An Improved Uncertainty in Multi-Criteria Decision Making Model based on Type-2 Fuzzy TOPSIS



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I would like to dedicate this thesis to my loving family - Mom, Dad, Jefri and my children.

Declaration

The works in this thesis are based on research carried out at the Intelligent Modelling and Analysis (IMA) and Lab for Uncertainty in Data and Decision Making (LUCID) Research Groups, School of Compute Science, University of Nottingham, UK. No part of this thesis has been submitted elsewhere for any other degree or qualification and it all my own work unless referenced to the contrary in the text.

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Abstract

This thesis presents a detailed study about one of the Multiple Criteria Decision Making (MCDM) models, namely Technique for Order Preference by Similarity to Ideal Solution (TOPSIS), based on fuzzy set theory (FST) by focusing on improving modelling uncertain information provided by a group of decision makers (DMs).

An explorations of issues and limitations in current models of standard TOPSIS and fuzzy TOPSIS were made. Despite many variations of type-1 fuzzy TOPSIS (T1-TOPSIS) model, none of the studies explaining the details of the key stages of standard TOPSIS (non-fuzzy) and T1-TOPSIS are based on step-wise procedure. A detailed study was conducted which involve the process of identifying the limitations of standard TOPSIS and T1-TOPSIS. Based on this, a novel contribution on the comparison between these two models in systematic stepwise procedure was given. This study successfully identified and discussed the limitations, issues and challenges which have not been investigated sufficiently in the context of T1-TOPSIS model.

Based on this exploration, further investigation of multiple variants of the extension of fuzzy TOPSIS model for solving MCDM problem was made with the primary aim of detailing the steps involved. One challenge that has risen is that it is not straightforward to differentiate between the multiple variants of TOPSIS existing today. A systematic comparison was made between standard T1-TOPSIS model with the recent extended model to show the differences between both models and to provide context for their respective strengths and limitations both in the complexity of application and expressiveness of results. Based on the resulting

comparison, the differences in the steps implemented by these two Fuzzy TOPSIS models were highlighted throughout the worked example. Furthermore, this task highlights the ability of both models to handle different levels of uncertainty.

Following the exploration of issues and limitations of the current model, as well as a comparative study, a novel extension of type-2 fuzzy TOPSIS model is proposed in this thesis which suggests providing an interval-valued output to reflect the uncertainties and to model subjective information. The proposed model enables to uniquely captures input uncertainty (i.e., decision makers' preferences) in the decision-making outputs and provide a direct mapping of uncertainty in the inputs to outputs. By keeping the output values in interval form, the proposed model reduces the loss of information and maximises the potential benefit of using Interval Type-2 Fuzzy Sets (IT2 FSs).

To demonstrate the MCDM problems when a various level of uncertainty is introduced, a novel experimental method was proposed in this study. The primary aim is to explore the use of IT2 FSs in handling uncertainty based on the TOPSIS model. This experiment was conducted to show how the variation of uncertainty levels in the input affects the final outputs. An implementation of the proposed model to two different case studies was conducted to evaluate the proposed model. The proposed model for the first time generates an interval-valued output. As intervals can, for example, exhibit partial overlap, a novel extended measure is proposed to compare the resulting interval-valued output from various cases (i.e., overlapping and non-overlapping) of the interval with considering uncertainty.

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Nomenclature

Acronyms

CC	Closeness Coefficient
DM	Decision Maker
FN	Fuzzy Number
FST	Fuzzy Set Theory
gauss2mf	2-sided composite gaussian MF
gaussmf	Gaussian MF
gbell	Generalized bell shape
IT2 FS	Interval Type-2 Fuzzy Set
IT2 TFN	Interval Type-2 Triangular Fuzzy Number
IT2-TOPSIS	Interval Type-2 Fuzzy TOPSIS
MADM	Multiple-Attribute Decision Making
MCDM	Multi-Criteria Decision Making
MCDM	Multiple Criteria Decision-Making

MCGDM	Multiple Criteria Group Decision-Making
MF	Membership Function
MODM	Multiple-Objective Decision Making
smf	mirror-image MF
T1 FS	Type-1 Fuzzy Set
T1-TOPSIS	Type-1 fuzzy TOPSIS
T1 TFN	Type-1 Triangular Fuzzy Number
T2 FN	Type-2 Fuzzy Number
T2 FS	Type-2 Fuzzy Set
T2 MF	Type-2 MF
TOPSIS	Technique for Order Preference by Similarity to Ideal Solution
trapmf	Trapezoidal MF
trimf	Triangular MF

Notations

$[a_L, a_U]$	Lower and upper bound of interval $\bar{a}$
$\bar{a}$	Interval number a
$\bar{X}$	Standard interval-TOPSIS (non-fuzzy) decision matrix
$\dot{CC}_i$	Crisp Closeness Coefficient (CC) values
$\dot{W}$	Standard TOPSIS (non-fuzzy) weight decision matrix

$\dot{X}$	Standard TOPSIS (non-fuzzy) decision matrix
$mid(\bar{a})$	Midpoint of interval $\bar{a}$
$rad(\bar{a})$	Radius of interval $\bar{a}$
$w(\bar{a})$	Width of interval $\bar{a}$

Chapter 1

Introduction

Uncertainty is always present in the real world. In the case of information, for instance, uncertainty can be described as imprecise, vague or incomplete. Since uncertainty is ever-present, it is therefore necessary for any intelligent model to manage and take into account any uncertainty to maximize the usefulness of the system's model. In general, there are two established techniques to model uncertainty, namely, probability theory and fuzzy set theory. This thesis is concerned with modeling uncertainties with fuzzy sets (FSs) and Technique for Order Preference by Similarity to Ideal Solution (TOPSIS).

TOPSIS is a well established model which falls under multiple criteria decision-making (MCDM) methods. MCDM is the process of finding the best possible solution (e.g., action, choice, option) from a set of feasible alternatives with the consideration of multiple conflicting criteria. In most cases, it is regarded as a complex task to be interpreted quantitatively. This is especially true when involving more than one decision maker, which is known as multi-criteria group decision making (MCGDM) [4]. The information about criteria or attribute values is usually uncertain or fuzzy because the complexity of the current environment and the vagueness of subjective data is increased [5].

Fuzzy set theory (FST), introduced by Zadeh [6] is used to handle such uncertain environments and vagueness in information. Bellman and Zadeh [7] were the pioneers of

implementing type-1 fuzzy sets (known as fuzzy sets) with decision making models. The implementation enables systematic reasoning to represent linguistic terms numerically and handle uncertain human judgements. Later, it has been shown that type-2 fuzzy sets (T2 FSs) as an extension of type-1 fuzzy set (T1 FSs) have the ability to provide better, more fine-grained models of uncertainty.

This thesis presents a detailed study of one of the MCDM models, i.e. TOPSIS, based on FST. It focusses on improving modeling uncertain information provided by a group of decision makers (DMs) using Interval T2 FSs (IT2 FSs). IT2 FSs are a special type of (general) T2 FSs that are most widely used due to their reduction in computational complexity. While most of the research studies to date have claimed that they have successfully captured uncertainty in their model, one question arises, "How is this uncertainty reflected from the initial to the final stages of the decision making process?" In other words, how do the initial uncertainties (i.e., input of the model) reflect themselves in the final evaluation (i.e., output of the model) of the process. In an attempt to address this essential question, this thesis describes a novel extension of interval type-2 fuzzy TOPSIS (IT2 TOPSIS) model suggested to provide an interval-valued output to reflect the initial uncertainties to the final evaluation and to model subjective information. The proposed model uniquely captures uncertainty in the input (i.e., DM's preferences) and provides a direct mapping of uncertainty in the inputs to the outputs. As intervals can, for example, exhibit partial overlap, a measure is developed to compare the resulting interval-valued output from various cases (i.e., overlapping and non-overlapping) of the interval whilst considering uncertainty.

1.1 Background and motivation

Human decision making usually deals with making a decision in the presence of vague, incomplete or imprecise information. In addition to this uncertain information, the existence of multiple conflicting criteria in MCDM problem contribute to increase the complexity

of the process. Note that the central problem in most MCDM models is how to evaluate a set of alternatives concerning multiple criteria [8]. Within this central problem, many MCDM models have been proposed. The TOPSIS model [3] is currently one of the most popular models and has been shown to provide helpful outputs in various applications. This model was developed for solving MCDM problems with the basic principle to choose the alternative which has the shortest distance from the positive ideal solution (best) and the farthest distance from negative-ideal solution (worst). Later, Chen [1] extended the TOPSIS model to a MCGDM in a fuzzy environment using fuzzy numbers (specific types of fuzzy set) to represent information by considering uncertainty. It was motivated by the fact that crisp values cannot model real-life situations as such situations are frequently characterized by vague and uncertain information. Obviously, the focus of the extended version of TOPSIS was made to handle uncertainty. Many researchers have applied TOPSIS to solve fuzzy MCDM problems to date, for example [1, 9–11].

In recent years, a variety of extensions, including fuzzy extensions of TOPSIS have been proposed. A majority of these studies have only showed the applicability of their proposed model when implementing it in practice. Thus, it is a challenge to determine the limitations of the various models. The process of identifying the limitations, issues and challenges is crucial in order to provide a meaningful justification or a logical reason why a model addressing all the uncertainty should be chosen to ensure confidence in the recommended decision. Another challenge that has arisen is that it is not straightforward to differentiate between the multiple variants of TOPSIS that exist today. Despite many variations of TOPSIS models, none of the studies that have compared the key differences between the standard TOPSIS and fuzzy TOPSIS have fully explored each of the step-wise stages. In fact, most studies do not highlight the differences between these two models in clear detail. There is no detailed comparison of these key stages in a systematic stepwise manner, clearly highlighting

differences. Thus, it is necessary to analyse these two models by highlighting the capability of both models reflecting uncertainty from initial evaluation to the final stage.

In MCDM problems, there are elements of quantitative or qualitative criteria whereby the performance of each alternative was given by decision makers. In this case, decision makers express their preferences; they measure the performance of the alternatives with the help of the appropriate scale that categorised their indifferent preferences. Qualitative criteria are commonly low-dimensional which means the performance of the alternatives can only be expressed in several more or less verbal terms. On the other hand, quantitative criteria are usually one-dimensional in which the performance of the alternatives can be expressed in physical or monetary units on a one-dimensional scale. The dynamic performance measurement may easily lead to incomparability and inconsistency preferences of the alternatives [12]. Recently, Yuen [13] proposed two-dimensional scale to overcome this dynamic phenomenon. The scale measurement is different compared to the standard T1-TOPSIS model proposed by Chen [1]. It is used as a tool for decision makers to express their preferences and to model uncertainty within the rating process. Thus, the question arises as to whether this Yuen's model provides more benefits as compared to other existing T1-TOPSIS models. To date, there has been no reliable evidence and no comparative study that has shown the strength and limitations of the standard T1-TOPSIS compared with the recently model proposed by Yuen.

Zadeh [14] recognized the limitation of T1 FSs in that they represent a limited level of uncertainty, and so he introduced type-2 fuzzy sets (T2 FSs), in which the membership degree is no longer a crisp value within $[0, 1]$, but is itself a fuzzy set. While looking at how to capture uncertainty in the decision-making process, some researchers have extended T1-TOPSIS to use a type-2 model, as T2 FSs [14] are widely accepted to be powerful models for capturing uncertainty [15–17]. Furthermore, the T2 models aim to handle uncertain, vague and imprecise information as well as providing DMs with better solutions. A majority

of the recent T2-TOPSIS models were proposed to handle uncertainty in the decision-making process. These models assume that uncertain information is present in the input. In decision-making situations in which we have perfect information, we can usually rank alternatives with certainty. However, when uncertainty affects the information about the alternatives in initial evaluation process, we can intuitively expect challenges in ranking them in the sense that we may not be able to rank the alternatives with the same level of certainty. From a practical point of view, it is clear that any uncertainty affecting the ranking of alternatives should be taken into account during decision making. Additionally, the extra information provided by T2 FSs in existing models is largely lost, since the final output is usually resolved to a crisp value. Hence, the benefit of using such sets is not fully exploited, as the extra information that they could provide is not available to DMs in the decision making process. This thesis investigates the effect of uncertainty in the initial evaluation into the final evaluation by using a novel experimental method. This method uses blurring membership functions to simulate the introduction of uncertainty in the initial evaluation into the final evaluation. The effect of this blurring to the final evaluation is observed. Based on this experimental result, a novel extension of IT2-TOPSIS is proposed to handle the uncertainty which affects the final evaluation. This is done by using an interval number to represent the final evaluation of the IT2-TOPSIS model. An interval number can be used alternatively to reduce uncertainty in the TOPSIS model. Hence, it is intuitive to develop a TOPSIS model based on IT2 FSs which takes advantage of the additional information provided by the sets, in which the final result remains an interval.

Notable progress was made in a model proposed by Chen and Lee [18], in which Interval Type-2 fuzzy sets (IT2 FSs) are used to represent initial evaluation (i.e., ratings) in TOPSIS model. The authors determined ranking values using elements in a fuzzy decision matrix and from a crisp ranking. They then calculated the Closeness Coefficient (CC) values using both models and compared the results. The results from both models are represented in crisp

values. Other researchers, however, have focused on specific steps of the IT2-TOPSIS model. For example, in [19] the authors proposed an IT2-TOPSIS model in which the determination of a fuzzy positive ideal solution and a fuzzy negative ideal solution are achieved using IT2 FSs. Many other studies have proposed IT2-TOPSIS models (other studies will be discuss in detail in Chapter 2). However, none of the models developed thus far provide a systematic mapping between uncertainty levels in the MCDM inputs to the uncertainty in the outputs. Thus, it can be beneficial to consider a better T1-TOPSIS model which can provide systematic mapping of uncertain inputs to uncertain outputs to solve MCDM problems. This thesis shows how this mapping can be done systematically and provides a better handling of uncertainty in the process using an interval-valued output as a final result.

Very often, MCDM models encounter steps in which they are required to rank output values. A basic example is a conventional T1-TOPSIS model whose inputs are preferences on a finite set of alternatives $1,...,n$, and whose outputs are CC values as a crisp number. Choosing the best alternative needs the definition of an ordering of the CC values. If they are real-numbers, there is straight-forward ranking according to the standard properties of real numbers. However, when the results are in interval form rather than real number, the ranking procedure in the real number system cannot be applied. As intervals can exhibit partial overlap, a problem of interval comparison materializes. This problem has already been addressed in many studies, which were pioneered by Moore [20]. For other studies, see [21–24]. Even though different comparing interval methods have been proposed, they can only be applied in certain intervals conditions. Given that the interval outputs from the IT2-TOPSIS model developed here are completely arbitrary, an ordering method is required that deals with all forms of overlapping and non-overlapping intervals.

1.2 Aim and objectives

This thesis aims to develop a novel extension of IT2-TOPSIS model with particular attention to improve handling elements of decision-making process that are subject to uncertainty using fuzzy sets and numbers. More specifically, the proposed model focusses on the ways in which the initial uncertainties manifest themselves in the final evaluation. In order to do this, the following objectives were identified:

1. Identify the issues and limitations of the current models of TOPSIS by providing the details of the key stages involved in standard versions of TOPSIS (non-fuzzy) and T1-TOPSIS models (fuzzy). Implement the T1-TOPSIS model with the dataset provided by Chen [1] to demonstrate the capabilities of the model and identify the issues of the model. Propose a framework to extend the current model to solve the identified issues.

2. Explore multiple variants of the extension of T1-TOPSIS model for solving MCDM problems with a specific focus on detailing the steps involved. Make a systematic comparison between standard T1-TOPSIS model proposed by Chen [1] with the extension model from Yuen [13] in terms of what additional information is provided in Yuen's model to handle uncertainty.

3. Develop a novel extension model based on IT2-TOPSIS that can improve handling the elements of decision-making process that are subject to uncertainty using IT2 FSs by providing an interval-valued output as a final result.

4. Develop a novel experimental method that illustrates the effect of uncertainty introduced in the initial evaluation of the standard TOPSIS model to the final evaluation of the model. This experiment simulates the uncertainty by blurring the membership functions used in the initial evaluation and observing the effect of doing so in the final

evaluation. Apply the proposed model to different datasets; one from literature [1] and the other, a novel dataset collected as part of this research. Compare the behaviour of these two models in terms of reflecting the uncertainty throughout the models.

5. Given that the proposed model produces an interval number as final output, construct an extended measure to compare a pair of interval values and provide ranking for both intervals. Integrate the proposed measure with the proposed IT2-TOPSIS model and apply it to the novel dataset. Finally, observe and evaluate the framework based on the implementation on the novel dataset.

1.3 Contribution overview

The contributions of this thesis are as follows:

1. An independent analysis of various TOPSIS models and how they include fuzzy for uncertainty handling. A detail analysis of the identified issues and limitations of the standard versions of TOPSIS (non-fuzzy) and T1-TOPSIS (fuzzy) models is provided along with an overview of the key stages involved in both models. A systematic comparison in a stepwise manner is given to identify issues and limitations of both models. The implemented T1-TOPSIS model with dataset from Chen [1] is given and the capabilities of the model is demonstrated using the dataset. The findings of this analysis provide an overall picture of the development of TOPSIS research specifically in terms of handling uncertainty. This could help practitioners and scholars to identify and assess the efforts that have been exerted towards the advancements of research related to this field.

2. Explore various T1-TOPSIS (fuzzy) models to pick the best underlying base framework which include a systematic comparison between the standard T1-TOPSIS model with the recenty extended model. A framework that systematically compares both models

which utilized the new rating scale is proposed. This may give significant comparison results on what value the additional information can provide and in which situations it is warranted specifically in terms of handling uncertainty.

3. A novel extension model using fuzzy set is proposed to improve uncertainty handling. Uncertainty in this proposed model is processed through each step to produce an interval output. This proposed model establishes a logical path of development with the use of interval-values as a final result. This will help in developing new lines of research for the future and in advancing the use of this model in other applications.

4. A novel experimental method which simulate the introduction of uncertainty in the initial evaluation to the final evaluation is given to show that the uncertainty in initial evaluation affects the final evaluation of decision-making process. It is designed to deeply understand the relation of uncertainty in the input to the uncertainty in the output. This experimental method is based on sensitivity analysis which used data in literature review and real-world problem.

5. An extended measure that compares different interval-valued outputs based on different cases is proposed and integrated with the proposed IT2-TOPSIS model.

This research has contributed to 3 peer-reviewed conference papers and 1 journal paper that is under preparation. These publications are listed below.

- Madi, E. N., Garibaldi, J. M., & Wagner, C. (2018). Improved Uncertainty Input-output Mapping of Type-2 Fuzzy TOPSIS Model. In preparation for IEEE Transactions on Fuzzy Systems.

- Madi, E. N., Garibaldi, J. M., & Wagner, C. (2017). Exploring the Use of Type-2 Fuzzy Sets in Multi-criteria Decision Making based on TOPSIS. In IEEE International Conference on Fuzzy Systems (FUZZ-IEEE) (pp. 1–6). IEEE.

- Madi, E. N., Garibaldi, J. M., & Wagner, C. (2016). An Exploration of Issues and Limitations in Current models of TOPSIS and Fuzzy TOPSIS. In IEEE International Conference on Fuzzy Systems (FUZZ-IEEE) (pp. 2098–2105). IEEE.

- Madi, E. N., Garibaldi, J. M., & Wagner, C. (2015). A comparison between two types of fuzzy TOPSIS model. In IEEE International Conference on Systems, Man, and Cybernetics (pp. 291–297). IEEE.

1.4 Organisation of the thesis

The thesis is organised as follows:

- Chapter 2 provides the essential background and overview of the relevant literature. It provides a general definition and a discussion of several models used to handle uncertainty, some fundamental concepts of fuzzy set theory (FST), a brief overview of another alternative that is commonly used to model uncertainty, namely the interval number, and a review of several MCDM models. This chapter also gives an overview of the standard TOPSIS (non-fuzzy) and interval TOPSIS (crisp interval) models used in most of the current literature.

- Chapter 3 provides an exploration of issues and limitations of the current TOPSIS model by providing details of the key stages involved in standard versions of TOPSIS (non-fuzzy) and T1-TOPSIS (fuzzy) models. The implementation of the standard T1-TOPSIS model with the dataset provided by one of the studies in the literature, Chen [1], is given to demonstrate the capabilities of the models in solving this decision-making problem. Additionally, the potential solution of the framework is given and explained in the chapter. Next, an exploration of multiple variants of the extension of T1-TOPSIS model is presented with a specific focus on detailing the steps involved. A

systematic comparison between standard T1-TOPSIS proposed by Chen [1] and the extension version proposed by Yuen [13] is described in a step-wise manner.

- Chapter 4 proposing a novel extension of IT2-TOPSIS model which provides an interval-value as a final output. The details of the proposed model is explained in detail. Next, this chapter describes the experimental method which is developed to illustrate the effect of uncertainty introduced in the initial evaluation of the standard TOPSIS model to the final evaluation of the model. The implementation of the proposed model to different datasets is demonstrated in this chapter.

- Chapter 5 proposes an extension measure to compare two or more interval-valued outputs based on the IT2-TOPSIS model. The chapter also shows how the measure is applied by using several synthetic examples. The integration of the measure to the proposed IT2-TOPSIS model is shown in this chapter.

- Chapter 6 summarizes the research and highlights its contributions and limitations. The scope of future work is suggested, addressing the need of MCDM models to model uncertainty in a better way by providing interval-valued outputs to the DMs.

Chapter 2

Literature Review

2.1 Introduction

This chapter provides the necessary background of fundamental concepts in literature for the presentation and definitions of the methodology used in the thesis. These concepts underpin the new model proposed in this thesis. As stated in Chapter 1, the aim of this thesis is to develop a novel extension of IT2-TOPSIS model with particular attention to improve handling the elements of decision-making process that are subject to uncertainty using fuzzy sets and numbers. More specifically, the proposed model focusses on the ways in which the initial uncertainties manifest themselves in the final evaluation. This chapter begins with Section 2.2 which provides a general definition and a discussion of several models used to handle uncertainty. Section 2.3 presents the Fuzzy Set Theory, the theory that first handled uncertainty in a comprehensive manner, and which has now been widely used. Some important definitions and background of the theory are given including the properties of a type-1 fuzzy set (also known as just a 'fuzzy set'), fuzzy number, the so-called type-2 fuzzy set, which is an extension of the type-1 fuzzy set and linguistic variable. Section 2.4 provides a brief overview of another alternative that is commonly used to model uncertainty, namely the interval number. This interval number is used in this thesis to represent the output of the

proposed model. The basic terms and concepts as well as order relations are discussed in this section. Section 2.5 reviews several multiple criteria decision making (MCDM) models and their extension of fuzzy set theory in order to focus on modeling uncertainty to solve MCDM problems. The classification of MCDM models into two different groups, Multi-Attribute decision making (MADM) and Multi-Objective Decision Making (MODM) is also discussed. There are many MCDM models in the literature. One of the commonly used models is the Technique to Order Preference by Similarity to Ideal Solution (TOPSIS). Section 2.6 presents this model in detail, including its extension to a fuzzy version (fuzzy TOPSIS). Related works on TOPSIS, interval TOPSIS and fuzzy TOPSIS models are also discussed. The key stages of standard TOPSIS model (non-fuzzy) and interval TOPSIS (I-TOPSIS) are given in this section. Section 2.7 provides the details and key stages of a standard TOPSIS while Section 2.8 gives details overview of key stages of a T1-TOPSIS (fuzzy version). Finally, Section 2.9 provides the chapter summary.

2.2 Modelling Uncertainty

It is not surprising to see that uncertainty always exists in the human world. It is one of the key characteristics of any mathematical system that attempts to model the real world. It has a significant role in any attempts to maximize the usefulness of system models. In the context of MCDM applications, the definition of uncertainty given by Zimmermann [25] seems appropriate to be applied in real-world. The proposed definition of uncertainty is quoted from Zimmermann as follows:

> "Uncertainty implies that in certain situation a person does not dispose about information which quantitatively and qualitatively is appropriate to describe, prescribe or predict deterministically and numerically a system, its behaviour or other characteristics".

At the most basic level of understanding, uncertainty relates to the lack of certain knowledge about something. As mentioned in Chapter 1, MCDM is concerned with the selection of alternatives in the presence of conflicting criteria. In most decision problems, the process of evaluating the alternatives is usually described as a complex task with some unknown attributes or criteria leading to an uncertain condition. Such uncertainty is known as a distinct lack of certainty and may derive from various sources [26]. A variety of uncertainty forms therefore exists, for example as stated in [27]. A number of authors has suggested specific categories to differentiate various sources of uncertainty in the context of decision making [25, 27]. For example, in [27], it is stated that uncertainty may exist in three different stages of a model:

i. Uncertainties expressed during modelling (e.g., about what might happen or what could be done, about meaning / ambiguity);

ii. Uncertainties expressed during exploration of the models (e.g., arising from lack of knowledge, uncertainty about judgements);

iii. Uncertainty expressed during interpretation of the model (e.g., uncertainty about the depth of an analysis which needs to be conducted).

Mendel [28] classified two types of uncertainty: linguistic and random. Apparently, in the case of decision making, linguistic uncertainty may occur since the high majority of decision makers give their preferences in words, and words can mean different things to different people [29]. Besides, more uncertainties may occur in complex decision making contexts [30], such as:

i. Limited amounts of reliable information;

ii. Very limited numbers of reliable predictions of the characteristics, properties and behaviours of complex systems;

iii. Poorly defined goals, attributes and criteria that need to be considered in the decision process;

iv. Lack of knowledge in defining a number of factors and criteria.

Neglecting these uncertainties can lead to a subsequent decrease in the quality of decision. To address uncertainties in any decision-making model, one should know and decide on how to represent them. Several formats exist to express uncertainties particularly within MCDM models. However, this chapter will not attempt to review the various formats. According to [31], there are four common ways to express the uncertainty in the current literature. These are listed as follows:

i. *Probabilities*. The use of probabilities, probability distributions and their extensions, termed as degrees of belief.

ii. *Decision Weights*. The use of generalization of probabilities theory known as decision weights. A large number of papers suggest that when making decision, people differentially weight the importance of uncertain outcomes.

iii. *Explicit risk measure*. This measure aims to identify the impacts of uncertainties on preferences using at least one or a small number of summarised measure. These measures include variances, ranges and quantiles. For example, the problem of risk measurement in gambling activity. One can relate that the risk is increased with an increase in range, variance or expected loss of gamble, and it can be decreased with the addition of a constant positive amount to all outcomes of a gamble.

In this thesis, fuzzy set theory is used in the decision model TOPSIS to assist DMs in making a selection of alternatives with respect to multiple criteria. This theory will be explained in the next section. The theory is chosen in the main study as to consider several sources of uncertainty that cannot be modelled by other theories. For example, does an individual who

wants to buy a 'cost-effective' car, really understand what 'cost-effective' really means? The use of fuzzy set theory can help reduce this individual's lack of clarity and clarify his or her thoughts. Another reason leading to the application of fuzzy set theory in decision making model is the huge amount of linguistic information used to describe many situations. Such situation is known as ambiguity where the linguistic information used in the process may have different meanings to different people. Additionally, different people might have different preferences in different contexts [32]. These differences may occur due to differences in expertise, culture, academic level, etc.

Apart from the various sources of uncertainty, another important aspect that needs to be considered when managing uncertainty in decision making process is the types of available information [25]. These can include linguistic, interval-valued or numerical type of information. In this thesis, the proposed model considers the linguistic type of information. The information provided in the decision model is therefore given in a natural language instead of in a formal language [7]. Recent studies that applied fuzzy sets were interested in extending it to type-2 fuzzy sets (i.e., an extension of type-1 fuzzy setz) due to its ability in handling the linguistic uncertainty. This extended set which was originally proposed by Zadeh [14] consists of membership grades that are fuzzy instead of $[0, 1]$, as is usually the case in type-1 fuzzy sets. Section 2.3.3 provides both basic properties and details of these sets.

Several studies have been conducted which have demonstrated successfully that type-2 fuzzy sets are better in handling linguistic uncertainty. For example, Mendel [33] conducted a survey to observe how different people interpret a same word (referred as labels in the survey), without stating any specific context. The analysis of the survey showed that words can mean different things to different people, and it is this that causes the uncertainty. The result also showed that type-2 fuzzy set can model such uncertainties and type-1 fuzzy set can not incorporate such uncertainty. This was supported by [34] in which it is stated that

implementations of T2 FSs allow us to handle linguistic and numerical uncertainties in a better way. Also in [35], in which it was claimed that type-2 fuzzy sets have the ability to model linguistic information by stating:

> "Type-2 Fuzzy Sets (T2 FSs) allow for linguistic grade of membership, thus assisting in knowledge representation. They also offer improvement on inferencing with type-1 sets".

In addition to the claim stated previously, it was shown in [36] and [37], that T1 FSs have limitations in handling the linguistic uncertainties concerning the flexibility and consistency in experts' decision making. In contrast, T2 FSs' ability to handle the uncertainties and better model the human decision-making process was highlighted. Mendel [38] stated that the use of T1 FSs for words is scientifically incorrect, as fuzzy sets are inherently certain and cannot be used to model something that is uncertain (e.g. words). Wagner and Hagras [39] stated that the use of T2 FSs in approximate reasoning can only be defined numerically by approximate meaning. Hence, the use of T2 FSs may hold a greater promise in imitating human reasoning. Before the discussion on T2 FSs takes place, it is necessary to provide a background of T1 FSs to gain a better understanding of why and how the decision making model needs to undergo a transition from T1 to T2 FSs in terms of modeling uncertainty. The next section provides the necessary background and properties of this theory.

2.3 Fuzzy Set Theory

2.3.1 Type-1 Fuzzy Sets - Definition

T1 FSs (commonly known as fuzzy sets), were first introduced by Zadeh [6] as an extension of crisp sets to model the imprecision and uncertainty inherent in assigning memberships of elements to real-world sets (e.g., set of old people or the set of tall people). In a crisp set, the membership of an element in a set can be described as either belonging (full membership) or

not belonging to the set (no membership at all). Such membership can be represented using the membership function (MF) μ_A of A where [40–42]:

$$\mu_A(x) = \begin{cases} 1, & \text{if} \quad x \in A \\ 0, & \text{if} \quad x \notin A \end{cases} \tag{2.1}$$

Then, Zadeh [6] extended this definition by introducing T1 FSs with a concept of partial membership in which values range between 0 and 1. The non-membership grade can be obtained by one minus the membership grade [6, 43]. For example, the FS of tall people may be represented as:

$$tall = 0.2/160\ cm\ +\ 0.5/165\ cm\ +\ 1/170\ cm \tag{2.2}$$

where the $+$ sign denotes the collection of all points in the domain while $/$ sign links each element with its corresponding membership grade. Now each person can be thought of as having a membership grade within the range of 0 and 1. Based on (2.2), a 170 cm tall individual has a membership grade of 1 indicating that they are certainly in the set of tall people. A 165 cm tall individual on the other hand has a membership grade of 0.5 which means that they are considered half in the set and half not in the set of tall people.

The formal definition of a T1 FS is given by Zadeh [6] as:

Definition 1 *[6] A type-1 fuzzy set, A in X is characterized by a membership function $\mu_A(x)$ that determines the degree of membership of each element $x \in X$ and is defined as:*

$$A = \{(x, \mu_A(x)) | \forall x \in X\} \tag{2.3}$$

Thus, the nearer the value of $\mu_A(x)$ to 1, the higher the grade of membership x in A.

Alternatively, in a continuous function, the FS A is defined as [42]:

$$A = \int_X \mu_A(x)/x \tag{2.4}$$

or in a discrete function, FS A is defined as [42]:

$$A = \sum_X \mu_A(x)/x \tag{2.5}$$

where $\int$ and $\sum$ denote a collection of all admissible points $x \in X$. T1 FSs are two dimensional as shown in Figure 2.1. Once the membership function value is chosen, the uncertainty disappears because the membership degrees of T1FSs are completely precise.

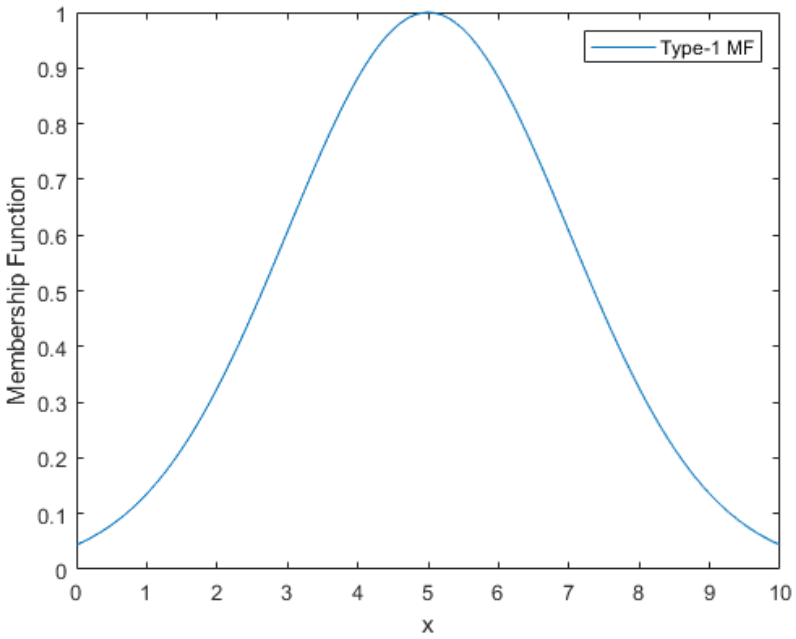


Fig. 2.1 A Gaussian type-1 membership function

2.3.2 Properties of Type-1 Fuzzy Sets

The universe of discourse (UoD) of fuzzy set is defined as the universal set of values over a base variable range. In the example of the set of tall people, the base variable is height and the UoD may be within the range of 155 cm to 180 cm.

The height, $h(A)$, of a FS A is the largest membership grade obtained by any element in that set. It can be defined by:

$$h(A) = \sup_{x \in X} \mu_A(x) \tag{2.6}$$

A T1 FS A is described as normal when at least one of its elements attain maximum possible of membership grade, that is $h(A) = 1$; it is subnormal when $h(A) < 1$ [40].

An α-cut of FS A is a crisp set A_α which consists of all the elements of the universal set X, characterized by a membership grade in A that is greater than or equal to the specified value of α. It can be formally defined as [44]:

$$A_\alpha = \{x | \mu_A(x) \geq \alpha\} \tag{2.7}$$

Another property that is important in FST is convexity. A FS is convex if and only if each of its α-cuts is a convex set. It can be written as [44]:

$$\mu_A(\lambda)x_1 + (1-\lambda)x_2 \geq \min[\mu_A(x_1), \mu_A(x_2)] \tag{2.8}$$

for all x_1 and $x_2 \in \mathbb{R}$ and all $\lambda \in [0,1]$, where min denotes the minimum operator. In general, the shape of the membership functions can be any shape, however several of the common shapes used in practice are triangular, trapezoidal, Sigmoid, Gaussian or S-shaped. An example of triangular and trapezoidal MFs are illustrated in Figure 2.2, where the former is denoted with A and latter is denoted with B.

A convex and normalized fuzzy set defined on $\mathbb{R}$ whose membership function is piecewise continuous is called fuzzy number. Real-world problems often involve many complex

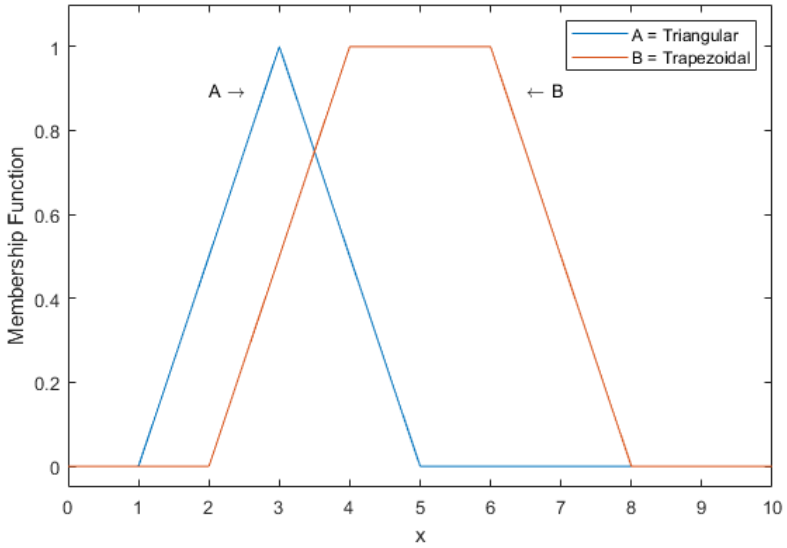


Fig. 2.2 Membership functions of shape (a) triangular, (b) trapezoidal

quantities of inaccurate information described in numerical values. Based on this problem, many people use words such as "around" and "about" in such cases. Consider when we want to measure the height of a person. We normally get a numerical value that includes a level of imprecision. There are many reasons that contribute to this imprecision. For example, it may be caused by measuring tool or by the person who took the measurements. As a result, a "precise value" (real number) h is chosen as the person's height. Formally, the expression "around" or "about" can be represented by a fuzzy subset A, in which the domain of the MF is the set of real numbers. Furthermore, in this case, the expectation that $\mu_A = 1$ is reasonable to make. In FST, this case can be modelled by using the concept of a fuzzy number.

Such concepts have already been used in many applications such as approximate reasoning, fuzzy control, decision making [45, 46] and optimization [47] where it can be used to characterized fuzzy variables. Dubois and Prade [48] described the fuzzy number A as any fuzzy subset of the real line $\mathbb{R}$ with membership function μ_A which holds the following properties:

i. μ is a continuous mapping from $\mathbb{R}$ to $[0, 1]$;

ii. $\mu_A(x) = 0, \forall x \in (-\infty, a]$;

iii. μ is strictly increasing on [a,b];

iv. $\mu_A(x) = 1, x \in [b, c]$;

v. μ is strictly decreasing on $[c, d]$;

vi. $\mu_A(x) = 0, \forall x \in [d, \infty)$;

where a, b, c and d are real numbers with $a \leq b \leq c \leq d$. The fuzzy number A can also be defined as:

Definition 2 *[48] The fuzzy number A (or type-1 fuzzy number) will have a membership function μ such that*

$$\mu_A(x) = \begin{cases} \mu_A^L(x) & \text{for } a \leq x \leq b \\ 1 & \text{for } b \leq x \leq c \\ \mu_A^R(x) & \text{for } c \leq x \leq d \\ 0 & \text{otherwise} \end{cases} \tag{2.9}$$

where $\mu_A^L(x)$ and $\mu_A^L(x)$ are left and right of membership functions of A, respectively. A can be denoted as (a, b, c, d).

Based on this definition, if $b = c$, type-1 triangular fuzzy number is obtained and can be denoted as (a, b, c), where $a \leq b \leq c$. It can be defined mathematically by:

Definition 3 *A type-1 triangular fuzzy number (T1 TFN) A with μ_A as its MFs is defined on $\mathbb{R}$ by*

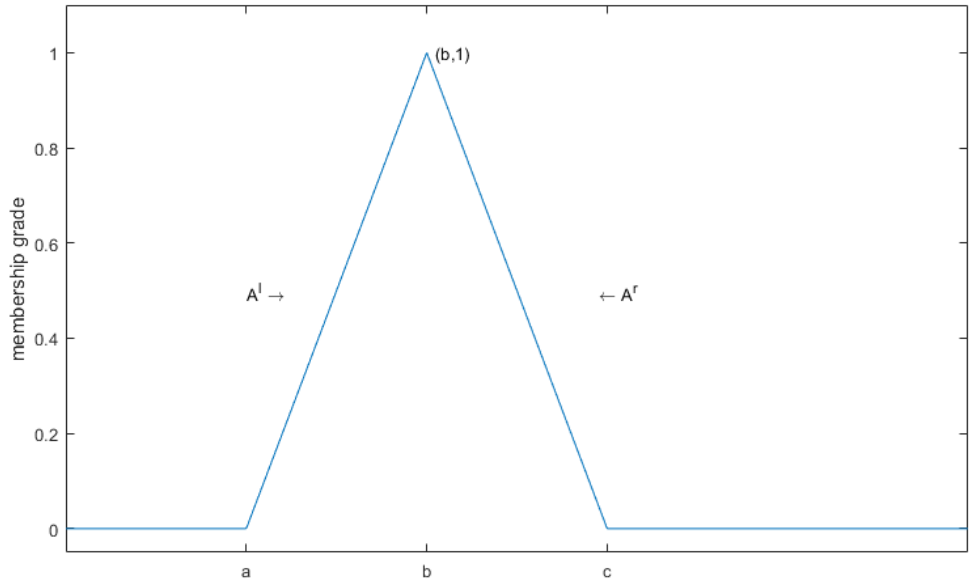


Fig. 2.3 Example of T1 TFN

$$\mu_A(x) = \begin{cases} 0, & x < a \\ \mu_A^L = \dfrac{x-a}{b-a}, & for\ a \le x \le b \\ \mu_A^R = \dfrac{c-x}{c-b} & for\ b \le x \le c \\ 0, & x > c \end{cases} \tag{2.10}$$

where a, b and c are real numbers.

An example of a T1 TFN is illustrated in Figure 2.3.

2.3.3 Type-2 Fuzzy Sets (T2 FSs)

The concept of a T2 FS was first introduced by Zadeh [14] as an extension of T1 FS. As mentioned in Section 2.3.1, a T1 FS is defined by a precise MF (crisp) where in reality, there is no uncertainty present. In contrast, T2 FSs are able to handle such uncertainties because their MFs are themselves fuzzy. While T1 FSs are two-dimensional, T2 FSs are

three-dimensional which it provides additional degrees of freedom that make it possible to directly model uncertainties. Dubois and Prade [43, pp. 256] discussed the problem of determining T1 membership functions and stated:

> To take into account the imprecision of membership functions, we may think of using type-2 fuzzy sets.

Type-2 fuzzy sets are considered very useful and important since the linguistic grades can be used instead of using crisp measure of fuzziness. Another advantage is highlighted by Yager [49] in his paper:

> The usefulness of fuzzy subsets of type II [type-2] is that it enables us to extend membership grades to linguistic values.

Hence, type-2 fuzzy sets can be used to have membership grades that are words rather than numbers in $[0,1]$. A T2 FS consists of two instances namely: General type-2 fuzzy set (GT2 FS) and Interval type-2 fuzzy set (IT2 FS). The mathematical definition of a GT2 FS is given as follows.

Definition 4 *[42] A GT2 FS $\tilde{A}$, is characterized by a type-2 membership function (T2 MF) $\mu_{\tilde{A}}(x,u)$, where $x \in X$ and $u \in J_x \subseteq [1,0]$ defined by:*

$$\tilde{A} = \{((x,u), \mu_{\tilde{A}}(x,u)) | \forall x \in X, \forall u \in J_x \subseteq [0,1]\} \tag{2.11}$$

where x is the primary variable, u is the secondary variable and J_x is the support of the secondary membership function in the third dimension of x.

The third dimension of x in GT2 FS provides an additional degree of freedom to directly model uncertainty [50]. The additional membership is defined as a T1 FS instead of a single point of μ. Therefore, GT2 FSs are characterized by more parameters than are MFs for type-1 FSs. However, the use of GT2 FS in practice has been limited due to the significant increase in

computational complexity involved in their implementation. Based on this limitation, Mendel [42] has introduced a concept known as the *footprint of uncertainty* (FOU) which simplifies the graphical description of the uncertainty modelled by any type-2 set. Although the FOU and GT2FS can be used simultaneously and can provide flexibility to handle uncertainty, their computations are still very complex and rarely used in practice. As a consequence, a simplified version of T2 FS is introduced, namely: interval type-2 fuzzy set (IT2 FS). The following two definitions formally express the definition of IT2FS and FOU:

Definition 5 *[51] When all* $\mu_{\tilde{A}}(x,u) = 1$ *in (2.11), then* $\tilde{A}$ *is an IT2 FS.*

Definition 6 *[51] Uncertainty in the primary memberships of an IT2 FS,* $\tilde{A}$*, consists of a bounded region named the footprint of uncertainty (FOU). It is the union of all primary memberships, i.e.,*

$$FOU(\tilde{A}) = \bigcup_{x \in X} J_x \tag{2.12}$$

The FOU is characterized in terms of upper and lower membership functions and can be defined as:

Definition 7 *[42] An upper membership function and lower membership function are two type- 1 membership functions that are bound for the FOU of type-2 fuzzy set* $\tilde{A}$*. The upper membership function is denoted as* $\mu_{\tilde{A}^U}(x), \forall x \in X$ *which is associated with upper bound of* $FOU(\tilde{A})$ *whilst lower membership function is denoted as* $\mu_{\tilde{A}^L}(x), \forall x \in X$ *which is associated with lower bound of* $FOU(\tilde{A})$*. Both upper and lower membership functions can be written as:*

$$FOU(\tilde{A}) = \bigcup_{x \in X} J_x = \{(x,y) : y \in J_x = [\mu_{\tilde{A}^U}(x), \mu_{\tilde{A}^L}(x)]\} \tag{2.13}$$

$$= [(\underline{\mu}_{\tilde{A}^U}(x), \bar{\mu}_{\tilde{A}^U}(x)),\ (\underline{\mu}_{\tilde{A}^L}(x), \bar{\mu}_{\tilde{A}^L}(x))] \tag{2.14}$$

where FOU is shown by shaded region. $(\underline{\mu}_{\tilde{A}^U}(x),\ \bar{\mu}_{\tilde{A}^U}(x))$ *and* $(\underline{\mu}_{\tilde{A}^L}(x), \bar{\mu}_{\tilde{A}^L}(x))$ *are the left and right limit membership function of* $\mu_{\tilde{A}^U}(x)$ *and* $\mu_{\tilde{A}^L}(x)$*.*

An example of IT2 TFs with trapezoidal MF shape is shown in Figure 2.4. It can be denoted as $\tilde{A} = [(a_U, b_U, c_U, d_U; h^U), (a_L, b_L, c_L, d_L; h^L)]$ and the MFs can be defined, for example, as follows:

$$\mu_{\tilde{A}^U}(x) = \begin{cases} \dfrac{x - a_U}{b_U - a_U} & x \in [a_U, b_U] \\ h^U & x \in [b_U, c_U] \\ \dfrac{d_U - x}{d_U - c_U} & x \in [c_U, d_U] \\ 0 & \text{Otherwise} \end{cases} \tag{2.15}$$

$$\mu_{\tilde{A}^L}(x) = \begin{cases} \dfrac{x - a_L}{b_L - a_L} & x \in [a_L, b_L] \\ h^L & x \in [b_L, c_L] \\ \dfrac{d_L - x}{d_L - c_L} & x \in [c_L, d_L] \\ 0 & \text{Otherwise} \end{cases} \tag{2.16}$$

where the upper membership value $h^U \in (0, 1]$, the lower membership value $h^L \in (0, 1]$ and $h^U \geq h^L$. In this example, when $b_U = c_U$ and $b_L = c_L$, the triangular MF shape is obtained. This thesis utilized triangular MF shape to represent the rating and weight of alternatives in the decision-making process.

The interval type-2 fuzzy set can reduce the burden of computations with the third dimension values equals to 1. The IT2 FS can be expressed graphically using FOU with two dimensional (2-D) plane. The FOU has the ability to capture the uncertainties where the bounded region of FOU indicates the amount of uncertainty that has been captured within the primary membership of T2FS. Hence, the wider the FOU, the more uncertain the primary memberships are.

Apart from type-2 fuzzy set, set A can be represented as type-2 fuzzy number which holds the following properties [52]:

i. The primary membership function is type-1 fuzzy number.

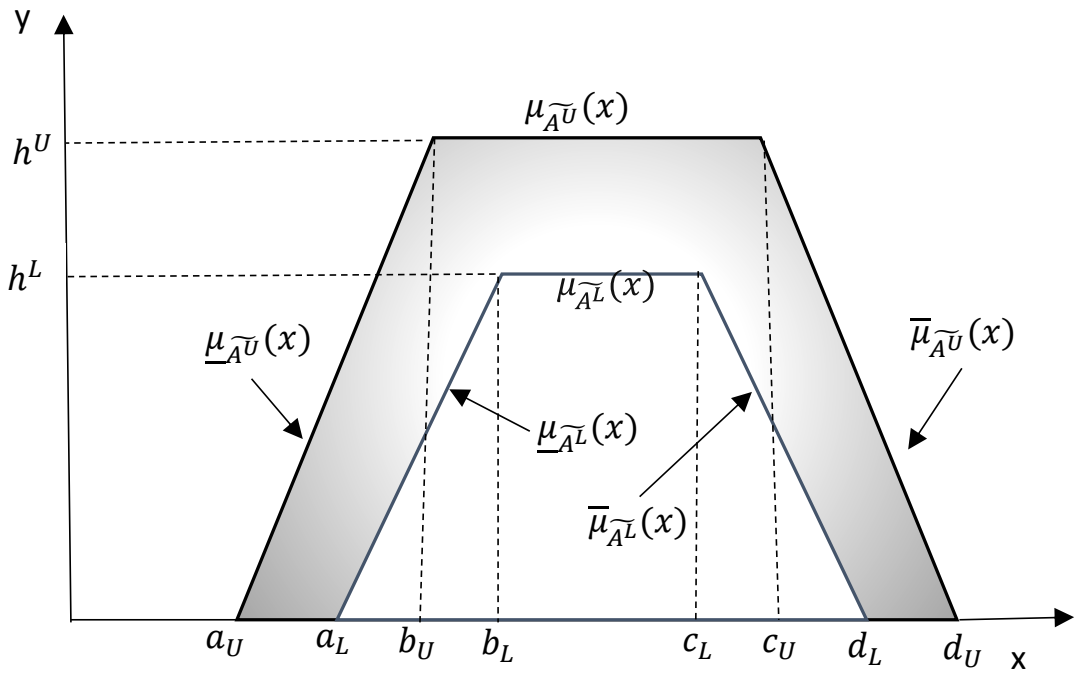


Fig. 2.4 An example of IT2 FS

ii. The lower $\underline{\mu}_{\tilde{A}}$ and upper $\bar{\mu}_{\tilde{A}}$ membership functions are upper semi-continuous.

iii. The membership function $\mu_{\tilde{A}}(x,u)$ satisfies $\mu_{\tilde{A}}(x,u) > 0$, if $\underline{\mu}_{\tilde{A}}(x) \leq u \leq \bar{\mu}_{\tilde{A}}(x)$, where $0 \leq \underline{\mu}_{\tilde{A}} \leq 1$ and $0 \leq \bar{\mu}_{\tilde{A}} \leq 1$.

Generally, type-2 fuzzy number is considered as the extension of a type-1 fuzzy number. Most recently, the definition of type-2 fuzzy number is given as:

Definition 8 *[42] $\tilde{A}$ is a type-2 fuzzy number (T2 FN) if and only if there exists a closed interval $[a,b] \neq 0$ for each UMF $(\bar{\mu}_{\tilde{A}})$ and LMF $(\underline{\mu}_{\tilde{A}})$ such that:*

$$\mu_{\tilde{A}}(x,u) = \begin{cases} 1, & \text{for } x \in [a,b],\ u \in J_x \subseteq [0,1] \\ \mu_A^L(x,u) & \text{for } x \in (-\infty,a],\ u \in J_x \subseteq [0,1] \\ \mu_A^R(x,u) & \text{for } x \in [b,\infty),\ u \in J_x \subseteq [0,1] \end{cases} \tag{2.17}$$

where $\mu_A^L : (\infty,a] \to F([0,1]), u \in J_x \subseteq [0,1]$ is monotonic non-decreasing and continuous from the right, and $\mu_A^R : [b,\infty) \to F([0,1]), u \in J_x \subseteq [0,1]$ is monotonic non-increasing and continuous from the left.

The proposed model in this thesis used interval type-2 fuzzy number to represent the input which described linguistic variable in the model. Next section presents the concept of linguistic variable which exist in the fuzzy set theory.

2.3.4 Linguistic Variable

The concept of a linguistic variable was introduced by Zadeh in 1973 where the definition of linguistic variable is a variable whose values are sentences or words in natural or artificial language [53]. The paper highlighted the need to use an approximate value to describing behaviour systems that are too complex and too ill-defined to apply any precise mathematical expression. Then, an extended version of linguistic variables was presented in 1975 [14, 54, 55].

Formally, Zadeh's [54] definition of linguistic variables is as follows:

Definition 9 *A linguistic variable is a quintuple $(L,T(L),U,S,M)$ such that:*

- *L is a variable name,*
- *$T(L)$ is a finite term set or labels of L, that is the collection of its linguistic values,*

- *U is a universe of discourse,*
- *S is the syntactic rule to generate the terms in $T(L)$,*
- *M is the semantic rule associated with each linguistic label X, that is the meaning of $M(X)$, where $M(X) \subset U$.*

To illustrate the concept of a linguistic variable, let consider the concept of *age*. This concept cannot be characterized precisely by most of the people since different people have different preferences on the same concept (i.e., *age*). However, by employing fuzzy sets, we can approximately describe this concept as follows (quoted from Klir and Yuan [40]):

- *Age* is a linguistic variable in which values are described in a linguistic label, $T(Age) = \{very\ young, young, middle\ age, old, very\ old\}$.
- The linguistic labels are characterized by fuzzy sets on the universal set $U \subset \mathbb{R}$, namely the *operating domain*.
- The domain represents the base variable in which the values are real numbers within a specific range.

Figure 2.5 shows an example of the linguistic variable *age* on the universal set, $U = [0, 100]$, known as base variable illustrating age in years. The MFs are used for the labels of *very young. young, middle age, old, very old*. The concept of linguistic variables can be considered as an important role in various applications, for example in financial, control and management systems. Such applications utilize the concept of linguistic variables to represent, for example, confidence, risk and profit. Apart from fuzzy set has been used as an alternative to represent information, an interval number can be used as a way to represent information. The next section provides an overview of basic concepts in the interval numbers theory.

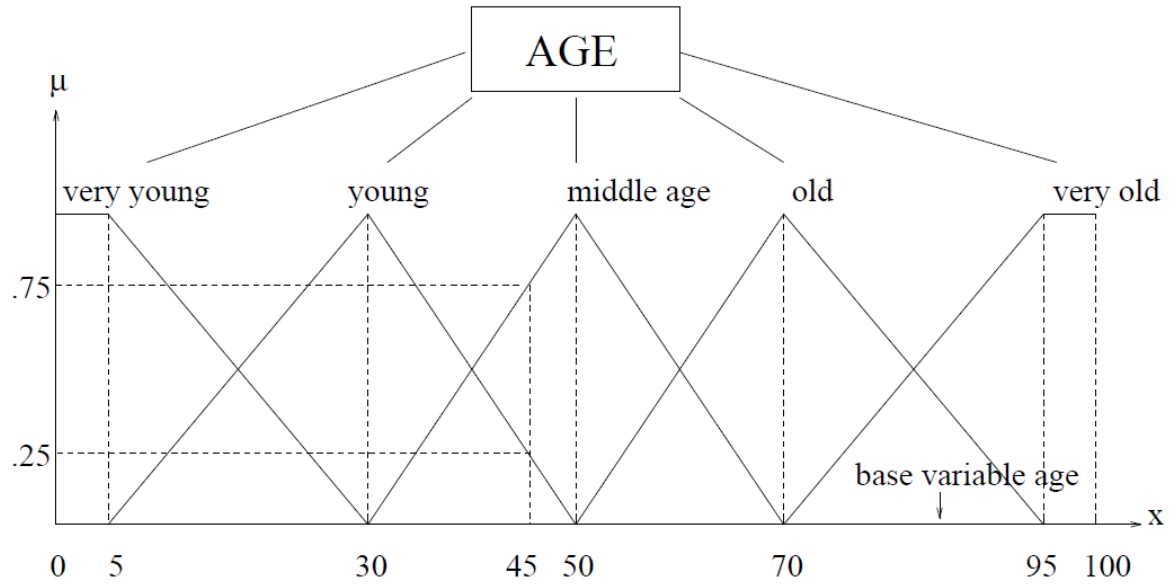


Fig. 2.5 Linguistic terms which represent the variable *age*. Figure is adapted from [2]

2.4 Interval Number

The concept of interval number represents the range values of confidence at one unique level, that is either at level 0 or 1. This confidence level is termed, *level of presumption* by Kaufmann and Gupta [56]. The maximum level of presumption in interval number is defined at level 1 while the minimum presumption at level 0. Let us assume in one situation, it has uncertain values and the values belong to a set of real numbers. In practice, it is possible to represent such values in interval by specifying its maximum value (known as upper bound) and its minimum value (known as lower bound). Then, such values can be represented inside a closed interval of real number $\mathbb{R}$, that is an interval number of $\mathbb{R} : [a_L, a_U]$. This representation of value is defined as an interval of confidence since one is certain that the value is greater or equal to a_L and smaller or equal to a_U. It is described as a generalization of real numbers and are subsets of $\mathbb{R}$ (the set of real numbers). Just as real numbers have an associated arithmetic and mathematical analysis, interval numbers have a distinct interval arithmetic and interval analysis. A formal definition of interval number is given as:

Definition 10 *[57] A closed interval number A is defined as*

$$\bar{a} = [a_L, a_U] = x : a_L \leq x \leq a_U,\ x \in \mathbb{R} \tag{2.18}$$

where $[a_L, a_U]$ *are the lower and upper bounds of the interval* $\bar{a}$*, respectively.*

The values of a_L and a_U are usually finite. However, in some cases, it can be infinite such that $a_L = -\infty$ and/or $a_U = \infty$. There exists another useful representation other than closed interval. Several commonly used notations that represent different conditions in interval are listed below:

- $]a_L, a_U]$ or $(a_L, a_U]$; left-open interval;
- $[a_L, a_U[$ or $[a_L, a_U)$; right-open interval;
- $]a_L, a_U[$ or (a_L, a_U); open interval.

In this thesis, the closed interval is considered to represent the final output of the proposed decision-making model. Hence, in this thesis, the term *interval* means *closed interval*. The next section gives an overview of the basic terms and concepts of interval number that are mainly used in this thesis.

2.4.1 Basic Terms and Concepts on interval number

This section provides a necessary fundamental background of basic terms and concepts useful in interval analysis. These terms and concepts are mainly used in this thesis to represent the resulting decision-making output from the proposed model. This thesis uses small letter with bar as a notation for an interval number to distinguish it with the notation of fuzzy number (i.e., used as bold capital letter).

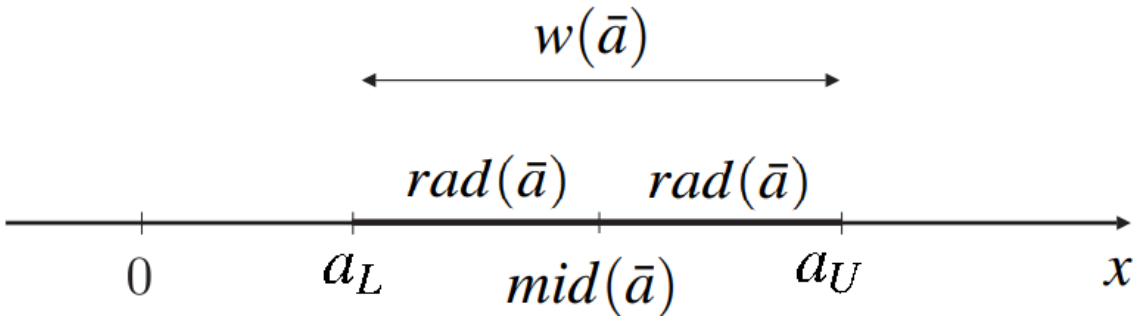


Fig. 2.6 Width, radius, and midpoint of an interval $\bar{a}$

The width of interval $\bar{a} = [a_L, a_U]$, $w(\bar{a})$ can be obtained by:

$$w(\bar{a}) = a_U - a_L \geq 0 \tag{2.19}$$

Note, each real number $a \in \mathbb{R}$ can be expressed as an interval number $[a,a]$ with 0 width.

Interval $\bar{a}$ can be represented alternatively using its midpoint $mid(\bar{a})$ and its radius $rad(\bar{a})$, $\langle mid(\bar{a}), rad(\bar{a})\rangle$ and can be calculated respectively as:

$$mid(\bar{a}) = \frac{1}{2}(a_U + a_L) \tag{2.20}$$

$$rad(\bar{a}) = \frac{1}{2}(a_U - a_L) \tag{2.21}$$

These three basic concepts are illustrated in Figure 2.6.

As intervals can, for example, exhibit partial overlap, a measure is developed to compare the resulting interval-valued output from various cases (i.e., overlapping and non-overlapping) of the interval by considering uncertainty. The measure is necessary in decision-making practice as at the end of the process, the decision has to be made and this measure can be used as a guideline to compare as well as to order several resulting interval-values output. The detail of this proposed measure will be discussed in Chapter 6. The following section reviews studies in the literature on order relation for comparing intervals which are mainly used in this thesis. Specifically, the section presents the background studies of order relation

for comparing intervals which exist in literatures. Chapter 6 will demonstrate its usefulness in the proposed decision making model.

2.4.2 Order Relations for Intervals

In a real number system, any real numbers are ordered by the relation $<$. This relation is *transitive*, that is if $a < b$ and $b < c$, then $a < c$ for any a, b and $c \in \mathbb{R}$. Then, the relation can be extended to interval number. By using the same notation, the order relation $<$ can be defined as:

$$\bar{a} < \bar{b} \tag{2.22}$$

means that interval $\bar{a}$ is less than or smaller than interval $\bar{b}$. For example, consider two interval numbers as $\bar{a} = [0,1]$ and $\bar{b} = [2,3]$. The transitive relation can still be defined as shown below:

$$\bar{a} < \bar{b} \text{ and } \bar{b} < \bar{c} \longrightarrow \bar{a} < \bar{c} \tag{2.23}$$

Another transitive order relation for intervals can be defined as a set inclusion:

$$\bar{a} \subseteq \bar{b} \text{ if and only if } b_L \leq a_L \text{ and } a_U \leq b_U \tag{2.24}$$

For example, consider two interval numbers as: $\bar{a} = [1,3]$ and $\bar{b} = [0,3]$. Thus, it can be seen that $[1,3] \subseteq [0,3]$. Note that not every pair of intervals consideration can be compared under a set of inclusion as some elements in the first interval may not be in the second interval. This situation is known as *partial ordering* [58]. Other methods for comparing interval will be discussed further in Chapter 5. The next section presents the related studies of multi-criteria decision making (MCDM) model in the literature.

2.5 Multi-Criteria Decision Making (MCDM) and Fuzzy MCDM models

Over the past four decades, many MCDM models have been developed, and their number continues to grow. Based on the surveys conducted by Aruldoss [59], MCDM is one of the dominant models for capturing the optimal choice for convoluted decision making problems. The models have been successfully implemented in various contexts, such as business, medical and management. The comprehensive reviewed of various MCDM models can be seen in [60–62]. The model aims to provide a structure in assisting decision makers involving a number of alternatives, criteria, factors or variables. MCDM is also referred to as Multi-Criteria Decision Analysis (MCDA) and Multi-Dimensions Decision Making (MDDM) [59].

Bellman and Zadeh [7] introduced the concept of 'decision making in a fuzzy environment', which means the constraints (criteria or attributes) and/or the goals that represent the class of alternatives have boundaries that are not sharply defined. Motivated by their concept, many studies have been published on MCDM which can be categorized into two main groups:

- Multiple-Attribute Decision Making (MADM)
- Multi-Objective Decision Making (MODM)

MADM is a model that makes selections among elements in a set of actions with the presence of multiple, commonly conflicting attributes in a discrete event. It is also formed with a finite number of a pre-selected set of alternatives and has discrete preference ratings [63]. It is developed to choose 'the best alternative' by considering the trade-off within a set of conflicting criteria [64]. The selection is made among the set of alternatives considering their attributes. It usually involves sorting and ranking to achieve the final ranking.

On the other hand, the MODM model is suitable for developing or structuring the model to achieve an optimal solution by considering multiple interactions among the given criteria. As compared to MADM, MODM has an infinite number of alternatives and the trade-off interaction is defined by a continuous function. Unlike MADM, MODM does not provide a set of pre-determined alternatives, but MODM develops and design such a set. Hence, the amount of alternatives may be significant [64].

There exists many MCDM methods proposed in the literature and so it may become a challenge for practitioners to choose a suitable one. A pioneer study from Ozernoy [65] provides outline procedures to help select an appropriate MCDM models. The main problem that exists relates to the term of MCDM itself, where it refers to a procedure for ordering the alternatives either to provide a rank-ordering of alternatives or to identify the most preferred alternative. The term 'most preferred alternative' can be used if the following conditions hold [65]:

i. the DMs are knowledgeable about the model's rationale and assumptions;

ii. the DMs can provide their own judgment comfortably without any pressure or difficulty;

iii. the model gives a final ranking order of the set of pre-determined alternatives.

In addition to that, Tzeng and Huang [63] suggested two significant factors that need to be taken into further consideration when choosing the 'most preferred alternative': preference/judgment structure and weights of criteria. The author further outlined the basic structure of handling an MCDM problem. Firstly, the process of identifying the problems by determining the number of attributes or criteria that exist in the problem. Secondly, the process of collecting and constructing the preferences by collecting useful data. Thirdly, the process of evaluating alternatives by determining the set of possible actions in which the aim of the process is achieved. Finally, the process of identifying the best alternative by choosing

the appropriate approach that enables the DMs to evaluate, outrank and enhance the possible actions [63].

The increasing complexity of the real-world makes it less reasonable for an individual to assess all related factors of an MCDM problem. Hence, many organizations implement Group Decision Making (GDM) which results' the introduction of a great deal of complexity in the analysis structure [66]. The integration of MCDM and GDM (termed as Multi-Criteria Group Decision Making (MCGDM)) aims to handle problems involving a set of feasible alternatives that are evaluated by multiple, conflicting and non-comparable criteria by a group of people. Thus, fuzzy set theory (FST) can be considered a powerful tool for handling such problems in GDM settings [67]. Recently, a comprehensive survey of fuzzy MCDM techniques was made by Mardani [60]. As a result, a large and growing body of literature has been proposed to integrate FST in decision making processes especially by introducing new technique using combination of FST with the existing MCDM techniques (e.g., Fuzzy TOPSIS [1, 68–70, 9], Fuzzy AHP [71–73], Fuzzy VIKOR [74]). Additionally, Ozernoy [65] stated that selecting the 'best' MCDM model can be considered as a problem that is essential as the structuring the decision situation.

2.5.1 MADM and Fuzzy MADM method

An MADM model is concerned with making selections among some courses of action in the presence of multiple, commonly conflicting attributes. Note that there are common occurrences to problems in everyday life that deal with MADM. For example, one may choose to buy a house (out of several types of houses) depending on its location, price, design, etc. One may choose to buy a car (out of several cars) depending on its price, safety, etc. Hence, MADM problems are rather typical for any individuals or organizations.

MADM problem can be expressed in matrix format as follow:

$$\dot{X} = \begin{array}{c} \\ \mathscr{A}_1 \\ \mathscr{A}_2 \\ \vdots \\ \mathscr{A}_m \end{array} \begin{array}{c} \begin{array}{cccc} \mathscr{C}_1 & \mathscr{C}_2 & \cdots & \mathscr{C}_n \end{array} \\ \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{bmatrix} \end{array}, \tag{2.25}$$

where $\mathscr{A}_i,\ i = 1, \cdots,\ m$ are possible course of actions (known as alternatives); $\mathscr{C}_j,\ j = 1, \cdots, n$ are attributes with which alternative performance are measured; x_{ij} is the performance (known as rating) of alternative $\mathscr{A}_i$ with respect to attribute $\mathscr{C}_j$. There are numerous classical MADM models in the literature. Hwang and Yoon [3] presented a comprehensive survey of the classical MADM models. Although there were successful applications in the past, however, in most cases, the decision makers cannot provide a precise rating x_{ij}. The sources of imprecision may come from [75]:

i. **Unquantifiable information**. Some of the attributes or criteria can only be assessed by using linguistic terms such as *good, fair, bad,* etc.

ii. **Incomplete information**. The use of linguistic values such as "about 50 degrees of Celcius" but not in precise term "exactly 50 degrees of Celcius". The term "about" can be represented by using fuzzy set.

iii. **Non-obtainable information**. In certain contexts, the crisp data is costly or contain privacy of individuals. Thus, it is necessary to use approximation data which can be represented by fuzzy number.

iv. **Partial ignorance**. Sometimes, one knows only part of the facts which then contributes to partial ignorance of a phenomenon.

The classical MADM models cannot effectively solve the problems with these imprecisions. Thus, as discussed in Section 2.3, fuzzy set theory can be used to resolve these difficulties. The evaluation of an alternative given by a decision maker in fuzzy MADM problem may be represented by fuzzy sets. Thus, the final outcomes are expressed by fuzzy sets as well. In general, an integration of fuzzy set theory with the MADM model is used to solve discrete choice problems that aims to make a selection of the best alternative with respect to a finite number of criteria.

Based on previous works, the integration of fuzzy set theory with MADM can be seen at least on one of the conditions listed below:

1. Final ratings represented by fuzzy set,

2. Final ranking in which the attributes are represented by fuzzy set.

The pioneering work on fuzzy MADM was made by Bass and Kwakernaak [76] where they utilized fuzzy MFs to solve a discrete problem and compared the result with the probabilistic model. During the last three decades, many fuzzy MADM models have been proposed and reviewed [75, 77, 8, 78]. Several of these were proposed as an extension to existing MADM models, such as weighted averages or expected utilities using fuzzy interval weights [79],[80], Fuzzy AHP [81], Fuzzy TOPSIS [1, 68–70, 9], fuzzy Preference Ranking Organization METHods for Enrichment Evaluations (PROMETHE) [82, 83] and ELimination Et Choice Translating REality (ELECTRE) [59, 84]. Ribeiro [85] reviewed the main theories and methods used for MADM in a fuzzy environment and classified these methods into two phases: rating and ranking. A new technique was proposed to derive the importance weight of attributes based on scaling techniques and on the case-based paradigm.

Based on Kahraman [64], there are four most frequently used MADM methods based on fuzzy concepts; fuzzy Analytic Hierarchy Process (AHP), fuzzy TOPSIS, fuzzy outranking methods and fuzzy weighting methods. Some fundamental concepts and terminology are given below. They align with the notations presented in [3].

- *Alternative*: A set of discrete and finite number of alternatives from several to thousands which need to be analysed such as filtered, screened, selected and ranked. An alternative may refer to a set of actions or a set of persons or etc.

- *Multiple attributes*: Attributes can be referred to as goals, objectives or criteria. The number of attributes can be between several to around 350 [86].

- *Decision Weights*: Most MADM models require information regarding the relative importance of each attribute usually given by a set of weights that are normalized to sum to one.

- *Decision matrix*: A decision matrix D is an $(m \times n)$ matrix whose element x_{ij} indicates the performance rating of alternative i, A_i with respect to attribute j, X_j.

Recently, several studies have been proposed to implement T2 FSs in MADM as T2 FSs are widely accepted as powerful models for capturing uncertainty [15, 16]. One study [87] used interval T2 FS (IT2 FS) to express ratings in a decision matrix and provide crisp value for ranking. In the same study, a model was proposed to determine the weight of criteria based on an optimization using T2 FS information. There exists a study which combined other theory to be applied to T2 FSs information such as soft set theory to solve MADM problems [88]. An MADM method based on ranking values to solve MADM with the setting of GDM was introduced by Chen and Lee [89]. Then, based on this study, an improved version was proposed by Chen, Lee and Yang [90] which aims to reduce the complexity of calculation and uncertainty in the process. This thesis concerned with one of the established model in MADM, which is TOPSIS. In the next section, some related works on TOPSIS and various extension of TOPSIS are reviewed.

2.6 Technique for Order Preference by Similarity to Ideal Solution (TOPSIS)

One of the well-known MCDM models is the 'Technique for Order Preference by Similarity to Ideal Solution' (TOPSIS) which was introduced by Hwang and Yoon in 1981 [3]. The fundamental principle of TOPSIS is it is based on the distance where an alternative is considered as optimal if it has the shortest distance from a so-called *positive ideal*, i.e. the best solution, and the farthest distance from a so-called *negative ideal*, i.e. the worst solution. It is regarded as the model that is most straightforward and simple to implement in the MCDM field [91]. The model extensively uses attribute information, provides a cardinal ranking of alternatives, and does not require attribute preferences to be independent. Based on classical TOPSIS, several requirements need to be considered when applying this model: all attribute values must be numeric, either increasing or decreasing and all attributes need to have commensurable units [92]. This model has been successfully implemented in various applications and it continues to work satisfactorily [92]. Its implementation in real-world applications involves, for example, but not limited to, network analysis [93, 94], healthcare industry [95], supply chain management [96, 97], e-commerce [98], risk management [99], manufacturing systems [100] and many more.

In recent years, there are numerous attempts to propose, develop, apply and extend the TOPSIS model for solving MADM problems. The standard TOPSIS model can be performed using an objective way, such that the evaluation of alternative with respect to criteria is made using real number characterized as precise value. However, in practice, many MADM problems can only be performed in a subjective way, such that the main input data are obtained from preferences of DMs. A recent study compiling several existing modifications and extended models was conducted by Zavadskas et al. [101]. They presented an overview of the development of TOPSIS model to solve complicated problems from 2000 to 2015. The

reviewed studies aimed to improve and solve the identified problems. This section does not intend to give a detail review the various extended TOPSIS models. However, this section presents two of extensions of TOPSIS, interval TOPSIS and fuzzy TOPSIS models with the particular attention to observe how the model reflects the uncertainty in the input (i.e., initial evaluation) to the uncertainty in the output (i.e., final evaluation). This can be made by comparing the input of the model (i.e. rating of alternative) with the presentation of the output (i.e. the Closeness Coefficient) values. The next section presents a discussion on several related works which used an interval number to represent the ratings of alternatives at the initial evaluation and act as an input in the TOPSIS model.

2.6.1 Related work on Interval-TOPSIS (I-TOPSIS)

One of the earlier works on interval TOPSIS was made by Giove [102]. The interval number was used to represent information given by DMs as a rating of alternative with respect to criteria. Unlike in the standard TOPSIS which used precise number to represent ratings, this interval TOPSIS model used interval numbers as an alternative way to consider uncertainty among DMs when giving the assessment. However, the used of interval number means that the confidence level of DMs about the upper bound and lower bound of the initial evaluation are precise. Thus, it is possible that the final evaluation can be represented by either interval number or crisp number. Giove, however, presented the final evaluation of the model as an interval number. Then, the comparison of resulting two or more interval numbers were made using a method proposed by Sengupta and Pal [22]. This interval comparison method will be discussed in detail in Chapter 6.

Next, Jahanshahloo et al. [103, 104] also proposed the TOPSIS model to solve decision-making problems using interval data at the initial evaluation stage. The proposed model by Jahanshahloo was directly extended from a standard TOPSIS model introduced by Hwang and Yoon [3]. The information used in this model is interval number and the model proposed

several extensions to handle interval information in every step. The proposed interval TOPSIS model by Jahanshahloo is summarized as follows: Suppose $\mathscr{A}_1, \cdots, \mathscr{A}_m$ are m possible alternatives that the DMs need to choose from, $\mathscr{C}_1, \cdots, \mathscr{C}_n$ are n criteria to which alternative evaluation needs to be considered, and x_{ij} is the rating of alternative $\mathscr{A}_i$ with respect to criterion $\mathscr{C}_j$. This model assumes that the rating x_{ij} is unknown and only knows the rating within a range of values such that $x_{ij} \in [x_{ij^L}, x_{ij^U}]$. The problem with interval data is presented in matrix format as follows:

$$\overline{\mathbf{X}} = \begin{array}{c} \\ \mathscr{A}_1 \\ \mathscr{A}_2 \\ \vdots \\ \mathscr{A}_m \end{array} \begin{array}{c} \begin{array}{cccc} \mathscr{C}_1 & \mathscr{C}_2 & \cdots & \mathscr{C}_n \end{array} \\ \begin{bmatrix} [x_{11^L}, x_{11^U}] & [x_{12^L}, x_{12^U}] & \cdots & [x_{1n^L}, x_{1n^U}] \\ [x_{21^L}, x_{21^U}] & [x_{21^L}, x_{21^U}] & \cdots & [x_{2n^L}, x_{2n^U}] \\ \vdots & \vdots & \ddots & \vdots \\ [x_{m1^L}, x_{m1^U}] & [x_{m2^L}, x_{m2^U}] & \cdots & [x_{mn^L}, x_{mn^U}] \end{bmatrix} \end{array}, \tag{2.26}$$

$$\mathbf{W} = \begin{bmatrix} w_1 & w_2 & \cdots & w_n \end{bmatrix}, \tag{2.27}$$

where w_j is the weight of criterion $\mathscr{C}_j$, L and U denote the lower and upper bound of interval. The algorithm of interval TOPSIS proposed by Jahanshahloo et al. [103, 104] is listed below:

1. Normalized the rating, $x_{ij} \in [x_{ij^L}, x_{ij^U}]$ to $\bar{n}_{ij} \in [\bar{n}_{ij^L}, \bar{n}_{ij^U}]$ by

$$\bar{n}_{ij^L} = \frac{x_{ij^L}}{\sqrt{(x_{ij^L})^2 + (x_{ij^U})^2}},\ j = 1, \cdots,\ m;\ i = 1,\ \cdots,\ n, \tag{2.28}$$

$$\bar{n}_{ij^U} = \frac{x_{ij^U}}{\sqrt{(x_{ij^L})^2 + (x_{ij^U})^2}},\ j = 1, \cdots,\ m;\ i = 1,\ \cdots,\ n, \tag{2.29}$$

2. Construct the weighted normalized interval decision matrix, $\bar{v}_{ij} = [\bar{v}_{ij^L}, \bar{v}_{ij^U}]$ as:

$$\bar{v}_{ij^L} = w_j \bar{n}_{ij^L},\ j = 1, \cdots,\ m;\ i = 1,\ \cdots,\ n, \tag{2.30}$$

$$\bar{v}_{ij^U} = w_j \bar{n}_{ij^U},\ j = 1, \cdots,\ m;\ i = 1,\ \cdots,\ n, \tag{2.31}$$

where w_j is the weight of criterion defined in (2.27).

3. Identify positive and negative ideal solutions as:

$$\bar{V}^+ = \{\bar{v}_1^+, \cdots, \bar{v}_n^+\} = \{(\max_j \bar{v}_{ij^U} | i \in \mathbb{B}), (\min_j \bar{v}_{ij^L} | i \in \mathbb{C})\}, \tag{2.32}$$

$$\bar{V}^- = \{\bar{v}_1^-, \cdots, \bar{v}_n^-\} = \{(\min_j \bar{v}_{ij^L} | i \in \mathbb{B}), (\max_j \bar{v}_{ij^U} | i \in \mathbb{C})\} \tag{2.33}$$

where $\mathbb{B}$ and $\mathbb{C}$ are associated with benefit and cost criteria, respectively. Note that the values of ideal solution are now reduced to real number instead of interval number.

4. Calculate the distance of each alternative from positive ideal solution using n-dimensional Euclidean distance as:

$$\bar{d}_j^+ = \left(\sum_{i \in \mathbb{B}} (\bar{v}_{ij^L} - \bar{v}_i^+)^2 + \sum_{i \in \mathbb{C}} (\bar{v}_{ij^U} - \bar{v}_i^+)^2 \right)^{\frac{1}{2}},\ j = 1, \cdots,\ m. \tag{2.34}$$

$$\bar{d}_j^- = \left(\sum_{i \in \mathbb{B}} (\bar{v}_{ij^U} - \bar{v}_i^-)^2 + \sum_{i \in \mathbb{C}} (\bar{v}_{ij^L} - \bar{v}_i^-)^2 \right)^{\frac{1}{2}},\ j = 1, \cdots,\ m. \tag{2.35}$$

5. Determine closeness coefficient ($\dot{CC}_i$) in order to rank the alternatives. $\dot{CC}_i$ is obtained by

$$\dot{CC}_i = \frac{\bar{d}_j^-}{(\bar{d}_j^+ + \bar{d}_j^-)} \tag{2.36}$$

Obviously, the $\dot{CC}_i$ values are reduced to real number instead of interval number. Thus, based on $\dot{CC}_i$ values, the ranking order of alternatives are determined based on real number where the best one is chosen based on descending order.

This extended model has been used as a basis for other interval TOPSIS models which used interval-number as input data. For example, Yue [105] extended the model by proposing

an additional step to determine the weights of DMs. This additional step was performed after the CC values were obtained. This procedure was proposed to avoid any "false" or "biased" DM in the process. Note, the CC values in this model was obtained as crisp number instead of interval number. Dymova et al. [106] was critical about the model presented by Jahanshahloo et al. and Yue since the positive and negative ideal solutions are presented by crisp numbers instead of interval numbers. Furthermore, the ideal solutions in the previous two models are not always attainable in the decision matrix. Thus, they proposed an extended model which solves the aforementioned problems. The presented CC values of their model are in interval number and the probabilistic-based comparison method is used to determine the final ranking of alternatives. All the models reviewed here are based on interval number as an input data at the initial evaluation step. However, in practice, it is usually difficult for the DMs to determine precisely the upper and lower bound of the interval number. Hence, the fuzzy set information is suitable to handle such imprecise situation. Dubois [107] stated in his paper several roles of fuzzy sets can play in the context of decision-making field. The next section will present several related studies on standard TOPSIS model and the transition of the model to the fuzzy version which specifically in aiming to handle the uncertainty within the decision-making process.

2.6.2 Related work on standard and fuzzy version of TOPSIS models

Numerous studies on TOPSIS have been published, for example, Behzadian et al. [92] presents a state-of-the-art survey of applications, while Opricovic and Tzeng [108], and Zanakis et al. [109] present comparison studies with other MADM methods. Recently, the trend of research on TOPSIS has changed towards employing the hybrid TOPSIS rather than the stand-alone TOPSIS. These hybridizations have made the standard TOPSIS model more illustrative and usable when handling practical and theoretical problems. Based on the surveys conducted by Behzadian et al. [92], the fuzzy set seems to be the most common

hybrid theory with TOPSIS. For instance, one of the findings in this survey is that more than half of the TOPSIS publications utilized linguistic variables and fuzzy numbers to handle problems with imprecise information. Thus, it is reasonable to suggest that Fuzzy TOPSIS is currently one of the most popular models and has worked satisfactorily in various application areas for the last 18 years (e.g., in [106],[110],[111],[112] and [113]).

Since then, many fuzzy TOPSIS models were developed in the literature: Chen [1] transform Hwang and Yoon's [3] classical TOPSIS into a fuzzy environment where linguistic variables were used to assess the weights of all criteria and the ratings of each alternative with respect to each criterion. This aforementioned study has been cited and used by more than 2000 studies in the last 18 years. Later, Tsaur et al. [114], integrate AHP, FSs and TOPSIS to evaluate the quality service of airlines. AHP was used to retrieve the importance weight of critera while FNs were used to represent preferences of decision makers (DMs) about the chosen airline. These fuzzy evaluations were then transformed to crisp number at the pre-processing level. In the context of robot selection, Chu and Lin [9] proposed an extension on TOPSIS based on FSs to select the robot in one of manufacturing companies. The extension was based on the objective and subjective criteria which are integrated in dimensionless indices to ensure the compatibility between the values of objective criteria and the linguistic ratings of subjective criteria. Then, similar with the Chu and Lin work, the fuzzy information then is changed to a crisp number to determine the ranking of alternatives based on TOPSIS models. Wang and Elhag [115] presented a TOPSIS model based on fuzzy alpha level sets (A_α) and provided a non-linear programming (NLP) solution method where the fuzzy element in ratings of alternatives remained in the same form. Then, an extended version of the model was made where there is no conversion from fuzzy to crisp number is required as all the evaluations remain in fuzzy data.

Wang and Lee [116] generalized TOPSIS for fuzzy MCDM by introducing a new operator to determine the ideal solution values which serves as a reference point where the so-called

positive ideal solution is a collection of best alternatives, and the so-called *negative ideal solution* is a collection of worst attributes. Later, Ashtiani et al. [117] and Chen and Tsao [118] extended the TOPSIS method to interval-valued fuzzy numbers while Boran [119] introduced an intuitionistic fuzzy TOPSIS method for the supplier selection problem. Recently, Yuen [13] proposed an extended fuzzy TOPSIS model based on the introduction of a 'two-dimensional' scale to overcome the dynamic phenomenon which may exist among DMs in rating processes. This scale is different compared to the scale in the standard fuzzy TOPSIS model proposed by Chen [1] in order to allow the capture of additional information from DMs. However, DMs preferences are often vague and decision-relevant information is imperfect as it is based on perceptions [120]. Furthermore, it was shown in [121] that variations in decision making among DMs exist and that variation was suggested to be modelled using fuzzy sets. Several attempts have been made to handle variations related to uncertain, imprecise and vague information. Some studies also recognized ambiguity problems when DMs used certain linguistic terms to describe their assessments.

Recently, the study in this field shows a growing interest to implement T2 FSs with a TOPSIS model as numerical and T1 FS assessments cannot adequately handle uncertainty and impreciseness. Chen and Lee [122] presented an extended TOPSIS model which utilized Interval Type-2 Fuzzy Sets (IT2 FSs) with trapezoidal MFs. A new step has been introduced to identify the ideal solution where the point is employed by maximum ranking of a decision matrix value for positive ideal solution and minimum ranking value for negative ideal solution. More recently, Sang and Liu [123] argued against the result presented by Chen and Lee [122] as the computation of ranking leads to significant information loss due to the fact that the fuzzy set is converted to crisp number in the beginning of the process. Hence, they proposed an analytical solution to IT2 FSs-based TOPSIS model where they employed the algorithm proposed in [124] and fractional non-linear programming to reach the final evaluation. This model maintains the IT2 FSs formats in the whole process. A summary of the most recent

work in IT2-TOPSIS is given in Table 2.1. In the next two sections, a detailed discussion about the key stages in standard TOPSIS and T1-TOPSIS are given.

Table 2.1 Recent IT2-TOPSIS Models

Authors	Description Method	Other models combined or compared
Yaakob et al. [94]	Fuzzy Network TOPSIS (FN-TOPSIS).	Fuzzy Network
Liao [125]	T2-TOPSIS with new distance measure and degree optimism.	Degree optimism
Nehi & Keikha [126]	IT2-TOPSIS with Choquet integral.	Choquet integral
Qin et al. [127]	IT2 TOPSIS based on an α-cut concept.	Alpha-cut
Wu et al.[93]	IT2-TOPSIS with fuzzy weighted average to combine the input.	Fuzzy weighted average
Mein Xie [128]	IT2-TOPSIS based in [129] and performs a sensitivity analysis.	-
Sang and Liu [123]	IT2-TOPSIS with nonlinear programming (NLP).	Nonlinear programming (NLP)
Dymova et al. [130]	IT2 TOPSIS based on α-cut	Alpha-cut
Baykasoglu & Golcuk [131]	IT2 TOPSIS with DEMATEL	DEMATEL
Chen [132]	IT2-TOPSIS based likelihood-comparison approach	Probability / Likelihood theory
Zyoud [133]	Provide bibliometric survey on AHP and TOPSIS.	AHP

2.7 Key Stages on standard TOPSIS model

This section provides details on a standard TOPSIS model using a stepwise procedure introduced by Hwang and Yoon [3, 134]. The standard MADM problem can be concisely

expressed in a decision matrix format as

$$\dot{\mathbf{X}} = \begin{array}{c} \\ \mathscr{A}_1 \\ \mathscr{A}_2 \\ \vdots \\ \mathscr{A}_m \end{array} \begin{array}{c} \begin{array}{cccc} \mathscr{C}_1 & \mathscr{C}_2 & \cdots & \mathscr{C}_n \end{array} \\ \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{pmatrix} \end{array}, \tag{2.37}$$

$$\dot{\mathbf{W}} = \begin{bmatrix} w_1 & w_2 & \cdots & w_n \end{bmatrix}, \tag{2.38}$$

where $\mathscr{A}_{i=1,\cdots,m}$ are alternative names, $\mathscr{C}_{j=1,\cdots,n}$ are criteria/attributes that are measured, x_{ij} is the rating of alternative $\mathscr{A}_i$ with respect to criterion $\mathscr{C}_j$ and w_j is the weight of criterion $\mathscr{C}_j$.

In the standard TOPSIS, it is assumed that each attribute in the decision matrix $\dot{\mathbf{X}}$ takes either a monotonically increasing or monotonically decreasing utility. In other words, the larger the attribute outcome is, the greater the preference for the 'benefit' criteria and the less the preference for the 'cost' criteria. Furthermore, any outcome which is expressed in a non-numerical way should be quantified through the appropriate scaling technique. For example, using a rating scale on certain points on '$0-10$' or '$0-100$' by a group of DMs. Since all criteria cannot be assumed to be of equal importance, the method receives a set of weights from the decision makers and can be expressed by matrix $\dot{\mathbf{W}}$ shown as (2.38) using the weight of importance scale.

Figure 2.7 shows the stepwise TOPSIS procedure from Hwang and Yoon [3]. After forming an initial decision matrix $\dot{\mathbf{X}}$, the procedure starts by normalizing the decision matrix. This step aims to transform the various attribute dimensions into non-dimensional attributes, which allows comparison across the attributes. There are many ways to normalize the matrix

as described by Chen and Hwang [86]. Here, the normalization step is made by taking the outcome of each criterion divided by the norm of the total outcome vector of the criterion at hand, as shown in Figure 2.7. For Step 2, a set of weights expressed in matrix $\dot{\mathbf{W}}$ given by DMs are accommodated to the decision matrix $\dot{\mathbf{X}}$ by multiplying each column of the matrix $\dot{\mathbf{X}}$ (i.e., r_{ij} value) with its associated weight w_j. Then, in Step 3, two ideal points known as Positive Ideal Solution (A^+, PIS) and Negative Ideal Solution (A^-,NIS) are defined.

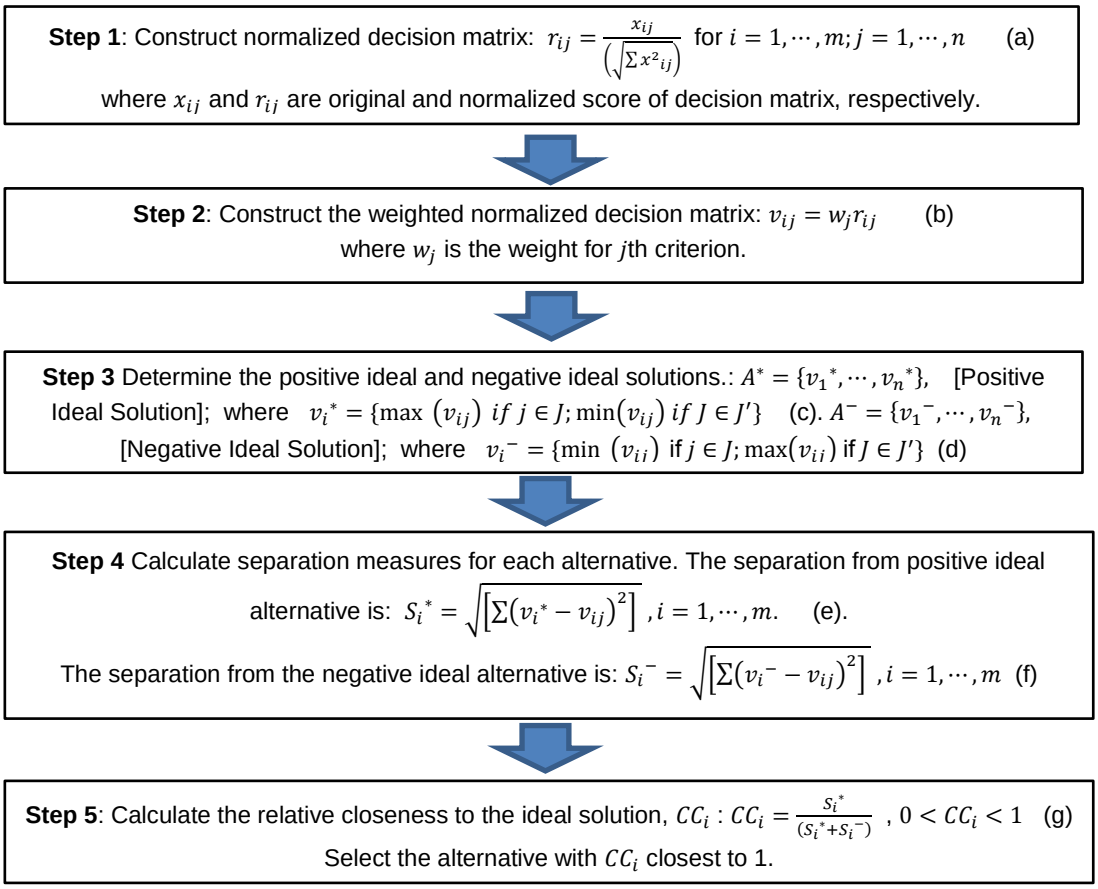


Fig. 2.7 Key stages of standard TOPSIS model [3]

Step 4 is a calculation distance between each alternative with the ideal solution using n-dimensional Euclidean technique. Then, in the final step, the relative Closeness Coefficient (CC_i) of alternative (A_i) with respect to A^+ is calculated. It is clear that $CC_i = 1$ if $\mathscr{A}_i = A^+$ and $CC_i = 0$ if $\mathscr{A}_i = A^-$. An alternative A_i is closer to A^+ as CC_i approaches 1. Next, an

alternative can now be ranked according to the descending order of CC_i. Overall, TOPSIS makes full use of attribute information, provides a cardinal ranking of alternatives, and does not require attribute preferences to be independent [86]. An improved version of the TOPSIS model which incorporates FST was developed to improve uncertainty and impreciseness in real-world information. Some related works have already been discussed in Section 2.6.2. In the next section, a detailed discussion about the key stages in the standard Fuzzy TOPSIS model (which termed in this thesis as T1-TOPSIS) is provided.

2.8 Key Stages on T1-TOPSIS model

The extension of the TOPSIS model to the fuzzy version by Chen [1] can be considered as one of the pioneer works in this field. In this work, type-1 triangular fuzzy numbers (T1 TFNs) were used to present the linguistic terms when DMs were asked to give a rating of each alternative. The T1 TFN is most widely used in decision making because of its intuitive MFs and computational simplicity [135, 136, 40]. A definition of T1 TFN has been given in Definition 2. Based on this, many researchers started to explore and enhance this initial T1-TOPSIS model. In this section, we provide the stepwise procedure of the original T1-TOPSIS model from Chen [1].

The main difference between standard TOPSIS as compared to T1-TOPSIS lie in the information given by DMs. Note, the development of T1-TOPSIS aims to model uncertainty, impreciseness and ambiguity in real-world information. Hence, instead of using crisp numbers in the rating value of x_{ij}, the T1-TOPSIS model uses T1 TFNs to represent them. For example, for rating *Good*, the associated T1 TFN is $(7,9,10)$. In contrast, in the standard TOPSIS model, the value of *Good* might be the crisp value 9. Next, the step-by-step algorithm of T1-TOPSIS is given [1].

Let $\mathscr{A}$ be the set of m alternatives being considered, in which the elements $\mathscr{A}_{i=1,\cdots,m}$ are alternative names. Let $\mathscr{C}$ be the set of n criteria used to evaluate the alternatives, in which

the elements $\mathscr{C}_{j=1,\cdots,n}$ are criterion name. These criteria $\mathscr{C}$ can be grouped into two sets $\mathbb{B}$ and $\mathbb{C}$ to indicate a collection of 'benefit criteria'– for which higher values are better and 'cost criteria' – for which lower values are better, respectively, and where $\mathbb{B} \cap \mathbb{C} = \emptyset$ and $\mathbb{B} \cup \mathbb{C} = \mathscr{C}$. Finally, let $\mathscr{D}$ be the set of k DMs in which the elements $\mathscr{D}_{e=1,\cdots,k}$ are DM's index name.

Firstly, each alternative is rated with respect to each criterion by each DM. These ratings, each of which is a term with a corresponding a T1 TFN, can be concisely expressed in a matrix $\mathbf{X}_{e=1,\cdots,k}$ each of size $m \times n$, such that X_{ij} is a T1 TFN, $X_{ij} = (a_{ij^X}, b_{ij^X}, c_{ij^X})$. Then, an average rating matrix $\mathbf{X}$ is represented as follows:

$$\mathbf{X} \doteq [X_{ij}]_{m\times n} \doteq \begin{bmatrix} X_{11} & X_{12} & \cdots & X_{1n} \\ X_{21} & X_{22} & \cdots & X_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ X_{m1} & X_{m2} & \cdots & X_{mn} \end{bmatrix} \tag{2.39}$$

where $\mathbf{X}$ can be formed by calculating each element as the average of each of the T1 TFNs obtained by:

$$\mathbf{X} = \frac{1}{k} \sum_{e=1}^{k} X_e \tag{2.40}$$

For example, X_{12} is the average rating across all DMs of alternative 1 with respect to criterion 2. The average of T1 TFNs is based on their vertex and defined in Chen [1].

The DMs also provide their perceived importance of each criterion as a term selected from the 'importance' linguistic variable. Each term is a T1 TFN.

The average weight over all DMs for the criteria can be expressed as the matrix $\mathbf{W}$

$$\mathbf{W} = [w_j] = \begin{bmatrix} w_1 & w_2 & \cdots & w_n \end{bmatrix} \tag{2.41}$$

formed with elements $w_j = (a_{jW}, b_{jW}, c_{jW})$ which are the average of T1 TFNs over the DMs.

Once the matrices **X** and **W** are formed, T1-TOPSIS can be performed, summarized as described in steps 1-6 below.

Step 1: Construct the fuzzy normalized decision matrix, $\mathbf{R} \doteq [R_{ij}]_{m\times n}$, by:

$$R_{ij} = \begin{cases} \left(\dfrac{a_{ij^X}}{c^*_{j^X}}, \dfrac{b_{ij^X}}{c^*_{j^X}}, \dfrac{c_{ij^X}}{c^*_{j^X}}\right), & \text{for} \quad j \in \mathbb{B} \\ \\ \left(\dfrac{a^-_{j^X}}{c^X_{ij}}, \dfrac{a^-_{j^X}}{b^X_{ij}}, \dfrac{a^-_{j^X}}{a^X_{ij}}\right), & \text{for} \quad j \in \mathbb{C}, \end{cases} \tag{2.42}$$

where $c^*_{j^X} = \max_i c_{ij^X}$ and $a^-_{j^X} = \min_i a_{ij^X}$ for $i = 1, \cdots, m$ and $j = 1, \cdots, n$.

Step 2: Construct a weighted normalized fuzzy decision matrix, $\mathbf{V} \doteq [V_{ij}]_{m\times n}$, where:

$$V_{ij} = R_{ij} \times W_j \tag{2.43}$$

and $V_{ij} = (a_{ij^V}, b_{ij^V}, c_{ij^V})$ are T1 TFNs.

Step 3: Determine the Fuzzy Positive Ideal Solution (FPIS, A^+) and Fuzzy Negative Ideal Solution (FNIS, A^-) by:

$$A^+ = [v_1^+, \cdots, v_n^+] \tag{2.44}$$

$$A^- = [v_1^-, \cdots, v_n^+] \tag{2.45}$$

where $v_j^+ = (1 \quad 1 \quad 1)$ and $v_j^- = (0 \quad 0 \quad 0)$, $j = 1, \cdots, n$.

Step 4: Calculate the distance (d_i^+) of V_{ij} from A^+ and the distance (d_i^-) of V_{ij} from A^- by:

$$d_i^+ = \sum_{j=1}^{n} d(V_{ij}, v_j^+) \tag{2.46}$$

$$d_i^- = \sum_{j=1}^{n} d(V_{ij}, v_j^-) \tag{2.47}$$

where $d(\cdot,\cdot)$ represents the distance between two T1 TFNs according to the vertex method proposed by Chen [1]. Let $A_1 = (a_1, b_1, c_1)$ and $A_2 = (a_2, b_2, c_2)$ be two T1 TFNs. The distance calculation of two T1 TFNs is shown in (2.48):

$$d(A_1, A_2) = \sqrt{\frac{1}{3}[(a_1 - a_2)^2 + (b_1 - b_2)^2 + (c_1 - c_2)^2]} \tag{2.48}$$

Step 5: Calculate the Closeness Coefficient, $\ddot{CC}_i$, of each alternative using (3.11).

$$\dot{CC}_i = \frac{d_i^-}{d_i^- + d_i^+}, \quad \text{for} \quad i = 1, \cdots, m. \tag{2.49}$$

Step 6: Determine the rank order of the alternatives and select the best one. An alternative $\mathscr{A}_i$ is closer to the FPIS (A^+) and farther from FNIS (A^-) as CC_i approaches 1. Thus, a higher $\dot{CC}$ is considered a higher rank.

2.9 Summary

In this chapter, the necessary background of fundamental concepts in the literature for the presentation and definitions of the methodology used in this thesis was provided. The chapter briefly discussed the concept of modelling uncertainty within the MCDM context. Some existing models that handle uncertainties were also discussed in the beginning of the chapter. The next section provided the fundamental background of fuzzy set theory including the properties of T1 FSs, fuzzy numbers, T2 FSs and linguistic variable. The section that follows discussed another method which used interval number to handle uncertainty in decision making process. The basic terms, concepts and several basic order relations were discussed in this section. Next, several MCDM models including MADM in the literature were reviewed. Then, the TOPSIS model as well as its extension to interval and fuzzy version were described in detail. Finally, the key stages of standard TOPSIS and T1-TOPSIS were outlined. In the next chapter, the first objective of this thesis, which is the identification of the issue and

limitations of the current TOPSIS and T1-TOPSIS model will be addressed. The details of the key stages involved will be compared by providing a demonstration using the dataset from Chen [1] to highlight the identified issues. Then, a potential framework will be proposed to solve the identified issues.

Chapter 3

An Exploration of Issues and Limitations in Standard and Fuzzy TOPSIS

3.1 Introduction

In recent years, a variety of extensions, including fuzzy extensions of TOPSIS have been proposed. Besides the many variations of standard TOPSIS, one family of extensions employing fuzzy sets is referred to as fuzzy TOPSIS (known as T1-TOPSIS in this thesis). One challenge that has arisen is that it is not straightforward to choose between the multiple variants of TOPSIS existing today. This chapter presents the first and second contributions of this thesis that is a study in exploring of issues and limitations in current TOPSIS models and a comparative study between two types of T1-TOPSIS models. Firstly, Section 3.2 outlines several motivations for doing the first study in this chapter. The exploration study is conducted to identify several issues and limitations in both models. These can be used further as a base guideline to improve the existing model. The exploration study begins with Section 3.3 which provides the key stages of standard TOPSIS model introduced by Hwang and Yoon [3]. As the key stages of T1-TOPSIS have already been provided in Section 2.8, the study proceeds by detailing each step in both the standard TOPSIS model and T1-TOPSIS

model in Section 3.3. Then, the first contribution of this thesis which is the discussion about identification of the limitations, issues and challenges, is given in Section 3.4. Having identified the issues, limitations and challenges, this study proceeds with a suggestion of potential solution to solve one of the identified issues, that is uncertainty in reliability. Section 3.6 provides discussion on this exploratory study.

After exploring these issues and limitations, this chapter proceeds to the second study which is the comparison of two extended TOPSIS models, which are standard T1-TOPSIS introduced by Chen [1] and the recent one from Yuen [13] in Section 3.7. This is the second contribution of this thesis. This comparison is made to show the differences between both models and to provide context for their respective strengths and limitations both in complexity of application, expressiveness of results and the capability in handling uncertainty. The motivation of doing the comparative study are outlined and the step-wise stages for each model is discussed in this section. Then, the comparison between these two models are made in systematic step-wise manner. The implementation of dataset from Chen [1] to both models are used as demonstration to compare the models. The advantages and disadvantages of the additional information in Yuen's model is highlighted in detail. The observation is made as to whether these models capture any additional information which further can reduce uncertainty in the process. Section 3.8 provides discussion on this comparative study. The discussion focuses on the similarities and differences of the final evaluation of both models. Both resulting final evaluations (which are in the form of crisp numbers) are illustrated in Figure 3.6 and 3.7. Finally, Section 3.9 summarizes the two studies in this chapter to provide a comprehensive assessment of the strengths and weaknesses of the various approaches.

3.2 Motivation

An impressive variety of fuzzy decision making applications have been developed in the last two decades. Behzadian et al.[92] reviewed in particular applications of TOPSIS and

found that using fuzzy sets (FSs) seems to be the most commonly used model in TOPSIS. Specifically, while the standard TOPSIS model assumes that alternative ratings and criteria weights are crisp numbers, more than half of the TOPSIS publications (52.2%) utilized linguistic variables and FSs to handle problems with imprecise information. They also found that most TOPSIS publications (76 papers) were related to group decision making issues. An initial extension of the TOPSIS model for group decision-making under fuzzy environment by Chen [1] in 2000 can be considered as one of the early works in this area. Based on the work, many researchers started to explore and enhance this model. Various types of enhancements were made. For example, some proposed to use different types of fuzzy information (e.g., type-1 fuzzy sets [1], type-2 fuzzy sets [137], intuitionistic fuzzy sets [119], interval valued sets [118], etc.) in order to handle impreciseness of information in the problem.

Despite many variations of T1-TOPSIS models, none of the studies explaining the details of the key stages of standard TOPSIS and T1-TOPSIS are based on a systematic stepwise manner. Furthermore, most studies do not highlight the differences between these two models in clear detail. Thus, it is difficult to determine the limitations of the various models. The process of identifying the limitations is crucial in order to provide meaningful justification or logical reason why such models should be chosen, as well as to provide confidence in the recommended decision. This chapter provides the detail about the key stages involved in both standard TOPSIS and T1-TOPSIS in a systematic stepwise manner. Further, this chapter highlights clearly which stage are different between both models. In order to illustrate how the T1-TOPSIS model works, this study provides a real-world numerical example taken from [1]. Then, the study identifies and discusses the limitations, issues and challenges in the existing T1-TOPSIS model. The crucial and the identified issues are relate to concepts of *reliability*, *truth* and *meaning*. Here, this study only highlights one main issue, namely *reliability*, to discuss further in MADM problems. The following sections proceed to detailing each step

in both TOPSIS and T1-TOPSIS models by comparing the models based on systematic step-wise manner.

3.3 Detailing Each Step in TOPSIS and T1-TOPSIS

Section 2.7 presents the step-by-step TOPSIS model proposed by Hwang and Yoon [3] and Section 2.8 provides the steps of its enhanced version, T1-TOPSIS model, proposed by Chen [1]. This section provides detailed discussion about which step has been enhanced and the differences between these two models. As stated in Section 2.8, the T1-TOPSIS model uses T1 TFNs to represent the rating of alternatives. Recall that, a T1 TFN is a T1 FS when the properties of convex, normal, semi-continuous and bounded are hold [43]. To increase readability, this section summarises the key stages of TOPSIS and T1-TOPSIS presented in Section 2.7 and Section 2.8 in Figure 3.1 and Figure 3.2, respectively. Note that, highlighted boxes in Figure 3.1 indicates that the steps in standard TOPSIS model are different as compared to steps in T1-TOPSIS model in Figure 3.2. A brief overview about similarities and different steps for both models are highlighted next.

In the beginning of both models, the same matrix forms are used to present the rating of alternatives, namely the decision matrix **D** shown in (2.37) and matrix **W** shown in (2.38) for the weight of each criterion (defined in Chapter 2). However, instead of using crisp numbers in the rating value of x_{ij}, the T1-TOPSIS model uses T1 TFNs to represent them.

Based on Figure 3.2, the calculation in Step 1 is slightly different compared to Step 1 in Figure 3.1. The reason is that different type of information has been used to express human judgment given by DMs. In T-TOPSIS model, T1 TFNs are used, as opposed to crisp numbers in standard TOPSIS model. In the T1-TOPSIS model, the T1 TFNs for its associated MF is expressed as (a_{ij}, b_{ij}, c_{ij}), where a, b and c are the smallest possible value, the peak value and the largest possible value, respectively. The normalization step is needed to transform various criteria scales into a single comparable scale, and to preserve the

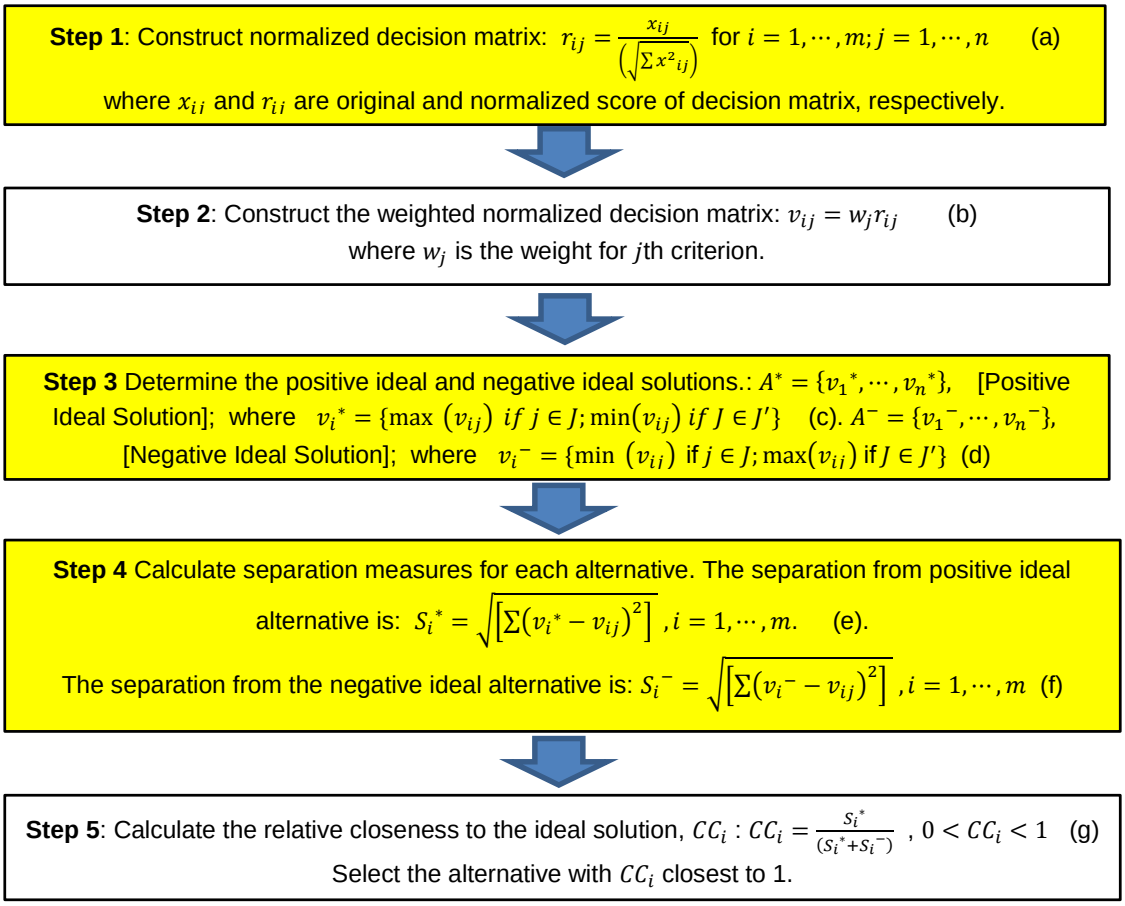


Fig. 3.1 Step-wise procedure of standard TOPSIS model [3]

Step 1: Construct normalized fuzzy decision matrix: $r_{ij} = \frac{a_{ij}}{c_j^*}, \frac{b_{ij}}{c_j^*}, \frac{c_{ij}}{c_j^*}$, for $j \in J$ (Benefit Criteria) OR $r_{ij} = \frac{a_j^-}{c_{ij}}, \frac{a_j^-}{c_{ij}}, \frac{a_j^-}{c_{ij}}$, for $j \in J'$ (Cost Criteria) ; where $c_j^* = \max_i c_{ij}$ if $j \in J$; and $a_j^- = \min_i a_{ij}$ if $j \in J'$.

Step 2: Construct the weighted normalized decision matrix: $v_{ij} = w_j r_{ij}$
where w_i is the weight for jth criterion.

Step 3 Determine the positive ideal and negative ideal solutions: $A^* = \{v_1^*, \cdots, v_n^*\}$, [Positive Ideal Solution], where $v_i^* = (1,1,1)$; $A^- = \{v_1^-, \cdots, v_n^-\}$, [Negative Ideal Solution]; where $v_i^- = (0,0,0)$

Step 4 Calculate separation measures for each alternative using vertex method. The separation from positive ideal alternative: $s_i^* = \sum_{j=1}^n d(v_{ij}, v_{ij}^*)$, $i = 1, \cdots, m$.
The separation from the negative ideal alternative is: $s_i^- = \sum_{j=1}^n d(v_{ij}, v_{ij}^-)$, $i = 1, \cdots, m$; where $d(\cdot, \cdot)$ represent the distance between two fuzzy numbers by vertex method:
Let $\tilde{m} = (m_1, m_2, m_3)$ and $\tilde{n} = (n_1, n_2, n_3)$ be two triangular fuzzy numbers (TFN). Distance calculation of these to TFN is

$$d(\tilde{m}, \tilde{n}) = \sqrt{\frac{1}{3}[(m_1 - n_1)^2 + (m_2 - n_2)^2 + (m_3 - n_3)^2]}$$

Step 5: Calculate the relative closeness to the ideal solution, CC_i : $CC_i = \frac{s_i^*}{(s_i^* + s_i^-)}$, $0 < CC_i < 1$
Select the alternative with CC_i closest to 1.

Fig. 3.2 Step-wise procedure of T1-TOPSIS model[1]

property of normalized T1 TFN belonging to $[0, 1]$. Then, Step 2 is the same as the standard TOPSIS used to compute the weighted normalized decision matrix, as shown in Step 2 of Figure 3.2. In Step 3, the fuzzy positive ideal solution (PIS) and fuzzy negative ideal solution (NIS) values are determined as $(1\ 1\ 1)$ and $(0\ 0\ 0)$, respectively. Both values are expressed in T1 TFNs form. This means the T1-TOPSIS model considers the ideal solution in the perfect condition where 1 means the perfect best point and 0 the perfect least point. In Step 4, the standard TOPSIS and T1-TOPSIS model have the same objective in which the separation measure is calculated for each alternative from both ideal solutions. In T1-TOPSIS, the separation measure is based on a vertex method in which each value in T1 TFN is considered simultaneously, as shown in Step 4 of Figure 3.2. Then, in the final step, the same calculation as Step 5 in Figure 3.1 is conducted to get the closeness coefficient to ideal solution (CC_i). An alternative can now be ranked according to the descending order of CC_i value.

A practical example which analyses the steps in T1-TOPSIS has already been presented in Section 3.7.5. The numerical example which implements the standard T1-TOPSIS presented in this chapter is the same as Chen's T1-TOPSIS in Section 3.7.5. Hence, this chapter proceeds with the main part of this study which is a process of identifying limitations, challenges and issues in T1-TOPSIS.

3.4 Limitations, Challenges and Issues in T1-TOPSIS

This section identifies the limitations, issues and challenges in the existing T1-TOPSIS model since its introduction in 2000 by Chen [1]. This study provides the list of basic characteristics of the T1-TOPSIS model in Table 3.1. Firstly, this model uses cardinal information in its analysis. The main process in T1-TOPSIS is to determine the distance between fuzzy PIS and fuzzy NIS for each alternative respectively. This model falls in the MADM category since attribute information is given. However, in existing T1-TOPSIS, there are no consistency and reliability checks, as explained below. These two characteristics

Table 3.1 Characteristics of the T1-TOPSIS model [1]

Characteristics	T1-TOPSIS
Category	Cardinal information
Core process	Cardinal
Attribute	Given
Weight elicitation	Given
Consistency check	None
Reliability check	None
Reflection of uncertainty from initial to final evaluation	None
No. of attributes accommodated	Many more
No. of alternatives accommodated	Many more
Others	Compensatory operation

are very important in any decision making process as it may lead to misleading results. Additionally, there is no reflection of uncertainty at the initial stage of evaluation to the final evaluation. These can be considered as fundamental aspects that need to be handled to produce a better decision. Section 3.4.1 will discuss these issues in detail. The basic T1-TOPSIS model can accommodate many attributes and alternatives at the same time and so it can be described as a 'compensatory operation' technique.

3.4.1 What is Lacking in the Literature?

Based on initial work on T1-TOPSIS model by Chen [1] as described in Section 3.3, the DMs judgement or preferences are described in FSs using T1 TFNs. Such sets are used in order to handle any linguistic uncertainty which may exist in the preferences of DMs. Majority of T1-TOPSIS related studies, the linguistic uncertainty is known as the 'fuzzy environment'. Most enhancements of T1-TOPSIS were made based on improving the capability in handling linguistic uncertainty. For example, instead of using ordinary FSs information (T1 FSs),

some researchers attempted to use type-2 sets (T2 FSs) [14] in the belief that they better represent any linguistic uncertainties [138].

Such uncertainty can be considered as fuzziness in information. However, real-world information is not only characterized by fuzziness. A subtly different characteristic in real-world information is that of 'partial reliability' [139]. Thus, it is not adequate to use only fuzzy number in handling fuzziness, but there is also a need for other information that can formally describe the partial reliability characteristic in real-world information. For example, DMs can provide a rating as *Good* without stating the confidence level in the source of information. In the literature, some researchers have proposed tools to assess reliability of information, for example the concepts of fuzzy measures [140] and fuzzy integrals [141]. A practical application of fuzzy measures and fuzzy integrals can be seen in [142], which deals with partially reliable data in a crowd-sourcing context. Zadeh introduced a new concept which he termed as 'Z-number' [143] to deal with reliability of information. It is an ordered pair $Z = (A, B)$ of fuzzy numbers used to describe a value of variable X, where A is an imprecise constraint on values of X, and B is an imprecise estimation of reliability of A, which can be viewed as a value of probability measure of A [139]. Further details of reliability and Z-numbers can be found in [139] and [143].

It is absolutely clear that various uncertainties other than linguistic uncertainty exist in MCDM problems. As an example, there may exist uncertainty due to missing information. For example, in the numerical example presented in Section 3.7.5, the DMs naturally do not provide ratings with a certain numerical value (i.e., crisp values) but would rather provide a prediction such as:

"The rating of candidate 1 (A1) with respect to Criteria 1 (C1) is *Medium Poor*".

It can be assumed that this DM is *certainly* sure about his/her preferences. Consider one more example given by a DM:

> "The rating of candidate 1 (A1) with respect to Criteria 1 (C1), *probably* is *Medium Poor*".

In this sentence, the DM seems unsure about the given rating in this particular event, by explicit use of the word *probably* in the preference statement. Thus, in order to take such situations into account, the concept of restriction should be used together with describing real-world information. It is suggested in [107] to use possibility distributions to represent the state of events in imprecise real-world information. However, the *probability* distribution of an event also needs to be considered. This will indicate to what degree does the *Medium Poor* rating in this example fit the DM's perception about *Medium Poor*.

Commonly, the rating process is constructed from a sentence what is known as a 'proposition' in natural language. When fuzzy information is used, the proposition is now known as a fuzzy proposition. The natural language used in the fuzzy proposition may contain rating levels, known as fuzzy predicates (e.g., *Good, Poor*), and / or quantifiers (e.g., *Very*). Zadeh [144] highlighted that the truth and meaning of a proposition drawn from a natural language may be represented as a restriction. A restriction R, on variable X, or $R(X)$, may be viewed as an answer to a question of the form: What is the rating for X? Thus, the variable X must be clear in order to answer the question. What really matters is not understanding of meaning the variable X, but precisiation of meaning. This is Zadeh's method of representation which has a formal mathematical definition which can then be accurately used in computation. Precisiation of meaning is a prerequisite to reasoning and computation with information described in natural language [144]. Thus, this study believes that in relation to the reliability aspect of information, it is important to take into account the truth and meaning which related to ambiguity concepts of a proposition. It can be described as a restriction of the existing T1-TOPSIS model. This can be done in an initial step of T1-TOPSIS model prior to carrying out the usual Step 1. The meaning concept in T1-TOPSIS can be implemented in situations in which the DMs faced uncertainty in determining the rating for each alternative. The meaning

of rating in T1-TOPSIS model can be achieved by a precisiation step. The detail of this step is not discussed here due to limited space. Please refer to [144] for more detail.

In the concept of truth degree, there is a close relationship with the concept of meaning, as described in the previous paragraph. To measure the truth value of any proposition in natural language, it is necessary to understand the meaning of the proposition. As stated in previous paragraph, to assess the meaning of proposition, you need to precisiate the proposition. For example, in the T-TOPSIS numerical application in Section 3.7.5, a rating proposition may exist as

"The rating of A1 with respect to C1 is *Certainly Good*."

In this proposition, the information consist of three relations: *Good* [rating], *Certainly* [proportion], and *Population* [A1 with respect to C1, rating].

The overall meaning of that proposition can be described as the possibility distribution associated with the fuzzy set *Good*. Informally, it is the possibility that A_1 is *Good*, and is equal to the grade of membership of A_1 in *Good*. In the T1-TOPSIS application example in Section 3.7.5, a rating from DM_1 for A_1 with respect to C_1 is *Medium Good* which can have the degree of truth, $\mu_{truth} = 0.9$. Assuming that the precisiation step has already been taken, then the resultant meaning is the value of 6.5. Then, the truth degree can be assessed, calculated as 0.9 and shown graphically as in Figure 3.3. From this truth value, it is expected that the level of confidence for each DM can be measured and will lead to a reliable decision in the T1-TOPSIS model. This process can be implemented in an initial step of a T1-TOPSIS model which precedes the usual Step 1.

Overall, this study summarises that reliability, truth and meaning issues are not considered sufficiently in existing T-TOPSIS model. The next section will further explore briefly one issue only, that of reliability, due to limited space.

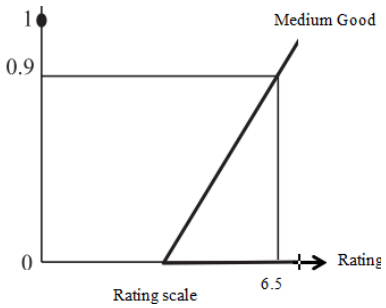


Fig. 3.3 Example of truth degree

3.4.2 Reliability of Information in T1-TOPSIS

Reliability of information can be viewed as to which extent we can rely on the source of the information and therefore the information itself. In other words, any approximation of values of interest, whether precise or not, is dependent on the confidence in the sources of information we deal with. Reliable information is dependable, trustworthy, unfailing, sure, authentic, genuine and reputable. Furthermore, a question may arise in real-world problem as "How reliable are the numbers which we deal with?". Thus, in a decision making process, the reputation of the sources of information is critical. As in the T1-TOPSIS model, all information is collected from humans as DMs. The partially reliable characteristic in real-world information has meant that the process of decision making based on T1-TOPSIS is a very challenging task. Therefore, the next section will suggest a potential solution which could be used in future to achieve reliable decisions.

3.5 A Potential Solution

In this section, with regard to the reliability issue, this study proposes a potential solution which can address the imprecision and (un-)reliability of information at the same time. Please note that only the original T1-TOPSIS model introduced by Chen [1] will be used as a base method. The proposed framework is shown in Figure 3.4.

The proposed framework starts with an initial step in which the DM gives a rating for each alternative with respect to each criterion. Suppose there is one DM evaluating k alternatives with respect to m criteria. Let the decision matrix $\mathbf{D} = d_{ij}$, where d_{ij} is a Z-number, consisting of two T1 TFNs, one for rating and one for reliability of the DM, and where $i = 1, \cdots, k, j = 1, \cdots, m$. In this step, the information is in the form of FSs. Then, the same procedure is taken as shown in Step 1 (Figure 3.4). In Step 2, the weighted normalized fuzzy decision matrix is constructed by multiplying the previous values in the decision matrix with the associated weight given by the DM. Prior to Step 3, this study proposes to transform all the FSs information for rating and reliability into crisp numbers. This step is needed in order to measure the distance between ideal solution points and each alternative. By doing this, we get the accurate distance since we applied the n-Euclidean distance formula with the crisp number. Note that, this distance formula is impractical for FS values since the meaning of the FS information is different, as compared to crisp numbers. It should be noted that, the common practice in the T1-TOPSIS field is to treat fuzzy set information as a crisp number. Therefore, this may lead to significant loss of information and misleading results. Next, the following steps 4 and 5 are carried out, as in the standard TOPSIS method as shown in Figures 3.4 and 2.7. The result will be better in terms of reliability, accuracy and trustworthiness of the decision.

3.6 Discussion of Issues, Limitation and Challenges

In this study, a brief discussion of the key stages involved in both the standard TOPSIS introduced by Hwang and Yoon [3], and the T1-TOPSIS model introduced by Chen [1] was made, carefully detailing each step. This is the first work which has attempted to both provide details and to highlight the key differences between these two models in a systematic manner. In addition, this study has identified and discussed the limitations, issues and challenges which have not been investigated sufficiently in the context of T1-TOPSIS. Then,

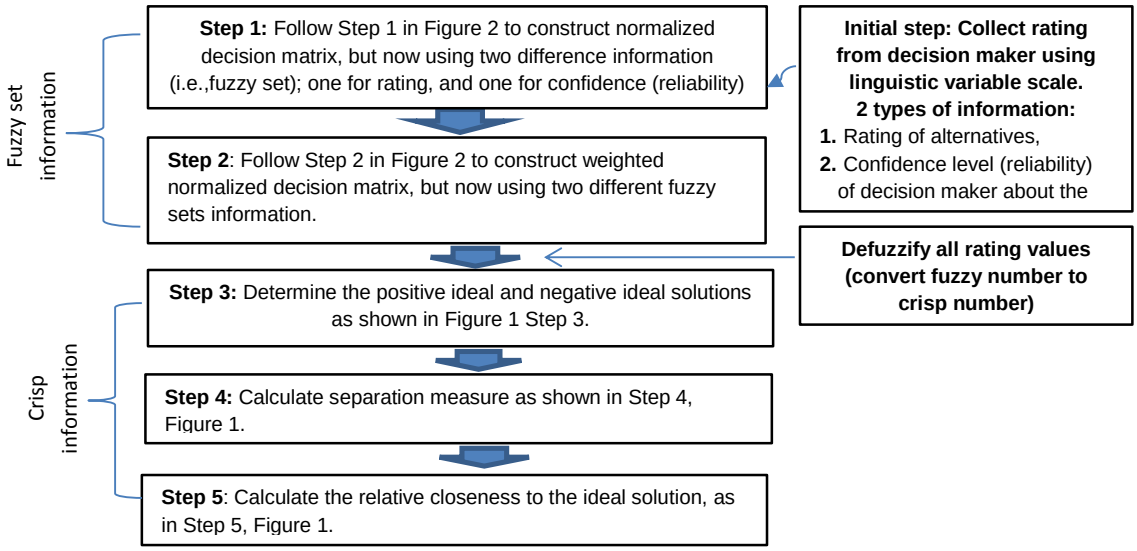


Fig. 3.4 Proposed new framework for Z-number based Fuzzy TOPSIS

this study focused on one main issue, that of *reliability*, which has been discussed previously. Following identification and clarification of this issue, this study has provided a potential solution which could result in better and more trustworthy decisions. In addition, the aim of this study is to provide the details of the TOPSIS and T1-TOPSIS as well as their issues, limitations and changes and to provide a novel contribution on comparison between these two models in systematic stepwise procedure. The next section presents a comparative study between the standard T1-TOPSIS model proposed by Chen [1] and the recently extended version by Yuen [13]. The comparative study is made to identify any additional benefit that the extended model can provide in terms of modeling uncertain information in the decision-making process.

3.7 A Comparison Between Two Types of T1-TOPSIS

In recent years, a variety of extensions to TOPSIS, including fuzzy extensions, have been proposed. One challenge that has risen is that it is not straightforward to differentiate between the multiple variants of TOPSIS existing today. Thus, in this section, a comparison between

the standard T1-TOPSIS model proposed by Chen [1] and the recently proposed TOPSIS extension by Yuen [13] is made. The purpose of this comparative study is to show the differences between both models and to provide context for their respective strengths and limitations both in complexity of application, expressiveness of results and the capability in handling uncertainty. A detailed synthetic numeric example and comparison of both models are provided.

3.7.1 Motivation of Comparative Study

Yuen proposed a new T1-TOPSIS model based on the introduction of a "two-dimensional" scale to overcome the dynamic phenomena which may exist among DMs in rating processes. This scale is different compared to the scale in the standard T1-TOPSIS model proposed by Chen [1] in order to allow the capture of additional information from DMs. However, DMs' preferences are often vague and decision-relevant information is imperfect as it is based on perceptions [120]. There exist variations in decision making among DMs and that can be modelled using fuzzy sets [145]. Based on reviews made in literature, it shows that except for Yuen's model [13], the other extended models focus mainly on the type of information involved at the initial evaluation processes (e.g., type-1 fuzzy number, interval-valued number and intuitionistic fuzzy valued) and the improvement of the step for the definition of the *positive ideal solution* (best) and *negative ideal solution* (worst) alternatives in the T1-TOPSIS model. However, there is little work focussing on the quality of input (preferences) given by DMs in the first (rating) step.

Usually, in standard TOPSIS, DMs use an ordinary scale (e.g., Likert scale) to represent their evaluation. Later, in Chen's work [1], a new scale was introduced based on type-1 triangular fuzzy numbers (T1 TFNs) which was subsequently used in T1-TOPSIS model proposed by other researchers in the last 18 years. However, a question remains as to whether the T1 TFNs based scale can be applied to every question in MCDM problems. Yuen [13]

developed a new scale to be used in the model, requiring two-fold rating and additional algorithms, increasing both data collection and computational complexity. If both models provide comparable results with comparable inputs, a question arises about how Yuen's model differs from Chen's simpler model, and what the effect of Yuen's secondary scale is. Furthermore, are the additional information provided by new scale in Yuen's model can handle uncertainty better than Chen's standard T1-TOPSIS model. To date, there has been no comparative study that shows the strength and limitations of both models.

In this section, for the sake of simplicity, this thesis refer to the standard T1-TOPSIS as Chen's T1-TOPSIS and the recently introduced model as Yuen's T1-TOPSIS. A comparison is made theoretically as well as in an applied example to show the differences and similarities between these two models. In particular, the comparison study of both models can provide comparable results under specific circumstances. In this numerical example, a synthetic data is used to show the implementation and enable the comparison of both models. Through the example this study shows the different steps involved to solve the same MCDM problems in both models. The importance weight of criteria used in this study is taken from Chen [1] and applied in both models. The details on this matter will be discussed in the Section 3.7.5.

3.7.2 Chen's T1-TOPSIS Model

Let $\mathscr{A}$ be the set of m alternatives being considered, in which the elements $\mathscr{A}_{i=1,\cdots,m}$ are alternative names. Let $\mathscr{C}$ be the set of n criteria used to evaluate the alternatives, in which the elements $\mathscr{C}_{j=1,\cdots,n}$ are criterion name. These criteria $\mathscr{C}$ can be grouped into two sets $\mathbb{B}$ and $\mathbb{C}$ to indicate a collection of 'benefit criteria'– for which higher values are better and 'cost criteria' – for which lower values are better, respectively, and where $\mathbb{B} \cap \mathbb{C} = \emptyset$ and $\mathbb{B} \cup \mathbb{C} = \mathscr{C}$. Finally, let $\mathscr{D}$ be the set of k DMs in which the elements $\mathscr{D}_{e=1,\cdots,k}$ are DM's index name.

Firstly, each alternative is rated with respect to each criterion by each DM. These ratings, each of which is a term with a corresponding a T1 TFN, can be concisely expressed in a matrix $\mathbf{X}_{e=1,\cdots,k}$ each of size $m \times n$, such that X_{ij} is a T1 TFN, $X_{ij} = (a_{ij^X}, b_{ij^X}, c_{ij^X})$. Then, an average rating matrix $\mathbf{X}$ is represented as follows:

$$\mathbf{X} \doteq [X_{ij}]_{m\times n} \doteq \begin{bmatrix} X_{11} & X_{12} & \cdots & X_{1n} \\ X_{21} & X_{22} & \cdots & X_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ X_{m1} & X_{m2} & \cdots & X_{mn} \end{bmatrix} \tag{3.1}$$

where $\mathbf{X}$ can be formed by calculating each element as the average of each of the T1 TFNs obtained by:

$$\mathbf{X} = \frac{1}{k}\sum_{e=1}^{k} X_e \tag{3.2}$$

For example, X_{12} is the average rating across all DMs of alternative 1 with respect to criterion 2. The average of T1 TFNs is based on their vertex and defined in Chen [1].

The DMs also provide their perceived importance of each criterion as a term selected from the 'importance' linguistic variable. Each term is a T1 TFN.

The average weight over all DMs for the criteria can be expressed as the matrix $\mathbf{W}$

$$\mathbf{W} = [w_j] = \begin{bmatrix} w_1 & w_2 & \cdots & w_n \end{bmatrix} \tag{3.3}$$

formed with elements $w_j = (a_{j^W}, b_{j^W}, c_{j^W})$ which are the average of T1 TFNs over the DMs.

Once the matrices $\mathbf{X}$ and $\mathbf{W}$ are formed, T1-TOPSIS can be performed, summarized as described in steps 1-6 below.

Step 1: Construct the fuzzy normalized decision matrix, $\mathbf{R} \doteq [R_{ij}]_{m\times n}$, by:

$$R_{ij} = \begin{cases} \left(\dfrac{a_{ij^X}}{c^*_{j^X}}, \dfrac{b_{ij^X}}{c^*_{j^X}}, \dfrac{c_{ij^X}}{c^*_{j^X}} \right), & \text{for} \quad j \in \mathbb{B} \\ \\ \left(\dfrac{a^-_{j^X}}{c^X_{ij}}, \dfrac{a^-_{j^X}}{b^X_{ij}}, \dfrac{a^-_{j^X}}{a^X_{ij}} \right), & \text{for} \quad j \in \mathbb{C}, \end{cases} \tag{3.4}$$

where $c^*_{j^X} = \max_i c_{ij^X}$ and $a^-_{j^X} = \min_i a_{ij^X}$ for $i = 1, \cdots, m$ and $j = 1, \cdots, n$.

Step 2: Construct a weighted normalized fuzzy decision matrix, $\mathbf{V} \doteq [V_{ij}]_{m \times n}$, where:

$$V_{ij} = R_{ij} \times W_j \tag{3.5}$$

and $V_{ij} = (a_{ij^V}, b_{ij^V}, c_{ij^V})$ are T1 TFNs.

Step 3: Determine the Fuzzy Positive Ideal Solution (FPIS, V^+) and Fuzzy Negative Ideal Solution (FNIS, V^-) by:

$$V^+ = [v_1^+, \cdots, v_n^+] \tag{3.6}$$

$$V^- = [v_1^-, \cdots, v_n^+] \tag{3.7}$$

where $v_j^+ = (1,\ 1,\ 1)$ and $v_j^- = (0,\ 0,\ 0)$, $j = 1, \cdots, n$.

Step 4: Calculate the distance (d_i^+) of v_{ij} from V^+ and the distance (d_i^-) of v_{ij} from V^- by:

$$d_i^+ = \sum_{j=1}^{n} d(v_{ij}, v_j^+) \tag{3.8}$$

$$d_i^- = \sum_{j=1}^{n} d(v_{ij}, v_j^-) \tag{3.9}$$

where $d(\cdot, \cdot)$ represents the distance between two T1 TFNs according to the vertex method proposed by Chen [1]. Let $m = (m_1, m_2, m_3)$ and $n = (n_1, n_2, n_3)$ be two T1 TFNs. The

distance calculation of two T1 TFNs is shown in (3.10):

$$d(m,n) = \sqrt{\frac{1}{3}[(m_1 - n_1)^2 + (m_2 - n_2)^2 + (m_3 - n_3)^2]} \tag{3.10}$$

Step 5: Calculate the Closeness Coefficient, $\dot{CC}_i$, of each alternative using (3.11).

$$\dot{CC}_i = \frac{d_i^-}{d_i^- + d_i^+}, \quad \text{for} \quad i = 1, \cdots, m. \tag{3.11}$$

Step 6: Determine the rank order of the alternatives and select the best one. An alternative $\mathscr{A}_i$ is closer to the FPIS (V^+) and farther from FNIS (V^-) as CC_i approaches 1. Thus, a higher $\dot{CC}$ is considered a higher rank.

3.7.3 Yuen's T1-TOPSIS Model

Yuen [13] proposed that the Compound Linguistic Ordinal Scale (CLOS) model be integrated into the T1-TOPSIS to improve Chen's existing T1-TOPSIS model. The CLOS model is concerned with defining the fuzzy linguistic terms used in the TOPSIS model and was developed to design a sufficient number of systematic fuzzy linguistic terms in matrix form. It consists of the distributions of the linguistic terms known as Compound Linguistic Variables (CLVs). Yuen aimed to fill a gap of lack of theory and convenient rules to build a large scale which involved linguistic terms that may be characterized by ambiguity. Furthermore, the qualitative scales often regarded as 'soft' or 'weak scales' [13]. In development of CLOS, two algorithms were proposed by Yuen, known as the *Hedge-Atom-Linguistic Representation Algorithm* to build two dimensional scale items and the *Fuzzy Normal Distribution Algorithm*.

Unlike the classical rating model which is the single step rating process, CLOS uses a double rating process in which a DM chooses a 2-tuple term in two stages. Yuen [146] stated that the advantage of CLOS is it is an ideal rating interface for addressing the problem of the rating dilemma. The linguistic terms collection, known as CLV is used as the main

component in the CLOS and it was extended from the linguistic variable concept introduced by Zadeh [14] (see Definition 9 in Section 2.3.4).

The CLV definition was proposed by Yuen as:

Definition 11 *[146] A Compound Linguistic Variable (CLV)* $(\mathcal{N}, T(\mathcal{N}), X, G, M)$ *is defined on X as a real number domain where* $\mathcal{N}$ *is a linguistic variable with its term set* $T(\mathcal{N})$ *expressed in matrix form. The* $T(\mathcal{N})$ *is built by syntactic rules G and semantic rules M which mapped* $\mathcal{N}$ *into X. Every linguistic term* α *in T characterized by the continuous or discrete elements x generated by G and M in X.*

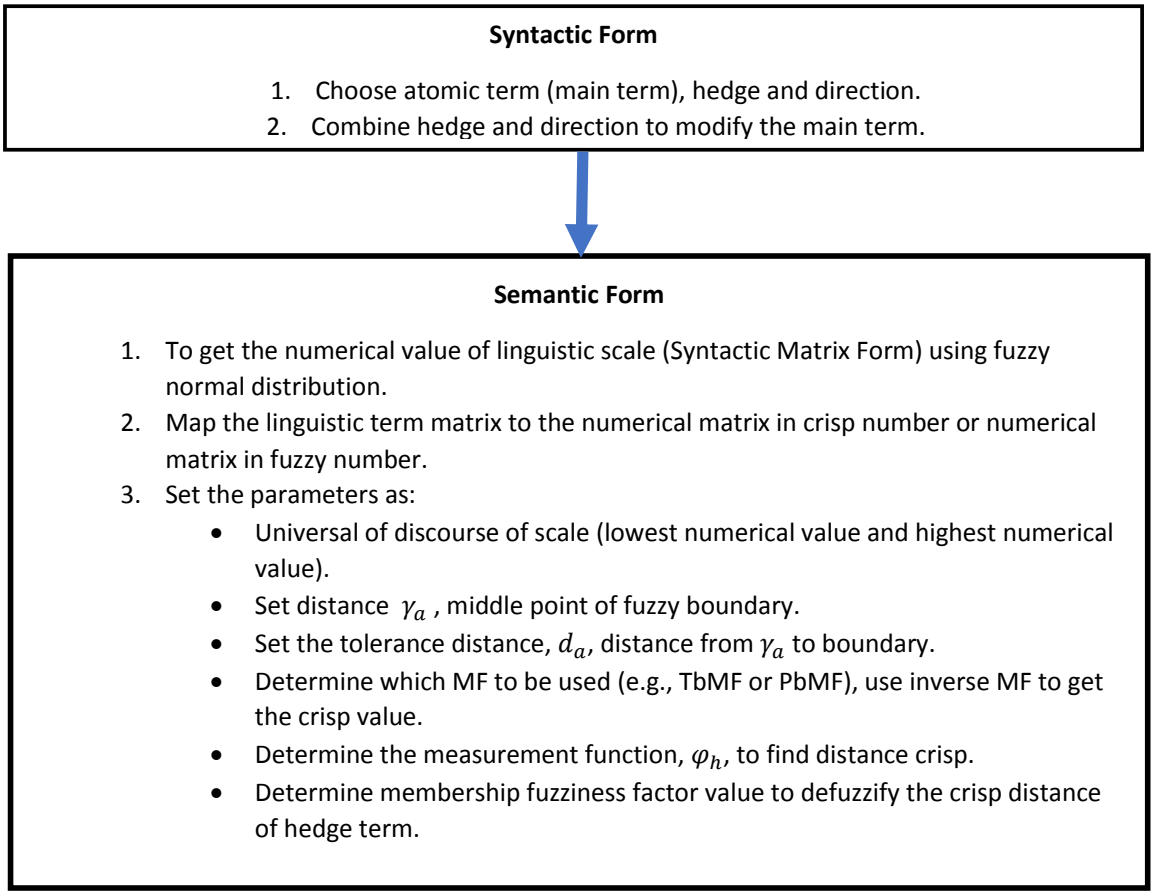


Fig. 3.5 Process of development the Compound Linguistic Scale (CLOS)

The whole process of developing the new scale by Yuen can be summarized as shown in Figure 3.5. The CLOS was built in matrix form known as matrix CLV, which is two-

dimensional to represent the collection of linguistic terms. This matrix is used by the DM for ratings in the decision making process. In the first stage, which is a syntactic form, the terms used in the matrix CLV are defined as follows:

Definition 12 *Term Set, $T(\mathscr{N})$: The syntactic pattern of term set T is a quintuple, (V_h, V_d, V_a, G_N). A term set T of linguistic variable $\mathscr{N}$ is syntactically mapped from V_h, V_d, V_a with the syntactic relation $G_{\mathscr{N}}$ as:*

$$G_{\mathscr{N}} : (V_h, V_d, V_a) \rightarrow T(\mathscr{N}) = \mathscr{N} \tag{3.12}$$

where $\rightarrow$ denotes "be linguistically mapped to", and $G_{\mathscr{N}}$ is the syntactic relationship function for $\mathscr{N}$.

Definition 13 *A vectorized atomic linguistic term V_a is used to describe a statement or measure the objects in the initial sense. V_a is written in the form of:*

$$V_{a_n} = [v_{a_i}]_{j=1}^{n} = [v_{a_1}, \cdots, v_{a_n}], \tag{3.13}$$

where $\leq_{j=1}^{n} (v_{a_j}) \equiv v_{a_1} \leq \cdots \leq v_{a_n}, v_a \in V_a$. n is an odd number and larger than or equal to three.

For example:

$$V_a = [Poor,\ Weak,\ Fair,\ Good,\ Excellent]$$

where $Poor < Weak < Fair < Good < Excellent$.

Definition 14 *A hedge term V_h is the strategies for adjusting the linguistic quantity in V_a. It can be written as:*

$$V_h = [v_{h_i}]_{i=1}^{n} = [v_{h_1}, \cdots, [v_{h_n}] \tag{3.14}$$

where $\leq_{i=1}^{n} (v_{h_i})$ and $v_h \in V_h$.

For example,

$$V_h = [Little,\ Quiet,\ Much]$$

where $Little < Quiet < Much$.

Definition 15 *A directional term V_d is used to indicate the direction of V_h, whether it is negative v_d^-, natural v_d^θ or positive preferences v_d^+. It can be written as:*

$$V_d = [v_{d_i}]_{i=1}^3 = [v_d^-,\ v_d^\theta, v_d^+] \tag{3.15}$$

with $\leq_{i=1}^3 (v_{d_i})$. A static point, v_d^θ is the separation line to divide the positive domain and negative domain of linguistic terms.

For example,

$$V_d = [Below,\ Absolutely,\ Above] \tag{3.16}$$

where $Below < Absolutely < Above$.

Definition 16 *The combination of hedge term and direction term is known as syntactic term V_{hd}. Let $V_{hd} = [v_{hd_i}]_{i=1}^m$ with $\leq_{i=1}^m$, which consists of v_{hd}^-, v_{hd}^θ and v_{hd}^+, where v_{hd}^- is a negative hedge linguistic term, v_{hd}^+ is a positive hedge linguistic term and v_{hd}^θ is a static hedge linguistic term, respectively. Then, the syntactic function can be written as:*

$$G_{V_{hd}} : (V_h, V_d) \rightarrow V_{hd} \tag{3.17}$$

where $V_{hd} = [v_{hd}^-,\ v_{hd}^\theta,\ v_{hd}^+]$.

For example:

$$V_{hd} = [Much\ Below,\ Quiet\ Below,\ Little\ Below,\ Absolutely,\ Quiet\ Above,\ Much\ Above]$$

V_{hd} can be used with V_a to form a new meaning.

Definition 17 *A compound linguistic term, α, can be derived using syntactic equation, and the term is represented as:*

$$\alpha = G_\alpha(v_h, v_d, v_a) = G_\alpha(v_{hd}, v_a) \tag{3.18}$$

where v_h, v_d and v_a are any members from V_h, V_d, V_a respectively. To extend the notation:

$$\alpha_{ij} \triangleq v_{hd_i} \oplus v_{aj},\ \alpha_i \triangleq v_{hd_i} \oplus V_a = [v_{hd_i} \oplus V_{a_j}]_{j=1}^{n},\ \alpha^j \triangleq v_{hd} \oplus v_{a_j} = [v_{hd_i} \oplus V_{a_j}]_{i=1}^{m} \tag{3.19}$$

For all $i \in (1, \cdots, m),\ j \in (\cdots, n)$ where cardinal numbers $m = |V_{hd}|$ and $n = |V_a|$.

For example:

$$\begin{aligned}\alpha^3 =& [\textit{Much Below Fair},\ \textit{Quiet Below Fair},\ \textit{Little Below Fair},\ \textit{Absolutely Fair},\\ & \textit{Little Above Fair},\ \textit{Quiet Above Fair},\ \textit{Much Above Fair}]\end{aligned}$$

There is an exceptional case for the maximum value. For example, *Absolutely Excellent*. Since it is a maximum level value, thus from a semantic view, no term *Much Above Excellent* is defined. These maximum cases are denoted as $\emptyset$ in the CLOS matrix. This can be written in the following proposition.

Proposition 1 *In the semantic aspect, the Normalized Syntactic Function is:*

$$G_{\alpha_{ij}}(v_{hd_i}, v_{a_j}) \triangleq \begin{cases} \emptyset & j = 1\ \&\ i \in \left\{1, \cdots, \left(\frac{(m+1)}{2}\right)\right\} \\ v_{hd_i} \oplus v_{a_j} & j \neq 1,\ n\ \&\ \forall i \\ \emptyset & j = n\ \&\ i \in \left\{\left(\frac{(m+1)}{2}\right), \cdots, m\right\} \end{cases} \tag{3.20}$$

Based on the Proposition 1, the CLV matrix can be formed.

Example 1 *Provided that, $V_a = [$Poor, Weak, Fair, Good, Excellent$]$, $V_d = [$Below, Absolutely, Above$]$ and $V_h = [$Little, Quiet, Much$]$. The Compound Linguistic Variable (CLV) can be obtained by:*

Form $V_{hd} = [$*Much Below, Quiet Below, Little Below, Absolutely, Little Above, Quiet Above, Much Above*$]$. *The cardinal of* $V_{hd} = m = 7$, *while the cardinal of* $V_a = n = 5$. *Thus, cardinal* $T(N) = N_{m,n} = N_{7,5} = m(n-1)+1 = 7(5-1)+1 = 29$. *Thus,* T_{mn} *can be written as:*

$$\mathcal{N}_{7,5} = \begin{array}{c} \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \end{array} \begin{bmatrix} \emptyset & MB\text{-}W & MB\text{-}F & MB\text{-}G & QA\text{ - }E \\ \emptyset & QB\text{-}W & QB\text{-}F & QB\text{-}G & QB\text{ - }E \\ \emptyset & LB\text{-}W & LB\text{-}F & LB\text{-}G & LB\text{-}E \\ A\text{ - }P & A\text{ - }W & A\text{ - }F & A\text{-}G & A\text{-}E \\ LA\text{ - }P & LA\text{ - }W & LA\text{ - }F & LA\text{-}G & \emptyset \\ QA\text{ - }P & QA\text{ - }W & QA\text{ - }F & QA\text{ - }G & \emptyset \\ MA\text{ - }P & MA\text{ - }W & MA\text{ - }F & MA\text{ - }G & \emptyset \end{bmatrix} \tag{3.21}$$

Next, a fuzzy normal distribution is used to obtain the numerical value for the linguistic term in the CLV matrix. Basically, this step can be considered as a mapping of the linguistic term matrix to the numerical matrix in crisp or fuzzy form. Based on the stepwise procedure of the semantic form in Figure 3.5, the following numerical value is obtained for Example 1.

$$\mathcal{N}_{7,5} = \begin{array}{c} \\ Much\ Below \\ Quite\ Below \\ Little\ Below \\ Absolutely \\ Little\ Above \\ Quite\ Above \\ Much\ Above \end{array} \begin{bmatrix} Poor & Weak & Fair & Good & Excellent \\ \emptyset & (1.00,\ 1.27,\ 1.54) & (2.00,\ 2.27,\ 2.54) & (3.00,\ 3.27,\ 3.54) & (4.00,\ 4.27,\ 4.54) \\ \emptyset & (1.40,\ 1.60,\ 1.79) & (2.40,\ 2.60,\ 2.79) & (3.40,\ 3.60,\ 3.79) & (4.40,\ 4.60,\ 4.79) \\ \emptyset & (1.67,\ 1.83,\ 2.00) & (2.67,\ 2.83,\ 3.00) & (3.67,\ 3.83,\ 4.00) & (4.67,\ 4.83,\ ,5.00) \\ (1,\ 1,\ 1) & (2,\ 2,\ 2) & (3,\ 3,\ 3) & (4,\ 4,\ 4) & (5,\ 5,\ 5) \\ (1,\ 1.17,\ 1.33) & (2.00,\ 2.17,\ 2.33) & (3.00,\ 3.17,\ 3.33) & (4.00,4.17,4.33) & \emptyset \\ (1.20,\ 1.40,\ 1.59) & (2.20,\ 2.40,\ 2.59) & (3.20,\ 3.40,\ 3.59) & (4.20,\ 4.40,\ 4.59) & \emptyset \\ (1.46,\ 1.73,\ 2.00) & (2.46,\ 2.73,\ 3) & (3.45,\ 3.73,\ 4) & (4.45,\ 4.73,\ 5) & \emptyset \end{bmatrix} \tag{3.22}$$

Based on this, the new scale was developed by Yuen [13] and the scale was used in the standard T1-TOPSIS model. The author stated that the proposed T1-TOPSIS can be considered as an improved version of Chen's T1-TOPSIS model. Even though Yuen developed the new scale to be used in the T1-TOPSIS model, it requires development of a two-dimensional preference scale in which various algorithms are needed to ensure it captures the problem and hence it needs more computational calculations. Thus, the question arises as

to whether this Yuen's model provides benefits as compared to other existing T1-TOPSIS model. The next section will answer this question through reviews and comparison of these two models.

3.7.4 Comparison of Chen's and Yuen's T1-TOPSIS

In the previous section, two types T1-TOPSIS models were described in detail. In this section, the step by step comparison of both models is summarised in Table 3.2. Note that Yuen's introduced a new step in the beginning of the process which is followed by a double rating process using a new, different linguistic variable scale, as opposed to Chen's single rating process. Other than the rating process and identifying the ideal solution step, the other steps follow Chen's T1-TOPSIS. The main goal of this comparative study is to provide context for the strengths and limitations of both models. Three steps differ between these two models. The differences are in the Steps 1, 2 and 6. In Step 1, both models use linguistic scales employing T1 TFNs to express the linguistic terms. While Chen's model only uses a one-dimensional scale, in Yuen's model, a two-dimensional scale is used to capture the linguistic terms. An example of both scales is given in Tables 3.3 and 3.4 respectively. In Table 3.3, the linguistic terms used contain no hedges, while in Table 3.4, Yuen's introduced hedges are shown.

The next step which is different is in Step 2. In this step, the DMs are asked to give a rating of each alternative with respect to each criterion based on the scale given in Step 1. Chen's rating process is a single rating process as the scale used in this model is one-dimensional scale. The DMs only choose one linguistic term to evaluate the alternatives. In contrast, Yuen's introduces a double rating process in this step. This process requires each DM to choose the main term shown in the columns of Table 3.4 and choose a suitable hedge as shown in the same table. An example of main terms as in Table 3.4 are: 'Poor', 'Medium Poor', 'Fair', 'Medium Good' and 'Good'. In the second stage, the DMs need to choose the

Table 3.2 Comparison of Chen's and Yuen's T-TOPSIS Model

Steps	Chen's T-TOPSIS	Yuen's T-TOPSIS
Scale	Use T1 TFNs One-dimensional scale (Table 3.3)	Use T1 TFNs Scale from Fuzzy Normal Distribution Two-D scale (Table 3.4)
Rating / Weighting	Single rating process	Double rating process
Aggregation	Weighted arithmetic mean	Same as Chen's model
Normalization	Normalized value decision matrix to get the value within range $[0, 1]$.	Same as Chen's model.
Weighted normalized	Structure weighted normalized fuzzy decision matrix.	Same as Chen's method.
Ideal solution	Defined ideal solution as a perfect value: Positive ideal solution (PIS) = (1,1,1); Negative ideal solution (NIS) = (0,0,0)	Defined maximum value (positive); Defined minimum value (negative).
Distance measurement	Defined vertex method	Same as Chen's model
Closeness Coefficient (CC)	Calculate CC alternative from ideal solution.	Same as Chen's model.
Ranking	Higher CC is better alternative.	Same as Chen's model.

Table 3.3 Linguistic variables for the evaluation of alternatives and weighting of criteria in Chen's T1-TOPSIS model

(a) Linguistic scale for weight of criteria				
Low (L)	Medium Low (ML)	Medium (M)	Medium High (MH)	High (H)
(0, 0.1, 0.3)	(0.1, 0.3, 0.5)	(0.3, 0.5, 0.7)	(0.5, 0.7, 0.9)	(0.7, 0.9, 1.0)
(b) Linguistic scale for rating of alternatives				
Poor (P)	Medium Poor (MP)	Fair (F)	Medium Good (MG)	Good (G)
(0,1,3)	(1, 3, 5)	(3, 5, 7)	(5, 7, 9)	(7, 9, 10)

second term which consists of hedges, e.g., 'Much Below', 'Quite Below', 'Little Below', 'Absolutely', 'Little Above', 'Quite Above' and 'Much Above'. Thus, Yuen's model basically collects more information from DMs than Chen's model. Note that regardless of the scale used, an important factor in any MCDM model is to produce a consistent ranking order of alternatives in a final result.

The last different step between both models is the definition of an ideal solution in Step 6. In Chen's model, the author defines the ideal solution and negative ideal solution as a perfect value on criteria where the values are always $(1,1,1)$ and $(0,0,0)$ respectively. On the other hand, Yuen defines the ideal solution as a maximum of the decision matrix value and the negative ideal solution as a minimum of the decision matrix value. The definition of the positive ideal solution, (r^*) in Yuen's model is shown in (3.23) and the negative ideal solution (r^-) is shown in (4.1):

$$r^* = (r^*_{j_a}, r^*_{j_b}, r^*_{j_c}) = \text{argmax}(r_{ij_a}, r_{ij_b}, r_{ij_c}) \tag{3.23}$$

$$r^- = (r^-_{j_a}, r^-_{j_b}, r^-_{j_c}) = \text{argmax}(r^a_{ij}, r^b_{ij}, r^c_{ij}) \tag{3.24}$$

where $(r_{j_a}, r_{j_b}, r_{j_c})$ is a T1 TFN in the decision matrix.

Table 3.4 Linguistic scale for rating of alternatives in Yuen's T1-TOPSIS model

	Poor (P)	Medium Poor (MP)	Fair (F)	Medium Good (MG)	Good (G)
Much Below	null	(0, 0.68, 1.36)	(2.5, 3.18, 3.86)	(5, 5.68, 6.36)	(7.5, 8.18, 8.86)
Quite Below	null	(1.01, 1.5, 1.98)	(3.51, 4, 4.48)	(6.01, 6.5, 6.98)	(8.51, 9, 9.48)
Little Below	null	(1.67, 2.08, 2.5)	(4.17, 4.58, 5)	(6.67, 7.08, 7.5)	(9.17, 9.58, 10)
Absolutely	(0, 0, 0.42)	(2.08, 2.50, 2.92)	(4.58, 5, 5.42)	(7.08, 7.5, 7.92)	(9.58, 10, 10)
Little Above	(0, 0.42, 0.83)	(2.50, 2.92, 3.33)	(5, 5.42, 5.83)	(7.5, 7.92, 8.33)	null
Quite Above	(0.52, 1, 1.49)	(3.02, 2.5, 3.99)	(5.52, 6, 6.49)	(8.02, 8.5, 8.99)	null
Much Above	(1.14, 1.82, 2.5)	(3.64, 4.32, 5)	(6.14, 6.82, 7.5)	(8.64, 9.32, 10)	null

The other steps are the same for both models, starting from Step 3 (aggregation), Step 4 (normalization), Step 5 (weighted normalized decision matrix), Step 7 (distance measurement), Step 8 (closeness coefficient) and Step 9 (ranking). For Step 3, both models use (3.4) to get an average value of rating of alternatives from the DMs. Next, the normalization step (Step 4) used (3.4) to get the normalized value of decision matrix. Then, Step 5 uses (3.5) to get the weighted normalized decision matrix value. Step 7 and 8 use (3.10) and (3.11) respectively to get the distance measurement and the closeness coefficient (CC). Then both models used the same ranking step where the higher the CC value, the better the alternative is. In order to demonstrate both models, the next section show an MADM problem with synthetic data to highlight the differences between these two models.

3.7.5 Numerical Example

In this section, an example is used to demonstrate the implementation of both models. This example is designed using synthetic data to enable the systematic comparison of the models using the same MADM problem. The goal for both models is to get the ranking order of alternatives. Specifically, we focus on comparing the following aspects:

1. The scale used in both models (Section 3.7.6);
2. DM rating aggregation and normalization (Section 3.7.7);
3. The definition of the ideal solution (Section 3.7.8);

For each case, a comparison for both models based on the synthetic MADM problem is provided.

3.7.6 The Scale Used in Both Models

In Chen's model, the scales used to represent the linguistic variable for weighting of criteria and rating of alternatives are shown in Table 3.3 (a) and (b) respectively. Chen assigns a

Table 3.5 Linguistic rating of alternatives and importance weight of criteria - Chen's and Yuen's model

		(a) Chen's rating			(b) Yuen's rating		
Criteria	Alternatives	$\mathscr{D}_1$	$\mathscr{D}_2$	$\mathscr{D}_3$	D_1	D_2	D_3
$\mathscr{C}_1$	$\mathscr{A}_1$	MG	G	MG	Ab-MG	Ab-G	Ab-MG
	$\mathscr{A}_2$	G	G	MG	Ab-G	Ab-G	Ab-MG
	$\mathscr{A}_3$	MG	MG	F	Ab-MG	Ab-MG	Ab-F
	Weight	H	VH	MH	H	VH	MH
$\mathscr{C}_2$	$\mathscr{A}_1$	G	MG	F	Ab-G	Ab-MG	Ab-F
	$\mathscr{A}_2$	G	G	G	Ab-G	Ab-G	Ab-G
	$\mathscr{A}_3$	MG	G	G	Ab-MG	Ab-G	Ab-G
	Weight	VH	VH	VH	VH	VH	VH
$\mathscr{C}_3$	$\mathscr{A}_1$	F	MG	MG	Ab-F	Ab-MG	Ab-MG
	$\mathscr{A}_2$	G	G	MG	Ab-G	Ab-G	Ab-MG
	$\mathscr{A}_3$	G	MG	MG	Ab-G	Ab-MG	Ab-MG
	Weight	VH	H	H	VH	H	H

value based on intuitive and logical sense without a complex design process. This numerical example simulate the case of three DMs being asked to rate the alternatives and to provide importance weights for the criteria. This example uses three benefit criteria, ($\mathscr{C}_1$, $\mathscr{C}_2$, $\mathscr{C}_3$) and three alternatives ($\mathscr{A}_1$, $\mathscr{A}_2$, $\mathscr{A}_3$). The collective rating of alternatives is shown in Table 3.5 (a) together with importance weights of the criteria.

It is worth noting that in this step, Yuen's method collects more information as each DM is required to give two-fold ratings. Firstly, as shown in Table 3.4, the DMs need to choose one of the main linguistic terms (e.g., 'Poor', 'Fair', etc.) and secondly, the DMs need to choose the hedges (e.g., 'Much Below', 'Quite Below', etc.). In contrast, Chen's model uses a direct rating process and following Table 3.3, the DMs only need to choose one linguistic term (e.g., 'Poor', 'Fair', etc).

For numerical example, this study generates comparable data as could realistically be collected for the same problem (as explored above for Chen's model) from the same DMs. Using the same number of DMs, alternatives and criteria as in Chen's example in the previous part, this study simulate the case of DMs being asked to rate the alternatives. For the weights of the criteria, the same values as in Chen's example are used. For the rating of alternatives, Yuen's model uses the scale in Table 3.4. For comparison purposes, the synthetic ratings of alternatives is the same as in Chen's model shown in Table 3.5, except for the addition of hedges. This study purposefully seek to compare both models using an example with data that is as similar as possible. Clearly, however, Yuen's model draws its main benefits from the additional degrees of freedom imparted by the hedges. Since there are no hedges in Chen's model, this study adopt the 'Absolutely' hedge throughout in order to establish a reasonable basis for the comparison of both models. The result is a rating of alternatives of Yuen's model as shown in Table 3.5, column (b).

3.7.7 DM Rating Aggregation and Normalization

Following rating, for both models, the next step is to calculate the average of all the preferences values to form a fuzzy decision matrix using (3.2), as explained in Section 3.7.2. For this example, the results of the process for Chen's model are shown in Table 3.6 (a) and for Yuen's model, in Table 3.6 (b). The table show an average of all DMs' preferences for each model. Post aggregation, a normalization step is applied, where both models used the same technique (3.4).

This step is taken to facilitate the computational problems inherent to the presence of the different units in the decision matrix and aims to obtain comparable scales. In this example, both models use all the benefit criteria. Then, for the overall weighted normalized decision matrix, both models use the same technique shown shown as (3.5). In this step, each value from the previous normalized matrix is multiplied by its associated weight as shown in Table

Table 3.6 Fuzzy decision matrix - Chen's and Yuen's method

(a) Chen's model			
	$\mathscr{C}_1$	$\mathscr{C}_2$	$\mathscr{C}_3$
$\mathscr{A}_1$	(5.67, 7.67, 9.33)	(5.00, 7.00, 8.67)	(5.67, 6.33, 8.33)
$\mathscr{A}_2$	(6.33, 8.33, 9.67)	(7.00, 9.00, 10.00)	(6.33, 8.33, 9.67)
$\mathscr{A}_3$	(4.33, 6.33, 8.33)	(6.33, 8.33, 9.67)	(5.67, 7.67, 9.33)
(b) Yuen's model			
$\mathscr{A}_1$	(7.91, 8.33, 8.61)	(7.08, 7.50, 7.78)	(6.25, 6.67, 7.09)
$\mathscr{A}_2$	(8.75, 9.17, 9.31)	(9.58, 10.00, 10.00)	(8.75, 9.17, 9.31)
$\mathscr{A}_3$	(6.25, 6.67, 7.09)	(8.75, 9.17, 9.31)	(7.91, 8.33, 8.61)
Weight	(0.70, 0.90, 1.00)	(0.90, 1.00, 1.00)	(0.77, 0.93, 1.00)

3.6 (a). The result of the weighted normalized decision making for Chen's and Yuen's model are presented in Table 3.7. The next step is the definition of the ideal solution which will be discussed in the next section.

3.7.8 Definition of the Ideal Solution

In Chen's model, the perfect value is used. For our numerical example, the Fuzzy Positive Ideal Solution (FPIS) is defined as in (3.25) while the Fuzzy Negative Ideal Solution (FNIS) as in (3.26):

$$V^+ = [(1\ 1\ 1), (1\ 1\ 1), (1\ 1\ 1)] \tag{3.25}$$

$$V^+ = [(0\ 0\ 0), (0\ 0\ 0), (0\ 0\ 0)] \tag{3.26}$$

On the other hand, in Yuen's model, the definition of FPIS and FNIS are established as shown in (3.27) and (3.28) respectively. In this example, the values of FPIS and FNIS are

Table 3.7 Weighted normalized decision matrix - Chen's and Yuen's model

	(a) Chen's model		
	$\mathscr{C}_1$	$\mathscr{C}_2$	$\mathscr{C}_3$
$\mathscr{A}_1$	(0.41, 0.71, 0.97)	(0.45, 0.70, 0.87)	(0.45, 0.61, 0.86)
$\mathscr{A}_2$	(0.46, 0.77, 0.10)	(0.63, 0.90, 1.00)	(0.50, 0.80, 1.00)
$\mathscr{A}_3$	(0.31, 0.59, 0.86)	(0.57, 0.83, 0.97)	(0.45, 0.74, 0.97)
	(b) Yuen's model		
$\mathscr{A}_1$	(0.60, 0.81, 0.93)	(0.64, 0.75, 0.78)	(0.52, 0.67, 0.76)
$\mathscr{A}_2$	(0.66, 0.89, 1.00)	(0.86, 1.00, 1.00)	(0.72, 0.92, 1.00)
$\mathscr{A}_3$	(0.47, 0.64, 0.76)	(0.79, 0.92, 0.93)	(0.65, 0.83, 0.93)

shown in (3.27) and (3.28) respectively:

$$r^{+} = [(0.66, 0.89, 1.00), (0.86, 1.00, 1.00), (0.72, 0.92, 1.00)] \tag{3.27}$$

$$r^{-} = [(0.47, 0.64, 0.76), (0.64, 0.75, 0.78), (0.52, 0.67, 0.76)] \tag{3.28}$$

These values arise from Table 3.7 (b), by which the definition of this ideal solution is the maximum value for each column that is associated to each criteria. For example, to get the ideal solution for Criteria 1 $\mathscr{C}_1$ as $(0.66, 0.89, 1.00)$ shown in (3.27), one must refer to the weighted normalised decision table shown in Table 3.7 (b). The first column is associated with $\mathscr{C}_1$. Then, one finds the maximum value (peak) to get the ideal solution for $\mathscr{C}_1$. Argument Max in (3.23) results in the maximum value as $(0.66, 0.89, 1.00)$ associated with Alternative $\mathscr{A}_2$ in Table 3.7 (b). The next step is to find the distance between the ideal solution and each of the alternatives. Both models use the same technique as shown in (3.10) to establish this distance. For Chen's model, the distance of each alternative in an example is shown in Table 3.8 (a) together with the closeness coefficient (CC_i) value and ranking of alternatives. The (CC_i) values for both models also use the same technique shown in (3.11).

Table 3.8 Distance, the *CC* and the rank of alternatives - Chen's and Yuen's model

	(a) Chen's model				(b) Yuen's model			
	$dist_i^+$	$dist_i^-$	CC_i	Rank	$dist_i^+$	$dist_i^-$	CC_i	Rank
$\mathscr{A}_1$	1.15	2.09	0.65	3	0.54	0.15	0	3
$\mathscr{A}_2$	0.87	2.43	0.74	1	0	0.69	0.92	1
$\mathscr{A}_3$	1.09	2.19	0.67	2	0.34	0.38	0.31	2

Finally, the ranking of alternatives was determined using the same concept. In Chen's model, alternative $\mathscr{A}_2$ is the best one followed by $\mathscr{A}_3$ and $\mathscr{A}_1$. As shown in Table 3.8, both models produce the same ranking of alternatives: Alternative 2 ($\mathscr{A}_2$) is ranked the best, followed by Alternative 3 ($\mathscr{A}_3$) and Alternative 1 ($\mathscr{A}_1$).

3.8 Discussion of Comparative Study

For the overall ranking, both models give the same ranking order as expected based on the synthetic data. Table 3.8 (a) shows that Chen's closeness coefficient (CC_i) values are close to each other, while Table 3.8 (b) shows Yuen's CC_i values have a large gap - differentiating them. Figures 3.6 and 3.7 show the distance measurement of both methods. Each corner, e.g. $\mathscr{A}_1$, in both figures represents each alternative in both simulated case studies. The red square points represent the distance measurement between FNIS with each alternative while the blue square points indicate the distance measurements between FPIS with each alternative. For example, in Fig. 3.6, $\mathscr{A}_1$ has a distance measurement of 2.09 for FNIS and has 1.15 distance measurement for FPIS. Table 3.8 provides all these values.

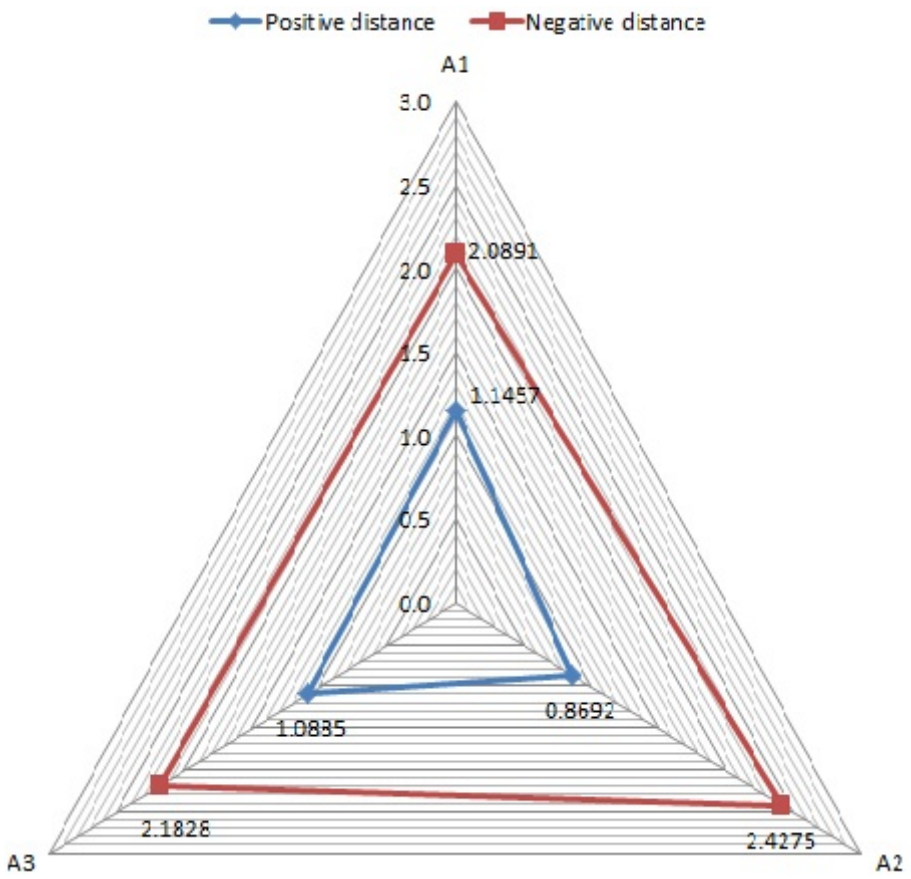


Fig. 3.6 Graphical representation of distance measurement - Chen's model

3.9 Summary

This chapter has presented the first and second contributions of this thesis: that is, a study in exploring of issues and limitations in the standard TOPSIS model and a comparative study in stepwise manner between standard T1-TOPSIS by Chen [1] and the extended T1-TOPSIS by Yuen [13]. Firstly, the motivation of doing an exploration study of issues and limitations of the standard and T1-TOPSIS model were outlined. Based on this, the reviewed of several studies was made and identified that there was none of the studies explaining the details of the key stages of standard TOPSIS and T1-TOPSIS based on a systematic stepwise manner. Several issues, for examples relate to reliability of the preferences given by DMs and the ability of the model in handling and reflecting uncertainty from initial to the final process

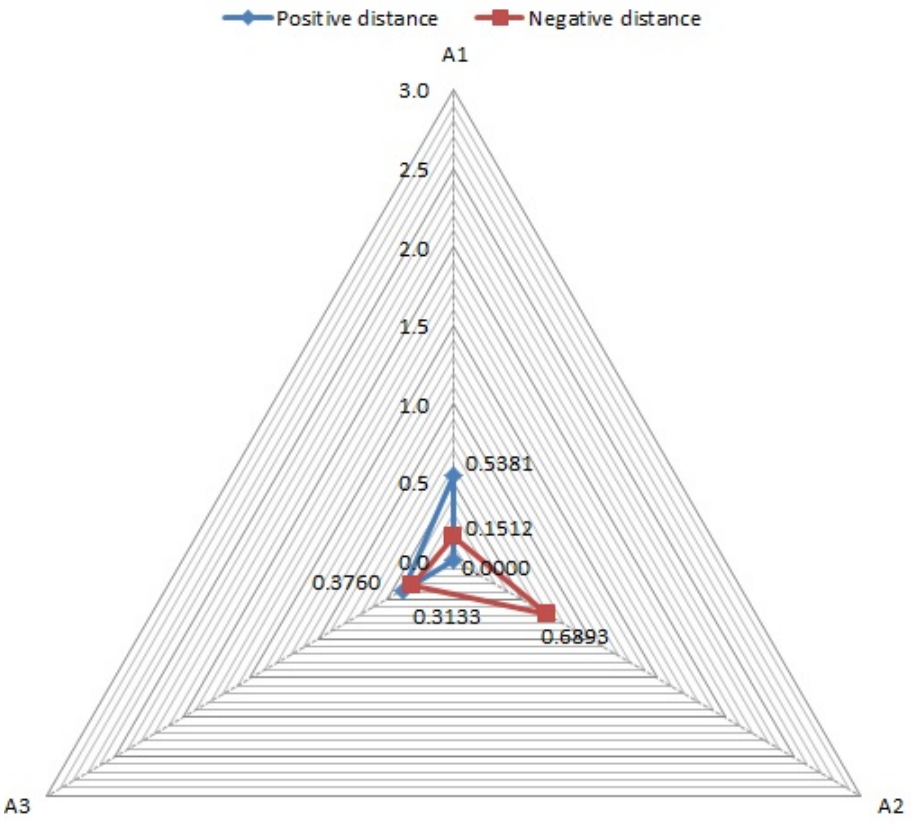


Fig. 3.7 Graphical representation of distance measurement - Yuen's model

were highlighted. These can be used further as a base guideline to improve the existing model. The key stages of standard TOPSIS and T1-TOPSIS models were briefly discussed. This is the first work which has attempted to both provide details and to highlight the key differences between these two models in a systematic manner. Following identification and clarification of these issues, this study provided a potential solution which could result in better and more trustworthy decisions. Thus, the aim of this study is to provide the details of the TOPSIS and T1-TOPSIS models as well as their issues and limitations were successfully described. The first part of this chapter addresses the first objective of this thesis.

The second part of this chapter concerned with a comparison study between two types of T1-TOPSIS models (i.e., Chen's and Yuen's model). The implementation of both models was demonstrated in detail using a synthetic numerical example. The synthetic data in this example was specifically designed to enable a direct comparison between both models, i.e.

the secondary scale employed by Yuen's model was largely neutralised by using only the neutral 'Absolute' hedge. This study was taken to focus on the difference in mechanisms in both models, rather than on the presence of additional information gathered from DMs in Yuen's model. Based on the resulting comparison, the differences in the steps implemented by these two T1-TOPSIS models were highlighted throughout the worked example, where they arise in the scale, rating and ideal solution definition steps. As was intuitively expected, the final ranking order of alternatives shows the same result for both models. Thus, there is necessary to explore the effect of varying individual hedges in Yuen's model and showing how the additional information captured in these hedges provides different outputs compared to Chen's model. Clearly, as Yuen's model is significantly more work-intensive, both computationally and more importantly in terms of expert time, it is important to establish what value the additional effort can provide and in which situations it is warranted. Based on this study, in the next chapter, an experimental method will be proposed in which various synthetic data are used to simulate various uncertainties in the decision-making process. The expected outcome of this experimental method is that it may provide insight into what value the additional information can provide in terms of reducing the uncertainty in the process.

Chapter 4

The Proposed IT2-TOPSIS Model with Interval-valued Outputs

4.1 Introduction

This chapter is comprised of two main parts. The first part of the chapter describes the proposed extension model of T1-TOPSIS to IT2-TOPSIS, building on the model introduced by Chen [1] that was initially discussed in Chapter 3. Section 4.2 gives several motivations for why this model is needed. Several existing studies are reviewed and the needs to implement this model are highlighted. This section aims to introduce the model in a stepwise manner and to show how the models provide a systematic mapping between uncertainty levels in the inputs to the uncertainty in the outputs. Section 4.3 introduces the proposed model in detail. The model uses IT2 TFNs to model input data and for the first time generates interval-valued CC outputs denoted as $\overline{CC}$. It was designed to directly map uncertainty in inputs to uncertainty in decision outputs within multi-criteria decision making (MCDM) problems in the TOPSIS model.

The second part of the chapter, starting from Section 4.4, describes an initial set of experiments that introduces a number of different levels of uncertainty using IT2 Triangular

FNs (IT2 TFNs) which are associated with linguistic labels (e.g. Good, Fair, etc.) and are produced by blurring initial T1 TFNs. By conducting the experiments this study specifically explores whether any (and which) relationships exist between the level of uncertainty (amount of blurring) and the TOPSIS result, i.e. the CC interval. Section 4.5 describes the implementation of the proposed model in two different case studies used to evaluate the model. Section 4.6 provides a discussion about the results of each case study. Finally, Section 4.7 summarizes this chapter and provides its relation to the next chapter.

4.2 Motivation

Recall that in Chapter 3, several issues and limitations were highlighted. One of them is related to the existing models for which there is no systematic mapping between uncertainty levels in the inputs to the uncertainty in the outputs. In decision-making situations in which we have perfect information, we can usually rank alternatives with certainty. However, when uncertainty affects the information about the alternatives, we can intuitively expect challenges in ranking them in the sense that we may not be able to rank the alternatives with the same level of certainty. From a practical point of view, it is clear that any uncertainty affecting the ranking of alternatives should be taken into account during decision making.

IT2-TOPSIS is recognized as a useful model in addressing MCDM problems by majority of the studies in this field. However, the additional information provided by IT2 FSs in the models reviewed here is largely lost, since the final output (the CC) is still a crisp value. Thus, the benefit of using such sets is not fully exploited, as the extra information that they could provide is not available to DMs in the decision making process. Hence, it is intuitive to develop a TOPSIS model based on IT2 FSs which takes advantage of the additional information provided by the sets, where the final result remains an interval.

Many researchers have applied TOPSIS to solve fuzzy MCDM problems in the past [1, 9–11, 69, 68]. Current IT2-TOPSIS models commonly use IT2 TFNs to represent

ratings and weights given by DMs. However, these models are limited to providing a crisp output to be used by DMs for ranking [147, 137, 90, 148, 123, 130, 149, 125, 150]. In this proposed IT2-TOPSIS model, IT2 TFNs are used to capture uncertainty in the input and provide interval-values in the output. While looking at how to capture uncertainty in the process, some researchers have extended T1-TOPSIS to an interval type-2 model, as IT2 FSs [54, 151, 51] are widely accepted as powerful models for capturing uncertainty [15, 16]. While the previously reviewed fuzzy MCDM models in Chapter 2 have been highly successful in a number of applications, they do not effectively deal with uncertainty in the input to the uncertainty in the output. None of the models developed thus far provide a systematic mapping between uncertainty levels in the MCDM inputs to the uncertainty in the outputs. This model therefore intends to provide a solution for the aforementioned mapping problem.

To the best of the author's knowledge, no current IT2-TOPSIS model is designed to capture uncertainty and represent it within its decision support outputs. Instead, the models give a crisp output, which in turn is used to create rankings. However, as these rankings are interpreted with certainty, they effectively remove any inherent uncertainty within the decision support system. It is this potential pitfall we are seeking that the model proposed in this chapter seeks to address. In an earlier work by Madi [152], the study originally showed that one could produce an interval-valued output based on the IT2-TOPSIS model. This earlier work presents a detailed preliminary study to illustrate the benefits of the proposed model as opposed to current literature.

The proposed model provides a framework for capturing uncertainty in the output of decision making based on the TOPSIS model and provides a systematic mapping of uncertain inputs to uncertain outputs. Specifically, this study proposes a novel extension to IT2-TOPSIS, which enables the direct mapping of uncertainty in the inputs to uncertainty in the outputs. The latter is achieved by developing an IT2-TOPSIS model that generates an interval-valued

output (i.e., CC) which captures the uncertainty. This makes it possible to employ recent research on the appropriate ranking of interval-valued data for the ranking of the alternatives. Next section will introduce the proposed model in detail.

4.3 The Proposed Model

Consider a decision-making problem for the T1-TOPSIS model, denoting alternatives ($\mathscr{A}_{i=1,\cdots,m}$), criteria ($\mathscr{C}_{j=1,\cdots,n}$) and DMs ($\mathscr{D}_{e=1,\cdots,k}$). As stated earlier, in the proposed IT2-TOPSIS model, IT2 TFNs are used for linguistic rating scales. DMs still provide two sets of information (rating of alternatives and weight of each criterion) by using linguistic terms, modelled by IT2 TFNs. The linguistic terms are represented using seven IT2 TFNs as detailed in Table 4.1 and illustrated in Figure 4.4. The details of the proposed model are explained next.

Table 4.1 Linguistic terms for the evaluation of alternatives and weighting of criteria using IT2 TFNs

	IT2 TFNs	
Terms for Ratings / Weighting	LMF	UMF
Very Poor (VP) / Very Low (VL)	(0, 0, 0.5)	(0, 0 ,1.5)
Poor (P) / Low (L)	(0.5, 1, 2.5)	(-0.5, 1, 3.5)
Medium Poor (MP) / Medium Low (ML)	(1.5, 3, 4.5)	(0.5, 3, 5.5)
Fair (F) / Medium (M)	(3.5, 5, 6.5)	(2.5, 5, 7.5)
Medium Good (MG) / Medium High (MH)	(5.5, 7, 8.5)	(4.5, 7, 9.5)
Good (G) / High (H)	(7.5, 9, 9.5)	(6.5, 9, 10.5)
Very Good(VG) / Very High (VH)	(9.5, 10, 9.5)	(8.5, 10, 10.5)

Following the same procedure of the T1-TOPSIS model in Section 2.8, each alternative is rated with respect to each criterion by each DM. An example of linguistic terms that each

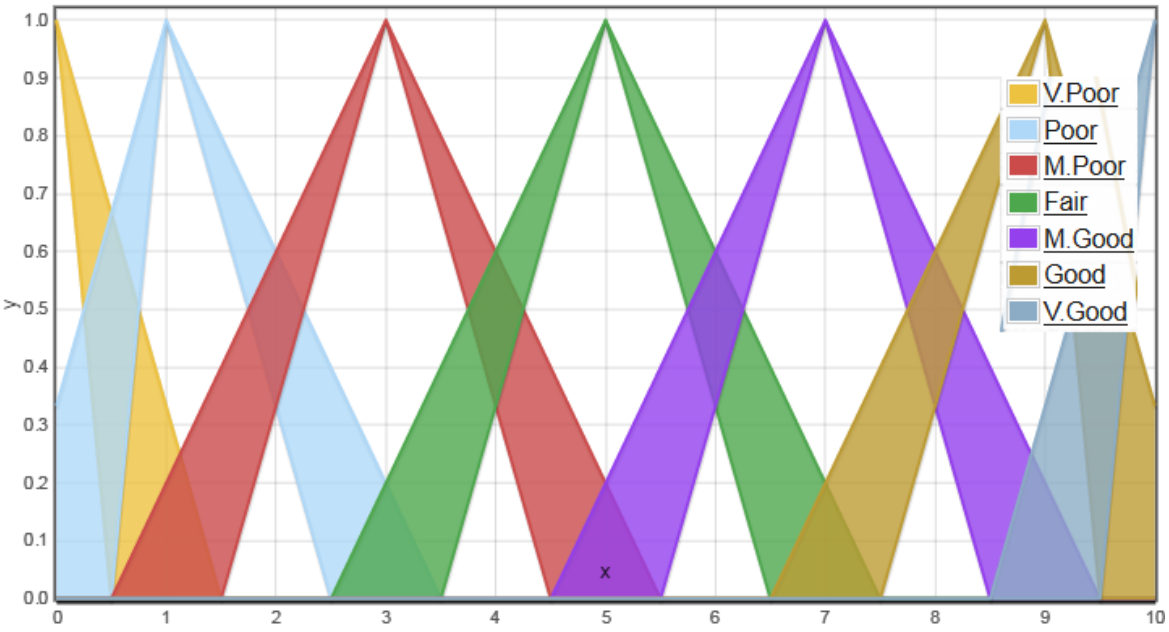


Fig. 4.1 Seven linguistic terms represented by IT2 TFNs

DM can use to provide their ratings of alternatives is shown in Table 4.1, where the parameter associated with each term is an IT2 TFN.

Let $\tilde{\mathbf{X}}_e$ denote the matrix of IT2 TFN ratings provided by a DM (e) with lower and upper membership functions (LMF, UMF) defined by the parameters, $(a_{e_{ij}^L}, b_{e_{ij}^L}, c_{e_{ij}^L})$ and $(a_{e_{ij}^U}, b_{e_{ij}^U}, c_{e_{ij}^U})$, respectively. Recall back that i is the index of alternative, $i = 1, \cdots, m$ and j is the index of criteria, $j = 1, \cdots, n$. The IT2 TFN (defined by its lower and upper MF) of one alternative (i) with respect to one criterion (j),i.e.,

$$\tilde{X}_{ij} \doteq (X_{ij^L}, X_{ij^U}) \doteq \left((a_{X_{ij}^L}, b_{X_{ij}^L}, c_{X_{ij}^L}), (a_{X_{ij}^U}, b_{X_{ij}^U}, c_{X_{ij}^U})\right)$$

among all DMs are obtained by (4.1) and (4.2), respectively. The aggregated ratings for all combinations of alternatives and criteria are jointly represented in decision matrix $\tilde{\mathbf{X}}$ shown in (4.5) with $m \times n$ elements.

$$(a_{X_{ij}^L}, b_{X_{ij}^L}, c_{X_{ij}^L}) = \left(\min_e(a_{e_{ij}^L}),\ \frac{1}{k}\sum_{e=1}^{k} b_{e_{ij}^L},\ \max_e(c_{e_{ij}^L}) \right) \tag{4.1}$$

$$(a_{X_{ij}^U}, b_{X_{ij}^U}, c_{X_{ij}^U}) = \left(\min_e(a_{ij^U}),\ \frac{1}{k}\sum_{e=1}^{k} b_{ij^U},\ \max_e(c_{ij^U}) \right) \tag{4.2}$$

The weight of criteria given by each DM, (e), with LMF and UMF are defined by the parameters $\left(a_{W_{e_j}^L}, b_{W_{e_j}^L}, c_{W_{e_j}^L}\right)$ and $\left(a_{W_{e_j}^U}, b_{W_{e_j}^U}, c_{W_{e_j}^U}\right)$, respectively. With the same procedure, the aggregated weights of criteria, $\tilde{W}_j \doteq (W_j^L, W_j^U) \doteq \left((a_{W_j^L}, b_{W_j^L}, c_{W_j^L}), (a_{W_j^U}, b_{W_j^U}, c_{W_j^U})\right)$ obtained by (4.3) and (4.4) are presented in matrix $\tilde{\mathbf{W}}$ shown as (4.6) with $j = 1, \cdots, n$ elements.

$$(a_{W_j^L}, b_{W_j^L}, c_{W_j^L}) = \left(\min_e(a_{We_j^L}),\ \frac{1}{k}\sum_{e=1}^{k} b_{We_j^L},\ \max_e(c_{We_j^L}) \right) \tag{4.3}$$

$$(a_{W_j^U}, b_{W_j^U}, c_{W_j^U}) = \left(\min_e(a_{We_j^U}),\ \frac{1}{k}\sum_{e=1}^{k} b_{We_j^U},\ \max_e(c_{We_j^U}) \right) \tag{4.4}$$

The aggregated ratings and weights are represented by the matrices ($\tilde{\mathbf{X}}$) and ($\tilde{\mathbf{W}}$), respectively, as follows:

$$\tilde{\mathbf{X}} \doteq [\tilde{X}_{ij}]_{m\times n} = \begin{bmatrix} \tilde{X}_{11} & \tilde{X}_{12} & \cdots & \tilde{X}_{1n} \\ \tilde{X}_{21} & \tilde{X}_{22} & \cdots & \tilde{X}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{X}_{m1} & \tilde{X}_{m2} & \cdots & \tilde{X}_{mn} \end{bmatrix} \tag{4.5}$$

$$\tilde{\mathbf{W}} \doteq [\tilde{W}_j]_{1\times n} = \begin{bmatrix} \tilde{W}_1 & \tilde{W}_2 & \cdots & \tilde{W}_n \end{bmatrix} \tag{4.6}$$

Step 1: Construct the normalized fuzzy decision matrix, $\tilde{\mathbf{R}} \doteq (R_{ij^L}, R_{ij^U})$. For a T1-TOPSIS model, (2.42) is used in the normalization process where (a, b, c) are the T1 TFN parameters. In IT2-TOPSIS this procedure is extended to become (4.7) and (4.8) where

$\tilde{R}_{ij} \doteq (R_{ij^L}, R_{ij^U}) \doteq \Big((a_{R_{ij}^L},\ b_{R_{ij}^L},\ c_{R_{ij}^L}), (a_{R_{ij}^U},\ a_{R_{ij}^U},\ a_{R_{ij}^U})\Big)$ are IT2 TFNs. $\tilde{\mathbf{R}}$ is given as:

$$\tilde{\mathbf{R}} = (\tilde{R}_{ij})_{m\times n} = (R_{ij^L}, R_{ij^U}) \tag{4.7}$$

where

$$\tilde{R}_{ij} = \begin{cases} \left(\left(\dfrac{a_{X_{ij}^L}}{c_{X_j^{L*}}}, \dfrac{b_{X_{ij}^L}}{c_{X_j^{L*}}}, \dfrac{c_{X_{ij}^L}}{c_{X_j^{L*}}}\right), \left(\dfrac{a_{X_{ij}^U}}{c_{X_j^{U*}}}, \dfrac{b_{X_{ij}^U}}{c_{X_j^{U*}}}, \dfrac{c_{X_{ij}^U}}{c_{X_j^{U*}}}\right)\right), \\ \text{where } j \in \mathbb{B}, \\ \left(\left(\dfrac{a_{X_j^{L-}}}{c_{X_{ij}^L}}, \dfrac{a_{X_j^{L-}}}{b_{X_{ij}^L}}, \dfrac{a_{X_j^{L-}}}{a_{X_{ij}^L}}\right), \left(\dfrac{a_{X_j^{U-}}}{c_{X_{ij}^U}}, \dfrac{a_{X_j^{U-}}}{b_{X_{ij}^U}}, \dfrac{a_{X_j^{U-}}}{a_{X_{ij}^U}}\right)\right), \\ \text{where } j \in \mathbb{C}, \end{cases} \tag{4.8}$$

and $(c_{X_j^{L*}},\ c_{X_j^{U*}})$ is defined as $\max_i\ (c_{X_{ij}^L}, c_{X_{ij}^U})$, and $(a_{X_j^{L-}}, a_{X_j^{U-}})$ is defined as $\min_i\ (a_{X_{ij}^L},\ a_{X_{ij}^U})$. $\mathbb{B}$ and $\mathbb{C}$ denoted as benefit and cost criteria, respectively.

Step 2: Construct a weighted normalized fuzzy decision matrix, $\tilde{\mathbf{V}}$ by multiplying the normalized value, $\tilde{R}_{ij}$, by the weight of criteria $\tilde{W}_j$:

$$\tilde{\mathbf{V}} \doteq (\tilde{V}_{ij})_{m\times n} \doteq (V_{ij^L}, V_{ij^U}),\ i = 1,\cdots,m,\ \ j = 1,\cdots,n \tag{4.9}$$

where $V_{ij^L} = R_{ij^L} \times W_{j^L}$ and $V_{ij^U} = R_{ij^U} \times W_{j^U}$.

Step 3: Define the fuzzy positive ideal solution (FPIS) $\tilde{V}^+$ and fuzzy negative ideal solution (FNIS) $\tilde{V}^-$ for each alternative. The $\tilde{V}^+$ and $\tilde{V}^-$ are defined as follows:

$$\tilde{V}^+ \doteq (\tilde{v}_j^+) \doteq \Big((v_{1^L}^+, v_{2^L}^+, \cdots, v_{j^L}^+), (v_{1^U}^+, v_{2^U}^+, \cdots, v_{j^U}^+)\Big) \tag{4.10}$$

$$\tilde{V}^- \doteq (\tilde{v}_j^-) \doteq \Big((v_{1^L}^-, v_{2^L}^-, \cdots, v_{j^L}^-), (v_{1^U}^-, v_{1^U}^-, \cdots, v_{j^U}^-)\Big) \tag{4.11}$$

where $v_{j^L}^+ = v_{j^U}^+ = (1,\ 1,\ 1)$ and $v_{j^L}^- = v_{j^U}^- = (0,\ 0,\ 0);\ j = 1,\cdots,n.$

Step 4: Determine the distance between each weighted normalized value (V_{ij^L}, V_{ij^U}) (from Step 2), with respect to the FPIS $(V_{j^L}^+, V_{j^U}^+)$ and FNIS $(V_{j^L}^-, V_{j^U}^-)$ (from Step 3). The proposed approach extends (3.9) and (3.10) to (4.12) and (4.13). Note that both equations now generate interval-valued, rather than numeric, outputs as the distances between the IT2 TFNs are captured by the individual distances between the upper and lower MFs as follows:

$$\bar{d}_i^+ \doteq [d_{i^L}^+, d_{i^U}^+] = \left[\sum_{j=1}^{n} d^L(V_{ij^L}, V_{j^L}^+), \sum_{j=1}^{n} d^U(V_{ij^U}, V_{j^U}^+)\right] \tag{4.12}$$

$$\bar{d}_i^- \doteq [d_{i^L}^-, d_{i^U}^-] = \left[\sum_{j=1}^{n} d^L(V_{ij^L}, V_{j^L}^-), \sum_{j=1}^{n} d^U(V_{ij^U}, V_{j^U}^-)\right] \tag{4.13}$$

To calculate the distance between two IT2 TFNs, based on vertex distance, (2.48) is extended to (4.14) and (4.15). Let

$$\tilde{A}_1 = ((a_{1^L}, b_{1^L}, c_{1^L}), (a_{1^U}, b_{1^U}, c_{1^U})),$$

$$\tilde{A}_2 = ((a_{2^L}, b_{2^L}, c_{2^L}), (a_{2^U}, b_{2^U}, c_{2^U}))$$

be two IT2 TFNs, where $a_{i^L} \leq b_{i^L} \leq c_{i^L}$, $a_{i^U} \leq b_{i^U} \leq c_{i^U}$ and $b_{i^L} = b_{i^U}$; $i = 1, 2$. The vertex based distance between $\tilde{A}_1$ and $\tilde{A}_2$ are defined as follows:

$$d^L(\tilde{A}_1, \tilde{A}_2) = \sqrt{\frac{1}{3}[(a_{1^L} - a_{2^L})^2 + (b_{1^L} - b_{2^L})^2 + (c_{1^L} - c_{2^L})^2]} \tag{4.14}$$

$$d^U(\tilde{A}_1, \tilde{A}_2) = \sqrt{\frac{1}{3}[(a_{1^U} - a_{2^U})^2 + (b_{1^U} - b_{2^U})^2 + (c_{1^U} - c_{2^U})^2]} \tag{4.15}$$

Based on (4.14) and (4.15), the distance s of $d^U(V_{ij^U}, V_{j^U}^+)$, $d^U(V_{ij^U}, V_{j^U}^-)$, $d^L(V_{ij^L}, V_{j^L}^+)$ and $d^L(V_{ij^L}, V_{j^L}^-)$ are obtained. Moving to Step 5, this study build on the interval-valued nature of the distances as follows. Note that, the $\bar{d}_i^+, \bar{d}_i^-$ values indicate the distance between the weighted normalized value (V_{ij^L}, V_{ij^U}) and FPIS $(\tilde{V}^+)$ as well as FNIS $(\tilde{V}^-)$.

Step 5: Establish the (now also) interval-valued Closeness Coefficient ($\overline{CC}_i$) for each alternative $\mathscr{A}_i$. To achieve this, (3.11) is extended to (4.19), maintaining track of the upper and lower values. This equation applies the division operator established in interval arithmetic [153] defined as follows: Given two closed intervals, $\bar{x} = [x_L, x_U]$ and $\bar{y} = [y_L, y_U]$, the following interval operations are defined:

$$\begin{aligned} \bar{x} \times \bar{y} = & [\min\{(x_L * y_L), (x_L * y_U), (x_U * y_L), (x_U) * (y_U)\}, \\ & \max\{(x_L * y_L), (x_L * y_U), (x_U * y_L), (x_U) * (y_U)\}] \end{aligned} \tag{4.16}$$

$$\frac{1}{x} = \left[\frac{1}{\bar{x}_U}, \frac{1}{\bar{x}_L}\right] \text{ if } \bar{x}_L > 0 \text{ or } \bar{x}_U < 0 \tag{4.17}$$

$$\bar{x} \div \bar{y} = \bar{x} \times \frac{1}{\bar{y}} \tag{4.18}$$

Based on the standard interval arithmetic operation, the interval-valued CC ($\overline{CC}$) is calculated by

$$\overline{CC}_i \doteq [CC_{iL}, CC_{iU}] = \frac{[d_{iL}^-, d_{iU}^-]}{[d_{iL}^-, d_{iU}^-] + [d_{iL}^+, d_{iU}^+]} = \frac{\bar{d}_i^-}{\bar{d}_i^- + \bar{d}_i^+},\ i = 1, \cdots, m. \tag{4.19}$$

$$\overline{CC}_i = \frac{\bar{d}_i^-}{\bar{d}_i^- + \bar{d}_i^+},\ i = 1, \cdots, m \tag{4.20}$$

where the inequality $\overline{CC}_{iL} \leq \overline{CC}_{iU}$ is holds. The interval-valued $\overline{CC}$ indicates the relative distance between two values which described in step 4.

Step 6: The ranking order of alternatives is determined by using interval-valued ranking techniques. The latter is itself a complex problem, especially when intervals are overlapping,

and this chapter does not focus on the details. Several examples of promising techniques are demonstrated in [154, 22, 155, 47, 102].

In general, the principle steps are the same as seen in [1] and some of IT2-TOPSIS works. The key difference between the proposed model and other IT2-TOPSIS models begins in aggregation step,that is pre-normalization step (Step 1) and in Step 4 where uncertainty in the output of step 3 is captured by maintaining the upper and lower values and then converting them to interval-valued form. As mentioned earlier, in the proposed model maintaining T2 upper and lower values through step 1 to 3 and convert all T2 information to interval-valued through step 4 to 6. Therefore, keeping the resulting T2 information and interval-values provides an alternative interpretation of the TOPSIS process.

Note, it is expected that anyone who concerned about uncertainty in the information in the input should implement this proposed model and look into potential overlapping outputs before making any decision. When using the current, existing T2-TOPSIS models, one could risk making a poor decision. Thus, the model represents a valuable addition and provides an alternative to the models established in the literature having a different interpretation of the steps of the TOPSIS model. This can be an alternative model to enhanced the decision-making process and provide more informed decisions while considering the condition of uncertain inputs involved at the beginning of the process.

The proposed model can be an alternative model to enhanced the decision-making process and provide more informed decisions while considering the condition of uncertain inputs involved at the beginning of the process. In the next section, a sensitivity analysis will be introduced showing how the outputs are affected by variations in uncertainty of the inputs, and how this relates to the behaviour of discrete, classical TOPSIS models. An experiment is developed and is explained in the next section whereby the original T1 TFNs from standard T1-TOPSIS [1] are expanded to IT2 TFNs by a systematic blurring technique. A series of values denoted δ are used to simulate the level of uncertainty where $\delta \in [0,1]$. The

details of this experiment will be discussed in the next section. In addition, to evaluate the proposed model, in the same section, a detailed demonstration of its behaviour on the basis of a sensitivity analysis — showing how the outputs are affected by variations in uncertainty of the inputs, and how this relates to the behaviour of discrete, classical TOPSIS models is given. As intervals can, for example, exhibit partial overlap, a measure is developed and explained in chapter 5 to compare the resulting interval-valued output from various cases (i.e., overlapping and non-overlapping) of the interval with considering uncertainty.

4.4 Evaluation of the proposed IT2-TOPSIS model with interval-valued Output

4.4.1 Experimental method

This section presents the experimental method developed in this study which was used to simulate the case when a series of uncertainty levels, δ are introduced to the input of IT2-TOPSIS. The systematic blurring technique is used to expand the footprint of uncertainty (FOU) from standard T1 TFNs to IT2 TFNs that represent the input data. The next section briefly describes a methodology (originally introduced in [152]) for the systematic blurring of T1 TFNs such as shown in Fig. 4.2 to generate IT2 TFNs (Fig. 4.4).

The delta values used in this experiment are:

$$\delta = (0.10, 0.13, 0.14, 0.145, 0.15, 0.20, 0.30, 0.40, 0.50, 0.60).$$

Each delta value is used to shift the MFs used in the T1-TOPSIS model (Fig. 4.2) both left and right as shown in Fig. 4.3. The CC values that are the output of the process are in interval-valued form, thus preserving additional information compared to crisp outputs. Thus, the uncertainty information encapsulated in the IT2 FNs and reflected in the CC intervals is

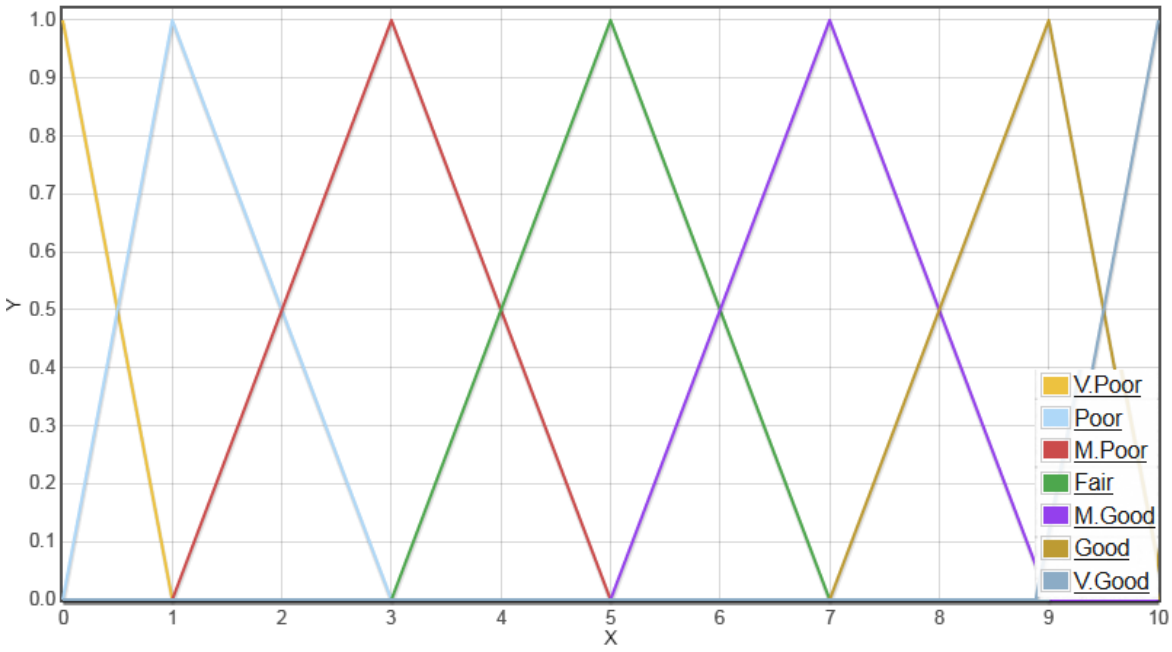


Fig. 4.2 T1 TFNs for rating scale

available to DMs to guide their decision making process. The next section provides further details on how T1 TFNs are 'expanded' to IT2 TFNs using an example.

In order to demonstrate the behaviour of the proposed IT2 extension of TOPSIS, it is helpful to be able to compare the effects on TOPSIS outputs of employing IT2 TFNs with different size FOUs, from type-1 (no FOU) to IT2 TFNs with very broad FOUs. The blurring of T1 TFNs is used to simulate different degrees of uncertainty that may be present in the real world inputs of a given TOPSIS application. The construction of IT2 TFNs to represent uncertainty is explained next.

Let A be a T1 TFN with parameters denoted (a,b,c). The value δ is introduced to the value a, denoted δ_a, and value c, denoted as δ_c. Both δ values determine the degree of blurring applied to the standard T1 TFNs used in the T1-TOPSIS model (Fig. 4.2) in both directions, as illustrated in Fig. 4.3. Now, an IT2 TFNs is established and its FOU is bounded by lower and upper values denoted $(a_{e_{ij}^L}, b_{e_{ij}^L}, c_{e_{ij}^L})$ and $(a_{e_{ij}^U}, b_{e_{ij}^U}, c_{e_{ij}^U})$, respectively. This FOU simulates the uncertainty that may exist in the input data of TOPSIS real-world applications.

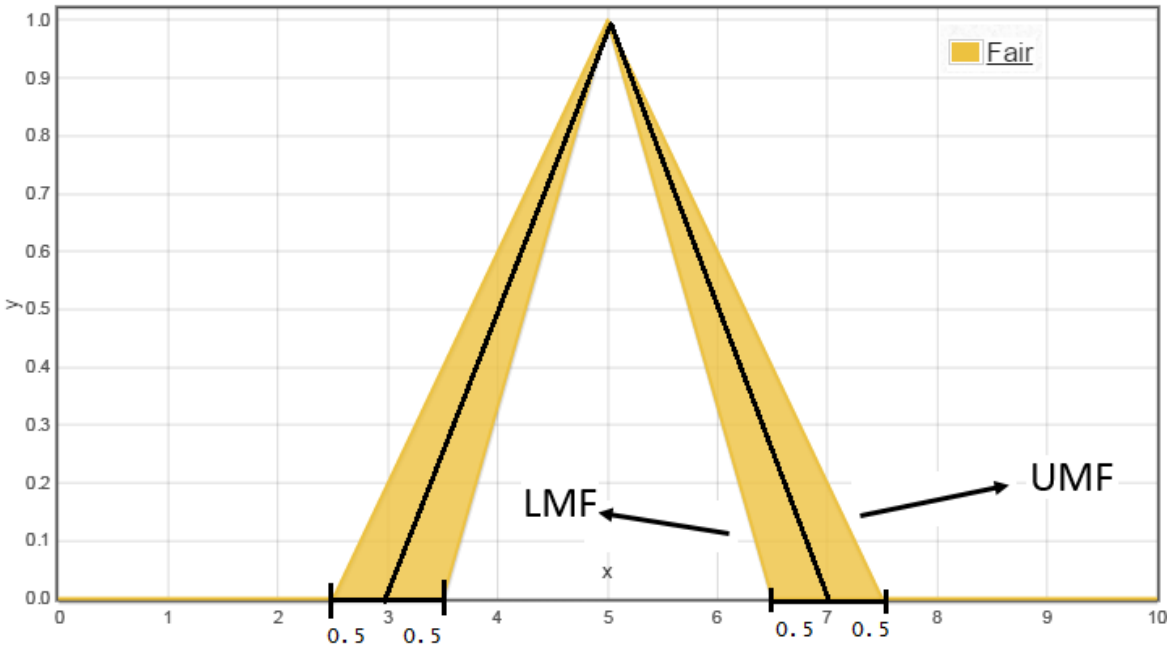


Fig. 4.3 IT2 TFN for label *Fair*, an example of blurring $\delta = 0.50$

This study defines the *uncertainty intervals* as $[a-\delta_a, a+\delta_a]$ and $[b-\delta_b, b+\delta_b]$. Let $c-a=l$. The apex of the triangle (b) is defined as $b=\frac{l}{2}$. The left vertex of the triangle on the x-axis ranges from $a-\delta_a$ to $a+\delta_a$. Formally, the region of uncertainty in the left-hand side of the triangle is defined at vertices: $(a-\delta_a,0)$,$(a+\delta_a,0)$ and $(\frac{l}{2},1)$. The region of uncertainty in the right-hand side of the triangle is defined at vertices: $(b-\delta_b,0)$,$(b+\delta_b,0)$ and $(\frac{l}{2},1)$.

This blurring procedure is adopted to capture the uncertainty of the FOU [34, 138]. The area of the FOU increases (decreases) as the uncertainty increases (decreases), and when the FOU collapses to a T1 FS, uncertainty disappears. Post-blurring, the set of pre-determined linguistic terms (e.g., 'Very Good' and 'Good'), now modelled with IT2 TFNs is used in the evaluation in the TOPSIS model, i.e. different instances of TOPSIS which are identical except for different δ can be compared.

This experiment shows the systematic mapping between the uncertainty in the input and the uncertainty in the output of the TOPSIS process. There are three types of outputs that are observed here:

i. Crisp CC output ($\dot{CC}$),

ii. Interval-valued CC output ($\overline{CC}$),

iii. Center of $\overline{CC}$ outputs.

The $\dot{CC}$ is determined by implementing a T1-TOPSIS model, and the output of the other two types are determined with the proposed IT2-TOPSIS model. The next section presents the details of these three output representations.

4.4.2 Three Forms of CC Outputs

Recall that the proposed IT2-TOPSIS model introduced in Section 4.3 provides interval CC values ($\overline{CC}$) as the final output. Additionally, the experiment is developed in this section to show the systematic mapping between uncertainty in inputs and uncertainty in the output. The effect of introducing different level of uncertainty in the input to the output is observed by conducting this experiment. There are three different forms of output based on the implementation of T1-TOPSIS and T2-TOPSIS models. These three output forms are used to observe the effect of introducing uncertainty to the input in two case studies that will be describe in Section 4.5. In the next three sections, the details for each form of output are described.

Crisp output ($\dot{CC}$) from a T1-TOPSIS model

The implementation of standard T1-TOPSIS and T2-TOPSIS models in all studies in the current literature provides a crisp CC output as the final result to determine the ranking of the alternatives. In this experiment, this study implements a standard T1-TOPSIS model based

on a model proposed by Chen [1] to get a crisp CC output. Later, the figure representing a case study result, the crisp output $\dot{CC}$ is labelled as "$\triangle$''and is used for comparison purposes. The T1-TOPSIS model used is shown in Section 2.8.

Interval-valued output ($\overline{CC}$) from proposed IT2-TOPSIS model

The second form of output in this experiment known as interval-valued CC ($\overline{CC}$) which is the result of the proposed model presented in Section 4.3. Such intervals are later illustrated by an interval line in figures representing a case study result. The $\overline{CC}$ is produced when a series of δ values ranging from $[0,1]$ are introduced to standard T1 TFNs as described in Section 4.4. For each δ value, the IT2-TOPSIS model gives different intervals which are associated with different alternatives. The detailed implementation of the proposed model is described in the Section 4.4.3.

4.4.3 Example when $\delta = 0.50$ is introduced to T1 TFNs

Figure 4.3 shows an example of the effect of the direct expanding technique on a T1 TFN, here, representing the linguistic label *Fair*. The figure shows the original T1 TFN and the resulting IT2 FN bounded by an Upper Membership Function (UMF) and a Lower Membership Function (LMF). Next, this step is repeated for the other linguistic labels in the rating scale. Each fuzzy label is defined with an associated FN value. For example, the linguistic label *Fair* in Fig. 4.3, is defined as $((2.5,5,7.5),(3.5,5,6.5))$, where the first and second elements capture upper and lower values, respectively. Note that the IT2 TFNs generated are based on the original T1 TFNs from Fig. 4.2. The remaining labels for the linguistic ratings scale now become IT2 TFNs as shown in Fig. 4.4. The new scale can be defined with IT2 TFNs as shown in Table 4.2. The DMs then use the linguistic label in that table to give ratings for each alternative and weight for each criterion.

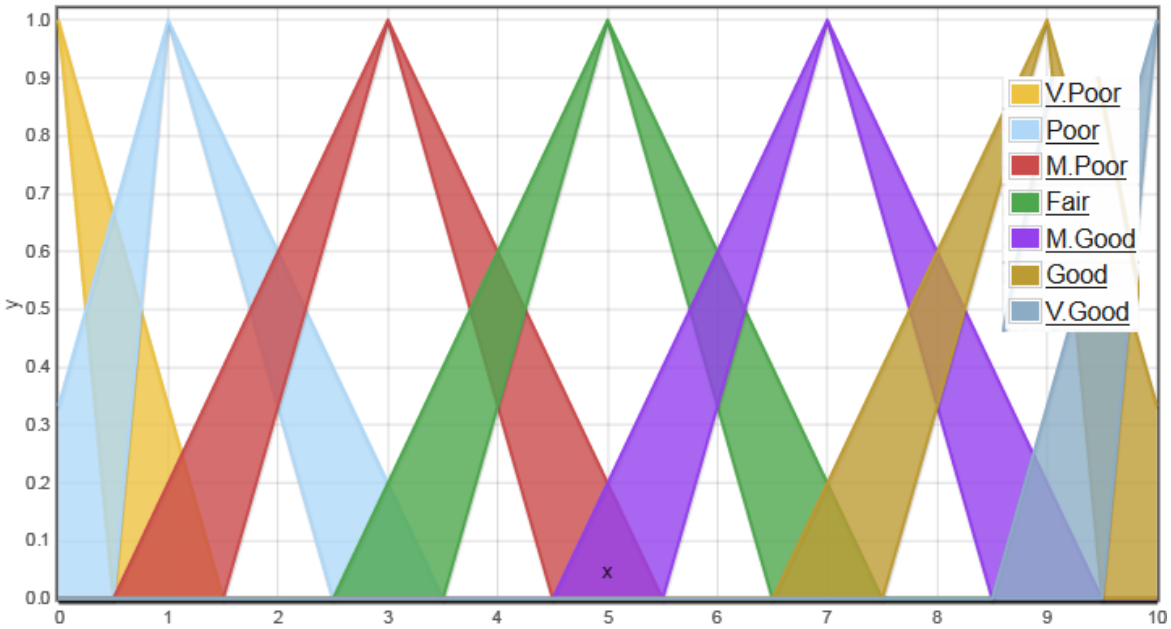


Fig. 4.4 IT2 Fuzzy Linguistic Scale For Rating of alternatives and weight of criteria when $\delta = 0.50$

Finally, the T1-TOPSIS procedure as shown in Section 2.8 is applied to get the rank of each alternative. Then, applied blurring technique according to specific δ value. To recap, the procedure starts by collecting the ratings from DMs and calculating the average for each rating value. The results are represented as IT2 TFNs that are captured in a decision matrix, $\tilde{\mathbf{X}}$, which has two elements for each rating, $\tilde{x}_{ij}$, namely for the upper and lower values. Then, those values are processed separately in each step as described in Section 2.8. Since the uncertainty is preserved in the model (by considering the footprint of uncertainty, i.e. the upper and lower values, separately) from the start to the end of procedure, the final CC values can be expressed in interval form based on the two individual outputs. In order to observe the effect of uncertainty introduced in the input of decision-making process to the output CC values, the implementation of the proposed model is made and will be demonstrate in the next section.

Table 4.2 Linguistic labels for the evaluation of alternatives and Weighting of criteria using IT2 TFNs

	IT2 TFNs	
Terms for Ratings / Weighting	Lower value	Upper value
Very Poor (VP) / Very Low (VL)	(0,0,0.5)	(0,0,1.5)
Poor (P) / Low (L)	(0.5,1,2.5)	(-0.5,1,3.5)
Medium Poor (MP) / Medium Low (ML)	(1.5,3,4.5)	(0.5, 3, 5.5)
Fair (F) / Medium (M)	(3.5,5,6.5)	(2.5,5,7.5)
Medium Good (MG) / Medium High (MH)	(5.5,7,8.5)	(4.5,7,9.5)
Good (G) / High (H)	(7.5,9,9.5)	(6.5,9,10)
Very Good (VG) / Very High (VH)	(9.5,10,9.5)	(8.5,10,10)

4.5 Implementation of the proposed model in two case studies

In this section, an analysis of two different case studies using the proposed IT2-TOPSIS model is presented and described. The first case study is one published by Chen [1], and the second is a novel case study concerned with the selection of students' food outlets during lunchtime. This section explains and presents the results of each case study in the subsections that follow. This study observes the effect of introducing uncertainty to the final output and provides a detailed interpretation of the systematic mapping between the inputs and outputs of the proposed interval IT2-TOPSIS model.

4.5.1 Case study 1: Chen's case study (dataset from Chen [1])

This section provides a detailed demonstration of the application of the IT2-TOPSIS to the dataset published by Chen [1], producing three types of output as explained in Section 4.4.2. Initially, a preliminary investigation is conducted using the proposed IT2-TOPSIS model as described in Section 4.3 and examined the three forms of CC outputs. The delta values used

in this preliminary work are: $\delta = (0.10, 0.13, 0.14, 0.145, 0.15, 0.20, 0.30, 0.40, 0.50, 0.60)$. In the previous chapter and early work in [152], it is shown that the model can produce an interval-valued output using the proposed IT2-TOPSIS model. This section demonstrates and evaluates the proposed model in two different case studies to show the effect of varying uncertainty in inputs on the output of a TOPSIS model more clearly. Firstly, to get the first form of output (a crisp CC ($\dot{CC}$)), the study applies the T1-TOPSIS model presented in Section 2.8 to Chen's dataset.

Chen's case study is summarized as follows: Suppose that a software company wants to hire a systems analysis engineer. After preliminary screening, three applicants $\mathscr{A}_1$, $\mathscr{A}_2$ and $\mathscr{A}_3$ are retained for further evaluation. A committee of three decision-makers, $\mathscr{D}_1$, $\mathscr{D}_2$ and $\mathscr{D}_3$ is formed to conduct an interview and to select the most suitable applicant. Five desirable criteria are considered; emotional steadiness ($\mathscr{C}_1$); oral communication skills ($\mathscr{C}_2$), personality ($\mathscr{C}_3$), past experience ($\mathscr{C}_4$) and self-confidence ($\mathscr{C}_5$). The DMs use the linguistic scale in Table 4.3 to give ratings and provide the weightings of criteria. Both ratings and weightings are presented in Table 4.4. The step-by-step procedure to get the crisp output for Chen's dataset is provided in the next section.

Implementation of T1-TOPSIS model to case study 1 to get crisp CC ($\dot{CC}$)

Based on the T1-TOPSIS model in Section 2.8, the linguistic evaluation shown in Table 4.4 is converted into the T1 TFN parameters shown in Table 4.3 to construct the fuzzy decision matrix, $\mathbf{X}_{e=1,2,3}$, where e is the index of the DM. Three decision matrices from three DMs are shown as follows:

$$\mathbf{X}_1 = \begin{bmatrix} MG & G & F & VG & F \\ G & G & G & VG & VG \\ MG & MG & G & G & G \end{bmatrix} \tag{4.21}$$

Table 4.3 Linguistic variables for the evaluation of alternatives and weighting of criteria using T1 TFNs [1]

(a) Rating scale	
Very Poor (VP)	(0, 0, 1)
Poor (P)	(0, 1, 3)
Medium Poor (MP)	(1, 3, 5)
Fair (F)	(3, 5, 7)
Medium Good (MG)	(5, 7, 9)
Good (G)	(7, 9, 10)
Very Good (VG)	(9, 10, 10)
(b) Weighting scale	
Very Low (VL)	(0, 0, 0.1)
Low (L)	(0, 0.1, 0.3)
Medium Low (ML)	(0.1, 0.3, 0.5)
Medium (M)	(0.3, 0.5, 0.7)
Medium High (MH)	(0.5, 0.7, 0.9)
High (H)	(0.7, 0.9, 1)
Very High (VH)	(0.9, 1, 1)

Table 4.4 Ratings of all alternatives and weightings of criteria given by decision makers

		DM's rating					DM's rating		
Criteria	Alternatives	$\mathscr{D}_1$	$\mathscr{D}_2$	$\mathscr{D}_3$	Criteria	Alternatives	$\mathscr{D}_1$	$\mathscr{D}_2$	$\mathscr{D}_3$
$\mathscr{C}_1$	$\mathscr{A}_1$	MG	G	MG	$\mathscr{C}_4$	$\mathscr{A}_2$	VG	G	VG
	$\mathscr{A}_2$	G	G	MG		$\mathscr{A}_2$	VG	VG	VG
	$\mathscr{A}_3$	MG	MG	F		$\mathscr{A}_3$	G	VG	MG
Weight $\mathscr{C}_1$		H	VH	MH	Weight $\mathscr{C}_4$		VH	VH	VH
$\mathscr{C}_2$	$\mathscr{A}_1$	G	MG	F	$\mathscr{C}_5$	$\mathscr{A}_1$	F	F	F
	$\mathscr{A}_2$	G	G	G		$\mathscr{A}_2$	VG	MG	G
	$\mathscr{A}_3$	MG	G	G		$\mathscr{A}_3$	G	G	MG
Weight $\mathscr{C}_2$		VH	VH	VH	Weight $\mathscr{C}_5$		M	MH	MH
$\mathscr{C}_3$	$\mathscr{A}_1$	F	MG	MG					
	$\mathscr{A}_2$	G	G	MG					
	$\mathscr{A}_3$	G	MG	MG					
Weight $\mathscr{C}_3$		VH	H	H					

$$\mathbf{X}_2 = \begin{bmatrix} G & MG & MG & G & F \\ G & G & G & VG & MG \\ MG & G & MG & VG & G \end{bmatrix} \tag{4.22}$$

$$\mathbf{X}_3 = \begin{bmatrix} MG & F & MG & VG & F \\ MG & G & MG & VG & G \\ F & G & MG & MG & MG \end{bmatrix} \tag{4.23}$$

The weight of each criterion given by each DM is represented in the matrix $\mathbf{W}_{e=1,2,3}$ as follow:

$$\mathbf{W}_1 = [H \;\; VH \;\; VH \;\; VH \;\; M] \tag{4.24}$$

$$\mathbf{W}_2 = [VH \;\; VH \;\; H \;\; VH \;\; MH] \tag{4.25}$$

$$\mathbf{W}_3 = [MH \;\; VH \;\; H \;\; VH \;\; MH] \tag{4.26}$$

All the linguistic terms in matrices $\mathbf{X}_{e=1,2,3}$ and $\mathbf{W}_{e=1,2,3}$ are converted into numerical values using parameters of T1 TFNs presented in Table 4.3.

Then, an average rating matrix $\mathbf{X}$, can be formed by calculating each element as the average of each T1 TFNs, calculated using (3.2). Table 4.5 presents the average rating among all DMs denoted as matrix $\mathbf{X} \doteq [X_{ij}]$.

Table 4.5 T1-TOPSIS decision matrix for case study 1

	$\mathscr{C}_1$	$\mathscr{C}_2$	$\mathscr{C}_3$	$\mathscr{C}_4$	$\mathscr{C}_5$
$\mathscr{A}_1$	(5.7,7.7,9.3)	(5,7,9)	(5.7,7.7,9)	(8.33,9.67,10)	(3,5,7)
$\mathscr{A}_2$	(6.3,8.3,9.7)	(9,10,10)	(8.3,9.7,10)	(9,10,10)	(7,9,10)
$\mathscr{A}_3$	(6.3,8,9)	(7,9,10)	(7,9,10)	(7,9,10)	(6.3,8.3,9.7)
W_j	(0.7,0.9,1)	(0.9,1,1)	(0.77,0.93,1)	(0.9,1,1)	(0.43,0.63,0.83)

For example, for element X_{11} in the Table 4.5, the value $(5.7, 7.7, 9.3)$ is calculated using (3.2) as follows: From Table 4.4, the ratings given for $\mathscr{A}_1$ by considering $\mathscr{C}_1$ from three DMs are:

$\mathscr{D}_1 = MG\ (5\ \ 7\ \ 9)$, $\mathscr{D}_2 = G\ (7\ \ 9\ \ 10)$ and $\mathscr{D}_3 = MG\ (5\ \ 7\ \ 9)$

By using (3.2), an average value is calculated by: $(5+7+5)/3 = 5.7$, $(7+9+7)/3 = 7.7$ and $(9+10+9)/3 = 9.3$. Hence, element $X_{11} = (5.7, 7.7, 9.3)$ is a T1 TFN. The other elements are calculated with the same procedure and represented in Table 4.5.

For the weights of criteria, the average can be calculated the same way as X_{ij}, where W_j denotes the weight of the jth criterion. The average weight of criteria is shown in Table 4.5. For example, the weight of $\mathscr{C}_5$ in the last column given by three DMs is:

$\mathscr{D}_1 = M(0.3,0.5,0.7)$, $\mathscr{D}_2 = MH(0.5,0.7,0.9)$ and $\mathscr{D}_3 = MH(0.5,0.7,0.9)$.

The average of all weights for $\mathscr{C}_5$, W_5 across DMs is:

$W_5 = ((0.3+0.5+0.5)/3, 0.5+0.7+0.7)/3, 0.7+0.9+0.9)/3) = (0.43,0.63,0.83)$

Once the elements of matrices **X** and **W** are established, the T1-TOPSIS can be performed as described in Steps 1–6 next.

Step 1: Construct the fuzzy normalized decision matrix, $\mathbf{R} \doteq [R_{ij}]_{m\times n}$, where $i = 3$ and $j = 3$. For this case study, all of the criteria are categorized as benefit criteria (ie., $\mathbb{B}$). The fuzzy normalized decision matrix is presented in Table 4.6.

Table 4.6 Fuzzy normalized decision matrix: Case study 1

	$\mathscr{C}_1$	$\mathscr{C}_2$	$\mathscr{C}_3$	$\mathscr{C}_4$	$\mathscr{C}_5$
$\mathscr{A}_1$	(0.59,0.79,0.96)	(0.5,0.7,0.9)	(0.57,0.77,0.9)	(0.83,0.97,1)	(0.3,0.5,0.7)
$\mathscr{A}_2$	(0.65,0.86,1)	(0.9,1,1)	(0.83,0.97,1)	(0.9,1,1)	(0.7,0.9,1)
$\mathscr{A}_3$	(0.65,0.82,0.93)	(0.7,0.9,1)	(0.7,0.9,1)	(0.7,0.9,1)	(0.63,0.83,0.97)

Step 2: Construct a weighted normalized fuzzy decision matrix, $\mathbf{V} \doteq [V_{ij}]_{m\times n}$, using (2.43); the values are presented in Table 4.7.

For example, to get the value of $V_{11} = (0.41,0.71,0.96)$, multiply R_{11} from Table 4.6 with $W_1 = (0.7,0.9,1)$ from Table 4.5.

Step 3: Determine the Fuzzy Positive Ideal Solution (FPIS, V^+) and Fuzzy Negative Ideal Solution (FNIS, V^-) by:

$$V^+ = [v_1^+, \cdots, v_n^+] \tag{4.27}$$

Table 4.7 Fuzzy weighted normalized decision matrix: Case study 1

	$\mathscr{C}_1$	$\mathscr{C}_2$	$\mathscr{C}_3$	$\mathscr{C}_4$	$\mathscr{C}_5$
$\mathscr{A}_1$	(0.41,0.71,0.96)	(0.45,0.7,0.9)	(0.44,0.72,0.9)	(0.75,0.97,1)	(0.13,0.32,0.58)
$\mathscr{A}_2$	(0.46,0.77,1)	(0.81,1,1)	(0.64,0.9,1)	(0.81,1,1)	(0.3,0.57,0.83)
$\mathscr{A}_3$	(0.46,0.74,0.93)	(0.63,0.9,1)	(0.54,0.84,1)	(0.63,0.9,1)	(0.27,0.52,0.81)

$$V^- = [v_1^-, \cdots, v_n^+] \tag{4.28}$$

Step 4: Calculate the distance (d_i^+) of V_{ij} from V^+ and the distance (d_i^-) of V_{ij} from V^- using (3.8) and (3.9), respectively. The result is presented in Table 4.8.

Table 4.8 The distance measurement – Case study 1

	d_i^+	d_i^-
$\mathscr{A}_1$	1.9434	3.44964
$\mathscr{A}_2$	1.2664	4.11843
$\mathscr{A}_3$	1.6034	3.80864

An example calculation for the value of $d_1^+ = 1.9434$ for $\mathscr{A}_1$ across all criteria is as follows:

d_1^+ for $\mathscr{C}_1$ vs $\mathscr{A}_1$: $\left(\frac{1}{3}((0.41-1)^2+(0.71-1)^2+(0.96-1)^2)\right)^{\frac{1}{2}} = 0.3784$

d_1^+ for $\mathscr{C}_2$ vs $\mathscr{A}_1$: $\left(\frac{1}{3}((0.45-1)^2+(0.7-1)^2+(0.9-1)^2)\right)^{\frac{1}{2}} = 0.3663$

d_1^+ for $\mathscr{C}_3$ vs $\mathscr{A}_1$: $\left(\frac{1}{3}((0.44-1)^2+(0.72-1)^2+(0.9-1)^2)\right)^{\frac{1}{2}} = 0.3676$

d_1^+ for $\mathscr{C}_4$ vs $\mathscr{A}_1$: $\left(\frac{1}{3}((0.44-1)^2+(0.72-1)^2+(0.9-1)^2)\right)^{\frac{1}{2}} = 0.1471$

d_1^+ for $\mathscr{C}_5$ vs $\mathscr{A}_1$: $\left(\frac{1}{3}((0.13-1)^2+(0.32-1)^2+(0.58-1)^2)\right)^{\frac{1}{2}} = 0.6840$

Hence, the value in d_1^+ in Table 4.8 is the total value for $\mathscr{A}_1$ to each criterion. The total distance is calculated as follows:

$d_1^+ = 0.3784 + 0.3663 + 0.3676 + 0.1471 + 0.6840 = 1.9434$

Step 5: Calculate the Closeness Coefficient, $\dot{CC}$, of each alternative using (3.11). The CC value for each $\mathscr{A}_i$ is calculated as follows:

$$\dot{CC}_1 = \frac{3.44964}{(3.44964 + 1.9434)} = 0.6396$$

$$\dot{CC}_2 = \frac{4.1184}{(4.1184 + 1.2664)} = 0.7648$$

$$\dot{CC}_3 = \frac{3.8086}{(3.8086 + 1.6034)} = 0.7037$$

Step 6: Determine the rank order of the alternatives, and select the best one. An alternative $\mathscr{A}_i$ is closer to the FPIS (V^+) and farther from FNIS (V^-) as CC_i approaches 1. Thus, a higher CC is considered a higher rank. The ranking order for this case study is presented in Table 4.9.

Table 4.9 The $\dot{CC}$ values and the rank of alternatives for T1 TOPSIS model - Case study 1

	CC_i	Rank
$\mathscr{A}_1$	0.6396	3
$\mathscr{A}_2$	0.7648	1
$\mathscr{A}_3$	0.7037	2

Note, this section did not show the example implementation proposed for an IT2-TOPSIS model to Chen's dataset. The implementation of the dataset to the proposed IT2-TOPSIS model will be presented in the next section using case study 2. The result of the implementation the proposed IT2-TOPSIS model using Chen's dataset will be presented in Section 4.5.3.

4.5.2 Case study 2: Selection of Food Outlets During Lunchtime – Real-world case study, novel dataset

In the decision-making paradigm, an important part is how we model various types of data in an accurate way. One of the well-known techniques to collect subjective data is by conducting a survey with a group of people. Thus, eliciting information and data from surveyed groups of people is a key challenge in most decision-making techniques. The collection of opinions or perceptions given by a group of people in a survey can be used to model and analyze the

dynamics of the decision-making process, as well as identify any relationship between these data with the perception model.

In order to evaluate the proposed IT2-TOPSIS model, this study surveyed a group of students in the University of Nottingham, UK (see Appendix A for survey questions). The survey focuses on how university students make a decision based on given alternatives (e.g., different food outlets) with respect to multiple criteria (e.g., food price, service speed). The purpose of this study is to examine the decision making process involved in this task and to understand the different ways people make a decision. Furthermore, this study observes the effect of uncertainty in the input to the output in this particular decision-making problem. In the survey, students were asked to provide their opinion on a set of criteria.

The survey consists of four parts; Part I - basic information gathering; Part II - opinions on word labels; Part III - performance ratings for each food outlet using linguistic labels; and Part IV - performance ratings of each food outlet with respect to each criteria using an ellipse. In Jubilee Campus, University of Nottingham, UK, there are three main food outlets (two main cafeterias and one shop) which sell food from 8 am to 5 pm, Monday to Friday. These food outlets are used as a set of the alternatives. The food outlets are; Cafe A ($\mathscr{A}_1$), Cafe B ($\mathscr{A}_2$) and Shop C ($\mathscr{A}_3$). In this case study, three criteria are used; average price of food ($\mathscr{C}_1$), speed of service ($\mathscr{C}_2$) and the taste of the food ($\mathscr{C}_3$). There were 25 respondents acting as DMs in this survey, all of whom are University of Nottingham students. Further details on the survey are given in Table 4.10.

Table 4.10 Details on survey: Selection of food outlets during lunchtime

Descriptions	Details of description
Number of participants:	25
Location of survey:	Jubilee Campus, University of Nottingham, UK
Date of survey:	March 2017
Age range of participants:	18 - 45

Based on Table 4.10, 25 students from University of Nottingham, UK were chosen randomly and classified as decision makers (DMs). The group of DMs consists of local and foreign students. The IT2-TOPSIS model will be used to determine which food outlet (i.e., alternatives) is most preferred by considering multiple and conflicting criteria. Obviously, since it is a multi-dimensional MADM problem, the DMs commonly decide which alternative they should choose by considering each criterion. Based on the DMs' preferences and assessment, the model is used to analyse and generate an interval-valued output. The next two sections will provide the details of the survey and its analysis.

Details of survey

Human beings always have to make a decision, be it simple or complex. The capability of making a good decision is characterized by how much uncertainty is involved in the process. One simple decision that a person often makes is choosing a food outlet to buy lunch from. As a student, some criteria can be considered very important and given a higher weight than others. For example, some students may think about choosing the outlet that offers the lowest price as a main consideration. Thus, the process of combining all the criteria to make a decision can be complex and crucial.

The proposed IT2-TOPSIS model provides a decision output, namely interval-valued Closeness Coefficient (denoted as $\overline{CC}$), for the three alternatives as follows:

- Cafe A ($\mathscr{A}_1$),
- Cafe B ($\mathscr{A}_2$),
- Shop C ($\mathscr{A}_3$)

The DMs then rated each alternative with respect to each criterion. Three criteria were used in this study. They are:

- Average price of food (the average cost per meal during lunchtime for each $\mathscr{A}$),

- Speed of service (the service provided by staff at each $\mathscr{A}$),
- Taste of food (average quality of taste of meals during lunchtime for each $\mathscr{A}$)

Each criterion was assigned a set of linguistic term as summarized in Table 4.11.

Table 4.11 Criteria represented by linguistic variables and associated linguistic terms

Criteria	Terms
Average price of food ($\mathscr{C}_1$)	Very Low (VL)
	Low (L)
	Medium Low (ML)
	Fair (F)
	Medium High (MH)
	High (H)
	Very High (VH)
Speed of service ($\mathscr{C}_2$)	Very Slow (VS)
	Slow (S)
	Medium Slow (MS)
	Fair (F)
	Medium Fast (MF)
	Fast (FS)
	Very Fast (VF)
Taste of food ($\mathscr{C}_3$)	Very Bad (VB)
	Bad (B)
	Medium Bad (MB)
	Fair (F)
	Medium Good (MG)
	Good (G)
	Very Good (VG)

In this case study, an example DMs' evaluation for alternative $\mathscr{A}_1$ with respect to criterion $\mathscr{C}_1$ is shown in Table 4.12. The other evaluations (e.g., $\mathscr{A}_2$ versus $\mathscr{C}_1$, etc.) are shown in

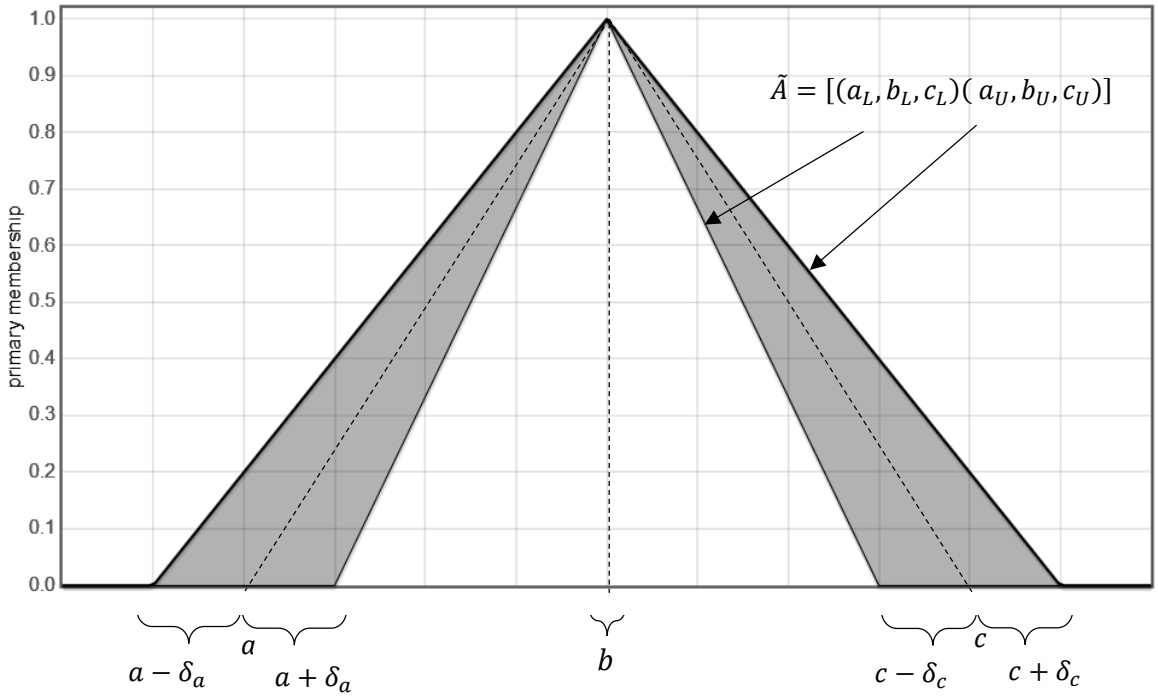


Fig. 4.5 Systematic blurring of $\tilde{A}$ to construct IT2 TFNs

Appendix B. In the following section, this study follows the steps of the proposed IT2-TOPSIS explained in Section 4.3 to aid in the selection of food outlets during lunchtime.

Analysis according to the proposed IT2-TOPSIS model

This section presents the steps involved in the proposed IT2-TOPSIS model to determine the most preferred alternative (i.e., optimal choice), with uncertainty level $\delta = 0.1$ given to the input. The average rating for each alternative with respect to each criterion is summarized in matrix $\tilde{\mathbf{X}}$. Note, the proposed model is built on the basis of T1-TOPSIS. Recall that in order to construct interval type-2 triangular fuzzy numbers (IT2 FNs), a delta (δ) value is used to expand the T1-TFNs in both directions, as illustrated in Figure 4.5 and explained in Section 4.4. This technique creates an FOU which then simulates the uncertainty that may exist in input data in real-world MADM problems.

Based on Figure 4.5, the uncertainty intervals are defined as follows:

Table 4.12 A collective evaluation of $\mathscr{A}_1$ with respect to $\mathscr{C}_1$ given by $\mathscr{D}_{e=1,\cdots,25}$

	Criterion 1 ($\mathscr{C}_1$): Average price of food		
	How do you rate the price of the food in terms of good value of money?	How do you rate the price of the food in terms of competition?	How do you rate the price of the food shown in the menu?
$\mathscr{D}_1$	H	MH	H
$\mathscr{D}_2$	ML	F	H
$\mathscr{D}_3$	ML	F	H
$\mathscr{D}_4$	MH	H	MH
$\mathscr{D}_5$	MH	H	H
$\mathscr{D}_6$	MH	H	H
$\mathscr{D}_7$	FAIR	MH	H
$\mathscr{D}_8$	MH	H	VH
$\mathscr{D}_9$	MH	FAIR	H
$\mathscr{D}_{10}$	ML	H	VH
$\mathscr{D}_{11}$	H	MH	H
$\mathscr{D}_{12}$	MH	H	H
$\mathscr{D}_{13}$	MH	H	H
$\mathscr{D}_{14}$	MH	H	H
$\mathscr{D}_{15}$	MH	H	H
$\mathscr{D}_{16}$	H	MH	H
$\mathscr{D}_{17}$	MH	FAIR	H
$\mathscr{D}_{18}$	H	MH	L
$\mathscr{D}_{19}$	MH	MH	VH
$\mathscr{D}_{20}$	MH	H	H
$\mathscr{D}_{21}$	FAIR	FAIR	H
$\mathscr{D}_{22}$	MH	MH	H
$\mathscr{D}_{23}$	MH	FAIR	FAIR
$\mathscr{D}_{24}$	MH	FAIR	VH
$\mathscr{D}_{25}$	MH	MH	H

- $[a-\delta_a,\ a+\delta_a]$,
- $[c-\delta_c,\ c+\delta_c]$

where the region of uncertainty on the left hand side of the triangle can be defined at vertices: $(a-\delta_a,0)$, $(a+\delta_a,0)$ and $\left(\frac{l}{2},1\right)$ while the region of uncertainty in the right-hand side of the triangle is defined at vertices: $(c-\delta_c,0)$, $(c+\delta_c,0)$ and $(\frac{l}{2},1)$. If $c-a=l$, then $b=\frac{l}{2}$ is the apex of the triangle.

The average ratings across all DMs are converted to IT2 TFNs and presented in the decision matrix $\tilde{\mathbf{X}}$, where $\tilde{X}_{ij}$ is the element of the matrix that indicates the average rating of alternative i with respect to criterion j. The decision matrix $\tilde{\mathbf{X}}$ is represented in Table 4.13.

Table 4.13 Decision matrix $\tilde{\mathbf{X}}$ for case study 2 when $\delta=0.1$

	$\mathscr{C}_1=\tilde{X}_{i1}$		$\mathscr{C}_1=\tilde{X}_{i2}$	
$\mathscr{A}_1$	((2.30, 5.36, 8.30)	(2.10, 5.36, 8.50))	((2.30, 5.66, 8.50)	(2.10, 5.66, 8.70))
$\mathscr{A}_2$	((3.10, 7.70, 9.70)	(2.90, 7.70, 9.90))	((2.30, 6.11, 8.90)	(2.10, 6.11, 9.10))
$\mathscr{A}_3$	((2.30, 6.38, 9.70))	(2.10, 6.38, 9.90))	((3.50, 7.50, 9.90)	(3.30 7.50 10.10))
	$\mathscr{C}_1=\tilde{X}_{i3}$			
$\mathscr{A}_1$	((2.30, 5.36, 8.30)	(2.10, 5.36, 8.50))		
$\mathscr{A}_2$	((5.90, 9.18, 9.90)	(5.70 9.18 10.10))		
$\mathscr{A}_3$	((2.30, 6.38, 9.70)	(2.10, 6.38, 9.90))		

For simplicity, the average weight of criteria given by DMs are shown in (4.29) as matrix $\tilde{\mathbf{W}}$ where the column indicates criterion j.

$$\tilde{\mathbf{W}}=[\tilde{W}_3]=[0.290,\ \ 0.343,\ \ 0.367] \tag{4.29}$$

Once the matrices $\tilde{\mathbf{X}}$ and $\tilde{\mathbf{W}}$ are formed, the proposed IT2-TOPSIS model can be performed as summarized in Figure 4.6.

The step-by-step procedure is shown as follow:

Step 1: Determine the normalized fuzzy decision matrix $\tilde{\mathbf{R}}$ using (4.7) and (4.8) as:

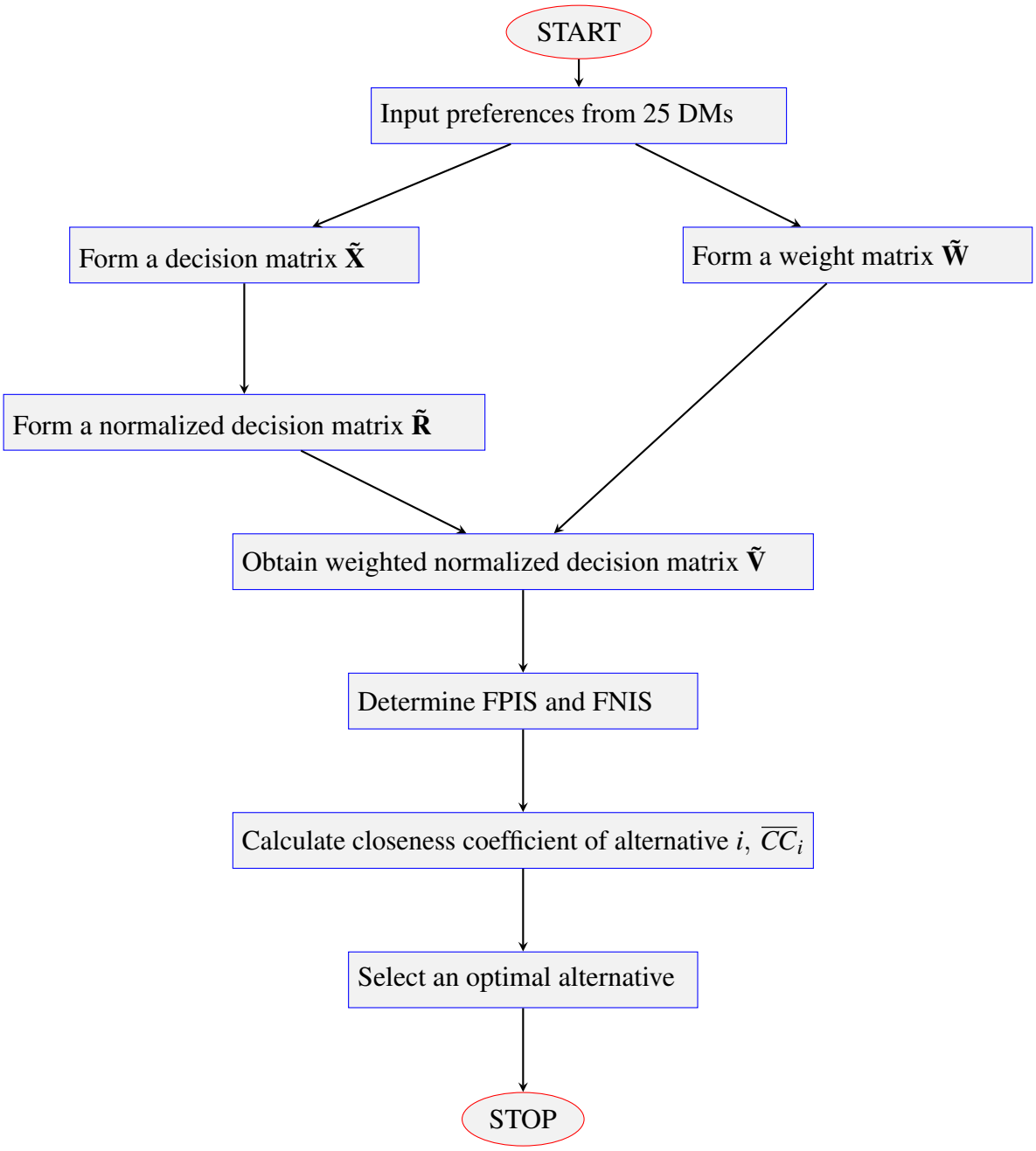


Fig. 4.6 Implementation of the proposed IT2-TOPSIS model for case study 2

$$\tilde{\mathbf{R}} = [\tilde{R}_{33}] = \begin{bmatrix} \tilde{R}_{11} & \tilde{R}_{12} & \tilde{R}_{13} \\ \tilde{R}_{21} & \tilde{R}_{22} & \tilde{R}_{23} \\ \tilde{R}_{31} & \tilde{R}_{32} & \tilde{R}_{33} \end{bmatrix} \tag{4.30}$$

The matrix elements $\tilde{R}_{ij}$ are IT2 TFNs and are represented in Table 4.14.

Table 4.14 Fuzzy normalized decision matrix: Case study 2

	$\mathscr{C}_1 = \tilde{R}_{i1}$		$\mathscr{C}_2 = \tilde{R}_{i2}$	
$\mathscr{A}_1$	((0.2371, 0.5526, 0.8557)	(0.2121, 0.5414, 0.8586))	((0.2323, 0.5713, 0.8586)	(0.2079 0.5600 0.8614))
$\mathscr{A}_2$	((0.3196,0.7934,1.0000)	(0.2929, 0.7774, 1.0000))	((0.2323, 0.6168, 0.8990)	(0.2079, 0.6046, 0.9010))
$\mathscr{A}_3$	((0.2371, 0.6573, 1.0000)	(0.2121, 0.6440, 1.0000))	((0.3535, 0.7572, 1.0000)	(0.3267, 0.7422, 1.0000))
	$\mathscr{C}_3 = \tilde{R}_{i3}$			
$\mathscr{A}_1$	((0.2323, 0.5414, 0.8384)	(0.2079, 0.5307, 0.8416))		
$\mathscr{A}_2$	((0.5960, 0.9277, 1.0000)	(0.5644, 0.9093, 1.0000))		
$\mathscr{A}_3$	((0.2323, 0.6440, 0.9798)	(0.2079, 0.6313, 0.9802))		

Step 2: Construct the weighted normalized fuzzy decision matrix, $\tilde{\mathbf{V}}$, using (4.9). In this case study, the average weight of criteria given by all DMs are shown as (4.29). Then, to get the element of matrix $\tilde{\mathbf{V}}$, multiply each criterion weight (i.e. $\tilde{W}_j$) with the normalized rating value $\tilde{R}_{ij}$ and construct a weighted normalized fuzzy decision matrix $\tilde{\mathbf{V}}$, shown as (4.31) and Table 4.15.

$$\tilde{\mathbf{V}} = [\tilde{V}_{33}] = \begin{bmatrix} \tilde{V}_{11} & \tilde{V}_{12} & \tilde{V}_{13} \\ \tilde{V}_{21} & \tilde{V}_{22} & \tilde{V}_{23} \\ \tilde{V}_{31} & \tilde{V}_{32} & \tilde{V}_{33} \end{bmatrix} \tag{4.31}$$

Step 3: Determine the Fuzzy Positive Ideal Solution (FPIS) $\tilde{V}_j^+$ and Fuzzy Negative Ideal Solution (FNIS) $\tilde{V}_j^-$ as $\tilde{V}_j^+ = ((1\ 1\ 1)(1\ 1\ 1))$ and $\tilde{V}_j^- = ((0\ 0\ 0)(0\ 0\ 0))$.

Step 4: Based on (4.12) and (4.13), calculate the distance between each weighted normalized value (from Step 2), (V_{ij^L}, V_{ij^U}) with respect to the FPIS $(V_{j^L}^+, V_{j^U}^+)$ and FNIS

Table 4.15 Fuzzy weighted normalized decision matrix: Case study 2

	$\mathscr{C}_1 = \tilde{V}_{i1}$		$\mathscr{C}_2 = \tilde{V}_{i2}$	
$\mathscr{A}_1$	((0.0687, 0.1601, 0.2480)	(0.0615, 0.1569, 0.2488))	((0.0798, 0.1962, 0.2948)	(0.0714, 0.1923, 0.2957))
$\mathscr{A}_2$	((0.0926,0.2299,0.2898)	(0.0849, 0.2253, 0.2898))	((0.0798, 0.2118, 0.3087)	(0.0714, 0.2076, 0.3093))
$\mathscr{A}_3$	((0.0687, 0.1905, 0.2898)	(0.0615, 0.1866, 0.2898))	((0.1214, 0.2600, 0.3433)	(0.1122, 0.2548, 0.3433))
	$C_3 = \tilde{V}_{i3}$			
$\mathscr{A}_1$	((0.0852, 0.1986, 0.3076)	(0.0763, 0.1947, 0.3088))		
$\mathscr{A}_2$	((0.2186, 0.3403, 0.3669)	(0.2071, 0.3336, 0.3669))		
$\mathscr{A}_3$	((0.0852, 0.2363, 0.3595)	(0.0763, 0.2316, 0.3596))		

$(V_{j^L}^-, V_{j^U}^-)$, established in Step 3. For example, the distance $\overline{d}_{1^L}^+$ between $\tilde{V}_{11^L}$ and $V_{j^L}^+$ is calculated as follows:

$$\overline{d}_{1^U}^+ = \left(\frac{1}{3}((0.0687-1)^2 + (0.1601-1)^2 + (2480-1)^2)\right)^{\frac{1}{2}} = 0.8443$$

Next, using (4.13), the distance between $\tilde{V}_{11^U}$ and FNIS $(V_{j^U}^+)$ is calculated as follows:

$$\overline{d}_{1^U}^+ = \left(\frac{1}{3}((0.0615-1)^2 + (0.1569-1)^2 + (0.2488-1)^2)\right)^{\frac{1}{2}} = 0.8477$$

Then, with the same procedure, the total distance for all criteria is calculated.

Step 5: Calculate the relative closeness coefficient (CC) of each alternative using (4.19). For example, the CC value for Alternative 1 ($\mathscr{A}_1$) is calculated as follows:

$$\overline{CC}_{1^U} = \frac{0.6016}{(0.6016+2.467)}$$
$$= 0.1961$$

$$\overline{CC}_{1^L} = \frac{0.5966}{(0.5966+2.479)}$$
$$= 0.1940$$

Thus, for $\mathscr{A}_1$, the interval-valued CC is $[0.54489, 0.5450]$, where the property of $\overline{CC}_{1^L} \leq \overline{CC}_{1^U}$ is is not necessary to hold as the CC_i indicated the distance between two values which described in step 4, Section 4.3. Table 4.16 shows the result for case study 2 as well as its ranking according to average value between lower and upper bound values.

Table 4.16 Closeness Coefficient ($\overline{CC}$) values for all alternatives in Case Study 2 for blurring $\delta = 0.10$

Alternative i	CC_i	Ranking based on centre of interval
$\mathscr{A}_1$	$[0.1940, 0.1961]$	3
$\mathscr{A}_2$	$[0.2439, 0.2475]$	1
$\mathscr{A}_3$	$[0.2292, 0.2321]$	2

4.5.3 Results

Previously, two case studies were described in detail. Then, the previous two sections showed the implementation of T1-TOPSIS model to case study 1 and the implementation of the proposed IT2-TOPSIS to a novel dataset, case study 2, when $\delta = 0.1$ is introduced. Note that both models, T1 and IT2-TOPSIS, are used to evaluate the problems presented by two case studies. The T1-TOPSIS result is used in this study for comparison purposes. This section further presents the results for both implementations of TOPSIS models to both case studies.

For the implementation of the T1-TOPSIS model to case study 1 (Chen's dataset) the results are obtained and shown in Table 4.9 (i.e., $\delta = 0$) and Figure 4.7a, denoted with symbol '$\triangle$' in the figure. With a similar procedure, the implementation of the T1-TOPSIS model to Case Study 2 (novel dataset) is made and the result is shown in Table 4.18 and Figure 4.7b,

denoted with '△' symbol. Next, for the implementation of the proposed IT2-TOPSIS model to Case Study 1, the result is shown in Table 4.9 (i.e., $\delta = 0$) and for case study 2, in Table 4.18. Note that the different symbols are used in Figure 4.7 to represent the three types of CC outputs. The three different outputs are described in detail in Section 4.4.2.

Table 4.17 Result of blurring various δ values for Case Study 1

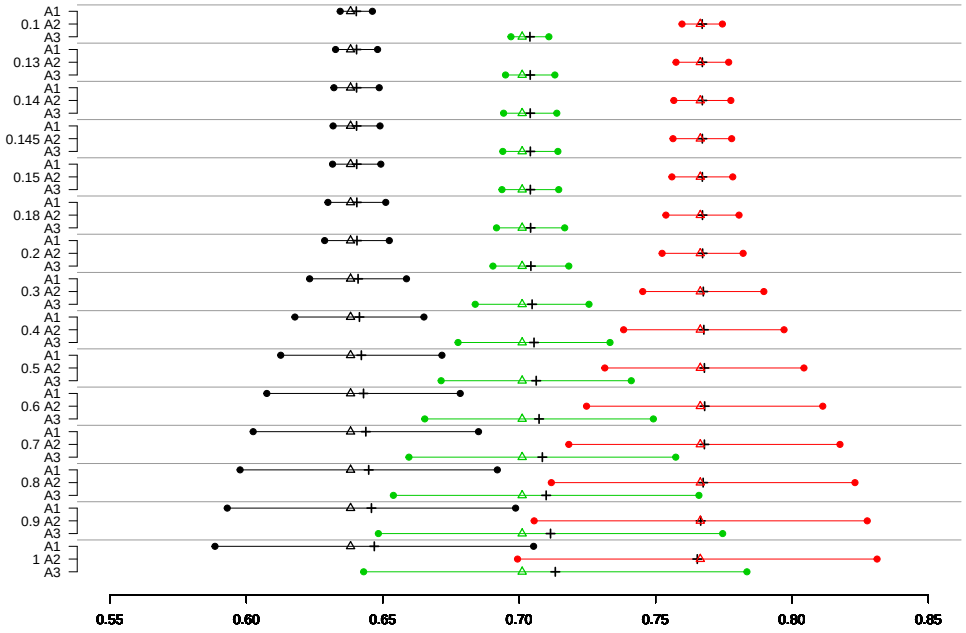
$\mathscr{A}_i$	$\delta = 0$	$\delta = 0.1$	$\delta = 0.13$	$\delta = 0.14$
$\mathscr{A}_1$	0.6382	$[0.6344, 0.6462]$	$[0.6327, 0.6481]$	$[0.6321, 0.6487]$
$\mathscr{A}_2$	0.7664	$[0.7597, 0.7745]$	$[0.7575, 0.7768]$	$[0.7567, 0.7776]$
$\mathscr{A}_3$	0.7011	$[0.6970, 0.7109]$	$[0.6950, 0.7131]$	$[0.6943, 0.7138]$
	$\delta = 0.145$	$\delta = 0.15$	$\delta = 0.18$	$\delta = 0.20$
$\mathscr{A}_1$	$[0.6318, 0.6490]$	$[0.6316, 0.6493]$	$[0.6299, 0.6511]$	$[0.6287, 0.6524]$
$\mathscr{A}_2$	$[0.7564, 0.7779]$	$[0.7560, 0.7783]$	$[0.7538, 0.7806]$	$[0.7524, 0.7821]$
$\mathscr{A}_3$	$[0.6940, 0.7142]$	$[0.6937, 0.7145]$	$[0.6917, 0.7167]$	$[0.6904, 0.7182]$
	$\delta = 0.30$	$\delta = 0.40$	$\delta = 0.50$	$\delta = 0.60$
$\mathscr{A}_1$	$[0.6232, 0.6587]$	$[0.6178, 0.6651]$	$[0.6126, 0.6717]$	$[0.6075, 0.6784]$
$\mathscr{A}_2$	$[0.7453, 0.7897]$	$[0.7383, 0.7971]$	$[0.7314, 0.8044]$	$[0.7247, 0.8113]$
$\mathscr{A}_3$	$[0.6839, 0.7256]$	$[0.6776, 0.7333]$	$[0.6714, 0.7411]$	$[0.6654, 0.7492]$
	$\delta = 0.70$	$\delta = 0.80$	$\delta = 0.90$	$\delta = 1.0$
$\mathscr{A}_1$	$[0.6025, 0.6851]$	$[0.5977, 0.6920]$	$[0.5930, 0.6987]$	$[0.5885, 0.7053]$
$\mathscr{A}_2$	$[0.7182, 0.8176]$	$[0.7118, 0.8231]$	$[0.7055, 0.8276]$	$[0.6994, 0.8312]$
$\mathscr{A}_3$	$[0.6596, 0.7574]$	$[0.6539, 0.7659]$	$[0.6484, 0.7746]$	$[0.6430, 0.7835]$

4.6 Discussion

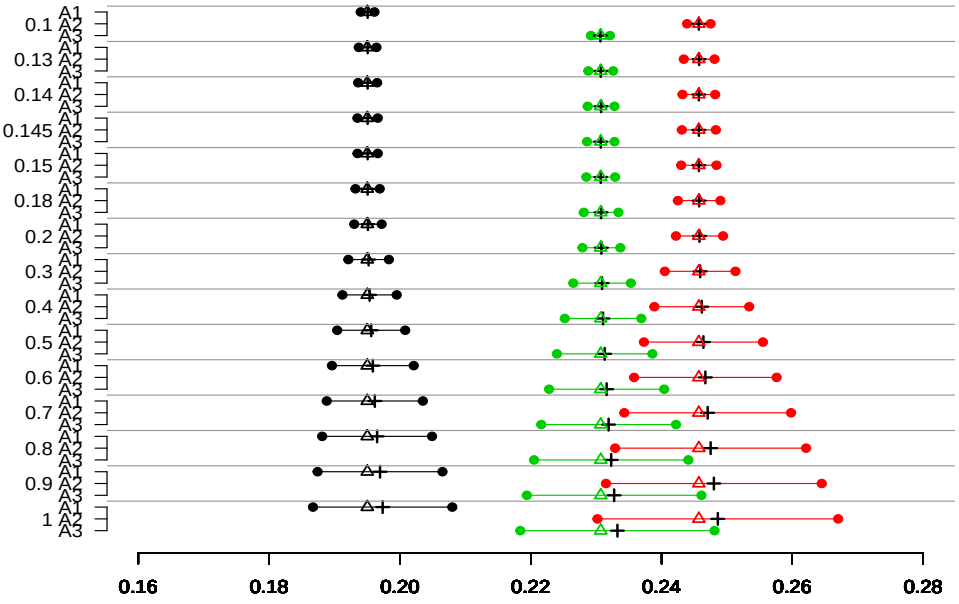
This work is interested in observing three types of outputs and how they are affected when uncertainty is introduced to the inputs (see Section 4.4). There is no doubt that when there is uncertainty in the input, one should reflect upon it and consider it in the decision making process. In the MCDM field, TOPSIS is used widely by many researchers to solve the MCDM problems. However, so far, there is still no way of mapping varying levels of uncertainty in the input to varying levels of uncertainty in the output. Thus, in this study,

Table 4.18 Result of blurring various δ values for Case Study 2

$\mathscr{A}_i$	$\delta = 0$	$\delta = 0.1$	$\delta = 0.13$	$\delta = 0.14$
$\mathscr{A}_1$	0.1950	$[0.1940, 0.1961]$	$[0.1937, 0.1964]$	$[0.1936, 0.1965]$
$\mathscr{A}_2$	0.2457	$[0.2439, 0.2475]$	$[0.2434, 0.2481]$	$[0.2432, 0.2482]$
$\mathscr{A}_3$	0.2307	$[0.2292, 0.2321]$	$[0.2288, 0.2326]$	$[0.2287, 0.2328]$
	$\delta = 0.145$	$\delta = 0.15$	$\delta = 0.18$	$\delta = 0.20$
$\mathscr{A}_1$	$[0.1935, 0.1966]$	$[0.1935, 0.1966]$	$[0.1932, 0.1969]$	$[0.1930, 0.1972]$
$\mathscr{A}_2$	$[0.2431, 0.2483]$	$[0.2430, 0.2484]$	$[0.2425, 0.2490]$	$[0.2422, 0.2494]$
$\mathscr{A}_3$	$[0.2286, 0.2328]$	$[0.2285, 0.2329]$	$[0.2281, 0.2334]$	$[0.2279, 0.2337]$
	$\delta = 0.30$	$\delta = 0.40$	$\delta = 0.50$	$\delta = 0.60$
$\mathscr{A}_1$	$[0.1921, 0.1983]$	$[0.1912, 0.1995]$	$[0.1904, 0.2008]$	$[0.1896, 0.2021]$
$\mathscr{A}_2$	$[0.2405, 0.2513]$	$[0.2389, 0.2534]$	$[0.2373, 0.2555]$	$[0.2358, 0.2576]$
$\mathscr{A}_3$	$[0.2265, 0.2353]$	$[0.2252, 0.2369]$	$[0.2240, 0.2386]$	$[0.2228, 0.2404]$
	$\delta = 0.70$	$\delta = 0.80$	$\delta = 0.90$	$\delta = 1.0$
$\mathscr{A}_1$	$[0.1888, 0.2035]$	$[0.1881, 0.2049]$	$[0.1874, 0.2065]$	$[0.1867, 0.2080]$
$\mathscr{A}_2$	$[0.2343, 0.2598]$	$[0.2329, 0.2621]$	$[0.2315, 0.2645]$	$[0.2302, 0.2670]$
$\mathscr{A}_3$	$[0.2216, 0.2422]$	$[0.2205, 0.2441]$	$[0.2194, 0.2461]$	$[0.2184, 0.2481]$



(a) $\overline{CC}$ for Case study 1: Chen's dataset



(b) $\overline{CC}$ for Case study 2: Selection of food outlets

Fig. 4.7 The resulting CC intervals for all case studies. '△' indicates $\dot{CC}$ from the Type-1 model, and '+' indicates the centre of $\overline{CC}$. A black line is given for $\mathscr{A}_1$, a red line for $\mathscr{A}_2$ and a green line for $\mathscr{A}_3$

blurred T1 TFNs are used to simulate different degrees of uncertainty and shows how to map such uncertainty to the outputs using our proposed model. This study has demonstrated this mapping using two different case studies.

For each case study, three alternatives (options) that need to be ranked from the best to the worst alternative are used. However, when uncertainty is present in the input, it can be seen clearly from Fig. 4.7 that no trivial ranking of the alternatives can be made. Particularly, when the resulting intervals are observed for each case study, there are always intervals that overlap. DMs should consider this overlap as an indication that they should not rank the alternatives with unwarranted certainty. For example, referring to Fig. 4.7b (from case study 2), to determine a ranking order of alternative $\mathscr{A}_1$ (black solid line) and $\mathscr{A}_2$ (red solid line), as well as with $\mathscr{A}_3$ at $\delta = 0.7$ onwards, one should take extra caution, especially when the degree of overlapping between two intervals is high. We may face a non-trivial ranking in this case. The ordering of alternatives is no longer: $\mathscr{A}_3 > \mathscr{A}_2 > \mathscr{A}_1$, meaning that $\mathscr{A}_3$ is preferable to $\mathscr{A}_2$ is preferable to $\mathscr{A}_1$. The ordering could be: $\mathscr{A}_3 \simeq \mathscr{A}_2 > \mathscr{A}_1$, which means $\mathscr{A}_3$ is similar or equal to $\mathscr{A}_2 > \mathscr{A}_1$. Another possible ordering is: $\mathscr{A}_3 \simeq \mathscr{A}_2 > \mathscr{A}_1$, which means $\mathscr{A}_3$ is proportional to $\mathscr{A}_2 > \mathscr{A}_1$. Thus, the proposed model provides an alternative interpretation for DMs and they shouldn't necessarily rank the alternatives in the same order as they would in the crisp value case.

In Fig. 4.7, it is shown that it is not straightforward to rank the alternatives when uncertainty is present in the input and reflected in the output. To date, people have used T2-TOPSIS to produce crisp outputs, losing information, which has allowed for undue certainty in ranking. However, in this study, a method using T2-TOPSIS is shown, accounting for the level of uncertainty that may exist. Practitioners using the method should look at the interval-valued output and the potential for overlapping between intervals. If we ignore the uncertainty in the inputs and outputs by creating a crisp output, the risk of making a bad decision may incur. By providing the interval-valued outputs, the DMs will be more cautious

and more aware of the outputs' characteristics, especially when intervals start to overlap with each other. Thus, the DMS can avoid making a bad decision when provided with an interval-valued output, especially when there are overlapping intervals.

The model presented here is a complete demonstration on how uncertainty in the input should be mapped systematically to the output in the decision-making process. This study visualized such a mapping by applying it to two case studies. It can be seen clearly from Fig. 4.7 that the uncertainty in the inputs affects the interval-valued output as uncertainty is increased. Thus, it is an essential feature to provide interval-valued outputs to the DMs. Then, the DMs face a complex problem where they need to rank the interval-valued outputs in order to make an optimal decision. Interval ranking problems have already been discussed in the literature, for example [104, 154, 156, 157, 22, 102, 47]. These papers provide different methods to rank interval-valued numbers that are dependent upon the context of the application. In the next chapter, a brief discussion about interval ranking problems is presented. A modified and extended measure to rank a pair of intervals will also be proposed in the next chapter.

4.7 Summary

This chapter described the proposed extension model of T1-TOPSIS, building on the model introduced by Chen [1]. The model uses IT2 TFNs for modeling input data and generates interval-valued closeness coefficient ($\overline{CC}$), in turn providing additional information to DMs. Current IT2-TOPSIS models commonly use IT2 TFNs to represent ratings and weights given by DMs and provide a crisp output to be used by DMs for the ranking of alternatives [147, 137, 90, 148, 123, 130, 149, 125, 150]. To date, people have used T2-TOPSIS to produce crisp outputs $\dot{CC}$, losing information, which has allowed for undue certainty in ranking. However, this thesis proposed a model that uses T2-TOPSIS, accounting for the level of uncertainty that may exist. Practitioners using the model should look at the interval-

valued output and the potential for overlapping between intervals. If practitioners ignore the uncertainty in the inputs and outputs by creating a crisp output, they may incur the risk of making a poor decision. By providing the interval-valued outputs, this makes the DMs more cautious and more aware of the outputs' characteristics, especially when intervals start to overlap with each other. It is expected that by providing DMs with an interval-valued output, they can avoid making a bad decision, especially when there exists overlapping intervals.

Next, this chapter introduced the experimental method that was conducted to observe the effect that introducing uncertainty in the input has on the output of a decision-making model. The numerical example utilizing the dataset from Chen [1] was used to simulate cases in which uncertainty is present in the input. This is related to the third objective in this thesis where the effects of introducing uncertainty in the input to the output were observed by developing a novel experimental method. Based on this, there exist several effects that lead to changes in the uncertainty output.

The fourth objective of this thesis is presented in this chapter by implementation of the proposed model on two different datasets. The first dataset was from the literature (i.e., Chen [1]) and the other dataset was from a novel case study collected in 2017 at the University of Nottingham. The details of the novel case study were also presented.The results of both implementations were presented in Section 4.5.3 of the chapter. It is shown that there is a direct relationship between the amount of uncertainty present in the input represented by MFs and the size of the CC intervals. To date, no other studies have proposed the use of interval CC values with the implementation of IT2 TOPSIS model. By keeping the CC values in interval form, the proposed model reduces the loss of information and maximizes the potential benefit of using IT2 FSs. While the interpretation of the resulting intervals and specifically, their respective widths, is not trivial in a decision making context, the study has established a systematic way to linking input uncertainty to output uncertainty in TOPSIS, which itself provides a rich basis for future study. As intervals can, for example, exhibit

partial overlap, it is necessary to use appropriate methodology in order to rank the resulting interval-valued CC. In other words, by providing the DMs with interval-valued outputs, there is a potential for having overlapping intervals that will impact on how ranking of alternatives should be performed. The ranking of interval-values is known as a complex problem and already established in the literature. For example, [104, 154, 156, 157, 22, 102, 47] provide complete discussion on how interval-values can be ranked. However, there exist limitations in applying these existing methods. The limitations of the existing methods will be discussed in the next chapter. Then, the next chapter will introduce an extended measure for ranking a pair of intervals.

Chapter 5

An Extended Measure for Comparing Interval Numbers

5.1 Introduction

This chapter presents a study that is expanded from Chapter 4, which aims to compare a pair of interval-valued number and provide ranking for the number. In the previous chapter, a proposed IT2-TOPSIS model provides interval number as an output for MCDM problem. However, in practice, the MCDM model need to provide ranking order of alternatives in order to assist DM in making decision. In this chapter, an extended measure is proposed to provide a rank for interval number. Section 5.2 provides the motivation and related work of this study. Several related works are reviewed and some problems are highlighted. Based on the real case study presented in the previous chapter, there is a need to extend the existing method on comparing a pair of interval numbers. Section 5.3 introduces the proposed extended measure used to compare a pair of interval values in this study. Finally, Section 5.4 provides summary of the chapter.

5.2 Motivation and related work

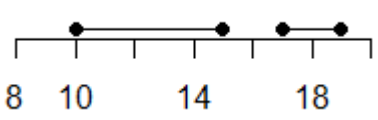
In the previous chapter, the proposed IT2-TOPSIS model with an interval output was described in detail. In practice, the use of interval coefficients in certain contexts may be more useful than using a crisp number. Hence, the proposed IT2-TOPSIS model which provides an interval output values can clearly indicate to what extent of tolerance or region a parameter can possibly take [22]. However, there exists one key question related to the interval output in the proposed model, which is how to compare several intervals in order to rank and make a final decision. In Section 2.4.2, several fundamental background on comparing interval are given. This section presents several existing and related works on comparing interval.

There are three different relations associated with three different cases that may need to be considered when comparing two intervals as illustrated in Figure 5.1. These are:

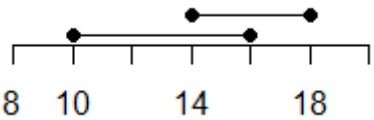
i. Case 1: Non-overlapping intervals

ii. Case 2: Partially overlapping intervals

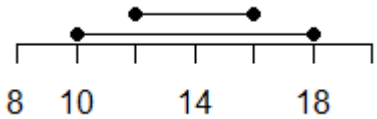
iii. Case 3: Nested intervals



(a) Case 1: Non-overlapping intervals



(b) Case 2: Partial overlapping intervals



(c) Case 3: Nested intervals

Fig. 5.1 Different cases of intervals

A study on comparing intervals was introduced in 1979, in which the pioneer work is from Moore [20]. Two order relations were introduced as follows:

i. $\bar{a} < \bar{b}$ iff $\bar{a}_R < \bar{b}_U$,

ii. $\bar{a} \subseteq \bar{b}$ iff $\bar{b}_L \leq \bar{a}_L$ and $\bar{a}_R \leq \bar{b}_R$

Clearly, the first transitive order relation "$<$" is applicable only for Case 1 (i.e., non-overlapping intervals) while the second relation is the generalization of the definition of subsets for intervals, which describes the interval $\bar{a}$ as nested in $\bar{b}$, but it cannot make a value-based ordering of intervals.

Ishibuchi and Tanaka [158] improved the definition by Moore [20] by introducing the following order relations. Instead of using only lower and upper bound of intervals, they also used the midpoint ($mid(\bar{a})$) and the radius ($rad(\bar{a})$) of an interval. The definitions of $mid(\bar{a})$ and $rad(\bar{a})$ are given in (2.20) and (2.21), respectively. Additionally, they defined the relations for maximization and minimization conditions. Order relations for maximization problems are defined as follows [158]:

Definition 18 *If $\bar{a} = [\bar{a}_L, \bar{a}_U]$ and $\bar{b} = [\bar{b}_L, \bar{b}_U]$ are two interval profits, then the order relation $\leq_{LU}$ for maximization problems is defined as:*

$$\bar{a} \leq_{LU} \bar{b} \text{ iff } \bar{a}_L \leq \bar{b}_L \text{ and } \bar{a}_U \leq \bar{b}_U \tag{5.1}$$

$$\bar{a} <_{LU} \bar{b} \text{ iff } \bar{A} \leq \bar{b} \text{ and } \bar{a} \neq \bar{b} \tag{5.2}$$

Definition 19 *Let $\bar{a} = \langle mid(\bar{a}), rad(\bar{a})\rangle$ and $\bar{b} = \langle mid(\bar{b}), rad(\bar{b})\rangle$ be two intervals defined by their midpoint and radius. The order relation $\leq_{CW}$ for maximization problems is defined as*

$$\bar{a} \leq_{CW} \bar{b} \text{ iff } mid(\bar{a}) \leq mid(\bar{c}) \text{ and } rad(\bar{a}) \geq rad(\bar{b}) \tag{5.3}$$

$$\bar{a} <_{CW} \bar{b} \text{ iff } \bar{a} \leq \bar{b} \text{ and } \bar{a} \neq \bar{b} \tag{5.4}$$

Order relations for minimization problems are defined as follows:

Definition 20 *If $\bar{a} = [\bar{a}_L, \bar{a}_U]$ and $\bar{b} = [\bar{b}_L, \bar{b}_U]$ are two intervals costs, then the order relation $\leq^*_{LU}$ for maximization problems is defined as:*

$$\bar{a} \leq^*_{LU} \bar{b} \text{ iff } \bar{a}_L \leq \bar{b}_L \text{ and } \bar{a}_U \leq \bar{b}_U \tag{5.5}$$

$$\bar{a} <^*_{LU} \bar{b} \text{ iff } \bar{a} \leq^*_{LU} \bar{b} \text{ and } \bar{a} \neq \bar{b} \tag{5.6}$$

Definition 21 *Let $\bar{a} = \langle mid(\bar{a}), rad(\bar{a})\rangle$ and $\bar{b} = \langle mid(\bar{b}), rad(\bar{b})\rangle$ be two intervals defined by their midpoint and radius. The order relation $\leq^*_{CW}$ for minimization problems is defined as*

$$\bar{a} \leq^*_{CW} \bar{b} \text{ iff } mid(\bar{a}) \leq mid(\bar{b}) \text{ and } rad(\bar{a}) \geq rad(\bar{b}) \tag{5.7}$$

$$\bar{a} <^*_{CW} \bar{b} \text{ iff } \bar{a} \leq^*_{CW} \bar{b} \text{ and } \bar{a} \neq \bar{b} \tag{5.8}$$

Two additionally definitions are given by Ishibuchi and Tanaka as follows.

Definition 22 *For maximization problems, the order relation $\leq_{LC}$ is defined by*

$$\bar{a} \leq_{LC} \bar{b} \text{ iff } \bar{a}_L \leq \bar{b}_L \text{ and } mid(\bar{a}) \leq mid(\bar{b}) \tag{5.9}$$

$$\bar{a} <_{LC} \bar{b} \text{ iff } \bar{a} \leq_{LC} \bar{b} \text{ and } \bar{a} \neq \bar{b} \tag{5.10}$$

Definition 23 *For minimization problems, the order relation $\leq^*_{RC}$ is defined by*

$$\bar{a} \leq^*_{RC} \bar{b} \text{ iff } \bar{a}_L \leq \bar{b}_L \text{ and } mid(\bar{a}) \leq mid(\bar{b}) \tag{5.11}$$

$$\bar{a} <^*_{RC} \bar{b} \text{ iff } \bar{a} \leq^*_{RC} \bar{b} \text{ and } \bar{a} \neq \bar{b} \tag{5.12}$$

Ishibuchi and Tanaka's definitions can compare Case 1 and Case 2 very well. However, for Case 3 intervals, these can only be applied partially. For example, consider Case 3 intervals as follows:

$$A = \bar{a} = [0, 10] = \langle 5, 5\rangle$$

$$B = \bar{b} = [4,8] = \langle 6,2 \rangle$$

These two intervals are shown in Figure 5.2. Ishibuchi and Tanaka's method is partially applicable for this interval case where it can define the order relation for the maximization case but not for the minimization case. For the maximization case, the interval B is preferred to interval A. For the minimization case, the method is not applicable to define any order relation between these two intervals. Even though it successfully defines order relations in the maximization case, the method only provides a preference ordering instead of a value-based ordering [22].

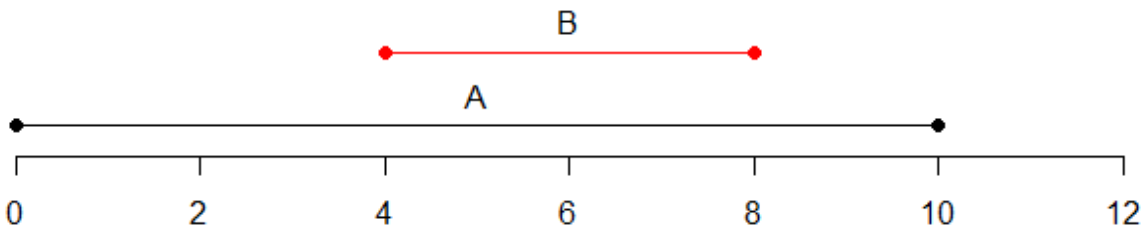


Fig. 5.2 An example of partially applicable interval case of using Ishibuchi and Tanaka's method

Later, Sengupta and Pal [22] introduced two different indices that can give a value-based indicator to compare intervals. The first one is the acceptability index $acc(\bar{a})$ and the second one is the fuzzy preference ordering. This chapter only considers the $acc(\bar{a})$ index that will be implemented in the numerical example presented in the next section. The definition of the index is given as follows [22]:

Definition 24 *Let $\mathbb{IR}$ be the set of all closed intervals on the real line $\mathbb{R}$. The acceptability function, $acc(\bar{a}) : \mathbb{IR} \times \mathbb{IR} \to [0,\infty)$, for the intervals $\bar{a},\bar{b} \in \mathbb{R}$ with $mid(\bar{a}) \leq mid(\bar{b})$ is denoted by $acc(\bar{a},\bar{b})$ or $acc_{\leq}(\bar{a},\bar{b})$ and is defined by*

$$acc(\bar{a},\bar{b}) = \frac{mid(\bar{b}) - mid(\bar{a})}{rad(\bar{b}) + mid(\bar{a})}, rad(\bar{b}) + rad(\bar{a}) \neq 0 \quad and \quad mid(\bar{b}) \geq mid(\bar{a}) \tag{5.13}$$

$acc(\bar{a},\bar{b})$ indicates a grade of acceptability of "$\bar{a}$ is smaller to the $\bar{b}$". Further interpretations of this index are given based on the position of the midpoint and radius of interval $\bar{b}$ with respect to interval $\bar{a}$ as follows:

$$acc(\bar{a},\bar{b}) \begin{cases} = 0, & \text{if} \quad mid(\bar{a}) = mid(\bar{b}), \\ = m : (0 < m < 1) & \text{if} \quad mid(\bar{a}) < mid(\bar{b}) \quad \& \quad a_U > b_U, \\ \geq 1, & \text{if} \quad mid(\bar{a}) < mid(\bar{b}) \quad \& \quad a_U \leq b_L, \end{cases} \tag{5.14}$$

From (5.14), if $acc(\bar{a},\bar{b}) \geq 1$, for maximization problems, interval $\bar{b}$ is accepted as the larger interval with full satisfaction. If $0 < acc(\bar{a},\bar{b}) < 1$, then interval $\bar{b}$ is accepted as the larger interval with the grade of acceptability m. Lastly, if $acc(\bar{a},\bar{b}) = 0$, the index simply conclude that $\bar{a} = \bar{b}$, without considering different width within these two intervals.

Giove [102] extended the index proposed by Sengupta and Pal by providing a modification on the index as follows:

Definition 25 *If $\bar{a} = [a_L, a_U]$ and $\bar{b} = [b_L, b_U]$ are two non-negative interval numbers, with midpoint and radius represented as $\bar{a} = \langle mid(\bar{a}), rad(\bar{a})\rangle$ and $\bar{b} = \langle mid(\bar{a}), rad(\bar{b})\rangle$, respectively, the modified $acc(\bar{a},\bar{b})$ index, termed as $\xi_{\bar{a}>\bar{b}}$ is:*

$$\xi_{\bar{a}>\bar{b}} = \frac{mid(\bar{a}) - mid(\bar{b})}{rad(\bar{a}) - rad(\bar{b}) + 1}, \; mid(\bar{a}) \geq mid(\bar{b}). \tag{5.15}$$

Unlike the $acc(\bar{a})$ index by Sengupta and Pal, the modified index by Giove simultaneously compares crisp and interval numbers. The Giove's index is positive if the midpoint of the first interval is greater than the midpoint of the second interval, and vice versa. However, the index is null if both midpoints are same. The index decreases with the sum of two spreads.

Most recently, Ruan et al. [47] proposed an improved index based on Giove's index. The improved index not only can compare crisp and interval numbers simultaneously, but can

also reflect the degree of the decision maker's (DM) optimism. The definition of the Ruan's index is as follows.

Definition 26 *Let* $\bar{a} = [a_L, a_U]$ *and* $\bar{b} = [b_L, b_U]$ *are two non-negative interval numbers.*

$$\delta(\bar{a} > \bar{b}) = \frac{o(\bar{a}) - o(\bar{b})}{rad(\bar{a}) + rad(\bar{a}) + 1} = \frac{(\gamma a_L + (1-\gamma)a_U) - (\gamma b_L + (1-\gamma)b_U)}{rad(\bar{a}) + rad(\bar{b}) + 1} \tag{5.16}$$

where $o(\bar{a})$ and $o(\bar{b})$ are the perceived values of $\bar{a}$ and $\bar{b}$ according to the Hurwicz [159] optimism-pessimism states of DMs. The perceived value of $\bar{a}$, denoted $o(\bar{a})$, is defined as follows:

Definition 27

$$o(\bar{a}) = \gamma a_L + (1-\gamma)a_U \tag{5.17}$$

where γ *denotes the optimism degree of DMs,* $0 \leq \gamma \leq 1$*. When* $0 \leq \gamma < 0.5$*, the DM is pessimistic; when* $\gamma = 0.5$*, the DM is moderate; and when* $0.5 < \gamma \leq 1$*, the DM is optimistic.*

A numerical example that consists of all interval cases is given in this chapter to achieve a higher understanding of how comparing intervals works with Sengupta & Pal's, Giove's and Ruan's methods. Table 5.1 and Table 5.2 provide numerical examples of all pairs of intervals. Each column indicates each pair of the interval which belong to different types of interval cases. Figure 5.3 illustrates all the pairs used in the example.

Based on the values in Table 5.1 and Table 5.2, the first row (i.e., Sengupta and Pal's index) indicates that $\bar{b} < \bar{a}$. If the index $0 < acc(\bar{b}, \bar{a}) < 1$, it is acceptable that $\bar{b}$ is inferior/less than/less preferred/smaller than interval $\bar{a}$. If $acc(\bar{b}, \bar{a}) = 0$, as shown by Pair 6 in Table 5.2, then the condition $\bar{b} < \bar{a}$ is not accepted. Finally, if $acc(\bar{b}, \bar{a}) > 1$, the condition of $\bar{b} < \bar{a}$ is fully accepted.

The other two indices, Giove's and Ruan's indices, both indicate that $\bar{a} > \bar{b}$, which means that $\bar{a}$ is superior/greater than/most preferred than $\bar{b}$. Hence, if the index results in a positive

Table 5.1 Numerical example on various cases and methods: Pair 1 – Pair 3

Method	Pair 1 $\bar{a} = [10,20]$ $=\langle 15,5\rangle$ $\bar{b} = [11,11]$ $=\langle 11,0\rangle$	Pair 2 $\bar{a} = [10,20]$ $=\langle 15,5\rangle$ $\bar{b} = [21,23]$ $=\langle 22,1\rangle$	Pair 3 $\bar{a} = [10,20]$ $=\langle 15,5\rangle$ $\bar{b} = [20,22]$ $=\langle 21,1\rangle$
Sengupta & Pal $\left(acc(\bar{b} < \bar{a})\right)$	$\frac{15-11}{5+0} = 0.8$	-1.667	-1
Giove ($\xi_{\bar{a}>\bar{b}}$)	$\frac{15-11}{5-0+1} = 0.667$	-1.4	-1.2
Ruan ($\delta_{0.5}(\bar{a} > \bar{b})$)	$\frac{15-11}{6} = 0.667$	-1	-0.857

value, the condition $\bar{a} > \bar{b}$ is accepted. If the index results in a negative value, the condition $\bar{a} > \bar{b}$ is not accepted (i.e., $\bar{a} < \bar{b}$ may be accepted). However, similar to Sengupta and Pal's index, both Giove's and Ruan's indices also produce the value 0 when the two intervals have the same midpoint. This situation may occur in practice in complex decision making. Next section provides a sensitivity analysis to explore a problem that may occur when comparing a pair of nested intervals with the same midpoint. Then, an extended measure is proposed to solve the identified problem.

5.3 The proposed extended measure for comparing two intervals

As presented in the previous section, all selected methods to compare intervals cannot be applied when the intervals belong to Case 3 (Figure 5.1 (c)), specifically when both compared intervals have the same midpoint. Note, the acceptability index by these three papers (i.e., [22, 102, 47]) are based on the midpoint and radius of the intervals. Generally, all acceptability indices reviewed in the previous section hold the same fundamental rule:

Table 5.2 Numerical example on various cases and methods: Pair 4 – Pair 7

Method	Pair 4 $\bar{a} = [10,20]$ $=\langle 15,5\rangle$ $\bar{b} = [16,23]$ $=\langle 19.5,3.5\rangle$	Pair 5 $\bar{a} = [10,20]$ $=\langle 15,5\rangle$ $\bar{b} = [17,19]$ $=\langle 18,1\rangle$	Pair 6 $\bar{a} = [10,20]$ $=\langle 15,5\rangle$ $\bar{b} = [14,16]$ $=\langle 15,1\rangle$	Case 7 $\bar{a} = [10,20]$ $=\langle 15,5\rangle$ $\bar{b} = [12,16]$ $=\langle 14,2\rangle$
Sengupta & Pal $\left(acc(\bar{b} < \bar{a})\right)$	-0.529	-0.5	0	0.143
Giove ($\xi_{\bar{a}>\bar{b}}$)	-1.8	-0.6	0	0.25
Ruan ($\delta_{0.5}(\bar{a} > \bar{b})$)	-0.474	-0.429	0	0.125

the position of the midpoint of an interval compared to the other reference interval indicates whether the first interval is superior/greater than (inferior/less than) the other reference interval. In contrast, the radius of the superior (or inferior) interval compared to the other reference interval provides the acceptability degree to which the decision maker (DM) is satisfied with the superiority (or inferiority) of the other compared interval.

From the concept of statistics, the mean (midpoint of interval) and the spread (radius of interval) can be used as a basis for comparing intervals. Furthermore, it is our intuition to accept that an interval with a larger midpoint value and lower radius value is ranked higher. Thus, the fundamental idea from the acceptability index proposed by three papers in the previous section is promising to achieve the goal for comparing two intervals. However, since all the existing acceptability indices are not applicable when the midpoint of intervals are identical, it is necessary to have another measure that can be applied to compare these two intervals. Note, the problem of comparing interval numbers is a fundamental aspect and contribution of this chapter, which explores finding an appropriate measure to provide a ranking of any two interval numbers. Theoretically, interval numbers can only be partially ranked and hence cannot be compared in the ordinary sense [160]. However, if interval numbers are used in practical applications, for example in decision-making models, the

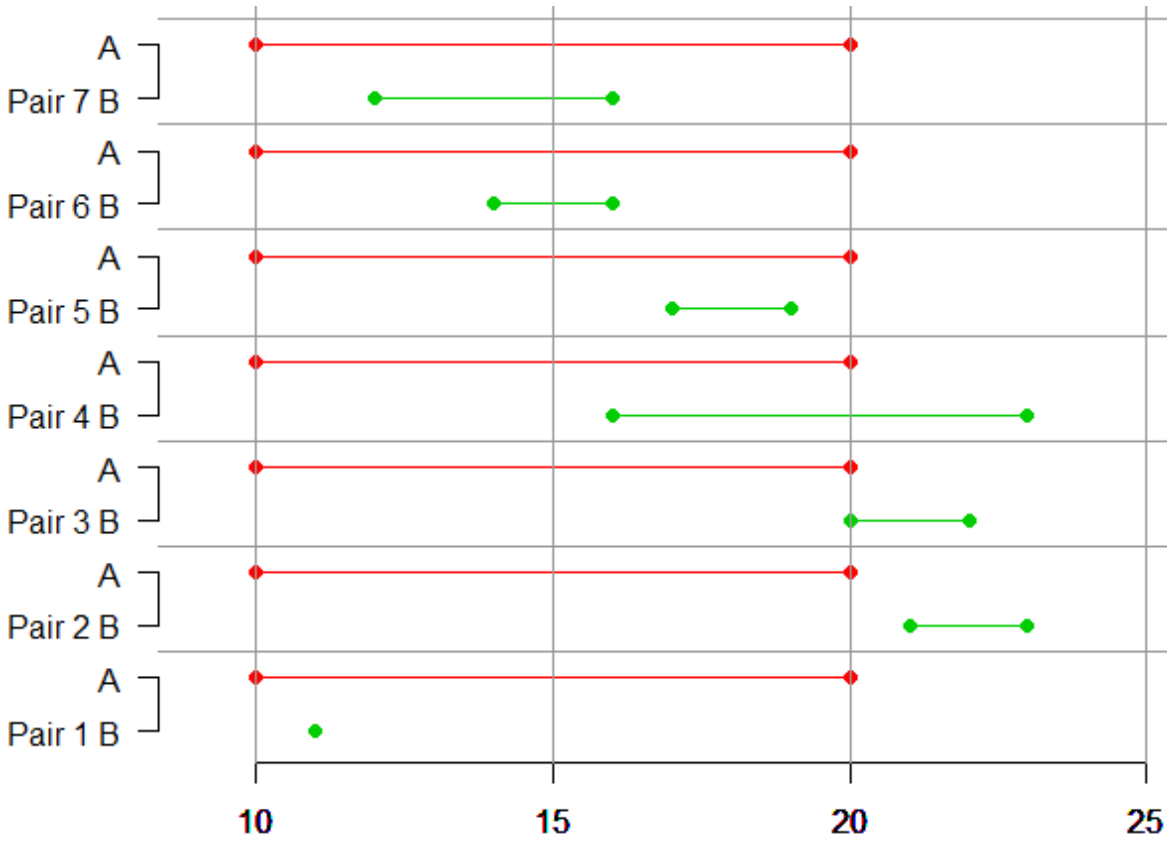


Fig. 5.3 Different cases of intervals, where $A = \bar{a}$ and $B = \bar{b}$

comparison of interval numbers becomes necessary. The next section will present the basis idea towards a proposed extended measure that improves the existing index measure for comparing two interval numbers.

The idea of the proposed measure is based on the concept of coefficient of variation (CV) in statistics. The concept of CV in statistics is used to measure the dispersion or relative variability between two different values. If applying these concepts to interval numbers, the statistical mean of data can be represented by the midpoint of the interval while statistical variation can be represented by the width of the interval. Hence, this study introduces a measure for comparing intervals that uses the concept of CV.

Previously, CV was used to compare two fuzzy numbers in [161]. As stated in [160], fuzzy numbers generalize both the concepts of a real number and a closed interval numbers.

Hence, the concept of CV can be applied to interval numbers that it indicates the ratio of the width of an interval to the midpoint of an interval. The definition of CV is given in [161] as follows:

$$CV = \frac{\sigma \ \text{(standard error)}}{\mu \ \text{(mean)}} \tag{5.18}$$

where $\mu \neq 0$ and $\sigma > 0$.

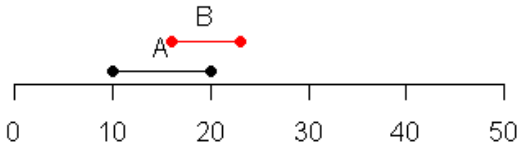
For comparing interval numbers with this CV concept, the midpoint and width of each interval is first obtained. Then the ratio using the basis of CV from each interval is calculated. Next, the ratio of interval bounds to the domain solution is calculated. However, to obtain the rank order of intervals, Wang et al. [162] states that the use of midpoints to compare or rank interval numbers is sometimes not convincing. Thus, we integrate the CV ratio based index with the probabilistic approach introduced by [163] for comparing intervals.

There are two types of non-trivial cases for comparing intervals: The first is the case of partially overlapping intervals (Case 2 in Figure 5.1 (b)) and the second is nested intervals (Case 3 in Figure 5.1 (c)). One of the common methods based on probabilistic approaches for comparing intervals is from Sevastjanov and Rog [163]. Let $\bar{a} = [a_L, a_U]$ and $\bar{b} = [b_L, b_U]$ be independent interval numbers and $a \in [a_L, a_U]$, $b \in [b_L, b_U]$ indicates random variables on these intervals. It is natural to assume that the values of the random variables a and b are uniformly distributed [163]. Then, there exist several subintervals that fall within these two intervals, such as $[a_L, b_L], [b_L, a_U], [a_U, b_U]$. These subintervals can be treated as a set of several independent random events. Assume that the events are denoted by

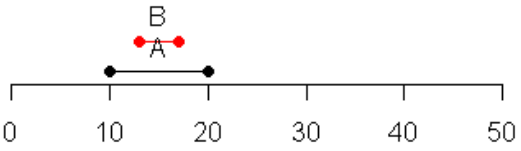
$$S_k : a \in \bar{a}_i, \ b \in \bar{b}_j, \ \text{ if } k = 1, \cdots, n \tag{5.19}$$

where $\bar{a}_i$ and $\bar{b}_j$ are certain subintervals of $\bar{a}$ and $\bar{b}$ such that $\bar{a} = \bigcup_i \bar{a}_i$, $\bar{b} = \bigcup_j \bar{b}_j$. Consider the two pairs of intervals illustrated in Figure 5.4 (a), where $A \doteq \bar{a} = [10, 20]$ and $B \doteq \bar{b} = [16, 23]$.

There are $n = 4$ events for this numerical example according to (5.19) which belong to Case 2 interval.



(a) Case 2: Partial Overlapping Intervals



(b) Case 3: Nested Intervals

Fig. 5.4 Overlapping and inclusion case of interval, where $A = \bar{a}$ and $B = \bar{b}$

The following subintervals are identified as four different events:

- $S_1 : a \in [a_L, b_L]$ & $b \in [a_U, b_U] = a \in [10, 16]$ & $b \in [20, 23]$
- $S_2 : a \in [a_L, b_L]$ & $b \in [b_L, a_U] = a \in [10, 16]$ & $b \in [16, 20]$
- $S_3 : a \in [b_L, a_U]$ & $b \in [b_L, a_U] = a \in [16, 20]$ & $b \in [16, 20]$
- $S_4 : a \in [b_L, a_U]$ & $b \in [a_U, b_U] = a \in [16, 20]$ & $b \in [20, 23]$

These subintervals represent the probabilities of events (i.e., $S_1, \cdots, S_4$) and these probabilities can be obtained as follows:

$$P(S_1) = \left(\frac{b_L - a_L}{a_U - a_L}\right) * \left(\frac{b_U - a_U}{b_U - b_L}\right) = \left(\frac{16 - 10}{20 - 10}\right) * \left(\frac{23 - 20}{23 - 16}\right) = 0.2571 \tag{5.20}$$

$$P(S_2) = \left(\frac{b_L - a_L}{a_U - a_L}\right) * \left(\frac{a_U - b_L}{b_U - b_L}\right) = \left(\frac{16 - 10}{20 - 10}\right) * \left(\frac{20 - 16}{23 - 16})\right) = 0.8571 \tag{5.21}$$

$$P(S_3) = \left(\frac{a_U - b_L}{a_U - a_L}\right) * \left(\frac{a_U - b_L}{b_U - b_L}\right) = \left(\frac{20-16}{20-10}\right) * \left(\frac{20-16}{23-16})\right) = 0.2286 \qquad (5.22)$$

$$P(S_4) = \left(\frac{a_U - b_L}{a_U - a_L}\right) * \left(\frac{b_U - a_U}{b_U - b_L}\right) = \left(\frac{20-16}{20-10}\right) * \left(\frac{23-20}{23-16})\right) = 0.1714 \qquad (5.23)$$

In this case (i.e., partial overlapping intervals), only a 2-order relation may occur: $\bar{a} < \bar{b}$ (i.e, $S_1,\ S_2,\ S_4$) or $\bar{a} = \bar{b}$ (i.e., S_3). Based on a probabilistic approach [163], it can be seen that:

$$P(S_1) + P(S_2) + P(S_3) + P(S_4) = 1$$

Thus, the 'strong' evidence of $\bar{a} < \bar{b}$ consists of events S_1, S_2 and S_4, while event S_3 can be treated as 'weak' evidence of $\bar{a} < \bar{b}$. Such weak evidence is caused by simultaneously equal relations $\bar{a} = \bar{b}$. Thus, the total probabilities of 'strong' evidence events can be summed up as follow:

$$t(\bar{a} < \bar{b}) = P(S_1) + P(S_2) + P(S_4) = 0.7714$$

The probability of 'weak' evidence of event S_3 is obtained by

$$t(\bar{a} < \bar{b},\ \bar{a} = \bar{b}) = P(S_3) = 1 - 0.7714 = 0.2286$$

Consider Case 3 (Fig. 5.4 (b)), where $\bar{a} = [10, 20]$ and $\bar{b} = [14, 16]$ with $n = 3$ events. With a similar procedure, the following subintervals are identified:

- $S_1 : a \in [a_L, b_L]$ & $b \in [b_L, b_U] = a \in [10, 14]$ & $b \in [14, 16]$
- $S_2 : a \in b_L, b_U]$ & $b \in [b_L, b_U] = a \in [14, 16]$ & $b \in [14, 16]$
- $S_3 : a \in [b_R, a_U]$ & $b \in [b_L, b_U] = a \in [16, 20]$ & $b \in [14, 16]$

In this case, three relations may occur $S1 : \bar{a} < \bar{b}$, $S_2 : \bar{a} = \bar{b}$ and $S_3 : \bar{a} > \bar{b}$. The probabilities are obtained with the procedure as discussed in the previous case as follows:

- $P(S_1) : \left(\frac{14-10}{20-10}\right) * \left(\frac{16-14}{16-14}\right) = 0.4$
- $P(S_2) : \left(\frac{16-14}{20-10}\right) * \left(\frac{16-14}{16-14}\right) = 0.2$
- $P(S_3) : \left(\frac{20-16}{20-10}\right) * \left(\frac{16-14}{16-14}\right) = 0.4$

Thus, for $(\bar{a} < \bar{a}) = 0.4$, $(\bar{a} = \bar{a}) = 0.2$ and $(\bar{a} > \bar{a}) = 0.4$. However, as can be clearly seen here, the probabilistic approach results in the same probability value for different interval order relations. Thus, it is necessary to improve this method for comparing intervals. Based on the concept of ratio CV and probabilistic concepts, this study further explores the behaviour of nested intervals, highlighting, in particular, when they are Case 3 intervals.

As discussed previously, the interval comparison method based on probabilistic approaches introduced in [163] cannot be applied when a pair of intervals have the same midpoint. This problem has already been demonstrated with the numerical example shown in Fig. 5.4 (b). The same problem occurs with other methods of comparing intervals reviewed in Section 5.2 (i.e., Sengupta and Pal's, Giove's and Ruan's methods). This study intends to improve the methods of comparing intervals by integrating the ratio and probabilistic concepts.

5.3.1 The Proposed Measure for Comparing Interval Numbers

Based on the background and problems discussed in the previous section, this section introduces an extended measure for comparing intervals. The following definitions are needed to achieve this.

Definition 28 *Given an interval $\bar{a} = [a_L, a_U]$, a ratio from the right bound of the interval to the right bound of the domain, denoted x_R, and the left bound of the interval to the left bound of the domain, denoted x_L, is obtained by*

$$T_x = \left(\frac{x_R}{x_L}\right) \tag{5.24}$$

Definition 29 *Given an interval $\bar{a} = \langle mid(\bar{a}), rad(\bar{a})\rangle$, which is represented by its midpoint and radius, respectively, a ratio of its right radius, denoted w_R, and its left radius, denoted w_L, is obtained by*

$$T_w = \left(\frac{w_R}{w_L}\right) \tag{5.25}$$

Definition 30 *The total ratio of the interval $\bar{a}$ to the domain is obtained by*

$$Tot(\bar{a}) = \left(\frac{x_R}{x_L}\right) * \left(\frac{w_R}{w_L}\right) = T_x * T_w \tag{5.26}$$

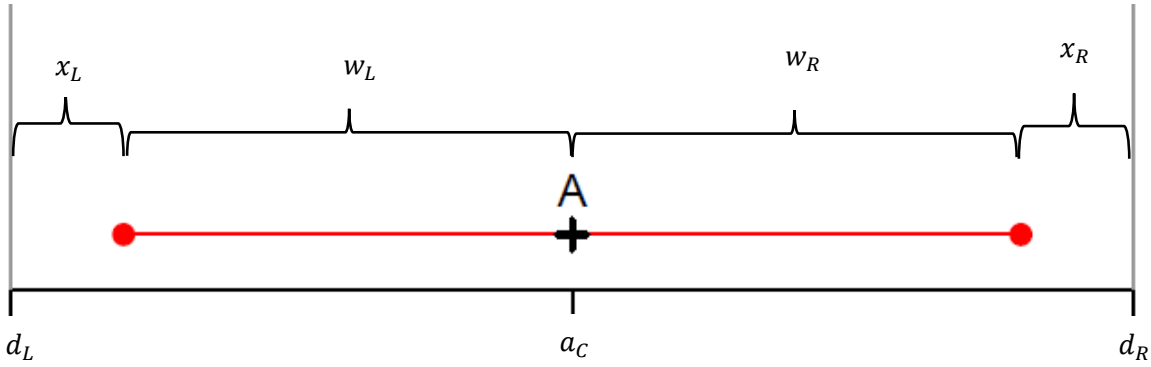


Fig. 5.5 Ratio interval $\bar{a}$ to the domain interval $\bar{d} = [d_L, d_U]$

The details of Definition 28 – 30 are illustrated in Fig. 5.5. These definitions are used to calculate the ratio of each interval, which is based on the radius compared with the ratio of the interval to the domain interval.

Let us assume that there are 10 pairs of different intervals that belong to Case 3, as presented in Table 5.3 and illustrated in Fig. 5.6. These intervals are bounded by $[0,20]$ as a domain of context. Each pair of intervals has the same midpoint value denoted as $mid(\bar{a})_i$ and $mid(\bar{b})_i$ for intervals $\bar{a}$ and $\bar{b}$, respectively.

Table 5.3 Ten pairs of nested intervals with the same midpoint, $mid(\bar{a})_i = mid(\bar{b})_i$

Pair i	Domain	$\bar{a}_i = [a_{li}, a_{Ui}]$	$mid(\bar{a})_i$	$\bar{b}_i = [b_{li}, b_{Ui}]$	$mid(\bar{b})_i$
1	$[0,20]$	[1, 11]	6	[2, 10]	6
2	$[0,20]$	[2, 12]	7	[3, 11]	7
3	$[0,20]$	[3, 13]	8	[4, 12]	8
4	$[0,20]$	[4, 14]	9	[5, 13]	9
5	$[0,20]$	[5, 15]	10	[6, 14]	10
6	$[0,20]$	[6, 16]	11	[7, 15]	11
7	$[0,20]$	[7, 17]	12	[8, 16]	12
8	$[0,20]$	[8, 18]	13	[9, 17]	13
9	$[0,20]$	[9, 19]	14	[10, 18]	14
10	$[0,20]$	[10, 20]	15	[11, 19]	15

Based on these 10 pairs of intervals used as a numerical example to describe the problem, this study further maps these 10 intervals to a graph that plots the midpoint of each interval to the total ratio of each interval. Figure 5.7 shows this mapping where the blue line indicates a mapping of the midpoints of all intervals $\bar{a}$ versus the total ratio of the same intervals, while the red line indicates a mapping of the midpoints of all intervals $\bar{b}$ versus the total ratio of the same intervals.

Based on Definition 28 – 30, the resulting numerical values of the total ratio mapped in Figure 5.7 are shown in Table 5.4. Based on the figure, the total ratio of both intervals decrease towards 0 as the midpoint values for each interval approaches to ∞. The total ratio of all intervals $\bar{a}$ in each pair are higher compared to interval $\bar{b}$ from midpoint value of 6 to 10. Then, the total ratio of interval $\bar{b}$ slightly increased at the midpoint value of 10 and is

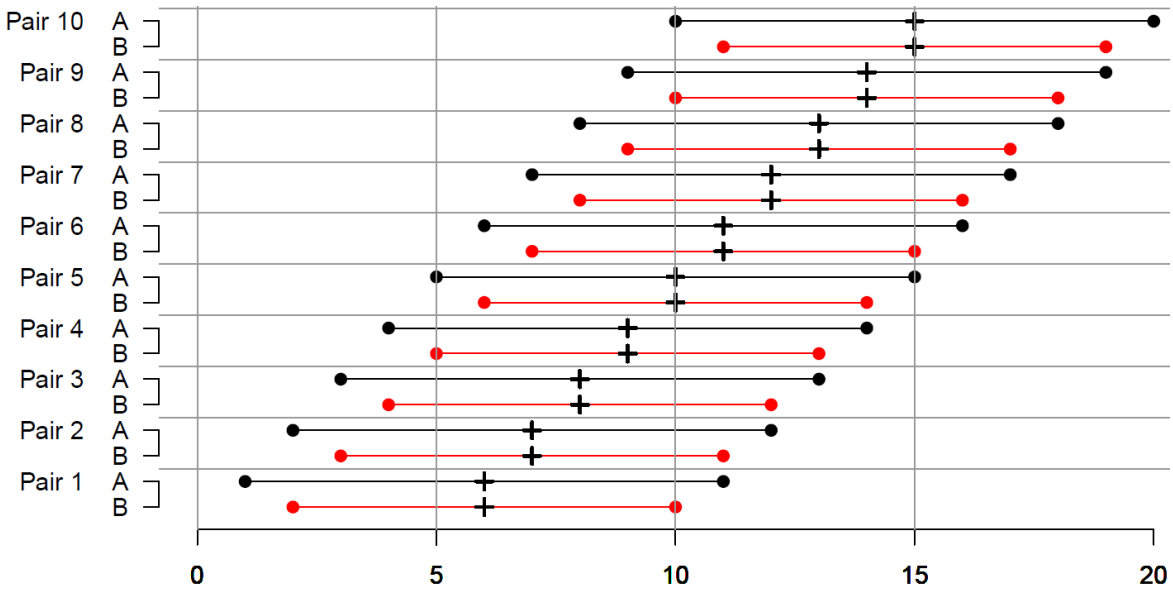


Fig. 5.6 10 pairs of intervals with the same midpoint, where $A = \bar{a}$ and $B = \bar{b}$

slightly higher than all intervals $\bar{a}$. Hence, to make a comparison between these two intervals, when the pair of intervals has the same midpoint, the ratio between these two intervals to the domain is necessary. Thus, based on numerical example for Case 3 in probabilistic approach illustrated in Fig. 5.4 (b), the approach has a problem when it produces same probability value for the order relation of $\bar{a} < \bar{b}$ and $\bar{a} > \bar{b}$. Thus, this study proposed to integrate the probabilistic approach with the ratio introduced in this section to solve such problem. Note that the measure is proposed only for solving the problem of comparing intervals that are nested. Other types of intervals can be compared solely either by the ratio method or the probabilistic method. The proposed measure is defined as follows:

Definition 31 *Let $\tau_>$ be an extended order relation between the intervals $\bar{a} = [a_L, a_U]$ and $\bar{b} = [b_L, b_U]$ on the real line $\mathbb{R}$, then for $mid(\bar{a}) = mid(\bar{b})$, a premise $(\bar{a}\tau_>\bar{b})$ which implies that interval $\bar{a}$ is preferred to interval $\bar{b}$ based on the numerical ratio. The term 'preferred to' is analogous to 'greater than'.*

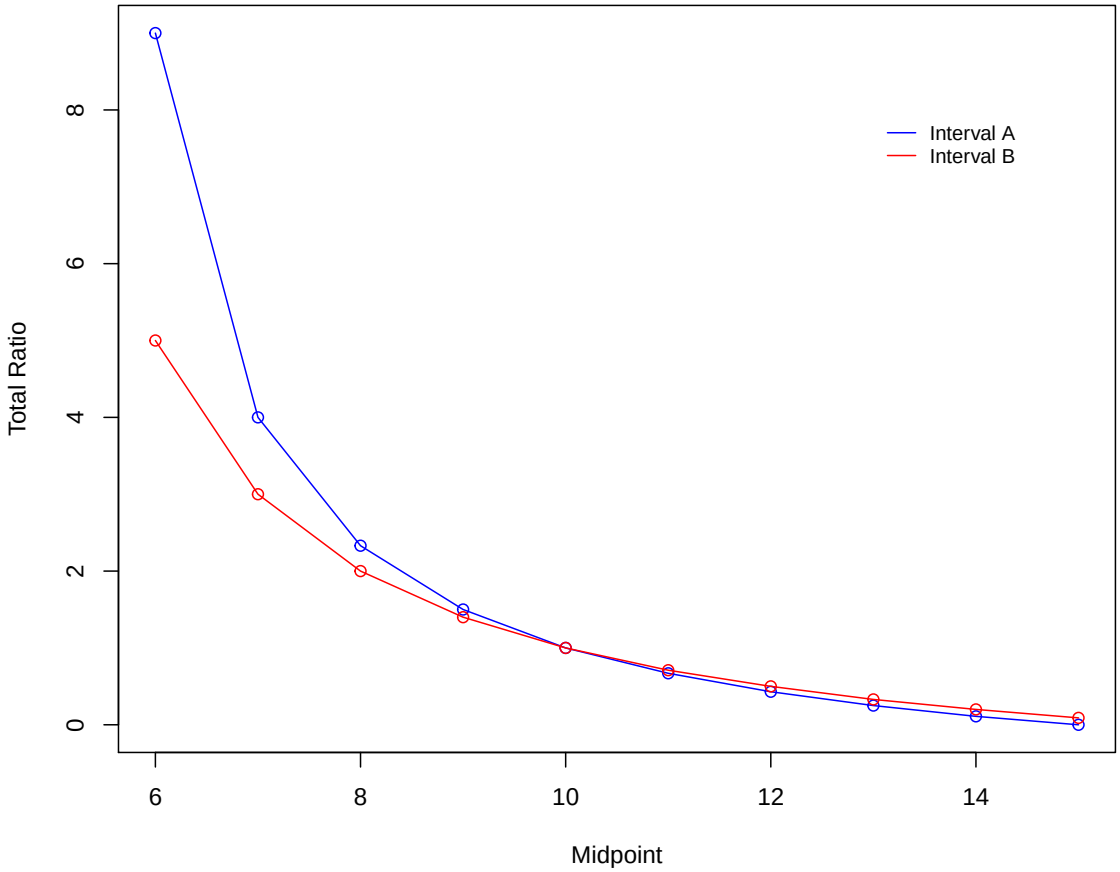


Fig. 5.7 10 pairs of intervals with the same midpoint, where $A = \bar{a}$ and $B = \bar{b}$

Definition 32 *Let I be the set of all closed intervals on the real line $\mathbb{R}$. The modified measure termed as the $\tau_>$ index and $\tau_> : I \times I \to [0, \inf)$ such that*

$$\tau_> \ \ or \ \ \tau(\bar{a}) = (Tot(\bar{a}) \times P(S_i)) \ \ and \ \ \tau(\bar{b}) = (Tot(\bar{b}) \times P(S_i)) \tag{5.27}$$

Then, compare these two interval by

$$\tau_> \ \ or \ \ \tau(\bar{a} > \bar{b}) = (Tot(\bar{a}) \times P(S_i)) \ \ \geq \ \ \tau(\bar{b}) = (Tot(\bar{b} \times P(S_i)) \tag{5.28}$$

where $Tot(\bar{a})$ and $Tot(\bar{a})$ is the ratio of the each interval to the domain as defined in Definition 28 – 30 and illustrated in Fig. 5.5,

Table 5.4 Numerical values of total ratio for each interval in 10 pairs of intervals

Pair i	Total ratio $\bar{a}_i$	Total ratio $\bar{b}_i$
1	9.00	5.00
2	4.00	3.00
3	2.33	2.00
4	1.50	1.40
5	1.00	1.00
6	0.67	0.71
7	0.43	0.50
8	0.25	0.330
9	0.11	0.20
10	0.00	0.09

This ratio is analogous with statistical CV ratio defined in (5.18). The final ratio of each interval is calculated by integrating the total ratio of $Tot(\bar{a})$, $Tot(\bar{b})$ and $P(S_i)$.

By using the same numerical example in the probabilistic Case 3 intervals (i.e., $\bar{a} = [10,20] = \langle 15,5 \rangle$ and $\bar{b} = [14,16] = \langle 15,1 \rangle$) shown in previous section, the probability that events $P(S_1) : \bar{a} < \bar{b}$ and $P(S_3) : \bar{a} > \bar{b}$ are the same is 0.4. Clearly, the extended measure is needed to further compared these two intervals. Hence, this study integrates the probabilistic approach and the total ratio of each interval is calculated to make a comparison between these two intervals. Based on the Definition 28 – 30, assuming that the domain of the problem is within the range of $[0,100]$, the ratio of interval $\bar{a}$, $Tot(\bar{a})$ is 0.8 and the ratio of interval $\bar{b}$, $Tot(\bar{b})$ is 6. The calculations for both ratio are shown in (5.29) and (5.30). Then, use the proposed measure (i.e., Definition 32) to compare these two intervals.

$$Tot(\bar{a}) = \left(\frac{100-20}{10-0}\right) * \left(\frac{20-15}{15-10}\right) = 0.8 \tag{5.29}$$

$$Tot(\bar{b}) = \left(\frac{100-16}{14-0}\right) * \left(\frac{16-15}{15-14}\right) = 6 \tag{5.30}$$

Based on the proposed measure, the probability of each interval is integrated with the ratio of each interval. The probability values of $(\bar{a} > \bar{b}) = (\bar{a} < \bar{b}) = 0.4$, thus

$$\tau(\bar{a} > \bar{b}) = P(\bar{a} > \bar{b}) \times Tot(\bar{a}) \geq P(\bar{a} < \bar{b}) \times Tot(\bar{b}) \tag{5.31}$$

Both $\tau(\bar{a})$ and $\tau(\bar{b})$ are calculated as follow:

$$\tau(\bar{a}) = 0.4 \times 0.8 = 0.32; \quad \tau(\bar{b}) = 0.4 \times 6 = 2.4 \tag{5.32}$$

From (5.32), $\tau(\bar{b}) \geq \tau(\bar{a})$, thus, for this numerical example, interval $\bar{b}$ is preferred to interval $\bar{a}$.

Based on this measure, the higher the values of the measure, the higher the ranking of the interval according to maximization case. Note that, this proposed measure is necessary when two intervals are belong to the group of Case 3 intervals (i.e., nested intervals).

5.4 Summary of chapter

This chapter presented a detailed study regarding the problem of comparing interval that exist in the context of decision-making. The first two sections of this chapter provided a brief discussion on the motivation and background of related work and theories. Then, a proposed extended measure was introduced and demonstrated with numerical examples. The proposed extended measure introduced in this chapter can be used as an alternative way to measure a pair of the interval which has the same midpoint. As explained at the beginning of this chapter, the existing methods which used *acceptability index* proposed in three different papers ([22, 102, 47] are not applicable to be implemented when the intervals have the same midpoint. Additionally, the probabilistic approach proposed in [163] also not applicable as demonstrated by the numerical example in this chapter. Thus, an extended measure was proposed in order to solve this problem. This measure can be used when the

intervals compared have the same midpoint. As the two case studies used in this thesis does not result in nested intervals, hence, the proposed measure is not implemented to compare the intervals. Thus, the final objective of this thesis is achieved.

Chapter 6

Conclusion

This chapter concludes all the contributions from the research presented in Chapters 3, 4 and 5. These three chapters present the five main stages of the study. The research begins with the investigation of the current T1-TOPSIS model in terms of its limitations and issues. Then, this research provides a detailed comparative study between the two main varieties of T1-TOPSIS model, followed by the implementation of both models using the same case study. Based on the issues and limitations identified, a novel extension of Interval Type-2 TOPSIS models was presented and applied to two different case studies. Then, an experiment was developed to evaluate the proposed model. As the proposed model provides interval outputs as a final result, an extended measure for comparing intervals was introduced. Finally, this chapter presents discussion of potential future research that can be explored arising from the findings of this thesis.

6.1 Contributions

This section summarises the key contributions of this thesis.

An independent analysis of various TOPSIS models and how they include fuzzy for uncertainty handling was carried out, including identifying issues and limitations of both the standard TOPSIS and fuzzy TOPSIS models.

The analysis of identifying issues and limitations of the standard TOPSIS model (non-fuzzy) with fuzzy TOPSIS model is to highlight the justifications of the transition. Various extensions of TOPSIS models have been proposed, including fuzzy extensions. One challenge that has risen is that it is not straightforward to differentiate between the multiple variants of TOPSIS existing today. Thus, this thesis provides comparison between the standard T1-TOPSIS model proposed by Chen [1] and the recently proposed model extended by Yuen [13]. The purpose of this comparative study is to show the difference between both models and to provide context for their respective strengths and limitations both in complexity of application and expressiveness of results. A detailed synthetic numeric example and comparison of both models is provided.

The various T1-TOPSIS (fuzzy) models were explored in order to identify the best underlying base framework, including a systematic comparison between Chen's standard T1-TOPSIS model and Yuen's recently extended model.

Based on the resulting comparison, the differences in the steps implemented by these two Fuzzy TOPSIS models were highlighted throughout the worked example. In this numerical example, the synthetic data is used to show the implementation and enable the comparison of both models. Through the example, this study shows the different steps involved to solve the same MCDM problems in both models.

A novel extension model using type-2 fuzzy sets has been proposed to improve uncertainty handling. In this proposed model uncertainty is propagated through each step to produce an interval output.

This thesis developed an alternative to the IT2-model, where the final CC values adopt an interval-valued form. As the first TIT2 that provides an interval-valued output to capture

output uncertainty, the proposed model is designed to reduce the loss of information and to maximize the benefit of using T2FSs. The initial results indicate substantial potential in further development and exploration of the proposed model and similar approaches, highlighting promising future work.

A novel experimental method to explore the relationship between uncertainty in the initial evaluation and the final evaluation has been provided, to show how the uncertainty in the initial knowledge base affects the final evaluation of the decision-making process.

A series of systematically designed experiments showed that increasing uncertainty in the T2 membership functions affects the interval-valued CC outputs. Specifically, the complex behaviour was highlighted in terms of the relationship between the uncertainty levels and the outputs, including non-symmetric and non-linear growth in the CC intervals in response to linearly growing levels of uncertainty.

An extended measure to allow comparison of different interval-valued outputs has been proposed, and integrated into the extended IT2-TOPSIS model.

In practice, the use of interval coefficients in certain contexts may be better than using a crisp number. The novel extended model presented in Chapter 6 produces interval outputs that need to be compared for ranking purposes. This thesis proposed a measure for comparing intervals in the context of the IT2-TOPSIS model and integrates the measure with the proposed IT2 fuzzy TOPSIS model.

6.2 Future Research

The areas that can be further developed are as follows:

i) More research studies could be undertaken which explore the potential of using alternative fuzzy set design methods, such as zSlices based general type-2 fuzzy sets

[164], intuitionistic fuzzy sets [165], to model further forms of uncertainty in MADM models. These study could focus on exploring the effect of varying the different level of uncertainty and showing how the additional information captured in these other design of fuzzy sets provides different outputs compared to the model proposed here. It is important to establish what value the additional effort may offer and in which situations it is warranted, especially in relation to the amount of computational resources and expert time needed.

ii) This thesis presents a simple case study to show how the proposed model works and to perform the sensitivity analysis. In the case study, the output (the final decision) only involves three interval values. However, in a real-world problem, the complexity of the problem may be much higher and can include many criteria and alternatives. Some criteria may be classified further as sub-criteria. Thus, in future, further investigations are needed to evaluate the proposed model when it involves more complex and large practical applications. For example, for problems that may comprise more than three criteria and more than three alternatives.

The increasing complexity of the problem may provide a significant impact on the decision-making process when the proposed model is implemented. The expected outcome will be many resulting intervals which are produced, potentially including very many interval overlapping conditions. The interpretation of the output in terms of final decision-making is unlikely to be only focused on the ranking, but also on the extents of various overlaps in these output intervals. Thus, more exploration and interpretation are likely to be needed to understand such complex results.

iii) Outside the field of fuzzy set theory, interval numbers are also considered a useful representation of data. There are TOPSIS models based on interval data, and further work could be carried out to explore the relationship between interval based TOPSIS, producing interval outputs, and the type-2 fuzzy set based TOPSIS featured in this work.

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Appendix A

Students' Lunchtime Food Outlet Survey

Students' Lunchtime Food Outlet Survey

Introduction

This survey is designed to assess your opinion regarding your selection of a food outlet during lunchtime by examining your decision-making process involved in this task. The purpose of this survey is to understand the different ways people select a food outlet during lunchtime. This survey consists of four parts which are: Part I to gather some basic information about you; Part II to gather your opinions; Part III for you to rate each food outlet with linguistic label; Part IV to gather your rating about each food outlet based on criteria with using an ellipse.

In Jubilee Campus, University of Nottingham, UK, there are three main food outlets (i.e., two main cafeterias and one shop) which sell food from 8am to 5pm every Monday to Friday. The food outlets are:

1. Café Terrazo (located in Exchange Building)
2. Café Aspire (located at Amenities Building)
3. SPAR shop (located next to Café Terrazo in Exchange Building).

Part I

In this part, we will start by asking a few questions about your general details. Please provide the following information by writing down the answer or tick the appropriate box.

1. Age:

☐	18-25	☐	26-30	☐	31-35	☐	35-40	☐	41-45	☐	45 onwards

2. Gender: ☐ Male ☐ Female

3. Nationality: _____

4. ☐ Postgraduate Student ☐ Undergraduate Student

5. How many days a week do you do each?

   Having lunch from one of the campus food outlets. _____

   I bring my lunch. _____

   I do not eat lunch. _____

6. How often do you visit one of these food outlets (Terrazo, Aspire or SPAR) during lunchtime?

☐	Never
☐	When meeting someone
☐	About once or twice a month
☐	About once a week
☐	Several times a week
☐	Everyday

Part II

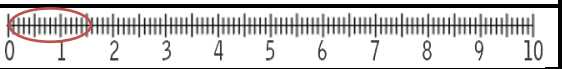
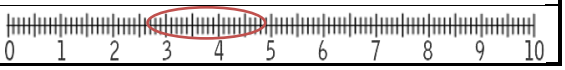
In this survey, three criteria will be used:

1. Average price of food
2. Speed of service
3. Taste of food

In this part, we want to ask your opinion about each criterion (i.e., average price of food, speed of service and taste of food).

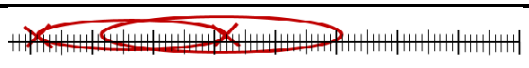
Please give your **opinion of what the range of values** best represent the **labels**. Please use an ellipse to indicate uncertainty / variability in your answer. The wider the ellipse the higher the uncertainty / variability in your answer. Table 1 shows an example of how a participant might express her / his opinion by using an ellipse to present the value.

Table 1*: Example of responses*

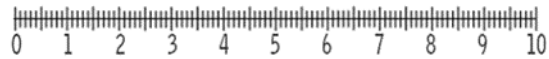
Question	Label	Example of responses (as a range)
How sweet is the cake?	Not sweet at all	
	Not sweet	

If you make a mistake and wish to change your opinion, make a cross at each end of the ellipse you would like to discard as shown in Table 2.

Table 2: A corrected response

Question	Label	Example of responses (as a range)
How sweet is the cake?	Not Sweet at all	

For each label, please provide your assessment of the appropriate value in each criterion.

Criteria	Label	Range associated with the label
Average price of lunch (For each row, select ranges between £0 to £10)	Very Low	**£** 
	Low	**£** 
	Medium Low	**£** 
	Fair	**£** 
	Medium High	**£** 
	High	**£** 
	Very High	**£** 
Speed of service (For each row, select ranges between 0 min to 10 mins)	Very Slow	**mins** 
	Slow	**mins** 
	Medium Slow	**mins** 
	Fair	**mins** 
	Medium Fast	**mins** 
	Fast	**mins** 
	Very Fast	**mins** 
Taste of food (For each row, select ranges between level 1 to level 10 where 0 = negative value, 10 = positive value)	Very Bad	**level** 
	Bad	**level** 
	Medium Bad	**level** 
	Fair	**level** 
	Medium Good	**level** 
	Good	**level** 
	Very Good	**level** 

Part III

Please answer all questions by using an ellipse.

1. How important are the following criteria (i.e., speed of service, average price of food and taste of food) to you when choosing a food outlet for lunch? Please draw an ellipse to show your answer within the range of *Unimportant* (0) to *Extremely Important* (10) criteria.

Criteria	Range associated with the importance of the criteria
Speed of service	Unimportant  Extremely Important
Average price of food	Unimportant  Extremely Important
Taste of food	Unimportant  Extremely Important

2. Please describe TWO (2) other criteria (if any) other than those listed in Question 1, which you would consider when choosing a food outlet.

Please TICK on specific rating that you think is the most represented by each food outlet with respect to each criterion based on the scale you have provided in Part II.

1. Café Terrazo

Criterion 1: Price of the Food							
How you rate the price of the food in term of good value of money?	Very Low ◯	Low ◯	Medium Low ◯	Fair ◯	Medium High ◯	High ◯	Very High ◯
How you rate the price of the food in term of competitive? (as opposed to competitor)	Very Low ◯	Low ◯	Medium Low ◯	Fair ◯	Medium High ◯	High ◯	Very High ◯
How you rate this food outlet in term of the frequency giving the discount on the price of food?	Very Low ◯	Low ◯	Medium Low ◯	Fair ◯	Medium High ◯	High ◯	Very High ◯
How you rate the price of the food shown in the menu in term of correctness?	Very Low ◯	Low ◯	Medium Low ◯	Fair ◯	Medium High ◯	High ◯	Very High ◯
How you rate the bill in this food outlet in term of erroneous?	Very Low ◯	Low ◯	Medium Low ◯	Fair ◯	Medium High ◯	High ◯	Very High ◯

Criterion 2: Speed of Service							
How you rate a speed of service when placing an order at this food outlet?	Very Slow ◯	Slow ◯	Medium Slow ◯	Fair ◯	Medium Fast ◯	Fast ◯	Very Fast ◯
How you rate a speed of service in term of food preparation (i.e. cooking, etc.,)?	Very Slow ◯	Slow ◯	Medium Slow ◯	Fair ◯	Medium Fast ◯	Fast ◯	Very Fast ◯
How you rate a speed of service when the food is serving?	Very Slow ◯	Slow ◯	Medium Slow ◯	Fair ◯	Medium Fast ◯	Fast ◯	Very Fast ◯
How you rate a speed of service when you request a food bill?	Very Slow ◯	Slow ◯	Medium Slow ◯	Fair ◯	Medium Fast ◯	Fast ◯	Very Fast ◯
How you rate a speed of service when you need assisting to get additional things (i.e. serviette, cutlery, etc.) from this food outlet?	Very Slow ◯	Slow ◯	Medium Slow ◯	Fair ◯	Medium Fast ◯	Fast ◯	Very Fast ◯

Criterion 3: Taste of Food							
How you rate the taste of food in this food outlet?	Very Bad ◯	Bad ◯	Medium Bad ◯	Fair ◯	Medium Good ◯	Good ◯	Very Good ◯
How you rate the food taste when the food is served cold (e.g. sandwiches, panini etc.) in this food outlet?	Very Bad ◯	Bad ◯	Medium Bad ◯	Fair ◯	Medium Good ◯	Good ◯	Very Good ◯
How you rate the food taste when ordering the pre-packed food?	Very Bad ◯	Bad ◯	Medium Bad ◯	Fair ◯	Medium Good ◯	Good ◯	Very Good ◯
How you rate the food taste when too many customers in this food outlet at the same time?	Very Bad ◯	Bad ◯	Medium Bad ◯	Fair ◯	Medium Good ◯	Good ◯	Very Good ◯
How you rate the food taste when different staff is preparing/serving the food?	Very Bad ◯	Bad ◯	Medium Bad ◯	Fair ◯	Medium Good ◯	Good ◯	Very Good ◯

2. Café Aspire

Criterion 1: Price of the Food							
How you rate the price of the food in term of good value of money?	Very Low ◯	Low ◯	Medium Low ◯	Fair ◯	Medium High ◯	High ◯	Very High ◯
How you rate the price of the food in term of competitive? (as opposed to competitor)	Very Low ◯	Low ◯	Medium Low ◯	Fair ◯	Medium High ◯	High ◯	Very High ◯
How you rate this food outlet in term of the frequency giving the discount on the price of food?	Very Low ◯	Low ◯	Medium Low ◯	Fair ◯	Medium High ◯	High ◯	Very High ◯
How you rate the price of the food shown in the menu in term of correctness?	Very Low ◯	Low ◯	Medium Low ◯	Fair ◯	Medium High ◯	High ◯	Very High ◯
How you rate the bill in this food outlet in term of erroneous?	Very Low ◯	Low ◯	Medium Low ◯	Fair ◯	Medium High ◯	High ◯	Very High ◯

Criterion 2: Speed of Service							
How you rate a speed of service when placing an order at this food outlet?	Very Slow ◯	Slow ◯	Medium Slow ◯	Fair ◯	Medium Fast ◯	Fast ◯	Very Fast ◯
How you rate a speed of service in term of food preparation (i.e. cooking, etc.,)?	Very Slow ◯	Slow ◯	Medium Slow ◯	Fair ◯	Medium Fast ◯	Fast ◯	Very Fast ◯
How you rate a speed of service when the food is serving?	Very Slow ◯	Slow ◯	Medium Slow ◯	Fair ◯	Medium Fast ◯	Fast ◯	Very Fast ◯
How you rate a speed of service when you request a food bill?	Very Slow ◯	Slow ◯	Medium Slow ◯	Fair ◯	Medium Fast ◯	Fast ◯	Very Fast ◯
How you rate a speed of service when you need assisting to get additional things (i.e. serviette, cutlery, etc.) from this food outlet?	Very Slow ◯	Slow ◯	Medium Slow ◯	Fair ◯	Medium Fast ◯	Fast ◯	Very Fast ◯

Criterion 3: Taste of Food							
How you rate the taste of food in this food outlet?	Very Bad ◯	Bad ◯	Medium Bad ◯	Fair ◯	Medium Good ◯	Good ◯	Very Good ◯
How you rate the food taste when the food is served cold (e.g. sandwiches, panini etc.) in this food outlet?	Very Bad ◯	Bad ◯	Medium Bad ◯	Fair ◯	Medium Good ◯	Good ◯	Very Good ◯
How you rate the food taste when ordering the pre-packed food?	Very Bad ◯	Bad ◯	Medium Bad ◯	Fair ◯	Medium Good ◯	Good ◯	Very Good ◯
How you rate the food taste when too many customers in this food outlet at the same time?	Very Bad ◯	Bad ◯	Medium Bad ◯	Fair ◯	Medium Good ◯	Good ◯	Very Good ◯
How you rate the food taste when different staff is preparing/serving the food?	Very Bad ◯	Bad ◯	Medium Bad ◯	Fair ◯	Medium Good ◯	Good ◯	Very Good ◯

3. SPAR Shop

Criterion 1: Price of the Food							
How you rate the price of the food in term of good value of money?	Very Low ◯	Low ◯	Medium Low ◯	Fair ◯	Medium High ◯	High ◯	Very High ◯
How you rate the price of the food in term of competitive? (as opposed to competitor)	Very Low ◯	Low ◯	Medium Low ◯	Fair ◯	Medium High ◯	High ◯	Very High ◯
How you rate this food outlet in term of the frequency giving the discount on the price of food?	Very Low ◯	Low ◯	Medium Low ◯	Fair ◯	Medium High ◯	High ◯	Very High ◯
How you rate the price of the food shown in the menu in term of correctness?	Very Low ◯	Low ◯	Medium Low ◯	Fair ◯	Medium High ◯	High ◯	Very High ◯
How you rate the bill in this food outlet in term of erroneous?	Very Low ◯	Low ◯	Medium Low ◯	Fair ◯	Medium High ◯	High ◯	Very High ◯

Criterion 2: Speed of Service							
How you rate a speed of service when placing an order at this food outlet?	Very Slow ◯	Slow ◯	Medium Slow ◯	Fair ◯	Medium Fast ◯	Fast ◯	Very Fast ◯
How you rate a speed of service in term of food preparation (i.e. cooking, etc.,)?	Very Slow ◯	Slow ◯	Medium Slow ◯	Fair ◯	Medium Fast ◯	Fast ◯	Very Fast ◯
How you rate a speed of service when the food is serving?	Very Slow ◯	Slow ◯	Medium Slow ◯	Fair ◯	Medium Fast ◯	Fast ◯	Very Fast ◯
How you rate a speed of service when you request a food bill?	Very Slow ◯	Slow ◯	Medium Slow ◯	Fair ◯	Medium Fast ◯	Fast ◯	Very Fast ◯
How you rate a speed of service when you need assisting to get additional things (i.e. serviette, cutlery, etc.) from this food outlet?	Very Slow ◯	Slow ◯	Medium Slow ◯	Fair ◯	Medium Fast ◯	Fast ◯	Very Fast ◯

Criterion 3: Taste of Food							
How you rate the taste of food in this food outlet?	Very Bad ◯	Bad ◯	Medium Bad ◯	Fair ◯	Medium Good ◯	Good ◯	Very Good ◯
How you rate the food taste when the food is served cold (e.g. sandwiches, panini etc.) in this food outlet?	Very Bad ◯	Bad ◯	Medium Bad ◯	Fair ◯	Medium Good ◯	Good ◯	Very Good ◯
How you rate the food taste when ordering the pre-packed food?	Very Bad ◯	Bad ◯	Medium Bad ◯	Fair ◯	Medium Good ◯	Good ◯	Very Good ◯
How you rate the food taste when too many customers in this food outlet at the same time?	Very Bad ◯	Bad ◯	Medium Bad ◯	Fair ◯	Medium Good ◯	Good ◯	Very Good ◯
How you rate the food taste when different staff is preparing/serving the food?	Very Bad ◯	Bad ◯	Medium Bad ◯	Fair ◯	Medium Good ◯	Good ◯	Very Good ◯

Please review the questions below, answering them using the same ellipse techniques as for Part II.

1. Café Terrazo

Criterion 1: Price of the Food		
How you rate the price of the food in term of good value of money?	VERY LOW	 VERY HIGH
How you rate the price of the food in term of competitive? (as opposed to competitor)	VERY LOW	 VERY HIGH
How you rate this food outlet in term of the frequency giving the discount on the price of food?	VERY LOW	 VERY HIGH
How you rate the price of the food shown in the menu in term of correctness?	VERY LOW	 VERY HIGH
How you rate the bill in this food outlet in term of erroneous?	VERY LOW	 VERY HIGH

Criterion 2: Speed of Service		
How you rate a speed of service when placing an order at this food outlet?	VERY SLOW	 VERY FAST
How you rate a speed of service in term of food preparation (i.e. cooking, etc.,)?	VERY SLOW	 VERY FAST
How you rate a speed of service when the food is serving?	VERY SLOW	 VERY FAST
How you rate a speed of service when you request a food bill?	VERY SLOW	 VERY FAST
How you rate a speed of service when you need assisting to get additional things (i.e. serviette, cutlery, etc.) from this food outlet?	VERY SLOW	 VERY FAST

Criterion 3: Taste of Food		
How you rate the taste of food in this food outlet?	VERY BAD	 VERY GOOD
How you rate the food taste when the food is served cold (e.g. sandwiches, panini etc.) in this food outlet?	VERY BAD	 VERY GOOD
How you rate the food taste when ordering the pre-packed food?	VERY BAD	 VERY GOOD
How you rate the food taste when too many customers in this food outlet at the same time?	VERY BAD	 VERY GOOD
How you rate the food taste when different staff is preparing/serving the food?	VERY BAD	 VERY GOOD

2. Café Aspire

Criterion 1: Price of the Food		
How you rate the price of the food in term of good value of money?	VERY LOW	 VERY HIGH
How you rate the price of the food in term of competitive? (as opposed to competitor)	VERY LOW	 VERY HIGH
How you rate this food outlet in term of the frequency giving the discount on the price of food?	VERY LOW	 VERY HIGH
How you rate the price of the food shown in the menu in term of correctness?	VERY LOW	 VERY HIGH
How you rate the bill in this food outlet in term of erroneous?	VERY LOW	 VERY HIGH

Criterion 2: Speed of Service		
How you rate a speed of service when placing an order at this food outlet?	**VERY SLOW**	0 1 2 3 4 5 6 7 8 9 10 **VERY FAST**
How you rate a speed of service in term of food preparation (i.e. cooking, etc.,)?	**VERY SLOW**	0 1 2 3 4 5 6 7 8 9 10 **VERY FAST**
How you rate a speed of service when the food is serving?	**VERY SLOW**	0 1 2 3 4 5 6 7 8 9 10 **VERY FAST**
How you rate a speed of service when you request a food bill?	**VERY SLOW**	0 1 2 3 4 5 6 7 8 9 10 **VERY FAST**
How you rate a speed of service when you need assisting to get additional things (i.e. serviette, cutlery, etc.) from this food outlet?	**VERY SLOW**	0 1 2 3 4 5 6 7 8 9 10 **VERY FAST**

Criterion 3: Taste of Food		
How you rate the taste of food in this food outlet?	**VERY BAD**	0 1 2 3 4 5 6 7 8 9 10 **VERY GOOD**
How you rate the food taste when the food is served cold (e.g. sandwiches, panini etc.) in this food outlet?	**VERY BAD**	0 1 2 3 4 5 6 7 8 9 10 **VERY GOOD**
How you rate the food taste when ordering the pre-packed food?	**VERY BAD**	0 1 2 3 4 5 6 7 8 9 10 **VERY GOOD**
How you rate the food taste when too many customers in this food outlet at the same time?	**VERY BAD**	0 1 2 3 4 5 6 7 8 9 10 **VERY GOOD**
How you rate the food taste when different staff is preparing/serving the food?	**VERY BAD**	0 1 2 3 4 5 6 7 8 9 10 **VERY GOOD**

3. SPAR Shop

Criterion 1: Price of the Food			
How you rate the price of the food in term of good value of money?	**VERY LOW**		**VERY HIGH**
How you rate the price of the food in term of competitive? (as opposed to competitor)	**VERY LOW**		**VERY HIGH**
How you rate this food outlet in term of the frequency giving the discount on the price of food?	**VERY LOW**		**VERY HIGH**
How you rate the price of the food shown in the menu in term of correctness?	**VERY LOW**		**VERY HIGH**
How you rate the bill in this food outlet in term of erroneous?	**VERY LOW**		**VERY HIGH**

Criterion 2: Speed of Service			
How you rate a speed of service when placing an order at this food outlet?	**VERY SLOW**		**VERY FAST**
How you rate a speed of service in term of food preparation (i.e. cooking, etc.,)?	**VERY SLOW**		**VERY FAST**
How you rate a speed of service when the food is serving?	**VERY SLOW**		**VERY FAST**
How you rate a speed of service when you request a food bill?	**VERY SLOW**		**VERY FAST**
How you rate a speed of service when you need assisting to get additional things (i.e. serviette, cutlery, etc.) from this food outlet?	**VERY SLOW**		**VERY FAST**

Criterion 3: Taste of Food		
How you rate the taste of food in this food outlet?	**VERY BAD**	 **VERY GOOD**
How you rate the food taste when the food is served cold (e.g. sandwiches, panini etc.) in this food outlet?	**VERY BAD**	 **VERY GOOD**
How you rate the food taste when ordering the pre-packed food?	**VERY BAD**	 **VERY GOOD**
How you rate the food taste when too many customers in this food outlet at the same time?	**VERY BAD**	 **VERY GOOD**
How you rate the food taste when different staff is preparing/serving the food?	**VERY BAD**	 **VERY GOOD**

Please return your completed consent form and survey to:
Elissa Madi, Room 38, School of Computer Science, Jubilee Campus, University of Nottingham.

Thank you for your participation, it is very much appreciated.

Appendix B

Case study 2: Ratings of alternatives with respect to each criterion

Appendix B (continued)

Ratings of Cafe A $\mathscr{A}_1$ given by 25 DMs with respect to each criterion

	Criterion 1: Price of food				Criterion 2: Speed of service				Criterion 3: Taste of food				
	How do you rate the price of the food in term of good value of money?	How do you rate the price of the food in term of competition?	How do you rate this food outlet in term of the frequency giving the discount on the price of food?	How do you rate the price of the food shown in the menu in term of correctness?	How do you rate the bill in this food outlet in term of erroneous?	How do you rate a speed of service when placing an order at this food outlet?	How do you rate a speed of service in term of food preparation (i.e. cooking, etc.,)?	How do you rate a speed of service when the food is serving?	How do you rate a speed of service when you request a food bill?	How do you rate a speed of service when you need assisting to get additional things (i.e. serviette, cutlery, etc.) from this food outlet?	How do you rate the taste of food in this food outlet?	How do you rate the food taste when the food is served cold (e.g. sandwiche s, panini etc.) in this food outlet?	How do you rate the food taste when ordering the pre-packed food?	How do you rate the food taste when too many customers in this food outlet at the same time?	How do you rate the food taste when different staff is preparing /serving the food?
Participant 1	H	MH	L	H	VL	F	MF	MF	F	F	F	MB	F	F	G
Participant 2	ML	F	VL	H	VL	S	F	MS	F	F	MG	F	MG	MG	F
Participant 3	ML	F	L	H	VL	F	F	MS	VF	VF	MG	F	MG	MG	MG
Participant 4	MH	H	VL	MH	ML	F	MF	F	VF	F	MG	F	MG	MG	MG
Participant 5	MH	H	VL	H	L	MF	MF	F	VF	VF	F	MG	F	MG	MG
Participant 6	MH	H	ML	H	L	FAST	MF	MF	VF	F	G	MG	MG	G	G
Participant 7	FAIR	MH	ML	H	L	FAST	MF	MF	VF	VF	FAIR	FAIR	FAIR	MB	MB
Participant 8	MH	H	H	VH	L	MF	MF	MF	F	VF	MG	G	G	G	G
Participant 9	MH	FAIR	L	H	L	FAST	MF	MF	VF	VF	MG	FAIR	FAIR	MG	FAIR
Participant 10	ML	H	MH	VH	VL	VF	F	MF	VF	VF	FAIR	MG	MG	MG	FAIR
Participant 11	H	MH	ML	H	L	FAIR	MS	FAIR	FAST	FAST	MG	FAIR	FAIR	FAIR	FAIR
Participant 12	MH	H	VL	H	VL	MF	FAIR	FAIR	FAST	VF	MG	FAIR	FAIR	MG	MG
Participant 13	MH	H	ML	H	L	MF	FAIR	FAIR	FAST	VF	MG	G	MG	MG	MG
Participant 14	MH	H	FAIR	H	VL	FAST	MF	MF	MF	VF	MG	MG	F	MG	MG
Participant 15	MH	H	FAIR	H	VL	MF	FAIR	FAIR	VF	VF	MG	FAIR	MG	MG	MG
Participant 16	H	MH	L	H	L	MF	FAIR	MF	FAST	FAST	MG	FAIR	FAIR	MG	MG
Participant 17	MH	FAIR	MH	H	L	FAST	MF	FAST	VF	VF	MG	FAIR	FAIR	FAIR	FAIR
Participant 18	H	MH	L	L	L	MF	MF	MF	VF	VF	MG	FAIR	FAIR	MG	MG
Participant 19	MH	MH	VL	VH	VL	MF	FAIR	FAIR	VF	VF	MG	MG	MG	MG	MG
Participant 20	MH	H	ML	H	VL	MF	MF	MF	FAIR	FAIR	G	MG	MG	MG	MG
Participant 21	FAIR	FAIR	ML	H	L	MF	FAIR	MF	VF	VF	MG	FAIR	FAIR	MG	MG
Participant 22	MH	MH	H	H	L	MF	MF	FAST	FAST	VF	FAIR	FAIR	MG	MG	FAIR
Participant 23	MH	FAIR	MH	FAIR	FAIR	MF	MS	FAIR	MS	MF	MG	G	MG	MG	G
Participant 24	MH	FAIR	H	VH	L	MF	FAST	FAST	VF	VF	MG	FAIR	FAIR	FAIR	MG
Participant 25	MH	MH	FAIR	H	VL	MF	MF	MF	FAST	VF	MG	FAIR	FAIR	MG	MG

Ratings of Cafe B $\mathscr{A}_2$ given by 25 DMs with respect to each criterion

	Criterion 1: Price of food					Criterion 2: Speed of service				Criterion 3: Taste of food				
	How do you rate the price of the food in term of good value of money?	How do you rate the price of the food in term of competition?	How do you rate this food outlet in term of the frequency giving the discount on the price of food?	How do you rate the price of the food shown in the menu in term of correctness?	How do you rate the bill in this food outlet in term of erroneous?	How do you rate a speed of service when placing an order at this food outlet?	How do you rate a speed of service in term of food preparation (i.e. cooking, etc.,)?	How do you rate a speed of service when the food is serving?	How do you rate a speed of service when you request a food bill?	How do you rate a speed of service when you need assisting to get additional things (i.e. serviette, cutlery, etc.) from this food outlet?	How do you rate the taste of food in this food outlet?	How do you rate the food taste when the food is served cold (e.g. sandwiches, panini etc.) in this food outlet?	How do you rate the food taste when ordering the pre-packed food?	How do you rate the food taste when too many customers in this food outlet at the same time?	How do you rate the food taste when different staff is preparing/ serving the food?
Participant 1	H	MH	L	VH	VL	MS	S	S	MF	MF	G	MB	MG	F	MB
Participant 2	F	MH	VL	MH	L	VS	S	S	VF	FR	G	F	G	F	F
Participant 3	MH	H	VL	H	L	MS	S	S	F	F	G	MG	MG	F	F
Participant 4	MH	H	L	MH	VL	MS	MS	S	MF	F	G	MG	F	MG	G
Participant 5	MH	H	VL	H	VL	F	FAIR	FAIR	F	F	G	MG	F	F	G
Participant 6	MH	H	ML	FAIR	VL	MF	FAIR	MS	VF	FAST	G	FAIR	FAIR	MG	MG
Participant 7	FAIR	ML	VL	H	VL	FAIR	FAIR	MS	VF	VF	VG	FAIR	FAIR	MG	MG
Participant 8	H	VH	VL	H	L	MS	MS	FAIR	VF	VF	G	MG	MG	G	G
Participant 9	H	VH	VL	VH	VL	MS	MS	S	VF	VF	VG	G	G	VG	VG
Participant 10	MH	VH	VL	VH	VL	MF	FAIR	FAIR	VF	VF	G	MG	MG	G	G
Participant 11	VH	H	VL	VH	VL	MS	FAIR	MS	VF	VF	MG	FAIR	FAIR	FAIR	FAIR
Participant 12	H	VH	VL	VH	VL	FAIR	MS	MS	VF	MS	G	MG	G	MG	MG
Participant 13	H	VH	VL	VH	VL	FAIR	MS	MS	MS	VF	FAIR	MG	F	F	F
Participant 14	H	VH	VL	H	VL	MF	FAIR	FAIR	FAST	VF	G	FAIR	FAIR	MG	MG
Participant 15	H	VH	VL	VH	VL	MS	FAIR	MS	VF	VF	G	MG	MG	MG	MG
Participant 16	VH	H	ML	H	VL	FAIR	FAIR	FAIR	FAST	VF	G	MG	FAIR	FAIR	G
Participant 17	H	H	H	VH	L	MF	FAIR	FAIR	FAST	VF	VG	G	MG	G	MG
Participant 18	H	H	ML	H	L	MF	FAIR	FAIR	VF	VF	VG	G	G	VG	VG
Participant 19	H	VH	VL	VH	L	FAIR	MS	MS	FAST	VF	VG	G	G	VG	G
Participant 20	H	VH	VL	VH	VL	MF	FAIR	FAIR	VF	FAST	G	G	MG	MG	MG
Participant 21	H	H	ML	VH	VL	FAIR	MS	MS	VF	FAST	VG	G	G	VG	G
Participant 22	H	VH	ML	H	VL	MS	MS	FAIR	FAST	VF	VG	G	G	G	G
Participant 23	H	MH	ML	H	VL	MS	FAIR	FAIR	FAIR	FAST	G	MG	MG	MG	VG
Participant 24	H	H	MH	VH	L	FAIR	FAIR	FAIR	FAST	VF	G	MG	MG	MG	MG
Participant 25	MH	MH	F	H	VL	MF	FAIR	FAIR	FAIR	FAST	MG	FAIR	FAIR	MG	MG

Appendix B (continued)

Ratings of Shop C $\mathscr{A}_3$ given by 25 DMs with respect to each criterion

	Criterion 1: Price of food					Criterion 2: Speed of service					Criterion 3: Taste of food				
	How do you rate the price of the food in term of good value of money?	How do you rate the price of the food in term of competition?	How do you rate this food outlet in term of the frequency giving the discount on the price of food?	How do you rate the price of the food shown in the menu in term of correctness?	How do you rate the bill in this food outlet in term of erroneous?	How do you rate a speed of service when placing an order at this food outlet?	How do you rate a speed of service in term of food preparation (i.e. cooking, etc.,)?	How do you rate a speed of service when the food is serving?	How do you rate a speed of service when you request a food bill?	How do you rate a speed of service when you need assisting to get additional things (i.e. serviette, cutlery, etc.) from this food outlet?	How do you rate the taste of food in this food outlet?	How do you rate the food taste when the food is served cold (e.g. sandwiches, paninies etc.) in this food outlet?	How do you rate the food taste when ordering the pre-packed food?	How do you rate the food taste when too many customers in this food outlet at the same time?	How do you rate the food taste when different staff is preparing /serving the food?
Participant 1	H	VH	L	H	VL	F	VF	VF	VF	VF	F	F	MG	G	F
Participant 2	MH	H	ML	MH	F	MF	F	VF	VF	VF	F	MG	G	MG	G
Participant 3	H	MH	MH	H	L	F	VF	VF	VF	VF	MG	MG	G	G	G
Participant 4	H	MH	H	H	L	F	F	VF	F	VF	G	MG	MG	MG	MG
Participant 5	L	MH	MH	H	L	FAST	VF	VF	VF	VF	MG	FAIR	MG	MG	G
Participant 6	MH	H	MH	H	VL	F	F	VF	F	VF	G	MG	FAIR	MG	G
Participant 7	FAIR	H	H	VH	L	F	F	VF	VF	VF	F	MB	FAIR	FAIR	FAIR
Participant 8	MH	MH	FAIR	MH	ML	FAST	FAST	MF	VF	VF	FAIR	MG	FAIR	FAIR	FAIR
Participant 9	ML	H	MH	VH	VL	VF	FAST	MF	VF	VF	FAIR	MG	MG	MG	FAIR
Participant 10	FAIR	MH	H	VH	L	FAST	VF	FAST	FAST	VF	G	VG	G	G	G
Participant 11	ML	H	MH	H	VL	VF	VF	VF	VF	VF	FAIR	MB	FAIR	FAIR	FAIR
Participant 12	FAIR	MH	FAIR	H	VL	VF	VF	VF	VF	VF	FAIR	MG	MG	MG	FAIR
Participant 13	ML	MH	VL	FAIR	L	FAST	MF	FAST	VF	VF	MB	FAIR	FAIR	MB	MB
Participant 14	ML	F	ML	H	F	FAST	VF	FAST	VF	VF	FAIR	MB	MB	FAIR	FAIR
Participant 15	MH	H	VH	VH	VL	VF	FAST	FAST	VF	VF	FAIR	FAIR	MB	FAIR	FAIR
Participant 16	H	MH	MH	VH	VL	VF	MF	MF	VF	FAST	FAIR	MB	MB	G	MG
Participant 17	FAIR	H	L	H	VL	FAST	MF	MF	FAST	MF	G	MG	MG	MG	MG
Participant 18	FAIR	MH	H	VH	L	FAST	MF	MF	MF	VF	MG	FAIR	FAIR	MG	MG
Participant 19	FAIR	FAIR	ML	FAIR	ML	FAST	MF	FAST	MF	MF	MG	FAIR	MG	FAIR	MG
Participant 20	MH	F	H	H	VL	VF	VF	VF	VF	VF	MG	FAIR	FAIR	MG	MG
Participant 21	H	VH	VH	VH	VL	VF	VF	VF	VF	VF	G	MG	MG	MG	MG
Participant 22	MH	MH	H	H	VL	FAST	FAST	MF	FAST	VF	MG	G	G	G	G
Participant 23	H	MH	FAIR	H	VL	FAST	FAST	FAST	VF	VF	FAIR	FAIR	FAIR	MG	FAIR
Participant 24	H	MH	MH	H	L	MF	MF	MF	FAST	FAST	MG	FAIR	FAIR	MG	MG
Participant 25	FAIR	ML	H	H	L	FAST	MF	FAST	FAST	FAST	G	MG	MG	MG	MG