Bargaining and Contribution Games with Deadlines

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Abstract

This thesis considers play in bargaining games subject to Endogenous Commitment and in contribution games with a sunk cost. In bargaining games, Endogenous Commitment (EC) describes a common feature in negotiation: once an offer is made, neither would the proposer offer nor would the respondent accept anything worse. Similarly, in contribution games, the notion of sunk cost implies an irrevocability similar to EC: it is impossible for either contributor to reduce his or her contribution, so far as the cost is sunk. Another similarity between the bargaining and contribution games in our thesis is that we assume (most of) them to be finite, meaning that there is a deadline effect: when approaching the deadline, the final negotiator/contributor has a stronger incentive to reach an agreement/complete the project. The deadline effect puts the final negotiator/contributor in a relatively weaker position. With these two similarities, the bargaining and contribution games in our thesis share some similar features.

In the first chapter, we conduct a literature review. In the second chapter, we study two player alternating finite/infinite bargaining games with Endogenous Commitment. In both cases, the outcomes are affected by the assumption of EC. In the third chapter, we apply Endogenous Commitment in bargaining games with protocols involving uncertainty. The settlement timings then exhibit a U-shaped pattern: players reach an agreement at the first or the last stage of the game. In the fourth chapter, we turn to contribution games with sunk cost and heterogeneous valuations. We show that a minor difference in valuation could affect the total welfare significantly.

In our thesis, we adopt several settings in all chapters. As all models include two players, we refer to player 1 as male and to player 2 as female for convenience. When no specific player is referred to, we use i and j to indicate the two players, assuming i to be male and j to be female. When player 1 makes an offer (in bargaining games) or makes a contribution (in contribution games) in stage t ($t \in [1, 2, ...T]$, T is the length of the game), we denote it as x_t; and when player 2 does so, we denote it as y_t. Similarly, we denote player i's and player j's choice as m_t and n_t.

Chapter 1

Introduction and Literature Review

1.1 Bargaining games

1.1.1 Bargaining games: introduction

Game theorists have studied bargaining since Nash 1950[47], 1953[48]. Schelling I960[54] explored the role, feasibility and persuasiveness of commitment tactics in bargaining situations. Commitments can be made in various ways:

> "In principle, one might evade extortion equally well by drugging his brain, conspicuously isolating himself geographically, getting his assets legally impounded, or breaking the hand that he uses in signing checks"[1],

and the seemingly suicidal commitments could actually benefit the negotiator. Muthoo discusses this topic in a series of papers.[2] In the second and third chapter of our thesis, we study a special type of commitment: in negotiation, each player commits to reject any offer less than the ones he has turned down before (Endogenous Commitment).

Stahl 1972[58] and Rubinstein 1982[52] provide two simple and elegant alternating offer bargaining models to use in the study of bargaining. In their models, two players

[1] Schelling 1960[54], Page 18.
[2] Muthoo 1990[42], 1992[43], 1995 a[44], 1995 b[45].

alternate in making offers; negotiation ends if either player accepts his rival's offer and continues otherwise. The difference is that in Stahl's model there is an exogeneously given deadline, while in Rubinstein's model the negotiation continues infinitely. In our second chapter, we study how applying Endogenous Commitment in these two models would change the equilibrium outcomes.

Binmore and Dasgupta 1987[4] Chapter 5 provides another bargaining protocol: instead of letting players alternate in making proposals, now they each have an equal chance to propose in every period. In our third chapter, we show that applying Endogenous Commitment in a finite horizon bargaining model can explain U-shaped settlement patterns if the bargaining protocol is stochastic (in some periods).

In Section 1.1.2, we introduce previous studies on Endogenous Commitment; in Section 1.1.3, we introduce previous studies of the deadline effect and arbitration on bargaining outcomes.

1.1.2 Endogenous Commitment

Endogenous Commitment (EC as an abbreviation), as a general phenomenon, has been studied in various literatures. Here we list several papers using EC as their main assumption. In these papers, EC is framed slightly differently; so we divide them into several sub-sections. We introduce the three explanations of EC in each section.

Negotiator's Prestige

The negotiator's prestige should be considered. If the negotiation is conducted by representatives, then those who accepted a poorer offer would suffer from reputation loss unless they can explain why they have accepted an offer less generous than the one they previously rejected. A mistake like this could ruin the negotiator's fame as a tough bargainer.

Combined with a deadline, Endogenous Commitment is introduced in Fershtman

and Seidmann 1993[23] (FS 1993 for short) to explain why negotiations are often not settled until the 11th hour: in a two player random offer bargaining game with Endogenous Commitment and a specific deadline (say, T), there always exists a threshold discount factor $\delta_T' < 1$ such that, for any $\delta > \delta_T'$, the unique sub-game perfect equilibrium of the game involves agreement reached at the deadline. They also show that when δ is small enough, an immediate agreement is reached; and the introduction of EC leads to a first-mover disadvantage. In contrast to Rubinstein 1982 for example, when $T = 2$ and $\delta \in (\frac{2}{3}, 3-\sqrt{5}]$, it is better for a player to respond than to propose.

FS 1993 describe Endogenous Commitment in the following way: the respondent cannot accept a lower share of the surplus than she has previously rejected. My thesis is based on FS 1993; the main difference is that they focus on extreme cases: δ small enough or $\delta \to 1$ and they use a random-proposer protocol; whereas we try to depict a more complete pattern here by extending the research to every discount factor $\delta \in (0,\ 1)$, and assume a fixed order of proposers (in Chapter 2).

FS 1993's main conclusion is extended in a series of papers by Calabuig, Cunyat, and Olcina. In their papers, they assume a similar but only partially irrevocable commitment.

Calabuig et al. 2002 a[8] adopt a commitment constraint mentioned in Muthoo 1999[46]: a player could accept a less generous offer, but by accepting it, would incur a finite and proportional (depending how low the offer is) cost if the offer is higher than a specific critical value, and incur an infinite cost if the offer is lower than the critical value. The revoking function $C(x_t^i, z_t, \epsilon, \lambda)$ for player i at period t is

$$C(x_t^i, z_t, \epsilon, \lambda) = \begin{cases} 0 & if\ z_t \leq x_t^i \\ \lambda(z_t - x_t^i) & if\ (1-\epsilon)z_t \leq x_t^i < z_t \\ \infty & if\ x_t^i < (1-\epsilon)z_t \end{cases},$$

where x_t^i is the current period received offer, z_t is the highest received offer, while $\epsilon \in [0, 1]$ and $\lambda \in [0, 1]$ respectively define the threshold value and the proportional scale parameter. With ϵ and λ, they define the degree of irrevocability, and a high

degree of irrevocability causes delay in equilibrium. In this sense, their result extends FS 1993, in which the highest level of irrevocability is assumed.

Calabuig et al. 2002 b[9] introduce EC into a finite horizon multiple-player bargaining model: one seller chooses which of the two buyers to bargain with in each period. The selected buyer is the proposer and the seller chooses to accept the offer or not. In the last period T, once the seller has chosen a buyer, a fair lottery determines the identity of the proposer. The seller has an EC constraint and the buyers do not. What's more, these two buyers are not identical: one buyer has a higher valuation than the other ($\frac{1}{2} < V_2 < V_1$). When the discount factor δ and V_2 are large enough, the equilibrium involves an immediate agreement. Therefore, delay could be eliminated by competition between players on the long side of the market. Another type of inefficiency might occur though: the good may not be sold to the buyer with the highest valuation.

Calabuig et al. 2004[10] use this (generalised) Endogenous Commitment assumption to study an uncertain instead of fixed deadline bargaining game. After every rejection, there is a chance that the negotiation would enter a deadline stage in which both players have an equal chance to make the final attempt with a take-it-or-leave-it offer.[3] They study the game in a buyer-seller situation and assume that only the seller's offers must satisfy the commitment. In their model, agreement could be reached immediately or when the game enters the deadline stage; and the commitment affects the outcome whether the degree of irrevocability is high or low.

In FS 1993, they assume that no respondent would like to accept a poorer offer; actually, no proposer would like to make a poorer offer either. Even if the poorer offer will be accepted, the proposer would face a difficult situation explaining why he couldn't persuade the respondent to accept the same offer earlier.

Unlike FS 1993, Schwartz and Wen 2007[55] define Endogenous Commitment by preventing either player from ever reducing demand, and they name such a constraint

[3]It is similar to the infinite horizon game with outside options in every stage as in Compte and Jehiel 2004[16].

"Good faith bargaining" (GFB) rule after Stahl 1972[58]. Adding such a rule to Fernandez and Glazer 1991[21] (FG 1991) model of firm-union bargaining would make the union's threat of striking incredible, and would thereby refine the multiple equilibria into a unique equilibrium with no striking. FG 1991 describes a firm-union wage negotiation model in which the union freely chooses between working at the status-quo wage and striking after each rejection. In other words, the union has an inside option. This inside option makes the union's threat to strike credible and therefore supports a range of equilibrium wages $w \in [\underline{w}, \overline{w}]$, generating multiple equilibria, including inefficient ones that involve a strike. GFB rule limits the firm's ability to make low wage offers and makes the option of not striking irresistible, thereby refining the equilibrium correspondence.

Reference Points

Dating back to Tversky and Kahneman 1991[59] (and further explored by Koszegi and Rabin 2006[35]), reference point theory is frequently used to explain the origin of Endogenous Commitment. The idea seems quite natural: the maximum received share is the reference point; as it is what one could have obtained, any smaller share would be seen as a loss.

Compte and Jehiel 2003 a[14] is one of the papers studying reference points in bargaining models. They adopt a "hybrid" bargaining protocol with alternating and random offers. The multiple stage game has many phases: players alternate in making offers within one phase, while there is a chance that, after each rejection, the game will enter a new phase, with both players suffering a loss and sharing an equal chance of becoming the first proposer. They introduce Endogenous Commitment by embedding a non-stationary preference structure into players' value functions: every time play enters a new phase, players update their reference points to the highest offer they rejected in the previous phase; any offer below the reference point would be valued negatively. In other words, in their model reference points only adjust when a salient event (change of bargaining phase) occurs. In equilibrium, there is an upper

bound on the extra generosity of the offer made in every period of the game, and the equilibrium path exhibits gradualism: until the agreement is reached, players make higher offers every time they propose. The simple intuition is that whenever too generous an offer is made, the receiver of the offer gains a strong incentive to refrain from making any offer and wait for the breakdown of current phase, in order to enter the new phase in which that very offer becomes his new reference point. Anticipating this, neither player would make too large, an offer and the agreement must be reached gradually.

Li 2007[36] also uses Endogenous Commitment to explain gradualism in bargaining. Like Compte and Jehiel 2003 a[14], the author adopts a non-stationary preference structure; but unlike most papers in this area, it is not the share of the rejected offer but the discounted value of the rejected offer that is considered to be the reference point: set players i's value function as $u_i(x_i,\ z_i^t) = \begin{cases} u_i(x_i) & x_i > z_i^t \\ -\epsilon & x_i < z_i^t \end{cases}$, and represent player i's preference by $U_i(x,t) = \delta_i^t u_i(x_i,\ z_i^t)$, where x_i is player i's agreed share of the pie, and z_i^t is the highest discounted value player i has rejected before stage t. With a simple alternating offer protocol, such a setting also generates gradualism in equilibrium.

Driesen, Perea and Peters 2012[18] introduce loss averse players in a Rubinstein alternating offer bargaining game, where the largest rejected offer is the reference point. They show that in equilibrium, an immediate agreement is still reached, but loss aversion is disadvantageous to each player. Their result coincides with our infinite horizon model in Chapter 2: if players are extremely loss averse, they would never accept a less favorable offer; in other words, they would behave in accordance with Endogenous Commitment.

No withdrawal

Another way to interpret Endogenous Commitment is that whenever an offer is made, it can't be withdrawn. In other words, all offers remain on the table forever. In this sense, it is pointless for a proposer to make a poorer offer; or a respondent to accept

a poorer offer.

The reason of no withdrawn is also straightforward: it may be too costly to do so, or it may not be an option at all. In western dining tradition, food is distributed before people actually start eating. Things are different in the Chinese dining tradition: people eat from the same bowl of food, in other words, they distribute the food while eating. Under this routine, there is certainly no reverse. This is an example of an exogenously enforced EC: instead of communication through a revocable gesture (like language), players simply make irrevocable gestures to "make the proposal".

Compte and Jehiel 2004[16] is one example. In their paper, all offers made are considered "concessions" and are naturally irrevocable. They study a model which generalizes both a bargaining game with Endogenous Commitment and an outside option in all stages, and a contribution game with possibilities of termination in all stages. They use this model to explain gradualism in negotiations and joint projects. In their model, the outside option payoffs (in a concession game) and termination payoffs (in a contribution game) are history dependent. Therefore, for any player, making a large concession/contribution gives his rival an opportunity to terminate the game right away and benefit from opting out. In equilibrium, both players slowly increase their actions to avoid sudden termination, exhibiting a gradualism phenomenon.

Lipman and Wang 2000[37], 2009[38] study switching costs in repeated games: any player who deviates from his previous choice incurs a cost. For example, if two players play yhe same prisoner's dilemma game repeatedly, then whoever changes his action in the new game incurs a cost. Caruana and Einav 2008[13] extend this research to dynamic games: in a one-shot game, players are allowed to change their minds as often as they wish before they make their actual moves, but each change incurs a cost which increases approaching the deadline. In equilibrium, the switching cost leads to strategic delay. Our research also has a dynamic structure, but instead of assuming a relatively small cost, we assume a unidirectional and infinitely large cost.

Although EC is not a widely accepted assumption in bargaining literature, it is not so uncommon in other areas. In auction games, we take non-decreasing bidding as default. In contribution games, at least half of the time we assume that the contributions made are sunk (which naturally implies irrevocability). A more detailed literature review of contribution games is introduced later in this chapter. Ausubel and Schwartz 1999[3] use EC as as an eligibility rule in multiple-unit auctions; Zissimos 2007[62] uses EC as a withdrawal of equivalent concessions in international trade. While in bargaining games offers are usually set to be revocable, we claim that irrevocable offers are no less plausible.

1.1.3 Delay and arbitration

Delay

Deadline effects are a common feature in finite horizon bargaining games: negotiators seem to have strong interest in reaching last minute agreements, even that earlier agreements are more favorable to all of them. Lab experiments provide strong evidence to support this.

Roth et al. 1988[51] is the first experimental paper to study the deadline effect in a lab, through a series of experiments respectively focusing on different variables: 1. Information (players' values in the lotteries) available to players; 2. Description of the values (in chips or in money); 3. Risk preferences; 4. Value of the lotteries (mean and variance). In all of the experiments conducted, the result is salient: "the distribution of agreements over time appears to be much less sensitive to the experimental manipulations than is the distribution of the terms of agreement", and "there is a striking concentration of agreements reached in the very last seconds".

Since then, more experiments have been carried out to explore reasons for the deadline effect. Guth et al. 2005[28] concentrate on three issues: the length of the game, the discount factor, and the bargaining protocol (alternating or not). Gneezy et al. 2003[27] study the deadline effect in repeated ultimatum games. Again and

again, the deadline effect proves to be robust.

Various theoretical explanations have been provided for the deadline effect.

Kennan and Wilson 1990[31] provide two categories of explanation: Informational Effects and Procedural Effects. Incomplete information provides an obvious explanation for delay: parties take time to compete and identify their values, in order to determine who is stronger (attrition model); a seller uses a "skimming" strategy to gradually test buyer's reservation price (screening model); both parties take longer to respond to the offers to show that they have little interest in making the deal (signaling model).[4] By contrast, Procedural Effects are caused by multiple equilibria in games with simultaneous proposals. Recall Fernandez and Glazer 1991[21] as an example: with the option of striking in each stage instead of working at the original wage level, even a rational union with complete information might choose to strike for several periods before they sign a new contract with the firm (in some of the multiple subgame perfect equilibria). The problem is that in such models, delay is possible instead of necessary. We focus on bargaining under complete information, and are more interested in an explanation indicating an inevitable delay rather than a possible one. Here are several papers with a similar objective.

We have introduced Fershtman and Seidmann 1993 already: in their paper irreversible commitment is suggest as an explanation of the deadline effect. Calabuig et al. 2006[11] extend the conclusion to an uncertain deadline, and show that like in FS 1993, negotiations always end at the deadline stage when players are patient enough. [5]

Ma and Manove 1993[40] include a transmission time for offers in the model: after each proposal, the offer takes an exogenously determined random time to arrive at the respondent; once any player decides to reject the offer from his rival, he can choose when he will make the counter-offer as well as its content. The deadline effect

[4] In their later paper Kennan and Wilson 1993[32], they summarize bargaining papers with incomplete information in more detail.

[5] The main model is similar to the one in Calabuig et al. 2004; but instead of assuming that only the seller has a partially Endogenous Commitment, here they assume that both the seller and the buyer have EC.

is clear in this model: for the proposer, with some luck, the offer is observed by the respondent just before the deadline. This would leave his rival no time to make a counter offer and would therefore induce his rival to accept anything. In other words, imperfect control over the timing of offers provides an incentive to strategically delay.

Jehiel and Moldovanu 1995[30] study a negotiation between one seller and multiple buyers. The purchase happens between the seller and one of the buyers, and trading the good could impose a positive or a negative externality on all other buyers. In each stage, the seller could choose to make an offer to one randomly chosen buyer or could choose to not make any offer at all; the randomly chosen buyer chooses between accepting or rejecting in response. It is conceivable that delay happens when there is a positive externality between buyers: all buyers prefer to let another buyer make the purchase; and someone will have to make the purchase when it gets closer to the deadline. However, even if the externality is negative, delay is still possible. If every player $j \neq i$ suffers when buyer i buys from the seller, then it is possible that all other buyers would have incentives to free-ride on each other to be the hero and prevent player i from buying. In other words, the free-riding incentive between negotiators could lead to delay.

Yildiz 2004[60] uses optimism to explain delay. When each player is certain that he has a better chance to be the final proposer and induce his rival to accept a deal beneficial to himself, it is impossible to reach an earlier, mutually more favorable agreement, as players' total expected payoff is too high to be offset by the time cost.

In our third chapter, we study the deadline effect when the bargaining protocol involves uncertainty, as in FS 1993. Unlike FS 1993 and any of the above mentioned papers, we further show that agreement generates a U-shaped settlement pattern in equilibrium: players settle at the beginning or at the end of the negotiation; there are no intermediate agreements.

Arbitration

In our third chapter, we study a bargaining model with an arbitrator determining the disagreement payoff beyond the deadline. Arbitration is hardly a new concept in bargaining. There are two sorts of arbitration: in conventional arbitration, an arbitrator is free to impose any settlement; in final-offer arbitration, the arbitrator must choose one player's final offer as the arbitration outcome. While there are many contributions on this topic, we will focus on the more recent and more closely related papers.

Within the framework of final-offer arbitration, Yildiz 2011[61] assumes that an arbitrator maximizes Nash's social welfare function after a sequential alternating offer process, and finds that the unique subgame-perfect equilibrium outcome coincides with the unique subgame-perfect equilibrium outcome in Rubinstein's alternating offer bargaining game.

Rong 2012[50] further extends this conclusion, proving that it still holds as long as the arbitrator is relatively "fair" (does not favor any party too much); but delay would occur if the arbitrator unfairly favors the first proposer.

In Chapter 3, we have a similar result: division of the pie in the arbitration stage might affect the timing of the agreement. The main difference between the earlier literatures and our chapter is that the disagreement payoffs were usually considered to be exogenously given; whereas in our model as in FS 1993, the arbitrator division depends on the history of offers, making the disagreement payoff endogenous.

1.2 Contribution games

1.2.1 Contribution games: introduction

In many ways, a contribution game is similar to a bargaining game. In a negotiation, every negotiator tries to make his rival accept an offer as small as possible, as long as they can reach a final agreement; in a joint project, every participant tries to induce his co-worker do as much work as possible, as long as the project is completed in

the end. In this type of games, all participants would be better off with a basic consensus, whilst they make effort to exploit each other on the basis of not breaking the consensus. While they have mutual interest in reaching the consensus, they have conflict interest the minute so long as consensus can be maintained. The balance between competition and cooperation is important.

The distinction between the bargaining games and the contribution games in our thesis is mostly structural. In the fourth chapter of the thesis, we study contribution games with sunk costs, implying that the contributions made are borne immediately and can't be taken back (similar to Endogenous Commitment). If we instead assume that the costs are borne upon the completion of the project and can be taken back if the project fails (non-sunk), but no one can reduce his committed contribution (upon completion), the games would be equivalent to the bargaining games with EC.[6] Another example of generalizing bargaining games and contribution games is Compte and Jehiel 2004[16].

In bargaining games, negotiators are usually assumed to have a fixed total revenue once an agreement is reached, and their profits merely depend on the concessions they made during the negotiation. By contrast, in contribution games we can easily further assume that contributors value the project differently from the start. In Chapter 4, we mainly focus on how heterogeneous valuations affect individual and social welfare in contribution games.

1.2.2 Overview

Before introducing our research, we briefly review the literature on dynamic contribution games.

In a static contribution game, all players make simultaneous, one-shot contributions without knowing how much anyone else contributes; in a dynamic contribution game, players make decisions having observed others' choices. With the development of behavioral economics in recent years, much attention has been paid to static contribution/public good games, and their various extensions. Meanwhile, dynamic

[6] Or as defined in Admati and Perry 1991[1], subscription games.

contribution games remain a relatively unexplored topic.

Why might contribution decisions differ in a dynamic setting as opposed to a static one? Schelling 1960[54] gives a good example:

> "...If each party agrees to send a million dollars to the Red Cross on condition the other does, each may be tempted to cheat if the other contributes first, and each one's anticipation of the other's cheating will inhibit agreement. But if the contribution is divided into consecutive small contributions, each can try the other's good faith for a small price. Furthermore, since each can keep the other on short tether to the finish, no one ever need risk more than one small contribution at a time. Finally, this change in the incentive structure itself takes most of the risk out of the initial contribution; the value of established trust is made obviously visible..."[7]

A dynamic environment makes promises and threats enforceable, and significantly reduces the risk of making the first step to establish trust: such trust is not due to ethical reasons, but on the basis of creating a win-win situation for all participants in the long run. However, while a dynamic environment makes cooperation more achievable, it also introduces a free-rider problem: when one player sees other players' contributions, he does not merely take it as a signal of trust; instead, may also take it as an opportunity to reduce his own share of the burden. Therefore, whether a dynamic environment is beneficial to individual and social welfare requires more detailed discussion.

Most papers on dynamic contribution games focus on one or several of the following aspects: the protocol of the games (simultaneous, alternating, and continuous, meaning all players make contributions continuously all along the path); the information available to contributors (whether an individual's past contribution or total past contribution are observable, whether every participant's valuation of the project is public knowledge); the reversibility of the actions (sunk or non-sunk cost); the dead-

[7] Schelling 1960, page 45.

line effect (infinite or finite horizon); and the payoff form of the project (binary or continuous). In most cases, they explore the efficiency of play under a combination of these conditions. The following table roughly summarizes some results.

	Protocol	Information	Actions	Horizon	Payoff	Efficiency
Admati & Perry 1991[8]	Alternating	Complete	Irreversible	Infinite	Binary	Inefficient
Admati & Perry 1991 [9]	Alternating	Complete	Reversible	Infinite	Binary	Efficient
Compte & Jehiel 2003 b	Alternating	Complete	Irreversible	Infinite	Binary	Efficient
Marx & Matthews 2000[10]	Simultaneous	Accumulated contribution observable	Irreversible	Finite	Linear	Efficient
Max & Matthews 2000[11]	Alternating	Complete	Irreversible	Finite	Binary	Efficient
Fershtman & Nitzan 1991	Continuous	Accumulated contribution unobservable	Irreversible	Infinite	Continuous	Inefficient
Fershtman & Nitzan 1991	Continuous	Accumulated contribution observable	Irreversible	Infinite	Continuous	More inefficient
Kessing 2007	Continuous	Complete	Irreversible	Infinite	Binary	Efficient

[8] Contribution games
[9] Subscription games
[10] Their main model
[11] Extension of AP 1991

Bonatti & Horner 2011	Continuous	Contributions unobservable, experimentation outcomes observable	Irreversible	Infinite	Uncertain	Inefficient
Bonatti & Horner 2011	Continuous	Contributions unobservable, experimentation outcomes observable	Irreversible	Endogenous	Uncertain	Efficient with deadline
Campbell et al. 2012	Continuous	Contributions unobservable, experimentation outcomes unobservable	Irreversible	Endogenous	Binary (uncertain)	Efficient with deadline
Georgiadis 2015	Continuous	Contributions unobservable, experimentation outcomes observable	Irreversible	Endogenous	Binary	Uncertain
Külpmann 2015	Continuous	Contribution unobservable, experimentation outcomes unobservable	Irreversible	Finite	Uncertain	Uncertain

1.2.3 Closely related papers

Our Chapter 4 is heavily based on Admati and Perry 1991[1] (AP 1991), Compte and Jehiel 2003 b[15] (CJ 2003) and Marx and Matthews 2000[41] (MM 2000) . We introduce these three papers in sequence first.

Admati and Perry 1991 start the literature of dynamic contribution games. They focus on the conditions for completion using an infinite horizon contribution game, in which two players alternate in making sunk contributions. The project is completed once total contributions reach W, yielding each player an immediate return of V.

Here is the intuition behind the solution: as the game has an infinite horizon, player j in stage $t+1$ would be in an identical situation to player i in stage t if player i contributes nothing in stage t. Hence in equilibrium, each player in each stage is indifferent between making his current contribution and making a zero contribution. To see this, consider the special case of a linear function. Suppose that the project is completed in stage t by player j. She must be indifferent between making her contribution y_t to earn a payoff of $V - y_t$, and making a zero contribution to earn a payoff of δV (in the next stage player i will complete the project and generate a payoff of V for her). Therefore, $y_t = (1-\delta)V$, where δ is a discount factor. With $y_t = (1-\delta)V$, we can further deduce that in stage $t-1$, player i must be indifferent between making her contribution x_{t-1} to earn a payoff of $\delta V - x_{t-1}$, and making a zero contribution to earn a payoff of $\delta^2 V - \delta^2 y_t$. Therefore, $x_{t-1} = (\delta - \delta^3)V$. Such a process continues recursively and we can find every contribution made on the equilibrium path: $(1-\delta)V$, $(\delta - \delta^3)V$, $(\delta^3 - \delta^5)V$... in periods V, $V-1$, $V-2$... . *Players make a total contribution equal to V to complete the project.*

Compare the total contribution players would like to make to complete the project V and the total contribution required to complete the project W. The main results are then easy to derive: the project would not be completed at all (with neither player contributing) unless a player is willing to complete the project by himself or herself ($V > W$); and when the project does get completed ($V < W$), the equilibrium involves delay: each player makes a series of small contributions in turn till the

project is completed. Indeed, if $W < 2V$ (in which case the total revenue exceeds the total cost in completing the project, i.e., the project is worth completing), equilibrium either involves failure (of the project) or delay.

According to the main result, we find that any project that one player can't complete by himself won't be completed by two players either. In other words, co-operation merely distributes the work between two players, but does not allow them to accomplish any new achievements. Why so? The authors attribute the reason to the sunk cost and introduce subscription games, which are similar to contribution games except that the cost is not sunk: players subscribe their intended contributions, which would only be borne when the project is completed. In this model, unlike in contribution games, any worth completing project ($W < 2V$) will be completed. What's more, players reach an early completion in two big steps, instead of reaching a late completion in many small steps (if they complete at all).

However, Admati and Perry's inefficient equilibrium in contribution games seems to be vulnerable in various aspects.

Compte and Jehiel 2003 b treat AP 1991 as a non-generic case of infinite horizon alternating contribution games due to their assumption that players value the project equally. Instead of assuming $V_1 = V_2$, they explore the equilibria where the valuations are heterogeneous.

With heterogeneous valuations, the intuition for solving AP 1991 model is no longer valid. It is conceivable that when the low valuing player would complete the project, the high valuing player would complete too; but when the high valuing player would complete the project, the low valuing player may not be willing to do so. The gradualism shown in the equilibrium in AP 1991 is the result of a stalemate between two homogeneous players: no one is more eager for an early completion; this stalemate no longer holds in CJ 2003. The high valuing player clearly has a stronger incentive to induce an early completion than the low valuing player.

The authors solve the model in two simple steps, starting in any subgame.

Denote M as the remaining contributions to be made to complete the project, and suppose that $V_i > V_j$. In the first step, they show that when M is in a certain

range, high valuing player i will complete the project but low valuing player j won't. This range covers from $(1-\delta)V_j$ to $(1-\delta)V_i$: for player i, delaying the project would cost him at least $(1-\delta)V_i$ and he is better off completing; for player j, anticipating that player i will complete in the next stage induces her to wait.

In the second step, they show that the upper bound on M can be extended up to V_i. They do so by induction. Suppose that $(1-\delta)V_i < M^* < V_1$ satisfies the property in the first step. If $M > M^*$ and M is close enough to M^* then player j could contribute $M - M^*$ to induce completion in the next stage; anticipating this, player i has no incentive to delay in exchange for player j's limited contribution.

Through above two steps, the authors show that if the project is worth completing (in this case, $W < V_1 + V_2$) then it would be completed. The player who values the project most would now make the final contribution and earn a zero payoff.

The difference between CJ 2003 and AP 1991 is clear: in AP 1991 each player knows that he can rely on his opponent to do his job in the next round, and both players have a strong incentive to free-ride on each other; in CJ 2003, one of the players can't do so because he has a higher valuation and has more to lose than his opponent. This heterogeneous valuations effect induces an early completion.

In our thesis we also consider heterogeneous valuations, which generates similar effects.

In their main model, Marx and Matthews 2000 replace an two-player alternating protocol with a N-player simultaneous protocol (in each stage of the game all players choose their contributions). By further considering the following three aspects, they study how the dynamic contribution game changes the completion pattern in comparison with the static (one-shot) contribution game.

The first aspect is players' knowledge of individual contributions after every round (observable if individual contributions are public information, unobservable if not). The second aspect is whether there is discounting in the game (when there is no discounting the project is short-term). The third aspect is the benefit function $f_i(X) = \begin{cases} \lambda_i X & for \quad X < W \\ V_i & for \quad X \geq W \end{cases}$, X is the accumulated contribution before the

deadline, λ_i is the player's marginal benefit from a non-completing contribution, where $\lambda_i = 0$ represents a binary benefit function and $\lambda_i W = V_i$ represents a continuous benefit function.

They come up with an interesting result: in the unobservable case, the project gets completed efficiently (in early stage) if players' benefit functions are similar. Here is their main result:

Theorem 2. Let g be a Nash equilibrium outcome. Then g is a PBE outcome if

(a) $c_1^* = c_i^*$ for all $i \in N$, or

(b) $c_1^* = c_2^*$, and both $g_1(t) \geq g_i(t)$ and $g_2(t) \geq g_i(t)$ for all $t \geq 0$ and $i = 3, ..., n$.

$c_i^* = \frac{V_i - \lambda_i W}{1 - \lambda_i}$ is player i's critical contribution (the maximum contribution he would like to make to complete the project); $g_i(t)$ is the contribution player i makes in stage t.

Given condition (a), with the punishment strategy of halting all future contributions by every non-deviator, the unilateral deviator will have to complete the project on his own as no one else has a stronger incentive than him to complete the project. Therefore, the symmetry in critical contribution c_i^* deters any unilateral deviation. Condition (b) states that with the same punishment strategy and every non-deviator believing that the unilateral deviator is player 1 or 2, every non-deviator among the remaining players (other than 1 and 2) expects "the deviator" 1 or 2 to complete the project on his own; the non-deviator among 1 and 2 expects the counterpart to complete the project on his own. As long as the two most incentivized players share a common critical contribution and make more contributions than others in every stage, unilateral deviation is unprofitable.

Intuitively, as long as that there is no specific player to wrap things up, two players with highest critical contribution have an incentive to free-ride on each other. The strong punishment strategy (all non-deviators make no further contributions) makes it unprofitable for any player to deviate: if someone does, the free-riding between player 1 and 2 would cause the project to fail; and that is not consistent with the interest of the deviator. Counter-intuitively, the possibility of free-riding could actually lead to a more efficient outcome. Notice that this theorem does not

hold if contributions are observable: while players 1 and 2 can't free-ride on each other, some player other than 1 or 2 would be able to free-ride on 1 and 2 without fearing that the project will fail.

In MM 2000, similarity between players could lead to a more efficient outcome. In our Chapter 4, we also show how the identities of players might affect the equilibrium outcome, but in a complete information environment. Unlike their result, similarity between players does not lead to a more efficient equilibrium.

Another important effect in our chapter is the deadline effect. In other words, unlike AP 1991 and CJ 2003, we study finite horizon contribution games.

MM 2000 mentioned this in their appendix as an extension of AP 1991. They show that if the model in AP 1991 has a deadline and the discount factor δ approaches 1, the project would be completed when $V < W < 2V$; the final contributor then earns 0. Therefore, instead of requiring $W < V$ (any single player would complete the project by himself) for the project to be completed in the infinite horizon case as in AP 1991, any project that is worth completing would be completed when there is a deadline. In other words, cooperation between two players would be more efficient with the help of a deadline. Clearly, the final contributor at the deadline would be in a weaker position than his rival; so he tries to avoid being put in such an awkward position. Consequently, the final contributor has a stronger incentive to induce an early completion.

In our chapter, we go through above mentioned model in detail, and extend it in three ways. First, we relax the restriction of δ approaching 1 and study all the cases of a large enough δ. (If δ is too small then it is conceivable that every player wants to complete as soon as possible.) We then show that this model has multiple equilibria, which is not the focus of MM 2000. Finally, we consider heterogeneous valuations, as in CJ 2003. We show that the multiplicity relies on homogeneous valuations; in this sense, many equilibria are not robust as players' valuation of the project can hardly be exactly the same. In our chapter, we provide two possible types of robust equilibria: one involves early completion and the other involves completion at the deadline.

1.2.4 Other papers

While the papers discussed in previous section are closely related to our model in the fourth chapter, some other papers are quite enlightening.

Continuous Contributions

Fershtman and Nitzan 1991[22] provides a different model of dynamic contribution games. They assume that a group of n agents make *continuous* contributions to a project. Set the accumulated contribution to be $K(t)$, the contribution rate of each player i to be $x_i(t)$, and a constant decay factor to be δ, the total contribution K accumulates over time according to $\overset{\circ}{K}(t) = dK/dt = \sum_{i=1}^{n} x_i(t) - \delta K(t)$; and the project provides a *continuous* payoff: $V_i(K, x_i) = \alpha f(K(t)) - C(x_i(t))$, where $f(.)$ is the production function, $C(.)$ is the cost function and α is the degree of the public good (a private good if $\alpha = \frac{1}{n}$, a pure public good if $\alpha = 1$, and an intermediate public good if $\frac{1}{n} < \alpha < 1$). Under these assumptions, they study the equilibrium when accumulated contributions are observable or unobservable. While there is free-rider problem existing and the equilibrium is not socially optimal in the latter case, the free-rider problem is even more acute in the former case. Here is a simple intuition: when previous actions are observable, a higher contribution today incentivizes lower contributions tomorrow. Anticipating this extends the free-riding problem into an inter-temporal version from a merely cross-sectional one. Essentially, within the continuous contribution structure and the continuous payoff structure, all efforts are strategic substitutable: the more player i contributes today, the less player $j \neq i$ is willing to contribute today, and the less all players (including i) are willing to contribute tomorrow.

Relatedly, Kessing 2007[33] assumes that payoffs are zero till completion instead of a flow payment, and comes up with the opposite result: individual contributions are now strategic complements rather than strategic substitutes. The intuition behind this difference is as follows: when the payoff is continuously divisible, making

a greater contribution now pays off immediately, increasing immediate utility and reducing marginal utility; when the payoff is discrete, making a greater contribution now accelerates completion and increases marginal utility.

In our chapter, we assume that benefits are zero till completion as in Kessing 2007. But in contrast to Kessing 2007, we also assume that players value the project differently and show that this may cause inefficiency in spite of the strategic complementary.

Deadline Effects

Deadlines are always an important factor in contribution games. A literature followed MM 2000, incorporating a deadline into dynamic provision of public good.

Bonatti and Horner 2011[5] (BH 2011) further explore the importance of the deadline in a model of strategic experimentation. They study moral hazard in teams through a continuous time model of effort provision towards a common project, whose success probability is uncertain and depends on an underlying state. When the project is "good", the probability of making a breakthrough depends on the total contribution made; when the project is "bad", there is no possibility of a breakthrough. The project ends if a breakthrough occurs. Effort is unobservable, but the experimentation outcome (a breakthrough happens or not) is publicly observable. In their model, an agent's current effort is a substitute for his own and also his opponents' future efforts,[12] and it causes procrastination: players are less willing to make their contributions early in the game. However, setting an appropriate deadline could restore efficiency: procrastination remains, but players are willing to take a last shot before they lose the opportunity.

Campbell, Edererz and Spinnewijnx 2012[12] take a slightly different path. In their model, a breakthrough does not end the game; instead it increases the value of the joint project. Each agent can choose unilaterally, at any point in time, to

[12]This differs from Kessing 2007 result because of the strategic experimentation setting: players update their beliefs depending on the previous history; so a higher current contribution implies a larger probability that the status is bad, and discourages future contributions.

implement the project.[13] If neither agent has implemented then the game ends at a finite deadline. By further assuming private information about the experimentation outcome and giving players the option to disclose their information to their partners, they show that the ability to conceal information allows a deadline to provide stronger incentives for effort than in the public information model and improves welfare, even though it introduces some inefficient delay. They also show that there is a unique optimal deadline that maximizes beneficial productive efforts while avoiding unnecessary delays.

Georgiadis 2017[25][14] studies the deadline effect from a new perspective: if the effort cost is convex (instead of linear as in earlier studies) then the socially optimal choice of contribution would naturally involve gradualism. With a similar structure as BH 2011, he introduces two effects. The front-loading effect originates from inter-temporal strategic complementarity, as in Kessing 2007: excessive early contributions might now be inefficient; the shirking effect originates from the general free-riding incentive: players contribute at a lower than socially optimal level. While implementing a deadline might mitigate the shirking effect in this setting, it would not alleviate the front-loading effect.

Külpmann 2015[34] constructs a more general model: by creating a "breakthrough effort distribution" (the current required total contribution given the accumulated contribution), he includes three types of strategic interaction: strategic substitutes as in BH 2011, strategic complements as in Georgiadis 2017, or strategic neutrals. Although different types of strategic interaction yield different behavior as illustrated in previous papers: strategic complements encourage future efforts while strategic substitutes discourage them; the assumption of deadline and convex costs lead to a consequence of procrastination, or, deadline effect, which is stronger than other effects when close to the deadline.

In each of above papers, all participants value the project identically (at least in their main models). The deadline effect combats free-riding incentive and usually

[13] Basically, they endogenize the uncertain underlying state of completing the project in BH 2011.

[14] Georgiadis, Lippman and Tang 2014[26], Cvitanic and Georgiadis 2016[17] also study the deadline effect in dynamic contribution games; but this paper is the most related one.

increases total welfare to some extent. In our chapter, we provide a rare example of a deadline slows down completion of the project when players value the project differently.

Chapter 2

Bargaining models with Endogenous Commitment I

Abstract

Traditional bargaining models assume that offers are revocable. In this chapter, we discuss the alternative assumption of completely irrevocable offers. We study the effect of the Endogenous Commitment in traditional finite and infinite alternating offer bargaining models.

2.1 Introduction

Bargaining is an important skill, particularly in China. Unlike in UK, where large supermarkets like Tesco or Sainsbury make take-it-or-leave-it offers, many families in China still purchase groceries from street vendors: the benefit from trade depends heavily on negotiation skills. After hundreds of years of feuds between vendors and housewives, certain interesting routines emerged.

In many cases, both sides bargain heatedly. Before the 20 minutes argument and the final agreement on the market price of one pound, an experienced vendor would probably start by demanding 10 pounds for one can of cola and an experienced housewife would most likely offer 10 pence. Street battle is no place for a rookie;

most players do recognize an unreasonable price when they see one, but such routines never vanish.

This phenomenon of extreme opening offers reveals certain unwritten, but truly existing commitment issues in bargaining.

On the one hand, the extreme opening offer creates a possibility, however small, that the opponent would accept an unreasonably high or low offer, and an accepted offer is non-retractable. Once the money and good exchange hands, the deal is sealed. Further analysis of this commitment problem can be found in Muthoo 1990[42], 1992[43], 1995 a[44], 1995 b[45], 1999[46]. In this chapter, we focus on the other commitment issue.

Besides taking a chance on one's opponent making a mistake, the main use of extreme opening offers is to create a large space for concessions. No one would admit failure; so the only way to keep the negotiation going after hearing a rejection from the opponent is to make a better offer to him. A series of concessions represents respect for the opponent and a willingness to make a deal; and the only way to make a concession without suffering from a loss is to open with an extreme offer which can be moderated. The hidden commitment here is that *when the proposer has made an offer then he is restricted to make better offers afterwards; when the respondent rejects an offer then he is restricted from accepting worse offers afterwards.* These are the two strategically equivalent descriptions of the Endogenous Commitment assumption we use in our thesis. This assumption was first made by Fershtman and Seidmann 1993[23].

In this chapter, we will study the effect of the Endogenous Commitment constraint in classical alternating offer bargaining games. We present the model in Section 2.2, and consider its implications for finite and infinite horizon bargaining games in Sections 2.3 and 2.4 respectively. We provide some comparative analysis in Section 2.5, and discuss our results in Section 2.6.

2.2 Model description

In this chapter, we study the effect of Endogenous Commitment by extending two classical bargaining models: Rubinstein's infinite horizon alternating offer bargaining model and Stahl's finite horizon alternating offer bargaining model. By comparing the original model and the extended model, we see how the introduction of Endogenous Commitment changes the equilibrium pattern and strategies.

We mainly discuss four models; here is the description of the first two.

Two impatient players $i = 1, 2$ are trying to allocate \$1 between them. Therefore, the set of all feasible agreement is $Z = \{(z,\ 1-z) :\ z \in [0, 1]\}$, in which z refers to 1's share of the total value and $1 - z$ refers to the share for player 2. The players have a common discount factor $\delta \in (0,\ 1)$; so we define each player's payoff as $U_1 = \delta^t u(z)$, $U_2 = \delta^t u(1-z)$: where $t \in [1, 2, 3...T]$ denotes the stage of the game at which agreement is reached, and T is the length of the game[1]. For simplicity, we focus on the linear case, $U_1 = \delta^t z$, $U_2 = \delta^t (1-z)$.

In this chapter, we will suppose that the two players alternate in acting as proposer or respondent. For example, at stage $t < T$, if it is player 1 who acts as the proposer, he makes an offer $x_t \in [0,\ 1]$, which player 2 could accept or reject. If the offer is accepted, then the game is over and the agreement is $\{1-x_t,\ x_t\}$. Otherwise the game continues to period $t + 1$, when it is player 2 who acts as the proposer, making an offer $y_{t+1} \in [0,\ 1]$, which player 1 could accept or reject. If the offer is accepted, then the game is over and the agreement is $\{y_{t+1},\ 1-y_{t+1}\}$. This process goes on until $t = T$. If T is finite and no offer has been accepted by the end of period T then the game ends with both players earning 0. We denote such a finite horizon bargaining game by Γ_F. Here we consider an exogenous disagreement payoff. The more complicated case with endogenous disagreement payoffs will be covered when we introduce a system of arbitration in Chapter 3 of the thesis. If $T = +\infty$, then negotiations would only end once players have reached an agreement. We denote such an infinite horizon bargaining game by Γ_I.

[1] We could extend our analysis to the asymmetric case where $\delta_1 \neq \delta_2$, but this would hardly affect the main conclusions.

Until now, we have defined standard alternating offer bargaining models akin to Rubinstein 1982[52] and Stahl 1972[58]. For future comparison, we provide the equilibrium patterns and strategies for Γ_I and Γ_F from the two papers :

Proposition 2.1. *When $T = +\infty$, there is a unique sub-game perfect equilibrium in the infinite-horizon alternating offer bargaining game Γ_I. In equilibrium, in any stage t player 1 offers $x_t = \frac{\delta}{1+\delta}$ as proposer and accepts an offer if and only if $y_t \geq \frac{\delta}{1+\delta}$ as respondent. Similar applies for player 2.*

Proposition 2.2. *When T is a positive integer, there is a unique sub-game perfect equilibrium in the finite-horizon alternating offer bargaining game Γ_F. In equilibrium, in any stage t player 1 offers $x_t = \delta\frac{1-(-\delta)^{T-t}}{1+\delta}$ as proposer and accepts an offer if and only if $y_t \geq \delta\frac{1-(-\delta)^{T-t}}{1+\delta}$. Similar applies for player 2.*

In spite of sharing the property of having a unique immediate agreement, equilibrium strategies in Γ_I are independent of t, while equilibrium strategies in Γ_F are depend on t. This distinction relies on the stationary structure of Γ_I and the non-stationary structure of Γ_F. To better understand this, we provide a sketch of the proof which is used in Shaked and Sutton 1984[56] for Proposition 2.1, based on the stationary structure of Γ_I :

Consider a sub-game starting in stage $t+1$. Without loss of generality suppose that player 1 is the proposer, and define M as the supremum of shares that player 1 can obtain in any SPE. Then in stage t, player 1 would accept any offer larger than δM, because if he rejects, he is bound to get a share no larger than M in future. Therefore, the most player 2 would offer to player 1 is δM, as player 2 would have an incentive to reduce the offer if it is more than δM. Hence, player 2 could guarantee a share equal to $1-\delta M$ in this stage (which is actually her infimum). Then in stage $t-1$, player 2 would accept no offer less than $\delta - \delta^2 M$, because if she rejects, she is able to get a share no smaller than $1-\delta M$ in the next stage. Therefore, the least

player 1 would offer to player 2 is $\delta - \delta^2 M$, and the supremum share for player 1 in this stage is $1 - \delta + \delta^2 M$.

But *the sub-game starting from* $t-1$ *is identical to the sub-game starting at* $t+1$, apart from a shrinkage of all payoffs by a factor of δ^2. Hence the supremum share that player 1 would obtain in stage $t-1$ and $t+1$ should be equal, i.e., $M = 1-\delta+\delta^2 M$. Similarly, the argument could be applied in finding the infimum share that player 1 would obtain: $N = 1 - \delta + \delta^2 N$. Obviously, the supremum and infimum of player 1's share coincide: $M = N = \frac{1}{1+\delta}$. Therefore the equilibrium shares are uniquely defined: the proposer receives $\frac{1}{1+\delta}$ and the respondent receives $\frac{\delta}{1+\delta}$. It is easy to show that this solution is indeed supported by a pair of strategies, so there is a unique sub-game perfect equilibrium outcome.

The italicized part of the proof in the previous paragraph indicates the stationarity of the structure of Γ_I. The identical structure of sub-games that start with the same proposer relies on two conditions: first, the previous history adds no restrictions on future strategies; second, all those sub-games have a common bargaining structure. Both conditions are satisfied in Γ_I as all sub-games have infinite remaining periods ahead, while things are clearly different in Γ_F. In Γ_F, the second condition is violated: sub-games starting from different periods of the game t have different numbers of remaining periods until the deadline; so players' strategies depend on the number of remaining periods $T - t$, as suggested in Proposition 2.2.

Γ_I and Γ_F provide simple and elegant models of general bargaining situations, and their tractability allows us to easily extend them. With the introduction of Endogenous Commitment, we capture the idea that the decisions made during negotiations will always depend on the history of the game; and an offer that is too generous would restrict the proposer's future flexibility. The plausibility of EC assumption is discussed in Chapter 1.

We now formally introduce Endogenous Commitment: whenever player i acts as the proposer, he is not allowed to offer a share lower than his last offer as a proposer: i.e., $\forall t \in [1,\ T]$, if player 1 proposes in period t then possible offers satisfy $x_t \in [x_{t-2},\ 1]$; if player 2 proposes in period t then possible offers satisfy

$y_t \in [y_{t-2},\ 1]$.[2]

Obviously, in equilibrium, we would have a series of increasing offers from both players: $... \leq x_{t-2} \leq x_t \leq x_{t+2} \leq ...$ and $... \leq y_{t-3} \leq y_{t-1} \leq y_{t+1} \leq ....$ In no circumstances would we have $x_t \leq x_{t'}$ or $y_t \leq y_{t'}$ when $t > t^{'}$.

Using this assumption in Γ_I and Γ_F , we define two new bargaining games: $\Gamma_I^{'}$ and $\Gamma_F^{'}$. The assumption constrains the proposer's behavior and guarantees history dependent strategies. We study $\Gamma_I^{'}$ in Section 2.3 and $\Gamma_F^{'}$ in Section 2.4 in detail.

The introduction of Endogenous Commitment changes the structure of the game dramatically, though the perfect information structure is kept. Endogenous Commitment turns the stationary structure in Γ_I into a non-stationary structure in $\Gamma_I^{'}$. In Γ_I, the stationary equilibrium strategy (offer $x_t = y_t = \frac{\delta}{1+\delta}$ and accept if and only if $x_t = y_t \geq \frac{\delta}{1+\delta}$) seems natural: every time player i gets to propose, he is in the same situation as before, so he makes the exact same offer. From another perspective, every time player i proposes, he is making a credible threat: I will not increase my offer, because I will always be in the same position when I am the proposer. If there is an offer that I won't make now, then I won't make this offer in future. The same applies to a respondent: every time player i responds, he is also in the same situation as before (if he receives the same offer), so he will always react the same. From another perspective, every time player i rejects, he is making following statement: as I will always be in the same position when receiving this offer again, I won't accept this offer in future if I don't accept this offer now.

In $\Gamma_I^{'}$ and $\Gamma_F^{'}$, these threats are no longer obviously valid because players will make decisions depending on histories. For example, in equilibrium the proposer might offer $x > x^{'}$ if the maximal offer he has made so far is $x^{'}$; while the maximal offer he is wiling to make is even higher than x.

Furthermore, due to the construction of Endogenous Commitment, in stage t, players' choices will be affected by players' choices in stage $t-1$ and $t-2$, but not by earlier ones. We can therefore define Markovian type sub-games in $\Gamma_I^{'}$ and $\Gamma_F^{'}$ as

[2]We denote that $x^- = y^- = 0$ for most of the chapter: in other words, neither player starts the negotiation with any exogenous pre-commitment. The $x^-,\ y^- > 0$ case will be briefly discussed in Section 2.5.

follows.

In $\Gamma_I^{'}$, every sub-game starting from the proposer at stage t has two state variables: m_{t-2} and n_{t-1}, m_{t-2} is the proposer's last offer made so far, and n_{t-1} is the respondent's last offer made so far. The proposer's actual choice $m_t \in [m_{t-2},\ 1]$ depends on the state variables $(m_{t-2},\ n_{t-1})$. Every sub-game starting from the respondent at stage t has two state variables: m_t and n_{t-1}, and the respondent's actual choice from $(accept\ m_t,\ reject\ m_t)$ depends on the state variables: $(m_t,\ n_{t-1})$. Notice that the number of stage t is not directly involved in the state variables in $\Gamma_I^{'}$.

In $\Gamma_F^{'}$, every sub-game starting from the proposer at stage t has three state variables: t, m_{t-2} and n_{t-1}. The proposer's actual choice $m_t \in [m_{t-2},\ 1]$ depends on the state variables $(t,\ m_{t-2},\ n_{t-1})$. Every sub-game starting from the respondent at stage t has three state variables: t, m_t and n_{t-1}, and the respondent's actual choice from $(accept\ m_t,\ reject\ m_t)$ depends on the state variables $(t,\ m_t,\ n_{t-1})$.

Comparing to $\Gamma_I^{'}$, in $\Gamma_F^{'}$ there is an extra state variable t indicating the number of remaining stages of the game $T - t$.

For convenience, in most part of this chapter we will use the following tie-breaking rule:[3]

Tie-Breaking Rule Whenever a proposer is indifferent between making an offer the respondent would accept or making an offer the respondent would reject, he would choose the former offer. Whenever the respondent is indifferent between accepting or rejecting an offer, he would choose to accept.

2.3 Infinite horizon alternating offer bargaining

In this section, we analyze the infinite horizon alternating offer bargaining model with Endogenous Commitment $\Gamma_I^{'}$.

$\Gamma_I^{'}$, as we defined earlier, is quite similar to Γ_I. In fact, they share the property

[3]The result in Section 2.3 does not rely on the tie-breaking rule, we use it anyway for simplification and consistency. The result in Section 2.4 relies on the tie-breaking rule, the situation without the tie-breaking rule will be discussed in Appendix 2.

of an immediate agreement. Also, it is not difficult to see that the unique equilibrium strategy in Γ_I itself does not violate the Endogenous Commitment constraint. Therefore, it is only natural that the equilibrium strategy in $\Gamma_I^{'}$ is similar with the equilibrium strategy in Γ_I.

In $\Gamma_I^{'}$, the equilibrium pattern is as follows.

Proposition 2.3. *When $T = +\infty$, there is a unique sub-game perfect equilibrium with immediate agreement in the infinite-horizon alternating offer bargaining game $\Gamma_I^{'}$. In equilibrium, player 1 offers $x_t = max\{x_{t-2},\ min\{\frac{\delta}{1+\delta},\ \delta(1-y_{t-1})\}\}$ and accepts player 2's offer y_t if and only if $y_t \geq min\{\frac{\delta}{1+\delta},\ \delta(1-x_{t-1})\}$. Player 2 offers $y_t = max\{y_{t-2},\ min\{\frac{\delta}{1+\delta},\ \delta(1-x_{t-1})\}\}$ and accepts player 1's offer x_t if and only if $x_t \geq min\{\frac{\delta}{1+\delta},\ \delta(1-y_{t-1})\}$.*

We prove the existence in Section 2.3.1 and the uniqueness in Section 2.3.2.

2.3.1 Existence

As $\Gamma_I^{'}$ is a discounted payoff game, it satisfies continuity at infinity (payoffs in very far future are not important), and we can prove that the strategies in Proposition 2.3 constitute a SPE using one shot deviation principle. Without loss of generality, let player 1 be the proposer and let player 2 be the respondent in stage $t-1$, we will show that, in stage $t-1$, neither player has an incentive to deviate in only one stage by finding their best response functions. As player 1 and player 2 use symmetric strategies, this analysis can easily be extended to every stage of the game, including those in which player 2 is the proposer and player 1 is the respondent.

In stage t, according to the given strategy combination, player 1 would accept any offer $y_t \geq min\{\frac{\delta}{1+\delta},\ \delta(1-x_{t-1})\}$ and reject any offer $y_t < min\{\frac{\delta}{1+\delta},\ \delta(1-x_{t-1})\}$; and player 2 would offer $y_t = max\{y_{t-2},\ min\{\frac{\delta}{1+\delta},\ \delta(1-x_{t-1})\}\}$. These strategies

generate a payoff pair of

$$\begin{cases} max\{y_{t-2},\ min\{\frac{\delta}{1+\delta},\ \delta(1-x_{t-1})\}\}, \\ 1-max\{y_{t-2},\ min\{\frac{\delta}{1+\delta},\ \delta(1-x_{t-1})\}\} \end{cases}$$

in stage t.[4]

We now look at stage $t-1$.

Foreseeing the payoffs in the next stage of the game, player 2 earns x_{t-1} if she accepts; and earns $\delta(1-max\{y_{t-2},\ min\{\frac{\delta}{1+\delta},\ \delta(1-x_{t-1})\}\})$ if she rejects.

Hence, player 2 has no incentive to deviate from accepting iff

$$x_{t-1} \geq \delta(1-max\{y_{t-2},\ min\{\frac{\delta}{1+\delta},\ \delta(1-x_{t-1})\}\}),$$

viz., accepting iff

$$x_{t-1} \geq min\{\frac{\delta}{1+\delta},\ \delta(1-y_{t-2})\},$$

which is exactly the choice according to the specified strategy.

$min\{\frac{\delta}{1+\delta},\ \delta(1-y_{t-2})\}$ is the minimal acceptable offer for player 2 in stage $t-1$; we denote it as $x^{\#}_{t-1}$ for convenience.

For player 1, anticipating a payoff of $max\{y_{t-2},\ min\{\frac{\delta}{1+\delta},\ \delta(1-x_{t-1})\}\}$ in next stage of the game given x_t rejected, he earns $1-x_{t-1}$ if $x_{t-3} \geq x^{\#}_{t-1}$; and earns

$$\begin{cases} 1-x_{t-1} \\ \delta max\{y_{t-2},\ min\{\frac{\delta}{1+\delta},\ \delta(1-x_{t-1})\}\} \end{cases} \quad if \quad \begin{array}{c} x^{\#}_{t-1} \leq x_{t-1}, \\ x_{t-3} \leq x_{t-1} < x^{\#}_{t-1} \end{array}$$

if $x_{t-3} < x^{\#}_{t-1}$.

To understand above formula and all the similar formulas in the remaining of the chapter: player 1 earns $1-x_{t-1}$ if $x_{t-3} \geq x^{\#}_{t-1}$; and if $x_{t-3} < x^{\#}_{t-1}$, player 1 earns $1-x_{t-1}$ if $x^{\#}_{t-1} \leq x_{t-1}$, and earns $\delta max\{y_{t-2},\ min\{\frac{\delta}{1+\delta},\ \delta(1-x_{t-1})\}\}$ if $x_{t-3} \leq x_{t-1} < x^{\#}_{t-1}$.

Therefore, player 1 has no incentive to deviate from following choices:

[4]Notice that the payoffs are all evaluated at the stage in which players make choices. The same principle applies to future similar situations.

$$x_{t-1} = x_{t-3}$$

if $x_{t-3} \geq x_{t-1}^{\#}$; and

$$\begin{cases} x_{t-1} = x_{t-1}^{\#} \\ x_{t-1} = x_{t-3} \end{cases} \quad if \quad \begin{array}{l} 1 - x_{t-1}^{\#} \geq \delta max\{y_{t-2},\ min\{\frac{\delta}{1+\delta},\ \delta(1 - x_{t-3})\}\}, \\ 1 - x_{t-1}^{\#} < \delta max\{y_{t-2},\ min\{\frac{\delta}{1+\delta},\ \delta(1 - x_{t-3})\}\} \end{array}$$

if $x_{t-3} < x_{t-1}^{\#}$.

When $x_{t-3} \geq x_{t-1}^{\#}$, knowing player 2 would accept any offer $x_{t-1} \geq x_{t-1}^{\#}$, any offer player 1 could possibly make would be accepted immediately and player 1 earns $1 - x_{t-1}$. Player 1 naturally make the minimal offer available: $x_{t-1} = x_{t-3}$. When $x_{t-3} < x_{t-1}^{\#}$, player 1 has two options: he could offer $x_{t-1} \geq x_{t-1}^{\#}$, which would be accepted by player 2 and leave him with a payoff of $1 - x_{t-1}$; or he could offer $x_{t-3} \leq x_{t-1} < x_{t-1}^{\#}$, which would be rejected by player 2 and leave him with the counteroffer from player 2 in the next stage of $max\{y_{t-2},\ min\{\frac{\delta}{1+\delta},\ \delta(1-x_{t-1})\}\}$. In both cases, player 1's payoff weakly decreases with x_{t-1}; so he would offer $x_{t-1} = x_{t-1}^{\#}$ and earn $1-x_{t-1}$, or offer $x_{t-1} = x_{t-3}$ and earn $\delta max\{y_{t-2},\ min\{\frac{\delta}{1+\delta},\ \delta(1-x_{t-3})\}\}$, depending which option generates a higher payoff.

As

$$min\{\frac{\delta}{1+\delta},\ \delta(1-y_{t-2})\}+\delta max\{y_{t-2},\ \frac{\delta}{1+\delta}\} = \begin{cases} \frac{\delta}{1+\delta} + \frac{\delta^2}{1+\delta} & when \quad y_{t-2} < \frac{\delta}{1+\delta} \\ \delta(1 - y_{t-2}) + \delta y_{t-2} & \quad\ \ y_{t-2} \geq \frac{\delta}{1+\delta} \end{cases} = \delta < 1,$$

we have

$$1-min\{\frac{\delta}{1+\delta},\ \delta(1-y_{t-2})\} > \delta max\{y_{t-2},\ \frac{\delta}{1+\delta}\} \geq \delta max\{y_{t-2},\ min\{\frac{\delta}{1+\delta},\ \delta(1-x_{t-3})\}\}. \tag{2.1}$$

We can therefore simplify player 1's best response function to

$$\begin{cases} x_{t-1} = x_{t-3} \\ x_{t-1} = x_{t-1}^{\#} \end{cases} \quad if \quad \begin{array}{l} x_{t-3} \geq x_{t-1}^{\#} \\ x_{t-3} < x_{t-1}^{\#} \end{array},$$

viz.,

$$x_{t-1} = max\{x_{t-3},\ min\{\frac{\delta}{1+\delta},\ \delta(1-y_{t-2})\}\},$$

which is the choice described by the strategy combination.

Similar analysis applies to every stage of the game, In light of the one deviation principle, the specified strategy combination is a sub-game perfect equilibrium of $\Gamma_I^{'}$.

2.3.2 Uniqueness

We now show that the SPE strategy profile described in Proposition 2.3 is unique. The following proof is heavily based on the approach used by Schwartz and Wen 2007[55]. In their paper, the outside option payoff (the payoff each player earns when they terminate the game before the deadline) is the threshold offer, while in our chapter, without an outside option, $\frac{\delta}{1+\delta}$, which is the "default equilibrium payoff" from Γ_I, is the threshold offer.

First of all, write the strategy combination in an alternate form for simplicity. When player 1 is the proposer, denote $x_{t-2} = x^{'}$, $y_{t-1} = y^{'}$, $x_t = x$, $y_{t+1} = y$. Similar goes for player 2 as proposer. In stage t player 1 offers

$$x = X(x^{'}, y^{'}) = \begin{cases} \frac{\delta}{1+\delta} & (x^{'}, y^{'}) \in R_1, \\ \delta(1-y^{'}) \quad when & (x^{'}, y^{'}) \in R_2, \\ x^{'} & (x^{'}, y^{'}) \in R_3 \cup R_4. \end{cases}$$

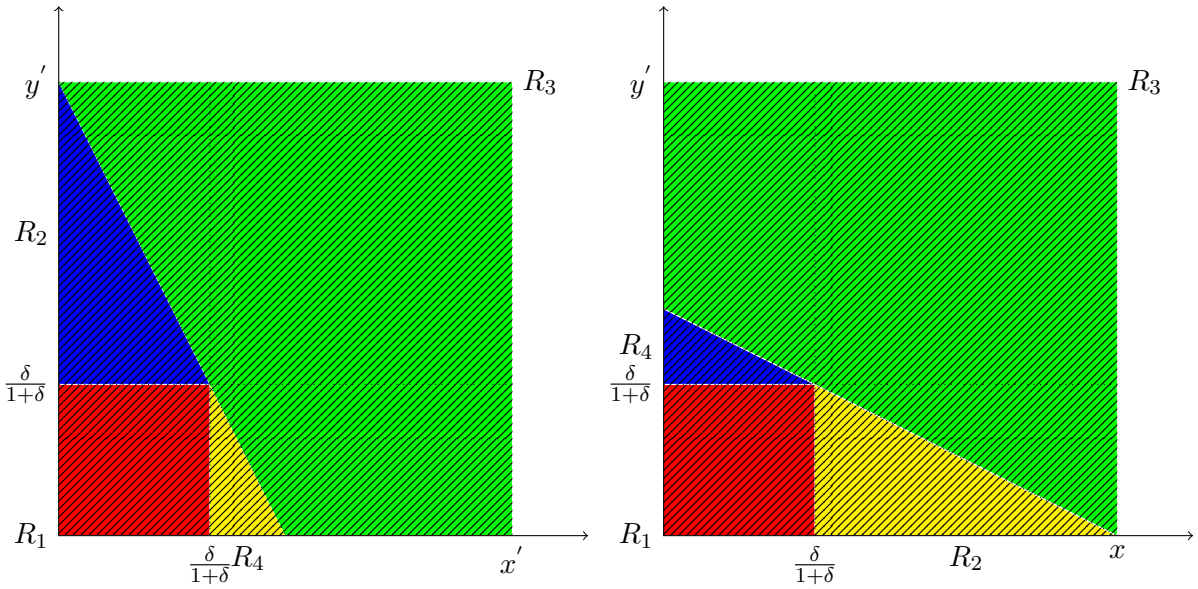
$R_1 = \{(x^{'}, y^{'}) \in [0,1]^2 :\ x^{'} \leq \frac{\delta}{1+\delta}\ and\ y^{'} \leq \frac{\delta}{1+\delta}\}$;

$R_2 = \{(x^{'}, y^{'}) \in [0,1]^2 :\ x^{'} \leq \delta(1-y^{'})\ and\ y^{'} > \frac{\delta}{1+\delta}\}$;

$R_3 = \{(x^{'}, y^{'}) \in [0,1]^2 :\ x^{'} > \delta(1-y^{'})\}$;

$R_4 = \{(x^{'}, y^{'}) \in [0,1]^2 :\ \frac{\delta}{1+\delta} < x^{'} \leq \delta(1-y^{'})\}$,

and player 2 accepts if and if $x \geq X(x^{'}, y^{'})$.



(In the left graph $x^{'}$ and $y^{'}$ are respectively on the horizontal and vertical axis, red area denotes R_1, blue area denotes R_2, green area denotes R_3 and yellow area denotes R_4. In the right graph x and $y^{'}$ are respectively on the horizontal and vertical axis, red area denotes R_1, yellow area denotes R_2, green area denotes R_3 and blue area denotes R_4.)

We already know from the existence proof that the set of SPE payoffs in this sub-game is non-empty. Let $M_i(m,\ n)$ and $m_i(m,\ n)$ respectively represent the supremum and infimum of player i' SPE shares in any stage t given the state $(m,\ n)$. Suppose that player 1 is the proposer and the state is $(x^{'}, y^{'})$, and player 1 offers x in this stage; then the state in the next stage is $(y^{'}, x)$. Then player 1's share in this stage is bounded between $m_1(x^{'}, y^{'})$ and $M_1(x^{'}, y^{'})$; while player 2's continuation share in next stage is bounded between $m_2(y^{'}, x)$ and $M_2(y^{'}, x)$.

Even in the best possible continuation for player 1, player 2 would reject if $x < \delta m_2(y^{'}, x)$. Player 1 could either make an acceptable or an unacceptable offer. In the best possible continuation, player 1 makes an acceptable offer that satisfies $x \geq \delta m_2(y^{'}, x)$ or makes an unacceptable offer that satisfies $x < \delta m_2(y^{'}, x)$. When $x < \delta m_2(y^{'}, x)$, player 1's continuation share could not exceed $\delta(1 - m_2(y^{'}, x))$. To summarize, we have

$$M_1(x^{'}, y^{'}) \leq max \begin{cases} \sup_{x \geq x^{'}} 1 - x & s.t. \quad x \geq \delta m_2(y^{'}, x), \\ \sup_{x \geq x^{'}} \delta(1 - m_2(y^{'}, x)) & \quad x < \delta m_2(y^{'}, x). \end{cases} \tag{2.2}$$

Similarly, even in the worst possible continuation for player 1, player 2 would accept if $x > \delta M_2(y^{'}, x)$; so

$$m_1(x^{'}, y^{'}) \geq max \begin{cases} \sup_{x \geq x^{'}} 1 - x & s.t. \quad x > \delta M_2(y^{'}, x), \\ \sup_{x \geq x^{'}} \delta(1 - M_2(y^{'}, x)) & \quad x \leq \delta M_2(y^{'}, x). \end{cases} \tag{2.3}$$

From the existence proof and the definition of $m_1(x^{'}, y^{'})$ and $M_1(x^{'}, y^{'})$, we can easily see that

$$m_1(x^{'}, y^{'}) \leq 1 - X(x^{'}, y^{'}) \leq M_1(x^{'}, y^{'}). \tag{2.4}$$

We are now going to prove the uniqueness of the equilibrium by proving that $m_1(x^{'}, y^{'}) = 1 - X(x^{'}, y^{'}) = M_1(x^{'}, y^{'})$.

Lemma 2.1. $M_1(x^{'}, y^{'}) \leq 1 - x^{'}$.

Proof.

We prove Lemma 2.1 by contradiction. If there exists an equilibrium in which player 1 gets a share $X > 1 - x^{'}$ then it must be realized when player 2 rejects player 1's offer x, because $1 - x \leq 1 - x^{'}$ imply that player 1 would get less than $1 - x^{'}$ if player 2 accepts. Hence, the equilibrium involves player 1 offering $x \geq x^{'}$ and player 2 rejects x. In this case, as player 1 earns X, player 2's continuation payoff is at most $1 - X$, which is less than $x^{'} \leq x$, so player 2 would be better off accepting x in the first place, which forms the contradiction. Therefore, player 1 can't get a share larger than $1 - x^{'}$.■

Using Lemma 2.1 and Equation 2.4, we can easily deduce that

$$M_1(x^{'}, y^{'}) = 1 - x^{'} = 1 - X(x^{'}, y^{'}) \; when \; (x^{'}, y^{'}) \in R_3 \cup R_4. \tag{2.5}$$

The rest of this proof is divided into the following six steps to pin down the value of $m_1(x^{'}, y^{'})$ and $M_1(x^{'}, y^{'})$ at $1 - X(x^{'}, y^{'})$ for all $(x^{'},\ y^{'}) \in [0,1]^2$.

Step 1 $m_1(x^{'}, y^{'}) = 1 - X(x^{'}, y^{'})$ for all $(x^{'}, y^{'}) \in R_2$.

For all $(x^{'}, y^{'}) \in R_2$ and $x \geq x^{'}$, $(y^{'}, x) \in R_3 \cup R_4$ in the next stage, so Equation 2.5 implies that $M_2(y^{'}, x) = 1 - y^{'}$. Equation 2.3 then implies that

$$m_1(x^{'}, y^{'}) \geq max \begin{cases} \underset{x \geq x^{'}}{sup} 1 - x & \\ \underset{x \geq x^{'}}{sup} \delta(1 - M_2(y^{'}, x)) & s.t. \end{cases} \begin{array}{l} x > \delta M_2(y^{'}, x), \\ x \leq \delta M_2(y^{'}, x). \end{array}$$

$$= max \begin{cases} \underset{x \geq x^{'}}{sup} 1 - x & \\ \underset{x \geq x^{'}}{sup} \delta y^{'} & s.t. \end{cases} \begin{array}{l} x > \delta(1 - y^{'}), \\ x \leq \delta(1 - y^{'}). \end{array}$$

$$= max \begin{cases} 1 - \delta(1 - y^{'}) \\ \delta y^{'} \end{cases}$$

$$= 1 - \delta(1 - y^{'}),$$

which fits the description of $X(x^{'}, y^{'})$ when $(x^{'}, y^{'}) \in R_2$. Combined with Equation 2.4, we have $m_1(x^{'}, y^{'}) = 1 - X(x^{'}, y^{'})$ for all $(x^{'}, y^{'}) \in R_2$.

Step 2 $m_1(x^{'}, y^{'}) = 1 - X(x^{'}, y^{'})$ for all $(x^{'}, y^{'}) \in R_3$.

For all $(x^{'}, y^{'}) \in R_3$ and $x \geq x^{'} > \delta(1 - y^{'})$, Lemma 2.1 implies that $M_2(y^{'}, x) \leq 1 - y^{'}$, so

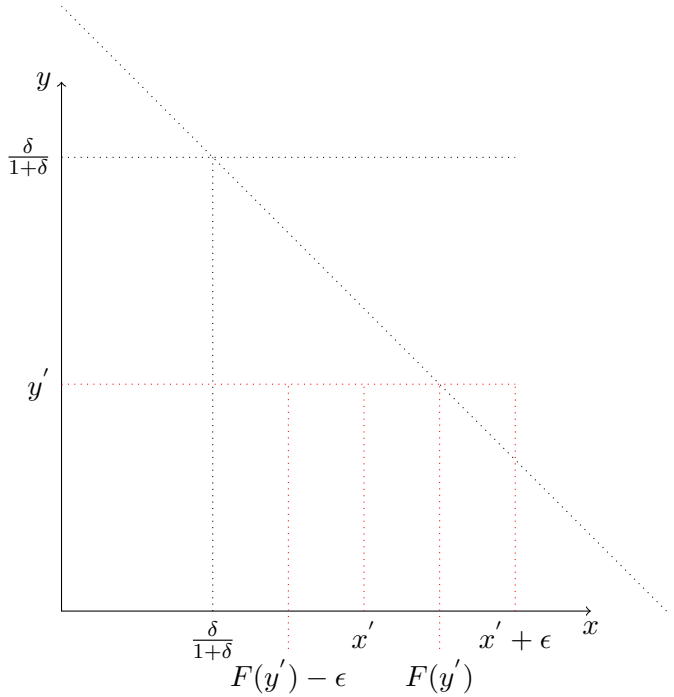
$$1 - x < 1 - \delta + \delta y^{'} \leq 1 - \delta + \delta(1 - M_2(y^{'}, x))$$

and therefore $x > \delta M_2(y^{'}, x)$. Player 2 would accept any offer of $x \geq x^{'}$. Equation 2.3 then implies that

$$m_1(x^{'}, y^{'}) \geq max \left\{ \underset{x \geq x^{'}}{sup} 1 - x \ s.t.\ x > \delta M_2(y^{'}, x) = 1 - x^{'}, \right.$$

which fits the description of $X(x^{'}, y^{'})$ when $(x^{'}, y^{'}) \in R_3$. Combined with Equation 2.4, we have $m_1(x^{'}, y^{'}) = 1 - X(x^{'}, y^{'})$ for all $(x^{'}, y^{'}) \in R_3$.

Step 3 $m_1(x^{'}, y^{'}) = 1 - X(x^{'}, y^{'})$ for all $(x^{'}, y^{'}) \in R_4$.



For all $y^{'} \in [0, 1]$, define the frontier of the states where player 1 offers $x = x^{'}$ as

$$F(y^{'}) = inf\{x^{'} : \ m_1(x, y) = 1 - x \ for \ all \ (x, y) \geq (x^{'}, y^{'})\}.$$

$F(y^{'})$ is well defined due to Steps 1 and 2: $m_1(x^{'}, y^{'}) = 1 - X(x^{'}, y^{'}) = 1 - x^{'}$ for all $(x^{'}, y^{'}) \in R_3$, so for every $y^{'}$, there exists a minimal $x^{'}$ such that $m_1(x, y) = 1 - x$ for all $(x, y) \geq (x^{'}, y^{'})\}$. If $F(y^{'}) \leq \frac{\delta}{1+\delta}$ for all $y^{'} \leq \frac{\delta}{1+\delta}$, then $m_1(x^{'}, y^{'}) = 1 - x^{'} = 1 - X(x^{'}, y^{'})$ on R_4.

To get a contradiction, suppose that $F(y^{'}) > \frac{\delta}{1+\delta}$ for some $y^{'} \leq \frac{\delta}{1+\delta}$, and choose an $\epsilon > 0$ small enough that

$$\epsilon < (1 - \delta^2)(F(y^{'}) - \frac{\delta}{1 + \delta}).$$

By definition of $F(y^{'})$, $\exists x^{'}$ such that $F(y^{'}) - \epsilon < x^{'} < F(y^{'})$, and $\exists (x, y) \geq (x^{'}, y^{'})$

such that $m_1(x,y) > 1-x$. This means that there is an SPE where player 2 would reject if player 1 made no concession at state $(x,\ y)$. Also by definition of $F(y^{'})$, player 1 can offer $x+\epsilon$ which would be accepted by player 2; so

$$m_1(x,\bar{y}) \geq 1-x-\epsilon \ for\ all\ \bar{y} \geq y. \tag{2.6}$$

At state $(x,\ y)$, if player 1 makes no concession at x and player 2 rejectes and then offers $\bar{y} \geq y$ in the following stage, player 2's payoff would be at most $\delta M_2(y,x) \leq \delta^2(1-m_1(x,\bar{y}))$ due to Equation 2.2 (as player 1 will not make a concession, it is obviously better for player 2 to make an acceptable offer)

$$\begin{aligned}
&\delta^2 - \delta^2 m_1(x,\bar{y}) \\
&\leq \delta^2 - \delta^2(1-x-\epsilon) \text{ due to Equation 2.6} \\
&= \delta^2 x + \delta^2\epsilon \\
&= x - (1-\delta^2)x + \delta^2\epsilon \\
&< x - (1-\delta^2)(F(y^{'})-\epsilon) + \delta^2\epsilon \text{ since } x \geq x^{'} > F(y^{'}) - \epsilon \\
&= x - (1-\delta^2)F(y^{'}) + \epsilon \\
&\leq x - (1-\delta^2)F(y^{'}) + (1-\delta^2)(F(y^{'}) - \tfrac{\delta}{1+\delta}) \\
&= x - (1-\delta)\delta \\
&< x.
\end{aligned}$$

Player 2 must therefore accept x at state $(x,\ y)$. This contradicts the initial supposition that $m_1(x,y) < 1-x$. Therefore, $F(y^{'}) > \frac{\delta}{1+\delta}$ is not valid: $F(y^{'}) \leq \frac{\delta}{1+\delta}$ for all $y^{'} \leq \frac{\delta}{1+\delta}$. We can then easily deduce that $m_1(x^{'},y^{'}) = 1-x^{'} = 1-X(x^{'},y^{'})$ for all $(x^{'},y^{'}) \in R_4$.

Step 4 $m_1(x^{'},y^{'}) = 1-X(x^{'},y^{'})$ for all $(x^{'},y^{'}) \in R_1$.

For all $(x^{'},y^{'}) \in R_1$, player 1 can offer $F(y^{'})$ to induce player 2 to accept. Therefore, $m_1(x^{'},y^{'}) \geq 1-F(y^{'}) \geq 1-\frac{\delta}{1+\delta} = 1-X(x^{'},y^{'})$. Combined with Equation 2.4, we have $m_1(x^{'},y^{'}) = 1-X(x^{'},y^{'})$ for all $(x^{'},y^{'}) \in R_1$.

Step 5 $M_1(x^{'},y^{'}) = 1-X(x^{'},y^{'})$ for all $(x^{'},y^{'}) \in R_2$.

For all $(x^{'},y^{'}) \in R_2$ and $x \geq x^{'}$, $(y^{'},x) \in R_3 \cup R_4$, we have $m_2(y^{'},x) = 1-y^{'}$ by Steps 2 and 3. Equation 2.2 implies that $\delta(1-m_2(y^{'},x)) = \delta y^{'}$; so

$$M_1(x^{'}, y^{'}) \leq max \begin{cases} \underset{x \geq x^{'}}{sup} 1 - x & \\ \underset{x \geq x^{'}}{sup} \delta(1 - m_2(y^{'}, x)) & s.t. \begin{array}{l} x \geq \delta m_2(y^{'}, x), \\ x < \delta m_2(y^{'}, x). \end{array} \end{cases}$$

$$= max \begin{cases} \underset{x \geq x^{'}}{sup} 1 - x & \\ \underset{x \geq x^{'}}{sup} \delta y^{'} & s.t. \begin{array}{l} x \geq \delta(1 - y^{'}), \\ x < \delta(1 - y^{'}). \end{array} \end{cases}$$

$$= max \begin{cases} 1 - \delta(1 - y^{'}) & as\ (x^{'}, y^{'}) \in R_2 \\ \delta y^{'} & \end{cases}$$

$$= 1 - \delta(1 - y^{'}).$$

Therefore, $M_1(x^{'}, y^{'}) \leq 1 - X(x^{'}, y^{'})$. Combined with Equation 2.4, we have $M_1(x^{'}, y^{'}) = 1 - X(x^{'}, y^{'})$ for all $(x^{'}, y^{'}) \in R_2$.

Step 6 $M_1(x^{'}, y^{'}) = 1 - X(x^{'}, y^{'})$ for all $(x^{'}, y^{'}) \in R_1$.

For $(x^{'}, y^{'}) \in R_1$ and any $x \geq x^{'}$, $(y^{'}, x) \in R_1 \cup R_2 \cup R_3$ we have

$$m_2(y^{'}, x) = \begin{cases} 1 - \frac{\delta}{1+\delta} & (y^{'}, x) \in R_1, \\ 1 - \delta(1 - x) \quad if & (y^{'}, x) \in R_2, \\ 1 - y^{'} & (y^{'}, x) \in R_3. \end{cases}$$

$$\geq \begin{cases} 1 - \frac{\delta}{1+\delta} & (y^{'}, x) \in R_1, \\ 1 - \delta + \frac{\delta^2}{1+\delta} \quad if & (y^{'}, x) \in R_2, \\ 1 - \frac{\delta}{1+\delta} & (y^{'}, x) \in R_3. \end{cases}$$

$$= \frac{1}{1 + \delta}.$$

Hence, $x^{'} \leq \frac{\delta}{1+\delta} < \delta m_2(y^{'}, x)$ when $(x^{'}, y^{'}) \in R_1$. Equation 2.2 then implies that

$$M_1(x^{'},y^{'}) \leq max \begin{cases} \underset{x \geq x^{'}}{sup} 1-x & \\ \underset{x \geq x^{'}}{sup} \delta(1-m_2(y^{'},x)) & \end{cases} \begin{matrix} s.t. & x \geq \delta m_2(y^{'},x), \\ & x < \delta m_2(y^{'},x). \end{matrix}$$

$$= max \begin{cases} \underset{x \geq x^{'}}{sup} 1-\delta m_2(y^{'},x) & \\ \underset{x \geq x^{'}}{sup} \delta - \delta m_2(y^{'},x)) & \end{cases} \begin{matrix} s.t. & x \geq \delta m_2(y^{'},x), \\ & x < \delta m_2(y^{'},x). \end{matrix}$$

$$\leq 1 - \frac{\delta}{1+\delta}.$$

Therefore, $M_1(x^{'},y^{'}) \leq 1-X(x^{'},y^{'})$. Combined with Equation 2.4, we have $M_1(x^{'},y^{'}) = 1 - X(x^{'},y^{'})$ for all $(x^{'},y^{'}) \in R_1$.

To sum up, we have shown that $m_1(x^{'},y^{'}) = 1 - X(x^{'},y^{'}) = M_1(x^{'},y^{'})$ for all $(x^{'},y^{'}) \in [0,1]^2$, which concludes the uniqueness proof.

2.4 Finite horizon alternating offer bargaining

In this section, we analyze the finite horizon alternating offer bargaining model with Endogenous Commitment.

Recall that Proposition 2.2 implies that in each stage t of Γ_F , each player accepts an offer equal to or larger than $\delta\frac{1-(-\delta)^{T-t}}{1+\delta}$.

Now $\frac{\delta\frac{1-(-\delta)^{T-t}}{1+\delta}}{\delta\frac{1-(-\delta)^{T-t-2}}{1+\delta}} = \frac{1-(-\delta)^{T-t}}{1-(-\delta)^{T-t-2}}$, where $(-\delta)^{T-t} > (-\delta)^{T-t-2}$ when $T-t$ is odd, and $(-\delta)^{T-t} < (-\delta)^{T-t-2}$ when $T-t$ is even. Therefore, for given T, either $x_t^{\#} > x_{t+2}^{\#}$ and $y_{t-1}^{\#} < y_{t+1}^{\#}$; or $x_t^{\#} < x_{t+2}^{\#}$ and $y_{t-1}^{\#} > y_{t+1}^{\#}$. In both cases, the equilibrium strategy in Γ_F violates the Endogenous Commitment constraint. Actually, the player who acts as the "last proposer" would reduce his proposals gradually, and the player who acts as the "last respondent" would increase his proposals gradually. Suppose that player 2 is the last proposer, following offers are made in the last stages in

equilibrium:

$x^{\#}_{T-5} = \delta - \delta^2 + \delta^3 - \delta^4 + \delta^5$,

$y^{\#}_{T-4} = \delta - \delta^2 + \delta^3 - \delta^4$,

$x^{\#}_{T-3} = \delta - \delta^2 + \delta^3$,

$y^{\#}_{T-2} = \delta - \delta^2$,

$x^{\#}_{T-1} = \delta$,

$y^{\#}_{T} = 0$.

It can easily be seen that $x^{\#}_{T-5} < x^{\#}_{T-3} < x^{\#}_{T-1}$, and $y^{\#}_{T-4} > y^{\#}_{T-2} > y^{\#}_{T}$.

While Γ_I and $\Gamma_I^{'}$ have similar equilibria, the SPEs of Γ_F and $\Gamma_F^{'}$ are quite different. Unlike the unique immediate agreement SPE in Γ_F, in $\Gamma_F^{'}$ the equilibrium pattern is as follows.

Proposition 2.4. *When T is a positive integer, there is a unique sub-game perfect equilibrium in the finite-horizon alternating offer bargaining game $\Gamma_F^{'}$. The player who is the last respondent always makes acceptable offers in every stage of the game; while the player who is the last proposer makes unacceptable offers when discount rate is large enough and his previous offer is small enough. The agreement is reached in either the first or second stage of the game, depending on T and δ.*

Proof.

Notice that, given the tie-breaking rule, equilibrium uniqueness is almost immediate consequence of Zermelo's Theorem. Following we provide a detailed description of the unique equilibrium strategy.

Let player 2 be the last proposer without loss of generality. The equilibrium strategy combination then satisfies that

- In any stage t such that $T - t$ is odd:

Denote $R[a] = \{\delta \,|\delta^a + \delta^{a+1} = 1\,, 0 < \delta < 1\}$, a refers to the stage of the game, $R[a]$ indicates a critical discount rate in each stage of the game. Player 2 accepts

if and only if $x_t \geq min\{\frac{\delta}{1+\delta},\ \delta(1-y_{t-1})\}$ when $0 < \delta \leq R[T-t-1]$; $x_t \geq min\{max\{\delta^{T-t}(1-y_{t-1}),\ \frac{\delta}{1+\delta}\},\ \delta(1-y_{t-1})\}$ when $R[T-t-1] < \delta < 1$. Notice that if $\delta \leq R[T-t-1]$ then $\delta^{T-t} \leq \frac{\delta}{1+\delta}$; so $min\{max\{\delta^{T-t}(1-y_{t-1}),\ \frac{\delta}{1+\delta}\},\ \delta(1-y_{t-1})\}$ becomes $min\{\frac{\delta}{1+\delta},\ \delta(1-y_{t-1})\}$. The term $\delta^{T-t}(1-y_{t-1})$ originates from the deadline effect, so the economic intuition is straightforward: the deadline effect kicks in when players are patient, and vanishes otherwise.

Player 1 offers $x_t = max\{x_{t-2},\ min\{\frac{\delta}{1+\delta},\ \delta(1-y_{t-1})\}\}$ when $0 < \delta \leq R[T-t-1]$; offers $x_t = max\{x_{t-2},\ min\{max\{\delta^{T-t}(1-y_{t-1}),\ \frac{\delta}{1+\delta}\},\ \delta(1-y_{t-1})\}\}$ when $R[T-t-1] < \delta < 1$.

Payoff pairs are then

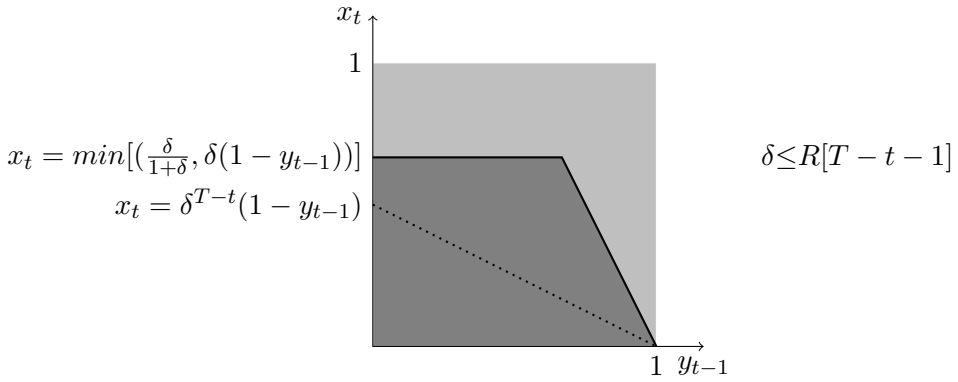
$$\begin{cases} 1 - max\{x_{t-2},\ min\{\frac{\delta}{1+\delta},\ \delta(1-y_{t-1})\}\}, \\ \quad max\{x_{t-2},\ min\{\frac{\delta}{1+\delta},\ \delta(1-y_{t-1})\} \end{cases}$$

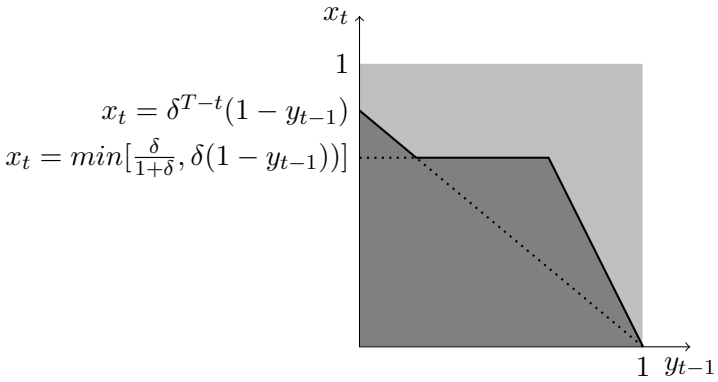
when $0 < \delta \leq R[T-t-1]$; and

$$\begin{cases} 1 - max\{x_{t-2},\ min\{max\{\delta^{T-t}(1-y_{t-1}),\ \frac{\delta}{1+\delta}\},\ \delta(1-y_{t-1})\}\}, \\ \quad max\{x_{t-2},\ min\{max\{\delta^{T-t}(1-y_{t-1}),\ \frac{\delta}{1+\delta}\},\ \delta(1-y_{t-1})\}\} \end{cases}$$

when $R[T-t-1] < \delta < 1$.

The figures below indicate the difference between small and large δ. Gray area denotes rejection by player 2, light gray area denotes acceptance.





$\delta > R[T - t - 1]$

(Player 2 accepts when $(x_t,\ y_{t-1})$ is in the gray area and rejects when in the light gray area.)

- In any stage t such that $T - t$ is even:

Player 1 accepts if and only if $y_t \geq min\{\frac{\delta}{1+\delta},\ \delta(1 - x_{t-1})\}$.

Player 2 offers $y_t = max\{y_{t-2},\ min\{\frac{\delta}{1+\delta},\ \delta(1-x_{t-1})\}\}$ when $R[T-t] < \delta < 1$ and $y_{t-2} \geq K[T-t, x_{t-1}]$, or when $0 < \delta \leq R[T-t]$; offers $y_t = y_{t-2}$ when $R[T-t] < \delta < 1$ and $y_{t-2} < K[T-t, x_{t-1}]$. $K[a, x_b] = min\{\frac{-1+\delta^a+\delta^{a+1}}{\delta^a+\delta^{a+1}},\ \frac{-1+\delta^a+\delta(1-x_b)}{\delta^a}\}$, indicating a condition that player 2 is indifferent between making the optimal acceptable offer and making the optimal unacceptable offer.

Payoff pairs are then

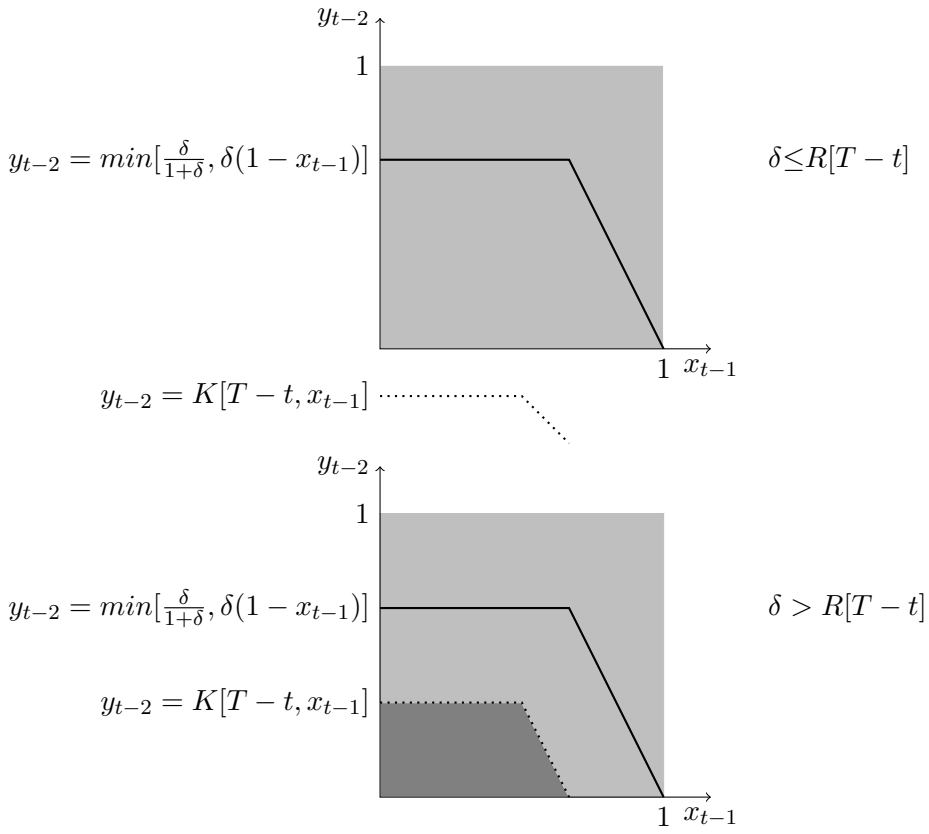
$$\begin{cases} max\{y_{t-2},\ min\{\frac{\delta}{1+\delta},\ \delta(1 - x_{t-1})\}\}, \\ 1 - max\{y_{t-2},\ min\{\frac{\delta}{1+\delta},\ \delta(1 - x_{t-1})\}\} \end{cases}$$

when $R[T - t] < \delta < 1$ and $y_{t-2} \geq K[T - t, x_{t-1}]$, or $0 < \delta \leq R[T - t]$; and

$$\begin{cases} \delta\{1 - max\{x_{t-1},\ min\{max\{\delta^{T-t-1}(1 - y_{t-2}),\ \frac{\delta}{1+\delta}\},\ \delta(1 - y_{t-2})\}\}, \\ \quad max\{x_{t-1},\ min\{max\{\delta^{T-t-1}(1 - y_{t-2}),\ \frac{\delta}{1+\delta}\},\ \delta(1 - y_{t-2})\}\}\} \end{cases}$$

when $R[T - t] < \delta < 1$ and $y_{t-2} < K[T - t, x_{t-1}]$.

The figures below indicate the difference between small and large δ. Gray area denotes player 2 offering $y_t = y_{t-2}$, light gray area denotes player 2 offering $y_t = max\{y_{t-2}, y_t^{\#}\}$. Notice that in the first graph $K[T-t, x_{t-1}] < 0$ and player 2 always offers $y_t = max\{y_{t-2}, y_t^{\#}\}$.



(Player 2 makes unacceptable offer when $(x_{t-1},\ y_{t-2})$ is in the gray area and acceptable offer when in the light gray area.)

Now we prove that this strategy profile does constitute a SPE of the game, using the following inductive hypothesis. Set

$$m_t = \begin{cases} x_t & when \quad T-t \text{ is odd}, \\ y_t & \quad\ \ T-t \text{ is even}. \end{cases}$$

Inductive Hypothesis 2.1. For every stage t and every state $(m_{t-2},\ n_{t-1},\ t)$, there are two sets of offers $a_t(n_{t-1}, t)$ and $A_t(m_{t-2}, n_{t-1}, t)$ with the following property: in every subgame equilibrium,

- The respondent accepts any offer m_t if and only if it belongs to $a_t(n_{t-1}, t)$, which is defined as $m_t \geq min\{\frac{\delta}{1+\delta},\ \delta(1-n_{t-1})\}$ if $0 < \delta \leq R[T-t-1]$ and $m_t \geq min\{max\{\delta^{T-t}(1-n_{t-1}),\ \frac{\delta}{1+\delta}\},\ \delta(1-n_{t-1})\}$ if $R[T-t-1] < \delta < 1$ when $T-t$ is odd; as $m_t \geq min\{\frac{\delta}{1+\delta},\ \delta(1-n_{t-1})\}$ when $T-t$ is even.

- The proposer repeats his last offer (defined as 0 in the first stage) if and only if $x_{t-1}, y_{t-1} \in A_t(m_{t-2}, n_{t-1}, t)$, where $A_t(m_{t-2}, n_{t-1}, t)$ is defined as $\emptyset$ when $T-t$ is odd, as $R[T-t] < \delta < 1$ and $y_{t-2} < K[T-t, x_{t-1}]$ when $T-t$ is even; the proposer otherwise makes the minimal offer possible in $a_t(n_{t-1}, t)$: $max\{m_{t-2}, a_t(n_{t-1}, t)\}$.

In Appendix 1, we show that IH 2.1 holds for the last three stages of the game. Here we show that IH 2.1 holds for stage t if it holds for stage $t+1$, $t < T-2$ in two steps.

Step 1 IH 2.1 holds for stage t if it holds for stage $t+1$ when $T-t$ is odd.

According to IH 2.1, payoffs in stage $t+1$ are:

$$\begin{cases} max\{y_{t-1},\ min\{\frac{\delta}{1+\delta},\ \delta(1-x_t)\}\}, \\ 1 - max\{y_{t-1},\ min\{\frac{\delta}{1+\delta},\ \delta(1-x_t)\}\} \end{cases}$$

when $R[T-t-1] < \delta < 1$ and $y_{t-1} \geq K[T-t-1, x_t]$, or $0 < \delta \leq R[T-t-1]$; and

$$\delta \begin{cases} 1 - max\{x_t,\ min\{max\{\delta^{T-t-2}(1-y_{t-1}),\ \frac{\delta}{1+\delta}\},\ \delta(1-y_{t-1})\}\}, \\ max\{x_t,\ min\{max\{\delta^{T-t-2}(1-y_{t-1}),\ \frac{\delta}{1+\delta}\},\ \delta(1-y_{t-1})\}\} \end{cases}$$

when $R[T-t-1] < \delta < 1$ and $y_{t-1} < K[T-t-1, x_t]$.

In stage t, player 2 compares the consequences of accepting and earning x_t immediately, with rejecting and earning

$$\delta(1 - max\{y_{t-1},\ min\{\frac{\delta}{1+\delta},\ \delta(1-x_t)\}\})$$

when $R[T-t-1] < \delta < 1$ and $y_{t-1} \geq K[T-t-1, x_t]$, or $0 < \delta \leq R[T-t-1]$; and

$$\delta^2 max\{x_t,\ min\{max\{\delta^{T-t-2}(1-y_{t-1}),\ \frac{\delta}{1+\delta}\},\ \delta(1-y_{t-1})\}\}$$

when $R[T-t-1] < \delta < 1$ and $y_{t-1} < K[T-t-1, x_t]$.

Therefore,

- When $0 < \delta \leq R[T-t-1]$, player 2 would accept iff

$$x_t \geq \delta(1 - max\{y_{t-1},\ min\{\frac{\delta}{1+\delta},\ \delta(1-x_t)\}\}),$$

viz., accept iff

$$x_t \geq min\{\frac{\delta}{1+\delta},\ \delta(1-y_{t-1})\}.$$

Notice that, in this and many similar situations, player 2 chooses to accept when she is indifferent according to the tie-breaking rule.

- When $R[T-t-1] < \delta < 1$ and $y_{t-1} \geq K[T-t-1, x_t]$, player 2 would accept iff

$$x_t \geq \delta(1 - max\{y_{t-1},\ min\{\frac{\delta}{1+\delta},\ \delta(1-x_t)\}\}),$$

viz., accept iff

$$x_t \geq min\{\frac{\delta}{1+\delta},\ \delta(1-y_{t-1})\}.$$

As $y_{t-1} \geq K[T-t-1, x_t]$, player 2 would accept if $\frac{-1+\delta^{T-t-1}+\delta(1-x_t)}{\delta^{T-t-1}} \leq x_t$ and $y_{t-1} \leq \frac{-1+\delta^{T-t-1}+\delta^{T-t}}{\delta^{T-t-1}+\delta^{T-t}}$, or $\frac{\delta}{1+\delta} \leq x_t$ and $\frac{-1+\delta^{T-t-1}+\delta^{T-t}}{\delta^{T-t-1}+\delta^{T-t}} < y_{t-1} \leq \frac{\delta}{1+\delta}$, or $\delta(1-y_{t-1}) \leq x_t$ and $\frac{\delta}{1+\delta} < y_{t-1}$; and would reject if $x_t < \frac{\delta}{1+\delta}$ and $\frac{-1+\delta^{T-t-1}+\delta^{T-t}}{\delta^{T-t-1}+\delta^{T-t}} \leq y_{t-1} \leq \frac{\delta}{1+\delta}$, or $x_t < \delta(1-y_{t-1})$ and $\frac{\delta}{1+\delta} < y_{t-1}$.

- When $R[T-t-1] < \delta < 1$ and $y_{t-1} < K[T-t-1, x_t]$, player 2 would accept iff

$$x_t \geq \delta^2 max\{x_t,\ min\{max\{\delta^{T-t-2}(1-y_{t-1}),\ \frac{\delta}{1+\delta}\},\ \delta(1-y_{t-1})\}\}.$$

As $y_{t-1} < K[T-t-1, x_t]$, player 2 would accept if $\delta^{T-t}(1-y_{t-1}) \leq x_t < \frac{-1+\delta^{T-t-1}+\delta(1-x_t)}{\delta^{T-t-1}}$ and $y_{t-1} < \frac{-1+\delta^{T-t-1}+\delta^{T-t}}{\delta^{T-t-1}+\delta^{T-t}}$; and would rejct if $x_t < \delta^{T-t}(1-y_{t-1})$ and $y_{t-1} < \frac{-1+\delta^{T-t-1}+\delta^{T-t}}{\delta^{T-t-1}+\delta^{T-t}}$.

Therefore, when $R[T-t-1] < \delta < 1$, combining the two possible conditions on y_{t-1}, player 2 would accept if $\delta^{T-t}(1-y_{t-1}) \leq x_t$ and $y_{t-1} \leq \frac{-1+\delta^{T-t-1}+\delta^{T-t}}{\delta^{T-t-1}+\delta^{T-t}}$, or $\frac{\delta}{1+\delta} \leq x_t$ and $\frac{-1+\delta^{T-t-1}+\delta^{T-t}}{\delta^{T-t-1}+\delta^{T-t}} < y_{t-1} \leq \frac{\delta}{1+\delta}$, or $\delta(1-y_{t-1}) \leq x_t$ and $\frac{\delta}{1+\delta} < y_{t-1}$; and would reject if $x_t < \delta^{T-t}(1-y_{t-1})$ and $y_{t-1} < \frac{-1+\delta^{T-t-1}+\delta^{T-t}}{\delta^{T-t-1}+\delta^{T-t}}$, or $x_t < \frac{\delta}{1+\delta}$ and $\frac{-1+\delta^{T-t-1}+\delta^{T-t}}{\delta^{T-t-1}+\delta^{T-t}} \leq y_{t-1} \leq \frac{\delta}{1+\delta}$, or $x_t < \delta(1-y_{t-1})$ and $\frac{\delta}{1+\delta} < y_{t-1}$. To simplify things: 2 would accept iff $x_t \geq min\{max\{\delta^{T-t}(1-y_{t-1}),\ \frac{\delta}{1+\delta}\},\ \delta(1-y_{t-1})\}$.

To sum up, player 2 accepts iff $x_t \geq min\{\frac{\delta}{1+\delta},\ \delta(1-y_{t-1})\}$ when $0 < \delta \leq R[T-t-1]$; and accepts iff $x_t \geq min\{max\{\delta^{T-t}(1-y_{t-1}),\ \frac{\delta}{1+\delta}\},\ \delta(1-y_{t-1})\}$ when $R[T-t-1] < \delta < 1$, as IH 2.1 asserts.

Now notice that player 1 earns the following if his offer x_t is rejected:

$$\delta max\{y_{t-1},\ min\{\frac{\delta}{1+\delta},\ \delta(1-x_t)\}\}$$

when $R[T-t-1] < \delta < 1$ and $y_{t-1} \geq K[T-t-1, x_t]$, or $0 < \delta \leq R[T-t-1]$; and

$$\delta^2(1-max\{x_t,\ min\{max\{\delta^{T-t-2}(1-y_{t-1}),\ \frac{\delta}{1+\delta}\},\ \delta(1-y_{t-1})\}\})$$

when $R[T-t-1] < \delta < 1$ and $y_{t-1} < K[T-t-1, x_t]$.

According to player 1's payoff in stage $t+1$ and the minimal acceptable offer $x_t^{\#}$ in stage t, we explore player 1's choices at stage t in four cases:

1. $0 < \delta \leq R[T-t-1]$,
2. $R[T-t-1] < \delta < 1$ and $\frac{\delta}{1+\delta} < y_{t-1}$,
3. $R[T-t-1] < \delta < 1$ and $\frac{-1+\delta^{T-t-1}+\delta^{T-t}}{\delta^{T-t-1}+\delta^{T-t}} < y_{t-1} \leq \frac{\delta}{1+\delta}$,
4. $R[T-t-1] < \delta < 1$ and $y_{t-1} \leq \frac{-1+\delta^{T-t-1}+\delta^{T-t}}{\delta^{T-t-1}+\delta^{T-t}}$.

Notice that if his offer x_t is rejected, player 1 earns $\delta max\{y_{t-1},\ min\{\frac{\delta}{1+\delta},\ \delta(1-x_t)\}\}$ in the previous three cases, $\delta^2(1-max\{x_t,\ min\{max\{\delta^{T-t-2}(1-y_{t-1}),\ \frac{\delta}{1+\delta}\},\ \delta(1-y_{t-1})\}\})$ in the fourth case. Player 2's minimal acceptable offer $x_t^{\#}$ equals to $min\{\frac{\delta}{1+\delta},\ \delta(1-$

$y_{t-1})\}$, $\delta(1-y_{t-1})$, $\frac{\delta}{1+\delta}$ and $\delta^{T-t-2}(1-y_{t-1})$ respectively in cases 1-4. Therefore, in these four cases player 1 earns different maximal payoffs when he makes acceptable and unacceptable offers. We now explore the four cases in sequence.

- When $0 < \delta \leq R[T-t-1]$, player 1 would offer

$$x_t = x_{t-2}$$

if $x_{t-2} \geq x_t^{\#}$; and

$$\begin{cases} x_t = x_t^{\#} \\ x_t \in x_{t-2}^{+} \end{cases} \quad if \quad \begin{array}{l} 1 - x_t^{\#} \geq \delta max\{y_{t-1},\ min\{\frac{\delta}{1+\delta},\ \delta(1-x_{t-2})\}\}, \\ 1 - x_t^{\#} < \delta max\{y_{t-1},\ min\{\frac{\delta}{1+\delta},\ \delta(1-x_{t-2})\}\} \end{array}$$

if $x_{t-2} < x_t^{\#}$, where $x_t^{\#} = min\{\frac{\delta}{1+\delta},\ \delta(1-y_{t-1})\}$ and

$$x_{t-2}^{+} = \{x_t \Big| \delta max\{y_{t-1},\ min\{\frac{\delta}{1+\delta},\ \delta(1-x_t)\}\} = \delta max\{y_{t-1},\ min\{\frac{\delta}{1+\delta},\ \delta(1-x_{t-2})\}\}\}.$$

x_{t-2}^{+} denotes the optimal unacceptable offer for player 1.

When $x_{t-2} \geq x_t^{\#}$, player 2 would accept any offer $x_t \geq x_{t-2}$; so player 1 makes the smallest possible offer: $x_t = x_{t-2}$.

When $x_{t-2} < x_t^{\#}$, player 1 can either make an acceptable offer or unacceptable offer. When he makes an acceptable offer, it is natural for him to minimize that offer, i.e., $x_t = x_t^{\#}$, gaining a payoff of $1 - x_t^{\#}$. When he makes an unacceptable offer, his payoff is $\delta max\{y_{t-1},\ min\{\frac{\delta}{1+\delta},\ \delta(1-x_t)\}\}$, weakly decreasing with x_t. The best he can do is to offer $x_t = x_{t-2}$: no other would yield him a payoff larger than $\delta max\{y_{t-1},\ min\{\frac{\delta}{1+\delta},\ \delta(1-x_{t-2})\}\}$. His choice between $x_t = x_t^{\#}$ and $x_t \in x_{t-2}^{+}$ depends on the comparison between $1-x_t^{\#}$ and $\delta max\{y_{t-1},\ min\{\frac{\delta}{1+\delta},\ \delta(1-x_{t-2})\}\}$.

Similar with Equation 2.1, we can easily find that

$$1 - x_t^{\#} \geq \delta max\{y_{t-1},\ min\{\frac{\delta}{1+\delta},\ \delta(1-x_{t-2})\}\}. \tag{2.7}$$

Therefore, player 1 would offer $x_t = max\{x_{t-2},\ min\{\frac{\delta}{1+\delta},\ \delta(1-y_{t-1})\}\}$.

- When $R[T-t-1] < \delta < 1$ and $\frac{\delta}{1+\delta} < y_{t-1}$, player 1 would offer

$$x_t = x_{t-2}$$

if $x_{t-2} \geq x_t^{\#}$; and

$$\begin{cases} x_t = x_t^{\#} \\ x_t \in x_{t-2}^{+} \end{cases} \quad if \quad \begin{array}{l} 1 - x_t^{\#} \geq \delta max\{y_{t-1},\ min\{\frac{\delta}{1+\delta},\ \delta(1-x_{t-2})\}\}, \\ 1 - x_t^{\#} < \delta max\{y_{t-1},\ min\{\frac{\delta}{1+\delta},\ \delta(1-x_{t-2})\}\} \end{array}$$

if $x_{t-2} < x_t^{\#}$,

where $x_t^{\#} = \delta(1 - y_{t-1})$ and

$$x_{t-2}^{+} = \{x_t \Big| \delta max\{y_{t-1},\ min\{\frac{\delta}{1+\delta},\ \delta(1-x_t)\}\} = \delta max\{y_{t-1},\ min\{\frac{\delta}{1+\delta},\ \delta(1-x_{t-2})\}\}\}.$$

The analysis when $0 < \delta \leq R[T-t-1]$ applies here, except that the minimal acceptable offer $x_t^{\#}$ is different: now

$$1 - \delta(1 - y_{t-1}) > \delta y_{t-1}$$

and

$$1-\delta(1-y_{t-1}) > 1-(1-y_{t-1}) = y_{t-1} > \frac{\delta}{1+\delta} > \frac{\delta^2}{1+\delta} \geq \delta min\{\frac{\delta}{1+\delta},\ \delta(1-x_{t-2})\}.$$

Therefore

$$1 - x_t^{\#} \geq \delta max\{y_{t-1},\ min\{\frac{\delta}{1+\delta},\ \delta(1-x_{t-2})\}\} \tag{2.8}$$

and player 1 would offer $x_t = max\{x_{t-2},\ \delta(1-y_{t-1})\}$.

- When $R[T-t-1] < \delta < 1$ and $\frac{-1+\delta^{T-t-1}+\delta^{T-t}}{\delta^{T-t-1}+\delta^{T-t}} < y_{t-1} \leq \frac{\delta}{1+\delta}$, player 1 would offer

$$x_t = x_{t-2}$$

if $x_{t-2} \geq x_t^{\#}$; and

$$\begin{cases} x_t = x_t^{\#} \\ x_t \in x_{t-2}^{+} \end{cases} if \begin{array}{l} 1 - x_t^{\#} \geq \delta max\{y_{t-1},\ min\{\frac{\delta}{1+\delta},\ \delta(1-x_{t-2})\}\}, \\ 1 - x_t^{\#} < \delta max\{y_{t-1},\ min\{\frac{\delta}{1+\delta},\ \delta(1-x_{t-2})\}\} \end{array}$$

if $x_{t-2} < x_t^{\#}$, where $x_t^{\#} = \frac{\delta}{1+\delta}$ and

$$x_{t-2}^{+} = \{x_t \Big| \delta max\{y_{t-1},\ min\{\frac{\delta}{1+\delta},\ \delta(1-x_t)\}\} = \delta max\{y_{t-1},\ min\{\frac{\delta}{1+\delta},\ \delta(1-x_{t-2})\}\}\}.$$

The analysis when $0 < \delta \leq R[T-t-1]$ applies here, except that the minimal acceptable offer $x_t^{\#}$ is different: now

$$\frac{1}{1+\delta} > \frac{\delta^2}{1+\delta} \geq \delta y_{t-1}$$

and

$$\frac{1}{1+\delta} > \frac{\delta^2}{1+\delta} \geq \delta min\{\frac{\delta}{1+\delta},\ \delta(1-x_{t-2})\}.$$

Therefore

$$1 - x_t^{\#} \geq \delta max\{y_{t-1},\ min\{\frac{\delta}{1+\delta},\ \delta(1-x_{t-2}\}\} \tag{2.9}$$

and player 1 would offer $x_t = max\{x_{t-2},\ \frac{\delta}{1+\delta}\}$.

- When $R[T-t-1] < \delta < 1$ and $y_{t-1} < \frac{-1+\delta^{T-t-1}+\delta^{T-t}}{\delta^{T-t-1}+\delta^{T-t}}$, player 1 would offer

$$x_t = x_{t-2}$$

if $x_{t-2} \geq x_t^{\#}$; and

$$\begin{cases} x_t = x_t^{\#} \\ x_t \in x_{t-2}^{+} \end{cases} if \begin{array}{l} 1 - x_t^{\#} \geq \delta^2(1 - max\{x_{t-2},\ min\{max\{\delta^{T-t-2}(1-y_{t-1}),\ \frac{\delta}{1+\delta}\},\ \delta(1-y_{t-1})\}\}), \\ 1 - x_t^{\#} < \delta^2(1 - max\{x_{t-2},\ min\{max\{\delta^{T-t-2}(1-y_{t-1}),\ \frac{\delta}{1+\delta}\},\ \delta(1-y_{t-1})\}\}) \end{array}$$

if $x_{t-2} < x_t^{\#}$, where $x_t^{\#} = \delta^{T-t}(1-y_{t-1})$ and

$$x_{t-2}^{+} = \left\{ x_t \;\middle|\; \begin{array}{c} \delta^2(1 - max\{x_t,\ min\{max\{\delta^{T-t-2}(1-y_{t-1}),\ \frac{\delta}{1+\delta}\},\ \delta(1-y_{t-1})\}\}) \\ = \delta^2(1 - max\{x_{t-2},\ min\{max\{\delta^{T-t-2}(1-y_{t-1}),\ \frac{\delta}{1+\delta}\},\ \delta(1-y_{t-1})\}\}) \end{array} \right\}.$$

The analysis when $0 < \delta \leq R[T-t-1]$ applies here, except that the minimal acceptable offer $x_t^{\#}$ and player 1's payoff from unacceptable offer is different: now

$$\delta^{T-t}(1-y_{t-1}) < \delta^2 max\{x_{t-2},\ \delta^{T-t-2}(1-y_{t-1})\},$$

which implies that

$$-\delta^{T-t}(1-y_{t-1}) > -\delta^2 max\{x_{t-2},\ \delta^{T-t-2}(1-y_{t-1})\},$$

therefore

$$1 - \delta^{T-t}(1-y_{t-1}) > \delta^2 - \delta^{T-t}(1-y_{t-1}) > \delta^2(1 - max\{x_{t-2},\ \delta^{T-t-2}(1-y_{t-1})\}).$$

Now

$$y_{t-1} < \frac{-1 + \delta^{T-t-1} + \delta^{T-t}}{\delta^{T-t-1} + \delta^{T-t}},$$

so

$$\delta^{T-t-2}(1-y_{t-1}) > \frac{1}{\delta(1+\delta)} > \frac{\delta}{1+\delta},$$

and

$$max\{\delta^{T-t-2}(1-y_{t-1}),\ \frac{\delta}{1+\delta}\} = \delta^{T-t-2}(1-y_{t-1}).$$

Hence,

$$min\{max\{\delta^{T-t-2}(1-y_{t-1}),\ \frac{\delta}{1+\delta}\},\ \delta(1-y_{t-1})\} = \delta^{T-t-2}(1-y_{t-1}),$$

so

$$\delta^2(1 - max\{x_{t-2},\ min\{max\{\delta^{T-t-2}(1-y_{t-1}),\ \frac{\delta}{1+\delta}\},\ \delta(1-y_{t-1})\}\})$$

$$= \delta^2(1 - max\{x_{t-2},\ \delta^{T-t-2}(1-y_{t-1})\}).$$

Therefore

$$1 - x_t^{\#} \geq \delta^2(1 - max\{x_{t-2},\ min\{max\{\delta^{T-t-2}(1-y_{t-1}),\ \frac{\delta}{1+\delta}\},\ \delta(1-y_{t-1})\}\}) \tag{2.10}$$

is always valid, player 1 offers $x_t = max\{x_{t-2},\ \delta^{T-t}(1-y_{t-1})\}$.

Combining the four cases discussed above: player 1 would offer $x_t = min\{\frac{\delta}{1+\delta},\ \delta(1-y_{t-1})\}$ when $0 < \delta \leq R[T-t-1]$; and would offer $x_t = min\{max\{\delta^{T-t}(1-y_{t-1}),\ \frac{\delta}{1+\delta}\},\ \delta(1-y_{t-1})\}$ when $R[T-t-1] < \delta < 1$, as IH 2.1 asserts.

Step 2 IH 2.1 holds for stage t if it holds for stage $t+1$ when $T-t$ is even.

According to IH 2.1, we can easily conclude that payoffs in stage $t+1$ are:

$$\begin{cases} 1 - max\{x_{t-1},\ min\{\frac{\delta}{1+\delta},\ \delta(1-y_t)\}\}, \\ \quad max\{x_{t-1},\ min\{\frac{\delta}{1+\delta},\ \delta(1-y_t)\}\} \end{cases}$$

when $0 < \delta \leq R[T-t-2]$; and

$$\begin{cases} 1 - max\{x_{t-1},\ min\{max\{\delta^{T-t-1}(1-y_t),\ \frac{\delta}{1+\delta}\},\ \delta(1-y_t)\}\}, \\ \quad max\{x_{t-1},\ min\{max\{\delta^{T-t-1}(1-y_t),\ \frac{\delta}{1+\delta}\},\ \delta(1-y_t)\}\} \end{cases}$$

when $R[T-t-2] < \delta < 1$.

In stage t player 1 compares the consequences of accepting and obtaining x_t

immediately, with rejecting and obtaining

$$\delta(1 - max\{x_{t-1},\ min\{\frac{\delta}{1+\delta},\ \delta(1-y_t)\}\})$$

when $0 < \delta \leq R[T-t-2]$; and

$$\delta(1 - max\{x_{t-1},\ min\{max\{\delta^{T-t-1}(1-y_t),\ \frac{\delta}{1+\delta}\},\ \delta(1-y_t)\}\})$$

when $R[T-t-2] < \delta < 1$.

Therefore,

- When $0 < \delta \leq R[T-t-2]$, player 1 would accept iff

$$y_t \geq \delta(1 - max\{x_{t-1},\ min\{\frac{\delta}{1+\delta},\ \delta(1-y_t)\}\}),$$

viz., accept iff

$$y_t \geq min\{\frac{\delta}{1+\delta},\ \delta(1-x_{t-1})\}.$$

- When $R[T-t-2] < \delta < 1$, player 1 would accept iff

$$y_t \geq \delta(1 - max\{x_{t-1},\ min\{max\{\delta^{T-t-1}(1-y_t),\ \frac{\delta}{1+\delta}\},\ \delta(1-y_t)\}\})$$

viz., accept iff

$$y_t \geq min\{\frac{\delta}{1+\delta},\ \delta(1-x_{t-1})\}.$$

To sum up, player 1 accepts iff $y_t \geq min\{\frac{\delta}{1+\delta},\ \delta(1-x_{t-1})\}$, as IH 2.1 asserts.

Now notice that player 2 earns the following if her offer y_t is rejected:

$$\delta max\{x_{t-1},\ min\{\frac{\delta}{1+\delta},\ \delta(1-y_t)\}\}$$

when $0 < \delta \leq R[T-t-2]$; and

$$\delta max\{x_{t-1},\ min\{max\{\delta^{T-t-1}(1-y_t),\ \frac{\delta}{1+\delta}\},\ \delta(1-y_t)\}\}$$

when $R[T-t-2] < \delta < 1$.

We now explore the two cases in sequence.

- When $0 < \delta \leq R[T-t-2]$, player 2 would offer

$$y_t = y_{t-2}$$

if $y_{t-2} \geq y_t^{\#}$; and

$$\begin{cases} y_t = y_t^{\#} \\ y_t \in y_{t-2}^{+} \end{cases} if \quad \begin{array}{c} 1 - y_t^{\#} \geq \delta max\{x_{t-1},\ min\{\frac{\delta}{1+\delta},\ \delta(1-y_{t-2})\}\}, \\ 1 - y_t^{\#} < \delta max\{x_{t-1},\ min\{\frac{\delta}{1+\delta},\ \delta(1-y_{t-2})\}\} \end{array}$$

if $y_{t-2} < y_t^{\#}$, where $y_t^{\#} = min\{\frac{\delta}{1+\delta},\ \delta(1-x_{t-1})\}$ and

$$y_{t-2}^{+} = \{y_t \Big| \delta max\{x_{t-1},\ min\{\frac{\delta}{1+\delta},\ \delta(1-y_t)\}\} = \delta max\{x_{t-1},\ min\{\frac{\delta}{1+\delta},\ \delta(1-y_{t-2})\}\}\}.$$

Similar with Equation 2.1, $1-y_t^{\#} \geq \delta max\{x_{t-1},\ min\{\frac{\delta}{1+\delta},\ \delta(1-y_{t-2})\}\}$. Therefore, player 1 would offer $y_t = max\{y_{t-2},\ min\{\frac{\delta}{1+\delta},\ \delta(1-x_{t-1})\}\}$.

- When $R[T-t-2] < \delta < 1$, player 2 would offer

$$y_t = y_{t-2}$$

if $y_{t-2} \geq y_t^{\#}$;

$$\begin{cases} y_t = y_t^{\#} \\ y_t \in y_{t-2}^{+} \end{cases} if \quad \begin{array}{c} 1 - y_t^{\#} \geq \delta max\{x_{t-1},\ min\{max\{\delta^{T-t-1}(1-y_{t-2}),\ \frac{\delta}{1+\delta}\},\ \delta(1-y_{t-2})\}\}, \\ 1 - y_t^{\#} < \delta max\{x_{t-1},\ min\{max\{\delta^{T-t-1}(1-y_{t-2}),\ \frac{\delta}{1+\delta}\},\ \delta(1-y_{t-2})\}\} \end{array}$$

if $y_{t-2} < y_t^{\#}$, where $y_t^{\#} = min\{\frac{\delta}{1+\delta},\ \delta(1-x_{t-1})\}$ and

$$y_{t-2}^{+} = \left\{ y_t \left| \begin{array}{c} \delta max\{x_{t-1},\ min\{max\{\delta^{T-t-1}(1-y_t),\ \frac{\delta}{1+\delta}\},\ \delta(1-y_t)\}\} = \\ \delta max\{x_{t-1},\ min\{max\{\delta^{T-t-1}(1-y_{t-2}),\ \frac{\delta}{1+\delta}\},\ \delta(1-y_{t-2})\}\} \end{array} \right. \right\}.$$

We now explore the condition

$$1 - y_t^{\#} \geq \delta max\{x_{t-1},\ min\{max\{\delta^{T-t-1}(1-y_{t-2}),\ \frac{\delta}{1+\delta}\},\ \delta(1-y_{t-2})\}\}. \quad (2.11)$$

When $\delta^{T-t-1}(1-y_{t-2}) \leq \frac{\delta}{1+\delta}$, viz., $y_{t-2} \geq \frac{-1+\delta^{T-t-2}+\delta^{T-t-1}}{\delta^{T-t-2}+\delta^{T-t-1}}$, Equation 2.11 can be rewritten as

$$1 - min\{\frac{\delta}{1+\delta},\ \delta(1-x_{t-1})\} \geq \delta max\{x_{t-1},\ min\{\frac{\delta}{1+\delta},\ \delta(1-y_{t-2})\}\},$$

which is similar with Equation 2.1.

When $\delta^{T-t-1}(1-y_{t-2}) > \frac{\delta}{1+\delta}$, viz., $y_{t-2} < \frac{-1+\delta^{T-t-2}+\delta^{T-t-1}}{\delta^{T-t-2}+\delta^{T-t-1}}$, Equation 2.11 can be rewritten as

$$1 - min\{\frac{\delta}{1+\delta},\ \delta(1-x_{t-1})\} \geq \delta max\{x_{t-1},\ \delta^{T-t-1}(1-y_{t-2})\}. \tag{2.12}$$

Now

$$1 > \delta = \delta(1-x_{t-1}) + \delta x_{t-1},$$

so

$$1 > min\{\frac{\delta}{1+\delta}\ \delta(1-x_{t-1})\} + \delta x_{t-1},$$

and

$$1 - min\{\frac{\delta}{1+\delta},\ \delta(1-x_{t-1})\} > \delta x_{t-1}.$$

As $1 - min\{\frac{\delta}{1+\delta},\ \delta(1-x_{t-1})\} > \delta x_{t-1}$, Equation 2.12 can be rewritten as

$$1 - min\{\frac{\delta}{1+\delta},\ \delta(1-x_{t-1})\} \geq \delta^{T-t}(1-y_{t-2}),$$

viz.,

$$y_{t-2} \geq min\{\frac{-1+\delta^{T-t}+\delta^{T-t+1}}{\delta^{T-t}+\delta^{T-t+1}}, \frac{-1+\delta^{T-t}+\delta(1-x_{t-1})}{\delta^{T-t}}\}.$$

Notice that $\frac{-1+\delta^{T-t}+\delta^{T-t+1}}{\delta^{T-t}+\delta^{T-t+1}} < 0$ when $\delta \leq R[T-t]$. As $R[T-t-2] < \delta < 1$ and $y_{t-2} < y_t^{\#}$: $1 - min\{\frac{\delta}{1+\delta},\ \delta(1-x_{t-1})\} \geq \delta^{T-t}(1-y_{t-2})$ when $R[T-t-2] < \delta \leq R[T-t]$ and $y_{t-2} < y_t^{\#}$, or when $R[T-t] < \delta < 1$ and $K[T-t, x_{t-1}] \leq y_{t-2} < y_t^{\#}$, where

$$K[T-t, x_{t-1}] = min\{\frac{-1+\delta^{T-t}+\delta^{T-t+1}}{\delta^{T-t}+\delta^{T-t+1}},\ \frac{-1+\delta^{T-t}+\delta(1-x_{t-1})}{\delta^{T-t}}\}.$$

Meanwhile,

$$1 - min\{\frac{\delta}{1+\delta},\ \delta(1-x_{t-1})\} < \delta^{T-t}(1-y_{t-2})$$

when $R[T-t] < \delta < 1$ and $y_{t-2} < K[T-t, x_{t-1}]$.

Also, notice that when $y_{t-2} < K[T-t, x_{t-1}]$,

$$y_{t-2} < \frac{-1+\delta^{T-t}+\delta(1-x_{t-1})}{\delta^{T-t}} < \frac{\delta^{T-t}-\delta x_{t-1}}{\delta^{T-t}} = \frac{\delta^{T-t-1}-x_{t-1}}{\delta^{T-t-1}},$$

so $x_{t-1} < \delta^{T-t-1}(1-y_{t-2})$.

Thus, if Equation 2.11 fails and player 2 is better off offering $y_t \in y_{t-2}^{+}$, as

$$\delta^{T-t-1}(1-y_{t-2}) > \frac{\delta}{1+\delta},$$

and

$$\delta^{T-t-1}(1-y_{t-2}) < \delta(1-y_{t-2}),$$

and

$$\delta^{T-t-1}(1-y_{t-2}) > x_{t-1},$$

player 2 earns

$$\delta max\{x_{t-1},\ min\{max\{\delta^{T-t-1}(1-y_{t-2}),\ \frac{\delta}{1+\delta}\},\ \delta(1-y_{t-2})\}\} = \delta^{T-t}(1-y_{t-2}).$$

Consequently, player 2 strictly prefers offering $y_t = y_{t-2}$.

Therefore, player 2 would offer $y_t = min\{\frac{\delta}{1+\delta},\ \delta(1-x_{t-1})\}\}$when $R[T-t-2] < \delta \leq R[T-t]$ and $y_{t-2} < y_t^{\#}$, or when $R[T-t] < \delta < 1$ and $K[T-t, x_{t-1}] \leq y_{t-2} < y_t^{\#}$; and would offer $y_t = y_{t-2}$ when $R[T-t] < \delta < 1$ and $y_{t-2} < K[T-t, x_{t-1}]$.

Combining the two cases discussed above: player 1 would offer $y_t = max\{y_{t-2},\ min\{\frac{\delta}{1+\delta},\ \delta(1-x_{t-1})\}\}$when $\delta \leq R[T-t]$ or when $R[T-t] < \delta < 1$ and $K[T-t, x_{t-1}] \leq y_{t-2} < y_t^{\#}$; and would offer $y_t = y_{t-2}$ when $R[T-t] < \delta < 1$ and $y_{t-2} < K[T-t, x_{t-1}]$, as IH 2.1 asserts.

This argument works for every stage $t < T-2$, combined with the result in Appendix 1, we can see that the IH 2.1 holds for the whole game.

IH 2.1 implies Proposition 2.4: $\Gamma_F^{'}$ has a unique sub-game perfect equilibrium, and the agreement is reached in either first or second stage of the game, depending on T and δ. This ends the proof. ■

In the remainder of this section, we explore the timing of agreement and payoffs in $\Gamma_F^{'}$ in more detail. Players' choices show a threshold property: when the discount rate is small, they behave alike in the infinite horizon model, as the deadline is relatively far away (comparing with the large discount rate condition); when the discount rate is large, the deadline effect is salient as the deadline is relatively near.

For any stage of the game $t \in [1, T-2]$, there is a threshold

$$\delta_t^* = \begin{cases} R[T-t-1] & when \quad T-t \text{ is odd number}, \\ R[T-t] & \quad\quad\quad T-t \text{ is even number}. \end{cases}$$

When $\delta \leq \delta_t^*$, each player's strategy and payoff is identical to those in stage t of $\Gamma_I^{'}$; and as δ_t^* weakly decreases with t, each player's strategy and payoff is identical to those in $\Gamma_I^{'}$ in all stages $t^{'} \leq t$. In this case, we say that stages $t^{'} \leq t$ fall into an infinite horizon loop (IHL).

When $\delta > \delta_t^*$, each player's strategy and payoff differs from those in stage t of $\Gamma_I^{'}$; and as δ_t^* weakly decreases with t, each player's strategy and payoff differs from those in $\Gamma_I^{'}$ at all stages $t^{'} > t$. In such a case, we say that stages $t^{'} > t$ fall into a finite horizon loop (FHL).

Define

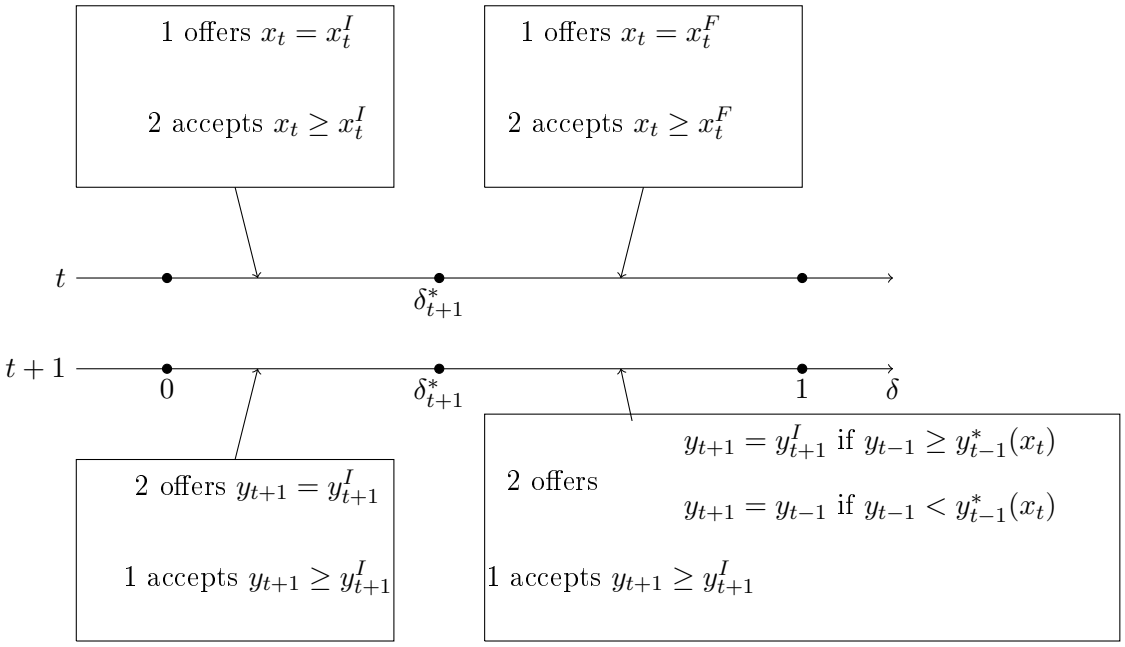
$$x_t^I = max\{x_{t-2},\ min\{\frac{\delta}{1+\delta},\ \delta(1-y_{t-1})\}\};$$

$$x_t^F = max\{x_{t-2},\ min\{max\{\delta^{T-t}(1-y_{t-1}),\ \frac{\delta}{1+\delta}\},\ \delta(1-y_{t-1})\}\};$$

$$y_t^I = max\{y_{t-2},\ min\{\frac{\delta}{1+\delta},\ \delta(1-x_{t-1})\}\}.$$

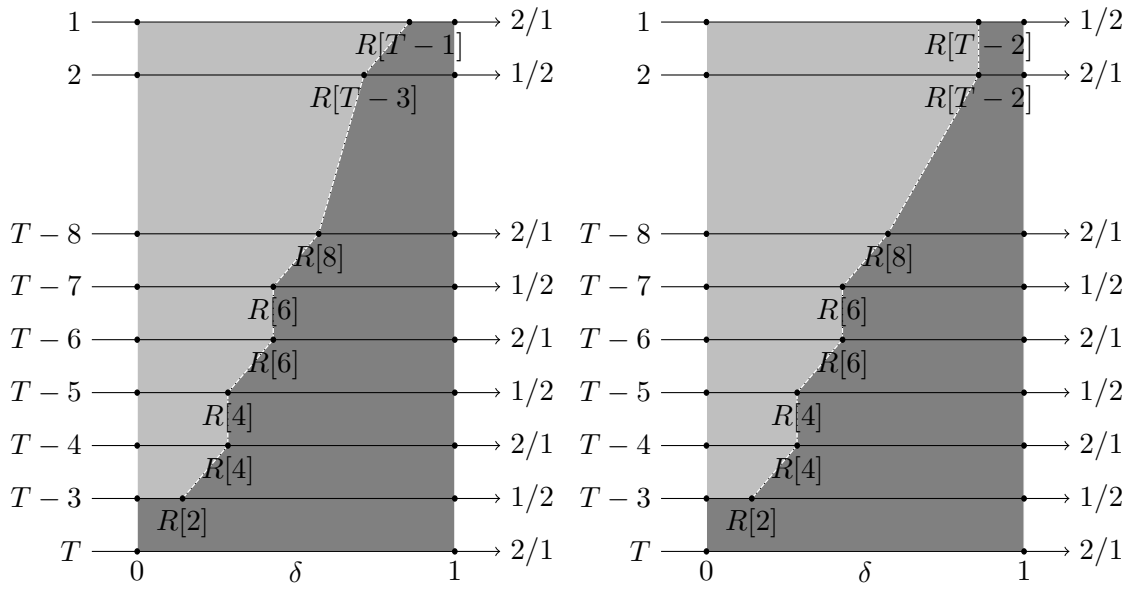
x_t^I, x_t^F and y_t^I are the acceptable offers under different conditions. Suppose, without loss of generality, that in stage t and $t+1$, players 1 and 2 are proposers respectively.

We can summarize players' choices as follows:



(The above segment indicates players' choices corresponding to different value of δ in stage t, the below segment in stage $t+1$.)

The next two figures represent the equilibrium path while T is odd and even (T is odd in the left graph and even in the right graph). $1/2$ means player 1 is the proposer, $2/1$ means player 2 is the proposer. The light gray area means that the equilibrium falls into the category of IHL and the dark gray area means the equilibrium falls into the category of FHL.



(The left graph shows players' choices when T is odd, the right graph shows players' choices when T is even.)

When $\delta \leq \delta_1^*$, the sub-game perfect equilibrium of $\Gamma_F^{'}$ falls into the category of IHL and the equilibrium outcome is identical with the unique SPE in $\Gamma_I^{'}$: the payoff pair is $\{\frac{\delta}{1+\delta},\ \frac{1}{1+\delta}\}$ when player 2 is the first proposer (T is odd), and $\{\frac{1}{1+\delta},\ \frac{\delta}{1+\delta}\}$ otherwise.

When $\delta > \delta_1^*$, the sub-game perfect equilibrium of $\Gamma_F^{'}$ falls into the category of FHL:

- If player 1 is the first proposer (T is even), then $\delta_1^* = R[T-2]$.

$\delta > \delta_1^*$ implies that $\delta^{T-2} + \delta^{T-1} > 1$, and therefore $\delta^{T-1} > \frac{\delta}{1+\delta}$.

Player 1 offers

$$x_1 = max\{x_{-1},\ min\{max\{\delta^{T-1}(1-y_0),\ \frac{\delta}{1+\delta}\},\ \delta(1-y_0)\}\}$$

$$= max\{0,\ min\{max\{\delta^{T-1},\ \frac{\delta}{1+\delta}\},\ \delta\}\}$$

$$= \delta^{T-1}$$

and player 2 would accept $x_1 = \delta^{T-1}$; so the payoff pair is

$$\{1 - \delta^{T-1},\ \delta^{T-1}\}.$$

- If player 2 is the first proposer (T is odd), then $\delta_1^* = R[T-1]$.

Player 2 offers $y_1 = y_{-1} = 0$ when $R[T-1] < \delta < 1$.

Player 1 would reject the offer and make an acceptable offer in the next stage of the game when $y_1 = 0$. $\delta > \delta_1^*$ implies that $\delta^{T-1} + \delta^T > 1$ and therefore $\delta^{T-2} > \frac{\delta}{1+\delta}$.

In the second stage of the game, player 1 offers

$$x_2 = max\{x_0,\ min\{max\{\delta^{T-2}(1-y_1),\ \frac{\delta}{1+\delta}\},\ \delta(1-y_1)\}\}$$

$$= max\{0,\ min\{max\{\delta^{T-2},\ \frac{\delta}{1+\delta}\},\ \delta\}\}$$

$$= \delta^{T-2}$$

and player 2 would accept $x_2 = \delta^{T-2}$; so the payoff pair is

$$\{1 - \delta^{T-2},\ \delta^{T-2}\},$$

with agreement in stage 2.

In conclusion, payoffs for players 1 and 2 in game $\Gamma_F^{'}$ are

	T is odd (2 is the first proposer)	T is even (1 is the first proposer)
$\delta \leq \delta_1^*$	$\{\frac{\delta}{1+\delta},\ \frac{1}{1+\delta}\}$	$\{\frac{1}{1+\delta},\ \frac{\delta}{1+\delta}\}$
$\delta > \delta_1^*$	$\{\delta - \delta^{T-1},\ \delta^{T-1}\}$	$\{1 - \delta^{T-1},\ \delta^{T-1}\}$

Notice that equilibria may be inefficient in $\Gamma_F^{'}$ when T is odd (so player 2 is the first proposer). This inefficiency originates from player 1's "last respondent" issue. Specifically, if the game reaches the last stage T, then player 1 would have very limited bargaining power as he would be willing to accept any positive offer. Player 1 is therefore keen to settle the dispute early, while player 2 is more tolerant of delay.

Thus, when player 2 is the first (and also the last) proposer, she might be willing to delay in stage one and wait to evaluate player 1's offer in stage two. We discuss the first and last proposer's advantage in the next section.

2.5 Comparative statics analysis

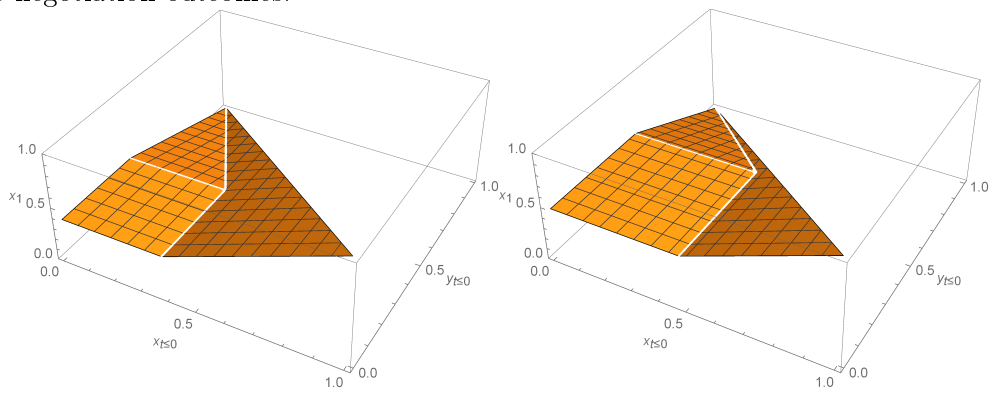
2.5.1 Infinite horizon case

Recall Section 2.3, the introduction of Endogenous Commitment does not affect the equilibrium outcome in infinite horizon alternating offer bargaining games when there is no pre-commitment: $x^- = y^- = 0$. Actually, it does not affect the equilibrium outcome as long as $x^- \leq \frac{\delta}{1+\delta}$ and $y^- \leq \frac{\delta}{1+\delta}$.

However, when $x^- > \frac{\delta}{1+\delta}$ or $y^- > \frac{\delta}{1+\delta}$, things would be slightly different.

The figures below represent the changes. The x axis (pointing southeast while starting from $(0, 0)$, the same below) represents the value of x^-, the y axis (pointing northeast) represents the value of y^-, and the z axis (pointing north, labeled x_1) represents player 1's equilibrium share. Player 1's share weakly increases with x^- and weakly decreases with y^-; player 2's share weakly decreases with x^- and weakly increases with y^-: the initial condition of the negotiation may affect the equilibrium outcome.

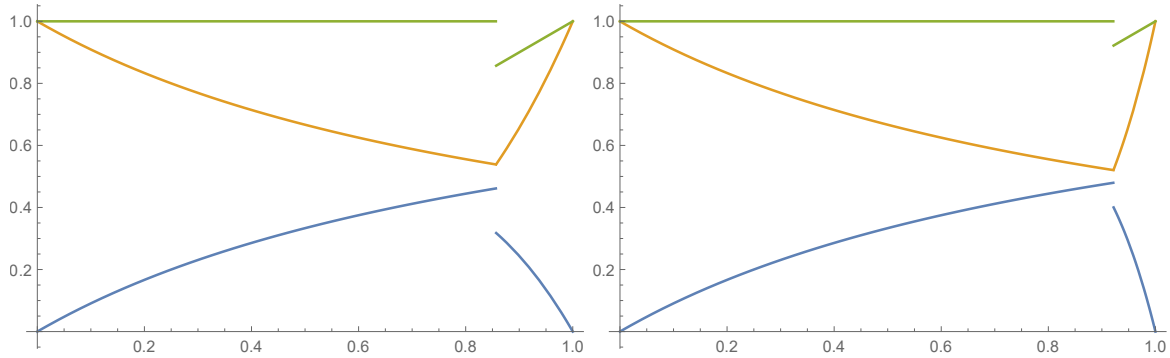
The left and right figure each has $\delta = 0.6$ and $\delta = 0.9$: the smaller δ is, the smaller the square part is, namely, the more likely the pre-negotiation offers affect the negotiation outcomes.



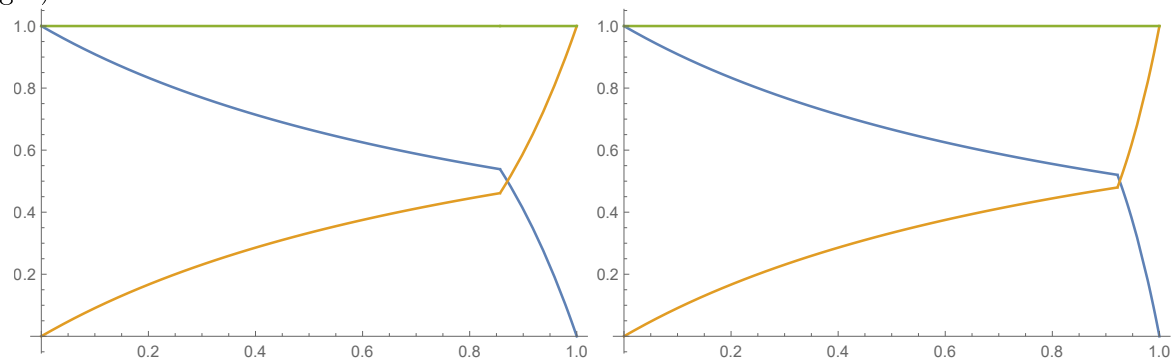
2.5.2 Finite horizon case

Recall Section 2.4, in $\Gamma_F^{'}$, payoffs depend on T and δ.

The two diagrams below represent the outcomes when $T = 5$ (left) and $T = 9$ (right):



The next two diagrams represent the outcomes when $T = 6$ (left) and $T = 10$ (right):



(In each figure, the x axis represents the value of δ; the y axis represents the value of the payoffs; the blue curve represents player 1's payoff; the orange curve represents player 2's payoff; and the green curve represents the aggregate payoff.)

We can draw several conclusions from these figures:

1. When T is odd, players' payoff curves reverse in direction after crossing the threshold δ: player 1 benefits from extreme (either large or small) δ and player 2 suffers from extreme δ. The aggregate payoff is independent of δ below the threshold, plummeting at the threshold, and then increasing with δ above the threshold.

   When T is even, players' payoff curves are monotonic: player 1 benefits from a small δ and player 2 benefits from a large δ; and aggregate payoffs do not

depend on δ.

2. When T is odd, player 1's payoff increases with T, player 2's payoff decreases with T, and aggregate payoff increases with T.

   When T is even, player 1's payoff increases with T, player 2's payoff decreases with T, and the aggregate payoff does not depend on T.

The diagrams clearly illustrate first and last mover advantages.

The first and last proposer are identical (player 2) when T is odd, and she always earns more than player 1. When $\delta < \delta^{\#}$, the first mover advantage is obvious: $\frac{\frac{1}{1+\delta}}{\frac{\delta}{1+\delta}} = \frac{1}{\delta}$, which is decreasing in δ. Beyond the threshold $\delta^{\#}$, the last mover advantage kicks in: the larger δ is, and the smaller T is, the more conspicuous the advantage is: $\frac{\delta^{T-1}}{\delta-\delta^{T-1}} = \frac{\delta^{T-2}}{1-\delta^{T-2}}$, which is increasing in δ and decreasing in T.

The first proposer is player 1 and last proposer is player 2 when T is even: player 1 now has a first mover advantage while player 2 has a last mover advantage. The first mover advantage is obviously stronger when δ is small, the last mover advantage is greater when δ is large and T is small.

2.6 Conclusion

The elegance of traditional bargaining models and their poor predictive power (cf. Ochs and Roth 1989[49]) have been bothering many economists. This chapter may help to reconcile them by offering another explanation to the inefficiency in negotiations. Endogenous Commitment is a natural phenomenon, but not yet a widely used assumption in bargaining models. With the introduction of Endogenous Commitment into classical bargaining models, we create two non-stationary, history-dependent alternating offer bargaining models with interesting properties. This chapter is far from the end of research on Endogenous Commitment: the models in this chapter merely serve as a benchmark for further exploration. In the next chapter, we will introduce random proposers (or, arbitration), obtaining more interesting results, which may explain U-shaped settlement patterns.

2.7 Appendix

2.7.1 Appendix 1 (IH 2.1 for last 3 stages of the game)

- If no agreement has been reached by stage T

Player 1 would obviously accept any offer $y_T \geq 0$ and player 2 would offer $y_T = y_{T-2}$; so the payoffs would be $\{y_{T-2},\ 1 - y_{T-2}\}$.

- If no agreement has been reached by stage $T-1$

Player 2 compares the payoff from rejecting: $\delta(1 - y_{T-2})$ with x_{T-1} and she chooses to accept iff $x_{T-1} \geq \delta(1 - y_{T-2})$. Therefore, player 1 would offer

$$x_{T-1} = x_{T-3}$$

if $x_{T-3} \geq \delta(1 - y_{T-2})$, and

$$\begin{cases} x_{T-1} = \delta(1 - y_{T-2}) \\ x_{T-1} = x_{T-3} \end{cases} \quad if \quad \begin{array}{l} 1 - \delta(1 - y_{T-2}) \geq \delta y_{T-2}, \\ 1 - \delta(1 - y_{T-2}) < \delta y_{T-2} \end{array}$$

if $x_{T-3} < \delta(1 - y_{T-2})$.

Notice that $1 - \delta(1 - y_{T-2}) \geq \delta y_{T-2}$, player 1 offers $x_{T-1} = max\{x_{T-3},\ \delta(1 - y_{T-2})\}$, which player 2 accepts.

The payoffs would be $\{1 - max\{x_{T-3},\ \delta(1 - y_{T-2})\}\ max\{x_{T-3},\ \delta(1 - y_{T-2})\}\}$.

- If no agreement has been reached by stage $T-2$

Player 1 compares the payoff from rejecting: $\delta(1 - max\{x_{T-3},\ \delta(1 - y_{T-2})\})$ with y_{T-2} and he chooses to accept iff

$$y_{T-2} \geq \delta(1 - max\{x_{T-3},\ \delta(1 - y_{T-2})\}),$$

viz., accept iff

$$y_{T-2} \geq min\{\frac{\delta}{1+\delta},\ \delta(1 - x_{T-3})\}.$$

Therefore, player 2 would offer

$$y_{T-2} = y_{T-4}$$

if $y_{T-4} \geq y^{\#}_{T-2}$, and

$$\begin{cases} y_{T-2} = y^{\#}_{T-2} \\ y_{T-2} = y_{T-4} \end{cases} \quad if \quad \begin{array}{l} 1 - y^{\#}_{T-2} \geq \delta max\{x_{T-3},\ \delta(1 - y_{T-4})\}, \\ 1 - y^{\#}_{T-2} < \delta max\{x_{T-3},\ \delta(1 - y_{T-4})\} \end{array}$$

if $y_{T-4} < y^{\#}_{T-2}$, where $y^{\#}_{T-2} = min\{\frac{\delta}{1+\delta},\ \delta(1 - x_{T-3})\}$.

Notice that

$$1 - min\{\frac{\delta}{1+\delta},\ \delta(1 - x_{T-3})\} \geq \delta max\{x_{T-3},\ \delta(1 - y_{T-4})\}$$

is valid when $0 < \delta \leq R[2]$ and $y_{T-4} < min\{\frac{\delta}{1+\delta},\ \delta(1 - x_{T-3})\}$, or $R[2] < \delta < 1$ and $K(2,\ x_{T-3}) < y_{t-1} < min\{\frac{\delta}{1+\delta},\ \delta(1 - x_{T-3})\}$, where

$$K(2,\ x_{T-3}) = min\{\frac{-1 + \delta^2 + \delta^3}{\delta^2 + \delta^3},\ \frac{-1 + \delta^2 + \delta(1 - x_{T-3})}{\delta^2}\};$$

and

$$1 - min\{\frac{\delta}{1+\delta},\ \delta(1 - x_{T-3})\} < \delta max\{x_{T-3},\ \delta(1 - y_{T-4})\}$$

is valid when $R[2] < \delta < 1$ and $y_{T-4} < K(2,\ x_{T-3})$.

Therefore player 2 offers

$$y_{T-2} = max\{y_{T-4},\ min\{\frac{\delta}{1+\delta},\ \delta(1 - x_{T-3})\}\}$$

when $0 < \delta \leq R[2]$, or $R[2] < \delta < 1$ and $K(2,\ x_{T-3}) < y_{t-1}$; and offers

$$y_{T-2} = y_{T-4}$$

when $R[2] < \delta < 1$ and $y_{T-4} < K(2,\ x_{T-3})$.

The payoffs would be

$$\begin{cases} max\{y_{T-4},\ min\{\frac{\delta}{1+\delta},\ \delta(1-x_{T-3})\}\}, \\ 1-max\{y_{T-4},\ min\{\frac{\delta}{1+\delta},\ \delta(1-x_{T-3})\}\} \end{cases}$$

when $0 < \delta \leq R[2]$, or $R[2] < \delta < 1$ and $K(2,\ x_{T-3}) < y_{t-1}$; would be

$$\delta \left\{ \begin{array}{c} 1-max\{x_{T-3},\ \delta(1-y_{T-4})\}, \\ max\{x_{T-3},\ \delta(1-y_{T-4})\} \end{array} \right\}$$

when $R[2] < \delta < 1$ and $y_{T-4} < K(2,\ x_{T-3})$.

2.7.2 Appendix 2 (The generic case without a tie-breaking rule)

In section 2.4, the unique SPE in Proposition 2.4 relies on a tie-breaking rule, without which, things are slightly different.

Here we show that when player 2 is the proposer, both players have a unique best response function; when player 1 is the proposer, player 2 responds uniquely, but player 1 has several indifferent choices. This does not affect the equilibrium outcome, except when δ has a specific value. Therefore, in generic cases, there is a unique SPE outcome, but there are multiple equilibria; in one special case, there are multiple SPE outcomes.

We now examine the choices made by the players when indifferent in more detail.

First let's look at the case when $T-t$ is odd: player 1 is the proposer and player 2 is the respondent.

Player 1 always makes an acceptable offer and is not indifferent; so we examine player 2's choice when indifferent.

Suppose that player 2 rejects the offer with a positive probability p when indifferent. As Equation 2.7, 2.8, 2.9 and 2.10 suggest, player 1 strictly prefers to make an acceptable offer. Therefore, player 1 would have an incentive to deviate and increase $x_t^{\#}$ by $\epsilon > 0$ and make sure that player 2 accepts the offer $x_t^{\#} + \epsilon$. In this case, there is no equilibrium as player 1 would always have an incentive to reduce ϵ. Therefore, there is a unique SPE with $p = 0$: player 2 accepts with probability 1

when indifferent.

We now look at the case when $T-t$ is even: player 2 is the proposer and player 1 is the respondent.

We first examine player 1's choice when indifferent. Player 1 faces this choice when indifferent in three possible circumstances: player 2 strictly prefers $y_t = y_t^{\#}$ than $y_t = y_{t-2}$; player 2 strictly prefers $y_t = y_{t-2}$ than $y_t = y_t^{\#}$; player 2 is indifferent between offering $y_t = y_t^{\#}$ and $y_t = y_{t-2}$.

In the first circumstance, things are similar to the case when $T-t$ is odd, player 1 rejects the offer with probability 0 when he is indifferent.

In the second circumstance, player 1's choice does not affect the equilibrium outcome.

In the third circumstance,, in which

$$1 - y_t^{\#} = \delta max\{x_{t-1},\ min\{max\{\delta^{T-t-1}(1-y_{t-2}),\ \frac{\delta}{1+\delta}\},\ \delta(1-y_{t-2})\}\},$$

viz., $y_{t-2} = K[T-t, x_{t-1}]$, we have following proposition showing that player 1 accepts for sure.

Proposition 2.5. *Player 1 would reject the offer with probability* 0 *when he is indifferent, given that player 2 is indifferent between making an acceptable or unacceptable offer.*

Proof.

If player 1 chooses to reject the offer with a positive probability p when he is indifferent then player 2 earns

$$\delta max\{x_{t-1},\ min\{max\{\delta^{T-t-1}(1-y_{t-2}),\ \frac{\delta}{1+\delta}\},\ \delta(1-y_{t-2})\}\}$$

when she offers $y_t = y_{t-2}$, and

$$\delta max\{x_{t-1},\ min\{max\{\delta^{T-t-1}(1-y_t^{\#}),\ \frac{\delta}{1+\delta}\},\ \delta(1-y_{t-2})\}\}(1-p)$$

$$+\delta max\{x_{t-1},\ min\{max\{\delta^{T-t-1}(1-y_{t-2}),\ \frac{\delta}{1+\delta}\},\ \delta(1-y_{t-2})\}\}p$$

when she offers $y_t = y_t^{\#}$.

Recall from Section 2.4 that when $\delta^{T-t-1}(1-y_{t-2}) \leq \frac{\delta}{1+\delta}$,

$$1-y_t^{\#} > \delta max\{x_{t-1},\ min\{max\{\delta^{T-t-1}(1-y_{t-2}),\ \frac{\delta}{1+\delta}\},\ \delta(1-y_{t-2})\}\}.$$

Therefore, when

$$1-y_t^{\#} = \delta max\{x_{t-1},\ min\{max\{\delta^{T-t-1}(1-y_{t-2}),\ \frac{\delta}{1+\delta}\},\ \delta(1-y_{t-2})\}\},$$

$\delta^{T-t-1}(1-y_{t-2}) > \frac{\delta}{1+\delta}$. Hence, $1-y_t^{\#} = \delta max\{x_{t-1},\ \delta^{T-t-1}(1-y_{t-2})\}$. Furthermore, $1-y_t^{\#} > \delta x_{t-1}$ for sure, so $\delta^{T-t-1}(1-y_{t-2}) > x_{t-1}$.

Therefore, when player 1 is indifferent,

$$\delta max\{x_{t-1},\ min\{max\{\delta^{T-t-1}(1-y_{t-2}),\ \frac{\delta}{1+\delta}\},\ \delta(1-y_{t-2})\}\} = \delta^{T-t-1}(1-y_{t-2}).$$

Because $y_{t-2} < y_t^{\#}$, and $\delta max\{x_{t-1},\ min\{max\{\delta^{T-t-1}(1-y_t),\ \frac{\delta}{1+\delta}\},\ \delta(1-y_t)\}\}$ weakly decreases with y_t, we can deduce that

$$\delta max\{x_{t-1},\ min\{max\{\delta^{T-t-1}(1-y_{t-2}),\ \frac{\delta}{1+\delta}\},\ \delta(1-y_{t-2})\}\}$$

$$< \delta max\{x_{t-1},\ min\{max\{\delta^{T-t-1}(1-y_t^{\#}),\ \frac{\delta}{1+\delta}\},\ \delta(1-y_{t-2})\}\}.$$

Therefore, player 2 prefers to offer $y_t = y_{t-2}$ rather than $y_t = y_t^{\#}$. Consequently, player 2 has an incentive to offer $y_t = y_{t-2}$ when player 1 rejects with positive probability p when indifferent.

Hence, player 1 earns

$$\delta(1-max\{x_{t-1},\ min\{max\{\delta^{T-t-1}(1-y_{t-2}),\ \frac{\delta}{1+\delta}\},\ \delta(1-y_{t-2})\}\})$$

by choosing to reject with positive probability p when indifferent, while he earns $y_t^{\#}$

by choosing to reject with with probability 0 when indifferent.

Notice that

$$1 - y_t^{\#} = \delta max\{x_{t-1},\ min\{max\{\delta^{T-t-1}(1 - y_{t-2}),\ \frac{\delta}{1+\delta}\},\ \delta(1 - y_{t-2})\}\},$$

so

$$y_t^{\#} = 1 - \delta max\{x_{t-1},\ min\{max\{\delta^{T-t-1}(1 - y_{t-2}),\ \frac{\delta}{1+\delta}\},\ \delta(1 - y_{t-2})\}\}$$

$$> \delta(1 - max\{x_{t-1},\ min\{max\{\delta^{T-t-1}(1 - y_{t-2}),\ \frac{\delta}{1+\delta}\},\ \delta(1 - y_{t-2})\}\}).$$

Therefore, player 1 would always accept with probability 1 when indifferent. ■

We now examine player 2's choice when indifferent. In stage t ($T - t$ is even), it makes no difference to player 2 whether she makes the acceptable or the unacceptable offer when indifferent because in each case she earns the same payoff of $1 - y_t^{\#}$. In stage $t-1$, as player 2 always earns the same payoff in stage t, player 1 offers $x_{t-1} = \delta(1 - y_t^{\#})$ and player 2 accepts it. Therefore, player 2 has no incentive to deviate from any choice she makes in state $(y_{t-2},\ x_{t-1})$ that satisfies $y_{t-2} = K[T - t, x_{t-1}]$: she could offer $y_t = y_t^{\#}$, $y_t = y_{t-2}$, or mix between them. This generates multiple equilibria in the model, but all equilibria have the same outcome.

In generic cases ($\delta \neq \delta_1^{\#}$), the game has a unique SPE outcome because for any stage $t > 1$, player 2's choices when indifferent in stage t do not affect the payoff in the previous stage $t - 1$. The only exception is when $\delta = \delta_1^{\#}$ and player 2 is the first proposer (player 2 is otherwise never indifferent): in this case player 2 would be indifferent between making an acceptable offer or not in the first stage of the game, she could offer $y_1 = y_1^{\#} = \frac{\delta}{1+\delta}$, $y_1 = 0$, or mix between them. Player 1 earns different payoffs correspondingly; so player 2's choices when indifferent would generate different SPE outcomes.

Chapter 3

Bargaining Models with Endogenous Commitment II

Abstract

While alternating offer bargaining models with Endogenous Commitment are studied in our second chapter, we now explore an extension: what would happen if instead of alternating offers, we include uncertainty in the bargaining protocol. We show that when the bargaining protocol involves randomization, the settlement pattern is U-shaped: players either reach an agreement in the initial stage of the game when they are less patient, or reach an agreement in the final stage of the game when they are more patient. No intermediate agreements are possible.

3.1 Introduction

Delayed agreements are a widespread phenomenon in real-life negotiations. A new wage agreement is frequently the outcome of a strike and loss of production; a peace settlement is often reached at the expense of a war. While a mutually more favored agreement almost always exists, in certain cases a delayed and inefficient agreement is simply inevitable due to various reasons. While asymmetric information is what most research focuses on, we may nevertheless also explain delay in the absence of asymmetric information.

Many negotiations require agreement by some deadline, which is either given exogenously (e.g. settlement negotiations must end when the court case starts) or chosen by the negotiators themselves (e.g. the sunset clauses which Congress has written into legislation). Negotiations with a deadline often go to the wire, as illustrated by the recurrent difficulties in renewing lapsed legislation in the US. Settlement bargaining suggests an even starker structure: negotiators either agree at the outset or reach agreement on the courtroom steps, despite the continuing cost of legal representation. This phenomenon is known as a "U-shaped settlement pattern" in Spier 1992[57]. Similar evidences are seen in other areas other than in settlement bargaining. Take financial budget setting for example. In most cases negotiation would not end until very close to the deadline, or even sometimes surpass it: from 1988 to 2007, among 48 continental states in US, 23% of all budgets were approved after the fiscal year deadline: cf. Anderson et al. 2012[2]. Not only field data, but experimental evidence also suggests that such patterns are not a special feature of settlement bargaining. Instead, the deadline effect is salient in various bargaining situations (Gneezy et al. 2003[27], Guth et al. 2005[28], and Roth et al. 1988[51]).

U-shaped settlement patterns present a challenge to economic theory because conventional bargaining models predict that impatient negotiators would agree immediately, irrespective of whether there is a deadline or not. The best illustration is given through the model of alternating offer bargaining games in Stahl 1972[58] and Rubinstein 1982[52]. Bargaining models which incorporate incomplete information about a rival have explained why negotiation may last inefficiently long (e.g. Hart, 1989[29]); but they seem ill suited to explaining why delay takes the form of U-shaped settlement patterns. Fuchs and Skrzypacz 2013[24] is a recent related contribution. In their paper, both the buyer and the seller have private information, and the seller has a limited commitment power: to keep the current offer for a limited number of periods. Other papers attribute delay to multilateral negotiation: allocating negotiators with different roles as insiders and outsiders as in Shaked and Sutton, 1984[56], or assuming a group of "passive" players negotiating with an "active" player in sequence and the implementing of project requires unanimous agreements as in Cai 2000[6], 2003[7]. Unlike the above mentioned papers, we intend to remain in the

two player complete information structure while explaining the delay phenomenon and the U-shaped settlement pattern. The main assumptions and models we use in this chapter are closely related to those in Chapter 2.

The main assumption we use to explain U-shaped settlement pattern is still Endogenous Commitment, which was introduced to the literature by Fershtman and Seidmann 1993[23]. In a randomizing offer finite horizon bargaining model, they assumed that each negotiator is committed to reject any offer that is worse than the offer he has previously rejected, and showed that the negotiation lasts till the deadline whenever negotiators are patient enough. On that level, we intend to construct similar models to explain U-shaped settlement patterns: negotiators agree immediately if they are impatient enough, and go to the wire otherwise.

In this chapter, we mainly use an (almost) unilateral offer model to capture the phenomenon of U-shaped settlement pattern. In Section 3.2, we introduce the basic setting of the model and justify the assumptions. In Section 3.3, we present our main model and in Section 3.5 we apply it to a pre-trial setting. In Section 3.4, we present a three stage randomizing offer model as an extension. Section 3.6 concludes.

3.2 Framework

3.2.1 Basic structure and notation

In this chapter the framework is similar to the one in Chapter 2: two players $i = 1, 2$ bargain over division of a fixed-size pie, one player's gain is equivalent to the other player's lose as long as the agreement time is fixed. Time is valuable to players and a common discount factor is denoted as $\delta < 1$;[1] so both of them desire to reach an agreement as early as possible. The model has T stages and the agreement might be reached in any one of them: $t \in [1, ..., T]$. Beyond stage T, in other words, the deadline, each player earns zero payoffs.

In our main model (Sections 3.3 and 3.5), we assume that player 1 is the proposer

[1]The common discount factor is not necessary; similar conclusions remain when players have different discount factors.

in most periods. Specifically: in any stage $t \in [1, ..., T-1]$, player 1 is the proposer and makes an offer $x_t \in [0, 1]$ which player 2 can accept or reject. If the offer is accepted then the game is over and the agreement is $\{1-x_t, x_t\}$; players' payoffs are then $\delta^t\{1-x_t, x_t\}$. In the last stage of the game, we assume that both players have an equal probability to be the proposer. If player 1 is still the proposer, it is similar to the previous stages (except that now $t = T$). If player 2 is the proposer then she offers $y_T \in [0, 1]$ which player 1 could accept or reject: if the offer is accepted then the agreement is $\{y_T, 1-y_T\}$ and players' payoffs are $\delta^T\{y_T, 1-y_T\}$; if the offer is rejected then negotiation fails and both player earn a zero payoff.

Through constructing the model above, we are mainly explaining the following intuition: when the bargaining game has an endogenous disagreement payoff instead of an exogenous one, making a more generous offer to the respondent is more costly for the proposer; so it might not be worthwhile for the proposer to make an acceptable offer.

3.2.2 Main assumptions

Endogenous Commitment

Like in Chapter 2, Endogenous Commitment is the core assumption here. A less desirable offer from proposer to respondent is not possible: in other words $x_t \geq x_{t-1}$ for any t (as player 2 makes only one offer in this game Endogenous Commitment applies only to player 1). There are two straightforward ways to interpret this assumption: 1. The respondent is committed to reject any subsequent offer containing a smaller share of the pie; 2. The proposer is committed not to make less generous offer. These two explanations are formally equivalent. A detailed discussion of EC has been given in Chapter 1.

Endogenous disagreement payoff

In Chapter two, we saw that when we introduce Endogenous Commitment assumption into traditional alternating bargaining games, players' strategies become history dependent and the bargaining outcome may be different. In this chapter, we further

consider the situation in which players' disagreement payoffs are also history dependent. This assumption leads to the U-shaped settlement pattern in the model. We now explain why it is a robust assumption and how we apply this assumption in our model.

In traditional bargaining games, players obtain a fixed-value disagreement payoff (usually zero) when negotiation fails. In other words, their negotiating history no longer matters once the negotiation reaches an impasse. However, this might not always be the case. Even if the negotiation comes to a dead end, the negotiation process still provides a guidance for (possible) negotiations in future or for interim measures. Therefore, in this chapter we study endogenous disagreement payoff.

By constructing a T stage, (almost) unilateral offer model with exogenous disagreement payoff, we get a $T-1$ stage model with endogenous disagreement payoff. with in the final stage of the game, each player has an equal final chance to make an offer to his or her opponent. In the T stage model, due to the non-positive, fixed disagreement payoff, the respondent in the final stage will accept any non-negative offer. Therefore, we can easily deduce that in the sub-game equilibrium, in stage T, player 1 would offer $x_T = x_{T-1}$ (by Endogenous Commitment) and player 2 would accept; and player 2 would offer $y_T = 0$ and player 1 would accept[2]. With an equal probability of reaching an agreement of $\{1-x_{T-1},\ x_{T-1}\}$ or $\{0,\ 1\}$, players reach an (expected) agreement of $\{\frac{1-x_{T-1}}{2}, \frac{1+x_{T-1}}{2}\}$. Hence, the original T stage game with an exogenous disagreement payoff of $\{0,\ 0\}$ is equivalent to a $T-1$ stage game with an endogenous disagreement share of $\{\frac{1-x_{T-1}}{2}, \frac{1+x_{T-1}}{2}\}$.

We can think of the final stage T as an arbitration stage: if no agreement is reached in earlier stages, to avoid the failure of negotiation, an arbitrator could be introduced at the last minute to enforce a distribution between the two players. The arbitrator makes his decision referring to the negotiating history: the more someone was offered by his rival, the more he is allocated during the arbitration stage. The role of arbitrator could be played by a third party, or could be an arrangement by negotiators themselves before the negotiation starts. In section 3.5 we introduce a

[2]Player 1 is indifferent between accepting or not, player 2 would increase her offer slightly when player 1 rejects with positive probability; so player 1 accepts for sure in equilibrium. Equivalent arguments apply when player 2 is the respondent in previous stages.

pre-trial negotiation model in which an arbitrator is involved. In this chapter we introduce one function to map the set of terminal histories of the game to a pair of endogenous disagreement payoffs, in future, it would be interesting to explore other possibilities.

3.3 Unilateral offer model

In this section, we introduce the (almost) unilateral offer model.

3.3.1 No Endogenous Commitment

For comparison, we first look at this model without Endogenous Commitment.

We can easily solve the subgame-perfect equilibrium of this game using backward induction. In equilibrium, player 1 offers $x_t = \frac{\delta^{T-t}}{2}$ and player 2 accepts iff $x_t \geq \frac{\delta^{T-t}}{2}$ when $t \leq T-1$; player 1 offers $x_T = 0$ and accepts iff $y_T \geq 0$ while player 2 offers $y_T = 0$ and accepts iff $x_T \geq 0$ when $t = T$.

In the arbitration stage, both players take the advantage of being proposer and make the minimal offer they can make. In equilibrium, the agreement is reached in the first stage of the game and the agreement is $\{1 - \frac{\delta^{T-t}}{2}, \frac{\delta^{T-t}}{2}\}$.

3.3.2 With Endogenous Commitment

We now look at the model with Endogenous Commitment. We first exhibit the settlement pattern of this game via the following proposition.

Proposition 3.1. *There exists a threshold $\delta^{\#}$ such that players reach an agreement immediately when $\delta < \delta^{\#}$, and reach an agreement in the last stage when $\delta > \delta^{\#}$.*

Proposition 3.1 exhibits the U-shaped settlement pattern: early or late, but no intermediate.

Player 1, as the proposer in most of the game, is the "active" one in the negotiation. He could choose to make an acceptable offer to player 2 anytime early, or wait until the arbitration stage, in which he may be able to compel player 2 to accept his

final offer or may have to accept the only counter-offer. An early agreement is mutually favored due to time cost, but player 1 has to harm his own interest to induce player 2's acceptance with a generous enough offer; delay is costly for each of them and especially risky for player 1 because he may lose his proposer advantage. Player 1 faces a trade-off in every sub-game and in equilibrium, he makes his decision in the very beginning and sticks to it throughout the whole game. And his decision is polarized: either he induces an early agreement, or he waits until the end.

Player 2, as the respondent in most of the game, is the passive one in the negotiation. However, Endogenous Commitment gives her more bargaining power when confronting player 1.

Proof.

We prove Proposition 3.1 by induction.

Inductive Hypothesis 3.1. In stage $t \leq T-1$, player 2 accepts if $x_t \geq x_t^{\#}$ and rejects if $x_t < x_t^{\#}$. Player 1 offers $x_t = max\{x_{t-1},\ x_t^{\#}\}$ if $x_{t-1} \geq x_{t-1}^{*}$ and offers $x_t = x_{t-1}$ if $x_{t-1} < x_{t-1}^{*}$, where $x_t^{\#} = \frac{\delta^{T-t}}{2-\delta}$ and $x_{t-1}^{*} = 1 - \frac{2(1-x_t^{\#})}{\delta^{T-t}} = \frac{4-\delta}{2-\delta} - \frac{2}{\delta^{T-t}}$.

$x_t^{\#}$ is the minimal acceptable offer for player 2 in stage t; x_{t-1}^{*} is the triggering offer indicating that in stage t, player 1 makes an acceptable offer if and only if $x_{t-1} \geq x_{t-1}^{*}$.

IH 3.1 shows that both player 1 and player 2 have threshold strategies in this game. As anticipated, the respondent rejects an offer when it is low and accepts an offer when it is high. Also, it is no surprise that the proposer makes an acceptable offer if and only if the previous offer is large enough. From one perspective, a higher x_{t-1} restricts the options player 1 has in stage t because he is only allowed to make offers larger than x_{t-1}; from the other perspective, due to the endogenous disagreement payoff $\delta^{T-1}\{\frac{1-x_{T-1}}{2}, \frac{1+x_{T-1}}{2}\}$, a higher x_{t-1} implies a lower payoff for player 1 if the game is delayed to the arbitration stage.

We first show that IH 3.1 forms a unique sub-game perfect equilibrium in stage $t = T-1$.

In stage $T-1$, player 2 chooses between accepting x_{T-1} to earn a payoff of x_{T-1}, or reject it to earn a payoff of $\delta\frac{1+x_{T-1}}{2}$.[3] Comparing these two payoffs, we can easily see that there is a threshold offer $x^{\#}_{T-1} = \frac{\delta}{2-\delta}$ for player 2: she accepts any offer above or equal to it and reject any offer below it. We denote $x^{\#}_{T-1}$ as the minimal acceptable offer. Anticipating this, player 1 chooses between making an offer that player 2 will accept or not. If player 1 makes an acceptable offer, he is obviously better to make the minimal acceptable offer if possible: $x_{T-1} = max\{x_{T-2},\ x^{\#}_{T-1}\}$; if he makes an unacceptable offer, he is better to offer $x_{T-1} = x_{T-2}$ as his payoff in the next stage $\frac{1-x_{T-1}}{2}$ is strictly decreasing in x_{T-1}. Therefore, when $x^{\#}_{T-1} \geq x_{T-2}$, player 1 offers $x_{T-1} = x^{\#}_{T-1}$ if $1 - x^{\#}_{T-1} \geq \delta\frac{1-x_{T-2}}{2}$, viz., $x_{T-2} \geq 1 - 2\frac{1-x^{\#}_{T-1}}{\delta}$ and offers $x_{T-1} = x_{T-2}$ if $x_{T-2} < 1 - 2\frac{1-x^{\#}_{T-1}}{\delta}$; when $x^{\#}_{T-1} < x_{T-2}$, player 1 offers $x_{T-1} = x_{T-2}$. Both player 1 and player 2's choices satisfy the description in IH 3.1.

We now show that IH 3.1 forms a unique sub-game perfect equilibrium in stage $t-1$ if it does so in stages $t' \geq t$.

To do this, we first prove the following Lemma.

Lemma 3.1. *When IH 3.1 holds in stage $t' \geq t$, player 1 does not increase his offer in stage $t' > t$ if he does not increase his offer in stage t .*

Proof.

If $x_{t-1} < x^*_{t-1}$ in stage t, we can show that $x_{t'-1} < x^*_{t'-1}$ in all stages $t' > t$.

By IH 3.1, in stage t, if $x_{t-1} < x^*_{t-1}$, viz., $1 - x^{\#}_t < \delta^{T-t}\frac{1-x_{t-1}}{2}$, then player 1 repeats his last offer at each stage: $x_t = x_{t-1}$.

In stage $t+1$, we have a. $\delta^{T-(t+1)}\frac{1-x_t}{2} = \delta^{T-(t+1)}\frac{1-x_{t-1}}{2} > \delta^{T-t}\frac{1-x_{t-1}}{2}$; and b. $x^{\#}_{t+1} = \frac{x^{\#}_t}{\delta}$, viz., $1 - x^{\#}_{t+1} < 1 - x^{\#}_t$. Hence $1 - x^{\#}_{t+1} < 1 - x^{\#}_t < \delta^{T-t}\frac{1-x_{t-1}}{2} < \delta^{T-(t+1)}\frac{1-x_t}{2}$, viz., $x_t < x^*_t$, which indicates that player 1 offers $x_{t+1} = x_t$ in $t+1$ by IH 3.1.

Such a process continues in every stage $t' > t$, meaning that $x_{t'-1} < x^*_{t'-1}$ in every stage, and player 1 never increase his offer anymore. ■

[3] Payoffs are evaluated according to the current stage for convenience.

Suppose that IH 3.1 holds for all stage $t' \geq t$, we now use Lemma 3.1 to look at the bargaining outcome in stage t. If $x_{t-1} \geq x^*_{t-1}$, then by IH 3.1 player 1 makes the minimal acceptable offer and player 2 accepts it; so they reach an agreement of $\{1 - max\{x_{t-1},\ x^{\#}_t\},\ max\{x_{t-1},\ x^{\#}_t\}\}$. If $x_{t-1} < x^*_{t-1}$, then by IH 3.1 and Lemma 3.1 player 1 repeats his last offer and keeps doing this in all future stages until the arbitration stage, while player 2 repeatedly rejects it, leading to a payoff pair of $\delta^{T-t}\{\frac{1-x_{t-1}}{2},\ \frac{1+x_{t-1}}{2}\}$.

We now start analyzing players' choices in stage $t-1$.

Player 2 chooses between accepting or rejecting x_{t-1}. By IH 3.1, if $x_{t-1} \geq x^*_{t-1}$ then the negotiation ends in the next stage; if $x_{t-1} < x^*_{t-1}$ then the negotiation ends in the arbitration stage. Anticipating this, she earns a payoff of

$$\delta \begin{cases} max\{x_{t-1},\ x^{\#}_t\} & if \quad 1 - x^{\#}_t \geq \delta^{T-t}\frac{1-x_{t-1}}{2} \\ \delta^{T-t}\frac{1+x_{t-1}}{2} & \quad\ \ 1 - x^{\#}_t < \delta^{T-t}\frac{1-x_{t-1}}{2} \end{cases}$$

if she rejects x_{t-1}. We look at these two cases in sequence.

- If player 2 rejects $x_{t-1} \geq x^*_{t-1}$ then player 1 would offer $x_t = max\{x_{t-1},\ x^{\#}_t\}$ and player 2 would accept the offer in stage t; earning a payoff of $\delta max\{x_{t-1},\ x^{\#}_t\}$. Hence, player 2 accepts x_{t-1} iff $x_{t-1} \geq \delta max\{x_{t-1},\ x^{\#}_t\}$, viz., $x_{t-1} \geq \delta x^{\#}_t$.

  Notice that $x^*_{t-1} < \delta x^{\#}_t$:

$$x^*_{t-1} < \delta x^{\#}_t$$

$$\iff \frac{4-\delta}{2-\delta} - \frac{2}{\delta^{T-t}} < \delta\frac{\delta^{T-t}}{2-\delta}$$

$$\iff (4-\delta)\delta^{T-t} - (4-2\delta) < \delta(\delta^{T-t})^2$$

$$\iff \delta(\delta^{T-t})^2 - (4-\delta)\delta^{T-t} + (4-2\delta) > 0$$

$$\iff [\delta^{T-t+1} - (4-2\delta)](\delta^{T-t} - 1) > 0$$

$$\iff (4 - \delta^{T-t+1} - 2\delta)(1 - \delta^{T-t}) > 0.$$

  Therefore, in stage $t-1$, player 2 accepts if $x_{t-1} \geq \delta x^{\#}_t$ and rejects if $x^*_{t-1} \leq$

$x_{t-1} < \delta x_t^{\#}$.

- If player 2 rejects $x_{t-1} < x_{t-1}^*$ then player 1 never increase his offer and the negotiation enters the arbitration stage, generating player 2 a payoff of $\delta^{T-t+1}\frac{1+x_{t-1}}{2}$. Hence, player 2 accepts x_{t-1} iff $x_{t-1} \geq \frac{\delta^{T-t+1}}{2-\delta^{T-t+1}}$.
  Notice that $x_{t-1}^* < \frac{\delta^{T-t+1}}{2-\delta^{T-t+1}}$:

$$x_{t-1}^* < \frac{\delta^{T-t+1}}{2-\delta^{T-t+1}}$$

$$\iff \frac{4-\delta}{2-\delta} - \frac{2}{\delta^{T-t}} < \frac{\delta^{T-t+1}}{2-\delta^{T-t+1}}$$

$$\iff \frac{4-\delta}{2-\delta} < \frac{\delta\Delta}{2-\delta\Delta} + \frac{2}{\Delta},\ where\ \Delta \equiv \delta^{T-t}$$

$$\iff \frac{4-\delta}{2-\delta} < \frac{\delta\Delta^2+4-2\delta\Delta}{2\Delta-\delta\Delta^2}$$

$$\iff \frac{4-\delta}{2-\delta} < \frac{\delta(\Delta-1)^2+4-\delta}{-\delta(\Delta-\frac{1}{\delta})^2+\frac{1}{\delta}}$$

  As $\Delta = \delta^{T-t} < 1 < \frac{1}{\delta}$, when t increases, $T-t$ decreases, $\Delta = \delta^{T-t}$ increases, $\delta(\Delta-1)^2+4-\delta$ decreases and $-\delta(\Delta-\frac{1}{\delta})^2+\frac{1}{\delta}$ increases. Hence the right hand side of the inequality decreases with t. When $t = T$, $\Delta = 1$, $\frac{\delta(\Delta-1)^2+4-\delta}{-\delta(\Delta-\frac{1}{\delta})^2+\frac{1}{\delta}} = \frac{4-\delta}{2-\delta}$. Therefore, for any $t \leq T-1$, $\frac{\delta(\Delta-1)^2+4-\delta}{-\delta(\Delta-\frac{1}{\delta})^2+\frac{1}{\delta}} > \frac{4-\delta}{2-\delta}$ and the inequality holds. Therefore, in stage $t-1$, player 2 rejects if $x_{t-1} < x_{t-1}^*$.

Combining the two cases: in stage $t-1$, player 2 accepts if $x_{t-1} \geq \delta x_t^{\#}$ and rejects if $x_{t-1} < \delta x_t^{\#}$.

Now player 1 compares between making an acceptable offer or not. If he makes an acceptable offer, then it is in his interest to make the minimal acceptable offer: $x_{t-1}^{\#} = \delta x_t^{\#} = \frac{\delta^{T-t+1}}{2-\delta}$ and earn a payoff of $1 - x_{t-1}^{\#}$ if possible. If he makes an unacceptable offer, he would never increase his offer again and player 2 would keep rejecting that offer until the game enters the arbitration stage. In this case it is in player 1's best interest to make the minimal offer as possible: $x_{t-1} = x_{t-2}$, earning a payoff of $\delta^{T-t+1}\frac{1-x_{t-2}}{2}$. Therefore, he offers $x_{t-1} = min\{x_{t-2},\ x_{t-1}^{\#}\}$ if $1 - x_{t-1}^{\#} \geq \delta^{T-t+1}\frac{1-x_{t-2}}{2}$, viz., $x_{t-2} \geq x_{t-2}^*$; offers $x_{t-1} = x_{t-2}$ if $x_{t-2} < x_{t-2}^*$.

To sum up, both players' choices satisfy the description of IH 3.1 in $t-1$.

We have established that IH 3.1 forms a unique sub-game perfect equilibrium in stage $T-1$, and it forms a unique SPE in stage $t-1$ if it forms a unique SPE in stage t, we know that the game has a unique SPE with the strategy described in IH 3.1. Therefore, in the first stage of the game, if $1-\frac{\delta^{T-1}}{2-\delta} \geq \frac{\delta^{T-1}}{2}$ then player 1 make an acceptable offer for player 2; if $1-\frac{\delta^{T-1}}{2-\delta} < \frac{\delta^{T-1}}{2}$ then player 1 never makes any positive offer and the settlement is reached in the arbitration stage. Denote $\delta^{\#}$ to be the threshold discount factor solving $1-\frac{\delta^{T-1}}{2-\delta} = \frac{\delta^{T-1}}{2}$, negotiation ends early when $\delta < \delta^{\#}$ and ends late when $\delta > \delta^{\#}$. This ends the proof of Proposition 3.1. ■

We learn two things from Proposition 3.1: the settlement pattern in this model is U-shaped and has a threshold property.

The intuition for the U-shaped settlement is straightforward. Suppose that in equilibrium player 1 offers and player 2 accepts an offer of $x_t = x > x_{t-1}$ in any stage $1 < t < T$.[4] If player 1 deviates and offers $x_{t-1} = x$ in the previous stage, player 2 would accept it: if player 1's best option in stage t is to offer $x_t = x$ when the previous offer is $x_{t-1} < x$ and his option set is $x_t \geq x_{t-1}$, his best option in stage t would be the same when the previous offer is $x_{t-1} = x$ and his option set is $x_t \geq x$; for player 2 herself, in stage t she would accept $x_t = x$ anyway because player 1's choice in stage $t-1$ does not affect her decision in stage t. Anticipating player 2's acceptance, player 1 has an incentive to shift the accepted offer x to an earlier stage. Therefore, there is no intermediate agreement in this game: either player 1 makes an acceptable offer in the very beginning, or he waits until the final stage in which he does not necessarily makes the proposal.

The intuition for the threshold property is as follows. At the very beginning of the game, player 1 has two options: 1. make a series of offers that would generate the same payoff for player 2 throughout the game until the arbitration stage; 2. wait until the arbitration stage and maximize his payoff at that moment by offering

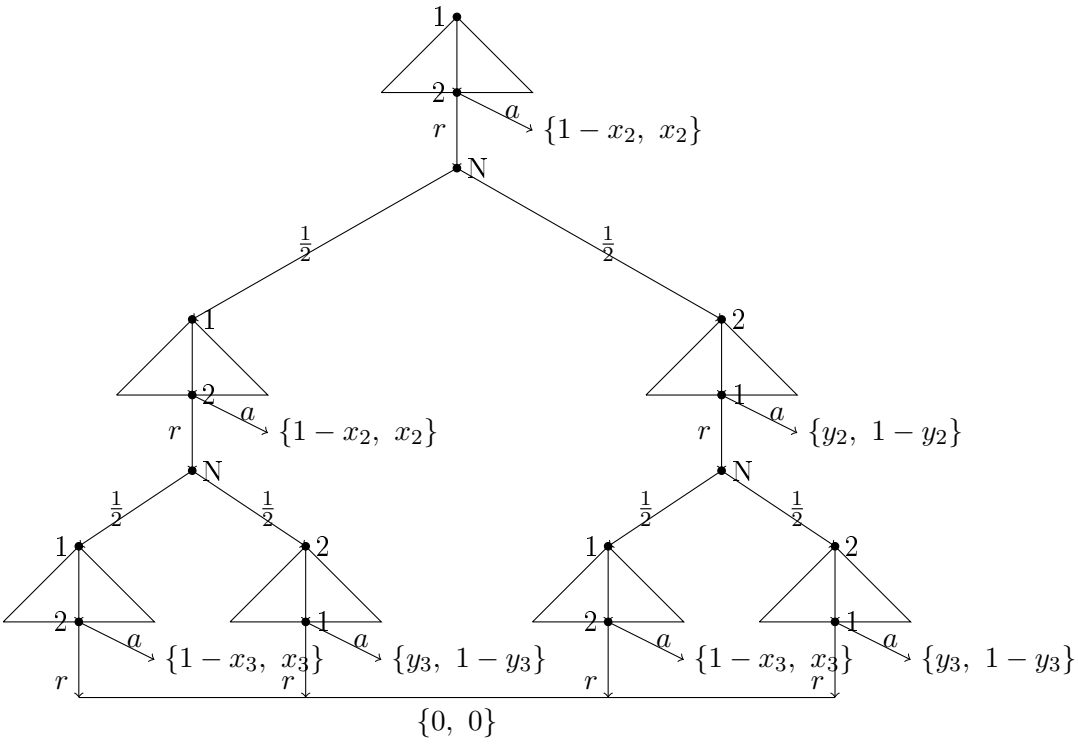
[4] If $x = x_{t-1}$ then player 2 will have an incentive to accept the offer earlier and the strategy combination can't be an equilibrium.

0 at all previous stages. Player 2, with no ability to make an offer, is indifferent between accepting player 1's exploiting offer in any stage of the game (when player 1 chooses the first option). However, if she decides to wait, then in the arbitration stage she gets a chance to be the proposer and seal a deal that favors herself instead. Therefore, when the discount rate gets higher, player 2 is more inclined to wait unless she receives a more attractive offer. For player 1, when the discount rate gets higher, he simultaneously faces a more demanding request from player 2 and a lower timing cost; so he tends to choose the second option instead of the first one.

3.4 Randomizing offer model

The unilateral offer bargaining model in Section 3.3 restricts one player to be the proposer and the other to be the respondent in most of the game. It is a simplification of the randomizing offer protocol, which was firstly proposed in Binmore and Dasgupta 1987[4]. In this section, we provide an example to show that the main conclusion remains when we allow both players to propose.

In a randomizing offer bargaining model, player 1 proposes with probability $p \in (0, 1)$ at each stage of the game. Therefore, instead of appearing only in the final stage of the game in the form of arbitration (in the unilateral offer model), uncertainty is in every stage of the game in a randomizing offer model. Here we provide an example with $T = 3$, and players always have an equal probability of proposing. Without loss of generality, we suppose that player 1 is the first proposer.



Proposition 3.2. *There exists a threshold $\delta^{\#}$ such that players reach an agreement immediately when $\delta < \delta^{\#}$, and reach an agreement in the last stage when $\delta > \delta^{\#}$.*

The proof of Proposition 3.2 is in the Appendix. Notice that Proposition 3.2 is identical with Proposition 3.1, although the two models are different.

Following are the SPE strategies.

In the first stage player 1 offers $x_1 = \frac{2\delta-\delta^2}{4-2\delta-\delta^2}$ when $\delta < \delta^{**}$, [5] and offers $x_1 = 0$ when $\delta > \delta^{**}$; player 2 accepts iff $x_1 \geq \frac{2\delta-\delta^2}{4-2\delta-\delta^2}$ when $\delta < \delta^{*}$,[6] and accepts iff $x_1 \geq \frac{4\delta^2-\delta^3}{8-4\delta-2\delta^2+\delta^3}$ when $\delta > \delta^{*}$.

In the second stage, if player 1 is the proposer then he offers $x_2 = \frac{\delta}{2-\delta}$ when $x_1 \geq \frac{-4+6\delta-\delta^2}{2\delta-\delta^2}$, and offers $x_2 = x_1$ when $x_1 < \frac{-4+6\delta-\delta^2}{2\delta-\delta^2}$; 2 accepts iff $x_2 \geq \frac{\delta}{2-\delta}$. If 2 is the proposer then she offers $y_2 = \frac{\delta}{2-\delta}(1-x_1)$ when $x_1 \geq \frac{-4+6\delta-\delta^2}{\delta^2}$ and offers $y_2 = 0$ when $x_1 < \frac{-4+6\delta-\delta^2}{\delta^2}$; 1 accepts iff $y_2 \geq \frac{\delta}{2-\delta}(1-x_1)$.

[5] δ^{**} is the unique solution of $8-8\delta-4\delta^2+2\delta^3+\delta^4=0]$ when $0<\delta<1$.
[6] δ^{*} is the solution of $8-8\delta-2\delta^2+\delta^3=0]$ when $0<\delta<1$.

In the third stage, if 1 is the proposer in the second stage and the third stage then 1 offers $x_3 = x_2$ and 2 accepts iff $x_3 \geq 0$. If 1 is the proposer in the second stage and 2 is the proposer in the third stage then 2 offers $y_3 = 0$ and 1 accepts iff $y_3 \geq 0$. If 2 is the proposer in the second stage and 1 is the proposer in the third stage then 1 offers $x_3 = x_1$ and 2 accepts iff $x_3 \geq 0$. If 2 is the proposer in the second and third stage then 2 offers $y_3 = y_2$ and 1 accepts iff $y_3 \geq 0$.

We can see several things from the SPE strategies above: like in the unilateral offer model, players have endogenous disagreement shares:$\{\frac{1-x_2}{2}, \frac{1+x_2}{2}\}$ if player 1 is the proposer in the second stage and $\{\frac{1-x_1+y_2}{2}, \frac{1+x_1-y_2}{2}\}$ if player 2 is. Also, players have threshold strategies.

Take the strategies in the second stage for example. Player 1 makes an acceptable offer only if his previous offer x_1 is high enough: as in the unilateral offer model, a higher previous offer restricts his choices in the current stage and reduces his disagreement share. What is not shown in the unilateral offer model is that player 2 also makes an acceptable offer only if her rival's previous offer x_1 is high enough. The intuition is as follows. When player 1's previous offer is higher, he has a stronger incentive to avoid the arbitration stage; so he is more prone to accept a lower offer from player 2. Therefore, although a higher x_1 increases player 2's disagreement share in the arbitration stage, the lowering of the minimal acceptable offer induces her to reach an early agreement.

In equilibrium, 1 offers $x_1 = \frac{2\delta-\delta^2}{4-2\delta-\delta^2}$ and 2 accepts the offer in the first stage when $\delta < \delta^{**}$; neither player makes any positive offers in stages one and two, the final respondent accepts the zero offer $x_3 = y_3 = 0$ when $\delta > \delta^{**}$. The settlement pattern is still U-shaped and exhibits a threshold property.

3.5 Application and analysis

In the (almost) unilateral offer model in Section 3.3, we denote the final stage as an "arbitration" stage, in which each player has one last chance to facilitate agreement;

and naturally, they do grab the chance. Another way to interpret this model is to treat it as a $T-1$ stages pre-trial negotiation model, in which player 1 is always the "active" proposer and player 2 is always the "passive" respondent. If no agreement could be reached in time (until stage $T-1$), an arbitrator is introduced to avoid the long lasting and catastrophic court time. In the arbitration stage, neither player has any decision making power: they must accept the ruling.

We can imagine that there are two types of arbitrator: one has access to the negotiation history and will most likely take it into consideration; the other one has no access and hence will probably have no better option than dividing the pie equally. The question here is whether it is acceptable to give arbitrators the right to access the negotiation history.

Although the arbitrator can always gather some information (the failure of negotiation sometimes tells part of the story), leaking of the negotiation history gives the arbitrator a direct guidance in making decisions; and this in turn affects players' choices during the negotiation.

For example, in a criminal trial, the prosecutor may sometimes make an overly generous offer either to save costs or to avoid the risk of losing, in spite of the danger of showing weakness to the defendant. However, this gesture may be interpreted as lacking of confidence in the evidence to make the conviction if it is admissible in court, hence guaranteeing a loss for the prosecutor. Anticipating this, the prosecutor would not make this move at the beginning. Meanwhile, for the defendant, nothing would be more convincing in proving innocence than to reject a plea bargain (suppose we believe that the innocent ones are less willing to plea); to exclude the negotiation history may weaken the defendant's bargaining power.

In our model, suppose that player 1 is the active prosecutor and player 2 is the passive defendant. We use the model without EC to indicate that the negotiation history is inadmissible and the model with EC to indicate that the negotiation history is admissible. In the arbitration stage, if there is no EC constraint then the arbitrator chooses a distribution of $\{\frac{1}{2},\ \frac{1}{2}\}$ (each player offers 0 to the other); if there is an EC constraint, the arbitrator sees the prosecutor's offer and takes that offer as a reference to choose a distribution of $\{\frac{1-x_{T-1}}{2},\ \frac{x_{T-1}}{2}\}$ (player 1 offers x_{T-1} and player 2 offers

0). In equilibrium, if there is no EC constraint (history inadmissible), prosecutor offers $\frac{\delta^{T-1}}{2}$ and an agreement is reached in the first stage of the game; if there is an EC constraint (history admissible), the prosecutor offers $\frac{\delta^{T-1}}{2-\delta}$ and an agreement is reached in the first stage of the game when both players are impatient enough; while the prosecutor never makes a positive offer and the arbitrator divides the pie in half when both players are patient enough.

To sum up players' equilibrium payoffs in one table:

No EC	$\delta < \delta^{\#}$	$\delta > \delta^{\#}$
prosecutor's payoff	$1 - \frac{\delta^{T-1}}{2}$	$1 - \frac{\delta^{T-1}}{2}$
defendant's payoff	$\frac{\delta^{T-1}}{2}$	$\frac{\delta^{T-1}}{2}$
Total payoff	1	1

EC	$\delta < \delta^{\#}$	$\delta > \delta^{\#}$
prosecutor's payoff	$1 - \frac{\delta^{T-1}}{2-\delta}$	$\frac{\delta^{T-1}}{2}$
defendant's payoff	$\frac{\delta^{T-1}}{2-\delta}$	$\frac{\delta^{T-1}}{2}$
Total payoff	1	δ^{T-1}

Comparing these two cases, we can easily see how allowing the negotiating history to be presented affects the outcome: 1. It benefits the defendant in most cases: $\frac{\delta^{T-1}}{2-\delta} > \frac{\delta^{T-1}}{2}$ when δ is low, and the same payoff of $\frac{\delta^{T-1}}{2}$ when δ is high; 2. It has an adverse effect on the prosecutor: $1 - \frac{\delta^{T-1}}{2-\delta} < 1 - \frac{\delta^{T-1}}{2}$ when δ is low, and $\frac{\delta^{T-1}}{2} < 1 - \frac{\delta^{T-1}}{2}$ when δ is high; 3. It generates a total welfare loss when both players are patient enough: in this case the agreement is delayed and the total payoff is δ^{T-1} instead of 1.

Generally speaking, allowing the negotiating history to be presented benefits those who are more "passive" (respondent) in the negotiation process, helping them to regain some bargaining power against the "active" ones (proposer). However, this pressure on the proposer could backfire and delay the agreement, in which case total welfare decreases and no one is better off than before.

3.6 Conclusion

In this chapter, we construct a model with Endogenous Commitment and an ensuing endogenous disagreement payoff. In this model, the equilibrium has a U-shaped settlement pattern and satisfies a threshold property. In most traditional complete information bargaining models, the delayed agreement could be anticipated, and therefore would be replaced by earlier mutually favorable terms. (cf. Kennan and Wilson 1993[32]) However, this might not be true here. The greater the concession a player makes, the more bargaining power he gives to his rival, a mutually favorable result may not exist. When the proposer tries to provide a mutually more favorable offer now to avoid a lose-lose situation in future, the proposing of this offer breaks the balance, putting the respondent in an advantageous position, and hence allows the respondent to reject this offer. Imagine that player 2 is extremely patient (has a discount rate equal to 1), then no offer could possibly satisfy her, as 1. any generous offer would only mean an even more generous offer in the last stage of the negotiation due to the Endogenous Commitment; and 2. no matter how generous the offer is, it does not exceed her own proposal. For any x_t, as $x_t \leq x_T \leq 1$ and $y_T = 0$, $x_t \leq \frac{1+x_t}{2} \leq \frac{1+x_{T-1}}{2}$ and she would always reject x_t.

U-shaped settlement pattern was explained in relation to pre-trial negotiations in incompletely information models (cf. Spier 1992[57]). Our model provides a different perspective in exploring the interesting intuition in pre-trial negotiation analysis. Endogenous Commitment might be spontaneous, but there is no doubt it would be much reinforced if the court allows the offers in the pre-trial negotiation to be recorded as evidence. Therefore, the bargaining model in our chapter with/without EC constraint could be seen as a comparison between the allowing or forbidding use of the negotiation history as evidence. Permission benefits the defendant while harming the prosecutor, reduces the generosity of the offers, and increases the settlement rate.

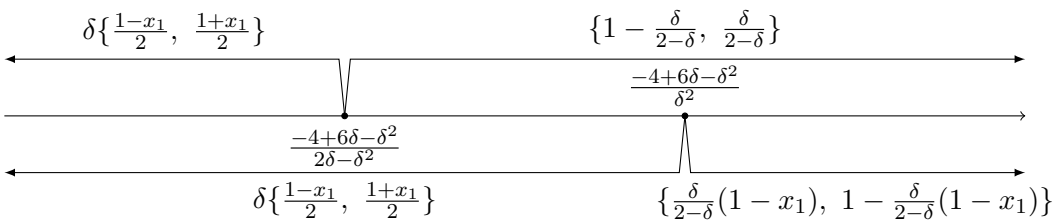
3.7 Appendix

In the third stage, both players make the minimal non-negative offer consistent with Endogenous Commitment, as their rivals would have no choice but to accept any non-negative offer. Hence, if player 1 is the proposer in the second stage then 1 offers $x_3 = x_2$ and player 2 accepts it, while player 2 offers $y_3 = 0$ and player 1 accepts it; if player 2 is the proposer in the second stage then player 1 offers $x_3 = x_1$ and player 2 accepts it, while player 2 offers $y_3 = y_2$ and player 1 accepts it. The sub-game equilibrium are payoffs are $\{\frac{1-x_2}{2},\ \frac{1+x_2}{2}\}$ and $\{\frac{1-x_1+y_2}{2},\ \frac{1+x_1-y_2}{2}\}$.

In the second stage, we have two cases. When player 1 is the proposer, player 2 earns x_2 by accepting x_2 and earns $\delta\frac{1+x_2}{2}$ by rejecting x_2; so she accepts if $x_2 \geq \frac{\delta}{2-\delta}$ and rejects if $x_2 < \frac{\delta}{2-\delta}$. Anticipating this, player 1 chooses between making acceptable or unacceptable offers: he earns $1 - x_2$ by offering $x_2 \geq \frac{\delta}{2-\delta}$, and earns $\delta\frac{1-x_2}{2}$ by offering $x_2 < \frac{\delta}{2-\delta}$; his best option in these two cases are $x_2 = \frac{\delta}{2-\delta}$ and $x_2 = x_1$ (due to the decreasing payoffs). Hence, he compares $1 - \frac{\delta}{2-\delta}$ and $\delta\frac{1-x_1}{2}$, offering $x_2 = \frac{\delta}{2-\delta}$ if $x_1 \geq \frac{-4+6\delta-\delta^2}{2\delta-\delta^2}$ and offering $x_2 = x_1$ if $x_1 < \frac{-4+6\delta-\delta^2}{2\delta-\delta^2}$. Notice that $\frac{\delta}{2-\delta} > \frac{-4+6\delta-\delta^2}{2\delta-\delta^2}$. Therefore, the payoffs are $\{1 - \frac{\delta}{2-\delta},\ \frac{\delta}{2-\delta}\}$ if $x_1 \geq \frac{-4+6\delta-\delta^2}{2\delta-\delta^2}$ and $\delta\{\frac{1-x_1}{2},\ \frac{1+x_1}{2}\}$ if $x_1 < \frac{-4+6\delta-\delta^2}{2\delta-\delta^2}$.[7]

When player 2 is the proposer, player 1 earns y_2 by accepting y_2 and earns $\delta\frac{1-x_1+y_2}{2}$ by rejecting y_2; so he accepts if $y_2 \geq \frac{\delta}{2-\delta}(1-x_1)$ and rejects if $y_2 < \frac{\delta}{2-\delta}(1-x_1)$. Anticipating this, player 2 chooses between making acceptable or unacceptable offers: she earns $1 - y_2$ by offering $y_2 \geq \frac{\delta}{2-\delta}(1 - x_1)$ and earns $\delta\frac{1+x_1-y_2}{2}$ by offering $y_2 < \frac{\delta}{2-\delta}(1 - x_1)$; her best option in these two cases are $y_2 = \frac{\delta}{2-\delta}(1 - x_1)$ and $y_2 = 0$ (due to the decreasing payoffs). Hence, she compares between $1 - \frac{\delta}{2-\delta}(1-x_1)$ and $\delta\frac{1+x_1}{2}$, offering $y_2 = \frac{\delta}{2-\delta}(1 - x_1)$ if $x_1 \geq \frac{-4+6\delta-\delta^2}{\delta^2}$ and offering $y_2 = 0$ if $x_1 < \frac{-4+6\delta-\delta^2}{\delta^2}$. The payoffs are $\{\frac{\delta}{2-\delta}(1-x_1),\ 1 - \frac{\delta}{2-\delta}(1-x_1)\}$ if $x_1 \geq \frac{-4+6\delta-\delta^2}{\delta^2}$ and $\delta\{\frac{1-x_1}{2},\ \frac{1+x_1}{2}\}$ if $x_1 < \frac{-4+6\delta-\delta^2}{\delta^2}$.[8]

[7] Notice that $\frac{-4+6\delta-\delta^2}{2\delta-\delta^2}$ is negative when $\delta < 3 - \sqrt{5}$, in which case $x_1 < \frac{-4+6\delta-\delta^2}{2\delta-\delta^2}$ does not hold.
[8] Again, $x_1 < \frac{-4+6\delta-\delta^2}{2\delta-\delta^2}$ does not hold when $\delta < 3 - \sqrt{5}$.



In the first stage, player 2 earns x_1 by accepting x_1, and her payoff from rejecting x_1 can be decomposed into three cases: agreement is reached in the final stage and she gets a share of $\frac{1+x_1}{2}$ if $x_1 < \frac{-4+6\delta-\delta^2}{2\delta-\delta^2}$; agreement is reached in the next two stages with equal probability and she gets a share of $\frac{\delta}{2-\delta}$ in next stage or $\frac{1+x_1}{2}$ in the third stage if $\frac{-4+6\delta-\delta^2}{2\delta-\delta^2} < x_1 \leq \frac{-4+6\delta-\delta^2}{\delta^2}$; agreement is reached in the next stage and she gets a share of $\frac{\delta}{2-\delta}$ or $1-\frac{\delta}{2-\delta}(1-x_1)$ if $\frac{-4+6\delta-\delta^2}{\delta^2} \leq x_1$. Therefore, her expected payoff from rejecting is

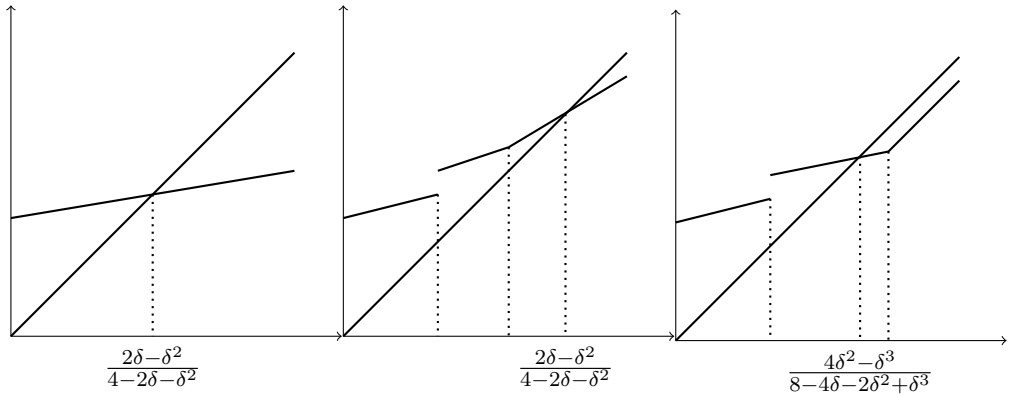
$$\begin{cases} \delta^2\frac{1+x_1}{2} & x_1 < \frac{-4+6\delta-\delta^2}{2\delta-\delta^2} \\ \frac{\delta}{2}(\frac{\delta}{2-\delta}+\delta\frac{1+x_1}{2}) & if \quad \frac{-4+6\delta-\delta^2}{2\delta-\delta^2} \leq x_1 < \frac{-4+6\delta-\delta^2}{\delta^2} \\ \frac{\delta}{2}(\frac{\delta}{2-\delta}+1-\frac{\delta}{2-\delta}(1-x_1)) & \frac{-4+6\delta-\delta^2}{\delta^2} \leq x_1 \end{cases},$$

while player 1's expected payoff is

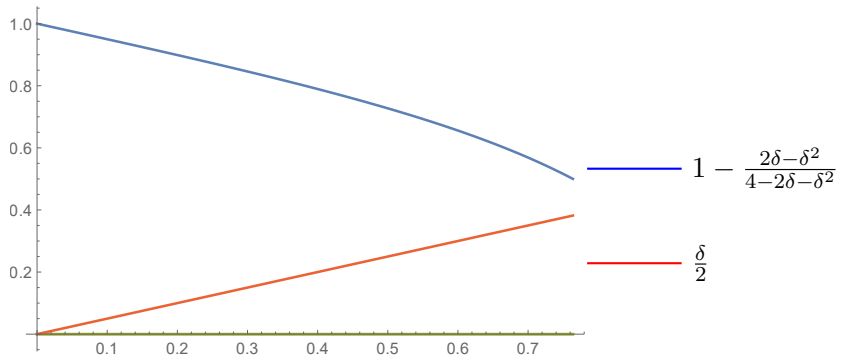
$$\begin{cases} \delta^2\frac{1-x_1}{2} & x_1 < \frac{-4+6\delta-\delta^2}{2\delta-\delta^2} \\ \frac{\delta}{2}(1-\frac{\delta}{2-\delta}+\delta\frac{1-x_1}{2}) & if \quad \frac{-4+6\delta-\delta^2}{2\delta-\delta^2} \leq x_1 < \frac{-4+6\delta-\delta^2}{\delta^2} \\ \frac{\delta}{2}(1-\frac{\delta}{2-\delta}+\frac{\delta}{2-\delta}(1-x_1)) & \frac{-4+6\delta-\delta^2}{\delta^2} \leq x_1 \end{cases}.$$

Comparing her expected payoff from rejecting with accepting x_1: she accepts iff $x_1 \geq \frac{2\delta-\delta^2}{4-2\delta-\delta^2}$ when $\delta < \delta^*$,[9] and accepts iff $x_1 \geq \frac{4\delta^2-\delta^3}{8-4\delta-2\delta^2+\delta^3}$ when $\delta^* < \delta$. As shown in the graphs below, 2 rejects $x_1 \in [0,\ \frac{-4+6\delta-\delta^2}{2\delta-\delta^2})$ irrespective of δ, rejects any $x_1 \in (\frac{-4+6\delta-\delta^2}{2\delta-\delta^2},\ \frac{-4+6\delta-\delta^2}{\delta^2})$ if $3-\sqrt{5} < \delta < \delta^*$, and accepts any $x_1 \in [\frac{-4+6\delta-\delta^2}{\delta^2},\ 1]$ if $\delta^* < \delta$.

[9] Notice that δ^* is the unique solution of $8-8\delta-2\delta^2+\delta^3=0$ when $0<\delta<1$. $\delta^* > 3-\sqrt{5}$.

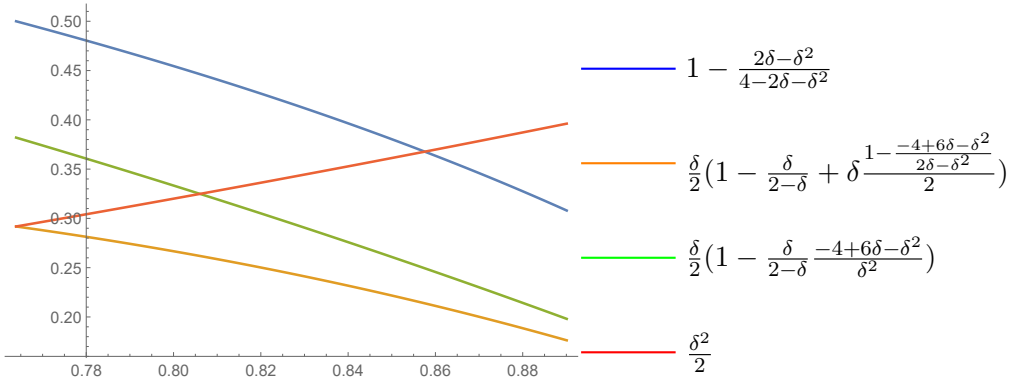


We now at the three cases in sequence. When $\delta < 3-\sqrt{5}$, player 1 earns $1-x_1$ by offering $x_1 \geq \frac{2\delta-\delta^2}{4-2\delta-\delta^2}$ and earns $\frac{\delta}{2}(1-\frac{\delta}{2-\delta}+\frac{\delta}{2-\delta}(1-x_1))$, viz., $\frac{\delta}{2}(1-\frac{\delta}{2-\delta}x_1)$ by offering $x_1 < \frac{2\delta-\delta^2}{4-2\delta-\delta^2}$. His best option in the previous two cases are respectively $x_1 = \frac{2\delta-\delta^2}{4-2\delta-\delta^2}$ and $x_1 = 0$ due to decreasing payoffs. Comparing $1-\frac{2\delta-\delta^2}{4-2\delta-\delta^2}$ and $\frac{\delta}{2}$: he offers $x_1 = \frac{2\delta-\delta^2}{4-2\delta-\delta^2}$ if $\delta < 3-\sqrt{5}$.

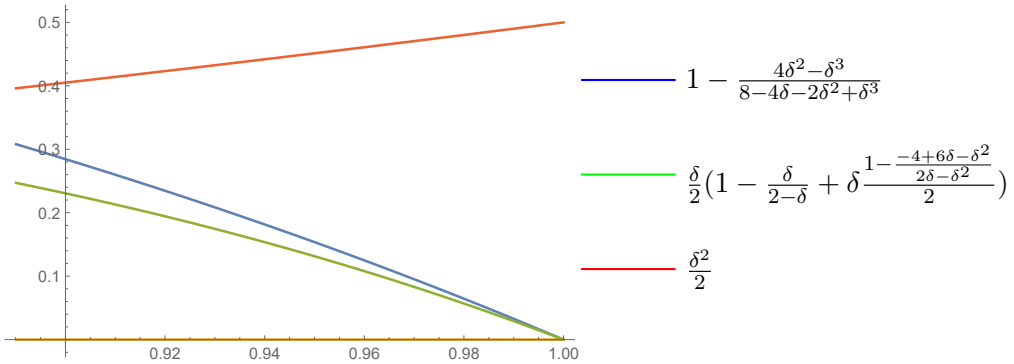


When $3-\sqrt{5} < \delta < \delta^*$, player 1 earns $1-x_1$ by offering $x_1 \geq \frac{2\delta-\delta^2}{4-2\delta-\delta^2}$. By offering $x_1 < \frac{2\delta-\delta^2}{4-2\delta-\delta^2}$, he earns $\delta^2\frac{1-x_1}{2}$ if $x_1 < \frac{-4+6\delta-\delta^2}{2\delta-\delta^2}$, earns $\frac{\delta}{2}(1-\frac{\delta}{2-\delta}+\delta\frac{1-x_1}{2})$ if $\frac{-4+6\delta-\delta^2}{2\delta-\delta^2} < x_1 < \frac{-4+6\delta-\delta^2}{\delta^2}$, and earns $\frac{\delta}{2}(1-\frac{\delta}{2-\delta}x_1)$ if $\frac{-4+6\delta-\delta^2}{\delta^2} < x_1 < \frac{2\delta-\delta^2}{4-2\delta-\delta^2}$. His best option in these four cases are respectively $x_1 = \frac{2\delta-\delta^2}{4-2\delta-\delta^2}$, $x_1 = 0$, $x_1 = \frac{-4+6\delta-\delta^2}{2\delta-\delta^2}$ and $x_1 = \frac{-4+6\delta-\delta^2}{\delta^2}$ due to decreasing payoffs. Comparing $1-\frac{2\delta-\delta^2}{4-2\delta-\delta^2}$, $\frac{\delta^2}{2}$, $\frac{\delta}{2}(1-\frac{\delta}{2-\delta}+\delta\frac{1-\frac{-4+6\delta-\delta^2}{2\delta-\delta^2}}{2})$ and $\frac{\delta}{2}(1-\frac{\delta}{2-\delta}\frac{-4+6\delta-\delta^2}{\delta^2})$: he offers $x_1 = \frac{2\delta-\delta^2}{4-2\delta-\delta^2}$ if $1-\frac{2\delta-\delta^2}{4-2\delta-\delta^2} > \frac{\delta^2}{2}$, viz., $\delta < \delta^{**}$[10], and offers $x_1 = 0$ if $\delta \geq \delta^{**}$.

[10]Recall that δ^{**} is the solution of $8-8\delta-4\delta^2+2\delta^3+\delta^4 = 0$, $3-\sqrt{5} < \delta^{**} < \delta^*$.



When $\delta > \delta^*$, player 1 earns $1 - x_1$ by offering $x_1 \geq \frac{4\delta^2-\delta^3}{8-4\delta-2\delta^2+\delta^3}$. By offering $x_1 < \frac{4\delta^2-\delta^3}{8-4\delta-2\delta^2+\delta^3}$, he earns $\delta^2\frac{1-x_1}{2}$ if $x_1 < \frac{-4+6\delta-\delta^2}{2\delta-\delta^2}$ and earns $\frac{\delta}{2}(1-\frac{\delta}{2-\delta}+\delta\frac{1-x_1}{2})$ if $\frac{-4+6\delta-\delta^2}{2\delta-\delta^2} < x_1 < \frac{4\delta^2-\delta^3}{8-4\delta-2\delta^2+\delta^3}$. His best option in each case is respectively $x_1 = \frac{4\delta^2-\delta^3}{8-4\delta-2\delta^2+\delta^3}$, $x_1 = 0$ and $x_1 = \frac{-4+6\delta-\delta^2}{2\delta-\delta^2}$ due to decreasing payoffs. Comparing $1-\frac{4\delta^2-\delta^3}{8-4\delta-2\delta^2+\delta^3}$, $\delta^2\frac{1-0}{2}$, and $\frac{\delta}{2}(1-\frac{\delta}{2-\delta}+\delta\frac{1-\frac{-4+6\delta-\delta^2}{2\delta-\delta^2}}{2})$: he offers $x_1 = 0$ if $\delta > \delta^*$.



Chapter 4

Inefficiency in Contribution games

Abstract

In this chapter we discuss inefficiency in contribution games with different protocols. We mainly focus on the delay problem in alternating contribution games caused by players valuing the project differently. Any heterogeneity in valuation could cause a severe welfare loss.

4.1 Introduction

When a project requires joint work from several agents, it is foreseeable that there would be a free-rider problem. The literature has explored this issue from various perspectives. This chapter focuses on one perspective: how heterogeneity of players' benefit from the project might affect whether and when the project is completed, in other words, the efficiency. In the main part of the chapter we use a finite horizon alternating contribution game with two impatient players to demonstrate the following result: if the player who contributes last values the project most then the project would be completed as quickly as possible: that is, the equilibrium play is efficient. By contrast, if the penultimate contributor values the project most then the project may not be completed till the deadline. A small change in the relative valuations could generate a large change in the payoffs. We also use a finite horizon randomiz-

ing contribution game to exhibit other types of inefficiency in a contribution game: failure of completion and waste of contributions. In Section 4.2 we solve the main alternating contribution model and in Section 4.3 we analyze the result. In Section 4.4 we present the randomizing model. Section 4.5 concludes.

4.2 Alternating model

There are two players $i = 1, 2$ working on a time-limited project. If their total contributions reach a certain amount W by the deadline then the project is completed and each player i receives a return of $V_i > 0$ right away: V_i is player i's valuation of the project; if the total contributions are less than W by the deadline then the project fails and both players receive returns of 0. Contributions by 1 and 2 in period t are respectively denoted as x_t and y_t, $t \in \{1, 2, 3, ...T\}$. T is the total number of stages.

1 and 2 make (sunk) contributions in sequence and we assume that 2 acts as the last contributor without loss of generality, hence 1 and 2 are not identical in our model. The deadline is set at a fixed stage $T \geq 4$[1], so that 1 is the first contributor if T is even, 2 is the first contributor when T is odd. Both players' returns and contributions are discounted by a discount factor $\delta \in (0, 1)$ through each stage. Player 1 and 2 earn $\delta^t V_1 - \sum_{s \leq t} \delta^s x_s$ and $\delta^t V_2 - \sum_{s \leq t} \delta^s y_s$ when the project is completed in stage t.

Define M_t as the remaining contribution required to complete the project in stage t. In each stage, M_t is the difference between the total contribution required for completion (W) and the accumulated contributions up to period t.

In this chapter we use sub-game perfect equilibrium as the solution concept. When we use the term equilibrium we refer to SPE.

Assumption 4.1. $max\{V_1,\ V_2\} < W < min\{V_1 + \delta V_2,\ \delta V_1 + V_2\}$.

[1] We try to enrich the set of possible completion patterns: to allow 1 or 2 to complete the project before the deadline, and 2 to complete at the deadline. Any $T \leq 3$ fails to include these three types: notice that no one can complete the project without the other's cooperation. Take $T = 3$ as an example: as the sequence of making the contribution is 2, 1, 2, player 2 can't complete the project before the deadline because it means that she has to do everything by herself.

This assumption ensures that the project is worth-completing, but only by two players. The definition of worth-completing is that the project could bring positive total payoff for players without violating each player's individual rationality.

This description is covered by the right side of the inequality. By assuming $W < min\{V_1 + \delta V_2,\ \delta V_1 + V_2\}$, we want to make sure that it is always profitable for participants to complete the project. Think of the case $\delta \to 1$ for convenience: players could never make total contributions larger than $V_1 + V_2$ because that would definitely violate at least one individual's rationality (induce a negative payoff for at least one of them). Therefore, when $W > V_1 + V_2$, it is impossible for the project to be completed. As individual rationality allows no total contributions exceeding own valuation of the project, taking time discounting into consideration, requiring W to be less than $V_1 + \delta V_2$ and $\delta V_1 + V_2$ allows for two possible completion patterns: 2 after 1 make their maximal contribution to complete the project, or 1 after 2 do the same thing. The case of W between $V_1 + \delta V_2$ and $\delta V_1 + V_2$ allows only one of the two above mentioned completion patterns and is therefore less interesting.

Furthermore, in this chapter we would like to focus on situations where cooperation is necessary for completion of the project. By assuming $W > max\{V_1,\ V_2\}$, no one is able to gain a positive payoff without the other one's collaboration; so no one completes alone. Our main results do not change significantly when $W < max\{V_1,\ V_2\}$.

We have set the upper and lower bound of the total contribution required for completion W, we now take a further step to look at the hidden assumption about the discounting rate δ.

If $V_1 > V_2$ then Assumption 4.1 becomes $V_1 < W < \delta V_1 + V_2$, which implies that $(1 - \delta)V_1 < V_2$, viz., $\delta > 1 - \frac{V_2}{V_1}$. Intuitively, if the discount factor is low enough, delay won't be an option. The left hand side of $(1 - \delta)V_1 < V_2$ is the minimal loss of waiting one stage for 1, and the right hand side is the maximal contribution from 2 (in the best case scenario, 2 makes the full contribution in the next stage). If, instead, $(1 - \delta)V_1 > V_2$, then it is conceivable that 1 would never choose to delay and seek help from 2, which means that there is no room for cooperation and the

project will not be completed. Similar analysis applies when $V_1 < V_2$.

Therefore, through Assumption 4.1, we focus our attention on the meaningful cases, which require δ to be reasonably large. We further increase the lower-bound of the discount factor in Section 4.2.3.

In this chapter, we explore how heterogeneity of players' valuations affects the time of completion; that is, the efficiency of the contribution games. Recalling that 2 is the final contributor, we explore three cases: 1 and 2 have the same valuation, 2 is the high valuing player and 2 is the low valuing player respectively in Sections 4.2.1-4.2.3. In Section 4.3, we compare the results in Sections 4.2.2 and 4.2.3. We discuss the origin of the differences briefly in the next paragraph.

There are two factors in the finite horizon alternating contribution games affecting players' bargaining power: a) *Deadline effect*: knowing that in the end 2 would always wrap things up, 1 has a stronger incentive to delay than 2; so the final contributor is in a weaker position, as in MM 2000. b) *Value effect*: the low valuing player does not want to complete because the cost of waiting for him is less than for the high valuing player (this will be shown in detail in Claim 4.3), i.e., whoever values the project more has a stronger incentive to complete earlier, so the high valuing player is in a weaker position, as in CJ 2003.

In Section 4.2.2 these two effects work in the same direction and in Section 4.2.3 they work in opposite directions. In the latter case delay occurs.

Before we start any formal analysis, we first introduce two Claims for future use.

Claim 4.1. *In equilibrium player i completes the project immediately when $M_t \leq (1-\delta)V_i$ in stage t.*

The intuition of Claim 4.1 is straightforward: either player completes the project when the cost of doing so is low. The tricky part is the indifference choice: the equilibrium is not valid when the indifference choice involves uncertainty.

Proof.

Take $i = 1$. Notice that through contributing M_t, 1 earns a payoff of V_1 immediately; otherwise, the best consequence is that 2 completes in the next stage and 1 earns a payoff of δV_1. Hence 1 would choose to complete when $M_t < (1-\delta)V_1$.

1 might be indifferent between these two options when $M = (1-\delta)V_1$. In such a case, if 1 chooses to complete with probability less than 1, 2 has an incentive to increase her contribution by $\epsilon \approx 0$ in the previous stage to make sure 1 completes: in such a case she guarantees a payoff of δV_2 instead of at most $\delta^2 V_2$, and 2 has an incentive to reduce ϵ infinitely. In this case, player 2's best response is not well defined and therefore player 1 would choose to complete when $M_t = (1-\delta)V_1$ in equilibrium.

A similar proof works for 2, hence the claim is valid. ■

Notice that this claim is valid for both $V_1 = V_2$ and $V_1 \neq V_2$.

Claim 4.2. *There is no equilibrium in which player i completes the project when $M_t < (1-\delta)V_i$ in stage t.*

While Claim 4.1 tells us that player i is willing to complete as long as $M_t \leq (1-\delta)V_i$, it is conceivable that if player i completes in stage t in equilibrium, in the previous stage player j contributes as less as possible, just enough to guarantee that M_t is small enough so that player 1 still completes in stage t.

Proof.

Take $i = 1$ for example. Suppose that 1 completes the project with $x_t < (1-\delta)V_1$ in stage t. If $y_{t-1} > 0$, then in stage $t-1$, 2 has an incentive to reduce her contribution by Claim 4.1: 1 would complete as long as $M_t \leq (1-\delta)V_1$, and 2 earns a payoff $\delta V_2 - y_{t-1}$ as long as 1 completes. If $y_{t-1} = 0$, then in stage $t-2$, 1 has an incentive to increase his contribution by M_t as it is better for him to complete the project in stage $t-2$ instead of in stage t without 2's contribution. Hence the supposition fails; and a similar proof works for $i = 2$. ■

We now start the formal analysis of the finite horizon alternating contribution model we described above. We study three cases in three sub-sections. In Section 4.2.1, we show that there are multiple equilibria in this model when $V_1 = V_2$; in Section 4.2.2, we show that when $V_1 < V_2$, the game has a unique SPE involving an early completion; and in Section 4.2.3, we show that when $V_1 > V_2$, the unique SPE involves completion either at the beginning or at the deadline.

4.2.1 $V_1 = V_2 = V$

In this sub-section, we consider cases in which valuations are symmetric. MM 2000 considers the limiting case of $\delta \to 1$ and reach the conclusion that the final contributor gets a payoff of 0 while the other contributor gets a payoff of $2V - W$. Here we explore a more general case by relaxing the constraint on δ.

Players with the same valuation in a contribution game is a special case and it could lead to interesting results. As shown in AP 1991, in an infinite horizon contribution game, homogeneous valuations lead to gradualism in equilibrium. The simple intuition is that knowing everyone to be the same, the contributors naturally free-ride on each other: when they know that if one should do something, whoever comes after him should do the same; so no one wants to be a hero. Instead, the smart thing is to make the minimal effort to show good faith in exchange of the next player doing the same thing, just enough to keep the process going. In the finite horizon contribution game we bring up here, things are slightly different. Even if the next contributor has the same valuation as the current one has, he or she is in a different position because he or she is closer to the deadline. However, homogeneous valuations in finite horizon games could generate another issue: in some circumstances, one contributor might be completely indifferent between completing the project immediately and inducing his opponent to complete in the next stage. Unlike what happened in Claim 4.1, previous history would not affect the indifference choice here. And this directly leads to multiple equilibria.

Take a three-stage model for example.

In the final stage of the game, 2 would contribute $y_3 = M_3$ as long as $M_3 \leq V$. Notice that 2 is indifferent between completing or not when $M_3 = V$. If 2 completes the project with probability less than one then 1 would increase his contribution slightly in the previous stage to make sure 2 completes with certainty in equilibrium.

Hence, in the second stage of the game, when $(1-\delta)V \leq M_2 \leq V$, 1 is indifferent between completing the project by himself by contributing $x_2 = M_2$ and letting 2 complete the project in the third stage by contributing $x_2 = M_2 - (1-\delta)V$. (When $M_2 > V$, 1 couldn't possibly complete by himself, he contributes $x_2 = M_2 - V$ to induce 2 to complete in the next stage; when $M_2 < (1-\delta)V$, 1 completes by himself by Claim 4.1.) If 1 makes the previous choice, 2 earns a payoff of V; and if 1 makes the latter choice, 2 earns a payoff of $\delta V - \delta(1-\delta)V$, viz., $\delta^2 V$. Notice that *player 2's choice in the first stage does not affect player 1's indifference choice in the second stage.* 2's choice in the first stage would only change M_2, but for 1, $V - M_2 = \delta V - (M_2 - (1-\delta)V)$, the change of M_2 happens on both side of the equality and he remains indifferent.

Suppose that in the second stage, 1 chooses to complete with probability p when he is indifferent. Then in the first stage of the game, 2 expects her payoff to be $p\delta V + (1-p)\delta^3 V$ if she induces M_2 between $(1-\delta)V$ and V. Hence, 1's indifference choice p in the second stage causes 2's choices to differ in the first stage.

2 has several options in the first stage of the game: a. contributes $y_1 > W - (1-\delta)V$, inducing $M_2 < (1-\delta)V$ to make sure 1 completes in the second stage of the game; b. contributes $y_1 < W - V$, inducing $M_2 > V$ to make sure 1 does not complete in the second stage of the game; c. contributes $W - V \leq y_1 \leq W - (1-\delta)V$ and let player 1's indifference choice p determine her payoff.

Option a generates a negative payoff of $\delta V - y_1 < \delta V - (W - (1-\delta)V) = V - W < 0$. Option b generates a payoff of $0 - y_1 \leq 0$ (1 contributes $x_2 = M - V$, 2 contributes $y_3 = V$), reaching the maximum of 0 when $y_1 = 0$. Depending on p, option c generates a payoff of $p\delta V + (1-p)\delta^3 V - y_1 \leq (1 + p\delta + (1-p)\delta^3)V - W$, reaching the maximum when $y_1 = W - V$. Notice that $(1 + p\delta + (1-p)\delta^3)V - W \in [(1+\delta^3)V - W,\ (1+\delta)V - W]$ as $p \in [0,\ 1]$, and $(1+\delta^3)V - W$ might be larger or smaller than 0. When $(1+\delta^3)V < W < (1+\delta)V$, compare option b and c,

we can easily deduce that with different p, there are different equilibria. When $(1+p\delta+(1-p)\delta^3)V-W<0$, viz., $p<\frac{W-(1+\delta^3)V}{\delta-\delta^3}$, 2 chooses option b; otherwise, 2 chooses option c. However, when $p=\frac{W-(1+\delta^3)V}{\delta-\delta^3}$, 2 is in another indifference situation. It is conceivable that if we're considering an even more complicated model with $T>3$, 2's indifference choice here would cause a more severe multiplicity in previous stages.

In a general model with T large enough, we have following proposition.

Proposition 4.1. *When $V_1=V_2$, there are multiple equilibria.*

We do not provide a detailed proof for Proposition 4.1 in this section. Later in Section 4.2.2 and 4.2.3, we will show that there is a unique equilibrium when $V_1<V_2$ and when $V_1>V_2$. The proof in each section applies also to the case of $V_1=V_2$ when we choose the players' indifference choices accordingly, which means that there are at least two sub-game perfect equilibria when $V_1=V_2$ and it thereby proves the existence and the multiplicity in Proposition 4.1.

For now, we focus on the origin of the multiplicity in this model.

As shown in the simple three-stage model, there can be many indifference choices in this model and they cause a severe multiplicity equilibria issue. When players are indifferent between several options, they may either complete the project right now by themselves, or leave the work to be done in the future. Even the same player could make different choices at different nodes of the game.

We focus on the more "plausible" equilibria: the same player makes consistent choices throughout the whole game when he/she is indifferent. In Sections 4.2.2 and 4.2.3, by considering the possible perturbations of players' valuations of the project, limited amount of equilibria remain. And the surviving equilibria fit our definition of the "plausible" equilibria: one of the player always accelerates the completion and the other one always delays it. Following is a more detailed description.

The core of the indifference choice is that whenever player i could complete by

himself by contributing M_t, he has an alternative choice of contributing $M_t-(1-\delta)V$ to let his opponent complete in the next stage. Both options are equivalent for him: $V - M_t = \delta V - (M_t - (1-\delta)V)$.

However, things change when $V_1 \neq V_2$. Player i is better off completing by himself if $V_i - M_t > \delta V_i - (M_t - (1-\delta)V_j)$, viz., $V_i > V_j$; contrarily he is better off letting player j complete in the next stage if $V_i < V_j$. Intuitively, the high valuing player has a stronger incentive to accelerate completion. This *value effect*, combining with the *deadline effect,* determines the completion pattern of the contribution games in this chapter. The *value effect* induces the high valuing player to contribute more, and the *deadline effect* makes the final contributor contribute more.

If players have different valuations of the project then they make consistent choices about accelerating or delaying completion of the project. In Sections 4.2.2 ($V_1 < V_2$) and 4.2.3 ($V_1 > V_2$) we show that there is a unique equilibrium in each model.

4.2.2 $V_1 < V_2$

In this sub-section, we describe a play in which the last contributor values the project most. In this case, the deadline and value effect work in the same direction, each favoring 1. As the final contributor, 2 has a stronger incentive to prevent the game from entering the deadline stage; and as the high valuing player, 2 loses more compared to 1 with every stage passed. With no bargaining chips in hand, 2 has to make peach with the unfair distribution of efforts from the very beginning.

Proposition 4.2. *If $V_1 < V_2$ then there is a unique sub-game perfect equilibrium involving an early completion.*

Proof.

We prove Proposition 4.2 by induction.

Inductive Hypothesis 4.1. *In stage $t-1$, when it is 1's turn to move, he contributes $x_{t-1} = M_{t-1}$ if $M_{t-1} \leq (1-\delta)V_1$, and $x_{t-1} = max\{M_{t-1} - V_2,\ 0\}$*

if $M_{t-1} > (1-\delta)V_1$*; in stage* t*, when it is 2's turn to move, she contributes* $y_t = M_t$ *if* $M_t \leq V_2$*, and* $y_t = 0$ *if* $M_t > V_2$*.*

We first show that the strategies in IH 4.1 form a unique subgame equilibrium when $t = T$:

- In stage T, 2 completes the project as long as $M_T \leq V_2$.[2]
- In stage $T-1$, 1 has several options: complete by himself, induce 2 to complete in the next stage or let the project fail (if possible). While the project is worth-completing ($W < \delta V_1 + V_2$), and $M_{T-1} \leq W$, it is easy to see that the third option is never favored by 1. Hence, anticipating 2's response in stage T, 1 chooses between completing by himself and contributing just enough to make sure that $M_T \leq V_2$. Comparing these two options, 1 contributes $x_{T-1} = max\{M_{T-1} - V_2,\ 0\}$ when $M_{T-1} > (1-\delta)V_1$, and contributes $x_{T-1} = M_{T-1}$ when $M_{T-1} \leq (1-\delta)V_1$ by Claim 4.1.

Now, suppose that the strategies in Proposition 4.2 form a unique subgame equilibrium for every $t' > t$. We show that they form a subgame equilibrium in stage t. We do this in two steps:

- It is 2's turn to move in stage t ($T - t$ is even):

  If she does not complete in stage t and $M_{t+1} > V_2$ then 1 contributes $y_{t+1} = M_{t+1} - V_2$ in stage $t+1$; 2 completes the project in stage $t+2$ and earns 0. If she does not complete in stage t and $M_{t+1} \leq V_2$ then 1 contributes $y_{t+1} = 0$ in stage $t+1$; 2 completes the project in stage $t+2$ and earns $\delta^2(V_2 - M_{t+1})$. Suppose that $M_t \leq V_2$. If 2 does not complete in stage t then $0 < M_{t+1} \leq M_t \leq V_2$; 1 contributes nothing in stage $t+1$, so she contributes $y_t = M_t - M_{t+1} < M_t$ and $y_{t+2} = V_2 - M_{t+1}$, earning $\delta^2(V_2 - M_{t+1}) - (M_t - M_{t+1})$, which is strictly worse than completing in stage t and earning $V_2 - M_t$: $\delta^2(V_2 - M_{t+1}) - (M_t - M_{t+1}) < \delta^2(V_2 - M_t) < V_2 - M_t$. Therefore, 2 contributes $y_t = M_t$ and completes in stage t.

  Now suppose that $M_t > V_2$. If 2 contributes $y_t \geq M_t - V_2$, viz., $M_{t+1} = M_t -$

[2] 2 is indifferent if $M_T = T$. If 2 did not complete for sure then 1 would increase his contribution slightly in stage $T-1$; so 2 completes for sure in equilibrium.

$y_t \leq V_2$ then, like the case of $M_t \leq V_2$, she earns $\delta^2(V_2 - M_{t+1}) - (M_t - M_{t+1})$, which is worse than completing in stage t and earning $V_2 - M_t < 0$. If 2 contributes $y_t < M_t - V_2$, viz., $M_{t+1} = M_t - y_t > V_2$ then $x_{t+1} = M_{t+1} - V_2$; so 2 completes in stage $t+2$ with $y_{t+2} = V_2$. Clearly, in both cases 2 earns a negative payoff, so she is better off making no contribution in stage t and earning zero.

To sum up, 2 contributes $y_t = M_t$ if $M_t \leq V_2$ and $y = 0$ if $M_t > V_2$.

- It is 1's turn to move in stage $t-1$:

  If $M_t > V_2$ then $M_t > V_1$, so 1 can't complete by himself. As 2 completes as long as $M_{t+1} \leq V_2$, it makes no sense for 1 to delay the process of reducing M_t to V_2; so he contributes $x_t = M_t - V_2$ when $M_t > V_2$. When $(1-\delta)V_1 < M_t \leq V_2$, 2 completes in the next stage irrespective of 1's move in stage t; so he contributes $x_t = 0$, which is a better option than completing by himself: $\delta V_1 > V_1 - M_t$. When $M_t \leq (1-\delta)V_1$, he contributes $x_t = M_t$ by Claim 4.1.

  To sum up, 1 contributes $x_t = M_t$ if $M_t \leq (1-\delta)V_1$, and $x_t = max\{M_t - V_2,\ 0\}$ if $M_t > (1-\delta)V_1$.

In conclusion, the strategies in Proposition 4.2 form a unique subgame equilibrium in stage $T-1$ and T, and form a unique subgame equilibrium in stage t as long as they form a subgame equilibrium in stage $t' > t$, proving the uniqueness of SPE in Proposition 4.2.

With $M_1 = W$, and by Assumption 4.1: $V_2 < W < V_1 + \delta V_2$, in the first stage of the game, if it is 1 who makes the first contribution, he contributes $x_1 = W - V_2$ to induce 2 to complete in the second stage by contributing $y_2 = V_2$; if it is 2 who makes the first contribution, she contributes $y_1 = 0$, and the rest is the same as if 1 is the first contributor: $x_2 = W - V_2$, $y_3 = V_2$. The project is completed in an early stage of the game. This ends the proof of Proposition 4.2. ■

From Proposition 4.2 we can easily see that player 2 has a higher threshold ($M = V_2$) than player 1 ($M = (1-\delta)V_1$) in completing the project, which means

that she is more eager to get the work done soon. When the main beneficiary is given too important a role in a mutual project, it is conceivable that she will strive too much while others simply play along, and this is exactly what happens in this case. While the *deadline effect* and *value effect* are both working in the same direction, giving 2 a much stronger incentive in completing the project early than her opponent, 1 could easily exploit 2 by working at the minimum level, just enough to make sure that this joint project is still appealing for 2. From the good side, in spite of 2 getting exploited, the project is completed soon because 2 has accepted her inferior position in this relationship from the start and she would rather be exploited sooner than later.

4.2.3 $V_1 > V_2$

In this sub-section, we describe a play when the last contributor values the project least. In this case, the *deadline effect* and *value effect* work in opposite directions. As the final contributor, 2 has a stronger incentive in preventing the game from entering the deadline stage, as the high valuing player, 1 loses more than 2 with every stage passed. Now that both players have something to rely on, 2 is able to make it more pricey for 1 to generate an early completion. This puts 1 in a trade-off: is it better to contribute more in exchange for 2's cooperation in an early stage or is it better to wait until the last stage, in which 2 is powerless? The latter choice leads to a delay.

Proposition 4.3. *If $V_1 > V_2$ and $\delta > \sqrt[T-t]{\frac{V_1-V_2}{V_1}}$ then there is a unique sub-game perfect equilibrium involving either an early completion or a deadline completion.*

Proof.

Road-map of the proof: In Claim 4.3, we first show that there is no equilibrium in which 2 completes the project before the deadline. Then by Lemma 4.3, Inductive Hypothesis 4.2 and Claim 4.4, we show that 1 and 2 adopt threshold strategies in equilibrium: 1 has two thresholds, he completes immediately when M is in the lower range, induces a completion *in the stage after next* when M is the middle

range, and waits for the deadline completion when M is the upper range; 2 also has two thresholds, she completes immediately when M is in the lower range, induces completion in the next stage when M is in the middle range, and waits for the deadline completion when M is in the upper range.

In detail, in Lemma 4.3, we show that 1 does not complete when M is in the middle or upper range. (Notice that here the threshold is different from the one in Claim 4.1.) In IH 4.2, we show that 1 completes when M is in the lower range and how 2's choices varies with respect to M. (Notice that 2's lower threshold is not different with the one in Claim 4.1.) In Claim 4.4, we characterize 1's choices when M is in the middle or the upper range. With the full description of choices, we can see that in this model, the completion pattern has a threshold property: either an early completion occurs (when W is small), or a late completion occurs (when W is large).

As mentioned in the Road-map, we first show that 1 does not complete when M is large enough. To do this we first prove Lemma 4.1 and Claim 4.3.

Lemma 4.1. *2 never completes the project with $y_t = M_t > (1-\delta)V_2$ before the deadline (at stage $t < T$).*

Proof.

We prove Lemma 4.1 by contradiction. If 2 completes the project with $y = M > (1-\delta)V_2$ before the deadline then she earns $V_2 - M$. If 2 contributes $y = max\{0,\ M-(1-\delta)V_1\}$ instead of $y = M$ then 1 would complete the project in the next stage by Claim 4.1, earning $\delta V_2 - max\{0,\ M-(1-\delta)V_1\}$. As $V_2 - M < \delta V_2 - max\{0,\ M-(1-\delta)V_1\}$, 2 could profitably deviate. ■

Lemma 4.1 is the direct exhibition of the *value effect* in this model. While 1 is now the high valuing player, whenever 2 wishes to complete the project, she has the option of letting 1 complete it soon afterwards and this option is better for her. The only exception occurs when the game enters the final stage, where 2 is the final contributor and there is no one else to free ride on anymore.

Claim 4.3. *In equilibrium, 2 never completes the project before the deadline (at stage* $t < T$*).*

Proof.

Claim 4.3 follows from Claim 4.2 and Lemma 4.1. By Claim 4.2 we know that 2 does not complete in equilibrium when $M_t < (1-\delta)V_2$ at $t < T$; by Lemma 4.1 we know that 2 does not complete in equilibrium when $M_t > (1-\delta)V_2$ at $t < T$. Suppose that there is an equilibrium in which 2 completes with $M_t = (1-\delta)V_2$ at $t < T$. In stage $t-1$, it is better for 1 to make an extra contribution equal to $(1-\delta)V_2$ to accelerate completion by one stage and earn an extra payoff of $(1-\delta)V_1$. Hence the supposition fails, and there is no equilibrium in which 2 completes the project at $t < T$.■

Claim 4.3 implies that 1 could only induce 2 to complete the project at the deadline. Eliminating the possibility of 2 completing the project in $t < T$, we are left with two types of completion patterns: in equilibrium, either 1 completes the project before the deadline, or 2 completes the project at the deadline.

The project always gets completed in this model, because even if there is no previous contribution, Assumption 4.1 guarantees that in stage $T-1$, 1 always has an incentive to induce 2 to complete at the deadline, if he does not complete himself. Meanwhile, in equilibrium, if completion occurs at the deadline then there are no previous contributions: neither player makes any contributions until the last two stages, because whenever a positive contribution is made, it is better to delay it if it does not change the timing of the completion. If the completion time can't be brought forward, it makes no sense to make any early contributions. In equilibrium, 1 contributes $x_{T-1} = W - V_2$ and 2 completes with $y_T = V_2$.

Lemma 4.2. *If* $\delta > \sqrt[T-t]{\frac{V_1-V_2}{V_1}}$ *then 1 does not complete the project in stage* t *when* $M_t > (1-\delta^{T-t})V_1$.

Proof.

Through contributing $x_{t<T-1} = 0$ and $x_{T-1} = max\{M_{T-1} - V_2,\ 0\}$, 1 earns $(\delta V_1 - max\{M_{T-1} - V_2,\ 0\})\delta^{T-t-1}$. As $M_{T-1} \leq M_t$, 1 could guarantee a payoff of at least $(\delta V_1 - max\{M_t - V_2,\ 0\})\delta^{T-t-1}$ in stage t. Therefore, he would not complete in stage t if

$$V_1 - M < (\delta V_1 - max\{M - V_2,\ 0\})\delta^{T-t-1} \tag{4.1}$$

When $M > V_2$, Equation 4.1 becomes

$$V_1 - M < (\delta V_1 + V_2 - M)\delta^{T-t-1}$$

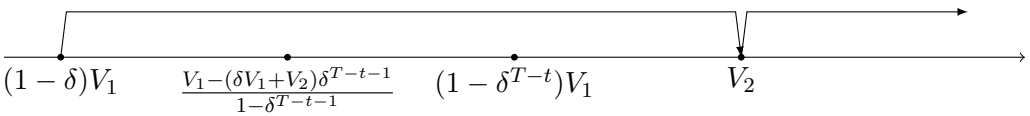
$$\iff M > \frac{V_1 - (\delta V_1 + V_2)\delta^{T-t-1}}{1 - \delta^{T-t-1}}.$$

When $(1-\delta)V_1 < M < V_2$, Equation 4.1 becomes

$$V_1 - M < (\delta V_1)\delta^{T-t-1}$$

$$\iff M > (1 - \delta^{T-t})V_1.$$

If $\delta > \sqrt[T-t]{\frac{V_1 - V_2}{V_1}}$, then $V_2 > \frac{V_1 - (\delta V_1 + V_2)\delta^{T-t-1}}{1-\delta^{T-t-1}}$ and $V_2 > (1-\delta^{T-t})V_1$; so Equation 4.1 becomes $M > (1-\delta^{T-t})V_1$. Therefore, if $\delta > \sqrt[T-t]{\frac{V_1 - V_2}{V_1}}$ then 1 does not complete the project in stage t when $M_t > (1 - \delta^{T-t})V_1$. ■



Lemma 4.1 shows that 1 is better off delaying completion when δ and M are large enough. It is an exhibition of the *deadline effect* and the intuition is straightforward: when the discount factor is high, delaying is less costly, 1's advantage around the deadline is more obvious; when the remaining contributions to completion is large, as established in Claim 4.3, 1 would have to complete by himself unless he is willing to delay; it therefore makes more sense to seek his opponent's collaboration instead.

$(1-\delta^{T-t})V_1$ is the threshold remaining contributions for 1: when M is above it, he delays completion; when M is below it, he completes immediately. The previous part is shown in Lemma 4.2 and the latter part would be shown in Inductive Hypothesis 4.2 below. Meanwhile, in IH 4.2, we show that 2 also has a threshold remaining contribution of $(1-\delta^{T-t})V_1+\delta V_2$: when M is above it, she delays completion; when M is below it, she induces completion in the next stage by contributing up to δV_2. Notice that 2 gets her payoff of V_2 in the next stage; so this threshold M indicates that she is willing to induce completion as long as it generates a non-negative payoff for her. This choice by 2 originates from the *deadline effect* again: anticipating an inferior position in the final stage, 2 is willing to make a large offer to avoid falling into the quagmire.

We now show that 1 completes when M is small and 2's choices satisfy a threshold property.

Inductive Hypothesis 4.2. For any stage $t < T$, if $\delta > \sqrt[T-t]{\frac{V_1-V_2}{V_1}}$, then 1 completes the project if $M_t \leq (1-\delta^{T-t})V_1$; 2 contributes $y_{t-1} = max\{M_{t-1} - (1-\delta^{T-t})V_1,\ 0\}$ to induce completion in stage t if $(1-\delta)V_2 < M_{t-1} \leq (1-\delta^{T-t})V_1 + \delta V_2$ and contributes $y_{t-1} = 0$ if $M_{t-1} > (1-\delta^{T-t})V_1 + \delta V_2$.

Proof.

We first show that IH 4.2 holds for the last stages, and then prove that it holds for earlier stages if it holds for later stages. Notice that IH 4.2 is written in terms of play in two stages; the deduction is also done in terms of two-stage play.

We first show that IH 4.2 is valid for $t = T-1$:

- In stage $T-1$, when $M_{T-1} \leq (1-\delta)V_1$, Claim 4.1 implies that 1 completes by contributing $x_{T-1} = M_{T-1}$ and earns $V_1 - M_{T-1}$, which satisfies IH 4.2. When $M_{T-1} > (1-\delta)V_1$, 1 contributes $x_{T-1} = max\{M_{T-1} - V_2,\ 0\}$ and induces completion in stage T. As $\delta V_1 - max\{M_{T-1} - V_2,\ 0\} \geq V_1 - M_{T-1}$, it is better to induce completion rather than completing himself.

- In stage $T-2$, when $M_{T-2} > (1-\delta)V_2$, 2 has two options: induce completion in stage $T-1$ or complete the project herself in stage T.

  The second option is not valid when $M_{T-2} \leq (1-\delta)V_1$ because $M_{T-1} \leq M_{T-2} \leq (1-\delta)V_1$, so 1 would complete in $T-1$ for sure. Hence 2 induces completion in stage $T-1$ by contributing $y_{T-2} = 0$.

  When $M_{T-2} > (1-\delta)V_1$, Claim 4.3 implies that 2 chooses between two options. In the first option, she contributes $y_{T-2} \geq M_{T-2} - (1-\delta)V_1$, and is then best off contributing $y_{T-2} = M_{T-2} - (1-\delta)V_1$. In the second option, 2 contributes $y_{T-2} = M_{T-2} - M_{T-1} \geq 0$, 1 then contributes $x_{T-1} = max\{M_{T-1} - V_2,\ 0\}$ and induces 2 to contribute $y_T = M_T = M_{T-1} - max\{M_{T-1} - V_2,\ 0\} = min\{V_2,\ M_{T-1}\}$. Therefore, 2 earns $\delta^2(V_2 - min\{V_2,\ M_{T-1}\}) - (M_{T-2} - M_{T-1})$, which is strictly increasing in M_{T-1}; so she is best off contributing $y_{T-2} = 0$.

  Comparing the payoffs from these two options, the first option generates a payoff of $\delta V_2 - (M_{T-2} - (1-\delta)V_1)$ and the second option generates a payoff of $\delta^2(V_2 - min\{V_2,\ M_{T-2}\})$. $\delta V_2 - (M_{T-2} - (1-\delta)V_1) \geq \delta^2(V_2 - min\{V_2,\ M_{T-2}\})$ becomes $\delta V_2 - (M_{T-2} - (1-\delta)V_1) \geq \delta^2(V_2 - M_{T-2})$, viz., $M_{T-2} \leq \frac{V_1 + \delta V_2}{1+\delta}$ when $M_{T-2} < V_2$; the inequality becomes $\delta V_2 - (M_{T-2} - (1-\delta)V_1) \geq 0$, viz., $M_{T-2} \leq (1-\delta)V_1 + \delta V_2$ when $M_{T-2} > V_2$. As $\frac{V_1 + \delta V_2}{1+\delta} > V_2$ and $(1-\delta)V_1 + \delta V_2 > V_2$, 2 induces completion in stage $T-1$ by contributing $y_{T-2} = M_{T-2} - (1-\delta)V_1$ when $M_{T-2} \leq (1-\delta)V_1 + \delta V_2$,[3] and she completes the project by herself in stage T by contributing $y_{T-2} = 0$ when $M_{T-2} > (1-\delta)V_1 + \delta V_2$, which satisfies the description in IH 4.2.

We now show that IH 4.2 holds for stage t if it holds for every stage $t' > t$.

- Let 1 move at stage t.

  Suppose that $M_t \leq (1-\delta^{T-t})V_1$. If 1 generates $M_{t+1} \leq (1-\delta)V_2$ and 2 completes in $t+1$ then 1 is better off making an extra contribution of M_{t+1} in stage

[3] Similarly, 2 is indifferent when $M_{T-2} = (1-\delta)V_1 + \delta V_2$. If 2 did not induce completion for sure then 1 would increase his contribution slightly in stage $T-3$; so 2 induces completion for sure in equilibrium.

t to accelerate completion by one stage: $(1-\delta)V_1 > (1-\delta)V_2 \geq M_{t+1}$. If 1 generates $M_{t+1} > (1-\delta)V_2$ ($x_t = M_t - M_{t+1}$) then $(1-\delta)V_2 < M_{t+1} \leq M_t \leq (1-\delta^{T-t})V_1 < (1-\delta^{T-t-2})V_1 + \delta V_2$,[4] and IH 4.2 implies that 2 contributes $y_{t+1} = max\{M_{t+1} - (1-\delta^{T-t-2})V_1,\ 0\}$ to induce completion in stage $t+2$: that is, $x_{t+2} = M_{t+1} - max\{M_{t+1} - (1-\delta^{T-t-2})V_1,\ 0\} = min\{(1-\delta^{T-t-2})V_1,\ M_{t+1}\}$. It is better for 1 to complete immediately by contributing $x_t = M_t$:

$$V_1 - M_t \geq \delta^2(V_1 - min\{(1-\delta^{T-t-2})V_1,\ M_{t+1}\}) - (M_t - M_{t+1})$$

$$\iff V_1 - M_t \geq \delta^2 max\{\delta^{T-t-2}V_1,\ V_1 - M_{t+1}\} - (M_t - M_{t+1})$$

$$\iff V_1 - M_{t+1} \geq \delta^2 max\{\delta^{T-t-2}V_1,\ V_1 - M_{t+1}\}$$

$$\iff V_1 - M_{t+1} \geq max\{\delta^{T-t}V_1,\ \delta^2(V_1 - M_{t+1})\}$$

$$\iff V_1 - M_{t+1} \geq \delta^{T-t}V_1$$

$$\iff M_t \leq (1-\delta^{T-t})V_1,$$

Therefore, 1 contributes $x_t = M_t$ to complete the project.[5] And this satisfies the description in IH 4.2.

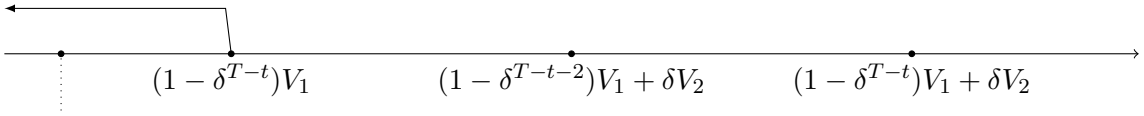
- Let 2 move at stage $t-1$.

  If $M_{t-1} \leq (1-\delta^{T-t-2})V_1 + \delta V_2$ then IH 4.2 implies that, if 2 does not induce completion in stage $t-1$ then she would induce completion in stage $t+1$. Consequently 1 has no incentive to contribute $x_t > 0$: it would be better for 2 to induce completion in stage $t-1$ by contributing $y_{t-1} = max\{M_{t-1} - (1-\delta^{T-t})V_1,\ 0\}$.

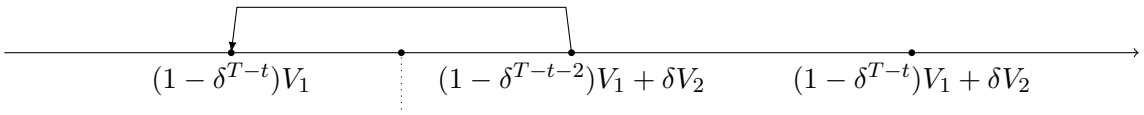
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[4] $\delta > \sqrt[T-t]{\frac{V_1-V_2}{V_2}} \iff (1+\delta^{T-t})V_2 > V_1 \iff (\delta^{T-t-2} - \delta^{T-t})V_1 < (\delta^{T-t-2} - \delta^{T-t})(1+\delta^{T-t})V_2$. $t \leq T-3 \iff \delta^{T-t} \leq \delta^3 \iff (\delta^{T-t-2} - \delta^{T-t})(1+\delta^{T-t}) < \delta(1-\delta^2)(1+\delta^3) < \delta(1-\delta^4) < \delta$. Hence, $(\delta^{T-t-2} - \delta^{T-t})V_1 < \delta V_2$, viz., $(1-\delta^{T-t})V_1 < (1-\delta^{T-t-2})V_1 + \delta V_2$.

[5] Similarly, 1 must complete for sure in equilibrium, else 2 would reduce M_t to induce completion in the previous stage.



When $M_{t-1} < (1-\delta^{T-t})V_1$, 1 will complete in the next stage irrespective of y_{t-1}. So it is best for 2 to contribute 0.



When $(1-\delta^{T-t})V_1 < M_{t-1} < (1-\delta^{T-t-2})V_1+\delta V_2$, 1 contributes nothing if $M_t > (1-\delta^{T-t})V_1$, so it is best for 2 to generate $M_t = (1-\delta^{T-t})V_1$.

- If $M_{t-1} > (1-\delta^{T-t-2})V_1+\delta V_2$ then we show that 2 always earns a non-positive payoff if she does not induce completion in stage t.

  Let 2 generate $M_t > (1-\delta^{T-t-2})V_1+\delta V_2$. Claim 4.3 implies that there are two possible types of completion: 1 completes the project in stage s such that $t < s < T$ or 2 completes the project in stage T. In either case, 2 earns a non-positive payoff. We first show that in the previous case (completion in stage s) 2 earns a non-positive payoff using the following Lemma.

Lemma 4.3. *If IH 4.2 holds for stages $t' > t$, $M_t > (1-\delta^{T-t-2})V_1+\delta V_2$ and 1 completes the project in stage $s > t$, then he contributes $x_{s-2} = M_t-(1-\delta^{T-s})V_1-\delta V_2$ and $x_s = (1-\delta^{T-s})V_1$, where 2 contributes $y_{s-1} = \delta V_2$.*

Proof.

If 1 completes in stage $s > t$ then $x_s = (1-\delta^{T-s})V_1$ and $y_{s-1} > 0$. If $y_{s-1} = 0$ then 1 would increase x_{s-2} by $(1-\delta^{T-s})V_1$ and complete the project in stage $s-2$. With $y_{s-1} > 0$, if $x_s < (1-\delta^{T-s})V_1$ then 2 would reduce y_{s-1} as in Claim 4.1 and there is no equilibrium; if $x_s > (1-\delta^{T-s})V_1$ then 1 would not complete by Lemma 4.2.

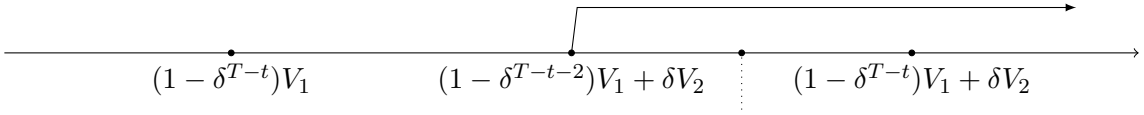
Furthermore, $x_{s-2} = M_t-(1-\delta^{T-s})V_1-\delta V_2$ and $y_{s-1} = \delta V_2$. 2 never contributes more than δV_2 as she does not complete by herself: the maximal contribution 1 could induce 2 to make is δV_2. IH 4.2 implies that 1 can induce 2 to contribute δV_2 by contributing the remaining part $M_t-(1-\delta^{T-s})V_1-\delta V_2$. It is clearly best for 1

to make this contribution as late as possible, given the fixed completion time. He therefore makes all his entire contribution in stage $s-2$ and induces 2 to contribute the rest in stage $s-1$. ■

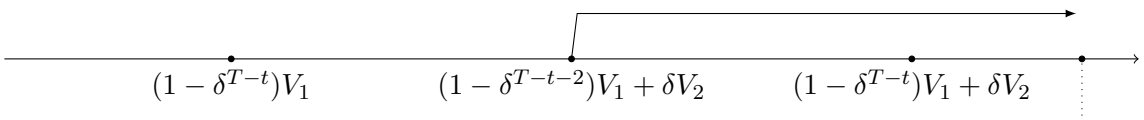
We now show that 2 earns a non-positive payoff if the project is completed in stage T. When discussing Claim 4.3 we mentioned that if 2 completes the project in stage T then 1 only contributes $x_{T-1} = W - V_2$ in stage $T-1$, and 2 only contributes $y_T = V_2$ in stage T. Notice that anticipating 2 to complete the project in stage T as long as $M_T \leq V_2$, 1 has no incentive to make any earlier contribution; and anticipating 1's behavior, neither does 2.

To sum up, whether 1 or 2 completes, 2 earns a non-positive payoff of $-y_{t-1}$ by generating $M_t > (1-\delta^{T-t-2})V_1 + \delta V_2$. Hence she contributes nothing if she does not induce completion in the next stage.

Therefore, if $M_{t-1} > (1-\delta^{T-t-2})V_1 + \delta V_2$, 2 earns zero payoff if she does not induce completion, and earns $\delta V_2 - (M_{t-1} - (1-\delta^{T-t})V_1)$ if she induces completion. Hence 2 induces completion in stage $t-1$ by contributing $y_{t-1} = M_{t-1} - (1-\delta^{T-t})V_1$ if $M_{t-1} \leq (1-\delta^{T-t})V_1 + \delta V_2$ [6] and contributes nothing if $M_{t-1} > (1-\delta^{T-t})V_1 + \delta V_2$. And this satisfies the description in IH 4.2.



When $(1-\delta^{T-t-2})V_1 + \delta V_2 < M_{t-1} < (1-\delta^{T-t})V_1 + \delta V_2$,
it is better for 2 to induce completion.



When $M_{t-1} > (1-\delta^{T-t})V_1 + \delta V_2$, it is better for 2 to delay.

To sum up, IH 4.2 holds for every stage $t < T$. This ends the proof of IH 4.2.■

[6] Similarly, 2 must induce completion for sure in equilibrium, else 1 would reduce M_{t-1} in the previous stage.

While we've already shown that each player completes when M is small enough by Claim 4.1, we pin down 1's lower bound of completing the project in IH 4.2 and it is higher than suggested in Claim 4.1. Also notice that 1's threshold M decreases over time; as that closer to the deadline it is, a better idea it seems to be to delay the project to the final stage due to the *deadline effect*. Correspondingly it is more likely for him to choose to delay.

Player 2, as the final contributor, suffers from a *deadline effect* throughout the whole game: every time she makes a choice, the primary goal is to induce completion to avoid delay, even at the cost of earning a zero payoff; because any alternative choice would only generate an even worse result for her. Therefore 2 adopts a simple threshold strategy: induce completion as long as it generates a non-negative payoff and contribute nothing otherwise.

By Lemma 4.2 and IH 4.2, we already know 1's choice when M is in the lower range and 2's choice in all ranges. The last thing missing is 1's choice when M is in the middle and high range. We solve this with following Claim.

Claim 4.4. *If* $\delta > \sqrt[T-t]{\frac{V_1-V_2}{V_1}}$ *and* $M_t > (1-\delta^{T-t})V_1$, *then 1 contributes* $x_t = max\{M_t - (1-\delta^{T-t-2})V_1 - \delta V_2,\ 0\}$ *and completes the project in stage* $t+2$ *if* $M_t \leq \frac{(1-\delta^{T-t-2})(V_1+\delta V_2)}{1-\delta^{T-t-1}}$ *and contributes* $x_t = 0$ *to delay completion till the final stage if* $M_t > \frac{(1-\delta^{T-t-2})(V_1+\delta V_2)}{1-\delta^{T-t-1}}$.

Proof.

In stage t, 1 contributes $x_t = 0$ when $M_t \leq (1-\delta^{T-t-2})V_1 + \delta V_2$. By Lemma 4.2 1 does not complete by himself; as $M_{t+1} < M_t \leq (1-\delta^{T-t-2})V_1 + \delta V_2$, IH 4.2 implies that 2 would contribute $y_{t+1} = max\{M_{t+1} - (1-\delta^{T-t-2})V_1,\ 0\}$ to induce completion in stage $t+2$ anyway, irrespective of x_t. Hence it makes no sense for 1 to make any contribution right now.

Now suppose that $M_t > (1-\delta^{T-t-2})V_1 + \delta V_2$. We have already shown in the proof of Lemma 4.3 that 1 would contribute $x_{s-2} = M_t - (1-\delta^{T-s})V_1 - \delta V_2$ and

$x_s = (1-\delta^{T-s})V_1$ to complete in some future stage $s > t$ if he chooses an early completion. We now show that $s = t+2$.

1's payoff from completing in stage $s > t$ is

$$\delta^{s-1}[V_1 - (1-\delta^{T-s})V_1] - \delta^{s-3}[M_t - (1-\delta^{T-s})V_1 - \delta V_2]$$

$$= \delta^{T-1}V_1 - \delta^{s-3}[M_t - (1-\delta^{T-s})V_1 - \delta V_2].$$

As

$$\frac{\delta^{s-3}[M_t - (1-\delta^{T-s})V_1 - \delta V_2]}{\delta^{s-1}[M_t - (1-\delta^{T-s-2})V_1 - \delta V_2]}$$

$$= \frac{M_t - (1-\delta^{T-s})V_1 - \delta V_2}{\delta^2[M_t - (1-\delta^{T-s-2})V_1 - \delta V_2]}$$

$$= \frac{\delta^{T-s}V_1 - (V_1 + \delta V_2 - M_t)}{\delta^{T-s}V_1 - \delta^2(V_1 + \delta V_2 - M_t)} < 1,$$

$\delta^{s-3}(M_t - (1-\delta^{T-s})V_1 - \delta V_2)$ increases with s; so 1's payoff from completing in stage s decreases with s. Therefore, 1 completes in stage $s = t+2$: he contributes $x_t = M_t - (1-\delta^{T-t-2})V_1 - \delta V_2$ and $x_{t+2} = (1-\delta^{T-t-2})V_1$ to complete in stage $t+2$ if he chooses an early completion.

On the other hand, 1 contributes nothing before stage $T-1$, and $x_{T-1} = M_t - V_2$[7] if he chooses a late completion.

Comparing these two options, 1 earns $\delta^2 V_1 - [M_t - (1-\delta^{T-t-2})V_1 - \delta V_2] - \delta^2(1-\delta^{T-t-2})V_1$ if he chooses an early completion, and earns $\delta^{T-t-1}[\delta V_1 - (M_t - V_2)]$ if he chooses a deadline completion.

$$\delta^2 V_1 - [M_t - (1-\delta^{T-t-2})V_1 - \delta V_2] - \delta^2(1-\delta^{T-t-2})V_1 \geq \delta^{T-t-1}[\delta V_1 - (M_t - V_2)]$$

$$\Leftrightarrow \delta^2 V_1 - M_t + (1-\delta^{T-t-2})V_1 + \delta V_2 - \delta^2 V_1 + \delta^{T-t}V_1 \geq \delta^{T-t}V_1 - \delta^{T-t-1}M_t + \delta^{T-t-1}V_2$$

$$\Leftrightarrow (1-\delta^{T-t-2})V_1 + (\delta - \delta^{T-t-1})V_2 \geq (1-\delta^{T-t-1})M_t$$

[7] $(1-\delta^{T-t-2})V_1 + \delta V_2 > V_2$, so $x_{T-1} = M_t - V_2 > 0$.

$$\Leftrightarrow (1-\delta^{T-t-2})(V_1+\delta V_2) \geq (1-\delta^{T-t-1})M_t$$

$$\Leftrightarrow M_t \leq \frac{(1-\delta^{T-t-2})(V_1+\delta V_2)}{1-\delta^{T-t-1}}.$$

Therefore, he contributes $x_t = M_t-(1-\delta^{T-t-2})V_1-\delta V_2$ if $M_t \leq \frac{(1-\delta^{T-t-2})(V_1+\delta V_2)}{1-\delta^{T-t-1}}$, and $x_t = 0$ if $M_t > \frac{(1-\delta^{T-t-2})(V_1+\delta V_2)}{1-\delta^{T-t-1}}$.

Combine the cases where $M_t \leq (1-\delta^{T-t-2})V_1 + \delta V_2$ and where $M_t > (1-\delta^{T-t-2})V_1 + \delta V_2$, we prove Claim 4.4.■

Claim 4.4 implies that when M is too large to be profitable for 1 to complete immediately, 1 makes a choice between early completion and late completion in every stage of the game. As we said before, throughout the whole game, 1 makes trade-offs: the *deadline effect* induces him to delay completion to the final stage, where 2 is in a much weaker position and 1 is able to exploit 2 thoroughly; the *value effect* induces him to facilitate an early completion, because he suffers a greater loss than 2 being the high valuing player. Notice that 2 makes a total contribution equal to δV_2 and V_2 in response to 1's two choices, which means 1 gets less help from 2 if he chooses an early completion and he gets more if he chooses a late one. When M is comparatively small then 1 is willing to contribute more and induce an early completion; when M is comparatively large then 1 has stronger incentive to endure the time cost in exchange for his rival's full contribution.

Combining Lemma 4.2, Inductive Hypothesis 4.2 and Claim 4.4, we now give a full description of players' SPE strategy (when $V_1 > V_2$ and $\delta > \sqrt[T-t]{\frac{V_1-V_2}{V_1}}$):

In stage $t < T-1$, if it is 1's move then he contributes $x_t = M_t$ if $M_t \leq (1-\delta^{T-t})V_1$, contributes $x_t = M_t - (1-\delta^{T-t-2})V_1 - \delta V_2$ if $(1-\delta^{T-t})V_1 < M_t \leq \frac{(1-\delta^{T-t-2})(V_1+\delta V_2)}{1-\delta^{T-t-1}}$ and contributes $x_t = 0$ if $M_t > \frac{(1-\delta^{T-t-2})(V_1+\delta V_2)}{1-\delta^{T-t-1}}$;[8] if it is 2's move then she contributes $y_t = M_t$ if $M_t \leq (1-\delta)V_2$, contributes $y_t = max\{M_t - (1-\delta^{T-t-1})V_1,\ 0\}$ if $(1-\delta)V_2 < M_t \leq (1-\delta^{T-t-1})V_1 + \delta V_2$ and contributes $y_t = 0$ if

[8] $\delta > \sqrt[T-t]{\frac{V_1-V_2}{V_1}} \Rightarrow \delta^{T-t} > \frac{V_1-V_2}{V_1} \Rightarrow \frac{1-\delta^{T-t-1}}{1-\delta^{T-t-2}}\delta^{T-t-1} > \frac{V_1-V_2}{V_1} \Rightarrow (1-\delta^{T-t-2})V_2 > (1-\delta^{T-t-1}-\delta^{T-t-2}+\delta^{2T-2t-2})V_1 \Rightarrow (1-\delta^{T-t-2})V_2 > (\delta^{T-t-3}-\delta^{T-t-1}-\delta^{T-t-2}+\delta^{2T-2t-2})V_1 \Rightarrow (1-\delta^{T-t-2})(V_1+\delta V_2) > (1-\delta^{T-t}-\delta^{T-t-1}+\delta^{2T-2t-1})V_1 \Rightarrow \frac{(1-\delta^{T-t-2})(V_1+\delta V_2)}{1-\delta^{T-t-1}} > (1-\delta^{T-t})V_1$

$M_t > (1-\delta^{T-t-1})V_1 + \delta V_2$.

In stage $T-1$, 1 contributes $y_{T-1} = M_{T-1}$ if $M_{T-1} \leq (1-\delta)V_1$, and contributes $y_{T-1} = max\{M_{T-1} - V_2,\ 0\}$ if $M_{T-1} > (1-\delta)V_1$.

In stage T, 2 contributes $y_T = M_T$ if $M_t \leq V_2$, and contributes $y_T = 0$ if $M_T > V_2$.

In Section 4.2.2 we have already shown that the above behavior constitutes the unique SPE in the final two stages, and through IH 4.2 and Claim 4.4, we know this is the unique SPE for the whole game.

In equilibrium, all choices are made at the very beginning of the game: if they choose an early completion, they complete as early as possible, and if they choose a deadline completion, they never change their minds.

When 1 is the first contributor, he induces completion in the third stage iff $W \leq \frac{(1-\delta^{T-3})(V_1+\delta V_2)}{1-\delta^{T-2}}$;[9] otherwise, he would delay completion till the final stage.[10] In detail: if $W \leq \frac{(1-\delta^{T-3})(V_1+\delta V_2)}{1-\delta^{T-2}}$ then $x_1 = W - (1-\delta^{T-3})V_1 - V_2$, $y_2 = V_2$, $x_3 = (1-\delta^{T-3})V$; if $W > \frac{(1-\delta^{T-3})(V_1+\delta V_2)}{1-\delta^{T-2}}$ then neither contributes before stage $T-1$, and $x_{T-1} = W - V_2$, $y_T = V_2$.

When 2 is the first contributor, she induces completion in the second stage if $W \leq (1-\delta^{T-2})V_1 + \delta V_2$;[11] induces completion in the fourth stage if $(1-\delta^{T-2})V_1 + \delta V_2 < W \leq \frac{(1-\delta^{T-4})(V_1+\delta V_2)}{1-\delta^{T-3}}$; and does not contribute till completion in stage T if $W >$

[9] Comparing $\frac{(1-\delta^{T-3})(V_1+\delta V_2)}{1-\delta^{T-2}}$ and V_1: $\frac{(1-\delta^{T-3})(V_1+\delta V_2)}{1-\delta^{T-2}} > V_1 \Leftrightarrow (1-\delta^{T-3})(V_1+\delta V_2) > (1-\delta^{T-2})V_1 \Leftrightarrow (\delta-\delta^{T-2})V_2 > (\delta^{T-3}-\delta^{T-2})V_1$, $W \leq \frac{(1-\delta^{T-3})(V_1+\delta V_2)}{1-\delta^{T-2}}$ is valid when the difference between V_1 and V_2 is small enough.

[10] Comparing $\frac{(1-\delta^{T-3})(V_1+\delta V_2)}{1-\delta^{T-2}}$ and $\delta V_1 + V_2$:

$$\begin{aligned}
&\frac{(1-\delta^{T-3})(V_1+\delta V_2)}{1-\delta^{T-2}} < \delta V_1 + V_2 \\
\Leftrightarrow\ &(1-\delta^{T-3})(V_1+\delta V_2) < (1-\delta^{T-2})(\delta V_1 + V_2) \\
\Leftrightarrow\ &(1-\delta^{T-3})V_1 + (\delta-\delta^{T-2})V_2 < (\delta-\delta^{T-1})V_1 + (1-\delta^{T-2})V_2 \\
\Leftrightarrow\ &(1-\delta-\delta^{T-3}+\delta^{T-1})V_1 < (1-\delta)V_2 \\
\Leftrightarrow\ &(1-\frac{\delta^{T-3}-\delta^{T-1}}{1-\delta})V_1 < V_2 \\
\Leftrightarrow\ &(1-\delta^{T-3}(1+\delta))V_1 < V_2.
\end{aligned}$$

As $\delta \geq \sqrt[T-3]{\frac{V_1-V_2}{V_1}}$, $(1-\delta^{T-3}(1+\delta))V_1 < (1-\frac{V_1-V_2}{V_1}(1+\delta))V_1 = V_2 - \delta V_1 + \delta V_2 < V_2$, $\frac{(1-\delta^{T-3})(V_1+\delta V_2)}{1-\delta^{T-2}} < \delta V_1 + V_2$, $W > \frac{(1-\delta^{T-3})(V_1+\delta V_2)}{1-\delta^{T-2}}$ is valid.

[11] Comparing $(1-\delta^{T-2})V_1 + \delta V_2$ and V_1,$V_1 < (1-\delta^{T-2})V_1 + \delta V_2 \Leftrightarrow \delta^{T-2}V_1 < \delta V_2 \Leftrightarrow \delta^{T-3}V_1 < V_2 \Leftrightarrow \delta < \sqrt[T-3]{\frac{V_2}{V_1}}$, as $\delta \geq \sqrt[T-3]{\frac{V_1-V_2}{V_1}}$, it is only valid when $\sqrt[T-3]{\frac{V_1-V_2}{V_1}} < \sqrt[T-3]{\frac{V_2}{V_1}}$, viz., $V_1 < 2V_2$.

$\frac{(1-\delta^{T-4})(V_1+\delta V_2)}{1-\delta^{T-3}}$. In detail: if $W \leq (1-\delta^{T-2})V_1 + \delta V_2$ then $y_1 = W - (1-\delta^{T-2})V_1$, $x_2 = (1-\delta^{T-2})V_1$; if $(1-\delta^{T-2})V_1 + \delta V_2 < W \leq \frac{(1-\delta^{T-4})(V_1+\delta V_2)}{1-\delta^{T-3}}$ then $y_1 = 0$, $x_2 = W - (1-\delta^{T-4})V_1 - V_2$, $y_3 = V_2$, $x_4 = (1-\delta^{T-4})V_1$; if $W > \frac{(1-\delta^{T-4})(V_1+\delta V_2)}{1-\delta^{T-3}}$ then neither contributes before stage $T-1$, and $x_{T-1} = W - V_2$, $y_T = V_2$.

In conclusion, if $V_1 > V_2$ then there is a unique SPE. In equilibrium the project gets completed in either stage two to four or gets completed at the deadline. This concludes the proof of Proposition 4.3. ■

From Proposition 4.3 we see that there is a threshold property in the completion pattern. When $W < W^*$, the project gets completed early; and when $W > W^*$, the project gets completed late. $W^* = \frac{(1-\delta^{T-3})(V_1+\delta V_2)}{1-\delta^{T-2}}$ when T is even, $W^* = \frac{(1-\delta^{T-4})(V_1+\delta V_2)}{1-\delta^{T-3}}$ when T is odd. Think of W as the workload for a group of workers to fulfill. Clearly, unlike in the case of $V_1 < V_2$ in Section 4.2.2, in this case the heavier workload causes a more intense relationship between players and makes it harder for them to cooperate earlier and reach an early completion. The intuition is not very straightforward: heavier workload makes it a necessity for both players to make their full contribution, and it is inconsistent with 2's willingness to use her valuation advantage and skip being exploited in the final stage: she makes a smaller total contribution if she is able to induce an early completion (although she still earns zero payoff). 1 does not necessarily enjoy using his final stage advantage either. Due to the time cost, he is willing to allow 2 to shirk and contribute more by himself to reach an early completion, but the heavy workload makes this choice impossible. It is natural that a heavier work takes a longer time to finish, but here we provide an alternative perspective in looking at this issue: the heavy burden might strengthen the free-riding incentive, making cooperation less efficient and causing delay. In a joint project, thing may look like going smoothly, but players are aready on the edge and a slight increase in work would cause the collapse of the whole project.

In Sections 4.2.2 and 4.2.3, we showed that there is a unique equilibrium in each

case.

If V_1 and V_2 are close enough then the previous analysis would apply to $V_1 = V_2$, with players' indifference choice fixed.

Take $V_1 < V_2$, $V_1 = V_2 - \epsilon$ and $\epsilon \to 0$, then the equilibrium in Section 4.2.2 could be seen as the equilibrium reached in the homogeneous valuations case when player 1 delays completion whenever indifferent and player 2 accelerates completion whenever indifferent.

Take$V_1 > V_2$, $V_1 = V_2 + \epsilon$ and $\epsilon \to 0$, then the equilibrium in Section 4.2.3 could be seen as the equilibrium reached in the homogeneous valuations case when player 1 accelerates completion whenever indifferent and player 2 delays completion whenever indifferent.

To sum up, there are at least two equilibria in the case of $V_1 = V_2$. This proves Proposition 4.1.

4.3 Optimal orders and analysis

In this section, we use the results above to compare individual and total welfare in the two cases when V_1 is greater or smaller than V_2. In the following analysis we assume both players have similar values: set $V_2 = V$, when $V_1 < V$, then $V_1 = V - \epsilon$ and $\epsilon \to 0$; when $V_1 > V$, then $V_1 = V + \epsilon$ and $\epsilon \to 0$,

Firstly, consider 1 as the first contributor (T is even).

- If $max\{V_1,\ V_2\} < W \leq \frac{(1-\delta^{T-3})(V_1+\delta V_2)}{1-\delta^{T-2}}$ then there will be an early completion irrespective of sign $(V_1 - V_2)$: if $V_1 > V_2$ then the project is completed in the third stage. 1 earns $(1-\delta^{T-3}+\delta^{T-1})V_1+\delta V_2-W \to (1+\delta-\delta^{T-3}+\delta^{T-1})V-W$ and 2 earns 0. If $V_1 < V_2$, the project is completed in the second stage. 1 earns $\delta V_1 + V_2 - W \to (1+\delta)V - W$ and 2 earns 0. Making the high valuing player contribute last Pareto-dominates the alternative choice.

- If $\frac{(1-\delta^{T-3})(V_1+\delta V_2)}{1-\delta^{T-2}} < W$ then the project is completed either at early stages or at the deadline. If $V_1 > V_2$ then the project is completed in the last stage: 1 earns $\delta^{T-1}(\delta V_1 + V_2 - W) \to \delta^{T-1}(1+\delta)V - W$ and 2 earns 0. If $V_1 < V_2$ then the payoffs are the same as when $max\{V_1,\ V_2\} < W \leq \frac{(1-\delta^{T-3})(V_1+\delta V_2)}{1-\delta^{T-2}}$: 1

earns $\delta(1+\delta)V-\delta W$ and 2 earns 0. Making the high valuing player contribute last Pareto-dominates making the low valuing player contribute last.

Hence, when 1 as the first contributor, it is worthwhile to let the higher valued player act as the final contributor from the perspective of social welfare. There is a jump in total payoff at $V_1 = V_2$.

Now let 2 be the first contributor (T is odd).

- If $max\{V_1,\ V_2\} < W \leq (1-\delta^{T-2})V_1 + \delta V_2$ then there is an early completion irrespective of sign $(V_1 - V_2)$: if $V_1 > V_2$ then the project is completed in the second stage. 1 earns $\delta^{T-1}V_1 \to \delta^{T-1}V$ and 2 earns $(1-\delta^{T-2})V_1+\delta V_2-W \to (1+\delta-\delta^{T-2})V-W$. If $V_1 < V_2$, the project is completed in the third stage. 1 earns $\delta(\delta V_1+V_2-W) \to \delta(1+\delta)V-\delta W$ and 2 earns 0. Making neither player contribute last Pareto-dominates making the other player contribute last.

- Suppose that $(1-\delta^{T-2})V_1+\delta V_2 < W \leq \frac{(1-\delta^{T-4})(V_1+\delta V_2)}{1-\delta^{T-3}}$. If $V_1 > V_2$ then the project is completed in the fourth stage. 1 earns $(\delta-\delta^{T-3}+\delta^{T-1})V_1+\delta^2 V_2-\delta W \to (\delta+\delta^2-\delta^{T-3}+\delta^{T-1})V-\delta W$ and 2 earns 0. If $V_1 < V_2$ then the payoffs are the same as when $max\{V_1,\ V_2\} < W \leq (1-\delta^{T-2})V_1+\delta V_2$: 1 earns $\delta(1+\delta)V-\delta W$ and 2 earns 0. Making the high valuing player contribute last Pareto-dominates making the low valuing player contribute last.

- Finally suppose that $\frac{(1-\delta^{T-4})(V_1+\delta V_2)}{1-\delta^{T-3}} < W$. If $V_1 > V_2$ then the project is completed in the last stage. 1 earns $\delta^{T-1}(\delta V_1+V_2-W) \to \delta^{T-1}(1+\delta)V-\delta^{T-1}W$ and 2 earns 0. If $V_1 < V_2$ then the payoffs are the same as when $max\{V_1,\ V_2\} < W \leq (1-\delta^{T-2})V_1+\delta V_2$: 1 earns $\delta(1+\delta)V-\delta W$ and 2 earns 0. Making the high valuing player contribute last Pareto-dominates making the low valuing player contribute last.

Things are similar to the case where 1 is the first contributor, except that 2 may earn a positive payoff now, and making the high valuing player contribute last does not necessarily Pareto-dominates.

For convenience, we list the total payoffs when $V_1 \to V_2 = V$ for comparison:

T is odd	$V_1 > V_2$	$V_1 < V_2$
$V < W \leq \frac{(1-\delta^{T-3})(1+\delta)}{1-\delta^{T-2}}V$	$(1+\delta-\delta^{T-3}+\delta^{T-1})V - W$	$(1+\delta)V - W$
$\frac{(1-\delta^{T-3})(1+\delta)}{1-\delta^{T-2}}V < W$	$\delta^{T-1}((1+\delta)V - W)$	$(1+\delta)V - W$
T is even	$V_1 > V_2$	$V_1 < V_2$
$V < W \leq (1-\delta^{T-2})V + \delta V$	$(1+\delta-\delta^{T-2}+\delta^{T-1})V - W$	$\delta((1+\delta)V - W)$
$(1-\delta^{T-2})V + \delta V < W \leq \frac{(1-\delta^{T-4})(1+\delta)}{1-\delta^{T-3}}V$	$(\delta+\delta^2-\delta^{T-3}+\delta^{T-1})V - \delta W$	$\delta((1+\delta)V - W)$
$\frac{(1-\delta^{T-4})(1+\delta)}{1-\delta^{T-3}}V < W$	$\delta^{T-1}((1+\delta)V - W)$	$\delta((1+\delta)V - W)$

Clearly, in most cases letting the high valuing player be the final contributor increases total welfare; the only exception occurs when T is even and W is small. It is also the only case in which the final contributor (player 2) breaks the zero payoff curse and earns a positive payoff from completion of the project. Except in this special case, when W is large enough, there is a jump of total welfare around $V_1 = V_2$.

This is an interesting observation from several perspective. 1. Recall the $V_1 = V_2$ case. As we mentioned before, if we suppose player's valuation to not be perfectly stable, then no stable equilibria involve middle-range completions: completion happens at the beginning or the end of the game. 2. It confirms the stereotype that joint projects usually needs a gate-keeper: someone needs to be elected to be responsible for the project. As long as it may disincentivise other group members, it could guarantee a higher efficiency. 3. In certain special cases, letting the low valuing player in the team act as the final contributor could work: the greater responsibility might incentivize him to do a better job instead of hiding behind others. However this could backfire and cause a total welfare loss; and this is more likely to happen when the work gets harder.

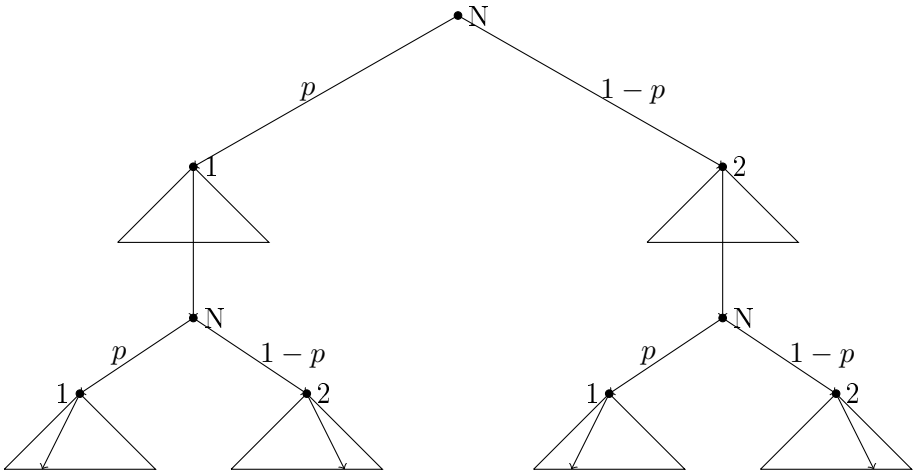
4.4 Randomizing model

In the main model we introduced an alternating contribution game in which completion is delayed. However delay is not the only possible type of inefficiency in contribution games. In this section we replace the alternating protocol with a randomizing protocol to exhibit other types of inefficiency.

If a project has the potential to bring more total revenue than total cost when it is done properly (without requiring anyone to violate individual rationality and earn a negative payoff), then we call this project worth-doing, the failure of the project is seen as a loss in social welfare. If a project fails with contributions made, then it is clearly worse than not contributing anything from the start; we may claim this "waste" to be inefficient also. However, nevertheless it is clearly ex-post inefficient when waste happens, it might be ex-ante efficient from the social perspective if waste happens with a positive probability,.

We now introduce a two-stage randomizing contribution game which exhibits above mentioned two types of inefficiency. All other features of the game are the same as in the main model; here we assume that $T = 2$, and in each stage there is a probability $0 < p < 1$ that 1 makes the contribution and a probability $0 < 1 - p < 1$ that 2 makes the contribution. Without loss of generality assume that $V_1 < V_2$.[12] Also assume that $V_2 < W < V_1 + \delta V_2$ (as in Assumption 4.1) so that the project is worth-doing by two players but not by either one of them. Hence, $(1 - \delta)V_2 < V_1$.

[12]Unlike in alternating protocol, players are homogeneous in the randomizing protocol except that they have their own valuations. Without the identity of a final contributor , it is meaningless to distinguish between high and low valuing players now.



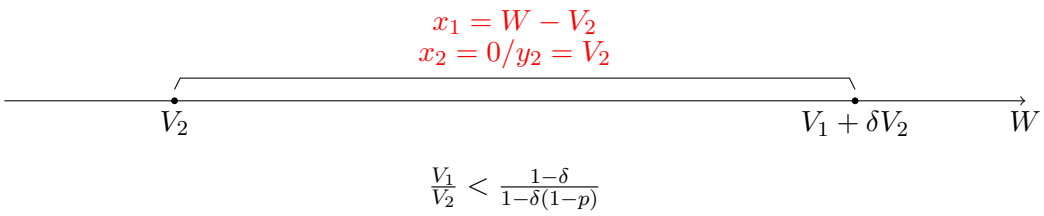
We solve the game through backward induction. Notice that in the first stage, Nature creates two possibilities: whether 1 or 2 is the first contributor, and players respond accordingly. We explore these two cases in sequence.

Case 1: 1 makes the first contribution.

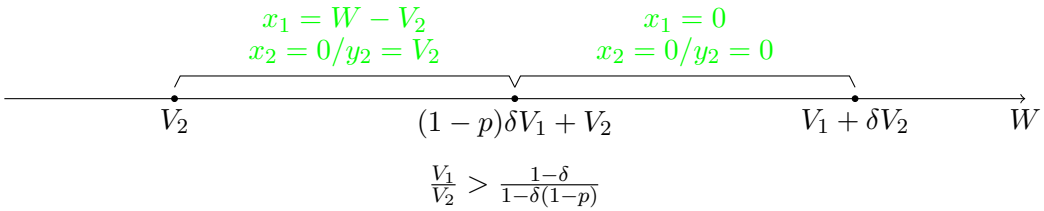
In this case, we find the unique sub-game perfect equilibrium[13] through a simple backward induction argument. Detail is provided in the Appendix.

In equilibrium, if $\frac{V_1}{V_2} < \frac{1-\delta}{1-\delta(1-p)}$, or $\frac{V_1}{V_2} > \frac{1-\delta}{1-\delta(1-p)}$ and $V_2 < W < (1-p)\delta V_1 + V_2$ then: in the first stage, 1 contributes $x_1 = W - V_2$; in the second stage, with probability p 1 contributes $x_2 = 0$ and the project fails; with probability $1-p$ 2 contributes $y_2 = V_2$ and the project gets completed.

If $\frac{V_1}{V_2} > \frac{1-\delta}{1-\delta(1-p)}$ and $(1-p)\delta V_1 + V_2 < W < \delta V_1 + V_2$ then neither player contributes anything and the project fails.



[13] Except when W and $\frac{V_1}{V_2}$ are in their threshold value, in these case there are multiple equilibria, we do not discuss them in here.



In the graph, red indicates a socially optimal result, green indicates a socially sub-optimal result.

We now examine the welfare gain and loss in this model. We have assumed $V_2 < W < V_1 + \delta V_2$ so that the project is worth-doing and that failure of completion is always inefficient. However, even if the project is completed, we can't guarantee that there is no welfare loss.

We first explore the maximal social welfare this project could bring in this protocol. From the social perspective, the maximal welfare must be reached when the project is completed immediately, in this case, by 1 in the first stage. However, this is clearly against 1's individual rationality as the project is not worth doing by 1 alone; so the project can only be completed in the second stage. With a fixed completion time given, it is socially favorable for the contributions to be made as late as possible due to the time discounting. It could be easily deduced that there are only two candidates for a socially optimum strategy profile here: 1 contributes $x_1 = W - V_1$ to guarantees completion of the project and 1 contributes $x_1 = W - V_2$ to make 2 the only one willing to complete the project. Any $x_1 > W - V_1$ would only add to the first stage contribution and reduce the second stage contribution while maintaining the completion time; any $x_1 < W - V_2$ would not be enough to induce completion at all; any $W - V_1 < x_1 < W - V_2$ is a mixture of the two possible maximum social welfare.

We now examine these two possible socially optimal strategy profiles. When 1 offers $W - V_1$ in the first stage, both players complete in the second stage, and total payoff equals $\delta(V_1 + V_2) - (W - V_1) - \delta V_1$; when 1 offers $W - V_2$, only 2 contributes in the second stage, and total payoff equals $(1-p)\delta(V_1 + V_2) - (W - V_2) - (1-p)\delta V_2$.

If $\frac{V_1}{V_2} > \frac{1-\delta}{1-\delta(1-p)}$ then the previous total payoff is larger, and completion for sure is socially optimal;[14] if $\frac{V_1}{V_2} < \frac{1-\delta}{1-\delta(1-p)}$ then the latter total payoff is larger, and completion with probability $1-p$ is socially optimal.

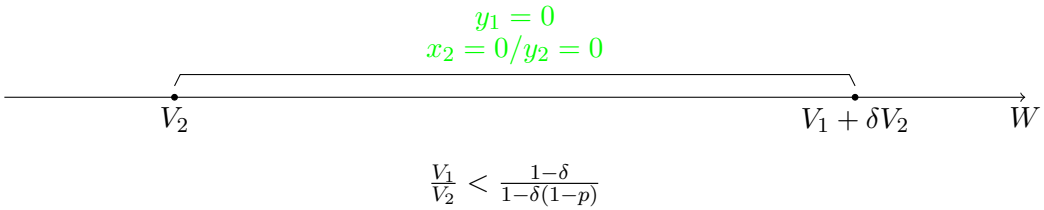
Combined with the equilibrium outcome, if $\frac{V_1}{V_2} < \frac{1-\delta}{1-\delta(1-p)}$ then the project is completed with probability $1-p$. The equilibrium path may involve waste: 1 makes a positive contribution in the first stage, but the project fails. But it maximizes players' total payoff and is socially optimal. If $\frac{V_1}{V_2} > \frac{1-\delta}{1-\delta(1-p)}$ then, if W is large then the project is completed with probability $1-p$ and the equilibrium path may also involve waste, and here it is socially sub-optimal; if W is large then the project fails, and the equilibrium is also inefficient.

Case 2: 2 makes the first contribution.

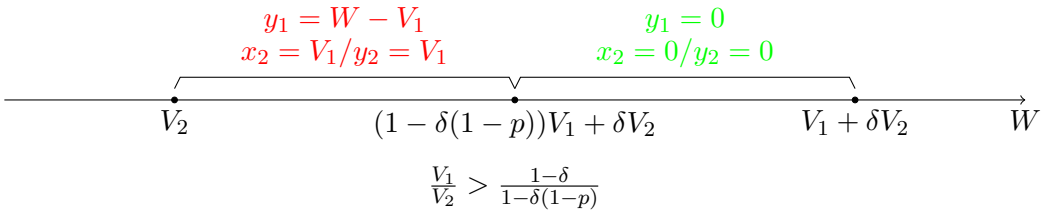
Similarly, we study this case using SPE. The arguments required to find the SPE are provided in the Appendix.

In equilibrium, if $\frac{V_1}{V_2} < \frac{1-\delta}{1-\delta(1-p)}$, or if $\frac{V_1}{V_2} > \frac{1-\delta}{1-\delta(1-p)}$ and $(1-\delta(1-p))V_1 + \delta V_2 < W < \delta V_1 + V_2$, then neither player contributes anything and the project fails.

If $\frac{V_1}{V_2} > \frac{1-\delta}{1-\delta(1-p)}$ and $V_2 < W < (1-\delta(1-p))V_1 + \delta V_2$, then in the first stage 1 contributes $x_1 = W - V_1$; in the second stage both players contribute V_1 and complete the project for sure.



[14] Notice that here completing for sure indicates that there is a probability of p that player 1 will break his individual rationality and earn a negative payoff. While the worth-doing property of the project implies the project can be completed without violating IR; the socially optimal way to complete the project here may involve violating IR.



Similar to Case 1, failure of completion is always inefficient, and whether completing for sure is socially optimal can't be guaranteed.

As analyzed in Case 1, the project is completed in the second stage when social welfare is maximized. Unlike in Case 1, 2 has no incentive to contribute $W - V_2$ in the first stage, so the only plausible socially optimal strategy is $y_1 = W - V_1$ and $x_2 = y_2 = V_1$.

Combined with the equilibrium outcome, if $\frac{V_1}{V_2} < \frac{1-\delta}{1-\delta(1-p)}$ then the project fails, which is clearly inefficient. If $\frac{V_1}{V_2} > \frac{1-\delta}{1-\delta(1-p)}$ then the project is completed for sure if W is small and the path is socially optimal; the project fails if W is large and the equilibrium is inefficient.

Combining Case 1 and 2, we have three possible equilibria: completion for sure, completion with a positive probability, or failure. While completion for sure is socially optimal (although it may involve violating individual rationality) and failure is not, completion with probability, in spite of involving "waste" which must be ex-post inefficient, might be ex-ante efficient: that is, socially optimal. The table below describes all the possibilities. In the table, efficient means socially optimal, inefficient means that the result is not socially optimal.

$\frac{V_1}{V_2} < \frac{1-\delta}{1-\delta(1-p)}$	$W < (1-p)\delta V_1 + V_2$	$W > (1-p)\delta V_1 + V_2$
1 is the first contributor	Completion with waste (efficient)	
2 is the first contributor	Failure (inefficient)	
$\frac{V_1}{V_2} > \frac{1-\delta}{1-\delta(1-p)}$	$W < (1-\delta(1-p))V_1 + \delta V_2$	$W > (1-\delta(1-p))V_1 + \delta V_2$
1 is the first contributor	Completion with waste (inefficient)	Failure (inefficient)
2 is the first contributor	Completion for sure (efficient)	Failure (inefficient)

Randomizing no doubt complicates things. While a worth-doing joint project might be well-designed and meant to benefit everyone, many problems may emerge when it is carried out. While total welfare is maximized with implementation of the project, someone might actually suffer a loss; with the potential to make everyone better, the project might end up not being completed at all; if things are arranged to make sure the project maximizes social welfare and no one would ever suffer a loss, nevertheless we are not able to dodge the bullet of possible waste. To sum up, there is no equilibrium in this model which is truly satisfying. Trade-offs have to be made.

4.5 Conclusion

In this chapter, we have mainly discussed finite horizon alternating contribution games with heterogeneous valuations. In the infinite horizon case, homogeneous valuations strengthen free-riding incentives and lead to a delayed completion; heterogeneous valuations eliminate this issue and lead to an almost immediate completion. In the finite horizon case, homogeneous valuations cause multiple equilibria; heterogeneous valuations eliminate all the intermediate completion equilibria and leave two extreme possibilities: early completion or deadline completion. Counter-intuitively, we show that around the homogeneous valuation there may be a jump in the total welfare: early completion happens when the final contributor values the project a bit more and deadline completion happens when the final contributor values the project a bit less. Even a tiny gap between valuations could cause a significant change in the timing of the completion. Furthermore, when valuations are fixed, a slight difference in W (the total contributions required to complete the project) could also produce the same effect. For the sake of efficiency, it is probably a good idea to let someone who values the project more to guard the gate, though that would almost ensure an over-load for the higher valued player. In certain circumstances, exchanging the role of the players and letting those who value the project less be the final contributor may help balancing the contribution distribution. We further discuss a randomizing contribution game with heterogeneous valuations. The uncertainty in protocol

allows us to include other types of inefficiency in our discussion besides delay. And it shows how hard it can be to carry out a worth-doing project without choosing between violating individual rationality, waste or failure.

4.6 Appendix

In Section 4.4, Case 1, we find the SPE using backward induction as follows.

In the second stage, if the project has not been completed then the two players choose between completing the project or not. Write M as the remaining contribution to complete the project in this stage. 1 would offer $x_2 = M$ to complete the project if $M < V_1$, would offer $x_2 = 0$ to let the project fail if $M > V_1$, and would be indifferent between these previous two options if $M = V_1$. 2 would offer $y_2 = M$ to complete the project if $M \leq V_2$;[15] and would offer $y_2 = 0$ to let the project fail if $M > V_2$.

Therefore, 1's payoff in the second stage is equal to

$$\begin{cases} V_1 - pM & M \leq V_1, \\ (1-p)V_1 \quad if & V_1 < M \leq V_2, \\ 0 & V_2 < M. \end{cases}$$

In the first stage, 1 has three options:

1. $y_1 = W - V_1$, in which case both players complete the project in the second stage;

2. $x_1 = W - V_2$, in which case only 2 completes the project in the second stage;

3. $x_1 = 0$, in which case the project fails.

[15] If $M = V_2$ then 1 lets the project fail. If 2 completes the project with probability q then, in the first stage, 1 gets a payoff equal to $\delta((1-p)qV_1) - x_1$; so he has an incentive to increase his contribution to make sure that $q = 1$.

1's payoffs from the three options are:

$$\begin{cases} \delta(V_1 - pV_1) - (W - V_1) & x_1 = W - V_1, \\ \delta(1-p)V_1 - (W - V_2) \quad if & x_1 = W - V_2, \\ 0 & 0. \end{cases}$$

Consequently:

1 induces completion with probability $1-p$ in the second stage if $W < (1-p)\delta V_1 + V_2$;

1 lets the project fail if $W > (1-p)\delta V_1 + V_2$. Notice that this is impossible when $(1-p)\delta V_1 + V_2 > V_1 + \delta V_2$, viz., when $\frac{V_1}{V_2} < \frac{1-\delta}{1-\delta(1-p)}$.

To sum up, if $\frac{V_1}{V_2} < \frac{1-\delta}{1-\delta(1-p)}$ then 1 induces completion with probability $1-p$ in the second stage.

If $\frac{V_1}{V_2} > \frac{1-\delta}{1-\delta(1-p)}$ then 1 induces completion with probability $1-p$ in the second stage if $V_2 < W < (1-p)\delta V_1 + V_2$, and lets the project fail if $W > (1-p)\delta V_1 + V_2$.

In Section 4.4, Case 2, we find the SPE using backward induction as follows.

In the second stage, if the project has not been completed then the two players choose between completing the project or not. Write M as the remaining contribution to complete the project in this stage, 1 would offer $x_2 = M$ to complete the project if $M \leq V_1$;[16] and would offer $x_2 = 0$ to let the project fail if $M > V_1$. 2 would offer $y_2 = M$ to complete the project if $M < V_2$, would offer $y_2 = 0$ to let the project fail if $M > V_2$, and would be indifferent between these previous two options if $M = V_2$.

[16] If $M = V_1$ then 2 completes the project. If 1 completes the project with probability q then, in the first stage, 2 gets a payoff of $\delta((pq + 1 - p)V_2 - (1-p)V_1) - y_1$; so she has an incentive to increase her contribution to make sure that $q = 1$.

Therefore, 2's payoff in the second stage is equal to

$$\begin{cases} V_2 - (1-p)M & M \leq V_1, \\ (1-p)V_2 - (1-p)M \quad if & V_1 < M \leq V_2, \\ 0 & V_2 < M. \end{cases}$$

In the first stage, 2 has two options:

1. $y_1 = W - V_1$, in which case both players complete the project in the second stage;

2. $y_1 = 0$, in which case the project fails.

Notice that unlike Case 1, here 2 has no incentive to contribute $y_1 = W - V_2$ for that only induces 2 to complete the project in the second stage. If 1 does not contribute then it is not worthwhile for 2 to complete the project by herself; she would rather let the project fail through offering $y_1 = 0$.

2's payoffs from the two options are:

$$\begin{cases} \delta(V_2 - (1-p)V_1) - (W - V_1) & y_1 = W - V_1, \\ 0 - 0 \quad if & y_1 = 0. \end{cases}$$

Consequently:

2 induces completion in the second stage if $W < (1 - \delta(1-p))V_1 + \delta V_2$;

2 lets the project fail if $W > (1-\delta(1-p))V_1 + \delta V_2$. Notice that this is impossible when $(1 - \delta(1-p))V_1 + \delta V_2 > V_1 + \delta V_2$, viz., when $\frac{V_1}{V_2} < \frac{1-\delta}{1-\delta(1-p)}$.

To sum up, if $\frac{V_1}{V_2} < \frac{1-\delta}{1-\delta(1-p)}$ then 2 lets the project fail.

If $\frac{V_1}{V_2} > \frac{1-\delta}{1-\delta(1-p)}$ then 2 induces completion in the second stage if $V_2 < W < (1 - \delta(1-p))V_1 + \delta V_2$, and she lets the project fail if $W > (1 - \delta(1-p))V_1 + \delta V_2$.

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