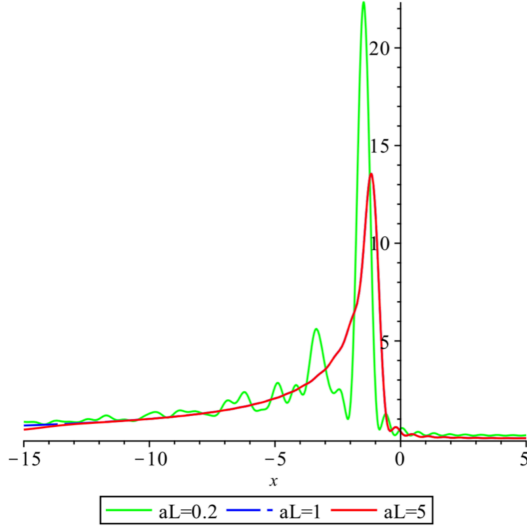
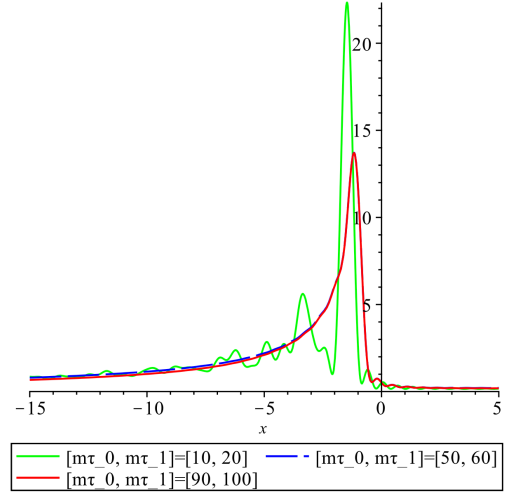
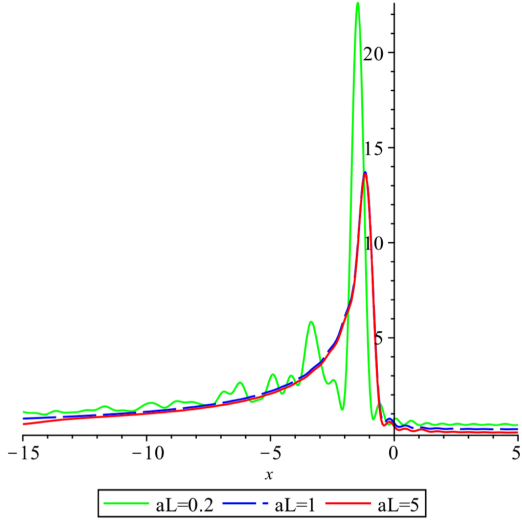
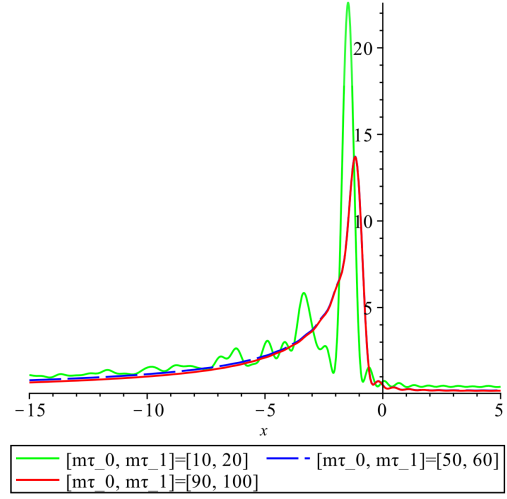
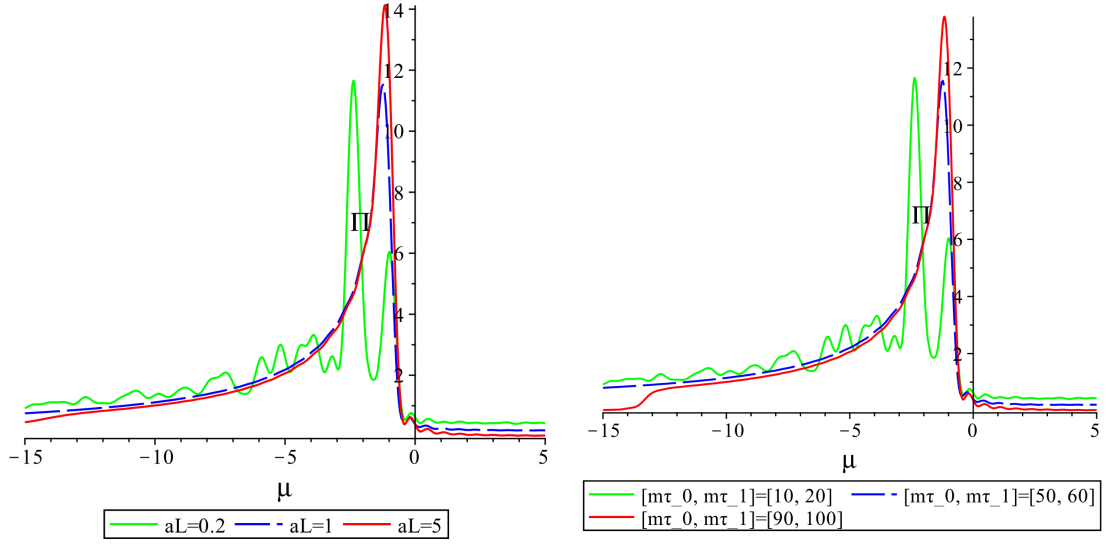
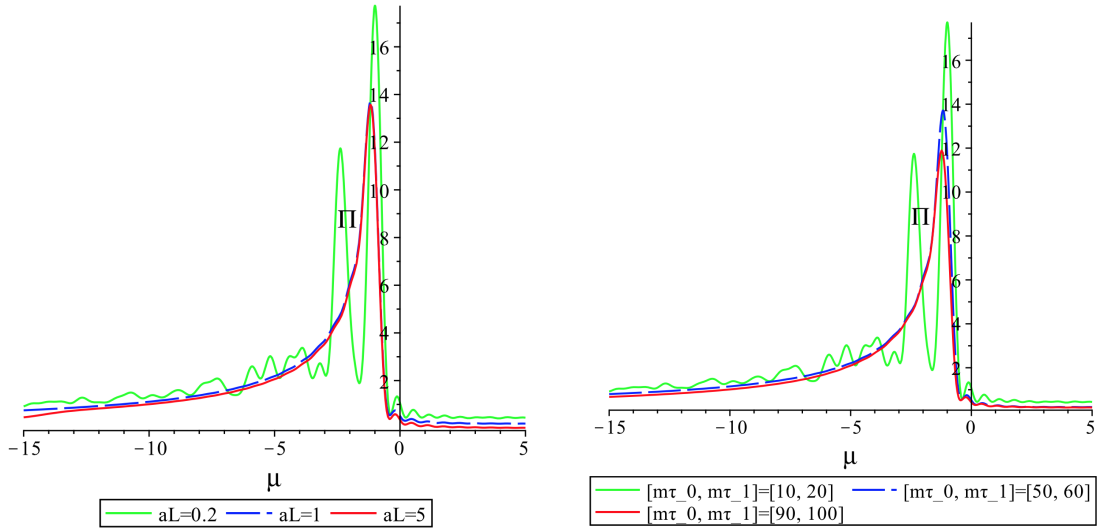

 (a) $m^2 \mathcal{F}_t^{out}$ for $[m\tau_0, m\tau_1] = [10, 20]$

 (b) $m^2 \mathcal{F}_t^{out}$ for $aL = 0.2$

 (c) $m^2 \mathcal{F}_t^{in}$ for $[m\tau_0, m\tau_1] = [10, 20]$

 (d) $m^2 \mathcal{F}_t^{in}$ for $aL = 0.2$

Figure 5.2: Cross-sectional plots of $m^2 \mathcal{F}_t^{out}$ and $m^2 \mathcal{F}_t^{in}$ for the twisted field (5.23) in, respectively, the *out* and *in* vacua for the comoving detector in terms of the dimensionless variable ω/m . Note: Π and μ in the axis labels stand, respectively, for $m^2 \mathcal{F}_t^{out}$ and ω/m . In 5.2(a) and 5.2(b) the green curve is the same in both plots. Similarly, in 5.2(c) and 5.2(d) the green curve is the same for both plots. Note that in In 5.2(c), the red curve partly masks the blue curve since they almost coincide.


 (a) $m^2 \mathcal{F}_u^{out}$ for $[m\tau_0, m\tau_1] = [10, 20]$

 (b) $m^2 \mathcal{F}_u^{out}$ for $aL = 0.2$

 (c) $m^2 \mathcal{F}_{u,\alpha,\beta}^{in}$ for $\sqrt{a}\alpha = 0$ and $\sqrt{a}\beta = \frac{\sqrt{\pi}}{2}$

 (d) $m^2 \mathcal{F}_{u,\alpha,\beta}^{in}$ for $\sqrt{a}\alpha = 0$ and $\sqrt{a}\beta = \frac{\sqrt{\pi}}{2}$

Figure 5.3: Cross-sectional plots of $m^2 \mathcal{F}_u^{out}$ and $m^2 \mathcal{F}_{u,\alpha,\beta}^{in}$ for the untwisted field in, respectively, the *out* and *in* vacua (5.29) for the comoving detector in terms of the dimensionless variable ω/m . Note: Π and μ in the axis labels stand, respectively, for $m^2 \mathcal{F}_u^{out}$ and ω/m . In 5.3(a) and 5.3(b) the green curve is the same in both plots. Similarly, In 5.3(c), where $[m\tau_0, m\tau_1] = [10, 20]$, and 5.3(d), where $aL = 0.2$, the green curve is again the same for both plots. Note that in 5.3(c), the red curve partly masks the blue curve since they almost coincide.

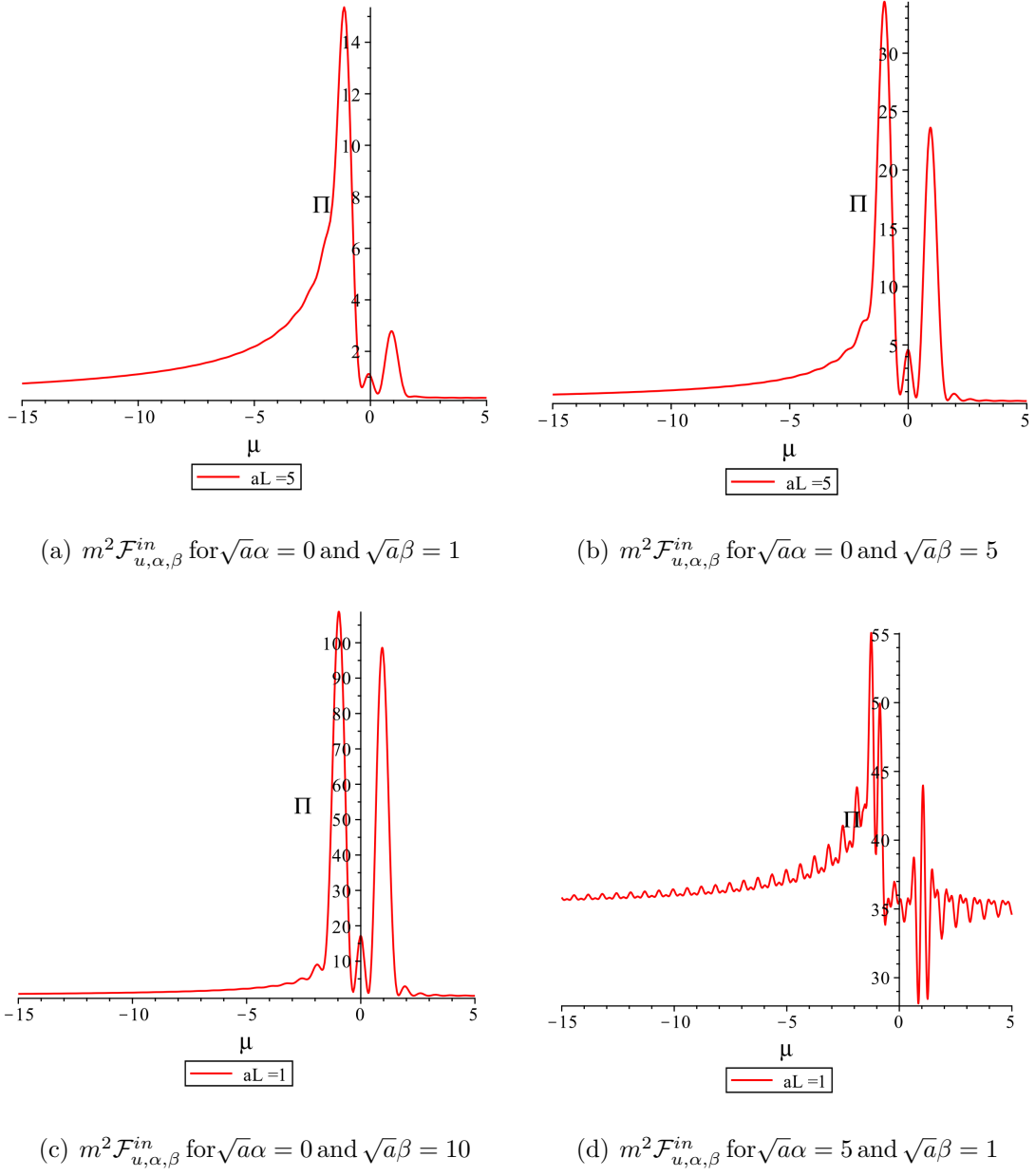


Figure 5.4: Cross-sectional plots of $m^2 \mathcal{F}_{u,\alpha,\beta}^{in}$ for the untwisted field in the *in*-vacuum for the comoving detector in terms of the dimensionless variable ω/m . We use the integral range $[m\tau_0, m\tau_1] = [10, 20]$ with $aL = 0.2$ for these plots. Note: Π and μ in the axis labels stand, respectively, for $m^2 \mathcal{F}_{u,\alpha,\beta}^{in}$ and ω/m . The values of α and β are stated below each plot.

Chapter 6

Conclusions

This thesis is concerned with quantum field theory in flat and curved spacetimes of arbitrary dimensions along with detecting bosons and fermions on these spacetimes.

6.1 Summary

In Chapter 1, we gave an overview of quantum field theory in curved spacetimes (QFTCS). In particular, we discussed that above the Planck scale, the gravitational field may be considered as a fixed classical background while the classical matter fields must be quantized according to quantum theory [1].

In Chapter 2 and Chapter 3, we have analysed the response of a spatially pointlike Unruh-DeWitt detector coupled linearly to the scalar density of a massless Dirac field in Minkowski spacetimes in dimension $d \geq 2$ and on the $(1+1)$ -dimensional flat static cylinder, allowing the detector's motion to remain arbitrary and allowing the detector to be switched on and off in an arbitrary smooth way. Working within first-order perturbation theory, we regularised the interaction by dropping an additive term that is technically ill-defined but formally proportional to the field's mass [37, 38].

In d -dimensional Minkowski, with the field in its Fock vacuum, we found that the response is identical to that of a detector coupled linearly to a massless scalar field in

$2d$ spacetime dimensions. For a uniformly linearly accelerated detector, this implies that the long time limit of the response exhibits the Unruh effect with a Planckian frequency dependence factor, for all d . While the Rindler power spectrum of the Dirac field is known to have a Planckian factor for odd d but a Fermi-Dirac factor for even d [35], the detector's response is Planckian for all d because the response is not proportional to the Rindler power spectrum but to the convolution of the Rindler power spectrum with itself.

In the special case of two-dimensional Minkowski, we saw that the detector's response has no infrared ambiguity. In this respect our detector differs from the detector coupled linearly to a massless scalar field, where in two dimensions the response is ambiguous due to the infrared ambiguity of the Wightman function [40].

On the $(1+1)$ -dimensional flat static cylinder, we found that the response distinguishes the Fock vacua of the field's oscillator modes for periodic and antiperiodic spin structures, and the zero mode that occurs for the periodic spin structure contributes to the response in a way that depends on the zero mode's initial state. We also provided a selection of analytic and numerical results for inertial and uniformly accelerated trajectories on the cylinder, recovering the $d = 2$ Minkowski results in the limit of large circumference.

In Chapter 4, we presented a detailed discussion of quantum field theory in curved spacetimes. Firstly, we discussed the quantization of the free Dirac field in an arbitrary curved, hyperbolic spacetime of arbitrary dimensions. Then, we gave what is intended to be a complete treatment for the Bogolubov transformation of the Dirac field as this is not as widely discussed as the Bogolubov transformation for the scalar field. Using the Bogolubov transformation we showed that the choice of a complete set of mode solutions is generally not unique and that different choices may involve different splits into positive and negative frequencies. As a result the fermionic Fock vacuum state is generally nonunique. Neither is the the notion of Fock particle states defined as eigenstates of the number operators. In general, the vacuum associated with a set of complete mode solutions to Dirac equation will

contain particles in another set of modes solution. Hence, the two sets do not share a common vacuum and will lead to a different number of particles. In short, the notion of particles in quantum field theory as excitations above the Fock vacuum of a free quantized field is a basis-dependent notion.

In Section 4.2.1 and Section 4.2.4, we discussed the quantization of the free Dirac field and the free scalar field for certain classes of cosmological spacetimes wherein the two-point functions have the Hadamard form. We showed that the Wightman functions for the Dirac field for the aforementioned classes of FRW spacetimes reduced to the flat spacetime counterparts in the limit where the scale factor $C(\eta) = 1$. In Section 4.2.5, we quantized the massive conformally coupled scalar field in the $(1 + 1)$ -dimensional FRW spacetimes with compactified spatial sections. There, we considered both twisted and untwisted fields, defined respectively, by imposing periodic and antiperiodic boundary conditions. While the quantization for the twisted field follows from the general quantization scheme, the untwisted field contains a conformal zero mode that arises whenever the effective mass vanishes at early or late times. More precisely we showed that this occurs whenever the zero-momentum mode solution of the scalar field's wave equation reduces to a conformal zero-mode in the corresponding asymptotic region(s). To deal with this issue, we developed a new quantization scheme for the zero-momentum mode. This new quantization scheme introduces a family of two real parameters for every zero-momentum mode with an associated two-real parameter set of *in/out* vacua. We then showed that the zero momentum started state's wave functional corresponds to a two-real-parameter set of Gaussian wave packets. Finally, we obtained explicit expression for the Wightman two-point functions in the *in* and *out* vacua.

In Section 4.3, we gave a brief review of the Unruh-DeWitt particle detector model coupled to a linear quantized scalar field. This particle detector can absorb or emit field quanta as it moves along its smooth, timelike trajectory, $x(\tau)$, parametrized by its proper time. In Section 4.3.1, we then established the equivalence between the adiabatic vacuum of infinite order and the conformal vacuum in

the massless limit. With the field in the conformal Fock vacuum, we then showed that the detector's response to an UDW particle detector coupled linearly to the scalar density of a massless Dirac field in the spatially flat FRW spacetimes in d -dimensions is identical to the response of a detector coupled to the massless scalar field in the spatially flat FRW spacetimes in $2d$ -dimensions.

In Chapter 5, we applied the formalism developed in Section 4.2.5 to the $(1+1)$ -dimensional spatially compactified Milne spacetime. In this case, the vanishing of the effective mass at early times causes the zero-momentum to become a conformal zero mode at early times while behaving as a positive frequency mode at late times. We obtained analytical results for the response functions for both the inertial comoving and non-comoving trajectories. We then presented numerical results for the comoving detector and showed that they agree with the response function of a massive detector in the $(1+1)$ -dimensional Minkowski in the limits where $aL \rightarrow \infty$ or when $m\tau_0 \rightarrow \infty$ with $\Delta T = \text{const}$. Moreover, we showed that when the in-vacuum state are chosen with the set of values for the two-real parameters $\alpha \neq 0$ and $\beta \neq \frac{1}{2}\sqrt{\frac{\pi}{a}}$, we observe, in addition to the expected de-excitation, spontaneous excitation of the comoving detector with probability of detection that peaks when $|\omega| = m$. This indicates that when the in-vacuum two-real parameter family of states is chosen far from the conventional Minkowski vacuum state, then this state contains particles. As a result, spontaneous excitation of the comoving detector occurs.

6.2 Overall Conclusions

The first set of broad conclusions of this thesis concerns the relation between the response of an Unruh-DeWitt detector coupled linearly to a scalar field and that of an Unruh-DeWitt detector coupled linearly to the scalar density of a fermion field. While there is well known literature on this relationship for uniform linear acceleration in Minkowski spacetime, we were able to show that for massless fields there

is a simple relationship for general worldlines in Minkowski spacetime of arbitrary dimension. A consequence in the special case of uniform linear acceleration is that the fermionic detector's response has a Planck-type factor in all spacetime dimensions: an erroneous statement to the contrary in the landmark work of Takagi [35] appears to have remained unnoticed until now.

Adapting from Minkowski spacetime to the $(1+1)$ -dimensional Minkowski cylinder, we were able to numerically demonstrate how the fermionic detector's excitations and de-excitations correspond to excitations of the field quanta in *pairs*, with the fermion's spin structure and (for the periodic spin structure) the state of the fermion's zero mode making a distinct imprint in the detector's response.

The second set of broad conclusions concerns the response of a detector in a cosmologically expanding spacetime. Some of these upshots were technical, including the identification of appropriate asymptotic initial and final states of the field, and the implementation of nontrivial spatial topology, for bosonic and fermionic fields. A conceptual highlight was, however, that our numerical work on a bosonic detector in the $(1+1)$ -dimensional spatially periodic Milne universe exhibited not only effects due to the local expansion but also effects due to the nontrivial spatial topology, and it allowed us to identify the regimes in which each effect dominates.

6.3 Future Work

There are many possible directions for future work on the topics discussed in this thesis. Further work for the finite-time detector's response in the $(1+1)$ -dimensional spatially compactified Milne spacetime could include obtaining numerics for the inertial non-comoving trajectory. We expect these plots would be different than the ones for the comoving trajectory. The extra parameters (mt_0, θ) contained in the formulas (5.31) and (5.33) would cause the peaks of spontaneous excitation and de-excitation in *in* and *out* vacuum to occur at values other than $|\omega| = m$ as is observed for the case of the comoving trajectory. Similar to what is observed for the inertial

comoving trajectory, we expect that for the non-comoving trajectory, spontaneous excitation of the detector would occur when $\alpha = 0$ and $\beta = \frac{1}{2}\sqrt{\frac{\pi}{a}}$. This would further confirm the observation made by Wald [6] that it is the interaction of the field in its vacuum state that is responsible for spontaneous de-excitation. A similar line of research would be to carry out the same analysis in the $(1+1)$ -dimensional spatially compactified de Sitter spacetime with line element

$$ds^2 = dt^2 - \frac{1}{H^2} \sinh^2(Ht) dy^2, \quad \text{where } (t, y) \in (t, y+L), L > 0$$

$$ds^2 = dt^2 - \frac{1}{H^2} \cosh^2(Ht) dy^2, \quad \text{where } (t, z) \in (t, z+2\pi).$$

In analogy with the computation for the Wightman two-point function (4.136), a further line of research on the spatially compactified Milne and other similar cosmological models would be to compute the *in* and *out* vacuum expectation values of the scalar field stress tensor operator for the untwisted field,

$$\langle \hat{T}_{\mu\nu} \rangle_u := \langle 0_L^{\text{osc}}, in/out | \otimes_{\alpha, \beta} \langle 0_L, in/out | \hat{T}_{\mu\nu} | 0_{in/out} \rangle_{\alpha, \beta} \otimes | 0_L^{\text{osc}}, in/out \rangle$$

and compare it with the expectation values for stress tensor operator for the twisted field,

$$\langle \hat{T}_{\mu\nu} \rangle_t := \langle 0_L, in/out | \hat{T}_{\mu\nu} | 0_L, in/out \rangle$$

One could then investigate how the compactified sections modify the long wavelength modes of the field as was analysed in [5] for the two-dimensional static cylinder embedded in Minkowski spacetime.

Part III

Appendices

Appendix A

Gamma matrices and basis spinors

In this appendix we record relevant properties of the gamma-matrices and the massive basis spinors in spacetime dimension $d \geq 2$. More detail can be found in [7].

The gamma-matrices γ^μ , $\mu = 0, 1, \dots, d-1$, are $N_d \times N_d$ matrices with

$$N_d = \begin{cases} 2^{d/2} & \text{for } d \text{ even,} \\ 2^{(d-1)/2} & \text{for } d \text{ odd,} \end{cases} \quad (\text{A.1})$$

satisfying

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \ , \quad (\text{A.2})$$

where on the right hand side we have suppressed the identity matrix $I_{N_d \times N_d}$. γ^0 is Hermitian, $\gamma^1, \dots, \gamma^{d-1}$ are anti-Hermitian, $\text{Tr}(\gamma^\mu) = 0$, and $\text{Tr}(\gamma^\mu \gamma^\nu) = N_d \eta^{\mu\nu}$. To find a representation for the γ -matrices, we first need to introduce an extra gamma matrix called the chiral matrix in $d = 2n$ even dimensions defined via

$$\gamma_{d+1} \equiv \eta \gamma^0 \gamma^1 \dots \gamma^{d-1} \quad (\text{A.3})$$

Where η is chosen by imposing the following properties

$$(\gamma_{d+1})^2 = I, \quad \gamma_{d+1}^\dagger = \gamma_{d+1} \ . \quad (\text{A.4})$$

This yields

$$\eta^2 = (-1)^{(d-2)/2} \implies \eta = \pm i^{(d-2)/2} \ . \quad (\text{A.5})$$

It is straightforward to show that

$$\{\gamma^\mu, \gamma_{d+1}\} = 0. \quad (\text{A.6})$$

Moreover since $(\gamma_{d+1})^2 = I$, we have $\gamma_{d+1} = \pm I$ or $\gamma_{d+1} = QDQ^{-1}$, where

$$D = \begin{pmatrix} \pm I & 0 \\ 0 & \mp I \end{pmatrix}, \quad (\text{A.7})$$

and Q is some invertible matrix in the space of $2n \times 2n$ matrices. Here the top sign (+ up, - down) corresponds to $\eta = -i^{(d-2)/2}$ while the bottom sign (- up, + down) corresponds to $\eta = +i^{(d-2)/2}$. I is the identity matrix in the space of $N \times N$ matrices with $N = 2^{n-1}$. Moreover, because the gamma matrices are unique up to a similarity transformation, $\gamma_{d+1} = D$ is a valid choice of a chiral representation in $d = 2n$ dimension. This choice is known in the literature as the Weyl representation. Clearly this choice exists in all even spacetime dimensions. From here on, we will stick to this representation. Moreover we will use the most widely used Weyl representation

$$\gamma_{d+1} = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}. \quad (\text{A.8})$$

Denote the set of gamma matrices in $d = 2n$ dimension by $\{\gamma^0, \gamma^1, \dots, \gamma^{2n-1}\}$. These matrices have dimension $N \times N$, given in (A.1), where $N = 2^{d/2} = 2^n$. In going from $d = 2n$ to $d = 2n + 1$, the set of gamma matrices have the same dimensions, $N = 2^{(d-1)/2} = 2^n$. Therefore, we can use the same set of gamma matrices as in $d = 2n$, with the extra gamma matrices defined in terms of the chiral matrix in $d = 2n$. Thus, the set of gamma matrices in $d = 2n + 1$ are $\{\gamma^0, \gamma^1, \dots, \gamma^{2n}\}$, where $\gamma^{2n} \equiv i\gamma_{2n+1}$.

To go from $d = 2n$ to $d = 2n + 2$ dimensional spacetimes, denote the gamma matrices in $d = 2n + 2$ by $\{\tilde{\gamma}^0, \tilde{\gamma}^1, \dots, \tilde{\gamma}^{2n+1}\}$. Given that we can easily construct the set of gamma matrices in $d = 2$, we can construct the higher dimensional gamma

matrices via the following recursive relations

$$\tilde{\gamma}^0 = \begin{pmatrix} 0 & I_{2n} \\ I_{2n} & 0 \end{pmatrix} \quad \tilde{\gamma}^1 = \begin{pmatrix} 0 & \gamma^0 \\ -\gamma^0 & 0 \end{pmatrix} \quad (\text{A.9a})$$

$$\tilde{\gamma}^{j+1} = \begin{pmatrix} 0 & -i\gamma^j \\ i\gamma^j & 0 \end{pmatrix}, \quad (j = 1, \dots, 2n-1) \quad (\text{A.9b})$$

$$\tilde{\gamma}^{2n+1} = \begin{pmatrix} 0 & \gamma_{2n+1} \\ -\gamma_{2n+1} & 0 \end{pmatrix}, \quad (\text{A.9c})$$

where we denote the $2^n \times 2^n$ identity matrix by I_{2n} . It can be easily verified that this set of gamma matrices satisfy the Clifford algebra (A.2). In $d = 2$ spacetime dimension, we define the set of gamma matrices in terms of the Pauli matrices $\{\gamma^0 = \sigma_1, \gamma^1 = i\sigma_2\}$. By this formulation, in $d = 3$ dimension, the set of gamma matrices are $\{\gamma^0 = \sigma_1, \gamma^1 = i\sigma_2, \gamma^2 = i\gamma_3 = i(\gamma^0\gamma^1) = -\sigma_3\}$. Here γ_3 is the chiral matrix in the 2-dimensional spacetime. We should remark that in the Weyl basis, in any dimension, the γ^0 matrix takes the form

$$\gamma^0 = \begin{pmatrix} 0 & I_{N_1 \times N_1} \\ I_{N_1 \times N_1} & 0 \end{pmatrix}, \quad N_1 = \begin{cases} 2^{(d-2)/2}, & \text{if } d \text{ is even} \\ 2^{(d-3)/2}, & \text{if } d \text{ is odd} \end{cases}. \quad (\text{A.10})$$

This turns out to be closely related to the number of spin states in d -dimensions as will be discussed at the end of this appendix.

Relevant Relations

Let $u^{(s)}(\mathbf{0})$ and $v^{(s)}(\mathbf{0})$ be eigenspinors of γ^0 such that

$$\gamma^0 u^{(s)}(\mathbf{0}) = u^{(s)}(\mathbf{0}), \quad (\text{A.11a})$$

$$\gamma^0 v^{(s)}(\mathbf{0}) = -v^{(s)}(\mathbf{0}), \quad (\text{A.11b})$$

with the orthonormality conditions

$$u^{(s)\dagger}(\mathbf{0})u^{(s')}(\mathbf{0}) = v^{(s)\dagger}(\mathbf{0})v^{(s')}(\mathbf{0}) = 2m\delta^{ss'} , \quad (\text{A.12a})$$

$$u^{(s)\dagger}(\mathbf{0})v^{(s')}(\mathbf{0}) = 0 , \quad (\text{A.12b})$$

where the helicity index s takes the values $s = 1, \dots, N_d/2$. The spinors $u^{(s)}(\mathbf{k})$ and $v^{(s)}(\mathbf{k})$ are defined by

$$u^{(s)}(\mathbf{k}) = \frac{\gamma^\mu k_\mu + m}{\sqrt{2m(k^0 + m)}} u^{(s)}(\mathbf{0}) , \quad (\text{A.13a})$$

$$v^{(s)}(\mathbf{k}) = \frac{-\gamma^\mu k_\mu + m}{\sqrt{2m(k^0 + m)}} v^{(s)}(\mathbf{0}) , \quad (\text{A.13b})$$

where $k^0 = (\mathbf{k}^2 + m^2)^{1/2}$. These spinors satisfy

$$(\gamma^\mu k_\mu - m)u^{(s)}(\mathbf{k}) = 0 , \quad (\text{A.14a})$$

$$(\gamma^\mu k_\mu + m)v^{(s)}(\mathbf{k}) = 0 . \quad (\text{A.14b})$$

The orthonormality conditions are

$$u^{(s)\dagger}(\mathbf{k})u^{(s')}(\mathbf{k}) = v^{(s)\dagger}(\mathbf{k})v^{(s')}(\mathbf{k}) = 2k^0\delta^{ss'} , \quad (\text{A.15a})$$

$$\bar{u}^{(s)}(\mathbf{k})u^{(s')}(\mathbf{k}) = -\bar{v}^{(s)}(\mathbf{k})v^{(s')}(\mathbf{k}) = 2m\delta^{ss'} , \quad (\text{A.15b})$$

$$\bar{u}^{(s)}(\mathbf{k})v^{(s')}(\mathbf{k}) = 0 , \quad (\text{A.15c})$$

and the completeness identities are

$$\sum_s u_a^{(s)}(\mathbf{k})\bar{u}_b^{(s)}(\mathbf{k}) = (\gamma^\mu k_\mu + m)_{ab} , \quad (\text{A.16a})$$

$$\sum_s v_a^{(s)}(\mathbf{k})\bar{v}_b^{(s)}(\mathbf{k}) = (\gamma^\mu k_\mu - m)_{ab} . \quad (\text{A.16b})$$

Remark A.1. *We can make the following remark about the properties of electrons and positrons in 3D spacetime. Because the number of spin states in a d -dimensional Minkowski spacetime depends on the dimension, according to (A.10), $N_1 = 2^{(d-2)/2}$ for even dimensional spacetime and $N_1 = 2^{(d-3)/2}$ for odd dimensional spacetime, which we show corresponds to the number of spin states. As is well known, the*

electron and positron can exist in two independent spin states ($N_1 = 2^{(4-2)/2} = 2$) in four dimensions (spin up and spin down). The spin is a two-level system with two allowed values for the spin components. However, in a three-dimensional spacetime, say a stationary two-dimensional crystal surface in an inertial frame, the electron and the positron have $N_1 = 2^{(3-3)/2} = 1$ spin state (up or down but not both). Thus, the spin state is a one level system with one allowed value.

Appendix B

$W^{(2,\bar{2})}$ in Minkowski vacuum

In this appendix we write out the correlation function $W^{(2,\bar{2})}(x, y)$ (2.5) in Minkowski spacetime in the Minkowski vacuum $|0\rangle$ in terms of the Wightman functions $S^\pm(x, y)$ (2.17). We treat the singular expression $S^-(x, x)$ here as a formal algebraic symbol but will address its interpretation in the main text.

Setting $|\Psi_0\rangle = |0\rangle$, (2.5) gives

$$\begin{aligned} W^{(2,\bar{2})}(x, y) &= \langle 0 | \bar{\psi}(x) \psi(x) \bar{\psi}(y) \psi(y) | 0 \rangle \\ &= \langle 0 | \bar{\psi}_a(x) \psi_a(x) \bar{\psi}_b(y) \psi_b(y) | 0 \rangle , \end{aligned} \quad (\text{B.1})$$

where each repeated spinor index is summed over.

We use the decomposition

$$\begin{aligned} \bar{\psi}(x) \psi(x) &= N[\bar{\psi}(x) \psi(x)] + \{\bar{\psi}_a^+(x), \psi_a^-(x)\} \\ &= N[\bar{\psi}(x) \psi(x)] + \text{Tr}[S^-(x, x)] , \end{aligned} \quad (\text{B.2})$$

where N stands for the Wick normal product of a fermionic field,

$$N[\bar{\psi}(x) \psi(x)] := \bar{\psi}_a^+(x) \psi_a^+(x) + \bar{\psi}_a^-(x) \psi_a^-(x) + \bar{\psi}_a^-(x) \psi_a^+(x) - \psi_a^-(x) \bar{\psi}_a^+(x) , \quad (\text{B.3})$$

and the last step in (B.2) uses (2.17). From (4.86) we have

$$\psi^+ |0\rangle = 0 = \langle 0 | \psi^- , \quad (\text{B.4a})$$

$$\bar{\psi}^+ |0\rangle = 0 = \langle 0 | \bar{\psi}^- , \quad (\text{B.4b})$$

which shows that $\langle 0|N[\bar{\psi}(x)\psi(x)]|0\rangle = 0$. As $\text{Tr}[S^-(x, x)]$ is proportional to the identity operator in the Fock space, we hence have

$$W^{(2,\bar{2})}(x, y) = \langle 0|N[\bar{\psi}(x)\psi(x)]N[\bar{\psi}(y)\psi(y)]|0\rangle + \text{Tr}[S^-(x, x)] \text{Tr}[S^-(y, y)] . \quad (\text{B.5})$$

For the first term in (B.5), we obtain

$$\begin{aligned} \langle 0|N[\bar{\psi}(x)\psi(x)]N[\bar{\psi}(y)\psi(y)]|0\rangle &= \langle 0|\bar{\psi}_a^+(x)\psi_a^+(x)\bar{\psi}_b^-(y)\psi_b^-(y)|0\rangle \\ &= \langle 0|\bar{\psi}_a^+(x)\psi_b^-(y)\psi_a^+(x)\bar{\psi}_b^-(y)|0\rangle \\ &= \langle 0|\{\bar{\psi}_a^+(x), \psi_b^-(y)\}\{\psi_a^+(x), \bar{\psi}_b^-(y)\}|0\rangle \\ &= S_{ba}^-(y, x)S_{ab}^+(x, y) \\ &= \text{Tr}[S^+(x, y)S^-(y, x)] , \end{aligned} \quad (\text{B.6})$$

first using (B.4), then anticommuting $\psi_b^-(y)$ past $\bar{\psi}_b^-(y)$ and $\psi_a^+(x)$, then using again (B.4) to replace $\bar{\psi}_a^+(x)\psi_b^-(y)$ and $\psi_a^+(x)\bar{\psi}_b^-(y)$ by anticommutators, and finally using the definition (2.17) of the Wightman functions $S^\pm$ and cyclicity of the trace.

Collecting, we have

$$W^{(2,\bar{2})}(x, y) = \text{Tr}[S^+(x, y)S^-(y, x)] + \text{Tr}[S^-(x, x)] \text{Tr}[S^-(y, y)] , \quad (\text{B.7})$$

from which (2.21) follows using the translational invariance of $S^-(x, y)$.

Appendix C

$W^{(2,\bar{2})}$ of the untwisted massless Dirac field

In this appendix we justify formula (3.26) for $W^{(2,\bar{2})}$ of the untwisted massless Dirac field on the $(1+1)$ -dimensional cylinder.

Starting with (3.26), inserting the split (3.13) and noting that terms with an unequal number of $\overline{\psi}^{\text{osc}}$ s and ψ^{osc} s vanish, we obtain

$$\begin{aligned} W^{(2,\bar{2})}(t, x; t', x') &= \langle \widetilde{\text{ZM}} | \overline{\psi}_a(t, x) \psi_a(t, x) \overline{\psi}_b(t', x') \psi_b(t', x') | \widetilde{\text{ZM}} \rangle \\ &= \Xi_1 + \Xi_2 + \Xi_3 + \Xi_4 + \Xi_5 + \Xi_6, \end{aligned} \quad (\text{C.1})$$

where

$$\Xi_1 = \langle \widetilde{\text{ZM}} | \overline{\psi}_a^{\text{osc}}(t, x) \psi_a^{\text{osc}}(t, x) \overline{\psi}_b^{\text{osc}}(t', x') \psi_b^{\text{osc}}(t', x') | \widetilde{\text{ZM}} \rangle, \quad (\text{C.2a})$$

$$\Xi_2 = \langle \widetilde{\text{ZM}} | \overline{\psi}_a^{\text{osc}}(t, x) \psi_a^{\text{osc}}(t, x) \overline{\psi}_b^{\text{zm}} \psi_b^{\text{zm}} | \widetilde{\text{ZM}} \rangle, \quad (\text{C.2b})$$

$$\Xi_3 = \langle \widetilde{\text{ZM}} | \overline{\psi}_a^{\text{osc}}(t, x) \psi_a^{\text{zm}} \overline{\psi}_b^{\text{zm}} \psi_b^{\text{osc}}(t', x') | \widetilde{\text{ZM}} \rangle, \quad (\text{C.2c})$$

$$\Xi_4 = \langle \widetilde{\text{ZM}} | \overline{\psi}_a^{\text{zm}} \psi_a^{\text{osc}}(t, x) \overline{\psi}_b^{\text{osc}}(t', x') \psi_b^{\text{zm}} | \widetilde{\text{ZM}} \rangle, \quad (\text{C.2d})$$

$$\Xi_5 = \langle \widetilde{\text{ZM}} | \overline{\psi}_a^{\text{zm}} \psi_a^{\text{zm}} \overline{\psi}_b^{\text{osc}}(t', x') \psi_b^{\text{osc}}(t', x') | \widetilde{\text{ZM}} \rangle, \quad (\text{C.2e})$$

$$\Xi_6 = \langle \widetilde{\text{ZM}} | \overline{\psi}_a^{\text{zm}} \psi_a^{\text{zm}} \overline{\psi}_b^{\text{zm}} \psi_b^{\text{zm}} | \widetilde{\text{ZM}} \rangle, \quad (\text{C.2f})$$

and each repeated spinor index is summed over.

For Ξ_1 , we may proceed as in the derivation of formula (B.7) in Appendix B. Dropping the ill-defined second term in the counterpart of (B.7), Ξ_1 reduces to $W_{\text{osc}}^{(2,\bar{2})}(t, x; t', x')$ as evaluated in subsection 3.0.3, with the result given in (3.18).

Ξ_6 reduces to $W_{\text{zm}}^{(2,\bar{2})}$ as evaluated in subsection 3.0.3, with the result (3.24).

Ξ_2 is proportional to $\text{Tr}[S_{ab}^{\text{osc}-}(t, x; t, x)]$, where $S_{ab}^{\text{osc}-}$ is given by (3.16b). This expression is not well defined because of the coincidence limit, but we interpret the expression as zero by the tracelessness of the gamma-matrices. Similarly, we interpret Ξ_5 as zero.

For Ξ_3 and Ξ_4 we have

$$\begin{aligned}
\Xi_3 + \Xi_4 &= \langle \widetilde{\text{ZM}} | \overline{\psi}_a^{\text{osc}}(t, x) \psi_b^{\text{osc}}(t', x') \psi_a^{\text{zm}} \overline{\psi}_b^{\text{zm}} | \widetilde{\text{ZM}} \rangle \\
&\quad + \langle \widetilde{\text{ZM}} | \psi_a^{\text{osc}}(t, x) \overline{\psi}_b^{\text{osc}}(t', x') \overline{\psi}_a^{\text{zm}} \psi_b^{\text{zm}} | \widetilde{\text{ZM}} \rangle \\
&= \langle 0^{\text{osc}} | \overline{\psi}_a^{\text{osc}}(t, x) \psi_b^{\text{osc}}(t', x') | 0^{\text{osc}} \rangle \langle \text{ZM} | \psi_a^{\text{zm}} \overline{\psi}_b^{\text{zm}} | \text{ZM} \rangle \\
&\quad + \langle 0^{\text{osc}} | \psi_a^{\text{osc}}(t, x) \overline{\psi}_b^{\text{osc}}(t', x') | 0^{\text{osc}} \rangle \langle \text{ZM} | \overline{\psi}_a^{\text{zm}} \psi_b^{\text{zm}} | \text{ZM} \rangle \\
&= i\partial_\mu G^{\text{osc}+}(t, x; t', x') \{ (\gamma^\mu)_{ba} \langle \text{ZM} | \psi_a^{\text{zm}} \overline{\psi}_b^{\text{zm}} | \text{ZM} \rangle + (\gamma^\mu)_{ab} \langle \text{ZM} | \overline{\psi}_a^{\text{zm}} \psi_b^{\text{zm}} | \text{ZM} \rangle \} \\
&= i\partial_\mu G^{\text{osc}+}(t, x; t', x') (\gamma^\mu)_{ab} \langle \text{ZM} | \{ \overline{\psi}_a^{\text{zm}}, \psi_b^{\text{zm}} \} | \text{ZM} \rangle \\
&= i\partial_\mu G^{\text{osc}+}(t, x; t', x') \text{Tr}(\gamma^0 \gamma^\mu) / L \\
&= (2i/L) \partial_t G^{\text{osc}+}(t, x; t', x') \\
&= L^{-2} \sum_{n=1}^{\infty} \left(e^{-2\pi i n(\Delta t + \Delta x - i\epsilon)/L} + e^{-2\pi i n(\Delta t - \Delta x - i\epsilon)/L} \right) \\
&= L^{-2} \left(\frac{1}{e^{2\pi i(\Delta t + \Delta x - i\epsilon)/L} - 1} + \frac{1}{e^{2\pi i(\Delta t - \Delta x - i\epsilon)/L} - 1} \right), \tag{C.3}
\end{aligned}$$

using (3.16), (3.17) and (3.19).

Collecting these results yields (3.26).

Appendix D

Inertial and uniformly accelerated detectors in $(1 + 1)$ Minkowski vacuum with Gaussian switching

In this appendix we evaluate the response of an inertial detector and a uniformly accelerated detector in $(1 + 1)$ Minkowski vacuum with a Gaussian switching.

Inertial detector

Inserting the inertial worldline (3.32) and the Gaussian switching (3.33) in (2.28) with $d = 2$, we may change variables by $\tau - \tau' = \sigma z$ and $\tau + \tau' = \sigma w$ and perform the Gaussian integral over w , with the result

$$\mathcal{F}(\Omega) = -\frac{1}{2\pi^2\sigma} H(\sigma\Omega) , \quad (\text{D.1})$$

where the function H of a real variable is defined by

$$H(\alpha) := \int_C dz \frac{\exp(-i\alpha z - z^2/4)}{z^2} , \quad (\text{D.2})$$

where the contour C follows the real axis from $-\infty$ to ∞ except for dipping into the lower half-plane near $z = 0$. Differentiating (D.2) twice and evaluating the Gaussian

integral gives $H''(\alpha) = -2\pi^{1/2}e^{-\alpha^2}$, and integrating this twice gives

$$H(\alpha) = \pi \left(\alpha \operatorname{erfc} \alpha - \frac{e^{-\alpha^2}}{\pi^{1/2}} \right) + A\alpha + B, \quad (\text{D.3})$$

where erfc is the error complement function [41] and A and B are constants.

To determine A and B , we deform the contour C in (D.2) to $z = u - i$ with $u \in \mathbb{R}$, which gives the estimate

$$|H(\alpha)| \leq e^{-\alpha+1/4} \int_{-\infty}^{\infty} du \frac{\exp(-u^2/4)}{u^2 + 1}, \quad (\text{D.4})$$

which shows that $H(\alpha) \rightarrow 0$ as $\alpha \rightarrow \infty$. The falloff of erfc at large positive argument then shows that $A = B = 0$ in (D.3).

Collecting,

$$\mathcal{F}(\Omega) = \frac{1}{2\pi\sigma} \left(\frac{e^{-\sigma^2\Omega^2}}{\pi^{1/2}} - \sigma\Omega \operatorname{erfc}(\sigma\Omega) \right). \quad (\text{D.5})$$

Uniformly accelerated detector

Substituting the uniformly accelerated trajectory (3.42) and the Gaussian switching (3.43) in (2.28) with $d = 2$, we change variables by $a(\tau - \tau') = 2z$ and $a(\tau + \tau' - 2\tau_0) = 2w$ and perform the Gaussian integral over w , with the result

$$\begin{aligned} \mathcal{F}(\Omega) &= -\frac{a}{4\pi^2} \int_{-\infty}^{\infty} \frac{dz}{\sinh^2(z - i\epsilon)} \exp\left(-\frac{z^2}{\sigma^2 a^2} - \frac{2i\Omega z}{a}\right) \\ &= \frac{ae^{-\pi\Omega/a}}{4\pi^2} \int_{-\infty}^{\infty} \frac{dz}{\cosh^2 r} \exp\left(-\frac{(r - i\pi/2)^2}{\sigma^2 a^2} - \frac{2i\Omega r}{a}\right), \end{aligned} \quad (\text{D.6})$$

where in the last equality we have deformed the contour to $z = r - i\pi/2$ with $r \in \mathbb{R}$.

Appendix E

First order relations for mode function $\chi_k^\pm$

In this section we compute various relations that depend on the general solutions to mode functions (4.68) and which are relevant for normalising Dirac field mode solutions (4.71) and for quantizing the Dirac field.

The zero-momentum basis spinors $u^s(0)$ and $v^s(0)$ given in (4.69) are chosen to be normalized according to

$$u^{(s)\dagger}(0)u^{(s')}(0) = v^{(s)\dagger}(0)v^{(s')}(0) = \delta^{ss'}, \quad (\text{E.1})$$

$$u^{(s)\dagger}(0)v^{(s')}(0) = v^{(s)\dagger}(0)u^{(s')}(0) = 0, \quad (\text{E.2})$$

and they satisfy the completeness relations

$$\sum_s u^s(0)u^{(s)\dagger}(0) = \frac{I + \gamma^0}{2}, \quad (\text{E.3a})$$

$$\sum_s v^s(0)v^{(s)\dagger}(0) = \frac{I - \gamma^0}{2}, \quad (\text{E.3b})$$

as well as the following identities

$$u^{(s)\dagger}(0)\gamma^i u^{(s)}(0) = 0, \quad (\text{E.4a})$$

$$v^{(s)\dagger}(0)\gamma^i v^{(s)}(0) = 0. \quad (\text{E.4b})$$

Duncan [75] showed that (4.68) can be re-expressed using the definitions of $D_\pm := i\partial_\mu \pm \mu_\eta$ as

$$D_\mp D_\pm \chi_k^{(\pm)}(\eta) = |\mathbf{k}|^2 \chi_k^{(\pm)}(\eta) \quad (\text{E.5})$$

And by applying $D_\mp$ to (E.5), we obtain

$$D_\pm D_\mp \left(D_\pm \chi_k^{(\pm)}(\eta) \right) = |\mathbf{k}|^2 \left(D_\pm \chi_k^{(\pm)}(\eta) \right), \quad (\text{E.6})$$

which reveals that $D_+ \chi_k^+(\eta)$ is a solution to the χ_k^- differential equation whereas $D_- \chi_k^-(\eta)$ is a solution to the χ_k^+ ODE of (4.68). We shall express this first order relations in the following way

$$D_+ \chi_k^+(\eta) = k \chi_k^-(\eta), \quad (\text{E.7a})$$

$$D_- \chi_k^-(\eta) = k \chi_k^+(\eta). \quad (\text{E.7b})$$

Several relevant relations that follow from (E.7) are given below

$$\begin{aligned} -ikW[\chi_k^{(-)}, \chi_k^{(+)*}] &= \left| \partial_\eta \chi_k^{(+)} \right|^2 + \omega_k^2(\eta) \left| \chi_k^{(+)}(\eta) \right|^2 \\ &\quad + i\mu(\eta) \left[\left(\partial_\eta \chi_k^{(+)} \right) \chi_k^{(+)*} - \left(\partial_\eta \chi_k^{(+)*} \right) \chi_k^{(+)} \right], \end{aligned} \quad (\text{E.8a})$$

$$\begin{aligned} ik \left(W[\chi_k^{(-)}, \chi_k^{(+)*}] \right)^* &= \left| \partial_\eta \chi_k^{(-)} \right|^2 + \omega_k^2(\eta) \left| \chi_k^{(-)}(\eta) \right|^2 \\ &\quad + i\mu(\eta) \left[\left(\partial_\eta \chi_k^{(-)*} \right) \chi_k^{(-)} - \left(\partial_\eta \chi_k^{(-)} \right) \chi_k^{(-)*} \right], \end{aligned} \quad (\text{E.8b})$$

$$\begin{aligned} -k^2 \chi_k^{(-)*} \chi_k^{(+)} &= - \left(\partial_\eta \chi_k^{(+)*} \right) \left(\partial_\eta \chi_k^{(-)} \right) + \mu^2(\eta) \chi_k^{(+)*}(\eta) \chi_k^{(-)}(\eta) \\ &\quad - i\mu(\eta) \left[\left(\partial_\eta \chi_k^{(+)*} \right) \chi_k^{(-)} + \left(\partial_\eta \chi_k^{(-)} \right) \chi_k^{(+)*} \right], \end{aligned} \quad (\text{E.8c})$$

$$-ikW[\chi_k^{(-)}, \chi_k^{(+)*}] = \left| \chi_k^{(+)}(\eta) \right|^2 + \left| \chi_k^{(-)}(\eta) \right|^2, \quad (\text{E.8d})$$

where the wronskian $W[\chi_k^{(-)}, \chi_k^{(+)*}]$ is defined as

$$W[\chi_k^{(-)}, \chi_k^{(+)*}] := \chi_k^{(-)} \partial_\eta \chi_k^{(+)*} - \chi_k^{(+)*} \partial_\eta \chi_k^{(-)}. \quad (\text{E.9})$$

Normalisation

Collecting (4.71) and (4.73), we have

$$\langle U_{\mathbf{k}}^{(s)}, U_{\mathbf{k}'}^{(s)'} \rangle = u^{(s)\dagger}(0) |A_{\mathbf{k}}(\eta)|^2 u^{(s)}(0) \delta^{d-1}(\mathbf{k} - \mathbf{k}'), \quad (\text{E.10a})$$

$$\langle V_{\mathbf{k}}^{(s)}, V_{\mathbf{k}'}^{(s)'} \rangle = v^{(s)\dagger}(0) |B_{\mathbf{k}}(\eta)|^2 v^{(s)}(0) \delta^{d-1}(\mathbf{k} - \mathbf{k}'), \quad (\text{E.10b})$$

$$\langle U_{\mathbf{k}}^{(s)}, V_{\mathbf{k}'}^{(s)'} \rangle = u^{(s)\dagger}(0) A_{\mathbf{k}}^\dagger(\eta) B_{\mathbf{k}}(\eta) v^{(s)}(0) \delta^{d-1}(\mathbf{k} - \mathbf{k}'), \quad (\text{E.10c})$$

where $|A_{\mathbf{k}}(\eta)|^2 := A_{\mathbf{k}}^\dagger(\eta) A_{\mathbf{k}}(\eta)$ and $|B_{\mathbf{k}}(\eta)|^2 := B_{\mathbf{k}}^\dagger(\eta) B_{\mathbf{k}}(\eta)$ with

$$A_{\mathbf{k}}^\dagger(\eta) := -i\partial_\eta \chi_k^{(+)*} + \mathbf{k} \cdot \boldsymbol{\gamma} \chi_k^{(+)*} + \mu(\eta) \chi_k^{(+)*}, \quad (\text{E.11a})$$

$$B_{\mathbf{k}}^\dagger(\eta) := i\partial_\eta \chi_k^{(-)*} - \mathbf{k} \cdot \boldsymbol{\gamma} \chi_k^{(-)*} + \mu(\eta) \chi_k^{(-)*}, \quad (\text{E.11b})$$

and where we used the identity

$$\int \frac{d^{d-1}k}{(2\pi)^{d-1}} e^{i(\mathbf{k}-\mathbf{k}') \cdot \mathbf{x}} = \delta^{d-1}(\mathbf{k} - \mathbf{k}'). \quad (\text{E.12})$$

Furthermore, we obtain

$$\begin{aligned} |A_{\mathbf{k}}(\eta)|^2 &= \left| \partial_\eta \chi_k^{(+)} \right|^2 + i\mathbf{k} \cdot \boldsymbol{\gamma} \left[\left(\partial_\eta \chi_k^{(+)*} \right) \chi_k^{(+)} + \left(\partial_\eta \chi_k^{(+)} \right) \chi_k^{(+)*} \right] \\ &\quad + i\mu(\eta) \left[\left(\partial_\eta \chi_k^{(+)} \right) \chi_k^{(+)*} - \left(\partial_\eta \chi_k^{(+)*} \right) \chi_k^{(+)} \right] + \omega_k^2(\eta) \left| \chi_k^{(+)} \right|^2, \end{aligned} \quad (\text{E.13a})$$

$$\begin{aligned} |B_{\mathbf{k}}(\eta)|^2 &= \left| \partial_\eta \chi_k^{(-)} \right|^2 + i\mathbf{k} \cdot \boldsymbol{\gamma} \left[\left(\partial_\eta \chi_k^{(-)*} \right) \chi_k^{(-)} + \left(\partial_\eta \chi_k^{(-)} \right) \chi_k^{(-)*} \right] \\ &\quad + i\mu(\eta) \left[\left(\partial_\eta \chi_k^{(-)*} \right) \chi_k^{(-)} - \left(\partial_\eta \chi_k^{(-)} \right) \chi_k^{(-)*} \right] + \omega_k^2(\eta) \left| \chi_k^{(-)} \right|^2, \end{aligned} \quad (\text{E.13b})$$

$$\begin{aligned} A_{\mathbf{k}}^\dagger(\eta) B_{\mathbf{k}}(\eta) &= - \left(\partial_\eta \chi_k^{(+)*} \right) \left(\partial_\eta \chi_k^{(-)} \right) - i(\mathbf{k} \cdot \boldsymbol{\gamma} \mu(\eta)) \left[\left(\partial_\eta \chi_k^{(+)*} \right) \chi_k^{(-)} \right. \\ &\quad \left. + \left(\partial_\eta \chi_k^{(-)} \right) \chi_k^{(+)*} \right] + (\mu^2(\eta) + 2\mathbf{k} \cdot \boldsymbol{\gamma} - k^2) \chi_k^{(+)*} \chi_k^{(-)}. \end{aligned} \quad (\text{E.13c})$$

Using the identities (E.4) along with substituting (E.8a) in (E.13a), (E.8b) in (E.13b) and (E.8c) in (E.13c), we obtain

$$|A_{\mathbf{k}}(\eta)|^2 = -ikW[\chi_k^{(-)}, \chi_k^{(+)*}], \quad (\text{E.14a})$$

$$|B_{\mathbf{k}}(\eta)|^2 = ik \left(W[\chi_k^{(-)}, \chi_k^{(+)*}] \right)^*, \quad (\text{E.14b})$$

$$A_{\mathbf{k}}^\dagger(\eta) B_{\mathbf{k}}(\eta) = -k^2 \left(\chi_k^{(-)*} \chi_k^{(+)*} - \chi_k^{(+)*} \chi_k^{(-)} \right). \quad (\text{E.14c})$$

Collecting (E.1), (E.10) and (E.14), the normalisation of the mode solutions reduces to

$$\langle U_{\mathbf{k}}^{(s)}, U_{\mathbf{k}'}^{(s)'} \rangle = -ikW[\chi_k^{(-)}, \chi_k^{(+)*}] \delta^{ss'} \delta^{d-1}(\mathbf{k} - \mathbf{k}'), \quad (\text{E.15a})$$

$$\langle V_{\mathbf{k}}^{(s)}, V_{\mathbf{k}'}^{(s)'} \rangle = ik \left(W[\chi_k^{(-)}, \chi_k^{(+)*}] \right)^* \delta^{ss'} \delta^{d-1}(\mathbf{k} - \mathbf{k}'), \quad (\text{E.15b})$$

$$\langle U_{\mathbf{k}}^{(s)}, V_{\mathbf{k}'}^{(s)'} \rangle = 0. \quad (\text{E.15c})$$

Thus, the normalisation of the mode solutions depends on our choice for the Wronskian. When we choose to normalise the wronskian via (4.74), the normalisation of the mode solutions reduces to (4.72).

Quantization

Collecting (4.71), (4.82) and (4.84) along with applying our choice of positive and negative frequency modes discussed in the paragraph below (4.79), we obtain

$$\begin{aligned} \{\psi_a(\eta, \mathbf{x}), \psi_b(\eta, \mathbf{x}')\} = & C^{\frac{1-d}{2}}(\eta) \int \frac{d^{d-1}k}{(2\pi)^{d-1}} e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')} \left[\left| \partial_\eta \chi_k^{(+in)} \right|^2 \right. \\ & + i\mu(\eta) \left[\left(\partial_\eta \chi_k^{(+in)} \right) \chi_k^{(+in)*} - \left(\partial_\eta \chi_k^{(+in)*} \right) \chi_k^{(+in)} \right] \\ & \left. + \omega_k^2(\eta) \left| \chi_k^{(+in)} \right|^2 \right]. \end{aligned} \quad (\text{E.16})$$

Substituting in the identity (E.8a), (E.16) reduces to

$$\{\psi_a(\eta, \mathbf{x}), \psi_b(\eta, \mathbf{x}')\} = C^{\frac{1-d}{2}}(\eta) \delta_{ab} \int \frac{d^{d-1}k}{(2\pi)^{d-1}} e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')} (-ik) W[\chi_k^{(-in)}, \chi_k^{(+in)*}]. \quad (\text{E.17})$$

Thereby, making the choice of the Wronskian given in (4.74), we recover (4.83b).

Appendix F

Conformal Zero Mode as a two-real parameter set of normalised Gaussian wave packets

In this appendix we show that the initial state of the conformal zero momentum mode discussed in (4.133) corresponds to a two-real parameter set of normalised gaussian wave packet. We follow a similar treatment to the massless zero mode of the scalar field on the static cylindrical spacetime given in [46]. Although the conformal mode can have an arbitrary mass, its effective mass vanishes, therefore a similar treatment is possible.

Let $\hat{Q}_s$ and $\hat{P}_s$ be the Shrödinger picture position and momentum operators corresponding to Q representing the conformal zero mode $\phi_{0,czm}(\eta)$ (see (4.133)) of the zero momentum component (4.130b) and P , its momentum conjugate, satisfying $[\hat{Q}_s, \hat{P}_s] = i$. Then the Heisenberg picture position operator reads

$$\hat{Q}_H(\eta) = \hat{Q}_s + \frac{\hat{P}_s}{L}\eta = \hat{\phi}_{0,czm}(\eta) = \frac{1}{L}\hat{a}\chi(\eta) + h.c. , \quad \chi(\eta) := \left(\alpha + \frac{i}{2\beta}\right) + \beta\eta , \quad (\text{F.1})$$

where the last equality follows from (4.130b) and (4.133). We have

$$\begin{aligned}\hat{Q}_s + \frac{\hat{P}_s}{L}\eta &= \frac{1}{L} \left[\hat{a} \left(\left(\alpha + \frac{i}{2\beta} \right) + \beta\eta \right) + \hat{a}^\dagger \left(\left(\alpha - \frac{i}{2\beta} \right) + \beta\eta \right) \right] \\ &= \frac{1}{L} \left[\beta\eta(\hat{a} + \hat{a}^\dagger) + \alpha(\hat{a} + \hat{a}^\dagger) + \frac{i}{2\beta}(\hat{a} - \hat{a}^\dagger) \right].\end{aligned}\quad (\text{F.2})$$

Therefore it follows from (F.2)

$$\hat{Q}_s = \frac{\alpha}{L}(\hat{a} + \hat{a}^\dagger) + \frac{i}{2L\beta}(\hat{a} - \hat{a}^\dagger), \quad (\text{F.3a})$$

$$\hat{P}_s = \beta(\hat{a} + \hat{a}^\dagger). \quad (\text{F.3b})$$

Combining (F.3a) and (F.3b), we obtained

$$\hat{Q}_s - \frac{\alpha\hat{P}_s}{L\beta} = \frac{i}{2L\beta}(\hat{a} - \hat{a}^\dagger). \quad (\text{F.4})$$

Thus, we have from (F.3b) and (F.4)

$$\hat{a} + \hat{a}^\dagger = \frac{1}{\beta}\hat{P}_s \quad (\text{F.5a})$$

$$\hat{a} - \hat{a}^\dagger = \frac{2L\beta}{i}\hat{Q}_s - \frac{2\alpha}{i}\hat{P}_s \quad (\text{F.5b})$$

Now collecting (F.5a) and (F.5b), we have

$$2\hat{a} = \frac{2L\beta}{i}\hat{Q}_s + \left(\frac{1}{\beta} - \frac{2\alpha}{i} \right) \hat{P}_s \quad (\text{F.6})$$

Using $\hat{P}_S = -i\partial/\partial q_s$, $\hat{Q}_S|q_s\rangle = q_s|q_s\rangle$ and the initial state of the zero momentum sector (4.135) $a_{0,\alpha,\beta}^{in/out}|0_L, in/out\rangle_{\alpha,\beta} = 0$, we obtain from (F.6)

$$\begin{aligned}\langle q_s|2\hat{a}|0_L, in/out\rangle_{\alpha,\beta} &= \langle q_s| \left(\frac{2L\beta}{i}\hat{Q}_s + \left(\frac{1}{\beta} - \frac{2\alpha}{i} \right) \frac{\partial}{i\partial q_s} \right) |0_L, in/out\rangle_{\alpha,\beta}, \\ &= \left(2L\beta q_s + \left(\frac{1}{\beta} + 2i\alpha \right) \frac{\partial}{\partial q_s} \right) \Psi_{\alpha,\beta,L}^{in/out}(q_s),\end{aligned}\quad (\text{F.7})$$

where we have included the superscript *in/out*, which we have suppressed before for clarity. The general solution to (F.7) is a Gaussian function given by

$$\Psi_{\alpha,\beta,L}^{in/out}(q_s) = N_{\alpha,\beta,L} \exp \left(-\frac{\beta^2 L}{1 + 2\alpha\beta i} q_s^2 \right), \quad (\text{F.8})$$

where $N_{\alpha\beta,L}$ is the normalisation factor. But because

$$\frac{\beta^2}{1+2\alpha\beta i} = \frac{\beta^2}{\sqrt{1+\gamma^2}} \frac{1+i\gamma}{\sqrt{1+\gamma^2}} = \frac{\beta^2 e^{i\delta_{\alpha,\beta}}}{\sqrt{1+\gamma^2}}, \quad \gamma := -2\alpha\beta, \quad (\text{F.9})$$

where

$$e^{i\delta_{\alpha,\beta}} = \frac{1-2i\alpha\beta}{\sqrt{1+4\alpha^2\beta^2}}. \quad (\text{F.10})$$

Therefore, we can rewrite the initial Fock state of the zero momentum sector as a wave functional

$$\Psi_{\alpha,\beta,L}^{in/out}(q_s) = N_{\alpha\beta,L} \exp\left(-\frac{L\beta^2 e^{i\delta_{\alpha,\beta}}}{\sqrt{1+4\alpha^2\beta^2}} q_s^2\right), \quad (\text{F.11})$$

where the normalisation factor is given by

$$N_{\alpha\beta,L} = \left(\frac{2\beta^2 L}{\pi\sqrt{1+4\alpha^2\beta^2}}\right)^{\frac{1}{4}}. \quad (\text{F.12})$$

Appendix G

Integral transformation for the finite-time detector's response of the massive scalar field

In this appendix we show how the integral (5.38) is obtained from (5.37).

Write

$$u = T_0 + v, \quad u' = T_0 + v', \quad \Delta T = T_1 - T_0, \quad (\text{G.1})$$

then (5.37) reads

$$\begin{aligned} 4m^2 \mathcal{F}(\omega) = & -i \int_0^{\Delta T} dv \int_0^v dv' H_0^{(2)}(v - v') \exp\left(-i \frac{\omega}{m} (v - v')\right) \\ & + i \int_0^{\Delta T} dv \int_v^{\Delta T} dv' H_0^{(1)}(v' - v) \exp\left(-i \frac{\omega}{m} (v' - v)\right). \end{aligned} \quad (\text{G.2})$$

Write $\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2$, where

$$4m^2 \mathcal{F}_1(\omega) = i \int_0^{\Delta T} dv \int_v^{\Delta T} dv' H_0^{(1)}(v' - v) \exp\left(-i \frac{\omega}{m} (v' - v)\right), \quad (\text{G.3a})$$

$$4m^2 \mathcal{F}_2(\omega) = -i \int_0^{\Delta T} dv \int_0^v dv' H_0^{(2)}(v - v') \exp\left(-i \frac{\omega}{m} (v - v')\right). \quad (\text{G.3b})$$

Invert the order of the integration in (G.3a) and rename $v \leftrightarrow v'$ to obtain

$$4m^2 \mathcal{F}_1(\omega) = i \int_0^{\Delta T} dv' \int_{v'}^{\Delta T} dv H_0^{(1)}(v - v') \exp\left(-i \frac{\omega}{m} (v - v')\right). \quad (\text{G.4})$$

Therefore, we have

$$4m^2 \mathcal{F}(\omega) = i \int_0^{\Delta T} dv \int_0^v dv' \left\{ iH_0^{(1)}(v-v') \exp\left(i\frac{\omega}{m}(v-v')\right) - iH_0^{(2)}(v-v') \exp\left(-i\frac{\omega}{m}(v-v')\right) \right\}. \quad (\text{G.5})$$

Write $v' = v - x$, $0 \leq x \leq v$ and express the Hankel functions $H_0^{(1)}$ ($H_0^{(2)}$) in terms of the Bessel function of the first and second kind to obtain

$$4m^2 \mathcal{F}(\omega) = -2 \int_0^{\Delta T} dv \int_0^v dx \left[J_0(x) \sin\left(\frac{\omega}{m}x\right) + Y_0(x) \cos\left(\frac{\omega}{m}x\right) \right]. \quad (\text{G.6})$$

Now we can collect (5.37) from (G.6).

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