At the edges of infinity and the finite

Charting a path to UV completion from number theory
to quantum fields and strings

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Thesis submitted to the University of Nottingham for the degree of
Doctor of Philosophy, July 2025

Statement of originality

Unless otherwise specified, I declare that the work presented in this thesis is my own. All sources have been appropriately acknowledged and referenced. No part of this thesis has been submitted previously for a degree or other qualification at this or any other university.

Details of collaborations and publications:

The work presented in this thesis is based on research carried out independently and in collaboration with others. It is largely based on the material published in the following papers:

[1] Antonio Padilla and Robert G. C. Smith. *Smoothed asymptotics: from number theory to QFT*. In Phys. Rev. D. 110. (2024) 025010 [arXiv:2401.10981].

[2] Antonio Padilla and Robert G. C. Smith. *Gauge invariance and generalised η regularisation*. In Phys. Rev. D 111 12 (2025) 125013 [arXiv:2412.12261].

[3] Robert G. C. Smith and Murdock Grewar. *η regularisation and the functional measure*. (2025). [arXiv:2505.01290].

These papers are [1–3] in the bibliography, respectively. In [3], which serves as the source material for Chapter 7, the idea of regularising the path integral measure originated from my co-author Murdock Grewar. This includes the general idea of the measure weighting and its preliminary regularisation, the iota function, and the insight that divergences may be discarded in the moduli space of theories since one may quotient out constant complex factors. The work presented builds on these ideas, developed along the lines of the broader theory of η regularisation, its number theoretic foundations, and other related concepts. Chapter 10 is based on ongoing research, and thus also on not yet published material. Finally, the paper [4] was published in collaboration with Yang Liu, Antonio Padilla, and Paul M. Saffin. However, the material presented therein has not been included in this thesis.

Abstract

This thesis explores the foundational presence of infinity at the nexus of theoretical physics and mathematics, developing a framework that connects number theory, quantum field theory, and string theory. Central to this investigation is the introduction and application of η regularisation, a novel technique inspired by smoothed asymptotics and the summation methods of Ramanujan. Starting with a rigorous treatment of divergent series, the thesis constructs a mathematical apparatus capable of extracting finite, physically meaningful values from divergent expressions.

Building on this, the η framework is extended to perturbative quantum field theory, offering deeper insight into a general theory of regularisation based on the preservation of gauge invariance. The approach is further generalised to the path integral formalism, extending η regularisation to operator language.

Tying together previous discussion, the culmination of this work is formulated in the context of ultraviolet finiteness in string theory, where the modular structure of perturbative string amplitudes at one-loop and their tropical limits are shown to induce a natural renormalisation of field theory integrals. We then discuss the hopeful idea of deriving classes of string theory induced regulators as multiplicative weights in the descendent low-energy field theory integration.

Overall, the bottom-up and top-down approach of this thesis suggests that the mechanisms by which quantum gravity might tame infinities may also, in turn, be fundamentally associated with number-theoretic phenomena. By tracing a conceptual arc from the infinite to the finite (and vice versa)—from divergent series to ultraviolet-complete theories—a fresh perspective on regularisation, modularity, and the unification of mathematics and physics is offered.

Acknowledgements

There are many people without whom this journey would not have been the same, and I wish to thank them sincerely.

First and foremost, I am profoundly grateful to Tony Padilla. From the time I showed up at your door as a first-year undergraduate, talking excitedly about string theory, to the completion of this thesis, you have shown unwavering belief in me. You allowed me the freedom to explore ideas across physics and mathematics without boundary or limitation—to pursue foundational questions, seek new mathematics, and chase nascent projects. Although my penchant for getting immersed in big, or long-term problems, occasionally came at the cost of short-term gains, I am thankful for your enduring patience and understanding. In my time studying under you, my love for physics only grew stronger. More than anything, I greatly appreciate the trust you placed in my ability to find my way, often on my own, and even, or especially, in the moments I had travelled far out into the technical wilderness of an obscure idea. While only a fraction of the work I began over the years appears in this thesis, if I eventually complete and publish any number of those projects, it will be thanks in large part to the time spent studying under you, and the intellectual freedom you afforded me to chase connections and pursue ideas. That meant everything.

Thank you also for your continuous professional mentorship. Though I may always be a hexagonal peg in the square frame of contemporary academic convention, you gave me a foundation on which to build. You're not only an exceptional supervisor, but also a good human being.

Secondly, to my wife and partner, Beth: I owe you my deepest thanks for your unwavering love, patience, belief, and support.

Finally, I would like to thank Cesc Cunillera for many discussions about strings and physics. I would also like to thank Paul Saffin, Pete Millington, Ed Copeland, Bobby

Acharya, Dimitri Skliros, and Benjamin Muntz, for helpful discussions over the years. Ed, I will remember fondly your stories about Joe Polchinski, and for our occasional discussions about strings. The short time I got to spend speaking with Bobby and Dimitri about stringy matters was one of the highlights of my time as a PhD student.

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1

Introduction: Sailing the boundless seas

How can it be that mathematics, being after all a product of human thought independent of experience, is so admirably appropriate to the objects of reality?

— ***Albert Einstein, 'Geometry and Experience' (1921)***

1.1 The infinite

Since the dawn of civilisation, humanity has sought to decode the cosmos through the universal language of mathematics, beginning with the elemental concept: *number*. Over millennia, this quest has evolved in step with the expanding reach of human insight. From the geometric alignments of Babylonian ziggurats [5] and the celestial calculations of Greek astronomers [6], to the abstract symmetries underlying particle physics and the frontier pursuits of quantum gravity—the history of science traces the ever-sharpening reflection of the universe in the mirror of mathematical thought.

As Galileo famously declared, nature is written in the language of mathematics [7]. And as Wigner later pondered in awe, the unreasonable effectiveness of mathematics in describing the natural world—or, more recently, the unreasonable effectiveness of physics in mathematics [8]—remains one of the great mysteries and triumphs of human understanding. The fundamental laws of physics can be captured in beautiful mathematical formulae. And yet, nature has also become a powerful source

of inspiration for even the most abstract realms of pure mathematics. These twin miracles of insight have become guiding lights in humanity's voyage across the deep, infinite sea of Platonic forms [9]. The further we voyage into these boundless waters in search of new islands of quantitative insight, the reflections of Galileo and Wigner resonate with renewed poetic force—in the uncanny way abstract mathematical structures appear, and reappear, with precision in the workings of the universe. It is as though, to reveal nature's intrinsic order, is to unveil the astonishing numerical patterns woven deep into the fabric of reality.

At the frontier of this intellectual voyage lies string theory—one of the most ambitious and conceptually rich developments in modern physics, and a cornerstone of 21st-century efforts to unify quantum mechanics with general relativity [10, 11]. Perhaps more than any theory before it, string theory embodies the profound union of mathematics and physics [8, 12]. What may have once seemed like two distinct realms—the elegant abstractions of pure mathematical thought, and the tangible mysteries of fundamental physical science—converge through subtle connection of diverse ideas and concepts forged over centuries. Here, at the frontier of fundamental physics, one sees that the interplay of mathematics and physics is not just well-motivated technical necessity—it is a source of aesthetic and intellectual beauty, in which the boundaries between physical law and mathematical structure dissolve, revealing an increasingly unified vision of reality. The vast sea of connections underlying this vision has matured to such an extent that one cannot now do physics without mathematics, nor mathematics without physics[1]. Together they are the ship with which humanity may use to sail forth deeper into the unknown.

Among the most enigmatic and persistent companions in this voyage, at least as the present thesis is concerned, is the concept of *infinity*. In the sea forms, infinity and the infinite are the metaphorical currents that guide us toward ever increasing physical insight. Mathematics is, in many respects, built on the infinite, with infinities present across foundational domains [14–18]. Cantor's theory of transfinite numbers introduced a hierarchy of infinities, distinguishing between countable and uncountable sets [19, 20]. Russell's paradox, which arises from considering self-referential infinite sets—such as the set of all sets that do not contain themselves—exposed the fragility

[1]To paraphrase Cumrun Vafa, who conveyed this wonderful observation quite eloquently in a recent interview [13]

of naïve set theory [21]. Generalising beyond sets, the concept of infinity now also plays a crucial role at the foundations of mathematics through modern developments of infinity category theory [22]. Here, even at one of the highest points in abstract mathematical thought, there still remains a common language with physics [23]. And in analysis, of which this work makes great use, the infinite permeates almost every structure, whether mathematical or physical: from the real number line to continuous manifolds, topologies, and metric spaces.

The philosophy behind this work, and many of its technical results, are based on the view that the language of physics *is* mathematics [8, 12], of which infinity has become a necessary tool in describing the laws governing nature [15–17, 24]. From Zeno's paradoxes and the infinitesimal calculus of Newton and Leibniz, to the divergences haunting perturbative quantum field theory (QFT) and the ultraviolet limits of string theory, infinity is both indispensable and elusive. Without it, many central structures—limits, derivatives, integrals, Hilbert spaces, and even the tower of string states—would collapse. But the infinite is also not always physically admissible. Its presence can be a source of physical ambiguity, and can often signal a breakdown in our understanding.

As physicists, how do we reconcile these observations and at the same time make sense of infinity as an indispensable concept? In this thesis, we seek to answer this question from a diversity of interrelated perspectives: from the humble foundations of elementary number theory to the mathematical heights of string theory amplitudes.

1.1.1 Non-technical motivations

Paradoxically, from the infinite it is often possible to extract the finite—a process as philosophically profound as it is mathematically delicate. To understand how infinity operates as an indispensable concept in physics, we must ask whether there exists a consistent logical framework to make sense of how infinite expressions can yield finite, and even physically meaningful, results. Among the many mathematical encounters with infinity, number theory provides some of the most striking and historically resonant examples of such phenomena [25], with various formal frameworks developed to explain or interpret rigorously the finite values emergent from infinite and divergent expressions. For this reason, we are motivated in this thesis to begin from the bottom-

up by considering a number-theoretic view of infinity—particularly through the study of infinite series and sums.

One such example is the Riemann zeta function $\zeta(s)$, which, while divergent at certain values, can also be assigned finite, meaningful values through analytic continuation [25–29]. In the more general study of divergent series [25], such as the sums of integer powers that are considered extensively in a later chapter, techniques like Cesàro and Abel summation similarly provide ways to assign consistent finite values to otherwise divergent quantities, except these methods do not require the use of analytic continuation. In fact, there are many different summation methods, each with their own strengths and weaknesses. The theory of summability aims to provide mathematical foundation to explain the formal manipulations of divergent series, and their logical consistency, such that meaningful sums may be rigorously defined.

Historically, the logic of summation of divergent series was ambiguous, if not obscure [30]. Over time, many efforts aided in uncovering logical justification and consistency of various summation methods [25, 31, 32]. For instance, generalised theories of summation—most notably the axiomatic frameworks of Hardy [25] and Knopp [33]—relate and classify various methods of summing divergent series, revealing important properties at the foundation of a theory of summability. These axiomatic approaches provide a partial unification, establishing a framework to compare and understand summation techniques such as Cesàro, Abel, and others. More recently, advanced frameworks like resurgence theory [34, 35] have sought to unify divergent series and trans-series, using algebraic and differential structures. Here, the concept of Borel summation serves as a key mathematical tool. But the interesting question of a fully generalised and unifying framework remains elusive.

Even though most divergent series are of number theoretic and analytic concern, among them are examples of fundamental physical importance. In fact, this goes for many different classes of number theoretic objects, including partition functions and, as we'll later discuss, modular forms—all of which deal with infinity in one way or another. Most importantly, there is a clear analogy, and at times a direct relation, between the study of summation methods and the theory of summation, and modern techniques in physics. Connecting these two pictures, we are led to ask in broadly general terms: in what way can number theoretic insights be expanded to calculations

of divergent quantities in physics? Is there a deeper framework through which these values, and the process of their extraction, might be understood uniquely in relation to physical observables?

Few figures in mathematics exemplify intuition about these sorts of questions more strikingly than Ramanujan (e.g., see [25, 36–39]). His conception of infinity was profoundly unorthodox: he treated infinite and divergent series not as formal or asymptotic objects, but as operationally meaningful expressions. Among his most well-known and controversial assertions is the identity [25, 37, 40]

$$1 + 2 + 3 + 4 + \cdots = -\frac{1}{12},$$

which is not valid in the usual sense of convergent series. It can, however, be understood in the context of the Riemann zeta function [25, 40]. Far from being a mere mathematical curiosity representative of some obscure island in the sea of abstract forms, Ramanujan's identity—like other similarly absurd identities for divergent series—can appear in physical calculations. To give one example, the assigned value $-1/12$ to the sum of natural numbers has a well known relation to the calculation of the Casimir effect [41, 42]—a measurable physical phenomenon—illustrating how nature may know something about the unique values attributed to divergent expressions.[2] Taking seriously such seemingly absurd expressions, an unavoidable question has to do with how one may understand rigorously their logical justification and consistency.

Furthermore, although Ramanujan lacked access to rigorous formalism, his method of assigning values to divergent series, now partly systematised as *Ramanujan summation* [25, 36–38], can be seen as anticipating modern techniques in quantum field theory and string theory. He also made profound contributions to the theory of modular forms and q-series, central structures in string theory [10, 43]. His investigations into the modular discriminant $\Delta(q)$, mock theta functions, and identities involving infinite q-series, such as [38]

$$\sum_{n=1}^{\infty} \frac{nq^n}{1-q^n},$$

[2]Interestingly, one see how standard summation methods that use traditional partial sums can fail to account for this experimentally observed value [25].

uncovered rich transformation properties under modular inversion, presaging the later use of modular invariance essential to the ultraviolet (UV) finiteness of string theory amplitudes [10]. Additionally, his analysis of the partition function $p(n)$—developed with Hardy via the circle method [25]—revealed exponential growth laws that parallel the statistical counting of microstates in string-theoretic black hole entropy [44]. In his long list of contributions, although often heuristic, Ramanujan's methods and insights have been found in a variety of contexts to be deeply meaningful—philosophically and scientifically—in the sense of engaging with infinity not as a boundary concept but as immediate chartable edges of the finite.

Ramanujan's intuition prompts a deeper line of questioning: if, in the first instance, we want to make sense of infinity as an indispensable concept, then how can we rigorously describe, as physicists, the emergence of the finite from the infinite? How are we to understand *generally* the unique finite values that emerge from divergent expressions? Is it possible to speak physically to Ramanujan's insights—that the emergence of unique finite values from divergent expressions might somehow resonate with the laws of nature? In other words, might it be that, in taming the infinite, we are doing more than performing formal tricks—that the assignment of unique finite values reflects something real?

Given that physicists have long sought to reconcile the formal utility of infinity with its physical inadmissibility, these questions are historically well motivated. Indeed, the development of sophisticated modern techniques in fundamental physics—namely, *regularisation* and *renormalisation* (see e.g., [41, 45–48])—were engineered precisely to tame divergent quantities while preserving physical consistency and predictive power. As already alluded, such sophisticated techniques have deep roots in number theory. But the meaning of a theory of regularisation remains somewhat opaque and obscure from a physical point of view. Additionally, the idea of developing a general theory of regularisation—and, for that matter, renormalisation—analogous to the development of a generalised theory of summation, presents a number of intriguing questions on its own that tie into the broader question of the infinite as a useful concept.

In many ways, such a space of enquiry is intimately tied into the development of QFT. Perturbative QFT is well established as a microscopic theory of the fundamental interactions, enjoying predictive power and stunning experimental success [49],

and yet the presence of divergences is well documented. As Dyson noted [50], the perturbative expansion used in quantum electrodynamics does not converge, even after renormalisation. Although this does not hinder its predictive power when the expansion parameter is small, it does raise concerns around the concepts upon which the theory is built. Many notable advances have been made in understanding the nature of such divergences at the foundations of perturbative QFT. For example, under the heading of resurgence theory [34, 35, 51], the usual perturbative expansion is replaced with a trans-series expansion, including non-perturbative instanton contributions. Connecting perturbative and non-perturbative expansions via Borel summation, resurgence frames the divergent series expansions in perturbative QFT and string theory in a more global light.

However, even at finite order, we encounter divergences from the integration over momenta running through loops. These problems were first identified for quantum electrodynamics by Oppenheimer [52] and later solved using the method of renormalisation developed by Tomonoga [53], Schwinger [54], and Feynman [55], and extended to non-abelian gauge theories by 't Hooft and Veltman [56]. As for general relativity, this works well as a perturbative QFT at scales below the Planck scale, but cannot be extended to arbitrarily high energies as the theory is known to be perturbatively non-renormalisable. Said another way, for general relativity, the number of counter-terms required to renormalise the divergences arising from loops is itself divergent!

Mathematically rigorous and axiomatic formulations of perturbative QFT have progressed significantly in recent decades, aiding our understanding on the nature of UV divergences in particular, with the view that all regularisation in QFT is really just an effort in obtaining sensible results for products of distributions [57]. But, with this progress and insight, not much has been said fundamentally about regularisation besides it being a useful, if not altogether *ad hoc*, tool.

With these considerations in mind, much of this thesis is motivated to develop a generalised theory of regularisation from the bottom-up, taking lessons directly from elementary and analytic number theory. In particular, we ask if there are clever ways to regularise divergent series that eliminate the divergences altogether, and, if so, whether they can be connected, via QFT, to the softening of scattering amplitudes at high energies in a fundamental microscopic theory of nature. Additionally, we also

seek to begin developing a top-down approach to a generalised theory of regularisation. In particular, we are interested in exploring whether the underlying structures of UV finite theories of quantum gravity might be connected, in some way, with the analogous structures of a generalised theory of regularisation in low-energy field theory.

Adopting Aristotle's philosophy in rejecting actual infinity [58, 59], at least in a natural context, one might hope that the correct theory of quantum gravity coupled to matter will in fact be free of all such divergences found in perturbative QFT. If this is the case, it may point towards a preferred way of regularising the divergences in the low energy field theory. String theory is certainly an ideal candidate in this regard, representing an important approach to unifying the standard model of particle physics and general relativity [10]. Exhibiting a wonderful synthesis of diverse mathematical and physical frameworks—including number theory, differential and algebraic geometry, topology, among others—with the structure of fundamental interactions, it provides a fundamental framework in which certain divergences—particularly in the UV regime—are naturally regulated: most notably via exponential damping in the modular invariance of the worldsheet theory [10, 43].

Furthermore, many technical approaches and research programmes serve as important sites of motivation in formulating our top-down approach. For instance, it is notable that in the Gross-Mende regime [60, 61] where the high energy, fixed angle behaviour of string scattering amplitudes is considered, the sum over all Riemann surfaces is dominated by a saddle point, and one finds direct evidence of exponential damping from the infinite tower of string states. Interestingly, with a view to understanding this from a simple particle theory perspective, Abel and others [62, 63] have constructed UV complete particle theories based on the Schwinger representation and worldline formalism. This includes particle theories with a tower of worldline internal degrees of freedom that mimic the softening behaviour of string amplitudes in the UV, from Gross-Mende saddle points to modular invariance [63]. In a more general setting, string field theory has revealed geometrically how such exponential damping comes from adding stubs to loop diagrams [64], which can also be translated to the point particles of QFT [65]. Finally, as another in a long list of examples, using the tropical amplitude formalism [66], it is observed directly how string theory provides UV completion of field theory amplitudes, particularly in how "stringy effects"

appear as weighted vertices in Feynman-like graphs upon taking the field theory limit $\tau_2 \to \infty$. Here, in the one-loop case, $\tau = \tau_1 + i\tau_2$ is the complex structure modulus of the genus one worldsheet, and τ_2 controls the length of the loop, playing the role of a (dimensionless) Schwinger proper time.

All of these insights, and others, beg an even deeper question about infinity than those already raised: according to what mechanisms does quantum gravity tame the infinite, if at all, and how does this relate to any possible UV completion of field theory? What insights might string theory provide in this regard? Put another way, in taking a top-down approach, do the key features of a regularisation inherited from string theory inform, or make contact with, a bottom-up number theoretic viewpoint? These questions would seem reasonable, since string theory amplitudes, often expressed through modular invariant partition functions and other modular structures common to analytic number theory, already suggest a deep interplay between quantum gravity, algebraic geometry, and a foundational theory of regularisation (see e.g., [10, 64–70]). The hopeful idea is that, in combining both a bottom-up and top-down viewpoint, a potentially unifying picture might begin to emerge: one in which the extraction of finite results from infinite expressions is not merely a calculational artifact, but potentially a deep structural feature of physical reality—across scales, from low-energy field theory to high energy regimes near enough the Planck scale.

1.2 Results and outline of this work

The work presented in this thesis is separated into three parts. Part I and Part II are based on the three papers [1–3], with the results presented in Part III, particularly Chapter 10, the content of ongoing research and a forthcoming paper.

In **Part I**, we begin with a brief review of the history and development of number theory, focusing particularly on the introduction of infinity as a concept, and then on infinite sums and divergent series in Chapter 2. This provides useful context for the discussion in Chapter 3, where we review basic definitions in number theory to be used throughout this thesis. Discussion includes formal definitions of infinite series, divergent series, and finally Ramanujan's summation method in relation to the Euler-Maclaurin summation formula.

We conclude **Part I** with Chapter 4, where we develop a novel regularisation prescription based on the method smoothed asymptotics originally proposed by Tao [40]. Termed η regularisation, we explore a rigorous approach to the regularisation of divergent number theoretic sums in which the Euler-Maclaurin summation formula serves as an essential tool. This helps us link discrete sums to integrals with Bernoulli number correction terms, highlighting the effectiveness in capturing the asymptotic behaviours of divergent series. We then develop a theory of "enhanced regulators", which are specialised smoothing functions designed to completely eliminate power-law divergences. An explicit example of such a regulator is $\eta(x) = e^{-x}\cos(x)$, which shows that the finite contribution to the divergent sum over the naturals $\sum_{n=1}^{\infty} n$ is in fact $-1/12$.

This rigorous exploration of divergent series and their regularisation establishes a strong mathematical foundation, which we then apply to divergent integrals in QFT in **Part II** of this thesis. Beginning with Chapter 5, a new ultraviolet regularisation scheme for loop integrals in QFT is introduced. We show how this general regularisation reveals interesting connections between the elimination of divergences in divergent series of powers and the preservation of gauge invariance in the regularisation of perturbative loop integrals. Remarkably, in the case of regularising the series of natural numbers so that it converges to minus one-twelfth, it is shown how this directly inspires a regularisation for non-abelian gauge theories coupled to Dirac fermions that preserves the Ward identity for the vacuum polarisation tensor and other higher point functions. To conclude, possible connection to Schwinger proper time integrals is also discussed.

In Chapter 6 we begin to develop more deeply a general (bottom-up) theory of regularisation, in which we generalise η regularisation in order to develop a framework for systematically studying regularisation of loops in perturbative QFT. This allows us to "solve" a set of gauge consistency conditions for families of gauge invariant regularisation schemes. We recover several known examples such as dimensional and denominator regularisations, as well as some more general solutions. We also offer the first insight into an η-inspired renormalisation, before studying anomalies in chiral theories in order to carefully describe how our formalism should be properly implemented.

Motivated by the analysis in the previous chapters, in Chapter 7 we consider Fujikawa's path integral formulation of the chiral anomaly and develop a generalised framework for systematically defining a regularised functional measure. This construction extends the η regularisation scheme to operator language, making the connection between spectral asymmetry and measure transformation fully explicit. Before recovering Fujikawa's expression for the chiral anomaly from the regularised measure, we explore the deeper number theoretic structure underlying the ill-defined spectral sum associated with the anomaly, interpreting it through the lens of smoothed asymptotics. Our approach in this chapter unifies two complementary perspectives: the analytic regularisation of Fujikawa and the topological characterisation given by the Atiyah–Singer index theorem. We further investigate how the measure transforms under changes to the regularisation scale and derive a function $\iota_E(\Lambda)$ that encodes this dependence, showing how its Mellin moments govern the appearance of divergences. Finally, we comment on the conceptual relationship between the regularised measure, η regularisation for loop integrals, and the generalised Schwinger proper-time formalism, with a particular focus on the two-dimensional Schwinger model.

In **Part III** of this thesis, we switch gears to the study of string theory amplitudes, with particular focus on the topic of UV completion. In Chapter 8, we present a cursory review of basic concepts in string theory, focusing efforts primarily in bosonic string theory. In passing, we comment on the possibility of a σ-invariant regularisation of the zero-point energy inspired by η regularisation. We then extend the discussion to string scattering amplitudes, building up the standard ideas that give definition to the stringy path integral and its relation to conformal field theory, before reviewing the general structure of tree-level and one-loop amplitudes for the closed bosonic string. Following this, in Chapter 9, we briefly review the modular structure of string amplitudes. This chapter serves to introduce key concepts pertaining to modular invariance and connections to analytic number theory, UV/IR mixing, UV completeness, and the field theory limit.

Finally, in Chapter 10, we study one-loop closed string amplitudes in the field theory limit. Upon presenting a general introduction, surveying various known methods for extracting field theory amplitudes from string theory, we turn to a review of the exciting and actively developing tropical amplitudes framework. Following the important work

by Tourkine [66], we discuss how the tropical amplitudes programme makes rigorous the connection between $\alpha' \to 0$ limit of string theory (correspondingly, the geometric limit $\tau_2 \to \infty$) and the recent field of mathematics called tropical geometry. Highlighting the way in which Feynman-like field theory graphs directly descend from string theory, emphasis is placed on the remarkable observation that string theory directly induces a renormalisation prescription (via the emergence of weighted vertices) for UV divergent field theory amplitudes. After reviewing the canonical example of the tropicalisation of the one-loop four-graviton Type II superstring amplitude, we turn our attention to calculating the tropicalisation of the genus one torus vacuum amplitude for the non-tachyonic, non-supersymmetric superstring. In the greater interest of deriving string theory regulator functions—motivated by the bottom-up analysis of η regularisation—we conclude this chapter by examining the challenges in deriving classes of string theory induced regulator functions. Complimenting the bottom-up approach discussed in *Part I* and *Part II* of this thesis, we offer several remarks based on ongoing research as to how multiplicative regulators directly descendent from string theory might appear in the *partial tropical limit*, including possible geometric interpretations in terms of dressed or smeared propagators, and also compare against insights from spectral and modular integral approaches (among others).

In Chapter 11, in addition to a number of concluding comments, several remarks are offered outlining future directions.

PART I

Number theory

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2

The conceptual evolution of infinity

One of the things that makes the history of science so endlessly fascinating is to follow the slow education of our species in the sort of beauty to expect in nature.

— ***Steven Weinberg, In Dreams of a Final Theory: The Scientist's Search for the Ultimate Laws of Nature (1992)***

Before discussing the main results of *Part I*, presented in Chapter 4, we introduce key ideas from number theory that underpin our analysis. We begin in this chapter with a brief historical review of the development of infinity as a concept. The history surveyed here is not simply a prelude, but a lens through which we can better understand the mathematical machinery developed in the subsequent work.

2.1 A brief history

As ancient civilisations first grappled with the regularities of the natural world, foundational ideas emerged that revealed the symbiotic relationship between mathematics and physical science (see, e.g., [30] and [71]). Since pre-Socratic times, a universal feature of this relationship has been a reliance on the concept of the infinite.

The story of the infinite in mathematics begins with roots in the finite. Dating back to at least 1800 BCE, Babylonian astronomers developed a sophisticated sexagesimal (base-60) numeral system and displayed advanced proficiency in arithmetic, geometry, algebra, and even solutions to quadratic and cubic equations [72]. Their mathematical

tools, though rooted in finite calculation, were instrumental in astronomical prediction and timekeeping. These efforts were recorded in detailed ephemerides, which included numerical tables used to predict eclipses and chart planetary movements. Among the oldest surviving documents from this period is the clay tablet Plimpton 322 [72], which lists what we now call *Pythagorean triples*—integer solutions (a, b, c) to the equation $a^2 + b^2 = c^2$. This early focus on integer relationships is a direct ancestor of modern number theory, which remains centrally concerned with the structure and properties of the integers.

While the Babylonians worked within a finite mathematical framework, they used rational numbers—an infinite system, as understood from a modern mathematical perspective. Furthermore, by around 2000 BCE, ancient Egyptian scribes were employing fractions—albeit restricted to sums of unit fractions (e.g., $1/2 + 1/4$) rather than arbitrary rational numbers. Although the ancient Egyptians did not conceive of such mathematics in the development of physics as a science, they applied many physical principles in engineering and daily life using these mathematical tools. From the concept of the seked, used to define the slope of a pyramid involving unit fractions, to weights and balances, time keeping, and even hydraulics and irrigation.

Although infinity was not conceived explicitly, from a modern lens it is easy to reinterpret even these early constructions as implicitly invoking infinite structures. For instance, a modern rational number such as $3/8$ can be formally understood as an equivalence class of integer pairs [9]

$$(3, 8),\ (-3, -8),\ (6, 16),\ (-6, -16),\ (9, 24),\ (-9, -24),\ \ldots. \tag{2.1}$$

Here, any pair (a, b) with $b \neq 0$ is equivalent to $(3, 8)$ if $a/b = 3/8$, capturing the notion that fractions represent classes of equivalent ratios. Thus, even such a simple object encodes an infinite set—a perspective unavailable to ancient mathematicians but essential to modern analysis.

It is in this context where, generalising the mathematical thought of the ancient Babylonians, the ancient Greeks gave rise to the formal problem of infinity. Thinkers such as Thales of Miletus [73] and Pythagoras of Samos [74] not only introduced the notion of mathematical proof but also fused practical calculation with metaphysical speculation [9, 75]. The Pythagoreans, in particular, saw numerical ratios as the basis

of harmony in music and geometry, treating numbers as the fabric of physical reality. But it was in the 6th century BCE when Anaximander [73–76], a tutor of Pythagoras, conjured up *apeiron*—an indefinite and limitless source from which everything is born and to which everything will return [16]. Later, the Eleatic philosophers, most notably Zeno of Elea, posed paradoxes that directly challenged the coherence of infinity [77, 78]. Zeno's *Dichotomy Paradox* and *Achilles and the Tortoise* confront the problem of motion in a universe infinitely divisible in space and time. These paradoxes stimulated centuries of reflection. In modern times, they have been reinterpreted and resolved within frameworks such as calculus, measure theory, and even quantum physics.

Plato [79] expanded this view through his *Theory of Forms*, positing that ideal mathematical entities underlie the observed world. In the dialogue *Timaeus*, he described the cosmos as constructed from geometric forms and mathematical ratios, foreshadowing the modern view of mathematics as the underlying language of nature.

However, the ancient Greeks did not fully embrace the infinite, with Aristotle accepting only the potential for infinity, rejecting it in actuality [59]. Despite this hesitancy to fully embrace the infinite as an actual mathematical entity, the ancient Greeks nevertheless developed powerful techniques that *conceptually* relied on infinite processes. The method of exhaustion, used by Eudoxus and formalised by Euclid and later Archimedes, approximated areas and volumes by inscribing sequences of polygons—a clear precursor to integral calculus. Archimedes' summation of infinite geometric series, such as

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2, \tag{2.2}$$

illustrates an early understanding of convergence, even if not formulated in modern rigorous terms. These methods showed that infinite processes could yield finite, useful results—a revolutionary idea still today.

2.1.1 Modern developments: calculus and the Basel problem

The 17th century marked a turning point [80, 81], with the independent development of calculus by Newton and Leibniz. Both made systematic use of infinite series in expressing functions and solving differential equations. Though concepts like

convergence were treated informally, the methods were extraordinarily successful in physics and mathematics alike.

These developments coincided with investigations into asymptotic expansions by various mathematicians, providing a broader framework for new systematic perspectives on infinity as a mathematical concept [82]. Most notably, in the late 17th and early 18th centuries, the Bernoulli brothers [83], among others, studied the behaviour of infinite series and laid significant groundwork for a more rigorous theory. Yet infinite sums and divergent series remained controversial—"the invention of the devil," as Abel [82] famously described them in the 19th century—treated more as curiosities than trustworthy mathematical objects. Around the same time, Gauss, perhaps the greatest mathematician of the modern era, warned that "the use of an infinite quantity as a completed one [...] is never permissible in mathematics. The infinite is only a *façon de parler*, where one is really speaking of limits where certain ratios come as close as one likes, while others are allowed to grow without restriction" [84].

Despite these warnings, an arguably pivotal moment in the study of infinite sums was when Euler solved one of the most intriguing mathematical puzzles of the time: *the Basel problem* [85]. First formulated by Leibniz and the Bernoulli brothers [86], the Basel problem concerned the infinite series whose terms are the reciprocal squares of the natural numbers

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + ... \tag{2.3}$$

In a display of his genius intuition, Euler broke with the view of his contemporaries and decided upon a radically new approach [85, 87, 88]. Using calculus to relate the discrete sum of an arbitrary function $\sum_n f(n)$ to the antiderivative of the form $\int dx f(x)$, Euler refit the problem into a new context and was able to prove that $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$.

In solving the Basel problem, Euler [85] would conclude a centuries long search for closed expressions for sums of numerical powers $\sum_{n=1}^{N} n^k \approx \int_0^N x^k \, dx$. (In fact, some examples of sums of numerical powers date as far back as the 6th century BCE, around the time of Greek intervention, as discussed in the next section). Since the sum and the integral provide first approximations to each other, in which, from elementary

calculus—understanding that the sum can be interpreted as the total area of rectangles forming a step graph along some curve, while the integral, when evaluated between limits, can be interpreted as the area under the same curve—Euler could imagine a bridge connecting continuous and discrete mathematics [86, 87]. This relation of the first approximation of a discrete sum and an integral proved an important step in what, today, we know as the Euler-Maclaurin summation formula [89–92].[1] (A fuller treatment of the classical Euler-Maclaurin summation formula is discussed in the next chapter. A generalisation of this formula is then presented in Chapter 5).

Despite Euler's many successes and triumphs working with infinite and divergent series, it was not until the 19th century that Augustin-Louis Cauchy [93] provided a rigorous definition of convergence and divergence, along with precise foundations for calculus in terms of limits, continuity, and real number structure. These developments culminated in the creation of real analysis and eventually in complex analysis, where the theory of infinite series and their analytic properties become essential.

2.2 Concluding remarks

In this chapter we briefly discussed the evolution of the concept of infinity. From the early insights of ancient civilisations—rooted in empirical needs and shaped by the philosophical rigour of the Greeks—the infinite developed into a central object of mathematical analysis. Whether through the method of exhaustion, the summation of geometric series, or the paradoxes of motion, the infinite emerged not only as a philosophical curiosity but also as a foundational tool for understanding continuity, change, and number. This indispensable mathematical tool became essential in the problem of reconciling discrete and continuous structures in Euler's solution of the Basel problem. This theme lies at the heart of the Euler-Maclaurin summation formula, which offers a powerful bridge between arithmetic and analysis. As we shall see in the following chapters, such ideas are not merely of historical interest but are central to modern number theory and its surprising applications in theoretical physics.

[1] For a historical review of the development of Euler's key ideas in this area, including his motivation to study the problem of interpolation and the connection to expressing the sum of a series using an integral, see [87, 88].

3

Introduction to number theory: A physicists view

If we compared the Bernoullis to the Bach family, then Leonhard Euler is unquestionably the Mozart of mathematics.

— ***Eli Maor, The Story of a Number (1994)***

In this chapter, we formalise several of the historical ideas introduced earlier, focusing on foundational concepts from modern number theory [25, 31, 32]: infinite series, convergence, divergence, and, finally, regularisation. We begin with a brief review of notation and conventions used throughout this thesis, followed by a review of infinite series and partial summation. We then discuss divergent series, laying the groundwork for a review of the Riemann zeta function. We conclude with a technical discussion on the Euler-Maclaurin summation formula and Ramanujan summation, which serve as important instruments for the results reported in Chapter 4.

3.1 Notation and conventions

In this work, we are primarily concerned with the use of elementary, analytic, and geometric number theory. We denote $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$ for the sets of integers, rational numbers, real numbers, and complex numbers, respectively. Conventionally, $\mathbb{N} = \{n \in \mathbb{Z} | n \geq 1\}$ is the set of natural numbers (i.e. positive integers), while $\mathbb{N}_0 = \{n \in \mathbb{Z} | n \geq 0\}$ is the set of non-negative integers. Unless otherwise specified, throughout this work

any reference to the natural numbers will mean $\mathbb{N}_0$.

In this thesis, we follow standard summation conventions, generally consistent with [25]. In most cases, we write sums as $\sum_{n\leq b} f(n)$, implicitly assuming $n \in \mathbb{N}_0$. When the upper limit depends on a large parameter N, we often consider asymptotic expressions as $N \to \infty$. In such cases, we use Big-O notation, where $O(N)$ denotes the asymptotic growth of a function in N, i.e., $f(N) = O(N)$ as $N \to \infty$.

We also frequently make use of classes of smooth functions [94]. In analysis, one understands the smoothness of a function as a property measured by the number of continuous derivatives. Given a function of class C^k, which is a function of smoothness of at least k, one defines C^k to be the class that has kth continuous derivatives in its domain. The class C^0 consists of continuous functions, whereas the class C^1 consists of all continuously differentiable functions. In general, we will frequently consider functions of class C^∞, which are infinitely differentiable and smooth to all orders.

3.2 Infinite series and partial sums

Number theory, broadly concerned with the properties and structures of integers, encompasses a variety of subfields, including algebraic, geometric, and analytic domains. A central theme in elementary number theory—and a key ingredient in the physical models discussed later—is the study of infinite series. An infinite series is an expression of the form

$$\sum_{n=0}^{\infty} a_n, \tag{3.1}$$

where $a_n \in \mathbb{C}$ for all $n \in \mathbb{N}_0$. Associated to each series is the sequence of partial sums $(s_N)_{N\in\mathbb{N}_0}$, where

$$s_N = \sum_{n=0}^{N} a_n. \tag{3.2}$$

According to Cauchy's [93] definition, an infinite series is said to be convergent if the sequence (s_N) of partial sums converges to some finite limit

$$\lim_{N\to\infty} s_N = s, \tag{3.3}$$

otherwise, the series is divergent. If the series is not convergent in the Cauchy sense, then it will generally be considered as one of two types of divergent series [18, 25]: (i) a series that grows in absolute value without limit, or (ii) a series that is bounded but whose sequence of partial sums does not approximate any specific value.

3.3 Divergent series

Let us begin with the following well known expressions for the finite sums of powers:

$$\sum_{n=1}^{N} n = \frac{1}{2}N + \frac{1}{2}N^2, \tag{3.4}$$

$$\sum_{n=1}^{N} n^2 = \frac{1}{6}N + \frac{1}{2}N^2 + \frac{1}{3}N^3, \tag{3.5}$$

$$\sum_{n=1}^{N} n^3 = \frac{1}{4}N^2 + \frac{1}{2}N^3 + \frac{1}{4}N^4. \tag{3.6}$$

These relationships date back more than two millennia. The first of them, corresponding to the sum of natural numbers, can be traced to the Pythagorean school in the 6th century BCE [86, 95]. Archimedes of Syracuse (circa 287-212 BCE), considered the greatest mathematician of antiquity, discovered the second relationship for a sum of the squares. The sum of cubes can be found in the work of Nicomachus of Gerasa (circa 60 -120 CE), along with the remarkable theorem that bears his name

$$\sum_{n=1}^{N} n^3 = \left(\sum_{n=1}^{N} n\right)^2.$$

The expressions for the three finite sums (3.4) to (3.6) are special cases of Faulhaber's formula [96]

$$\sum_{n=1}^{N} n^z = \frac{1}{z+1}\sum_{n=0}^{z}\binom{z+1}{n} B_n N^{z-n+1}, \tag{3.7}$$

where z is a positive integer and B_n are the Bernoulli numbers (following the convention

$B_1 = 1/2$), which are defined by the following recursion relation

$$\sum_{n=0}^{z} \binom{z+1}{n} B_n = z+1, \qquad B_0 = 1, \tag{3.8}$$

or, equivalently, from the exponential generating function

$$\frac{t}{1-e^{-t}} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}. \tag{3.9}$$

In the limit where we take $N \to \infty$, Faulhaber's formula will obviously break down. This is because the infinite series $s(z) = \sum_{n=1}^{\infty} n^z$ is known to be divergent in the Cauchy sense whenever $\Re(z) \geq -1$.

At high school we are taught to think of an infinite series $\sum_{n=1}^{\infty} a_n$ as the limit of its partial sums, $\lim_{N\to\infty} \sum_{n=1}^{N} a_n$. Partial sums are favoured because they allow us to perform standard arithmetic without issue. Of course, the method works well for convergent series, less so for divergent series. With the latter, the task is to develop a summation method that shares the most important properties of partial sums, but also allows the result to be generalised to infinity without giving a divergent answer. As mentioned previously, examples of conventional summation methods [25, 81, 97] include Cesàro summation. Here one computes the sequence of partial sums $s_N = \sum_{n=1}^{N} a_n$ and finds the limiting value of their average $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} s_n$. Another example is Abel summation, where one computes $\lim_{t\to 1-} \sum_{n=1}^{\infty} a_n t^n = s$, used in conjunction with Tauberian theorems [25, 98, 99] to demonstrate the convergence of a series. It is furthermore said that the series is Abel summable to s such that

$$\sum_{n=0}^{\infty} a_n = s \quad (A). \tag{3.10}$$

In his well known axiomatic study of summation methods, including the examples given above, Hardy [25] argues that any summation method should satisfy three properties: regularity, linearity, and stability. Regularity states that a summation method yields the known results for convergent series obtained using partial sums. For linearity, we require that $\sum_{n=1}^{\infty} \lambda a_n + \mu b_n = \lambda \sum_{n=1}^{\infty} a_n + \mu \sum_{n=1}^{\infty} b_n$. The third property of stability, namely $\sum_{n=1}^{\infty} a_n = a_1 + \sum_{n=1}^{\infty} a_{n+1}$, is considered less crucial

and known to fail in some important cases, including Ramanujan summation [100].

The real question is whether any meaning can be extracted from the finite results obtained by these methods. Although the formal treatment of divergent series remains an open question, a commitment to truncated partial sums ignores the fact that divergences can lead to inconsistent results in physical situations. Of course, such divergences may just be a reflection of a breakdown in the relevant mathematical description. However, in some cases, physically consistent results can be obtained by using summation methods that yield finite as opposed to infinite answers. Indeed, as highlighted in the introduction to this thesis, the infinite series of natural numbers appears in computations of the Casimir force [42], for example. As we will see in Chapter 8, it also appears in the critical dimension of bosonic string theory [10]. In both cases, it is necessarily interpreted as the seemingly absurd formula $\sum_{n=1}^{\infty} n = -1/12$ [25, 37].

3.4 Riemann zeta function

Although $\sum_{n=1}^{\infty} n = -1/12$ is often associated with Ramanujan [37], the result is best derived using zeta function regularisation.

When $\Re(z) > 1$, the Riemann zeta function is defined by the convergent series

$$\zeta(z) = \sum_{n=1}^{\infty} n^{-z}.$$

Of course, when $\Re(z) \leq 1$ the series is no longer convergent. However, we can analytically continue $\zeta(z)$ into $\Re(z) < 1$ using the following formula

$$\zeta(z) = 2^z \pi^{z-1} \sin\left(\frac{\pi z}{2}\right) \Gamma(1-z)\zeta(1-z), \tag{3.11}$$

which expresses the zeta function as a functional equation that includes the gamma function.

We may now identify the series given above, $S(z) = \sum_{n=1}^{\infty} n^z$ with $\zeta(-z)$, even when it is divergent, $\Re(z) \geq -1$. In particular, we obtain $S(1) = \sum_{n=1}^{\infty} n \overset{\zeta}{=} \zeta(-1) = -1/12$. More generally, to sum a divergent series $\sum_{n=1}^{\infty} a_n$ using zeta function regularisation, we identify a convergent series $\sum_{n=1}^{\infty} a_n^{-z}$ for z in some complex domain, then analytically

continue the result to $z = -1$.

3.5 Ramanujan and the Euler-Maclaurin summation formula

The majority of issues with divergent series come with the transition from partial sums to infinity, as notably exposed by Ramanujan [25, 37]. Although, as mentioned in the introduction, it does not appear that Ramanujan fully understood the validity of his asymptotic expansions, he was able to extract some remarkable results. In particular, he developed a way to study divergent series based upon the Euler-Maclaurin summation formula, often employing it in creative ways. The Euler-Maclaurin formula also plays an important role in the recent development of smoothed asymptotics as beautifully elucidated by Tao [40], which makes contact with some of Ramanujan's astonishing identities. Indeed, in Chapter 4, using Tao's methodology, we will begin to more rigorously understand how a divergent series of positive numbers could ever be identified with a finite negative number in a meaningful way, as Ramanujan intuited long ago. In the meantime, in this section we discuss Ramanujan's summation method and the Euler-Maclaurin formula.

In Chapter 2, we saw that Euler's solution to the Basel problem was an exercise from elementary calculus, which today is owed to how a sum and an integral provide first approximations to one other. In particular, the sum can be interpreted as the total area of rectangles forming a step graph along some curve, while the integral, evaluated between end points, can be interpreted as the area under the same curve. More formally, the Euler-Maclaurin summation formula provides an estimation of the sum $\sum_{n=a}^{b} f(n)$ in terms of the integral $\int_a^b \, dx \, f(x)$ and the derivatives of the function $f(x)$. As an estimate of how much the trapezoid rule fails, an error term is also included given by an integral involving Bernoulli polynomials. The formula can be written as [100]

$$\sum_{n=a}^{b} f(n) = \int_a^b dx f(x) + \frac{f(b)+f(a)}{2} + \sum_{k=2}^{m} \frac{B_k}{k!}(f^{(k-1)}(b) - f^{(k-1)}(a)) + R_m, \quad (3.12)$$

where a, b are integers such that $a \leq b$ and the integer $m \geq 2$. We denote the kth derivative $f^{(k)}(x) = \frac{d^k}{dx^k} f$. The sum on the right-hand side contains the non-vanishing Bernoulli numbers B_{2n} (recall that $B_{2n+1} = 0$ for $n \geq 1$). As these numbers grow asymptotically for large k as $B_{2n} \sim (-1)^{n+1} 4n^{2n} (\pi e)^{-2n} \sqrt{\pi n}$, this sum often diverges as $m \to \infty$, and is best treated as an asymptotic series. The corresponding error or remainder is given by

$$R_m = (-1)^{m+1} \int_a^b dx \ \frac{b_m(x)}{m!} f^{(m)}(x), \tag{3.13}$$

where $b_m(x)$ is the periodic function $B_m(x - \lfloor x \rfloor)$ given in terms of the Bernoulli polynomials. The latter are defined by the generating function

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}. \tag{3.14}$$

As emphasized in [40], the level of approximation offered by (3.12) is heavily dependent on the asymptotic behaviour of R_m. It is often the case that the expansion remains valid even after taking the limits $a \to -\infty$ and/or $b \to \infty$. Generally, the integral on the right-hand side can be evaluated in closed form in terms of elementary functions, even though the sum on the left-hand side cannot. In the best case scenario, all the terms in the asymptotic series can be expressed in terms of elementary functions.

The idea behind Ramanujan's summation method is to use the Euler-Maclaurin formula in the following asymptotic form

$$\sum_{n=1}^{N} f(n) \sim \int_a^N dx f(x) + \frac{f(N)}{2} + \sum_{k=2}^{\infty} \frac{B_k}{k!} f^{(k-1)}(N) + C(f, a), \tag{3.15}$$

where $C(f, a)$ is the so-called "constant of the series". This will depend on the choice of lower limit on the integral, first noted by Hardy [25]. We will return to this ambiguity in a moment. First, note that this constant is identified with the corresponding infinite series in the limit where $N \to \infty$ such that

$$\sum_{n=1}^{\infty} f(n) \overset{(\mathcal{R},a)}{=} C(f, a). \tag{3.16}$$

The $\mathcal{R}$ corresponds to Ramanujan, whereas the a reflects the ambiguity in choosing

the limits on the integral. To recover $C(f,a)$, we first define a sequence of constants $C_m(f,a)$ via the Euler-Maclaurin formula. In particular, we write

$$\begin{aligned}\sum_{n=1}^{N} f(n) &= \int_a^N dx f(x) \\ &+ \frac{f(N)}{2} + \sum_{k=2}^{m} \frac{B_k}{k!} f^{(2k-1)}(N) \\ &+ (-1)^m \int_N^\infty dx\, \frac{b_m(x)}{(2m)!} f^{(m)}(x) + C_m(f,a),\end{aligned} \tag{3.17}$$

so that

$$\begin{aligned}C_m(f,a) &= -\int_a^1 dx f(x) + \frac{f(1)}{2} - \sum_{k=2}^{m} \frac{B_k}{k!} f^{(k-1)}(1) \\ &+ (-1)^{m+1} \int_1^\infty dx\, \frac{b_m(x)}{m!} f^{(m)}(x).\end{aligned} \tag{3.18}$$

Assuming that $f \in \mathcal{C}^\infty$ and the integral above is convergent for sufficiently large values of m, one is able to show that $C_m(f,a)$ becomes independent of m at large m, settling down to $C(f,a)$. For $f(n) = n^z$, where z is a positive integer, it is sufficient to take any $m \geq z+1$ and evaluate the constant of the series, giving

$$\sum_{n=1}^{\infty} n^z \overset{(\mathcal{R},a)}{=} \frac{1}{z+1}(a^{z+1}-1) + \frac{1}{2} - \frac{1}{z+1} \sum_{k=2}^{z+1} \binom{z+1}{k} B_k = \frac{a^{z+1}}{z+1} - \frac{B_{z+1}}{z+1} \tag{3.19}$$

where we have used the recursion relation (3.8) for the Bernoulli numbers. Given that $\zeta(-z) = -\frac{B_{z+1}}{z+1}$ for positive integers, z, we see the connection of this result to zeta function regularisation in the limit where $a = 0$.

To better understand the role played by the arbitrary real number a in general, we first note that

$$C(f,b) - C(f,a) = \int_a^b dx f(x). \tag{3.20}$$

Furthermore, when the series $\sum_{n=1}^{\infty} f(n)$ is convergent, it is easy to see from the

asymptotic formula (3.15) that [100]

$$C(f,1) = \sum_{n=1}^{\infty} f(n) - \int_{1}^{\infty} dx f(x). \tag{3.21}$$

It immediately follows that $\sum_{n=1}^{\infty} f(n) = C(f,\infty)$, demonstrating the fact we should choose $a = \infty$ for the case of convergent series. For a divergent series with $f(n) = \sum_{z=0}^{s} c_z n^z$ a polynomial of degree s, it is clear that we must take $a = 0$ in order to match the result obtained via analytic continuation. When we develop the method of smooth asymptotics advocated by Tao [40] in Chapter 4, we will see how $C(f,0)$ also picks out the finite term that is independent of the choice of cut-off.

3.6 Concluding remarks

In this chapter, we have introduced core analytic tools from number theory that will underpin much of the technical developments to follow. By reviewing infinite and divergent series, zeta function regularisation, and the Euler-Maclaurin summation formula—alongside Ramanujan's summation method—we have quickly built a bridge between classical number theory and the more advanced technical developments presented in the next chapter. Indeed, the concepts reviewed thus far not only offer conceptual insight and historical context into long-standing mathematical puzzles. They also provide the necessary machinery for the development of a mathematical framework equipped with the ability to generalise how we view divergent series and integrals.

4

Smoothed asymptotics

[...] why are asymptotic theorems so much simpler than finite approximations? Infinity does not correspond to the popular image. It is a guiding light, a star that draws us to finite ways of thinking. God knows why.

— ***Stanslaw Ulam***

In this chapter, we begin the implementation of key ideas concerning the treatment of divergent series. Building on the foundational concepts introduced earlier, we develop a novel and highly general regularisation scheme based on the method of *smoothed asymptotics* [40]. This approach extends well beyond the limitations of classical partial summation and offers a more flexible analytic tool set. As we will demonstrate, this method yields asymptotic proofs of several remarkable identities for the sums of integer powers (3.4)–(3.6) attributed to Ramanujan, thereby further illuminating the deep structure underlying these divergent expressions.

4.1 Introduction

A recent and interesting approach to divergent series is the method of *smoothed asymptotics*, elegantly discussed by Tao [40]. For monomial series, this makes use of the Euler-Maclaurin summation formula and as such is closely related to Ramanujan's methods [25, 37].

The importance of smoothed asymptotics in understanding divergent series is best illustrated with an example. Consider again the infinite series of natural numbers,

$\sum_{n=1}^{\infty} n$, which, as already discussed, was famously assigned the seemingly absurd value of $-1/12$ by Ramanujan. The high school technique for evaluating infinite series is to take the limit of partial sums, which for the sum of natural numbers yields, $\sum_{n=1}^{N} n = \frac{1}{2}N(N+1)$. There is no sign of $-1/12$ in this expression, leading many to scoff at Ramanujan's claim. However, this is simply an artefact of the discontinuities that occur as we increase the cut-off N. To remedy this, we note that partial sums are really just infinite series where each term is weighted by a step function $\theta(n/N)$ that equates to unity for $n \leq N$ and vanishes for $n > N$. However, suppose we replace the step function with another regulator function, $\eta(n/N)$, where $\eta(x)$ is a bounded smooth function with compact support on the non-negative real line and with $\eta(0) = 1$ and $\eta(x) \to 0$ at large x. Tao shows that $\sum_{n=1}^{\infty} n\eta(n/N) = C_1[\eta]N^2 - 1/12 + \mathcal{O}(1/N)$, where $C_1[\eta] = \int_0^{\infty} dx\ x\eta(x)$ is the Mellin transform of the regulator function. As we will explicitly show, the same method can also be extended to regulator functions that are Schwartz class [94, 101].

Tao's results have several interesting features. The first concerns the divergence as $N \to \infty$. Unlike the partial sum, there is no linear divergence, while the quadratic divergence is dependent on the choice of regulator. This is in stark contrast to the finite term which is universal, returning Ramanujan's famous result. These features extend to infinite series of polynomials and are reminiscent of a well known result regarding UV divergences in QFT. In particular, when we compute the one loop effective action for a theory cut-off at some scale, Λ, the couplings associated with power law divergences are not universal. This is in contrast to the logarithmic divergences where the couplings *are* universal, much like the finite terms in regularising divergent series of polynomials using smoothed asymptotics. This overlap suggests that one loop divergences in QFT may be connected to divergent series.

The power law divergences that emerge from regularising a series with smoothed asymptotics can also be eliminated with a suitable choice of regulator. We call these *enhanced regulators*. For the series of natural numbers, an enhanced regulator is one for which the corresponding Mellin transform is vanishing $C_1[\eta] = 0$. As we will show, such enhanced regulators are relatively easy to find and we present several algorithms for finding them. A particularly elegant choice is the enhanced regulator $\eta(n/N) = e^{-\frac{n}{N}} \cos(n/N)$. When this regulator is used for the series of natural

numbers the result *converges* to $-1/12$ and there is no divergence whatsoever!

The rest of this chapter is structured as follows. In section 4.2, we review Tao's work on smoothed asymptotics, extending the analysis to include regulators that are Schwartz class and to series of general polynomials. In the following section, we then introduce the concept of enhanced regulators that eliminate divergences altogether, including one that is inspired by the Schwinger proper time formalism.

4.2 A theory of smoothed partial sums

When we use partial sums to sum an infinite series $\sum_{n=1}^{\infty} a_n$, we truncate the series at some finite value N and then compute the sum $\sum_{n=1}^{N} a_n$ before taking $N \to \infty$. We can think of the partial sum as modifying the infinite series with a step function

$$\theta(x) = \begin{cases} 1 & x \leq 1 \\ 0 & x > 1 \end{cases} \tag{4.1}$$

such that

$$\sum_{n=1}^{\infty} a_n \to \sum_{n=1}^{\infty} a_n \theta\left(\frac{n}{N}\right) = \sum_{n=1}^{N} a_n. \tag{4.2}$$

In the case of a convergent series, the partial sum tends towards a unique finite value as we take larger and larger values of N. For a divergent series, we have seen how there exist alternative summation that yield finite answers. Unfortunately, there is no obvious trace of those finite values anywhere in the partial sum. This is readily seen in Faulhaber's formula (3.7) where we see no evidence for the finite value of $S(z) = \sum_{n=1}^{\infty} n^z$ obtained via analytic continuation. In [40], Tao argues that this is an artefact of the jump discontinuity in $\theta(x)$ and can be remedied by generalising $\theta(x)$ to a smooth regulator function, as opposed to a sharp cut-off.

To this end, we regularise the infinite series with a smooth function, $\eta(x)$, such that

$$\sum_{n=1}^{\infty} a_n \to \sum_{n=1}^{\infty} a_n \eta\left(\frac{n}{N}\right). \tag{4.3}$$

We assume that $\eta(x)$ is a smooth bounded function, defined for the non-negative real numbers, with $\eta(0) = 1$ and $\eta(x) \to 0$ at large x. Indeed, Tao assumes that $\eta(x)$ is a bump function—a smooth function with compact support, vanishing whenever $x \notin [0, 1]$. Here we will relax this condition a little, and allow $\eta(x)$ to be a Schwartz function. That means that $\eta(x)$ and all its derivatives are rapidly decreasing at large x, going to zero at infinity faster than x^α for all $\alpha < 0$.

For series that were already absolutely convergent, the smoothing by $\eta(x)$ does not affect the asymptotic value [40]. However, for divergent series, smoothing can significantly improve the convergence properties. For example, we can easily sum Grandi's series $\sum_{n=1}^{\infty}(-1)^{n-1}$ using a regulator function $\eta(x) = e^{-x}$, so that the corresponding series converges towards one half.

Let us consider the impact of smoothing on a divergent series $\sum_{n=1}^{\infty} f(n)$ where $f(n) = \sum_{z=0}^{s} c_z n^z$ is a polynomial of degree s. From the Euler-Maclaurin formula (3.12) applied to the function $g_N(x) = f(x)\eta(x/N)$ with $a = 0$ and $b = \infty$, we readily obtain the following expression

$$\sum_{n=1}^{\infty} g_N(n) = \int_0^{\infty} dx\ g_N(x) - \frac{g_N(0)}{2} - \sum_{k=2}^{m} \frac{B_k}{k!} g_N^{(k-1)}(0) + R_m, \tag{4.4}$$

where we have used the fact that $g_N(x)$ and all of its derivatives vanish at infinity and the remainder term is given by

$$R_m = (-1)^{m+1} \int_0^{\infty} dx\ \frac{b_m(x)}{m!} g_N^{(m)}(x). \tag{4.5}$$

As $g_N^{(k)}(x) = f^{(k)}(x)\eta(x/N) + \mathcal{O}(1/N)$, and given the polynomial form for $f(n) = \sum_{z=0}^{s} c_z n^z$, we choose $m \geq s + 2$ and find that

$$\sum_{n=1}^{\infty} f(n)\eta\left(\frac{n}{N}\right) = \sum_{z=0}^{s} c_z \left[C_z[\eta] N^{z+1} + \zeta(-z)\right] + \mathcal{O}(1/N), \tag{4.6}$$

where $C_z[\eta] = \int_0^{\infty} dx\ x^z \eta(x)$ is the Mellin transform of the regulator function. Here we have used the fact that $\zeta(-z) = -\frac{B_{z+1}}{z+1}$ for natural numbers z. The appearance of the zeta function follows in a very similar way to its appearance in equation (3.19). In choosing $m \geq s + 2$, we guarantee that the remainder term $R_m = \mathcal{O}(1/N)$. This is

not immediately obvious. In particular we find that

$$R_m = (-1)^{m+1} \sum_{z=0}^{s} c_z \int_0^\infty dx \, \frac{b_m(x)}{m!} \frac{d^m}{dx^m} \left[x^z \eta\left(\frac{x}{N}\right) \right] \tag{4.7}$$

$$= (-1)^{m+1} \sum_{z=0}^{s} c_z N^{z+1-m} \int_0^\infty dy \, \frac{b_m(Ny)}{m!} \frac{d^m}{dy^m} (y^z \eta(y)). \tag{4.8}$$

Now since η is Schwartz and b_m is bounded, it follows that the integrand is also Schwartz, and so the corresponding integral is bounded[1]. Since $m \geq s+2$, and s is finite, we immediately see that $R_m = \mathcal{O}(1/N)$.

For monomials $f(n) = n^z$, there is a single divergence as $N \to \infty$ in the corresponding expression for the regularised series (4.6), and there is a unique finite piece given by $\zeta(-z)$. This coincides with the result obtained using Ramanujan or analytic continuation. The connection to Ramanujan summation (with $a = 0$) is understood via the following identity derived directly from (4.6)

$$\sum_{n=1}^{\infty} n^z \overset{(\mathcal{R},0)}{=} C(f,0) = \lim_{N\to\infty} \left[\sum_{n=1}^{\infty} n^z \eta\left(\frac{n}{N}\right) - \int_0^\infty dx x^z \eta\left(\frac{x}{N}\right) \right] = \zeta(-z). \tag{4.9}$$

Using the method of smoothed asymptotics, we see how Ramanujan performed a delicate cancellation between infinities that can be rigorously understood. It is notable that the procedure is reminiscent of renormalisation in perturbative QFT.

4.3 Enhanced regulators

As we will explore in more detail in Chapter 5, another feature of equation (4.6) reminiscent of QFT is the regulator dependence in the power law divergences. The regulator dependence in the divergences of (4.6) raises the possibility that there are families of *enhanced* regulators for which the power law divergences vanish altogether, just as they do for dimensional regularisation [41, 46].

[1] If $F(x)$ is Schwartz then for each $n \geq 0$, $|x^n F(x)| < Q_n$ for finite constants Q_n. Consider the corresponding integral $\int_0^\infty dx F(x) = \int_0^1 dx F(x) + \int_1^\infty dx F(x)$. $F(x)$ is smooth so the first integral running over a finite range, $\int_0^1 dx F(x)$, is bounded. As for the second integral, note that $\left|\int_1^\infty dx F(x)\right| \leq \int_1^\infty dx |F(x)| < \int_1^\infty dx Q_n/x^n = Q_n/(n-1)$. Taking $n \geq 2$ we immediately see that $\int_1^\infty dx F(x)$ is bounded. It follows that $\int_0^\infty dx F(x)$ is bounded.

This is indeed the case: an enhanced regulator is one for which the Mellin transform $C_z[\eta] = \int_0^\infty dx\ x^z\eta(x)$ vanishes for integer values of $z \geq 0$. More precisely, for all natural numbers z, we define an *enhanced regulator of order* z to be a regulator function $\eta_{[z]}(x)$ for which $\int_0^\infty dx\ x^z\eta_{[z]}(x) = 0$. We immediately see from (4.6) that if we use an enhanced regulator of order z to regularise the monomial series $S(z) = \sum_{n=0}^\infty n^z$, there are no divergences whatsoever. Indeed, the regularised series converges to the finite value $\zeta(-z)$ as $N \to \infty$, without any need to renormalise.

An elegant example of an enhanced regulator of order z is given by the function

$$\eta_{[z]}(x) = e^{-x\cot\left(\frac{\frac{\pi}{2}-\theta}{z+1}\right)}\frac{\cos(x+\theta)}{\cos\theta}, \tag{4.10}$$

where $0 < \theta < \frac{\pi}{2}$ and z is any natural number. One can readily check that we have $\int_0^\infty dx\ x^z\eta_{[z]}(x) = 0$ and so the smoothed series $\sum_{n=1}^\infty n^z\eta_{[z]}\left(\frac{n}{N}\right)$ converges to $\zeta(-z)$. We can actually take θ to be zero, except in the case of $z = 0$ where non-vanishing θ is required in order to ensure that the regulator remains Schwartz. Note that for $\theta = 0$ and $z = 1$ we recover the rather beautiful enhanced regulator of order one, $\eta_{[1]}(x) = e^{-x}\cos x$, from which we infer that[2]

$$\lim_{N\to\infty}\sum_n ne^{-\frac{n}{N}}\cos\left(\frac{n}{N}\right) = -\frac{1}{12}. \tag{4.11}$$

Since $\int_0^\infty dx\ x^z e^{-x}\cos x = 2^{-\frac{1}{2}(1+z)}z!\cos\left(\frac{\pi}{4}(z+1)\right)$ it just so happens that $e^{-x}\cos x$ is also an enhanced regulator of order $z = 4m+1$ for any natural number m. Likewise, it turns out that $\eta_{[z]}(x) = e^{-x\cot\left(\frac{\pi}{2(z+1)}\right)}\cos x$ is also enhanced regulator of order $z' = z + 2(z+1)m$ for any natural number m and $z \geq 1$.

More generally, we can employ a few simple algorithms for finding enhanced regulators of any given order. The first assumes that we already know an enhanced regulator of order zero, $\eta_{[0]}(x)$, with vanishing integral $\int_0^\infty dx\eta_{[0]}(x) = 0$. By changing variables from $x \to x^{z+1}$, we see that $\eta_{[z]}(x) = \eta_{[0]}(x^{z+1})$ is an enhanced regulator of

[2]To our best knowledge this remarkable regulator does not formally appear in the literature. However, in the process of this research, we found this formula appears on Wiki without citation or context. It is also the topic of discussion on this mathematics.stackexchange post, in which the author notices the nature of the function being Schwartz and also relates it to Tao's method of smooth summation.

order z. The second algorithm requires no previous knowledge of enhanced regulators: take any Schwartz function $\chi(x)$ defined on the positive real axis with the property that $\chi(0) = 0$ and $\chi^{(z+1)}(0) = 1$, then $\eta_{[z]}(x) = \chi^{(z+1)}(x)$ is an enhanced regulator of order z. To see this, we first note that since $\chi(x)$ is a Schwartz function, the same is trivially true of $\chi^{(z+1)}(x)$. Furthermore, since $\chi^{(z+1)}(0) = 1$, it is clear that $\chi^{(z+1)}(x)$ has all the properties of a regulator as defined at the beginning of this section. To see that it is an *enhanced* regulator, we compute the relevant Mellin transform

$$\int_0^\infty dx\ x^z \chi^{(z+1)}(x) = \left[\sum_{k=0}^{z} (-1)^k \frac{z!}{(z-k)!} x^{z-k} \chi^{(z-k)}(x)\right]_0^\infty = 0 \tag{4.12}$$

verifying the defining condition for an enhanced regulator of order z.

A third and final algorithm connects enhanced regulators to Schwinger's proper time formalism [102]. Indeed, for any natural number n, consider a regulator of the form

$$\lambda_n(x) = \frac{1}{(n-1)!} \int_0^\infty \frac{du}{u} \rho(u) (ux^2)^n e^{-ux^2}, \tag{4.13}$$

where the bounded function $\rho(u) \to 0$ faster than any power as $u \to 0$ and $\rho(u) \to 1$ as $u \to \infty$. The properties of $\rho(u)$ guarantee that this function is Schwartz with $\lambda_n(x) \to 1$ as $x \to 0^+$, as required for a regulator. After changing variables from x to ux^2, it is not difficult to show that the Mellin transform is given by

$$\int_0^\infty dx\ x^z \lambda_n(x) = \frac{\Gamma\left(n + \frac{z+1}{2}\right)}{2(n-1)!} \int_0^\infty du\ \rho(u) u^{-\left(\frac{z+3}{2}\right)}. \tag{4.14}$$

For any natural numbers $n \neq m$, we can now construct an enhanced regulator of order z as follows

$$\eta_{[z]}(x) = \frac{\frac{(n-1)!}{\Gamma\left(n+\frac{z+1}{2}\right)} \lambda_n(x) - \frac{(m-1)!}{\Gamma\left(m+\frac{z+1}{2}\right)} \lambda_m(x)}{\frac{(n-1)!}{\Gamma\left(n+\frac{z+1}{2}\right)} - \frac{(m-1)!}{\Gamma\left(m+\frac{z+1}{2}\right)}}. \tag{4.15}$$

This is clearly normalised so that it is classed as a regulator. To see that it is enhanced we simply check that the corresponding Mellin transform vanishes $\int dx\ x^z \eta_{[z]}(x) = 0$ using (4.14).

Although the enhanced regulators described above work well for summing specific monomial series, they are less suited to eliminating divergences for generic polynomial series. For example, the regulators given in equation (4.10) will yield convergent series for $\sum_n n^z \eta_{[z]}(n/N)$ but not for, say, $\sum_n (n^{z-1} + n^z)\eta_{[z]}(n/N)$. To remedy these problems for all natural numbers $r \leq s$, we introduce the notion of an enhanced regulator of order $[r, s]$ defined to be a regulator function $\eta_{[r,s]}(x)$ for which $\int_0^\infty dx\ x^z \eta_{[r,s]}(x) = 0$ for *all* natural numbers $z \in [r, s]$. It now follows from (4.6) that if we use an enhanced regulator of order $[0, s]$ to regularise the series over polynomials of degree s, there are no divergences whatsoever. The regularised series for the polynomial $f(n) = \sum_{z=0}^s c_z n^z$ converges to the finite value $\sum_{z=0}^s c_z \zeta(-z)$ as $N \to \infty$ without any need to renormalise. For the example given above, we find that $\lim_{N\to\infty} \sum_n (n^{z-1} + n^z)\eta_{[0,z]}(n/N) = \zeta(1 - z) + \zeta(-z)$ for $z \geq 1$.

We can generalise the second algorithm described above to find enhanced regulators of order $[r, s]$: take any Schwartz function $\chi(x)$ defined on the positive real axis with the property that $\chi^{(k)}(0) = 0$ for all natural numbers $k \in [0, s - r]$ and with $\chi^{(s+1)}(0) = 1$. It follows that $\eta_{[r,s]}(x) = \chi^{(s+1)}(x)$ is an enhanced regulators of order $[r, s]$. The argument for $\eta_{[r,s]}(x)$ being a regulator is the same as for $\eta_{[z]}(x)$. To see that it is enhanced at order $[r, s]$, we compute the Mellin transforms for all natural numbers $z \in [r, s]$ and see that they all vanish

$$\int_0^\infty dx\ x^z \chi^{(s+1)}(x) = \left[\sum_{k=0}^{z} (-1)^k \frac{z!}{(z-k)!} x^{z-k} \chi^{(s-k)}(x)\right]_0^\infty = 0. \tag{4.16}$$

As an example, we consider the case where $\chi(x) = e^{-x} \frac{x^{s+1}}{(s+1)!}$ which generates an enhanced regulator of order $[0, s]$ given by

$$\eta_{[0,s]}(x) = e^{-x} \sum_{k=0}^{s+1} \binom{s+1}{k} \frac{(-x)^k}{k!}. \tag{4.17}$$

The existence of enhanced regulators challenges the idea that divergent series of polynomials must be renormalised in order to recover their corresponding finite values. Indeed, they explicitly demonstrate a rigorous method of taking the series towards infinity without encountering any divergence. Of course, there is a magic of sort: the

enhanced regulators must always change sign and in just the right way in order for large positive contributions to be cancelled by large negative ones. It is here that we are reminded of the words of Euler who said "the quantities greater than infinity are also smaller than nothing and the quantities smaller than infinity also correspond to the quantities greater than nothing"[89].

4.4 Concluding comments

In this chapter, we have introduced the idea of η regularisation on the level of number theoretic sums. In doing so, we have extended Tao's work on smoothed asymptotics [40] in two technical ways: by enlarging the class of η regulators from bump functions to Schwartz functions, and by generalising from purely monomial series to polynomial series through the development of enhanced regulators, which we argue offers a tantalising hint at connections with perturbative QFT. As we'll see in Chapter 5, both cases exhibit regulator dependence of power law divergences, while the universal features of finite terms in Tao's study of divergent series mirror the universal features of logarithms in QFT. However, our analysis runs much deeper than this elegant analogy. We are beginning to build a theory of regularisation which we'll ultimately want to extend to the operator language of the path integral and, at the top of the mountain, associate with the UV regularisation of field theory integrals as derived directly from string theory. This will require substantial generalisation in the context of analytic number theory. But before we can glimpse such conceptual and theoretic heights, we must first focus on further developing η regularisation for divergent loop integrals in QFT and explore its implications.

PART II

Quantum field theory

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5

Divergent integrals in quantum field theory

Therefore, I think that the renormalisation theory is simply a way to sweep the difficulties of the divergences of electrodynamics under the rug.

— ***Richard P. Feynman, from his Nobel Lecture in 1965***

In this chapter, we build on the ideas discussed in Chapter 4 by extending the concept of η regularisation to the case of perturbative loop integrals. This allows us to identify η form factors as entering the integration via a modification of the integration measure, effectively acting as a weight in the integrand. These η weighted loop integrals then allow us to reveal a surprising connection between the elimination of divergences in divergent series of powers and the preservation of gauge invariance. As we will show both here and in subsequent chapters, the development of η regularisation will allow us to push the notion of regularisation to its maximum, and to study technical implications in a general gauge invariant way.

5.1 Introduction

Before extending the concept of η regularisation for number theoretic sums to an analysis of divergent integrals at one-loop in QFT, we first provide a few important comments. As will be elaborated upon in a following section, the key idea is to identify the integrand in (Euclidean) loop integrals as weighted by a regulator function $\eta(|k|/\Lambda)$,

where k is the Euclidean loop momentum, $|k| = \sqrt{k \cdot k}$ its Euclidean norm, and Λ is the cut-off. For this purpose, it is convenient to work directly with one fold irreducible loop integrals (ILIs) introduced by Wu [103] as the basic building block of all one particle irreducible graphs. When we implement η regularisation we see, as expected, that power law divergences are regulator dependent while logarithmic divergences are *universal*. As with divergent series and smoothed asymptotics, the power law divergences are multiplied by Mellin transforms of the regulator. By choosing enhanced regulators, we show that we can eliminate the power law divergences at will.

However, the real question is whether or not the elimination of power law divergences by enhanced regulators has any deeper meaning from a QFT perspective. Perhaps as one might have expected from dimensional regularisation, we find that the answer is yes. In a series of papers [103–108], Wu and others have derived a set of consistency relations that are necessary in order for the regulator to preserve gauge invariance. Requiring that these hold for η regularisation, we find that certain regulators must be enhanced. In other words, regulators that allow the infinite series of natural numbers to converge towards $-1/12$ are intimately connected to the preservation of gauge invariance at one-loop in a wide class of non-abelian gauge theories coupled to an arbitrary number of Dirac fermions.

The rest of this chapter proceeds as follows: in the next section we offer several comments on the similarities between divergences in effective field theory and the regularisation of number theoretic sums. We then formally introduce η regularisation in section 5.2.1 and study the implications for irreducible loop integrals in section 5.2.2. In section 5.2.3 we implement a set of gauge consistency conditions, demonstrating the role of enhanced regulators in preserving gauge invariance. We comment explicitly on the connection to Schwinger proper time in section 5.2.4. In section 5.3, we offer several concluding comments.

5.2 Divergent integrals

Divergent integrals are encountered with unsettling regularity in interacting QFTs. As revealed by Tomonoga [53], Schwinger [54] and Feynman [55] in the context of quantum electrodynamics and by 't Hooft and Veltman for non-abelian gauge theories

[56], for renormalisable theories, these divergences can be carefully regularised and absorbed into a renormalisation of a finite number of couplings. The origin of these divergences is understood to be a consequence of pushing the corresponding theory too hard, and assuming it to apply even at the shortest distance scales, or equivalently, at the highest energies. Thanks to the Applequist-Carazzone decoupling theorem, the effects of very heavy particles on low energy scattering processes are hidden inside a renormalisation of the low energy couplings [109]. Said another way, even though our QFTs encounter divergences that betray limitations on their validity at all scales, they can still be made to work well as effective field theories at low energies.

It is interesting to note parallels between divergences in effective field theory and the characteristic behaviour we see whenever we regularise a divergent series with a smooth cut-off. Indeed, it is well known that power law divergences in QFT are regulator dependent in stark contrast to the universal prediction obtained from the logarithmic divergences. Although not universal, power law divergences do play a role in the correct matching to UV physics in the Wilsonian EFT. They can also indicate an unwelcome sensitivity to the details of the UV completion, leading to problems with naturalness in high energy physics and cosmology [110].

In order to develop the analogy with our divergent series, it is instructive to examine power law and logarithmic divergences in QFT in a little more detail. Closely following the discussion presented in [111], we consider the Wilson action $S_\Lambda[\varphi]$ for modes lighter than some scale Λ defined according to the path integral [112]

$$e^{iS_\Lambda[\varphi]} = \int_{k>\Lambda} D[\varphi] e^{iS[\varphi]}. \tag{5.1}$$

This generically contains power law and logarithmic divergences, and can be written as

$$S_\Lambda[\varphi] = \int d^4x \left[\sum_i \Lambda^{4-d_i} g_i(\Lambda) \mathcal{O}_i(x) + \ln(\Lambda/\mu) \sum_a g_a(\Lambda) \mathcal{O}_a(x) + \dots \right], \tag{5.2}$$

where $\mathcal{O}_i$ are relevant operators of dimension $d_i = 0, 2$ and $\mathcal{O}_a$ are marginal operators of dimension 4. The ellipsis denote finite terms which may also depend on μ, an arbitrarily chosen mass scale. Now consider the Wilson action, $S'_\Lambda[\varphi]$, defined at some

lower scale, $\Lambda' < \Lambda$, and given by

$$e^{iS'_\Lambda[\varphi]} = \int_{\Lambda'<k<\Lambda} D[\varphi] e^{iS_\Lambda[\varphi]}. \tag{5.3}$$

This is obviously independent of the choice of Λ, but highly dependent on Λ' and correspondingly takes the form

$$S_{\Lambda'}[\varphi] = \int d^4x \left[\sum_i \Lambda'^{4-d_i} g_i(\Lambda') \mathcal{O}_i(x) + \ln(\Lambda'/\mu) \sum_a g_a(\Lambda') \mathcal{O}_a(x) + \ldots \right]. \tag{5.4}$$

It follows that $S'_\Lambda = S_\Lambda + \Delta\Gamma$ where $\Delta\Gamma$ is the contribution from integrating out intermediate modes $\Lambda' \leq k \leq \Lambda$. Focussing on the relevant and marginal operators, we must have

$$\Delta\Gamma = \int d^4x \left[\sum_i c_i(\Lambda', \Lambda) \mathcal{O}_i(x) + \sum_a c_a(\Lambda', \Lambda) \mathcal{O}_a(x) + \ldots \right], \tag{5.5}$$

where the contribution from the intermediate modes $c_i(\Lambda', \Lambda) = \Lambda'^{4-d_i} g_i(\Lambda') - \Lambda^{4-d_i} g_i(\Lambda)$ completely cancels the power law divergences in passing from one Wilson action to another. This is just Wilsonian renormalisation in action—the coefficients of power-like divergences in the momentum cut-off get cancelled in the regularisation procedure (see [113] for elegant discussions on this point). In contrast, the cancellation of the Λ dependence for the logarithmic couplings yields $c_a(\Lambda', \Lambda) = \ln(\Lambda'/\mu) \sum_a g_a(\Lambda') - \ln(\Lambda/\mu) \sum_a g_a(\Lambda')$. In order for the μ to consistently drop out to leading order, we see that there must be a universal property $g_a(\Lambda) \sim g_a(\Lambda') \sim g_a$.

In a similar vein, if we compute the one loop effective action

$$e^{i\Gamma_{\text{eff}}[\varphi]} = \int_{k<\Lambda} D[\delta\varphi] e^{iS_\Lambda[\varphi+\delta\varphi]} \tag{5.6}$$

so that $\Gamma_{\text{eff}}[\varphi] = S_\Lambda[\varphi] + \Delta\Gamma_{\text{eff}}[\varphi]$, we find that any power law divergences generated in $\Delta\Gamma$ are cancelled by the corresponding terms in the Wilson action, S_Λ. In contrast, the couplings associated with logarithmic divergences can survive as a universal feature of the effective action, necessarily independent of Λ.

These universal features of logarithmic divergences, in contrast to power law divergences, are strongly reminiscent of the results we saw in the previous chapter. When we regularised a divergent series with a smoothed cut-off, we saw how it was the finite terms that were universal, yielding the results obtained via analytic continuation, in contrast to the power law divergences that were regulator dependent. How far can we push the similarities? For example, having introduced the enhanced regulators to eliminate divergences appearing in number theory, it is tempting to ask if they can also play an interesting role in the loop divergences arising in QFT and whether this can be linked to the absence of divergences in finite theories. This question refers also to our broader interest in taking a top-down approach to a generalised theory of regularisation, where, in the introduction of this thesis, the reluctance of string theory to travel deep into the UV was mentioned as an important motivation.

5.2.1 η regularisation in QFT

To control the UV divergences that plague perturbative QFT, several regularisation schemes are included in the literature. A sharp momentum cut-off connects intuitively with Wilson's understanding of the renormalisation group [114–117] but is known to break translational invariance and, perhaps more importantly, the gauge invariance of the underlying theory. The standard way to regularise loop integrals whilst preserving gauge invariance is dimensional regularisation [56]. However, in analytically continuing the dimensionality of the spacetime, we run into difficulties when defining quantities unique to four dimensions (such as the γ_5 matrix).

For infinite series, the method of smoothed asymptotics amounts to replacing an infinite sum with a weighted infinite sum

$$\sum_{n=0}^{\infty} \# \to \sum_{n=0}^{\infty} \eta\left(\frac{n}{N}\right) \#. \tag{5.7}$$

The generalisation of this to divergent loops is straightforward: working in Euclidean signature, we simply replace the loop integral with a weighted loop integral

$$\int d^4k \# \to \int d^4k \; \eta\left(\frac{|k|}{\Lambda}\right) \#, \tag{5.8}$$

where $|k|$ is the norm of the Euclidean four-momentum and Λ is a cut-off scale. Of course, if η were just a step function, this would be nothing more than a sharp momentum cut-off. However, inspired by the method of smoothed asymptotics discussed in the previous chapter, we shall assume that η is a smooth regulator, corresponding to a Schwartz function that equates to unity at the origin.

The idea of smoothing the cut-off is, of course, nothing new although in most applications, this is implemented at the level of the propagator. For example, in his derivation of the exact renormalisation group equation, Polchinski makes use of a modified propagator that is exponentially damped at high momenta [45]. As a modification of the integral measure, η regularisation is similar to dimensional regularisation albeit with the advantage that it does not require us to analytically continue the dimension of spacetime. However, the proposed scheme has most in common with the smooth operator regularisation method inspired by [118] and developed in [119, 120] using Schwinger proper time [102]. Gauge invariance is also preserved in this case, as it is with dimensional regularisation. This is because the regularisation is transferred to the proper time integral, leaving the (gauge invariant) momentum integral unchanged. For a recent application of the smooth operator regularisation and further discussion on symmetry preservation, see [121].

Does η regularisation also preserve gauge invariance? Generically, it seems obvious that it will not. Just as we saw for sharp momentum cut-offs, gauge invariance is typically broken as soon as we regularise the momentum integral directly. However, if η regularisation has any chance of opening up a better understanding of how divergences are absent in string theory, it must, at the very least, admit a *preferred* set of regulators that preserve gauge invariance. This may seem unlikely at first glance, although as we will see in a moment, it will be possible to preserve gauge invariance and key to that are the enhanced regulators that eliminate divergences. We will also see how this connects to regularisation of Schwinger proper time integrals.

5.2.2 Irreducible loop integrals and eliminating divergences

To develop η regularisation in more detail, it is convenient to introduce the concept of irreducible loop integrals (ILIs) [103, 104, 107]. In general, n-fold ILIs are defined as the n-loop integrals for which there are no longer the overlapping factors $(k_i - k_j + p)$

in the denominator of the integrand and no factors of the scalar momentum k^2 in the numerator [107]. (Focusing on regularising UV divergences at one-loop, a detailed discussion of higher loops is postponed to future work). It was shown in [103] that upon use of the Feynman parameter method, all one-loop perturbative Feynman integrals of the one-particle irreducible graphs can be evaluated as the following one-fold ILIs in Minkowski spacetime:

$$I_{-2\alpha}(\mathcal{M}^2) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2+\mathcal{M}^2)^{2+\alpha}}, \tag{5.9}$$

$$I^{\mu\nu}_{-2\alpha}(\mathcal{M}^2) = \int \frac{d^4k}{(2\pi)^4} \frac{k^\mu k^\nu}{(k^2+\mathcal{M}^2)^{3+\alpha}}, \tag{5.10}$$

$$I^{\mu\nu\rho\sigma}_{-2\alpha}(\mathcal{M}^2) = \int \frac{d^4k}{(2\pi)^4} \frac{k^\mu k^\nu k^\rho k^\sigma}{(k^2+\mathcal{M}^2)^{4+\alpha}}, \tag{5.11}$$

where the subscript (-2α) labels the power counting dimension (of energy-momentum) with $\alpha = -1$ and $\alpha = 0$ corresponding to quadratic and logarithmically divergent integrals. The mass term $\mathcal{M}^2 = \mathcal{M}^2(m_1^2, p_1^2, \dots)$ is a function of Feynman parameters, external momenta, p_i and corresponding mass scales, m_i. Note that $k^2 = g_{\mu\nu}k^\mu k^\nu$ where the metric $g_{\mu\nu}$ is written with mostly positive signature.

The ILIs of tensor type are related to scalar integrals in the usual way:

$$I^{\mu\nu}_{-2\alpha}(\mathcal{M}^2) = \frac{1}{4} g^{\mu\nu} \int \frac{d^4k}{(2\pi)^4} \frac{k^2}{(k^2+\mathcal{M}^2)^{3+\alpha}}, \tag{5.12}$$

$$I^{\mu\nu\rho\sigma}_{-2\alpha}(\mathcal{M}^2) = \frac{1}{4!} S^{\mu\nu\rho\sigma} \int \frac{d^4k}{(2\pi)^4} \frac{k^4}{(k^2+\mathcal{M}^2)^{4+\alpha}}, \tag{5.13}$$

where $S^{\mu\nu\rho\sigma} = g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}$ is a totally symmetric tensor of rank four. Notice that these relations are merely a consequence of the symmetry of the momentum integral under orthogonal transformations of k: in Minkowski signature this is Lorentz invariance, while after Wick rotation it becomes ordinary rotational invariance in Euclidean momentum space. It is in this sense that we refer to these relations as a consequence of rotational symmetries of the four-dimensional space, which should not be confused with the gauge consistency relations derived in [103, 104, 107]. As we will see later, the latter can be accommodated by using different η's for regularising different ILIs.

When implementing the η regularisation, we Wick rotate the integrals to Euclidean signature $k_0 \to ik_4$ and insert a factor of $\eta(|k|/\Lambda)$ in the integrand. This yields $I^{\cdots}_{-2\alpha} \to iJ^{\cdots}_{-2\alpha}[\eta]$, where the form of η need not be universal for all one-fold ILIs, at least in principle. In particular, we now have that

$$J_{-2\alpha}[\eta](\mathcal{M}^2) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2+\mathcal{M}^2)^{2+\alpha}} \eta\left(\frac{|k|}{\Lambda}\right), \tag{5.14}$$

$$J^{\mu\nu}_{-2\alpha}[\eta](\mathcal{M}^2) = \frac{1}{4} g^{\mu\nu} \int \frac{d^4k}{(2\pi)^4} \frac{k^2}{(k^2+\mathcal{M}^2)^{3+\alpha}} \eta\left(\frac{|k|}{\Lambda}\right), \tag{5.15}$$

$$J^{\mu\nu\rho\sigma}_{-2\alpha}[\eta](\mathcal{M}^2) = \frac{1}{4!} S^{\mu\nu\rho\sigma} \int \frac{d^4k}{(2\pi)^4} \frac{k^4}{(k^2+\mathcal{M}^2)^{4+\alpha}} \eta\left(\frac{|k|}{\Lambda}\right), \tag{5.16}$$

where the integration is over four dimensional Euclidean space in each case. Making use of partial fractions, the regularised tensor ILIs can be written explicitly in terms of scalar counterparts

$$J^{\mu\nu}_{-2\alpha}[\eta](\mathcal{M}^2) = \frac{1}{4} g^{\mu\nu} \left[J_{-2\alpha}[\eta](\mathcal{M}^2) - \mathcal{M}^2 J_{-2(\alpha+1)}[\eta](\mathcal{M}^2) \right], \tag{5.17}$$

and

$$\begin{aligned} J^{\mu\nu\rho\sigma}_{-2\alpha}[\eta](\mathcal{M}^2) = \frac{1}{4!} S^{\mu\nu\rho\sigma} \Big[& J_{-2\alpha}[\eta](\mathcal{M}^2) \\ & - 2\mathcal{M}^2 J_{-2(\alpha+1)}[\eta](\mathcal{M}^2) + \mathcal{M}^4 J_{-2(\alpha+2)}[\eta](\mathcal{M}^2) \Big]. \end{aligned} \tag{5.18}$$

To understand the role of divergences, it is now sufficient to study the scalar integrals of the form (5.14). After integrating out the three-sphere, these can be written as

$$J_{-2\alpha}[\eta](\mathcal{M}^2) = \frac{1}{8\pi^2} \int_0^\infty dk \ \frac{k^3}{(k^2+\mathcal{M}^2)^{2+\alpha}} \eta\left(\frac{|k|}{\Lambda}\right). \tag{5.19}$$

For $\alpha > 0$ the integrals are convergent as $\Lambda \to \infty$ and one readily obtains

$$J_{-2\alpha}[\eta](\mathcal{M}^2) \sim \frac{1}{16\pi^2 \alpha(1+\alpha)\mathcal{M}^{2\alpha}}. \tag{5.20}$$

For $\alpha \leq 0$, the integrals diverge as $\Lambda \to \infty$, where they take the following asymptotic

form

$$J_0[\eta](\mathcal{M}^2) \sim \frac{1}{8\pi^2}\left[\ln(\Lambda/|\mathcal{M}|) + \gamma[\eta] - \frac{1}{2}\right], \tag{5.21}$$

$$J_2[\eta](\mathcal{M}^2) \sim \frac{1}{8\pi^2}\left[\Lambda^2 C_1[\eta] - \mathcal{M}^2\left(\ln(\Lambda/|\mathcal{M}|) + \gamma[\eta]\right)\right], \tag{5.22}$$

and

$$J_{2s+4}[\eta](\mathcal{M}^2) \sim \frac{1}{8\pi^2}\left[\sum_{z=0}^{s}\binom{s}{z} C_{2z+3}[\eta]\mathcal{M}^{2(s-z)}\Lambda^{2z+4}\right] \tag{5.23}$$

for any natural number s. These expressions are valid provided η is a regulator: a smooth Schwartz function with $\eta(0) = 1$. Here we have also defined the finite integral

$$\gamma[\eta] = -\int_0^\infty dx\ \eta'(x)\ln x, \tag{5.24}$$

and recall that $C_z[\eta] = \int_0^\infty x^z\eta(x)$ is the corresponding Mellin transform for any natural number z.

We immediately see the parallels with the divergent series in the previous chapter. Power law divergences are regulator dependent and weighted by the corresponding Mellin transform. These can always be eliminated with a judicious use of enhanced regulators. Logarithmic divergences are independent of the regulator, just as the finite terms were in the divergent series. This universal form for the log divergences agrees with the corresponding terms obtained by other regularisation methods, such as dimensional regularisation. Of course, for our QFT integrals, there are also finite terms, although unlike those that appear in the divergent series, they are regulator dependent. This is a direct consequence of the logarithm and the correction to the finite term that arises when we rescale the cut-off Λ. Indeed, for any regulator $\eta(x)$ and real number $\lambda > 0$, we can define another regulator $\eta_\lambda(x) = \eta(\lambda x)$ which is equivalent to rescaling the cut-off. It then follows that $\gamma[\eta_\lambda] = \gamma[\eta] - \ln\lambda$ while $C_z[\eta_\lambda] = \lambda^{-1-z}C_z[\eta]$. These relations will be useful when we consider the role of gauge invariance in the next section.

5.2.3 Gauge invariant η regularisation

To investigate how gauge invariance is affected by η regularisation, we follow [103, 104, 107] and consider a general gauge theory where the gauge group has dimension d_G and where N_f Dirac spinors Ψ_n ($n = 1, \ldots, N_f$) are interacting with the Yang Mills field A^a_μ ($a = 1, \ldots, d_G$). Such a theory is described by a Lagrangian

$$\mathcal{L} = \bar{\psi}_n(i\gamma^\mu D_\mu - m)\psi_n - \frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a, \tag{5.25}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f_{abc} A^b_\mu A^c_\nu, \qquad D_\mu\psi_n = (\partial_\mu + igT^a A^a_\mu)\psi_n, \tag{5.26}$$

and T^a are the generators of the gauge group whose commutator $[T^a, T^b] = if^{abc}T^c$ defines the structure constants f^{abc}. A careful computation of the vacuum polarisation for the gauge field at one-loop yields an expression of the form [103, 104, 107]

$$\Pi^{ab}_{\mu\nu}(p) = \Pi^{(g)ab}_{\mu\nu}(p) + \Pi^{(f)ab}_{\mu\nu}(p), \tag{5.27}$$

where p^μ is the external momentum. Here $\Pi^{(g)ab}_{\mu\nu}(p)$ are the pure Yang Mills contributions coming from gauge field loops and ghost loops. $\Pi^{(f)ab}_{\mu\nu}(p)$ are the contributions from fermion loops, arising from the interaction of the fermions with the gauge field. Gauge invariance is understood in terms of the Ward identities $p^\mu\Pi^{ab}_{\mu\nu} = \Pi^{ab}_{\mu\nu}p^\nu = 0$. Requiring this to hold for any gauge theory and with any number of fermions means that Ward identities should hold separately for the gauge field and fermionic contributions

$$p^\mu\Pi^{(g)ab}_{\mu\nu} = \Pi^{(g)ab}_{\mu\nu}p^\nu = 0, \qquad p^\mu\Pi^{(f)ab}_{\mu\nu} = \Pi^{(f)ab}_{\mu\nu}p^\nu = 0. \tag{5.28}$$

When the vacuum polarisation is computed in terms of the regularised ILIs, these generalised Ward identities impose strict consistency conditions on the regularisation scheme. There are also generalised Ward identities and corresponding consistency conditions associated with three and four point functions. Altogether, we find that for $\alpha = -1, 0, 1$ we must have (see Appendix A and [103, 104, 107, 121] for further

details)

$$I^{\mu\nu}_{-2\alpha}|_{\text{regularised}} \sim \frac{1}{2(\alpha+2)} g^{\mu\nu} I_{-2\alpha}|_{\text{regularised}}, \tag{5.29}$$

$$I^{\mu\nu\rho\sigma}_{-2\alpha}|_{\text{regularised}} \sim \frac{1}{4(\alpha+2)(\alpha+3)} S^{\mu\nu\rho\sigma} I_{-2\alpha}|_{\text{regularised}}, \tag{5.30}$$

in the asymptotic limit. We now wish to apply these conditions in the context of η regularisation. As mentioned earlier, it is important to note that, in principle, different ILIs use different regulators. At this stage, we do this in order to keep things sufficiently general. As we will see shortly, the Ward identities impose relations between different ηs for different loop topologies, which may point to a deeper underlying structure. We will investigate the meaning of these relations in more detail in our forthcoming work.

In the following we denote the regulators for scalar ILIs $I_{-2\alpha}$ by $\eta_{-2\alpha}$ and the regulators for the corresponding tensor ILIs of rank two and four by $\theta_{-2\alpha}$ and $\kappa_{-2\alpha}$ respectively. It follows that

$$I_{-2\alpha}|_{\text{regularised}} = iJ_{-2\alpha}[\eta_{-2\alpha}], \tag{5.31}$$

$$I^{\mu\nu}_{-2\alpha}|_{\text{regularised}} = iJ^{\mu\nu}_{-2\alpha}[\theta_{-2\alpha}], \tag{5.32}$$

$$I^{\mu\nu\rho\sigma}_{-2\alpha}|_{\text{regularised}} = iJ^{\mu\nu\rho\sigma}_{-2\alpha}[\kappa_{-2\alpha}]. \tag{5.33}$$

We now impose the conditions (5.29) and (5.30) making use of the decomposition formulae (5.17) and (5.18) giving

$$\frac{\alpha}{\alpha+2} J_{-2\alpha}\left[\tilde{\theta}_{-2\alpha}\right] \sim \mathcal{M}^2 J_{-2(\alpha+1)}[\theta_{-2\alpha}], \tag{5.34}$$

$$\frac{\alpha(\alpha+5)}{(\alpha+2)(\alpha+3)} J_{-2\alpha}\left[\tilde{\kappa}_{-2\alpha}\right] \sim 2\mathcal{M}^2 J_{-2(\alpha+1)}[\kappa_{-2\alpha}] - \mathcal{M}^4 J_{-2(\alpha+2)}[\kappa_{-2\alpha}], \tag{5.35}$$

as $\Lambda \to \infty$ and where

$$\tilde{\theta}_{-2\alpha} = \frac{(\alpha+2)\theta_{-2\alpha} - 2\eta_{-2\alpha}}{\alpha}, \qquad \tilde{\kappa}_{-2\alpha} = \frac{(\alpha+2)(\alpha+3)\kappa_{-2\alpha} - 6\eta_{-2\alpha}}{\alpha(\alpha+5)}. \tag{5.36}$$

Note that for $\alpha \neq 0$ the combination of regulators appearing in (5.36) are themselves regulators, being Schwartz and equating to unity at the origin, $\tilde{\theta}_{-2\alpha}(0) = \tilde{\kappa}_{-2\alpha}(0) = 1$. It remains to plug in the asymptotic expressions for the scalar ILIs given by (5.20),

(5.21), (5.22) and (5.23). For the convergent ILIs, with $\alpha \geq 1$, we can use the asymptotic formula (5.20) to show that the consistency relations hold automatically in the limit as $\Lambda \to \infty$, as, of course, they should. For $\alpha = 0$, the conditions (5.34) and (5.35) can be understood via the $\alpha \to 0$ limit, giving

$$J_0[\theta_0] - J_0[\eta_0] \sim \mathcal{M}^2 J_{-2}[\theta_0], \tag{5.37}$$

$$J_0[\kappa_0] - J_0[\eta_0] \sim 2\mathcal{M}^2 J_{-2}[\kappa_0] - \mathcal{M}^4 J_{-4}[\kappa_0]. \tag{5.38}$$

Plugging in the asymptotic formulae, we see that there are logarithmic divergences, although they cancel and the leftover finite parts yield the following constraints

$$\gamma[\theta_0] - \gamma[\eta_0] = \frac{1}{4}, \qquad \gamma[\kappa_0] - \gamma[\eta_0] = \frac{5}{12}. \tag{5.39}$$

For $\alpha = -1$, we get both quadratic and logarithmic divergences. The latter cancel when we impose the two consistency conditions (5.34) and (5.35), just as they did for $\alpha = 0$, and we are left with

$$\Lambda^2 C_1[\tilde{\theta}_2] + 2\mathcal{M}^2 \left[\gamma[\theta_2] - \gamma[\eta_2] - \frac{1}{4}\right] \sim 0, \tag{5.40}$$

$$\Lambda^2 C_1[\tilde{\kappa}_2] + \frac{3}{2}\mathcal{M}^2 \left[\gamma[\kappa_2] - \gamma[\eta_2] - \frac{5}{12}\right] \sim 0, \tag{5.41}$$

where we recall that $\tilde{\theta}_2 = 2\eta_2 - \theta_2$ and $\tilde{\kappa}_2 = \frac{3}{2}\eta_2 - \frac{1}{2}\kappa_2$. Cancellation of the quadratic divergences requires that $\tilde{\theta}_2$ and $\tilde{\kappa}_2$ are enhanced regulators of order one

$$C_1[\tilde{\theta}_2] = C_1[\tilde{\kappa}_2] = 0. \tag{5.42}$$

Here we see for the first time the connection between gauge invariance and the elimination of quadratic divergences in both divergent series and ILIs. The remaining finite parts in (5.40) and (5.41) yield constraints that are very similar to (5.39)

$$\gamma[\theta_2] - \gamma[\eta_2] = \frac{1}{4}, \qquad \gamma[\kappa_2] - \gamma[\eta_2] = \frac{5}{12}. \tag{5.43}$$

At this time, we have not been able to develop any further insight into the consistency conditions arising for the finite parts, given by (5.39) and (5.43). However, using the

fact that $\gamma[\eta_\lambda] = \gamma[\eta] - \ln\lambda$ and $C_z[\eta_\lambda] = \lambda^{-1-z}C_z[\eta]$ for any regulator $\eta_\lambda(x) = \eta(\lambda x)$ with a rescaled cut-off, we can arrive at a rather elegant solution to the full set of consistency conditions. For $\alpha \geq -1$, this is given by

$$\eta_{-2\alpha}(x) = \eta_{[1]}(x), \quad \theta_{-2\alpha}(x) = \eta_{[1]}(\lambda x), \quad \kappa_{-2\alpha}(x) = \eta_{[1]}(\mu x), \tag{5.44}$$

where $\lambda = e^{-1/4}$, $\mu = e^{-5/12}$, and $\eta_{[1]}(x)$ is any enhanced regulator of order one.

5.2.4 Connecting to Schwinger proper time

It was noted in [121] that regularised Schwinger proper time integrals provide an elegant framework for satisfying Wu's consistency relations (5.29) and (5.30). In particular, we write the regularised ILIs as follows

$$I_{-2\alpha}(\mathcal{M}^2)|_{\text{regularised}} = \frac{i}{(\alpha+1)!}\int_0^\infty \frac{d\tau}{\tau}\,\rho(\Lambda^2\tau)\tau^{2+\alpha}\int\frac{d^4k}{(2\pi)^4}e^{-\tau(k^2+\mathcal{M}^2)}, \tag{5.45}$$

$$I^{\mu\nu}_{-2\alpha}(\mathcal{M}^2)|_{\text{regularised}} = \frac{i}{(\alpha+2)!}\int_0^\infty \frac{d\tau}{\tau}\,\rho(\Lambda^2\tau)\tau^{3+\alpha}\int\frac{d^4k}{(2\pi)^4}k^\mu k^\nu e^{-\tau(k^2+\mathcal{M}^2)}, \tag{5.46}$$

$$I^{\mu\nu\rho\sigma}_{-2\alpha}(\mathcal{M}^2)|_{\text{regularised}} = \frac{i}{(\alpha+3)!}\int_0^\infty \frac{d\tau}{\tau}\,\rho(\Lambda^2\tau)\tau^{4+\alpha}\int\frac{d^4k}{(2\pi)^4}k^\mu k^\nu k^\rho k^\sigma e^{-\tau(k^2+\mathcal{M}^2)}, \tag{5.47}$$

where $\rho(u) \to 0$ faster than any power as $u \to 0$ and $\rho(u) \to 1$ as $u \to \infty$. To make contact with the current work, we note that we can write these formulae in the form of equations (5.31) to (5.33), provided we generalise the regulators as follows:

$$\eta_{-2\alpha}(x) \to \eta_{-2\alpha}(x,\mathcal{M}_\Lambda) = \frac{\int_0^\infty \frac{du}{u}\rho(u)[u(x^2+\mathcal{M}_\Lambda^2)]^{\alpha+2}e^{-u(x^2+M_\Lambda^2)}}{\int_0^\infty \frac{du}{u}\rho(u)[u\mathcal{M}_\Lambda^2]^{\alpha+2}e^{-uM_\Lambda^2}}, \tag{5.48}$$

$$\theta_{-2\alpha}(x) \to \theta_{-2\alpha}(x,\mathcal{M}_\Lambda) = \frac{\int_0^\infty \frac{du}{u}\rho(u)[u(x^2+\mathcal{M}_\Lambda^2)]^{\alpha+3}e^{-u(x^2+M_\Lambda^2)}}{\int_0^\infty \frac{du}{u}\rho(u)[u\mathcal{M}_\Lambda^2]^{\alpha+3}e^{-uM_\Lambda^2}}, \tag{5.49}$$

$$\kappa_{-2\alpha}(x) \to \kappa_{-2\alpha}(x,\mathcal{M}_\Lambda) = \frac{\int_0^\infty \frac{du}{u}\rho(u)[u(x^2+\mathcal{M}_\Lambda^2)]^{\alpha+4}e^{-u(x^2+M_\Lambda^2)}}{\int_0^\infty \frac{du}{u}\rho(u)[u\mathcal{M}_\Lambda^2]^{\alpha+4}e^{-uM_\Lambda^2}}, \tag{5.50}$$

where $u = \Lambda^2\tau$, $x = |k|/\Lambda$, and $\mathcal{M}_\Lambda = |\mathcal{M}|/\Lambda$. These do not have the precise form of the η regulators described earlier in this chapter, as the Λ dependence is spread between $x = |k|/\Lambda$ and $\mathcal{M}_\Lambda = |\mathcal{M}|/\Lambda$. We will consider much more general regulators in Chapter 6, but for now let us examine the form of these regulators in the limit where $\mathcal{M}_\Lambda \to 0$. In particular, note that in this limit, the denominators can generically be written as

$$\int_0^\infty \frac{du}{u}\rho(u)[u\mathcal{M}_\Lambda^2]^n e^{-uM_\Lambda^2} = \int_0^\infty \frac{dy}{y}\rho\left(\frac{y}{\mathcal{M}_\Lambda^2}\right) y^n e^{-y} \tag{5.51}$$

$$\sim \int_{0^+}^\infty \frac{dy}{y} y^n e^{-y} = (n-1)! \tag{5.52}$$

It follows that as $\mathcal{M}_\Lambda \to 0$, the regulators fall into the class described by equation (4.13), with

$$\eta_{-2\alpha}(x,0) = \lambda_{2+\alpha}(x), \qquad \theta_{-2\alpha}(x,0) = \lambda_{3+\alpha}(x), \qquad \kappa_{-2\alpha}(x,0) = \lambda_{4+\alpha}(x). \tag{5.53}$$

Using the expression (4.15), it is easy to verify that $\tilde{\theta}_2 = 2\eta_2(x,0) - \theta_2(x,0)$ and $\tilde{\kappa}_2 = \frac{3}{2}\eta_2(x,0) - \frac{1}{2}\kappa_2(x,0)$ are enhanced regulators of order one, consistent with the gauge consistency condition (5.42). Of course, this had to be true given the claim in [121] that regularised Schwinger proper time integrals automatically satisfy Wu's consistency relations but it serves as a nice check of our formalism.

We can also compute the finite integral (5.24) for each of the regulators in (5.53). Indeed, from (4.13), we change the integration variable to $y = ux^2$ and write

$$\lambda_n(x) = \frac{1}{(n-1)!}\int_0^\infty \frac{dy}{y}\rho(y/x^2) y^n e^{-y}, \tag{5.54}$$

to then find that

$$\gamma[\lambda_n] = \frac{1}{(n-1)!}\int_0^\infty dx \int_0^\infty dy \rho'(y/x^2)\frac{y^n}{x^3}e^{-y}(2\ln x) \tag{5.55}$$

$$= \frac{1}{2(n-1)!}\int_0^\infty dy \int_0^\infty du \rho'(u) y^{n-1} e^{-y}(\ln y - \ln u) \tag{5.56}$$

$$= \frac{1}{2}\left[\Psi(n) - \int_0^\infty du \rho'(u)\ln u\right], \tag{5.57}$$

where $\Psi(z) = \Gamma'(z)/\Gamma(z)$ is the digamma function and we have used the fact that $\rho(u) \to 0$ as $u \to 0$ and $\rho(u) \to 1$ as $u \to \infty$. It immediately follows that

$$\gamma[\theta_{-2\alpha}(x,0)] - \gamma[\eta_{-2\alpha}(x,0)] = \frac{1}{2\alpha+4}, \tag{5.58}$$

$$\gamma[\kappa_{-2\alpha}(x,0)] - \gamma[\eta_{-2\alpha}(x,0)] = \frac{2\alpha+5}{2\,(2+\alpha)\,(\alpha+3)}. \tag{5.59}$$

For the logarithmically divergent ILIs corresponding to $\alpha = 0$, the expressions (5.58) and (5.59) satisfy the finite parts of the consistency condition, given by (5.39). However, for the quadratically divergent ILIs corresponding to $\alpha = -1$, they do not agree with the corresponding consistency conditions (5.43). The difference is easily understood. Although [121] has shown that ILIs regularised using Schwinger proper time automatically satisfy Wu's consistency relations, recall that this is not exactly equivalent to η regularisation with regulators of the form $\eta(|k|/\Lambda)$ for which we derived the consistency conditions (5.42), (5.39) and (5.43). The equivalence only emerges in the limit $\mathcal{M}_\Lambda \to 0$, suggesting that in the limit of large Λ, only the leading order constraints should align. This is indeed the case. For quadratically divergent ILIs, the leading order constraint is (5.42) and this holds for the derived regulators, $\eta_2(x,0)$, $\theta_2(x,0)$ and $\kappa_2(x,0)$, whereas the subleading constraint (5.43) does not, exactly as expected. For logarithmically divergent ILIs, the leading order constraint is (5.39), which we have just seen to hold for the derived regulators, $\eta_0(x,0)$, $\theta_0(x,0)$ and $\kappa_0(x,0)$, just as it should.

5.3 Conclusions

In this chapter we have explored possible connections between the method of smoothed asymptotics in Chapter 4, and a new method for regularising divergent integrals in QFT. Inspired by Tao's work, we have dubbed this η *regularisation*. As a result, we have demonstrated a connection between the regularisation of divergent series and the preservation of gauge invariance at one-loop in a regularised QFT. We found that in order for η to preserve gauge invariance and satisfy the gauge consistency conditions, it must be an *enhanced regulator*, as defined in Chapter 4. Is this just a coincidence, or does it signal something deeper? Can it be that these first observations open a

possible window into how divergences are eliminated at high energies in a complete microscopic theory?

There is certainly much more to be learnt before we can answer this question. Indeed, it is notable that the consistency relations for preserving gauge invariance, at least in the context of η regularisation, also include a constraint on the finite terms, as well as the divergences. We have not been able to glean a deeper understanding of the meaning of these additional constraints, beyond noting that they are satisfied in the appropriate limit by η regulators inspired by Schwinger proper time integrals. Having said that, the link to Schwinger proper time integrals is intriguing, not least because of an intuitive connection between the worldline formulation of QFT and the field theory limit of string theory, which is a central topic of Part III of this thesis.

We also note how regularised Schwinger proper time techniques have been used to improve the behaviour of functional RG equations for gauge theories at the UV cut-off [122]. In the future, it would be interesting to see if a functional RG equation derived using enhanced η regulators enjoyed similar properties and to explore the implications for Weinberg's asymptotic safety program; however, such an analysis extends beyond the focus of this thesis.

Let us say a little more about higher loops and, in particular, difficulties associated with overlapping divergences that begin at two-loop order. Key insights in this regard were developed in the symmetry preserving regularisation scheme developed by Wu [103], which has much in common with our generalised approach. Although the extension of η regularisation to arbitrary loop order lies beyond the remit of the present thesis, we anticipate it will serve for an interesting analysis into the careful treatment of overlapping divergences and subdivergences. Moreover, at one-loop we have already seen that the ILIs encode the overall UV contribution and tensor structure of Feynman integrals. Regularisation then follows from a simple redefinition of the measure in the master integrals. At two-loops and higher, one must repeatedly use Feynman parametrisation to ensure no Feynman parameter integration contains UV divergences and that appropriate subtractions can be made to show all overlapping divergences are harmless. Generalising the ILI formalism, and using the α, β, γ technology first introduced by 't Hooft and Veltman [46], one can follow the procedures set out in [103], including key theorems based on the factorisation, subtraction, and reduction of

overlapping divergences. At n-loops, it is likely η weighted integrands and the measure of integration will be required to transform a certain way, with technical requirements different from those found in the loop regularisation by Wu [103].

6

Gauge invariance and generalised η regularisation

Nature seems to take advantage of the simple mathematical representations of the symmetry laws. When one pauses to consider the elegance and the beautiful perfection of the mathematical reasoning involved and contrast it with the complex and far-reaching physical consequences, a deep sense of respect for the power of the symmetry laws never fails to develop.

— ***Chen-Ning Yang, Selected Papers of Chen-Ning Yang II: With Commentaries (2013)***

Insights obtained in the previous chapter beg the question: is it possible to generalise η regularisation any further, and is a sufficiently general definition of η capable of capturing other regularisation schemes, like dimensional regularisation? In this chapter, we seek to answer these questions, and in doing so attempt to develop a general theory of regulators. We show that η regularisation can indeed be extended to a much broader class of regulators, going beyond the structure first inspired by Tao [40]. This allows us to capture all known methods of regularisation in physics and more beyond, leading to the derivation of a "master equation". These observations also inspire an examination of how η renormalisation is implemented, before turning attention to the study of chiral anomalies.

6.1 Introduction

As we have commented in previous chapters, there are a number of common regularisation schemes, all of which have have their advantages and disadvantages [42, 123]. When evaluating the strengths and weaknesses of a particular regularisation method, an important criterion is the preservation of gauge symmetry. All of the fundamental forces of nature—i.e., gravitational, electromagnetic, weak and strong forces—are governed by basic symmetry principles. For example, the electromagnetic, weak and strong forces are characterised by the gauge symmetry $U(1) \times SU(2) \times SU(3)$, while the gravitational force is governed by diffeomorphism invariance. In fact, any Lorentz invariant theory of massless fields of spin one or spin two must possess a gauge symmetry [41].

The calculation of perturbative quantum corrections can also give rise to quantum anomalies [48, 123, 124]. A quantum anomaly is the failure to preserve a classical symmetry in the process of quantisation and regularisation. This is not necessarily a problem if the classical symmetry is a global one. On the other hand, if the classical symmetry is local, a gauge anomaly generates a mismatch between the numbers of the degrees of freedom in the classical and quantum corrected theories.

Chiral anomalies are the prototypical example often used to probe the effects of regularisation, first discovered in the context of spinor electrodynamics. Classically, there are two conserved currents: an axial current associated with the chiral symmetry and a vector current associated with the gauge symmetry. In 1969 Adler [125] and Bell and Jackiw [126], showed in the computation of decay processes of a neutral pion to two photons that only the vector current is conserved after quantum corrections are taken into account. As a result, the conservation of the axial current is found to be anomalously broken.

Of course, not every regularisation scheme is well suited to studying a particular process for a particular QFT. For example, Pauli-Villars regularisation runs into issues with broken gauge invariance, in contrast to dimensional regularisation [56]. For its part, dimensional regularisation is hard to understand intuitively and is tricky to implement in chiral theories due the presence of the dimensionally dependent definition of the γ_5 matrix.

In this chapter, we study the problem of gauge invariance in a scheme independent way and in a fixed number of dimensions. The tools for doing this are inspired by the η regularisation scheme developed in Chapter 5. As we saw, just as smoothed asymptotics offers an intuitive and technically simple approach to the regularisation of divergent series, so η regularisation does the same for divergent loop integrals. This technical simplicity allowed us to reveal an unexpected relationship between the regularisation of divergent series in number theory and gauge invariance in QFT.

We shall generalise η regularisation just enough to capture most known regularisation schemes, whilst retaining enough simplicity to "solve" a set of gauge consistency conditions. These "solutions" correspond to regularisation schemes that preserve gauge invariance, like dimensional regularisation. We also study anomalies in chiral theories within this generalised framework. This allows us to demonstrate how η regularisation ought to be properly implemented in order to recover the correct results.

This is a first step towards a generalised theory of consistent regulators which may offer a tantalising glimpse of something more fundamental.

6.2 Generalised η regularisation

A systematic study of regularisation in QFT benefits from a canonical form for the diverging integrals that appear in a generic theory. At one-loop and for QFTs defined on Minkowski space, this canonical form is provided by the one-fold ILIs, as advocated by Wu [103, 104, 107] (see also discussion in earlier works [127]) and discussed in Chapter 5. In four spacetime dimensions, the ILIs of integer spin s are given by

$$I^{\mu_1\cdots\mu_s}_{-2\alpha}(\mathcal{M}^2) = \int \frac{d^4k}{(2\pi)^4} \frac{k^{\mu 1}\cdots k^{\mu s}}{\left(k^2+\mathcal{M}^2-i\epsilon\right)^{2+\frac{s}{2}+\alpha}}. \tag{6.1}$$

The subscript (-2α) once again labels the power counting dimension of energy-momentum with $\alpha=-1$ and $\alpha=0$ corresponding to quadratic and logarithmically divergent integrals. The mass term $\mathcal{M}^2=\mathcal{M}^2(m_1^2,p_1^2,\dots)$ is a function of Feynman parameters, external momenta p_i, and corresponding mass scales m_i. We define $k^2=g_{\mu\nu}k^\mu k^\nu$ where the metric $g_{\mu\nu}$ is written with mostly positive signature. As commented previously, to generalise this canonical form to n-loops, one can introduce

the n-fold ILIs for which there are no longer the overlapping factors $(k_i - k_j + p)$ in the denominator of the integrand and no factors of the scalar momentum k^2 in the numerator [107].

Following a similar strategy in Chapter 5, we implement a generalised form of η regularisation by Wick rotating to Euclidean signature $k_0 \to ik_4$ and inserting a regularisation factor of $\eta\left(\frac{k}{\mu}, \epsilon\right)$ into the integrand of the ILI, where k is now the norm of the Euclidean four-momentum, μ is a fixed mass scale, and ϵ is a dimensionless parameter that controls the rate at which the regulator damps the corresponding integrals. This yields $I^{\cdots}_{-2\alpha} \to iJ^{\cdots}_{-2\alpha}[\eta]$, where the η factor need not coincide for different ILIs of different spin. We will see why this must be the case in a moment. In any event, the regularised ILIs are given by

$$J^{\mu_1\cdots\mu_s}_{-2\alpha}[\eta](\mathcal{M}^2) = \int \frac{d^4k}{(2\pi)^4} \frac{k^{\mu}1 \cdots k^{\mu s}}{(k^2 + \mathcal{M}^2)^{2+\frac{s}{2}+\alpha}} \eta\left(\frac{k}{\mu}, \epsilon\right), \tag{6.2}$$

where the integration is over four-dimensional Euclidean space. Note that the rotational symmetry ensures that the ILIs of odd spin vanish identically.

As in Chapter 5, the role of the regulator function η is to suppress the ultraviolet divergences in the integrals. Recall that we previously took $\eta = \eta(k/\Lambda)$ where Λ is some regularisation scale and $\eta(x)$ was assumed to be a smooth function defined on the positive real line. For this to correspond to a good regulator, we further assumed that $\eta(x) \to 1$ as $x \to 0^+$ and that $\eta(x)$ was a Schwartz function, vanishing faster than any negative power of x as $x \to \infty$. In this chapter we explore a more general form for our regulator functions, namely $\eta\left(\frac{k}{\mu}, \epsilon\right)$ as described above. For $\eta(y, \epsilon)$ to be a good regulator function, we now require it to be smooth in y and to satisfy the following:

- in a neighbourhood of $\epsilon = 0$, we have $\eta(y, \epsilon) \approx 1 + \epsilon^{\delta} f(y)$ for some $\delta > 0$ and some smooth function $f(y)$;
- there exists $a, b, c > 0$, such that for $\epsilon \geq a$, we have $|\eta(y, \epsilon)| < cy^{-2b}$ as $y \to \infty$.

The first condition states that the regulator function tends smoothly to one as we remove the control parameter, ϵ. The second condition states that for ϵ greater than some critical value $a > 0$, the regulator dies off at infinity faster than some

negative power. Clearly this is a much weaker set of conditions than those imposed in Chapter 5. For example, we can now capture the form of the regulator expected from dimensional regularisation, for which $\eta(y,\epsilon) = y^{-2\epsilon}$. If we were to identify a regularisation scale, $\Lambda = \mu/\epsilon$, the class of regulator functions studied in the previous chapter correspond to the case where $\eta(y,\epsilon) = \eta(y\epsilon)$.

Whenever we compute a correlation function up to one-loop in a given QFT, we have to do some work to write it in terms of ILIs. This process is discussed in detail in Appendix A of [103] for generic vacuum polarisation functions for gauge fields. We also provide an explicit demonstration for the vacuum polarisation function in QED in section 6.4. To express a generic amplitude in terms of ILIs, we saw in the previous chapter that one must perform a series of standard manipulations, including Feynman parametrisation [41], which combines denominators into a single power, and momentum routing, which shifts the integration variable in loop momentum space. A subtle but crucial point arises when dealing with traces over gamma matrices or contractions involving the metric tensor and loop momenta. The decomposition into ILIs depends on whether such contractions are performed inside or outside the integral. It is essential that these contractions be carried out within the integrals. This is because metric tensors or traces of gamma matrices cannot, in general, be unambiguously pulled through divergent integrals. This was explicitly demonstrated in [128] for several regularisation schemes. We will see how it is also the case for gauge invariant η regularisation schemes in section 6.5.

As many of our manipulations are only valid if the integrals are properly regulated, these regularisations should be implicitly understood when arriving at the desired form of regularised ILIs. In this sense, our approach is similar in spirit to the implicit regularisation methods introduced in [127]. As stated above, some manipulations involve the use of Feynman parametrisation. In Appendix B, we explicitly show how the regulator function changes form as we undo a Feynman parametrisation.

The last step of Feynman parametrisation is always to shift the momentum integration. This might be a worry, since it is not at all obvious that momentum routing is allowed under η regularisation. In the last chapter, we allowed the regulators to differ for integrals of different spin. This was argued for on the grounds of generality, although we will now show that it was crucial to allow for momentum routing.

Consider the following regularised ILIs and impose momentum routing, with $\rho\left(\frac{k}{\mu},\epsilon\right)$ a general smooth regulator,

$$\begin{aligned} J^{\mu_1\cdots\mu_s}_{-2\alpha}[\rho_{-2\alpha}](\mathcal{M}^2) &= \int \frac{d^4k}{(2\pi)^4}\frac{k^{\mu_1}\cdots k^{\mu_s}}{\left(k^2+\mathcal{M}^2\right)^{2+\frac{s}{2}+\alpha}}\rho_{-2\alpha}\left(\frac{k}{\mu},\epsilon\right) \\ &= \int \frac{d^4k}{(2\pi)^4}\frac{(k+q)^{\mu_1}\cdots(k+q)^{\mu_s}}{\left((k+q)^2+\mathcal{M}^2\right)^{2+\frac{s}{2}+\alpha}}\rho_{-2\alpha}\left(\frac{|k+q|}{\mu},\epsilon\right) \\ &= \int \frac{d^4k}{(2\pi)^4}e^{q^\nu\frac{\partial}{\partial k^\nu}}\left[\frac{k^{\mu_1}\cdots k^{\mu_s}}{\left(k^2+\mathcal{M}^2\right)^{2+\frac{s}{2}+\alpha}}\rho_{-2\alpha}\left(\frac{k}{\mu},\epsilon\right)\right]. \end{aligned} \tag{6.3}$$

For this to be true for any value of q, we must have the following constraints for any integer value of $r,s\geq 0$

$$\int \frac{d^4k}{(2\pi)^4}\frac{\partial^s}{\partial k^{\nu_1}\cdots\partial k^{\nu_r}}\left[\frac{k^{\mu_1}\cdots k^{\mu_s}}{\left(k^2+\mathcal{M}^2\right)^{2+\frac{s}{2}+\alpha}}\rho_{-2\alpha}\left(\frac{k}{\mu},\epsilon\right)\right]=0. \tag{6.4}$$

Thanks to the rotational symmetry of the scalar integrand, this is trivially true for odd values of $r+s$. However, for even values, we obtain non-trivial relations between ILIs of different spin. For example, for $r=s=1$ and taking $\rho=\eta_{-2\alpha}$, we see that

$$\int \frac{d^4k}{(2\pi)^4}\frac{\partial}{\partial k^\mu}\left[\frac{k_\nu}{(k^2+\mathcal{M}^2)^{2+\alpha}}\eta_{-2\alpha}\left(\frac{k}{\mu},\epsilon\right)\right]=0. \tag{6.5}$$

For $\alpha\neq -2$, this yields the following relationship between the ILIs of spin two and spin zero

$$J^{\mu\nu}_{-2\alpha}[\theta_{-2\alpha}](\mathcal{M}^2)=\frac{1}{2(2+\alpha)}g^{\mu\nu}J_{-2\alpha}[\eta_{-2\alpha}](\mathcal{M}^2), \tag{6.6}$$

where

$$\theta_{-2\alpha}\left(y,\epsilon\right)=-\frac{(y^2+y_0^2)^{3+\alpha}}{2(2+\alpha)}\frac{\partial}{y\partial y}\left[\frac{\eta_{-2\alpha}\left(y,\epsilon\right)}{(y^2+y_0^2)^{2+\alpha}}\right]. \tag{6.7}$$

Similarly, for $r=s=2$ and again taking $\rho=\eta_{-2\alpha}$, we find

$$\int \frac{d^4k}{(2\pi)^4}\frac{\partial^2}{\partial k^{\mu_1}\partial k^{\mu_2}}\left[\frac{k_{\mu_3}k_{\mu_4}}{(k^2+\mathcal{M}^2)^{2+\alpha}}\eta_{-2\alpha}\left(\frac{k}{\mu},\epsilon\right)\right]=0. \tag{6.8}$$

For $\alpha \neq -2, -3$, this gives a complex relationship between the ILIs of spin four, spin two, and spin zero

$$J^{\mu_1\cdots\mu_4}_{-2\alpha}[\kappa_{-2\alpha}] = \frac{1}{2(3+\alpha)}\Big[J^{\mu_1\mu_3}_{-2\alpha}[\theta_{-2\alpha}]g^{\mu_2\mu_4} + J^{\mu_1\mu_4}_{-2\alpha}[\theta_{-2\alpha}]g^{\mu_2\mu_3} + J^{\mu_2\mu_3}_{-2\alpha}[\theta_{-2\alpha}]g^{\mu_1\mu_4} + J^{\mu_2\mu_4}_{-2\alpha}[\theta_{-2\alpha}]g^{\mu_1\mu_3} + J^{\mu_3\mu_4}_{-2\alpha}[\eta_{-2\alpha}]g^{\mu_1\mu_2}\Big] - \frac{1}{4(2+\alpha)(3+\alpha)}J_{-2\alpha}[\eta_{-2\alpha}]\left(g^{\mu_1\mu_3}g^{\mu_2\mu_4} + g^{\mu_1\mu_4}g^{\mu_2\mu_3}\right), \quad (6.9)$$

where

$$\kappa_{-2\alpha}(y,\epsilon) = \frac{(y^2+y_0^2)^{4+\alpha}}{4(2+\alpha)(3+\alpha)}\left(\frac{\partial}{y\partial y}\right)^2\left[\frac{\eta_{-2\alpha}(y,\epsilon)}{(y^2+y_0^2)^{2+\alpha}}\right]. \quad (6.10)$$

Combining (6.9) with (6.6), we can by-pass the ILIs of spin 2, and relate the ILIs of spin four and spin zero directly

$$J^{\mu_1\cdots\mu_4}_{-2\alpha}[\kappa_{-2\alpha}](\mathcal{M}^2) = \frac{1}{4(2+\alpha)(3+\alpha)}S^{\mu_1\cdots\mu_4}J_{-2\alpha}[\eta_{-2\alpha}](\mathcal{M}^2), \quad (6.11)$$

where $S^{\mu_1\cdots\mu_4} = g^{\mu_1\mu_2}g^{\mu_3\mu_4} + g^{\mu_1\mu_3}g^{\mu_2\mu_4} + g^{\mu_1\mu_4}g^{\mu_2\mu_3}$ is the symmetric product of metrics.

It is not too difficult to convince oneself that if $\eta_{-2\alpha}$ is a good regulator in the way we defined earlier, then the same is true of both $\theta_{-2\alpha}$ and $\kappa_{-2\alpha}$. Furthermore, we note that (6.6) and (6.11) are identical to the gauge consistency conditions derived by Wu [103, 104, 107] from generalised Ward identities in a generic gauge theory with a gauge group of arbitrary rank and an arbitrary number of Dirac spinors. This alignment is no coincidence, as discussed in detail in [127]. In Chapter 5, we imposed those gauge consistency conditions in η regularisation while allowing the regulator to differ for integrals of different spin and different dimension. We now see that this was a necessary ingredient, given the explicit form of $\theta_{-2\alpha}$ and $\kappa_{-2\alpha}$ presented in (6.7) and (6.10) derived using momentum routing.

We end this section with a brief comment on causality. As is well known, to preserve the causal structure, we should implement Feynman's $i\epsilon$ prescription prior to any Wick rotation from Minkowski to Euclidean signature, $k_0 \to ik_4$. In a generalised η

regularisation scheme, it should be possible to Wick rotate the integrand as long as the smooth regulator functions do not introduce any extra poles that block the path of the contour rotation.

6.3 Gauge invariant η regularisation

η regularisation provides us with a formalism to study regularisation schemes in an abstract way. To illustrate this, we consider the question of gauge invariance. It is well known that many popular regularisation schemes such as cut-off regularisation or Pauli-Villars can run into problems with broken gauge invariance. In this section, we will show how one can use the formalism of η regularisation to "solve" for regularisation schemes that preserve the underlying gauge symmetries.

Consider again a generic gauge theory where the gauge group has dimension d_G and where there are N_f Dirac spinors ψ_n ($n = 1, \ldots, N_f$) interacting with the Yang Mills field A^a_μ ($a = 1, \ldots, d_G$). This is described by a Lagrangian,

$$\mathcal{L} = \bar{\psi}_n(i\not{D} - m)\psi_n - \frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a, \tag{6.12}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f_{abc} A^b_\mu A^c_\nu, \qquad D_\mu \psi_n = (\partial_\mu + igT^a A^a_\mu)\psi_n, \tag{6.13}$$

and T^a are the generators of the gauge group whose commutator $[T^a, T^b] = if^{abc}T^c$ defines the structure constants f^{abc}. Following [103, 104, 129] and as discussed in Chapter 5, one can compute correlation functions at one-loop and impose a set of generalised Ward identities. For example, we recall that the vacuum polarisation of the gauge field takes the form,

$$\Pi^{ab}_{\mu\nu}(p) = \Pi^{(g)ab}_{\mu\nu}(p) + \Pi^{(f)ab}_{\mu\nu}(p), \tag{6.14}$$

where p^μ is the external momentum, $\Pi^{(g)ab}_{\mu\nu}(p)$ are the pure Yang Mills contributions coming from gauge field loops and ghost loops, and $\Pi^{(f)ab}_{\mu\nu}(p)$ are the contributions from fermion loops, coming from the interaction of the fermions with the gauge

field. As is well known, gauge invariance is given in terms of the Ward identities $p^\mu \Pi^{ab}_{\mu\nu} = \Pi^{ab}_{\mu\nu} p^\nu = 0$. However, since our interest lies in the regularisation scheme rather than the details of any individual theory, we require these Ward identities to hold for *any* gauge theory and with *any* number of fermions. This suggests that the Ward identities should hold separately for the gauge field and fermionic contributions

$$p^\mu \Pi^{(g)ab}_{\mu\nu} = \Pi^{(g)ab}_{\mu\nu} p^\nu = 0, \qquad p^\mu \Pi^{(f)ab}_{\mu\nu} = \Pi^{(f)ab}_{\mu\nu} p^\nu = 0. \tag{6.15}$$

One can apply a similar logic to the Ward identities derived from other correlation functions. In order to satisfy the resulting generalised Ward identities, we arrive at consistency conditions on the ILIs for $\alpha = -1, 0, 1$, which are those derived from momentum routing, in (6.6) and (6.11). Note that for $\alpha > 0$, the integrals are convergent and no regulator is required. In this case, the consistency conditions are satisfied trivially as, of course, they should be. For this, our remaining focus is on the case where $\alpha = 0, -1$, corresponding to logarithmic and quadratically divergent ILIs, respectively.

It is important to note that generic regularisation schemes do *not* necessarily satisfy the gauge consistency conditions. For example, consider the following quadratically divergent ILIs using cut-off regularisation

$$J_2^{\mu\nu}(\mathcal{M}^2) = \int^\Lambda \frac{d^4k}{(2\pi)^4} \frac{k^\mu k^\nu}{(k^2+\mathcal{M}^2)^2} = \frac{g^{\mu\nu}}{8\pi^2}\left(\frac{\Lambda^2}{8} - \frac{\mathcal{M}^2}{4}\ln\left(\frac{\Lambda^2}{\mathcal{M}^2}\right)\right) + \ldots, \tag{6.16}$$

and

$$J_2(\mathcal{M}^2) = \int^\Lambda \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2+\mathcal{M}^2)} = \frac{1}{8\pi^2}\left(\frac{\Lambda^2}{2} - \frac{\mathcal{M}^2}{2}\ln\left(\frac{\Lambda^2}{\mathcal{M}^2}\right)\right) + \ldots, \tag{6.17}$$

where the ellipses denote finite terms as $\Lambda \to \infty$. Since $J_2^{\mu\nu}(\mathcal{M}^2) \neq \frac{1}{2} g^{\mu\nu} J_2(\mathcal{M}^2)$ the gauge consistency conditions are clearly violated. This is well known: cut-off regularisation breaks gauge invariance.

However, as is also well known, gauge invariant regulators do exist. In this section, our goal is to present a systematic way of finding them using η regularisation. For a suitably chosen $\eta_{-2\alpha}$, we may be tempted to regard (6.7) and (6.10) as "solutions" of the gauge consistency conditions. However, these solutions are not particularly useful

or illuminating. The reason is that one is always free to shift each of the regulators as follows, without having any impact on the consistency conditions (6.6) and (6.11),

$$\eta_{-2\alpha}(y,\epsilon) \rightarrow \eta_{-2\alpha}(y,\epsilon) + Z^{(0)}_{-2\alpha}(y,\epsilon), \tag{6.18}$$

$$\theta_{-2\alpha}(y,\epsilon) \rightarrow \theta_{-2\alpha}(y,\epsilon) + Z^{(2)}_{-2\alpha}(y,\epsilon), \tag{6.19}$$

$$\kappa_{-2\alpha}(y,\epsilon) \rightarrow \kappa_{-2\alpha}(y,\epsilon) + Z^{(4)}_{-2\alpha}(y,\epsilon), \tag{6.20}$$

where the $Z^{(n)}$ for $n = 0, 2, 4$ are smooth functions defined on $y \geq 0$. These functions should vanish in the asymptotic limit, $Z^{(n)}(y,\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$. For sufficiently large $\epsilon > 0$, they should decay at large momenta, vanishing faster than some inverse polynomial as $y \rightarrow \infty$. These conditions ensure that the shifts given above preserve the properties of a good regulator defined in the previous section. However, in order to also protect the value of the ILIs, and by association the gauge consistency conditions, we further require

$$J_{-2\alpha}[Z^{(0)}_{-2\alpha}](\mathcal{M}^2) = 0,\ J^{\mu\nu}_{-2\alpha}[Z^{(2)}_{-2\alpha}](\mathcal{M}^2) = 0,\ J^{\mu_1\cdots\mu_4\nu}_{-2\alpha}[Z^{(4)}_{-2\alpha}](\mathcal{M}^2) = 0. \tag{6.21}$$

Note that a shift in, say, $\eta_{-2\alpha}$ of the form (6.18) will generate a shift of the form (6.19) and (6.20) in the $\theta_{-2\alpha}$ and $\kappa_{-2\alpha}$ under the definition (6.7) and (6.10).

All of this suggests that the consistency equations (6.6) and (6.11) are equations for the regulators that may be solved directly, allowing for solutions that do not necessarily take the specific form given by (6.7) and (6.10). This is the approach we will take to finding consistent regulators in a familiar form.

To this end, we use the rotational symmetry to write the regulated ILIs for spin two and spin four as scalar integrals,

$$J^{\mu\nu}_{-2\alpha}[\theta_{-2\alpha}](\mathcal{M}^2) = \frac{1}{4}g^{\mu\nu}\int\frac{d^4k}{(2\pi)^4}\frac{k^2}{(k^2+\mathcal{M}^2)^{3+\alpha}}\theta_{-2\alpha}\left(\frac{k}{\mu},\epsilon\right), \tag{6.22}$$

$$J^{\mu_1\cdots\mu_4}_{-2\alpha}[\kappa_{-2\alpha}](\mathcal{M}^2) = \frac{1}{4!}S^{\mu_1\cdots\mu_4}\int\frac{d^4k}{(2\pi)^4}\frac{k^4}{(k^2+\mathcal{M}^2)^{4+\alpha}}\kappa_{-2\alpha}\left(\frac{k}{\mu},\epsilon\right). \tag{6.23}$$

Plugging these expressions into the conditions (6.6) and (6.11) and rearranging a

little, we obtain the following integral constraints

$$\int_0^\infty dy\ \frac{y^3}{(y^2+y_0^2)^{2+\alpha}}\left[\frac{y^2}{y^2+y_0^2}\theta_{-2\alpha}-\frac{2}{2+\alpha}\eta_{-2\alpha}\right] = 0, \tag{6.24}$$

$$\int_0^\infty dy\ \frac{y^5}{(y^2+y_0^2)^{3+\alpha}}\left[\frac{y^2}{y^2+y_0^2}\kappa_{-2\alpha}-\frac{3}{3+\alpha}\theta_{-2\alpha}\right] = 0. \tag{6.25}$$

It will be useful to construct solutions to these constraints in the form

$$\frac{y^2}{y^2+y_0^2}\theta_{-2\alpha}-\frac{2}{2+\alpha}\eta_{-2\alpha} = \frac{(y^2+y_0^2)^{2+\alpha}}{y^3}\frac{\partial}{\partial y}\psi_{-2\alpha}(y,\epsilon), \tag{6.26}$$

$$\frac{y^2}{(y^2+y_0^2)}\kappa_{-2\alpha}-\frac{3}{(3+\alpha)}\theta_{-2\alpha} = \frac{(y^2+y_0^2)^{3+\alpha}}{y^5}\frac{\partial}{\partial y}\phi_{-2\alpha}(y,\epsilon), \tag{6.27}$$

where $\psi_{-2\alpha}(y,\epsilon)$ and $\phi_{-2\alpha}(y,\epsilon)$ are smooth functions defined on $y\geq 0$. For sufficiently large $\epsilon>0$, these functions should vanish at $y=0$ and as $y\to\infty$. Furthermore, in the limit where $\epsilon\to 0$, recall that all regulators tend to unity, from which we infer that

$$\psi_{-2\alpha}(y,0)=-\frac{1}{2(2+\alpha)}\frac{y^4}{(y^2+y_0^2)^{2+\alpha}},\ \phi_{-2\alpha}(y,0)=-\frac{1}{2(3+\alpha)}\frac{y^6}{(y^2+y_0^2)^{3+\alpha}}, \tag{6.28}$$

where we have ignored the possibility of an irrelevant additive constant in each case.

6.3.1 Explicit "solutions"

We are now ready to state explicit solutions to the consistency conditions (6.6) and (6.11) using the solution generating method we have just derived.

Tao inspired regularisation schemes:

We begin with the regularisation schemes explored in Chapter 4 and Chapter 5, for which the regulators have the form $\eta=\eta(\epsilon y)$. Introducing $x=\epsilon y$, the integral constraints (6.24) and (6.25) now read

$$\epsilon^{2\alpha}\int_0^\infty dx\ \frac{x^3}{(x^2+\epsilon^2y_0^2)^{2+\alpha}}\Big[\frac{x^2}{x^2+\epsilon^2y_0^2}\theta_{-2\alpha}-\frac{2}{2+\alpha}\eta_{-2\alpha}\Big]=0, \tag{6.29}$$

$$\epsilon^{2\alpha}\int_0^\infty dx\ \frac{x^5}{(x^2+\epsilon^2 y_0^2)^{3+\alpha}}\Big[\frac{x^2}{x^2+\epsilon^2 y_0^2}\kappa_{-2\alpha}-\frac{3}{3+\alpha}\theta_{-2\alpha}\Big]=0. \tag{6.30}$$

If we carry out an expansion in $\epsilon \ll 1$, we obtain

$$\begin{aligned}
&\epsilon^{2\alpha}\int_0^\infty dx\ x^{-1-2\alpha}\left[\theta_{-2\alpha}-\frac{2}{2+\alpha}\eta_{-2\alpha}\right]\\
&\quad+\epsilon^{2(\alpha+1)}y_0^2\int_0^\infty dx\ x^{-3-2\alpha}\left[-(3+\alpha)\theta_{-2\alpha}+2\eta_{-2\alpha}\right]\\
&\quad+\mathcal{O}\left(\epsilon^{2(\alpha+2)}\right)=0, \qquad (6.31)\\
&\epsilon^{2\alpha}\int_0^\infty dx\ x^{-1-2\alpha}\left[\kappa_{-2\alpha}-\frac{3}{3+\alpha}\theta_{-2\alpha}\right]\\
&\quad+\epsilon^{2(\alpha+1)}y_0^2\int_0^\infty dx\ x^{-3-2\alpha}\left[-(4+\alpha)\kappa_{-2\alpha}+3\theta_{-2\alpha}\right]\\
&\quad+\mathcal{O}\left(\epsilon^{2(\alpha+2)}\right)=0, \qquad (6.32)
\end{aligned}$$

where we have explicitly shown the terms which are non-vanishing as $\epsilon \to 0$, at least for $\alpha = 0, -1$. For quadratically divergent ILIs with $\alpha = -1$, requiring the divergent parts to vanish order by order in ϵ gives the following constraints

$$\int_0^\infty dx\ x\left[\theta_2-2\eta_2\right]=0, \qquad \int_0^\infty dx\ x\left[2\kappa_2-3\theta_2\right]=0, \tag{6.33}$$

$$\int_0^\infty dx\ \frac{\theta_2-\eta_2}{x}=0, \qquad \int_0^\infty dx\ \frac{\kappa_2-\theta_2}{x}=0. \tag{6.34}$$

Similarly, for logarithmically divergent ILIs with $\alpha = 0$, we obtain

$$\int_0^\infty dx\ \frac{\theta_0-\eta_0}{x}=0, \qquad \int_0^\infty dx\ \frac{\kappa_0-\theta_0}{x}=0. \tag{6.35}$$

The expressions (6.33) to (6.35) are exactly equivalent to the "enhanced regulator" conditions derived in Chapter 5. Recall that the vanishing Mellin transform given by (6.33) connects gauge invariance in this scheme to the suppression of divergences in number theory.

Dimensional regularisation:

We now turn our attention to a more familiar regularisation scheme and demonstrate how it can emerge in our formalism. In dimensional regularisation, the ILIs take the following form

$$\begin{aligned} J_{-2\alpha}|_{\text{dim reg}}[\eta_{-2\alpha}] &= \mu^{2\epsilon}\int\frac{d^{4-2\epsilon}k}{(2\pi)^{4-2\epsilon}}\frac{1}{(k^2+\mathcal{M}^2)^{2+\alpha}} \\ &= \int\frac{d^4k}{(2\pi)^4}f(\epsilon)\left(\frac{k}{\mu}\right)^{-2\epsilon}\frac{1}{(k^2+\mathcal{M}^2)^{2+\alpha}} \qquad (6.36) \\ J^{\mu\nu}_{-2\alpha}|_{\text{dim reg}}[\theta_{-2\alpha}] &= \mu^{2\epsilon}\int\frac{d^{4-2\epsilon}k}{(2\pi)^4}\frac{k^\mu k^\nu}{(k^2+\mathcal{M}^2)^{3+\alpha}} \\ &= \frac{\mu^{2\epsilon}}{4-2\epsilon}g_{\mu\nu}\int\frac{d^{4-2\epsilon}k}{(2\pi)^4}\frac{k^2}{(k^2+\mathcal{M}^2)^{3+\alpha}} \\ &= \frac{1}{4}g^{\mu\nu}\int\frac{d^4k}{(2\pi)^4}\frac{f(\epsilon)}{\left(1-\frac{\epsilon}{2}\right)}\left(\frac{k}{\mu}\right)^{-2\epsilon}\frac{k^2}{(k^2+\mathcal{M}^2)^{3+\alpha}}, \qquad (6.37) \end{aligned}$$

and

$$\begin{aligned} &J^{\mu_1\cdots\mu_4}_{-2\alpha}|_{\text{dim reg}}[\kappa_{-2\alpha}](\mathcal{M}^2) = \mu^{2\epsilon}\int\frac{d^{4k-2\epsilon}}{(2\pi)^{4-2\epsilon}}\frac{k^{\mu_1}\cdots k^{\mu_4}}{(k^2+\mathcal{M}^2)^{4+\alpha}} \\ &= \frac{1}{(4-2\epsilon)(6-2\epsilon)}S^{\mu_1\cdots\mu_4}\int\frac{d^{4-2\epsilon}k}{(2\pi)^{4-2\epsilon}}\frac{k^4}{(k^2+\mathcal{M}^2)^{4+\alpha}} \\ &= \frac{1}{4!}S^{\mu_1\cdots\mu_4}\int\frac{d^4k}{(2\pi)^4}\frac{f(\epsilon)}{\left(1-\frac{\epsilon}{2}\right)\left(1-\frac{\epsilon}{3}\right)}\left(\frac{k}{\mu}\right)^{-2\epsilon}\frac{k^4}{(k^2+\mathcal{M}^2)^{4+\alpha}}, \qquad (6.38) \end{aligned}$$

where

$$f(\epsilon) = \frac{(2\pi)^\epsilon\Omega_{3-2\epsilon}}{\Omega_3} \qquad (6.39)$$

and $\Omega_n = 2\pi^{\frac{n}{2}}/\Gamma\left(\frac{n}{2}\right)$ is the volume of a unit n-sphere. In other words,

$$\eta_{-2\alpha} = f(\epsilon)y^{-2\epsilon}, \qquad \theta_{-2\alpha} = \frac{f(\epsilon)y^{-2\epsilon}}{1-\frac{\epsilon}{2}}, \qquad \kappa_{-2\alpha} = \frac{f(\epsilon)y^{-2\epsilon}}{\left(1-\frac{\epsilon}{2}\right)\left(1-\frac{\epsilon}{3}\right)}. \qquad (6.40)$$

To see how this solution might have been obtained from the generating technique described above, we consider the smooth seed functions introduced in (6.26) and

(6.27), and make the following simple choice

$$\psi_{-2\alpha}(y,\epsilon) = -\frac{A(\epsilon)}{2(2+\alpha)}\frac{y^{4-2\epsilon}}{(y^2+y_0^2)^{2+\alpha}}, \qquad \phi_{-2\alpha}(y,\epsilon) = -\frac{B(\epsilon)}{2(3+\alpha)}\frac{y^{6-2\epsilon}}{(y^2+y_0^2)^{3+\alpha}}, \tag{6.41}$$

where the coefficients A, B depend on the regularisation parameter ϵ and are assumed to behave smoothly at the origin such that $A(0) = B(0) = 1$. It is easy to see that this choice satisfies all the conditions we impose on the seed functions above. It might also have been easily guessed as a deformation of the known solution at $\epsilon = 0$. Plugging (6.41) into (6.26) and (6.27), we find that

$$\begin{aligned}&\frac{y^2}{y^2+y_0^2}\theta_{-2\alpha} - \frac{2}{2+\alpha}\eta_{-2\alpha}\\ &= \frac{y^2}{y^2+y_0^2}A(\epsilon)y^{-2\epsilon} - \frac{2}{2+\alpha}A(\epsilon)\left(1-\frac{\epsilon}{2}\right)y^{-2\epsilon},\end{aligned} \tag{6.42}$$

$$\begin{aligned}&\frac{y^2}{(y^2+y_0^2)}\kappa_{-2\alpha} - \frac{3}{(3+\alpha)}\theta_{-2\alpha}\\ &= \frac{y^2}{(y^2+y_0^2)}B(\epsilon)y^{-2\epsilon} - \frac{3}{(3+\alpha)}B(\epsilon)\left(1-\frac{\epsilon}{3}\right)y^{-2\epsilon}.\end{aligned} \tag{6.43}$$

It is easy to read off the solution (6.40) provided we identify $A(\epsilon) = f(\epsilon)/\left(1-\frac{\epsilon}{2}\right)$ and $B(\epsilon) = f(\epsilon)/\left(1-\frac{\epsilon}{2}\right)\left(1-\frac{\epsilon}{3}\right)$. Notice that the solution generating technique correctly identifies the form of the regulators in (6.40) and their ratios.

Denominator regularisation:

Another gauge invariant regularisation scheme is denominator regularisation [130–133]. In this case the ILIs take the form:

$$\begin{aligned}&J_{-2\alpha}|_{\text{den reg}}[\eta_{-2\alpha}] = \mu^{2\epsilon}f_{(2+\alpha,-2\alpha)}(\epsilon)\int\frac{d^4k}{(2\pi)^4}\frac{1}{(k^2+\mathcal{M}^2)^{2+\alpha+\epsilon}}\\ &= \int\frac{d^4k}{(2\pi)^4}f_{(2+\alpha,-2\alpha)}(\epsilon)\left(\frac{k^2}{\mu^2}+\frac{\mathcal{M}^2}{\mu^2}\right)^{-\epsilon}\frac{1}{(k^2+\mathcal{M}^2)^{2+\alpha}},\end{aligned} \tag{6.44}$$

$$J^{\mu\nu}_{-2\alpha}|_{\text{den reg}}[\theta_{-2\alpha}] = \mu^{2\epsilon} f_{(3+\alpha,-2\alpha)}(\epsilon) \int \frac{d^4k}{(2\pi)^4} \frac{k^\mu k^\nu}{(k^2+\mathcal{M}^2)^{3+\alpha+\epsilon}}$$
$$= \frac{1}{4} g^{\mu\nu} \int \frac{d^4k}{(2\pi)^4} f_{(3+\alpha,-2\alpha)}(\epsilon) \left(\frac{k^2}{\mu^2} + \frac{\mathcal{M}^2}{\mu^2}\right)^{-\epsilon} \frac{k^2}{(k^2+\mathcal{M}^2)^{3+\alpha}}, \tag{6.45}$$

$$J^{\mu_1\cdots\mu_4}_{-2\alpha}|_{\text{den reg}}[\kappa_{-2\alpha}](\mathcal{M}^2) = \mu^{2\epsilon} f_{(4+\alpha,-2\alpha)}(\epsilon) \int \frac{d^4k}{(2\pi)^4} \frac{k^{\mu_1}\cdots k^{\mu_4}}{(k^2+\mathcal{M}^2)^{4+\alpha}}$$
$$= \frac{1}{4!} S^{\mu_1\cdots\mu_4} \int \frac{d^4k}{(2\pi)^4} f_{(4+\alpha,-2\alpha)}(\epsilon) \left(\frac{k^2}{\mu^2} + \frac{\mathcal{M}^2}{\mu^2}\right)^{-\epsilon} \frac{k^4}{(k^2+\mathcal{M}^2)^{4+\alpha}}. \tag{6.46}$$

In each case, the function $f_{(n,p)}(\epsilon)$ is introduced for integrals with whose denominator integrand has power n, after Feynman parametrisation, and whose integral has degree of divergence p. These functions should tend smoothly to unity as $\epsilon \to 0$ and are fixed by demanding that the Laurent expansion of the corresponding amplitude in ϵ converges for all finite $\epsilon > 0$. In other words, the amplitude should only have a simple pole at $\epsilon = 0$; $f_{(n,p)}(\epsilon)$ should cancel any UV poles that go as $1/(\epsilon - p/2)$ and any IR poles that emerge for $\epsilon \in \mathbb{N}^+$ even in the massless limit [131, 133]. The mass here refers to the physical masses in the system, as opposed to the mass $\mathcal{M}$ appearing in the ILIs, which also depends on external momenta and Feynman parameters. In practice, the procedure for computing the $f_{(n,p)}(\epsilon)$ requires us to also compute the integration over the Feynman parameters, as well as the regulated ILIs. In any event, we can read off the form of our η regulators as follows

$$\eta_{-2\alpha} = \frac{f_{(2+\alpha,-2\alpha)}(\epsilon)}{(y^2+y_0^2)^\epsilon},\ \theta_{-2\alpha} = \frac{f_{(3+\alpha,-2\alpha)}(\epsilon)}{(y^2+y_0^2)^\epsilon},\ \kappa_{-2\alpha} = \frac{f_{(4+\alpha,-2\alpha)}(\epsilon)}{(y^2+y_0^2)^\epsilon}. \tag{6.47}$$

Similar structures can also be derived from our formalism. To do so, we make the following choice for our seed functions

$$\psi_{-2\alpha}(y,\epsilon) = -\frac{A(\epsilon)}{2(2+\alpha)} \frac{y^4}{(y^2+y_0^2)^{2+\alpha+\epsilon}},$$
$$\phi_{-2\alpha}(y,\epsilon) = -\frac{B(\epsilon)}{2(3+\alpha)} \frac{y^6}{(y^2+y_0^2)^{3+\alpha+\epsilon}}, \tag{6.48}$$

where, as before, we assume that the coefficients A, B depend on the regularisation

parameter ϵ and are assumed to behave smoothly at the origin such that $A(0) = B(0) = 1$. Plugging this into (6.26) and (6.27), we find that

$$\begin{aligned}&\frac{y^2}{y^2+y_0^2}\theta_{-2\alpha} - \frac{2}{2+\alpha}\eta_{-2\alpha}\\ &= \frac{y^2}{y^2+y_0^2}\left[\frac{2+\alpha+\epsilon}{2+\alpha}\frac{A(\epsilon)}{(y^2+y_0^2)^\epsilon}\right] - \frac{2}{2+\alpha}\left[\frac{A(\epsilon)}{(y^2+y_0^2)^\epsilon}\right],\end{aligned} \tag{6.49}$$

$$\begin{aligned}&\frac{y^2}{(y^2+y_0^2)}\kappa_{-2\alpha} - \frac{3}{(3+\alpha)}\theta_{-2\alpha}\\ &= \frac{y^2}{y^2+y_0^2}\left[\frac{3+\alpha+\epsilon}{3+\alpha}\frac{B(\epsilon)}{(y^2+y_0^2)^\epsilon}\right] - \frac{3}{3+\alpha}\left[\frac{B(\epsilon)}{(y^2+y_0^2)^\epsilon}\right].\end{aligned} \tag{6.50}$$

This yields (6.47) provided we have the following consistency conditions

$$\begin{aligned}f_{(3+\alpha,-2\alpha)}(\epsilon) &= \frac{2+\alpha+\epsilon}{2+\alpha}f_{(2+\alpha,-2\alpha)}(\epsilon),\\ f_{(4+\alpha,-2\alpha)}(\epsilon) &= \frac{3+\alpha+\epsilon}{3+\alpha}f_{(3+\alpha,-2\alpha)}(\epsilon),\end{aligned} \tag{6.51}$$

and identify $A(\epsilon) = f_{(2+\alpha,-2\alpha)}(\epsilon)$ and $B(\epsilon) = f_{(3+\alpha,-2\alpha)}(\epsilon)$. These conditions seem to be violated by some of the $f_{(n,p)}(\epsilon)$ seen in the literature [131]. This seems to be because Horowitz advocates for a *minimal* choice of the $f_{(n,p)}(\epsilon)$, avoiding any additional pre-factors that do not affect the convergence properties of the amplitude. However, it seems that this is too restrictive. It is clear that the amplitude still converges for all finite $\epsilon > 0$ if we multiply $f_{(n,p)}(\epsilon)$ by some linear function of ϵ. Contrary to [132], our analysis suggests these non-minimal factors are necessary if denominator regularisation is to preserve gauge invariance.

Schwinger proper time:

The next gauge invariant regularisation method we will consider is Schwinger proper time. In the Schwinger representation of the propagator, the regularised ILIs take the

form:

$$J_{-2\alpha}|_{\rm Schwinger}[\eta_{-2\alpha}]$$
$$= \frac{1}{(1+\alpha)!}\int_0^\infty \frac{d\tau}{\tau}\ \rho(\mu^2\tau,\epsilon)\tau^{2+\alpha}\int \frac{d^4k}{(2\pi)^4}e^{-\tau(k^2+\mathcal{M}^2)}, \tag{6.52}$$

$$\tag{6.53}$$

$$J^{\mu\nu}_{-2\alpha}|_{\rm Schwinger}[\theta_{-2\alpha}]$$
$$= \frac{1}{(2+\alpha)!}\int_0^\infty \frac{d\tau}{\tau}\ \rho(\mu^2\tau,\epsilon)\tau^{3+\alpha}\int \frac{d^4k}{(2\pi)^4}k^\mu k^\nu e^{-\tau(k^2+\mathcal{M}^2)}, \tag{6.54}$$

$$\tag{6.55}$$

$$J^{\mu_1\cdots\mu_4}_{-2\alpha}|_{\rm Schwinger}[\kappa_{-2\alpha}]$$
$$= \frac{1}{(3+\alpha)!}\int_0^\infty \frac{d\tau}{\tau}\ \rho(\mu^2\tau,\epsilon)\tau^{4+\alpha}\int \frac{d^4k}{(2\pi)^4}k^{\mu_1}\cdots k^{\mu_4}e^{-\tau(k^2+\mathcal{M}^2)}, \tag{6.56}$$

where $\rho(u,\epsilon)\to 1$ as $\epsilon\to 0$. For sufficiently large $\epsilon>0$, we also require $\rho(u,\epsilon)\to 0$ as $u\to 0$, with the rate at which it vanishes being fast enough to render the integrals convergent. This gives the following form for the regulators

$$\eta_{-2\alpha} = \frac{1}{(1+\alpha)!}\int_0^\infty \frac{du}{u}\ \rho(u,\epsilon)\left[u(y^2+y_0^2)\right]^{2+\alpha}e^{-u(y^2+y_0^2)}, \tag{6.57}$$
$$\theta_{-2\alpha} = \frac{1}{(2+\alpha)!}\int_0^\infty \frac{du}{u}\ \rho(u,\epsilon)\left[u(y^2+y_0^2)\right]^{3+\alpha}e^{-u(y^2+y_0^2)}, \tag{6.58}$$
$$\kappa_{-2\alpha} = \frac{1}{(3+\alpha)!}\int_0^\infty \frac{du}{u}\rho(u,\epsilon)\left[u(y^2+y_0^2)\right]^{4+\alpha}e^{-u(y^2+y_0^2)}. \tag{6.59}$$

To see how this can emerge from our formalism, we first note that our boundary conditions (6.28) can be written in the following form

$$\psi_{-2\alpha}(y,0) = -\frac{y^4}{2(2+\alpha)!}\int_0^\infty \frac{du}{u}\ u^{2+\alpha}e^{-u(y^2+y_0^2)}, \tag{6.60}$$
$$\phi_{-2\alpha}(y,0) = -\frac{y^6}{2(3+\alpha)!}\int_0^\infty \frac{du}{u}\ u^{3+\alpha}e^{-u(y^2+y_0^2)}. \tag{6.61}$$

This inspires the following choice,

$$\psi_{-2\alpha}(y,\epsilon) = -\frac{y^4}{2(2+\alpha)!}\int_0^\infty \frac{du}{u}\rho(u,\epsilon)\; u^{2+\alpha}e^{-u(y^2+y_0^2)}, \tag{6.62}$$

$$\phi_{-2\alpha}(y,\epsilon) = -\frac{y^6}{2(3+\alpha)!}\int_0^\infty \frac{du}{u}\;\rho(u,\epsilon)u^{3+\alpha}e^{-u(y^2+y_0^2)}. \tag{6.63}$$

After plugging this into (6.26) and (6.27), we recover the desired regulators (6.57) to (6.59).

A general gauge invariant regulator:

Finally, we derive a very general gauge invariant regularisation scheme that captures all of the above and more by making the following choice for our seed functions

$$\psi_{-2\alpha}(y,\epsilon) = \psi_{-2\alpha}(y,0)\mathcal{F}(\vartheta,\varphi,\epsilon), \qquad \phi_{-2\alpha}(y,\epsilon) = \phi_{-2\alpha}(y,0)\mathcal{G}(\vartheta,\varphi,\epsilon), \tag{6.64}$$

where $\psi_{-2\alpha}(y,0)$ and $\phi_{-2\alpha}(y,0)$ are, of course, given by (6.28) and

$$\vartheta = \ln y^2,\;\; \varphi = \ln(y^2+y_0^2).$$

For sufficiently large $\epsilon > 0$, we require that $y^{-2\alpha}\mathcal{F}(\vartheta,\varphi,\epsilon)$ and $y^{-2\alpha}\mathcal{G}(\vartheta,\varphi,\epsilon)$ should vanish as $y\to\infty$. We also require that $\mathcal{F}(\vartheta,\varphi,0) = \mathcal{G}(\vartheta,\varphi,0) = 1$. Plugging this into (6.26) and (6.27), we find that

$$\frac{y^2}{y^2+y_0^2}\theta_{-2\alpha} - \frac{2}{2+\alpha}\eta_{-2\alpha} = \frac{y^2}{y^2+y_0^2}\left[\mathcal{F} - \frac{\partial_\varphi\mathcal{F}}{2+\alpha}\right] - \frac{2}{2+\alpha}\left[\mathcal{F} + \frac{\partial_\vartheta\mathcal{F}}{2}\right], \tag{6.65}$$

$$\frac{y^2}{(y^2+y_0^2)}\kappa_{-2\alpha} - \frac{3}{(3+\alpha)}\theta_{-2\alpha} = \frac{y^2}{y^2+y_0^2}\left[\mathcal{G} - \frac{\partial_\varphi\mathcal{G}}{3+\alpha}\right] - \frac{3}{3+\alpha}\left[\mathcal{G} + \frac{\partial_\vartheta\mathcal{G}}{3}\right]. \tag{6.66}$$

For consistency, $\mathcal{F} = \frac{1}{3}e^{-3\vartheta+(2+\alpha)\varphi}\partial_\vartheta\mathcal{H}$ and $\mathcal{G} = -\frac{1}{2+\alpha}e^{-3\vartheta+(2+\alpha)\varphi}\partial_\varphi\mathcal{H}$, giving

the following general form for gauge invariant regulators

$$\eta_{-2\alpha} = -\frac{1}{6}e^{-3\vartheta+(2+\alpha)\varphi}\left[\partial_\vartheta\mathcal{H}-\partial_\vartheta^2\mathcal{H}\right], \tag{6.67}$$

$$\theta_{-2\alpha} = -\frac{1}{3(2+\alpha)}e^{-3\vartheta+(2+\alpha)\varphi}\partial_\vartheta\partial_\varphi\mathcal{H}, \tag{6.68}$$

$$\kappa_{-2\alpha} = -\frac{1}{(2+\alpha)(3+\alpha)}e^{-3\vartheta+(2+\alpha)\varphi}\left[\partial_\varphi\mathcal{H}-\partial_\varphi^2\mathcal{H}\right]. \tag{6.69}$$

Note that $\mathcal{H}(\vartheta,\varphi,0)=e^{3\vartheta-(2+\alpha)\varphi}$. Meanwhile, for sufficiently large $\epsilon>0$, we require $e^{-3\vartheta+2\varphi}\partial_i\mathcal{H}(\vartheta,\varphi,\epsilon)\to 0$ at large $\vartheta\sim\varphi$.

6.4 η regularisation and renormalisation

To illustrate how renormalisation is implemented in η regularisation, we compute the one-loop vacuum polarisation in QED. This not only demonstrates how divergences are tamed by the η prescription but also allows us to explicitly derive the QED β function in an elegant fashion.

We start with the bare Lagrangian for QED,

$$\mathcal{L}=\bar{\psi}(i\not{D}-m)\psi-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{6.70}$$

where $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ and $D_\mu=\partial_\mu+ieA_\mu$. The photon self-energy is given by

$$i\Pi^{\mu\nu}(p)=-(-ie)^2\int\frac{d^4k}{(2\pi)^4}\,\mathrm{Tr}\left[\gamma^\mu\frac{i}{\not{k}-\not{p}-m+i\epsilon}\gamma^\nu\frac{i}{\not{k}-m+i\epsilon}\right]. \tag{6.71}$$

After Wick rotating to Euclidean signature, taking traces and performing Feynman parametrisations, we find that

$$\Pi^{\mu\nu}(p)=-4e^2\int\frac{d^4k}{(2\pi)^4}\int_0^1 dx\frac{2(k-p)^{(\mu}k^{\nu)}-g^{\mu\nu}[(k-p)\cdot k+m^2]}{[(k-xp)^2+\mathcal{M}^2]^2}, \tag{6.72}$$

where

$$\mathcal{M}^2=m^2+x(1-x)p^2$$

and the integral is taken over four dimensional Euclidean space. Using the rotational

symmetry of the integral, we find that it can be rewritten as

$$\begin{aligned}\Pi^{\mu\nu} = -4e^2 \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx \, \Big[2\mathcal{J}_2^{\mu\nu} - g^{\mu\nu}\mathcal{J}_2{}^{\lambda}{}_{\lambda} \\ - \Big[2x(1-x)\left(p^\mu p^\nu - p^2 g^{\mu\nu}\right) + \mathcal{M}^2 g^{\mu\nu}\Big]\mathcal{J}_0\Big]\end{aligned} \tag{6.73}$$

where

$$\mathcal{J}^{\mu\nu}_{-2\alpha} = \frac{(k-xp)^\mu (k-xp)^\nu}{\Big[(k-xp)^2 + \mathcal{M}^2\Big]^{3+\alpha}}, \qquad \mathcal{J}_{-2\alpha} = \frac{1}{\Big[(k-xp)^2 + \mathcal{M}^2\Big]^{2+\alpha}}. \tag{6.74}$$

We now put this in ILI form using the fact that $\mathcal{J}_2{}^{\lambda}{}_{\lambda} = \mathcal{J}_2 - \mathcal{M}^2\mathcal{J}_0$. This gives

$$\begin{aligned}\Pi^{\mu\nu} = -4e^2 \int_0^1 dx \, \Big[2J_2^{\mu\nu}[\theta_2](\mathcal{M}^2) - J_2[\eta_2](\mathcal{M}^2) g^{\mu\nu} \\ - 2x(1-x)\left(p^\mu p^\mu - p^2 g^{\mu\nu}\right) J_0[\eta_0](\mathcal{M}^2)\Big].\end{aligned} \tag{6.75}$$

If we apply the gauge consistency condition (6.6), the quadratically divergent terms cancel, leaving a logarithmic divergent result

$$\Pi^{\mu\nu} = -\left(p^\mu p^\mu - p^2 g^{\mu\nu}\right)\Pi(p^2) \tag{6.76}$$

with

$$\Pi(p^2) = -8e^2 \int_0^1 dx \; x(1-x) J_0[\eta_0](\mathcal{M}^2). \tag{6.77}$$

At one loop, the dressed photon propagator goes as

$$iD^{\mu\nu} = -\frac{i}{p^2}\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)\frac{1}{1-\Pi(p^2)}, \tag{6.78}$$

and the amplitude for electron scattering as

$$\mathcal{A} = (-ie^2) j_\mu(q_1, q_1')\left(-\frac{i}{p^2} g^{\mu\nu}\frac{1}{1-\Pi(p^2)}\right) j_\nu(q_2, q_2'), \tag{6.79}$$

where $j_\mu(q,q') = \bar{u}(q')\gamma_\mu u(q)$ are the currents for the external fields and $p = q_1 - q_1' = q_2' - q_2$ the momentum transfer.

We are now ready to renormalise. To this end, we introduce a renormalised charge $e_R(\Lambda) = e/\sqrt{1-\Pi(-\Lambda^2)}$ so that the amplitude scales as $e_R^2(\Lambda)/p^2$ near $p^2 = -\Lambda^2$. More generally, we find that

$$\mathcal{A} \sim \frac{e_R^2(\Lambda)}{p^2}\left(\frac{1-\Pi(-\Lambda^2)}{1-\Pi(p^2)}\right) \approx \frac{e_R^2(\Lambda)}{p^2}\frac{1}{1-\Pi_R(p^2)} \tag{6.80}$$

where, as usual, we work at leading order in e^2 (or equivalently e_R^2), so that

$$\Pi_{\rm R}(p^2) = \Pi(p^2) - \Pi(-\Lambda^2) = -8e_R^2\int_0^1 dx x(1-x)[J_0[\eta_0](\mathcal{M}) - J_0[\eta_0](\mathcal{M}_\Lambda)] \tag{6.81}$$

and $\mathcal{M}_\Lambda^2 = m^2 - \Lambda^2 x(1-x)$. Interestingly, the renormalised vacuum polarisation is essentially of a Pauli-Villars type structure. This is to be expected, because, predictably, we must included a difference in terms of the effective mass scale. Note that as we remove the regulator, this integral remains finite

$$\Pi_{\rm R}(p^2) \to -\frac{e_R^2}{2\pi^2}\int_0^1 dx x(1-x)\ln(\mathcal{M}_\Lambda^2/\mathcal{M}^2) \tag{6.82}$$

and vanishes for $p^2 = -\Lambda^2$.

In this particular example, we have seen both logarithmically and quadratically divergent ILIs. The latter dropped out thanks to the gauge consistency condition and the corresponding use of enhanced regulators. As for the remaining logarithmic ILIs, our renormalisation procedure amounts to a renormalisation of the ILIs

$$J_0[\eta_0](\mathcal{M}) \to J_0[\eta_0](\mathcal{M}) - J_0[\eta_0](\mathcal{M}_\Lambda), \tag{6.83}$$

giving rise to finite integrals even as we remove the regulator. Note that the 'counterterm ILI', $J_0[\eta_0](\mathcal{M}_\Lambda^2)$, has the same regularisation factor as its parent, but a different mass, set by the scale of the subtraction point.

We are now ready to compute the β function for the electron charge, using the fact that

$$e_R^{-2}(\Lambda) = e^{-2} - \chi(-\Lambda^2), \tag{6.84}$$

where we define $\chi(p^2) = \frac{\Pi(p^2)}{e^2} = -8\int_0^1 dx\ x(1-x)J_0[\eta_0](\mathcal{M}^2)$. Since the bare

coupling e is independent of the renormalisation scale, we find the following expression for the running of the renormalised coupling

$$\beta(e_R) \equiv \frac{de_R(\Lambda)}{d\ln\Lambda} = ae_R^3. \tag{6.85}$$

Now the prefactor, $a = -\Lambda^2\chi'(-\Lambda^2)$ is finite, even as we remove the regulators. Indeed, by explicit computation, we find that

$$\begin{aligned} a &\to -16\int_0^1 dxx(1-x)J_{-2}[1](\mathcal{M}_\Lambda^2)\Lambda^2 x(1-x) \\ &= \frac{1}{2\pi^2}\int_0^1 dxx(1-x)\left(1-\frac{m^2}{\mathcal{M}_\Lambda^2}\right), \end{aligned} \tag{6.86}$$

where we have used the fact that $J_{-2}[1](\mathcal{M}_\Lambda^2) = \frac{1}{32\pi^2\mathcal{M}_\Lambda^2}$. We see how the problem is reduced to one for convergent ILIs. For asymptotically high energies, $\Lambda \gg m$, we recover the well known result $a = \frac{1}{12\pi^2}$.

In general, the η regularisation procedure regulates each ILI independently with a damping factor that is smooth and Lorentz invariant. For gauge symmetric regulators, only logarithmic divergences remain. Counter-terms are now easily introduced via counter-term ILIs with the same structure, the same regularisation factor, but a different mass and opposite overall sign. What is nice given the subtraction of ILIs, is that η regularisation avoids all divergences whilst keeping explicit control of the running of the coupling. An advantage of this approach is the ability to maintain a clean separation of divergences from the physical mass/momentum scales. Furthermore, as usual in renormalisation, the choice of mass sets the subtraction point. As the residual divergence was logarithmic, the renormalised ILIs are guaranteed to be finite. Indeed, it is notable that we were able to compute the β function for QED without ever computing a divergent integral.

6.5 η regularisation and anomalies

In QFT, classical symmetries are often broken by quantum effects. In the case of a global symmetry, this is a physical effect and one that is always expected to occur if the theory is coupled to gravity [134]. Gauge anomalies, on the other hand, can be

fatal, since they allow new degrees of freedom to start propagating in the effective field theory, often leading to violations of unitarity. The classic playground for anomalies are chiral models in an even number of spacetime dimensions, in which a gauge field is coupled to massless Dirac fermions [123]. Classically, there are two conserved currents: a vector current associated with the gauge symmetry and an axial current associated with the chiral symmetry. In two dimensions, two point functions reveal both symmetries to be anomalous, although somewhat unusually, this does not lead to any inconsistencies [135]. In four dimensions, three point functions reveal a chiral anomaly, while the gauge symmetry remains intact [125, 126]. Of course, for non-abelian theories, the go-to regularisation scheme is dimensional regularisation [56] which is tricky to implement for chiral models. Indeed, the definition of the chiral model makes use of the γ_5 matrix whose properties are ambiguous when the number of dimensions is analytically continued (see, for example, [136–144]).

η regularisation has the advantage that the dimension of spacetime always remains fixed. Nevertheless, we have seen how it is capable of capturing the key ingredients of dimensional regularisation and other regularisation schemes that preserve gauge invariance. It is therefore important to better understand how the anomalies of the chiral model should be treated in our panoptic framework. Note that γ_5 ambiguities for regularisation schemes in fixed dimension have been considered in [128, 145, 146].

To simplify our analysis whilst still capturing the important physics, we consider the chiral Schwinger model. This is given by the Lagrangian (6.12) for a single massless Dirac fermion in two spacetime dimensions. Classically, the two conserved currents take the form: the vector current, $j^V_\mu = \bar{\psi}\gamma_\mu\psi$ and the axial current, $j^A_\mu = \bar{\psi}\gamma_\mu\gamma_5\psi$, where we define $\gamma_5 = -i\gamma^0\gamma^1$ in two dimensions. To investigate the anomalies, we consider correlation functions of these currents, specifically

$$\Pi^{VV}_{\mu\nu}(p) = \left\langle j^V_\mu(p) j^V_\nu(-p)\right\rangle, \Pi^{VA}_{\mu\nu}(p) = \left\langle j^V_\mu(p) j^A_\nu(-p)\right\rangle, \Pi^{AA}_{\mu\nu}(p) = \left\langle j^A_\mu(p) j^A_\nu(-p)\right\rangle. \tag{6.87}$$

The classical Ward identities give $p^\mu\Pi^{\cdots}_{\mu\nu} = p^\nu\Pi^{\cdots}_{\mu\nu} = 0$. These Ward identities are anomalous. It is well known that there is a gauge anomaly coming from the generation of a mass term for the gauge field [135]. While this results in the non-conservation of $\Pi^{VV}_{\mu\nu}$ at one-loop, in this particular case it does not lead to a violation of unitarity.

Here we will focus on the two point function with mixed vertices, $\Pi^{VA}_{\mu\nu}$ for which we expect only the axial part to be anomalous $p^\mu \Pi^{VA}_{\mu\nu} = 0,\ p^\nu \Pi^{AV}_{\mu\nu} \neq 0$. At one-loop, we have that

$$i\Pi^{VA}_{\mu\nu} = -\int \frac{d^2k}{(2\pi)^2} \mathrm{Tr}\left(\gamma_\mu \frac{i}{\not{k} - \not{p} + i\epsilon} \gamma_\nu \gamma_5 \frac{i}{\not{k} + i\epsilon}\right). \tag{6.88}$$

The first step in η regularisation is to cast this into an ILI form. To this end we perform a Wick rotation $k_0 \to ik_2$, $\gamma^0 \to -i\gamma^2$ and $\gamma_5 \to \gamma^1\gamma^2$ to Euclidean signature along with a Feynman parametrisation, giving

$$\Pi^{VA}_{\mu\nu} = \int \frac{d^2k}{(2\pi)^2} \int_0^1 dx \mathrm{Tr}\left(\gamma_5\gamma_\sigma\gamma_\mu\gamma_\rho\gamma_\nu\right) \frac{(k - xp + xp)^\sigma (k - xp + (x-1)p)^\rho}{\left[(k-xp)^2 + \mathcal{M}^2\right]^2}, \tag{6.89}$$

where $\mathcal{M}^2 = x(1-x)p^2$. At this stage, the temptation might be to pull the trace operator $\mathrm{Tr}\left(\gamma_5\gamma_\sigma\gamma_\mu\gamma_\rho\gamma_\nu\right)$ out of the integral and write the correlation function as a product of the trace operator and the corresponding ILIs. However, one cannot pull the trace through the divergent integral without leaving it unchanged [128]. As we will see, the correct thing to do is to leave trace operator *inside* the integral, decompose it in terms of the metric and express the final result in terms of the ILIs. To this end, we note that

$$\mathrm{Tr}\left(\gamma_5\gamma_\sigma\gamma_\mu\gamma_\rho\gamma_\nu\right) = 2(-\epsilon_{\sigma\mu}g_{\rho\nu} + \epsilon_{\sigma\rho}g_{\mu\nu} - \epsilon_{\sigma\nu}g_{\mu\rho} - \epsilon_{\mu\rho}g_{\sigma\nu} + \epsilon_{\mu\nu}g_{\sigma\rho} - \epsilon_{\rho\nu}g_{\sigma\mu}), \tag{6.90}$$

where $\epsilon_{\mu\nu}$ is the totally antisymmetric Levi-Civita tensor in two dimensions. Using the rotational symmetry of the integral, the correlation function can be written as

$$\Pi^{VA}_{\mu\nu} = -4\int \frac{d^2k}{(2\pi)^2} \int_0^1 dx\ \epsilon_{\sigma\nu}\left[\mathcal{J}_0^\sigma{}_\mu - \frac{1}{2}\mathcal{J}_0^\lambda{}_\lambda \delta^\sigma_\mu - x(1-x)\left(p^\sigma p_\mu - \frac{1}{2}p^2\delta^\sigma_\mu\right)\mathcal{J}_{-2}\right], \tag{6.91}$$

where, in two dimensions, we have

$$\mathcal{J}^{\mu\nu}_{-2\alpha} = \frac{(k-xp)^\mu (k-xp)^\nu}{\left[(k-xp)^2 + \mathcal{M}^2\right]^{2+\alpha}}, \qquad \mathcal{J}_{-2\alpha} = \frac{1}{\left[(k-xp)^2 + \mathcal{M}^2\right]^{1+\alpha}}. \tag{6.92}$$

To put this in ILI form, we once again use the fact that $\mathcal{J}_0^\lambda{}_\lambda = \mathcal{J}_0 - x(1-x)p^2\mathcal{J}_{-2}$

so that we can write

$$\begin{aligned}\Pi^{VA}_{\mu\nu} &= -4\int_0^1 dx\ \epsilon_{\sigma\nu}\Big[J^{\sigma}_{0\,\mu}[\theta_0](\mathcal{M}^2) - \frac{1}{2}J_0[\eta_0](\mathcal{M}^2)\delta^{\sigma}_{\mu} \\ &\quad -x(1-x)\left(p^{\sigma}p_{\mu} - p^2\delta^{\sigma}_{\mu}\right)J_{-2}\Big]. \end{aligned} \tag{6.93}$$

Here we have introduced the ILIs in two dimensions along with their corresponding regulators

$$J^{\mu_1\cdots\mu_2}_{-2\alpha}[\eta](\mathcal{M}^2) = \int \frac{d^2k}{(2\pi)^2}\frac{k^{\mu 1}\cdots k^{\mu s}}{\left[k^2+\mathcal{M}^2\right]^{1+\frac{s}{2}+\alpha}}\eta\left(\frac{k}{\mu},\epsilon\right). \tag{6.94}$$

Recall that the final ILI in (6.93) is convergent with $J_{-2}[1] = 1/4\pi\mathcal{M}^2$. We now use momentum routing in two dimensions to derive a consistency condition for the logarithmically divergent ILIs,

$$\int \frac{d^2k}{(2\pi)^2}\frac{\partial}{\partial k^{\mu}}\left[\frac{k_{\nu}}{k^2+\mathcal{M}^2}\rho\left(\frac{k}{\mu},\epsilon\right)\right] = 0 \implies J^{\mu\nu}_0[\theta_0](\mathcal{M}^2) = \frac{1}{2}J_0[\eta_0](\mathcal{M}^2)g^{\mu\nu}. \tag{6.95}$$

Plugging all of this into (6.93) we arrive at the final expression

$$\Pi^{VA}_{\mu\nu} = \frac{\epsilon_{\sigma\nu}}{\pi}\left(\hat{p}^{\sigma}\hat{p}_{\mu} - \delta^{\sigma}_{\mu}\right), \tag{6.96}$$

where $\hat{p}^{\mu} = p^{\mu}/p$. It is easy to check that this correlation function generates an axial anomaly but not a vector anomaly, consistent with our expectations,

$$p^{\mu}\Pi^{VA}_{\mu\nu} = 0, \qquad p^{\nu}\Pi^{VA}_{\mu\nu} = -\frac{\epsilon_{\mu\nu}p^{\nu}}{\pi}. \tag{6.97}$$

It is now clear η regularisation avoids some of the confusion and ambiguity associated with analytically continued dimensions in chiral theories. However, it is important that η regularisation is properly implemented. To recover the correct result for the anomaly, it was crucial for us to fully decompose the integrand in the correlation function into the integrands of the ILIs, taking all the relevant contractions with the metric *inside* the integral. Had we pulled the trace operator out of the integral before doing these decompositions and contractions, we would have got the wrong result.

To see this explicitly, consider what would happen had we adopted the wrong approach and pulled the trace operator through the integral before carrying out all the relevant contractions. We find that the correlation function would take the following form

$$\begin{aligned}\Pi^{VA}_{\mu\nu}|_{\text{wrong}} &\equiv \text{Tr}\left(\gamma_5\gamma_\sigma\gamma_\mu\gamma_\rho\gamma_\nu\right)\int\frac{d^2k}{(2\pi)^2}\int_0^1 dx\frac{(k-xp+xp)^\sigma(k-xp+(x-1)p)^\rho}{\left[(k-xp)^2+\mathcal{M}^2\right]^2}\\ &= \text{Tr}\left(\gamma_5\gamma_\sigma\gamma_\mu\gamma_\rho\gamma_\nu\right)\int\frac{d^2k}{(2\pi)^2}\int_0^1 dx\left[\mathcal{J}_0^{\sigma\rho}-x(1-x)p^\sigma p^\rho\mathcal{J}_{-2}\right]\\ &\to \text{Tr}\left(\gamma_5\gamma_\sigma\gamma_\mu\gamma_\rho\gamma_\nu\right)\int_0^1 dx\left[J_0^{\sigma\rho}[\theta_0](\mathcal{M}^2)-x(1-x)p^\sigma p^\rho J_{-2}(\mathcal{M}^2)\right],\end{aligned} \tag{6.98}$$

where in the last line we have introduced the ILIs and the corresponding regulators, θ_0, *before* carrying out the relevant contractions coming from the trace. This will be key in giving us a different answer to (6.93). After rewriting the trace using (6.90) and using the fact that $J_0{}^\lambda_\lambda[\theta_0](\mathcal{M}^2) = J_0[\theta_0](\mathcal{M}^2) - x(1-x)p^2J_{-2}(\mathcal{M}^2)$, we arrive at the following expression

$$\begin{aligned}\Pi^{VA}_{\mu\nu}\Big|_{\text{wrong}} &= -4\int_0^1 dx\,\epsilon_{\sigma\nu}\Big[J_0^\sigma{}_\mu[\theta_0](\mathcal{M}^2)-\frac{1}{2}J_0[\theta_0](\mathcal{M}^2)\delta^\sigma_\mu\\ &\quad -x(1-x)\left(p^\sigma p_\mu - p^2\delta^\sigma_\mu\right)J_{-2}\Big].\end{aligned} \tag{6.99}$$

This looks very similar to (6.93), albeit with one crucial difference: the choice of regulator in the second term corresponding to scalar ILI. The difference is entirely due to the fact that we have written the expression in ILI form before carrying out our contractions. The difference is also important. Recall that the gauge consistency condition is given by (6.5), and so

$$\Pi^{VA}_{\mu\nu}|_{\text{wrong}} = \Pi^{VA}_{\mu\nu} - 2\epsilon_{\mu\nu}\int_0^1 dx\,\left[J_0[\eta_0](\mathcal{M}^2)-J_0[\theta_0](\mathcal{M}^2)\right]. \tag{6.100}$$

Since $\eta_0 \neq \theta_0$ in general, we see explicitly how the order of operations in implementing η regularisation matters. To recover the right result for the anomalies, we

should take care to perform all the relevant contractions inside the integral before casting the result into its ILI decomposition. Of course, the underlying reason for this is that one cannot move metric contractions through a divergent integral without generating possible corrections [128].

6.6 Discussion

Originally inspired by regularisation techniques in number theory, η regularisation has provided us with a tool to investigate regularisation schemes in QFT in a systematic way, at least at one-loop. Here we have shown how it can be used to "solve" gauge consistency conditions, delivering old and new gauge invariant regularisation schemes. This includes dimensional regularisation, denominator regularisation, Schwinger proper time, and a generalisation that captures all of the above and more. We have also shown how the careful implementation of η regularisation yields the correct results for anomalies in chiral theories.

As mentioned at the conclusion of the previous chapter, the extension of η regularisation to higher loops is of paramount importance. Further examination of η regularisation at arbitrary loop order could potentially allow us to investigate unitarity, with the possibility of deriving and solving new consistency conditions associated with same. We would also like to use η regularisation to "solve" for regularisation schemes that preserve supersymmetry.

However, as mentioned several times before, our ultimate goal is to use η regularisation to gain a window into how nature protects us from infinity. For example, there are many robust arguments for why superstring theory is UV finite at each order [147–150], and there are reasons to believe why it should remain UV finite at higher loop order. When we carefully take the field theory limit, this UV finiteness should manifest itself in some way. As discussed in Part III of this thesis, this question was explored to some extent by Tourkine in [66], where the heavy string oscillator modes provide the appropriate counter-terms in the descendent field theory integration. Here we imagine another perspective, whereby the field theory inherits some sort of stringy regulator. Building from the bottom-up viewpoint we have been developing, from our generalised theory of regularisation we wish to ask the question: what does any such regulator

look like, and can it be generalised without spoiling its most important ingredients? Presumably, if string theory is consistent, a string inspired regulator should satisfy all the necessary consistency conditions anticipated from bottom up investigations like the one presented here. Furthermore, it is well known that worldsheet modular invariance is key to ensuring that string theory remains UV finite. This suggests that string inspired regulators might be expected to feature this in some way, perhaps through some subtle UV/IR mixing that leads to unexpected cancellations as in [151–153]. These are all topics of greater interest to explored in Part III.

In the meantime, we conclude Part II with a discussion on η regularisation of the path integral.

7

η regularisation and the functional measure

Since its inception [...], the path integral has been a physicist's dream and a mathematician's nightmare. To a physicist, the path integral provides a powerful and intuitive way to understand quantum mechanics [...]. The "sum over histories" offers a framework for tackling problems ranging from Feynman diagrams to lattice chromodynamics, from quantum cosmology to superfluid vortices to stock-option pricing. To a mathematician, the path integral is at best an ill-defined formal expression. It is some sort of vaguely integral-like object involving a "sum" over a badly specified collection of functions, having an undefined measure, and whose value is apparently determined by a group of unclear and perhaps incompatible limits that may or may not yield finite answers.

— ***Steven Carlip, Functional Integration: Action and Symmetries (2008).***

In this chapter, we shift gears from loop integrals to Fujikawa's path integral formulation of the chiral anomaly, and develop a generalised framework for systematically defining a regularised functional measure. This construction extends the η regularisation scheme discussed in Chapter 5 to operator language, making the connection between spectral asymmetry and measure transformation fully explicit. Before recovering Fujikawa's expression for the chiral anomaly from the regularised measure, we explore the deeper number-theoretic structure underlying the ill-defined spectral sum,

interpreting it through the lens of smoothed asymptotics. Our approach unifies two complementary perspectives: the analytic regularisation of Fujikawa and the topological characterisation given by the Atiyah–Singer index theorem. We then investigate how the measure transforms under changes to the regularisation scale and derive a function $\iota_E(\Lambda)$ that encodes this dependence, showing how its Mellin moments govern the appearance of divergences. Finally, we comment on the conceptual relationship between the regularised measure, η regularisation, and the generalised Schwinger proper-time formalism, with a particular focus on the two-dimensional Schwinger model.

7.1 Introduction

The path integral formulation of QFT provides a powerful framework for analysing the dynamics, symmetries, and topological properties of quantum fields [154–156]. By defining the path integral as a summation of all possible field configurations, weighted by the exponential of the classical action, it elegantly unifies quantum dynamics with classical principles, realising Feynman's original vision [157]. One notable advantage of this approach over operator-based methods [156] is the ability to make the connection between symmetries, conservation laws, and quantum anomalies transparent [48]. Its ability to seamlessly incorporate symmetries and constraints through functional determinants underscores its indispensability in the study of gauge theories [48], a topic central to modern physics. The path integral formalism is also particularly effective for exploring non-perturbative phenomena [154], where canonical methods can prove cumbersome or impractical.

Rigorously defining path integrals in physically relevant QFTs poses formidable technical difficulties [158]. Many of these difficulties arise from issues related to the definition of the integration measure [156, 158]. In standard quantum mechanics, this is less problematic, as the path integral can be rigorously formulated using measures defined on abstract Wiener spaces [159]. In QFT, however, similar attempts to rigorously define the measure have led to approaches such as the Osterwalder-Schrader axioms [158, 160, 161]. Although progress has been made, issues such as the convergence properties of integrals remain unresolved, and a fully rigorous definition

of the measure in the general case remains elusive.

Although these are primarily mathematical issues, they are far from irrelevant in physics. One area where considerations about the path integral measure play an important role is in the study of anomalies. In QFT, classical symmetries are often broken by quantum effects. As discussed in the last chapter, a quantum anomaly is defined as the failure of a classical symmetry to be preserved in the process of quantisation and regularisation. In the case of gauge anomalies, they can be fatal since they signal a breakdown of gauge invariance, which can cause violations of unitarity and introduce unphysical degrees of freedom in the effective field theory [134, 156, 162, 163]. In the path integral formulation, these gauge anomalies reveal subtle inconsistencies with classical symmetries in the functional measure when quantised [48].

As also discussed in the last chapter, the prototypical playground for studying anomalies is chiral gauge theory in an even number of spacetime dimensions, in which a gauge field is coupled to massless Dirac fermions [123]. As we reviewed, classically there are two conserved currents: a vector current associated with the gauge symmetry and an axial current associated with the chiral symmetry. In two dimensions, perturbative calculations show there are cases for certain QFTs where two-point functions reveal both symmetries can be anomalous, although somewhat unusually, this does not necessarily lead to any inconsistencies [135]. In other cases, depending on the structure of the theory, the chiral anomaly is a genuine gauge anomaly, meaning that it obstructs gauge invariance.

Furthermore, in four dimensions, if the chiral anomaly affects a global axial symmetry (like the baryon number in QCD or the axial symmetry in the Standard Model), it does not necessarily signal an inconsistency. If, on the other hand, the chiral anomaly were to affect a gauge symmetry (e.g., if a gauge theory has chiral fermions that lead to a gauge current anomaly), then it would be a gauge anomaly, and the theory would suffer from inconsistencies. In the simplest setting [125, 126] for the perturbative computation of the decay processes of a neutral pion to two photons, the three point functions in four dimensions revealed such an anomalous chiral current. However, the underlying gauge symmetry remained intact.

In all these cases of taking a perturbative approach to anomaly calculations, a key feature is the need for regularisation. The definition of chiral models makes

use of the γ_5 matrix, which, for certain regularisation schemes like dimensional regularisation, is the source of significant ambiguities [136, 138–141]. Specifically, in $d \neq 4$ dimensions, γ_5 is not uniquely defined, leading to multiple prescriptions for handling its anticommutation relations with other Dirac matrices. In general non-perturbative approaches, γ_5 can also cause ambiguities in operator ordering, while, in Fujikawa's path integral approach [48], a major subtlety in the spectral analysis has to do with γ_5 not being a trace-class operator.

The path integral approach developed systematically by Fujikawa [164, 165] and later by Fujikawa and Suzuki [48] to studying chiral anomalies, reveals classical symmetry breaking during quantisation much more explicitly than in perturbative approaches. In fact, Fujikawa's analysis has become fundamental in modern field theory, as it demonstrates the way in which anomalous effects of symmetry breaking arise directly from a non-trivial Jacobian associated with a change of variables in the path integral. A key observation made by Fujikawa is that the path integral measure itself breaks certain classical symmetries. Due to its explicit nature, the Fujikawa path integral method provides a conceptually clear framework for deriving anomalous identities and examining the effects of regularisation. This approach allows anomaly terms and associated UV divergences to appear early in the calculation, offering a transparent view of their origins.

Fujikawa's approach identifies the quantum anomaly as the determinant of the transformation in the space of paths, which for anomalous symmetries is nonzero. However, to compute these determinants, a regulator is required to ensure the calculation converges. This regulator is not a part of the original formulation of the path integral formalism, but something which is added on an ad-hoc basis at the time the anomaly is computed.[1] It is not clear, *a priori*, that anomalies computed in this way would be consistent with one another, as they are not derived from an initial regularised formulation of the path integral.

If one starts from the view that, roughly speaking, integration in the Feynman path integral is integration with respect to a pseudomeasure [168, 169], the primary issue is that one cannot construct the Feynman path integral as used in physics as

[1] In addition to Fujikawa's regularisation, time-slicing and zeta function regularisation are also commonly used [156]. Alternatively, proper-time regularisation is also used, which is closely related to the heat kernel regularisation and Fujikawa's smooth cut-off regularisation [118, 166, 167].

a mathematical representation of a standard Lebesgue integral [170, 171]. There are multiple ways to circumvent this issue, with each having its own advantages and suffering its own drawbacks. But any mathematical attempts at formalising a generalised measure must also address the need for regularisation in practical calculations. This raises an interesting question: how might one more rigorously construct a regularised path integral, as understood in its physics application?

For example, in [172] a formalisation of Feynman's original definition is considered, with the formal generalised Lebesgue-Feynman measure confirming the origin of quantum anomalies as given by Fujikawa and Suzuki [48]. However, the general role of renormalisation in the calculation of quantum anomalies was left open. Most recently [70], and perhaps most notably, work has been done at the foundations of perturbative QFT following the Wilsonian philosophy of renormalisation. Here, it is proposed that instead of attempting to define integrals over infinite-dimensional space such as $C^\infty(M)$, one should instead calculate path integrals over a finite-dimensional approximation $C^\infty(M)_{\leq\Lambda}$ of the infinite-dimensional space, using an energy cut-off scale Λ. The motivation, which we note is somewhat similar to the heat kernel approach in [48], is closer to the intuition utilised in practical models in physics, where one employs a regularisation scheme typically in the form of a high energy cut-off.

In this chapter we seek to answer the previously stated question, reproducing the chiral anomaly calculation in QED by formulating the theory with a functional measure that is regularised from the outset. Our approach is inspired by the generalised η regularisation developed on the level of perturbative loop integrals in Chapter 5, where smooth regulator functions enter as a modification of the integration measure. We show that in our approach to defining a regularised path integral measure, we effectively extend the notion of η regularisation for loop integrals to the functional integral. Our framework is shown to reproduce Fujikawa's results for the chiral anomaly in a general sense, capturing his original regularisation scheme. We also show how our generalised framework relates to recent efforts that calculate the chiral anomaly in two dimensions using η regularisation for perturbative loop integrals, while also commenting on the relation to the Schwinger proper-time formalism.

The rest of this chapter is organised as follows: in the next section we review Fujikawa's general formalism and calculation of the chiral anomaly. We emphasise

the way in which the chiral anomaly may be formally related to an infinite (divergent) series of the form $1+1-1-1+\ldots$, and show how for Fujikawa's choice of regulator the smoothed asymptotic method of η regularisation reveals this series to converge. We then discuss the physical meaning of this as it relates to the arrangement of eigenvalues in Fujikawa's path integral calculation. In Section 3, we introduce our generalised framework and define the regularised measure. We show that our formalism reproduces the regularised sum in the same way as Fujikawa's method, except that this regularised sum follows as an automatic consequence of regularising the functional measure from the outset. We then go on to show how the regularised measure may be related to the "'enhanced" regulators and the condition of vanishing Mellin transforms of the η regularisation scheme. Finally, in Section 4, we study the chiral anomaly for the two dimensional Schwinger model, and show that for the appropriate choice of regulator in the regularised measure we recover the same expression for the trace over Dirac indices as computed Chapter 6, offering further connection between the different frameworks.

7.2 A review of Fujikawa's path integral method and the chiral anomaly

To review Fujikawa's path integral method in the study of the chiral anomaly, we follow [48, 164, 165] and the conventions defined therein. We begin with the full Euclidean path integral for QED, incorporating both the Maxwell and Dirac actions as well as gauge-fixing through the Faddeev-Popov procedure. The generating functional in Euclidean space is given by

$$Z[J_\mu,\bar{\xi},\xi]=\int \mathcal{D}\Phi\,\exp\left\{-\frac{1}{\hbar}\int d^4x\left(\mathcal{L}_{\text{QED}}+J_\mu A^\mu+\bar{\xi}\psi+\xi\bar{\psi}\right)\right\}, \tag{7.1}$$

where

$$\mathcal{L}_{\text{QED}}=\left(\psi\gamma^\mu(\partial_\mu-eA_\mu)\psi-m\bar{\psi}\psi-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) \tag{7.2}$$

with $F_{\mu\nu}=(\partial_\mu A_\nu-\partial_\nu A_\mu)$ and source terms denoted as J_μ and ξ. The extended field measure $\mathcal{D}\Phi$ is defined to include the terms as a result of the Faddeev-Popov

gauge-fixing procedure to handle gauge redundancy. It is given by

$$\mathcal{D}\Phi = \mathcal{D}\bar{\psi}\mathcal{D}\psi \left(\prod_{\mu} \mathcal{D}A_{\mu}\right) \mathcal{D}B\mathcal{D}\bar{c}\mathcal{D}c \exp\left\{\frac{1}{\hbar}\int d^4x \left((\partial_{\mu}\bar{c})(\partial^{\mu}c) - (\partial_{\mu}B)A^{\mu}\right)\right\}. \quad (7.3)$$

The 'auxiliary field' B enforces the Landau gauge ($\partial_{\mu}A^{\mu} = 0$), which is an incomplete gauge-fixing, and the Faddeev-Popov ghosts $\bar{c}, c$ remove the remaining gauge redundancy [48]. The form of (7.1) already implies a rescaling of A^{α}_{μ} and $F^{\alpha}_{\mu\nu}$ to absorb factors of the gauge coupling constant [154], resulting in the matter part of the Lagrangian being quadratic in the field strength.

The Euclidean path integral (7.1) that Fujikawa considers is obtained by a Wick rotation $x^4 = -ix^0$ [48]. For the purpose of maintaining consistency with Fujikawa's notation and conventions, in this chapter we define the spacetime metric signature to be $g_{\mu\nu} = (-1, -1, -1, -1)$ with $A_4 = -iA_0$. The operator $\not{D} = \gamma^{\mu}(\partial_{\mu} + iA_{\mu})$ becomes a Hermitian operator $\not{D} = \gamma^{\mu}D_{\mu}$ in Euclidean space. We also note that for the Clifford algebra, $\gamma^0 \to -i\gamma^4$ and all γ^{μ} matrices become anti-Hermitian $(\gamma^{\mu})^{\dagger} = -\gamma^{\mu}$, with the anticommutation relation defined as $\{\gamma^{\mu}, \gamma^{\nu}\} = 2g^{\mu\nu}$. We also note the following conventions:

$$\gamma_5 = \gamma^4\gamma^1\gamma^2\gamma^3, \quad \gamma_5^2 = 1, \quad \gamma_5^{\dagger} = \gamma_5, \quad \{\gamma_5, \gamma^{\mu}\} = 0. \quad (7.4)$$

Following Fujikawa [48, 164], in the study of the chiral anomaly one is not required to give the precise form of the gauge field Lagrangian, and so we set aside discussion on the subtleties of gauge fixing and the precise form of the gauge field propagator. In fact, as Fujikawa showed, most important in the present context is the matter Lagrangian and the matter measure. It is thus sufficient for the purposes of the present discussion to consider the massless theory and suppress the source terms, and we include the Fadeev-Popov factor into $[\mathcal{D}A_{\mu}]$ whenever necessary. To simplify notation we use natural units $\hbar = 1$. Taking these steps, we are left with the simplified partition function for massive spinor QED as originally considered by Fujikawa [164]

$$Z = \int \mathcal{D}A_{\mu}\mathcal{D}\bar{\psi}\mathcal{D}\psi \, e^{S[A,\bar{\psi},\psi]}, \quad (7.5)$$

with the Lagrangian

$$\mathcal{L} = \bar{\psi}\left(i\not{D} - m\right)\psi - \frac{1}{4}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu}. \tag{7.6}$$

Fujikawa's formalism computes anomalies by examining the Jacobian determinant of the transformation of the field measure; here we follow [48] in particular. To facilitate the calculation of the determinant, the fermionic fields $\bar{\psi}, \psi$ are explicitly decomposed as summations over countable bases according to the free modes of propagation, i.e. according to the eigenfunctions of the Dirac operator $\not{D}$, viz.

$$\psi = \sum_n a_n \phi_n(x), \qquad \bar{\psi} = \sum_n \bar{b}_n \phi_n^\dagger(x)\,. \tag{7.7}$$

The fields $\bar{\psi}, \psi$ are Grassmann-number-valued, as the a_n and $\bar{b}_n$ are generators of a Grassmann algebra. The $\phi_n(x)$ are the spinor eigenfunctions of $\not{D}$:

$$\not{D}\phi_n = \lambda_n \phi_n\,. \tag{7.8}$$

The total path integral measure, denoted $\mathrm{d}\mu$, has its fermionic part decomposed viz.

$$d\mu \equiv \left[\mathcal{D}A_\mu(x)\right]\mathcal{D}\bar{\psi}\mathcal{D}\psi = \left[\mathcal{D}A_\mu(x)\right]\prod_n \mathrm{d}\bar{b}_n \mathrm{d}a_n\,. \tag{7.9}$$

Quantum anomalies are symmetry-breaking which is not expected in the classical theory, and Fujikawa's formalism attributes such anomalies to a change in the functional measure.

A *symmetry* may be defined as a transformation for which the correlation functions involving the transformed variables are unaltered by the transformation, e.g.

$$\langle \ldots \psi(x) \ldots \rangle = \langle \ldots \psi'(x) \ldots \rangle\,. \tag{7.10}$$

In the path integral formalism, this definition turns out to be equivalent to the requirement that the product of the measure and the integrand of the generating functional be invariant when the source terms $J, \bar{\xi}, \xi = 0$. In the case of the QED generating functional in (7.1), this amounts to the requirement that:

$$\mathcal{D}\bar{\psi}\mathcal{D}\psi\, \exp\left\{\int d^4x \mathcal{L}_{\mathrm{QED}}\right\} = \mathcal{D}\bar{\psi}'\mathcal{D}\psi'\, \exp\left\{\int d^4x \mathcal{L}'_{\mathrm{QED}}\right\}\,. \tag{7.11}$$

To study the symmetry properties of a local transformation $U(x)$ of the spinors $\bar{\psi}, \psi$, the transformed variables are introduced:

$$\psi'(x) = U(x)\psi(x)\,, \quad \bar{\psi}'(x) = \bar{\psi}(x)\bar{U}(x) \quad \text{where} \quad \bar{U}(x) = -\gamma^4 U^\dagger(x)\gamma^4\,, \tag{7.12}$$

where the transformation for the adjoint spinor $\bar{\psi}(x) = \psi'(x)^\dagger(-i\gamma^4)$ and the standard identity $\gamma^4\gamma^4 = -1$ is used. As can be seen, this implies the relation $\bar{\psi}(x) \to \bar{\psi}'(x) = \psi^\dagger(x)(-i\gamma^4)(-\gamma^4 U^\dagger(x)\gamma^4) = -i\psi'(x)^\dagger\gamma^4 U^\dagger(x)$.[2] In the case of the chiral anomaly, the transformation is defined by $U(x) = e^{i\gamma^5\alpha(x)}$, which gives $\bar{U}(x) = U(x) = e^{i\gamma^5\alpha(x)}$, and the transformed variables:

$$\psi'(x) = e^{i\gamma^5\alpha(x)}\psi(x)\,, \quad \bar{\psi}'(x) = \bar{\psi}(x)e^{i\gamma^5\alpha(x)}\,. \tag{7.13}$$

In operator form, the same equation reads

$$\langle x|\mathcal{U}|\psi\rangle = \langle x|e^{i\gamma^5\alpha(x)}|\psi\rangle\,, \quad \langle\bar{\psi}|\bar{\mathcal{U}}|x\rangle = \langle\bar{\psi}|e^{i\gamma^5\alpha(x)}|x\rangle\,. \tag{7.14}$$

The transformed Lagrangian $\mathcal{L}'_{\text{QED}}$ (which is obtained from $\mathcal{L}_{\text{QED}}$ by performing the substitution $\psi \mapsto \psi', \bar{\psi} \mapsto \bar{\psi}'$) is unchanged—this means that the chiral transformation is classically a symmetry. The quantum anomaly, which is the breaking of the classical symmetry when the theory is quantised, arises because the equality (7.11) does not hold, despite the Lagrangian on either side being equal. The inequality is down to the change in measure, i.e. the following measures are not the same:

$$\mathcal{D}\bar{\psi}\mathcal{D}\psi \neq \mathcal{D}\bar{\psi}'\mathcal{D}\psi'\,. \tag{7.15}$$

As a matter of notation, the measure on the left side of the equation is understood in a consistent manner with (7.11). Therefore, the transformed fields are expanded in the basis of eigenfunctions $\phi_n(x)$ of $\not{D}$ and expressed as a linear combination with coefficients given by Grassmann generators,

$$\psi' = \sum_n a'_n\phi_n, \qquad \bar{\psi}' = \sum_n \bar{b}'_n\phi_n^\dagger\,, \tag{7.16}$$

[2] If we had considered the full partition function (7.1) with the full functional measure derived using the Fadeev-Popov procedure, general local transformations would require further information.

and the measure is given by

$$\mathcal{D}\bar{\psi}'\mathcal{D}\psi' = \prod_n \mathrm{d}\bar{b}'_n \mathrm{d}a'_n \,. \tag{7.17}$$

This may be related to the original measure $\mathcal{D}\bar{\psi}\mathcal{D}\psi$ by expanding both sides of the equations $\psi' = \mathcal{U}\psi$ and $\bar{\psi}' = \bar{\psi}\bar{\mathcal{U}}$ viz.

$$\sum_n a'_n|\phi_n\rangle = \sum_n a_n \mathcal{U}|\phi_n\rangle \,, \qquad\qquad \sum_n \bar{b}'_n\langle\phi_n| = \sum_n \bar{b}_n\langle\phi_n|\bar{\mathcal{U}} \,. \tag{7.18}$$

For the first equation, matching the coefficients of $|\phi_n\rangle$ on either side yields a'_n as a combination of the a_n. A similar exercise is performed for $\bar{b}'_n$ in terms of $\bar{b}_n$. One finds that

$$\bar{b}'_n = \sum_m \langle\phi_m|\bar{\mathcal{U}}|\phi_n\rangle \bar{b}_m = \sum_m \bar{\mathcal{U}}^{\mathrm{T}}_{nm}\bar{b}_m \,, \qquad a'_n = \sum_m \langle\phi_n|\mathcal{U}|\phi_m\rangle a_m = \sum_m \mathcal{U}_{nm}a_m \,. \tag{7.19}$$

Since integration over Grassmann generators is defined by differentiation, the resulting Jacobian of the transformation is the inverse of that for conventional numbers. Therefore, using also $\det\bar{\mathcal{U}}^{\mathrm{T}}\det\mathcal{U} = \det\bar{\mathcal{U}}\det\mathcal{U} = \det\left(\bar{\mathcal{U}}\mathcal{U}\right)$, we find the relation between the measures

$$\mathcal{D}\bar{\psi}'\mathcal{D}\psi' = \det\left(\bar{\mathcal{U}}\mathcal{U}\right)^{-1}\mathcal{D}\bar{\psi}\mathcal{D}\psi \,. \tag{7.20}$$

It is now clear to see why the chiral transformation given by $\mathcal{U} = \bar{\mathcal{U}} = e^{i\gamma^5\alpha(x)}$ results in a change to the measure. Compare with a non-anomalous symmetry such as the gauge symmetry $\mathcal{U} = e^{i\alpha(x)}$, for which $\bar{\mathcal{U}} = e^{-i\alpha(x)}$ and $\bar{\mathcal{U}}\mathcal{U} = 1$; in that case the measures are equal, and so the symmetry is anomaly-free.

The factor $\det(\bar{\mathcal{U}}\mathcal{U})^{-1}$ is the Jacobian of the transformation. To compute it, we use the identity $\det(\exp(A)) = \exp(\mathrm{Tr}(A))$. The trace is a functional trace which is computed over the Dirac indices and the spatial indices, so it consists of a combined summation and integration. The Jacobian is found to be

$$(\det e^{2i\gamma^5\alpha(x)})^{-1} \;=\; \exp\left(-2i\sum_{n=1}^{\infty}\int d^4x\,\phi_n^\dagger(x)\gamma_5\alpha(x)\phi_n(x)\right). \tag{7.21}$$

The right-hand side is not well-defined because the operator $\bar{\mathcal{U}}\mathcal{U} = e^{2i\gamma^5\alpha(x)}$ is not *trace-class* [173, 174], which is to say it is not a compact operator on the Hilbert space whose singular values are summable (ensuring the trace is finite and well-defined). The anomaly is not well-defined until/unless the summation is regulated. In the next section, we will focus on the explicit structure of this sum in more detail. Important at this juncture is how Fujikawa resolves the ill-definedness of the summation by applying a regularisation scheme. The regulated form of the functional trace in (7.21) is defined in terms of a smooth regulating function $f : \mathbb{R} \to \mathbb{R}$ as follows[3]

$$\begin{aligned} &\lim_{\Lambda\to\infty} \exp\left(\mathrm{Tr}(f(\not{D}^2/\Lambda^2)2i\gamma^5\alpha(x))\right) \\ &= \lim_{\Lambda\to\infty} \exp\left(-2i\sum_{n=1}^{\infty}\int d^4x\phi_n^\dagger(x)f(\not{D}^2/\Lambda^2)\gamma_5\alpha(x)\phi_n(x)\right). \end{aligned} \tag{7.22}$$

The smooth function f applied to an operator, i.e. $f(\not{D}^2/\Lambda^2)$, is understood in the usual sense of functional calculus. Furthermore, f is required to satisfy $f(0) = 1$ and $\lim_{x\to\infty} f(x) = 0$. Fujikawa computes the regularised trace for an arbitrary choice of $\alpha(x)$, which involves lengthy and subtle computation [48]. The result is independent of f, provided $f(x)$ decays sufficiently quickly with x such that

$$\lim_{\Lambda\to\infty} \exp\left(\mathrm{Tr}(f(\not{D}^2/\Lambda^2)2i\gamma^5\alpha(x))\right) = \exp\left(-2i\int d^4x\,\alpha(x)\frac{e^2}{32\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}\right). \tag{7.23}$$

This agrees with the result for the chiral anomaly originally found by Adler, Bell, and Jackiw [125, 126]. It is straightforward to see [48] that when deriving

[3]There are several reasons for why this choice of regularisation is natural in the context of the functional trace (7.21), with the method further generalised by Umezawa [167]. Most importantly, $f(\not{D}^2/\Lambda^2)$ is clearly gauge invariant.As explained in [41, 124], although $f\left(D_\mu D^\mu/\Lambda^2\right)$ is also gauge invariant, and would result in no γ matrices being contained in the regulator and thus a vanishing quantity in the sum, the requirement that one must not only regularise the determinant but also the fermion propagator means the regulator function cannot be taken to be a function of $D_\mu D^\mu$. Moreover, since as we are technically dealing with fermions in an external gauge field A, the appropriate space on which we should compute the sum (or functional trace) is spanned by the eigenfunctions of $\not{D}$ and not $D_\mu D^\mu$. On the other hand, the choice $f\left(\not{\partial}^2/\Lambda^2\right)$ would break gauge invariance.

the Ward identities, the chiral (axial) current $j_A^\mu = i\bar{\psi}\gamma^\mu\gamma^5\psi$ associated with axial transformations is no longer conserved $\partial_\mu j_A^\mu = e^2/(16\pi^2)\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}$. Similar is true in the case of massive fermions with the exception that, classically, the axial current is not conserved $\partial_\mu J_5^\mu = 2im\bar{\psi}\gamma^5\psi$. The axial anomaly still contributes at the quantum level, with the non-conservation of the axial current taking the form $\partial_\mu J_5^\mu = 2im\bar{\psi}\gamma^5\psi + e^2/(16\pi^2)\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}$. In both cases, we see that the total charge is e times a topological invariant.

7.2.1 Spectral summation and smoothed asymptotics

In this section, we illustrate how the ill-definedness of the sum in (7.21) manifests when one tries to compute the chiral anomaly. In particular, we examine how the spectral summation [175–178] connects to smoothed asymptotics from a number theory perspective.

We begin by noting that the unregularised sum effectively looks like a trace over the chiral operator $\mathrm{Tr}(\gamma_5) = 0$, which gives zero. But as it is also weighted by the eigenfunctions $\phi_n(x)$, the total sum is a trace over an infinite number of states reflecting the infinite degrees of freedom in the definition of the measure (7.9). For the case that there is an eigenfunction $\phi_n(x)$ with eigenvalue $\lambda_n \neq 0$, then $\gamma^5\phi_n(x)$ is also an eigenfunction [179]. The reason for this follows from how $\gamma^\mu\gamma^5 = -\gamma^5\gamma^\mu$ such that [179]

$$\not{D}(\gamma^5\phi_n) = -\gamma^5\not{D}\phi_n = -\lambda_n\gamma^5\phi_n, \tag{7.24}$$

as clearly demonstrated by Fujikawa [48] by taking the basis set $\phi_n(x)$—which is required to diagonalise $\not{D}$—to simultaneously diagonalise γ_5. The key observation is that all nonzero eigenvalues come in pairs $\pm\lambda_n$ and the Dirac operator $\not{D}$ can be seen to have a symmetric spectrum. Although it may not be immediately obvious from looking at (7.21), when we make the summation explicit it becomes clear that the eigenvalues can be represented formally [179] by the sequence $\lambda_1, -\lambda_1, \lambda_2, -\lambda_2, \lambda_3, -\lambda_3, \ldots$, with each λ_n possessing a corresponding $-\lambda_n$. Furthermore, the chiral nature of γ_5 means that it switches eigenfunctions with opposite eigenvalues $\gamma_5\phi_n = \pm\phi_n$ [179]. The nonzero ϕ_n then naturally pair up with eigenvalues of ± 1 under the γ_5 operator, and

the functional trace becomes

$$(\det e^{2i\gamma^5})^{-1} = \exp\left(-2i\left(\underbrace{0+0+\cdots+0}_{\text{zero modes}}+\underbrace{1+(-1)+1+(-1)+\ldots}_{\text{paired modes}}\right)\right), \quad (7.25)$$

resembling a generalised alternating series [123, 165, 180].

Interestingly, for the case $\lambda_n = 0$ where we have the zero eigenvalues (i.e., zero modes), the operators $\not{D}$ and γ_5 can be simultaneously diagonalised since both ϕ_n and $\gamma^5\phi_n$ have the same zero eigenvalue. This follows from the fact that $(\gamma_5)^2 = 1$, and the possible eigenvalues of γ_5 are ± 1 [179]. For these reasons, famously the index of the Dirac operator is written as $\text{index}(\not{D}) = n_+ - n_-$, with n_+ the number of zero modes with positive γ_5 eigenvalue and n_- the number of zero modes with negative γ_5 eigenvalue [48, 164, 179, 180]. The zero modes are special [181], since they are the only modes that contribute asymmetrically to the trace. Hence [164]

$$\int d^4x \sum_{n=1}^{\infty} \phi_n^\dagger(x)\gamma_5\phi_n(x) = \int d^4x \sum_{n=1}^{\infty} \mid \phi_n \mid^2 h_n = \sum_{n\in \text{ zero modes}}^{\infty} h_n = n_+ - n_-, \quad (7.26)$$

where $h_n = \pm 1$ and

$$\text{index}(i\not{D}) = n_+ - n_- = \frac{1}{16\pi^2}\int d^4x\, \epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}, \quad (7.27)$$

which is the Atiyah-Singer index theorem [179, 182, 183].

For the case $\lambda_n \neq 0$ the unregularised sum clearly runs *over all eigenvalues* of $\not{D}$. The Dirac operator $\not{D}$ and γ_5 now cannot be simultaneously diagonalised [123]. Since for every eigenfunction ϕ_n with eigenvalue λ_n there is another eigenfunction ϕ_{-n} with eigenvalue $-\lambda_n$, and since ϕ_n and ϕ_{-n} are mixed by γ_5, the Dirac operator $\not{D}$ and γ_5 do not commute. Thus, as a result, the spectral summation over the eigenmodes in (7.25) is *conditionally convergent* [123]. And it is a property of such sums that the order of summation affects the outcome of that summation. Heuristically, it is in this manner that the anomaly is ill-defined.

In the naive summation, one might expect the series (7.25) organised in periods of

two $(1-1)+(1-1)+(1-1)+\ldots$ to cancel to zero. However, the introduction of the smoothed cut-off $f\left(\not{D}^2/\Lambda^2\right)$ breaks the naive symmetry, leading to a net chiral *asymmetry* [123, 165]. This is because the diagonalisation of $\not{D}$ that results in chiral asymmetry directly corresponds to a particular ordering of the summation [123], modifying the sum by damping the large eigenvalues (i.e., high energy modes) to make the series converge. Hence why, although the diagonalisation of $\not{D}$ reveals its spectral properties in terms of the pairings of eigenmodes, with exception of the zero modes [181], upon the use of Fujikawa's regularisation the result is the imposition of gauge invariance in the trace [123] but at the cost of the chiral (axial) symmetry becoming anomalous. Rather than the naive expectation that the trace would vanish $\mathrm{Tr}\,(\gamma_5)=0$, reflecting the symmetry in the eigenvalue contributions when the infinite series is organised in blocks of period two or period four $(1+1-1-1)+(1+1-1-1)+\cdots=0$, instead we have $\mathrm{Tr}\,(\gamma_5)\neq 0$.

To understand more deeply the role of Fujikawa's regularisation, consider the infinite series $1+1-1-1+\ldots$ which can be written as the following number theoretic sum[45]

$$\sum_{n=1}^{\infty}(-1)^{n(n-1)/2}. \tag{7.28}$$

Fujikawa's smooth cut-off $f\left(\not{D}^2/\Lambda^2\right)$ can be seen to effectively smooth out the sum of the eigenstates from $\not{D}$ in the sense defined in Chapter 4 and Chapter 5. In this way, Fujikawa's regularisation can be interpreted as an example of η regularisation and its smoothed asymptotics [1, 2]. To see this, note that for Fujikawa's choice of regularisation the regularised infinite sum takes the form

$$\sum_{n=1}^{\infty}(-1)^{n(n-1)/2}e^{-n^2/\Lambda^2}, \tag{7.29}$$

[4]This series also has the intuitive representation as the sum $\sum_{n=1}^{\infty}a_n$ with $a_n=\cos(n\pi/2)+\sin(n\pi/2)$.

[5]Thanks to Tony Padilla for discussions in relation to this series and its corresponding infinite sum.

from which, using the method of smoothed asymptotics, we infer

$$\lim_{\Lambda\to\infty}\sum_{n=1}^{\infty}(-1)^{n(n-1)/2}e^{-n^2/\Lambda^2}=1. \tag{7.30}$$

This is a key number theoretic insight into Fujikawa's formulation of the path integral. When regularising the trace (7.21), what one is actually doing is weighting the infinite sum in the sense described in Chapter 4 and as first introduced from a number theoretic perspective in [40]. One finds that the contributions from the nonzero modes cancel pairwise in the smoothed asymptotics (owing to the pairing of eigenstates of opposite chirality), leaving behind the net contribution from the unpaired zero modes.

In terms of the regularised functional trace (7.22) explicitly, let us demonstrate the example for $\alpha(x)=1$ for concreteness. Suppose there are k zero modes, and let us index the remaining paired modes by $\pm n$. Given the insertion of $f(\not{D}^2/\Lambda^2)$, for a fixed choice of Λ (i.e. before the limit is taken) one sees that the higher modes ϕ_n are indeed downweighted in the summation, with the regularised trace taking the form

$$\lim_{\Lambda\to\infty}\exp\left(-2i\left(\underbrace{0+\cdots+0}_{k\text{ zero modes}}+\underbrace{\frac{\lambda_n^2}{\Lambda^2}(1-1)+\frac{\lambda_{n+1}^2}{\Lambda^2}(1-1)+\ldots}_{\text{paired modes}}\right)\right)=1. \tag{7.31}$$

As the limit $\Lambda\to\infty$ is taken, the function $f(\not{D}^2/\Lambda^2)$ converges pointwise to 1 as expected. The overall effect of this regulation scheme is thus clearly equivalent to regularising the summation by applying a fixed ordering to its terms, and applying smooth asymptotics [1] to determine the result. In particular, the terms are ordered by increasing $|\lambda_n|$.

7.3 η regularisation of the functional measure

Mathematically, from the view of conventional measure theory (see for example [94, 184]), defining a measure on an infinite-dimensional space is highly non-trivial. Unlike in finite dimensions, where the Lebesgue measure provides a natural framework for

integration, functional integration in infinite dimensions faces fundamental challenges. In particular, key properties of the Lebesgue measure—such as translation invariance and countable additivity—are often difficult to maintain in functional integration in QFT and related contexts.

For practical purposes, Lebesgue integrability is directly analogous to the absolute convergence of an infinite series: just as absolute convergence ensures the order of summation does not affect the sum, Lebesgue integrability ensures an integral is well-defined without requiring regularisation. However, as discussed in the previous section, the functional trace appearing in the chiral anomaly computation is only conditionally convergent. This means that its result depends on the order of summation, introducing ambiguities in defining the trace. The key mathematical insight is that this issue is directly analogous to the study of non-integrable functions in Lebesgue theory, where improper integrals must be carefully regularised to produce finite results. Thus, just as conditionally convergent series require summation rules, the functional trace in the chiral anomaly calculation requires a well-defined regularisation scheme to control divergences.

Unlike conventional measures, the path integral measure does not necessarily satisfy all the axioms of the Lebesgue measure in a strict mathematical sense. As a result, alternative formulations such as the Wiener measure (e.g., for stochastic processes) or the Schwinger proper-time representation are sometimes employed in special cases. However, these approaches do not fully resolve the challenge of defining a general, well-behaved path integral measure in QFT [48, 184]. Furthermore, if we adopt the viewpoint that the path integral measure lacks a natural Lebesgue definition due to these issues with infinite-dimensional integration, then introducing a regularisation scheme serves a role analogous to ensuring Lebesgue integrability in finite-dimensional analysis. Specifically, instead of considering an unregularised functional measure, we can define a regularised measure incorporating a smoothing operator that ensures well-defined convergence and controls divergences systematically.

More formally, this analogy suggests defining an operator-valued regularisation scheme within the definition of the functional measure. That is, we are motivated to define a regulating function $\eta(\mathcal{O})$ which is then applied to the spectrum of the relevant operator $\mathcal{O}$ (e.g., the squared Dirac operator $\not{D}^2$) in order to dampen high-energy

modes and restore well-defined summation properties. This formulation naturally leads to an improved understanding of the role of smoothing in the functional determinant, linking the regularised measure to the trace-class properties of the relevant operators.

The regularised functional measure is defined as

$$\mathcal{D}\Phi \to \mathcal{D}\Phi|_{\text{Regularised}} = \mathcal{D}(\mathcal{R}_\eta \cdot \Phi), \tag{7.32}$$

with $\mathcal{R}_\eta$ the regulating operator. This regulating operator is then defined in terms of a smooth cut-off function $\eta(\mathcal{O})$ motivated by the η regularisation scheme [1, 2] such that

$$\mathcal{R}_\eta = \eta_\Lambda\left(\mathcal{O}\right), \tag{7.33}$$

where $\eta_\Lambda\left(\mathcal{O}\right)$ is a real analytic function $\eta : [0, \infty) \to [0, 1]$. More general than η regularisation for loop integrals, the smooth regulating function $\eta_\Lambda\left(\mathcal{O}\right)$ accepts an operator-valued input $\mathcal{O}$, where $\mathcal{O}$ is assumed to be a positive-definite (non-defective) linear operator on a separable Hilbert space $\mathcal{H}$ and has a discrete set of eigenvalues λ_n. This means that $\mathcal{O}$ is assumed to admit a spectral decomposition of the form

$$\mathcal{O} = \sum_n \lambda_n \left|\psi_n\right\rangle\!\left\langle\psi_n\right|, \tag{7.34}$$

where $\{|\psi_n\rangle\}$ is a complete orthonormal basis of eigenfunctions. In other words, for each n it follows $\mathcal{O}\left|\psi_n\right\rangle = \lambda_n \left|\psi_n\right\rangle$ and $\langle\psi_m|\psi_n\rangle = \delta_{mn}$. From this expansion, it also follows that for any orthonormal basis $\{|e_k\rangle\}$ of $\mathcal{H}$,

$$\mathrm{Tr}(\mathcal{O}) = \sum_k \langle e_k|\mathcal{O}|e_k\rangle = \sum_k \sum_n \lambda_n \left\langle e_k|\psi_n\right\rangle\!\left\langle\psi_n|e_k\right\rangle. \tag{7.35}$$

However, since $\{|\psi_n\rangle\}$ is a complete orthonormal basis, we find

$$\sum_k \langle e_k|\psi_n\rangle\langle\psi_n|e_k\rangle \;=\; 1 \quad \text{(resolution of the identity)}. \tag{7.36}$$

Hence, as a standard result, we arrive at the usual functional trace

$$\mathrm{Tr}(\mathcal{O}) \;=\; \sum_n \lambda_n \langle\psi_n|\psi_n\rangle, \tag{7.37}$$

which is the standard statement that the trace of the operator $\mathcal{O}$ is the sum of its eigenvalues. From Lidskii's theorem (i.e., spectral theorem), for the sum to be well-defined $\mathcal{O}$ must be at least a trace-class operator [173, 174] such that $\sum_n |\lambda_n| < \infty$. Many physically relevant operators fail to satisfy this condition, particularly unbounded operators such as differential operators (e.g., the Laplacian Δ, the Dirac operator $\not{D}$) and the chirality operator γ_5. In these cases, the naive sum of eigenvalues diverges, much like the ill-defined infinite sum encountered in Fujikawa's computation of the chiral anomaly.

The notion of an operator-valued smooth regulator (7.33) therefore directly generalises Fujikawa's regularisation, since we define $\eta_\Lambda(\mathcal{O})$ in terms of the eigenbasis of $\mathcal{O}$. This is directly analogous to the procedure when defining the unregularised Fujikawa measure (7.9) as reviewed earlier, and it will even allow us to explicitly capture Fujikawa's regularisation scheme. This is because, in an analogous way to Fujikawa's regularisation, the domain of $\eta(x)$ corresponds to the spectrum of the operator $\mathcal{O}$. For operators like $\not{D}^2$, the eigenvalues are non-negative. The codomain ensures that $\eta(x)$ serves as a smoothly regulating weight satisfying the properties that $\eta(0) = 1$ for small $x \to 0$ (coincidentally, $\lim_{\Lambda\to\infty}$) and $\eta(x) \to 0$ rapidly as $x \to \infty$ (coincidentally, $\lim_{\Lambda\to 0}$).

To make sense of the proposed generalised framework, recall the definition of the measure (7.9). The product signifies that each mode c_n contributes independently to the integration. For the general case of the regulated measure (7.32), the strategy is similar. Let the operator $\mathcal{O}$ be interpreted according to its operation on Φ in (7.32), and let $\mathrm{Tr}[\mathcal{O}]_{\mathcal{D}(\eta(\mathcal{O})\cdot\Phi)}$ denote the trace of $\mathcal{O}$ with respect to the regularised measure $\mathcal{D}(\eta(\mathcal{O})\cdot\Phi)$. Then, for a complex-valued field ϕ, we may express the decomposition of the regularised measure as

$$D(\eta(\mathcal{O})\cdot\phi) = \bigwedge_{n=1}^{\infty} d\phi'_n = \bigwedge_{n=1}^{\infty} d(\lambda_n\phi_n), \tag{7.38}$$

where $\{\phi'_n\}_{n=1}^{\infty}$ represents any standard orthonormal basis of the Hilbert space associated with the functional space of the field ϕ'. Here λ_n are the eigenvalues of the regulator accepting operator valued inputs and ϕ_n are the corresponding eigenfunctions. Note that, instead of the naive product appearing previously (7.9), the

wedge product is used in the generalised case since we are dealing with functional determinants and measures in an infinite-dimensional space [174].

Furthermore, we denote $\bigwedge_{n=1}^{\infty} d\phi'_n$ to describe the infinitesimal volume form constructed by taking the "products" of all infinitesimal differentials $d\phi'_n$ associated with the orthonormal basis components. We therefore consider the regularised measure to be over *all degrees of freedom* in the Hilbert space, except for the case $\lambda_j = 0$ implying that the zero modes of $\eta(\mathcal{O})$ are omitted. This formulation clearly aligns with standard approaches in functional analysis to defining orthonormal bases, where, it will be remembered, the basis functions $\{\phi_n\}$ diagonalise an operator.

Generally speaking, defining $\mathcal{R}_\eta = \eta_\Lambda(\mathcal{O})$, we treat $\mathcal{R}_\eta$ as a linear regulating operator that acts on the entire functional field space. We assume it is a self-adjoint operator and has a discrete spectral decomposition in the Hilbert space of field configurations. This construction also means that the action of $\eta_\Lambda(\mathcal{O})$ on field configurations Φ is determined by expressing Φ in the eigenbasis of $\mathcal{O}$. Since linear operators preserve eigenbasis action [48, 157, 185] for any eigenstate $|n\rangle$ of $\mathcal{O}$ with eigenvalue λ_n such that $\mathcal{O}|n\rangle = \lambda_n|n\rangle$, the same properties are satisfied in the regularised case. This follows from the fact that the regulator $\eta_\Lambda(\mathcal{O})$, by definition, operates by transforming the eigenvalues of $\mathcal{O}$; therefore, applying $\eta_\Lambda(\mathcal{O})$ to $|n\rangle$ simply modifies the eigenvalue.

For these reasons, analogous to Fujikawa's framework, given the regulator $\eta_\Lambda(\mathcal{O})$ is a real-valued function that acts on the operator $\mathcal{O}$, and given the eigenstates $|n\rangle$ and the corresponding eigenvalues λ_n of $\mathcal{O}$, the action of the regulator on the eigenstates is simply defined by

$$\eta_\Lambda(\mathcal{O}) \cdot |n\rangle = \eta_\Lambda(\lambda_n)\,|n\rangle. \tag{7.39}$$

Since $\eta_\Lambda(\mathcal{O})$ is defined consistently in terms of the spectral analysis, it is easy to see that this argument can be extended for the action of the regulator on any general state $|\Phi\rangle$ by linear combination of its action on the basis states

$$|\Phi\rangle = \sum_n c_n|n\rangle \quad \rightarrow \quad \eta(\mathcal{O})|\Phi\rangle = \sum_n c_n \eta_\Lambda(\lambda_n)|n\rangle, \tag{7.40}$$

assuming that this holds for any $|\Phi\rangle$ that belongs to the domain of $\eta(\mathcal{O})$, ensuring

the regulator is well-defined on the chosen Hilbert space. Since $\eta(\mathcal{O})$ suppresses high-energy modes, the resulting sum is absolutely convergent, thereby recovering the smoothed sum (7.29) for the choice of Fujikawa's regularisation.

7.3.1 Recovering the chiral anomaly

To check that our generalised framework is consistent with Fujikawa's path integral formalism, in this section we show that we can recover the correct expression for the chiral anomaly as reviewed in Section 7.2 and Section 7.2.1. As discussed, there are two complementary views of the chiral anomaly. In the standard Fujikawa approach, the regulator plays an essential role in damping UV divergences and breaking the spectral symmetry of the naive trace $\mathrm{Tr}(\gamma_5)$, thereby revealing the anomaly. In contrast, the index theorem approach shows that the anomaly localizes on the zero modes, where the regulator becomes trivial and the trace reduces to the index of the Dirac operator. These perspectives are reconciled by the fact that the anomaly is a topological quantity, independent of the regulator's detailed form. This picture naturally emerges in the case of the regularised measure.

For simplicity, and to directly access the structure of the anomaly, consider the density over field configurations Φ for the Euclidean path integral

$$P = \mathcal{D}\Phi \exp\left(S[\Phi]\right). \tag{7.41}$$

Since only the matter content contributes to the anomaly, this formal representation shall be sufficient. Once again assuming a massless theory, the regularised density takes the form

$$P_{\mathcal{R}_\eta} = \mathcal{D}\left(\mathcal{R}_\eta \cdot \Phi\right) \exp\left(S[\Phi]\right). \tag{7.42}$$

Let $U(x)$ be a local symmetry as defined in Section 7.2. We write $U(\Phi)$ to denote the field Φ transformed pointwise by the local operator $U(x)$ such that $[U(\Phi)](x) := U(x)\Phi(x)$. It acts as a transformation on Φ in both the regularised measure and the classical action such that

$$\mathcal{D}\left(R_\eta \cdot \Phi\right) \exp\left(S[\Phi]\right) = \mathcal{D}\left(R_\eta \cdot U(\Phi)\right) \exp\left(S[U(\Phi)]\right). \tag{7.43}$$

If $U(x)$ is linear, the fields and the action transform as $\Phi \to \Phi' = U(x)\Phi(x)$ and $S[\Phi] \to S[U(\Phi)]$, respectively. Clearly, in the non-regularised case, (7.41) transforms as $P \to P' = \mathcal{D}\left(U \cdot \Phi\right) \exp\left(S[U(\Phi)]\right)$. For the regularised measure, it is similarly straightforward to show that the regularised density (7.42) transforms as

$$\begin{aligned} P'_{\mathcal{R}_\eta} &= \mathcal{D}\left(\mathcal{R}_\eta \cdot U(\Phi)\right) \exp\left(S[U(\Phi)]\right) && (7.44) \\ &= \mathcal{D}\left(\mathcal{R}_\eta \cdot \Phi\right) \det\left(J\right) \exp\left(S[U(\Phi)]\right), && (7.45) \end{aligned}$$

where $(J) = \mathcal{R}_\eta U \mathcal{R}_\eta^{-1}$ is the transformation operator, whose determinant captures the change in the measure. The transformed action $S[U(\Phi)] = S[\Phi] + \Delta S$ with $\Delta S = S[U(\Phi)] - S[\Phi]$ explicitly contains the classical symmetry breaking. Therefore, we see that the regularised action is modified by two distinct contributions

$$P'_{\mathcal{R}_\eta} = \mathcal{D}\left(\mathcal{R}_\eta \cdot \Phi\right) \exp\left(S[\Phi]\right) \exp\left(\Delta S + \ln \det\left(J\right)\right), \tag{7.46}$$

with ΔS the change to the action functional and $\ln \det\left(J\right)$ is the Jacobian dependent term capturing the quantum anomaly resulting from the regularised measure. Rearranging terms in (7.46), we can rewrite the regularised density so that

$$P'_{\mathcal{R}_\eta} = P_{\mathcal{R}_\eta} \exp\left(\Delta S + \ln \det(J)\right), \tag{7.47}$$

where $P_{\mathcal{R}_\eta}$ is defined as in (7.42). Dividing both sides by $P_{\mathcal{R}_\eta}$ and taking the logarithm, we obtain the logarithmic variation of the regularised probability density

$$\ln\left(\frac{P'_{\mathcal{R}_\eta}}{P_{\mathcal{R}_\eta}}\right) = \Delta S + \ln \det(J). \tag{7.48}$$

Defining the variation $\Delta \ln P'_{\mathcal{R}_\eta} := \ln\left(\dfrac{P'_{\mathcal{R}_\eta}}{P_{\mathcal{R}_\eta}}\right)$ so that

$$\Delta \ln P'_{\mathcal{R}_\eta} = \Delta S + \ln \det(J), \tag{7.49}$$

the right-hand side can be seen to contain the sum of the classical symmetry breaking and the quantum symmetry breaking. Let $U(x)$ be small such that $U = \exp\left(i\alpha(x)T\right)$

for infinitesimal $\alpha(x)$, which is a function of position. As the trace is over field configurations, we have

$$\Delta \ln P'_{\mathcal{R}_\eta} = \Delta S + i \int d^4x\, \alpha(x) \mathrm{Tr}(\mathcal{R}_\eta^\dagger \mathcal{R}_\eta T), \tag{7.50}$$

via the standard identity $\det(e^{i\alpha(x)T}) = e^{i\alpha(x)\mathrm{Tr}(T)}$ giving the operator-theoretic form of the regularised trace as a local density integrated over spacetime.

The trace $\mathrm{Tr}(\mathcal{R}_\eta^\dagger \mathcal{R}_\eta T)$ is over internal (spinor and gauge) indices. Recalling $\mathcal{R}_\eta = \eta_\Lambda(\mathcal{O})$, we introduce a regulator for $\mathcal{R}$ and $\mathcal{R}^\dagger$ such that $\mathcal{R} = \eta(\mathcal{O})$ and $\mathcal{R}^\dagger = \tilde{\eta}(\mathcal{O})$. Following Fujikawa and Suzuki [48], the key observation is that from (7.50) we may thus proceed either by analysing ψ and $\bar{\psi}$ separately, or with the covariant calculation, which, at the end of the day, because of gauge covariance, amounts to the two regulators factorising into a single regulator function. In the covariant approach, this effectively amounts to a single insertion $\mathcal{R}_\eta \gamma_5$ upon specialising to the chiral transformation for $T = \gamma_5$.[6] A natural choice for the smooth cut-off is of course Fujikawa's regulator $\eta(\mathcal{O}) = e^{-\not{D}^2/\Lambda^2}$. Making this choice, it is straightforward to see that the regularised measure yields the standard expression for the chiral anomaly, as reviewed in Section 7.2. If, however, we instead follow a similar strategy as in Chapter 4, keeping the precise form of η general, it follows that we obtain for the regularised trace

$$\mathrm{Tr}[\mathcal{R}_\eta \gamma_5] = i \int d^4x\, \alpha(x) \mathrm{Tr}[\gamma_5 \langle x| \eta\left(\frac{\not{D}^2}{\Lambda^2}\right) |x\rangle]. \tag{7.51}$$

To recover the chiral anomaly for general η, we recall the results from Section 7.2.1. As discussed, by inserting the smooth regulator $\eta(\not{D}^2/\Lambda^2)$ we recognise the ill-defined trace can be written as

$$\mathrm{Tr}\left(\gamma_5 \eta\left(\frac{\not{D}^2}{\Lambda^2}\right)\right) = \sum_n \langle \phi_n | \gamma_5 \eta\left(\frac{\lambda_n^2}{\Lambda^2}\right) |\phi_n\rangle, \tag{7.52}$$

which weights the eigenvalue sum and breaks the symmetric spectrum. As $\Lambda \to \infty$, the regulator becomes 1 on the zero modes and 0 on the high eigenmodes, effectively

[6] Otherwise the regularised measure would suffer mathematical inconsistency and inconsistency with Fujikawa's original derivation of the chiral anomaly.

extracting the contribution of the zero modes. Furthermore, when $\Lambda \to \infty$, the regularised trace in the limit projects onto the kernel of $\not{D}^2$ because we demand $\eta(0) = 1$ ensuring the zero modes are included fully, and $\eta(\lambda_n^2/\Lambda^2) \to 0$ for $\lambda_n \neq 0$, so nonzero modes are suppressed. Thus, the trace becomes

$$\sum_{\lambda_n=0} \langle \phi_n | \gamma_5 | \phi_n \rangle = n_+ - n_- = \text{index}(\not{D}), \tag{7.53}$$

which recovers the topological identity (7.27) for the chiral anomaly. The same follows for $\mathcal{R}^\dagger \gamma_5$. Although the chiral anomaly is ultimately captured by the index of the Dirac operator, and hence by the contribution of the zero modes, this index arises through a regularised summation over all eigenmodes of the Dirac operator. In this way, the anomaly is "spread" across the spectrum in the regulated trace, even though the unregulated sum would vanish. Only in the $\Lambda \to \infty$ limit does the regulator act as a projection onto zero modes, at which point the anomaly becomes a topological invariant.

In conclusion, while the action $S[\Phi]$ may exhibit a classical symmetry, this symmetry is not necessarily preserved in the regularised measure $\mathcal{D}\left(\mathcal{R}_\eta \cdot \Phi\right)$. As demonstrated, our generalised regularisation framework not only reproduces the known expression for the chiral anomaly, but also provides a clear connection between the formal structure of the path integral, the spectral nature of the Dirac operator, and the topological content captured by the index theorem.

7.3.2 Studying the effects of the regularised measure

In η regularisation for loop integrals, we have seen in previous chapters that a key mechanism which ensures all divergences are completely killed comes from vanishing Mellin transforms of the smooth cut-off. In this section, we explore if there exists an analogous mechanism in the regularised measure (7.32).

To do this, our strategy is to probe the measure's transformation under a change to the regularisation scale Λ. We therefore start by varying the cut-off scale Λ, which will enable us to study the trace of the derivative of the regulator kernel. After some careful manipulations, we then systematically isolate the high-energy eigenvalue contributions and examine the behaviour of the regularised measure, particular as a means to probe

the analytic structure of the cut-off and whether any non-trivial cancellations occur. This approach is closely related to Fujikawa's [48], and it is expected that the physical result of the chiral anomaly is independent of the regularised measure.

Consider again the functional measure in the regularised density (7.42) for the case of a fermionic field ψ. Taylor expanding the regularising operator $\mathcal{R}_\eta(\Lambda+\epsilon) = \mathcal{R}_\eta + \epsilon\frac{\mathrm{d}\mathcal{R}_\eta}{\mathrm{d}\Lambda} + \mathcal{O}(\epsilon^2)$ in small ϵ, the regularised measure can be written as

$$\begin{aligned} P^{[\psi]}_{\mathcal{R}_\eta}(\Lambda+\epsilon) &= \mathcal{D}\left(\mathcal{R}_\eta\left(1+\epsilon\mathcal{R}_\eta^{-1}\frac{\mathrm{d}\mathcal{R}_\eta}{\mathrm{d}\Lambda}\right)\cdot\psi\right) \\ &= \mathcal{D}\left(\mathcal{R}_\eta\cdot\psi\right)\det\left(1+\epsilon\mathcal{R}_\eta^{-1}\frac{\mathrm{d}\mathcal{R}_\eta}{\mathrm{d}\Lambda}\right)^{-1}. \end{aligned} \tag{7.54}$$

For any change of variables the measure rescaling depends on the Jacobian determinant. In this case, the determinant captures the infinitesimal change in the regularised measure due to the variation of the cut-off scale. To write this in a more suggestive form, in the second line of (7.54) we recognise $\mathcal{D}\left(\mathcal{R}_\eta\cdot\psi\right)$ as $P_{\mathcal{R}_\eta}$, since we are only concerned here with the measure part of (7.42). Assuming $\epsilon\mathcal{R}_\eta^{-1}\frac{\mathrm{d}\mathcal{R}_\eta}{\mathrm{d}\Lambda} << 1$, which holds if $\mathcal{R}_\eta$ varies smoothly with Λ, we can use the standard expansion $\det(1+A) = \exp(\mathrm{Tr}[A])$ to rewrite the determinant. Taking these steps we get

$$P^{[\psi]}_{\mathcal{R}_\eta}(\Lambda+\epsilon) = P^{[\psi]}_{\mathcal{R}_\eta}\exp\left(-\mathrm{Tr}\left[\epsilon\mathcal{R}_\eta^{-1}\frac{\mathrm{d}\mathcal{R}_\eta}{\mathrm{d}\Lambda}\right]\right), \tag{7.55}$$

where we have included a minus sign in the exponential coming from the fact that the measure transformation includes a fermionic field ψ, and determinants in fermionic functional space are inverted. Since we are interested in studying the effects of the regularisation as the regulator is lifted, and since the variation of the regulator captures the change in the eigenvalue spectrum, we can rewrite the determinant purely in terms of $[d\mathcal{R}_\eta/d\Lambda]$. Then, taking the logarithm of both sides and rearranging terms, we obtain

$$\frac{\ln P^{[\psi]}_{\mathcal{R}_\eta}(\Lambda+\epsilon) - \ln P^{\psi}_{\mathcal{R}_\eta}}{\epsilon} = -\mathrm{Tr}\left[\frac{\mathrm{d}\mathcal{R}_\eta}{\mathrm{d}\Lambda}\right]. \tag{7.56}$$

It is important to emphasise that, on the left-hand side, we are varying only the

measure part. On the right-hand side, $\mathcal{R}_\eta$ is an operator acting in field space, so the trace must be defined with respect to an appropriate basis.

The next step is to observe that for small ϵ the left-hand side may be approximated by a first-order finite difference. We therefore arrive at the relation describing the change in the measure due to incremental change in the cut-off scale[7]

$$\frac{d}{d\Lambda} \ln P_{\mathcal{R}_\eta}^{[\psi]} = -\mathrm{Tr}\left[\frac{\mathrm{d}\mathcal{R}_\eta}{\mathrm{d}\Lambda}\right]. \tag{7.57}$$

Using the chain rule for logarithmic differentiation, we express the left-hand side as a ratio of logarithms, which aligns with structures commonly found in renormalisation group equations [186]. As a consequence of this rewriting we pick up a factor of Λ on the right-hand side. Since the fermionic path integral measure involves integration over both $\psi, \bar{\psi}$, the total contribution doubles. Considering this, we arrive at the relation defined in terms the function $\iota_E(\Lambda)$

$$\iota_E(\Lambda) := \frac{\mathrm{d}(\ln P_{\mathcal{R}_\eta}^{[\psi,\bar{\psi}]})}{\mathrm{d}(-\ln \Lambda)} = -2\Lambda \mathrm{Tr}\left[\frac{\mathrm{d}\mathcal{R}_\eta^2}{\mathrm{d}\Lambda}\right]. \tag{7.58}$$

To ensure consistency with the fermionic path integral structure, following a similar strategy as discussed in the previous section, we write $\mathcal{R}_\eta^2 = \mathcal{R}_\eta \mathcal{R}_\eta^\dagger$ and can thus proceed with the covariant calculation. We then set $\mathcal{R}_\eta = \eta(\mathcal{O}) = \eta\left(\not{D}^2/\Lambda^2\right)$ in (7.58), keeping the explicit form of the regulator general. We also note that (7.58) is beginning to look like a flow equation for the path integral measure under infinitesimal scaling of the cut-off. At this stage, the quantity $\iota_E(\Lambda)$ measures how sensitive the regularised measure is to changes in the cut-off.

Before calculating the trace, we use the chain rule again to rewrite $\iota_E(\Lambda)$ as

$$\iota_E(\Lambda) = \Lambda\,\mathrm{Tr}\left[\frac{\mathrm{d}\left(\eta\left(\frac{\not{D}^2}{\Lambda^2}\right)\right)}{\mathrm{d}\Lambda}\right] = -\frac{2}{\Lambda^2}\mathrm{Tr}\left[\eta'\left(\frac{\not{D}^2}{\Lambda^2}\right)\not{D}^2\right]. \tag{7.59}$$

Since $\eta(x)$ acts on the eigenvalues of $\not{D}^2$, the trace naturally transforms as

[7]Note, one can perform the same calculation for the case of a bosonic field, and a similar result follows analogously.

$\eta'\left(\frac{\not{D}^2}{\Lambda^2}\right)\not{D}^2$, ensuring that only eigenvalue contributions appear in the result.

Next, using Fujikawa's plane wave analysis [48], we may write

$$\iota_E(\Lambda) = -\frac{2}{\Lambda^2}\int \mathrm{d}^4x\langle x|\mathrm{Tr}\left[\eta'\left(\frac{\not{D}^2}{\Lambda^2}\right)\not{D}^2\right]|x\rangle \tag{7.60}$$

from which it follows

$$= -\frac{2}{\Lambda^2}\int \mathrm{d}^4x\int\frac{\mathrm{d}^4k}{(2\pi)^4}\langle x|\mathrm{Tr}\left[\eta'\left(\frac{\not{D}^2}{\Lambda^2}\right)\not{D}^2\right]|k\rangle\langle k|x\rangle$$

$$= -\frac{2}{\Lambda^2}\int \mathrm{d}^4x\int\frac{\mathrm{d}^4k}{(2\pi)^4}\langle k|x\rangle\langle x|\mathrm{Tr}\left[\eta'\left(\frac{\not{D}^2}{\Lambda^2}\right)\not{D}^2\right]|k\rangle$$

$$= -\frac{2}{\Lambda^2}\int \mathrm{d}^4x\int\frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{Tr}\left[\eta'\left(\frac{\not{D}_k^2}{\Lambda^2}\right)\not{D}_k^2\right]\cdot 1_x, \tag{7.61}$$

where 1_x is the resolution of identity in x-space.

In the transition from the second-last to last line in (7.61), any differential operators ∂_μ in the trace over fermion indices are replaced with $\partial_\mu + ik_\mu$, following plane-wave decomposition rules [48]. This substitution is valid for local operators, ensuring the Dirac operator in momentum space takes the form $\not{D}_k = \gamma^\mu(D_\mu + ik_\mu)$, which is standard in Fujikawa's approach.[8] This is the reason for the subscript in $\not{D}_k$. For concreteness, k_μ act like constants, not differential operators. They are merely the variables of integration referred to by the momentum integration d^4k.

Before proceeding, we may simplify (7.61) further by using change of variables $z = \not{D}_k^2/\Lambda^2$. Doing so gives

$$\iota_E(\Lambda) = -\frac{2}{\Lambda^2}\int \mathrm{d}^4x\int\frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{Tr}\left[\eta'(z)\,\Lambda^2 z\right]\cdot 1_x. \tag{7.62}$$

Using the identity $\not{D}^2 = D_\mu D^\mu + F$ and the gauge covariant decomposition $F = 1/4[\gamma^\mu,\gamma^\nu][D_\mu, D_\nu]$, we can expand the derivative operator such that

[8]The procedure of converting operators in this way is well-defined for local operators: express an operator as a sum of products of operators which are either differential (i.e. diagonal in k_μ basis) or non-differential (i.e. diagonal in x^μ basis). Then replace $\partial_\mu \mapsto \partial_\mu + ik_\mu$. In flat spacetime, a single coordinate patch covers the whole space, so this representation holds for all (flat) spacetime.

$$z = z_0 + \Delta, \text{ with } z_0 = -k^2/\Lambda^2, \text{ and } \Delta = \frac{2ik \cdot D + D^2 + F}{\Lambda^2}. \tag{7.63}$$

Assuming Δ is small, since the quadratic momentum term $-k^2/\Lambda^2$ dominates the argument of η' in the UV region, where $|k|$ is large, the rest of the terms can be seen to either scale sublinearly or are independent of k. Since η is smooth and its derivatives are smooth, we can therefore expand $\eta'(z)$ around $-k^2$ giving

$$\eta'\Big(z_0 + \Delta\Big) = \eta'\Big(z_0\Big) + \eta''\Big(z_0\Big)\,\Delta + \frac{1}{2}\eta'''\Big(z_0\Big)\,\Delta^2 + \cdots . \tag{7.64}$$

This is practically analogous to the heat kernel expansion [118]. Putting everything together, the product in the integrand (7.66) takes the form

$$\eta'\Big(z_0 + \Delta\Big)\,(z_0 + \Delta) = \left\{\eta'\Big(z_0\Big)(z_0 + \Delta) + \eta''\Big(z_0\Big)(z_0 + \Delta)\Delta + \cdots\right\}. \tag{7.65}$$

At leading order, the term $\eta'(z_0)\,z_0$ corresponds to free propagation and is independent of the external fields. The next non–vanishing contributions come from terms linear or quadratic in Δ that contain factors like D and F. Ultimately, to extract the chiral anomaly one requires terms quadratic in $F_{\mu\nu}$, which appear in the expansion of Δ and with Δ^2 also containing terms quadratic in D. However, since the goal is to study how the measure depends on the regulator—to determine whether the regulator causes any unexpected contributions or cancellations—it is sufficient to focus only on the linear terms in Δ. Thus, we obtain

$$\iota_E(\Lambda) = -2\int \mathrm{d}^4x \int \frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{Tr}\left[\eta'(z_0)z_0 + \Big[\eta'(z_0) + \eta''(z_0)z_0\Big]\,\Delta + \mathcal{O}(\Delta^2)\right]\cdot 1_x. \tag{7.66}$$

Following the procedure in [48], the next step is to exploit the symmetry of the momentum integration. We apply standard symmetric integration identities to replace products of $k_\mu \ldots k_\nu$ with products of $k^2 = k_\mu k^\mu g_{\mu\nu}$. Odd powers of k^μ vanish, meaning $\int \mathrm{d}^4k/(2\pi)^4\, k^\mu f(k^2) = 0$. Accordingly, the momentum integrals in (7.66) reduce to integrals of the general form

$$I(n,m) = \int \frac{d^4k}{(2\pi)^4}\, \eta^{(n)}\left(-\frac{k^2}{\Lambda^2}\right)\left(-\frac{k^2}{\Lambda^2}\right)^m, \tag{7.67}$$

which appears as a Mellin-like structure. As it turns out, this will be seen to not be a coincidence. Moreover, to extract the Λ dependence from (7.66), we perform a change of variables in the k-integration. Let $t = k^2/\Lambda^2$ so that $k = \sqrt{t}\Lambda$ and $dk = dt\Lambda/(2\sqrt{t})$. Then, for the leading term, we find

$$\int \frac{d^4k}{(2\pi)^4}\mathrm{Tr}\left[z_0\eta'(z_0)\right] = -\frac{\Lambda^4}{8\pi^2}\,\mathcal{M}_1\{\eta\}\mathrm{Tr}[1], \tag{7.68}$$

where we've used integration by parts and where $\mathcal{M}_1\{\eta\}$ is the Mellin transform of the regulator similar to what we observed in Chapter 5. For the next subleading term we get

$$\int \frac{d^4k}{(2\pi)^4}\mathrm{Tr}\left[\eta'(z_0)\Delta\right] = \frac{\Lambda^2}{16\pi^2}\mathcal{M}_0\{\eta\}\mathrm{Tr}[F]. \tag{7.69}$$

Altogether we therefore find

$$\iota_E(\Lambda) = \frac{1}{4\pi^2}\int \mathrm{d}^4x\left(\Lambda^4\mathcal{M}_1\{\eta\}\mathrm{Tr}[1] - \frac{\Lambda^2}{2}\mathcal{M}_0\{\eta\}\mathrm{Tr}[F] + \dots\right). \tag{7.70}$$

The leading terms in the limit $\Lambda \to \infty$ (as the regulator is lifted) are those proportional to Λ^4 and Λ^2. These terms are divergent and depend explicitly on the regulator η. We notice that the presence of the Mellin transforms resembles a structure similar to that found in Chapter 5 for which enhanced regulators ensured the divergences were completely killed. This observation serves as a nice motivation to draw deeper connection between different applications of η regularisation. Since we want $\iota_E(\Lambda) = 0$ to vanish, if we choose η to be enhanced for each respective Mellin integral, the overall integrand in (7.70) can be made to vanish.[9]

[9]In fact, what is likely required is the choice of a stricter class of "super-enhanced" regulators. The development of a theory of "super-enhanced" regulators extends beyond the remit of this work.

7.4 Bridging regularisation schemes: a unifying view

It was observed in [118] that generalised Schwinger proper time integrals provide an elegant framework for calculating anomalies. In [167], Umezawa showed that Fujikawa's regularisation is closely related to the Schwinger proper time approach in the coordinate representation. In previous chapters we have also seen, in addition to the large class of η regulators correctly capturing the chiral anomaly for the two dimensional Schwinger model, this general regularisation scheme is also closely connected to the generalised Schwinger proper time approach.

The purpose of the present section is to understand these distinct schemes from a unified viewpoint, particularly in how they relate to our generalised framework for the regularised path integral measure. For this purpose, the generalised Schwinger proper time framework is used as an important conceptual bridge.

We work in the Schwinger model, which is two dimensional quantum electrodynamics (QED_2) with massless Dirac fermions interacting with an abelian gauge field. The quantisation of the fermion field $\psi(x)$ follows from the familiar Lagrangian density

$$\mathcal{L} = \bar{\psi}(i\not{D})\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{7.71}$$

The Euclidean action is

$$S = \int d^2x \left[\bar{\psi}(i\not{D})\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right], \tag{7.72}$$

where $\bar{\psi} = \psi^\dagger\gamma^0$ is the Dirac adjoint spinor, $\not{D} = \gamma^\mu D_\mu = \gamma^\mu(\partial_\mu - ieA_\mu)$, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor. The fermionic functional integral can be performed exactly due to the simple structure of the Dirac operator in lower dimensions.

It is straightforward to see that, in our generalised framework with the regularised measure $\mathcal{D}(\mathcal{R}_\eta\psi(x))\mathcal{D}(\mathcal{R}_\eta^\dagger\bar{\psi}(x))$, we obtain the regularised Jacobian under a $2D$ chiral transformation [48]

$$\ln\left(\mathcal{R}_\eta J_5(\alpha)\mathcal{R}_\eta^\dagger\right) = -2i\sum_{n=1}\int d^2x\,\phi_n^\dagger(x)\alpha(x)\gamma_5\eta(\mathcal{O})\phi_n(x). \tag{7.73}$$

For the choice of Fujikawa's cut-off $\eta(\mathcal{O}) = \exp\left(-\not{D}^2/\Lambda^2\right)$ the chiral anomaly is given as

$$\mathrm{Tr}(\mathcal{R}_\eta\gamma_5\mathcal{R}_\eta^\dagger) = \frac{i}{2\pi}\int d^2x\,\epsilon^{\mu\nu}F_{\mu\nu}, \tag{7.74}$$

agreeing with established results [48]. Here, $\eta(x)$ is once again a smooth regulator satisfying the properties $\eta(0) = 1$ with rapid decay as $x \to \infty$.

The key point, returning to the regularised Jacobian (7.73), is that we can rewrite this expression for the regularised sum over modes directly in terms of the heat kernel. From the spectral decomposition, one may instead choose the heat kernel cut-off $\eta(\mathcal{O}) = \exp(-s\lambda_n^2)$ so that

$$\mathrm{Tr}(\gamma_5\eta(\mathcal{O})) = \lim_{s\to 0}\sum_n \phi_n^\dagger\gamma_5 e^{-sD^2}\phi_n, \tag{7.75}$$

in which the parameter s is the proper time. In the proper time representation, Fujikawa's regularisation just amounts to setting $s \sim 1/\Lambda^2$ [167]. One can also see this directly from the regularised functional trace of the form $\mathrm{Tr}(\gamma_5\exp(-\not{D}^2/\Lambda^2))$, where in the naive transformation to Schwinger proper time representation the regulator takes the form of a step function $\theta(s - 1/\Lambda^2)$ that picks out exactly one value of $s = 1/\Lambda^2$ in the proper time integration. The reason for this will become clear later on.

From (7.75), one can transform the expression for the regularised trace into the heat kernel representation

$$\mathrm{Tr}(\gamma_5\eta(\mathcal{O})) = \lim_{s\to 0}\int d^2x\,\mathrm{Tr}\left[\gamma_5 K(x,x,s)\right], \tag{7.76}$$

where $K(x,x';s)$ is the heat kernel associated with $\not{D}$ satisfying

$$\left(\frac{\partial}{\partial s} + \not{D}^2\right)K(x,x';s) = 0, \quad K(x,x',0) = \delta^2(x-x'). \tag{7.77}$$

In flat two dimensional Euclidean space, the heat kernel has a well-known asymptotic

expansion for small s in terms of Seeley-DeWitt coefficients [118]. This is given by

$$K(x,x;s) = \frac{1}{(4\pi s)} \sum_{n=0}^{\infty} s^n a_n(x,x), \tag{7.78}$$

where $a_n(x,x')$ are the Seeley-DeWitt coefficients that encode the local geometric properties the operator $\not{D}^2$. The leading term $a_0(x)$ is proportional to the identity, while the next term is $a_1(x,x)$ is proportional to $F_{\mu\nu}$ containing the first non-trivial contributions of the gauge field. Since $\mathrm{Tr}(\gamma_5\sigma^{\mu\nu}) = \epsilon^{\mu\nu}$, we recover the chiral anomaly (7.74). In fact, we may define the anomaly function in two dimensions explicitly as the limiting value of the heat kernel

$$\mathcal{A}(x) = \lim_{s\to 0} \mathrm{Tr}\left[\gamma_5 K(x,x;s)\right]. \tag{7.79}$$

To make better sense of these observations and to draw deeper connections, let us now compare with the generalised Schwinger proper time approach developed in [118], which is inspired by the heat kernel formalism. From the theory of massless Dirac fermions (7.71), one can instead consider the effective action written as a determinant of the Dirac operator

$$e^{-\Gamma_{\mathrm{eff}}[A]} = \int [\mathcal{D}\bar{\psi}\mathcal{D}\psi] e^{-\int d^2x \bar{\psi}(\not{D})\psi} = \det(\not{D}), \tag{7.80}$$

which, upon taking the logarithm, gives the standard fermionic determinant in two dimensions

$$\Gamma_{\mathrm{eff}}[A] = -\ln\det(\not{D}). \tag{7.81}$$

Using the identity $\ln\det(\not{D}) = \frac{1}{2}\mathrm{Tr}\ln(\not{D}^2)$, one can transform this expression for the effective action directly to the generalised Schwinger proper time representation such that

$$-\ln\det(\not{D}) = -\frac{1}{2}\int_0^\infty \frac{ds}{s}\int d^2x\, \mathrm{Tr}\, K(x,x;s), \tag{7.82}$$

where we note $(\not{D})^2 = \left(D^2 + \frac{1}{2}\,\sigma_{\mu\nu}F^{\mu\nu}\right)$ with $\sigma_{\mu\nu} = \frac{1}{2}[\gamma_\mu,\gamma_\nu]$. In this setting, the chiral anomaly is seen to be associated with a variation of the effective action. In the

heat kernel representation, one in fact writes

$$\delta\Gamma_{\text{eff}}[A] = -\frac{1}{2}\int_0^\infty \frac{ds}{s}\int d^2x\,\alpha(x)\,\text{Tr}(\gamma^5 K(x,x;s)), \tag{7.83}$$

where the trace is defined over Dirac and internal indices. Compared with (7.76), the main difference is that the UV divergence has been shifted completely to the proper time variable s. This means the s integration requires regularisation. Moreover, the general integral structure in (7.83) for anomaly calculations in proper time is based on the definition of a general family of Schwinger proper time integrals. This family of integrals, one for each $s \in [0,\infty)$, is given by

$$= -\frac{1}{2}\int_0^\infty \frac{ds}{s}\rho(\Lambda,s)\int d^2x\,\alpha(x)\,\text{Tr}(\gamma^5 K(x,x;s)), \tag{7.84}$$

with $\rho(\Lambda,s)$ a class of smooth regulator functions that regularises the proper time integration, ensuring convergence while preserving gauge invariance [118]. This regulator $\rho(\Lambda,s)$ is not as arbitrary as it may first appear. As we'll see, it is closely related to the class of η regulators [1] derived from irreducible loop integrals, and, consequentially, with the regularised measure in the path integral. Just the same, $\rho(\Lambda,s)$ must satisfy $\rho(s)\to 1$ for large s, and $\rho(s)\to 0$ as $s\to 0$. Common choices of ρ including proper time cut-off, Pauli-Villars, dimensional regularisation, and zeta function regularisation.

Given we are interested in a conceptual comparison, it is notable that equation (7.76) gives a regulated trace arising directly from the path integral measure (Fujikawa's approach), while the expression for $\delta\Gamma_{\text{eff}}$ in (7.84) represents the anomaly as a functional variation of the effective action, requiring integration over the proper time parameter s and insertion of the local chiral parameter $\alpha(x)$. Despite the structural differences, both formulations involve the same spectral content via $\text{Tr}(\gamma^5 K(x,x;s))$, and the regulator $\rho(\Lambda,s)$ provides a unified smoothing between these approaches.

The deeper point is that, as observed from (7.75), Fujikawa's exponential regularisation, which suppresses high eigenmodes in the spectral representation, can be approximated for the choice $\rho(\Lambda,s) = \theta(s - 1/\Lambda^2)$ in the Schwinger proper time representation [118]. In Schwinger proper time, one defines the Green function or effective action so that short time contributions $s \le 1/\Lambda^2$ are not integrated over,

or are effectively damped by $\rho(\Lambda, s)$. The difference in the two representations of the Fujikawa cut-off function is just the difference in the spectral and proper time descriptions of the same regularised theory.

Tying all of these approaches together, we note that from a momentum space perspective, the Euclidean heat kernel $\exp(-sD^2)$ can be expanded directly in terms of momentum eigenstates. For simplicity, from the fermionic determinant (7.82), using the plane wave basis one finds $\langle x|\exp(-sD^2)|x\rangle = \exp(-s(k^2+M^2))$. Hence, one can transform the position space integral to momentum space

$$\Gamma_{\text{eff}}^{Reg}[A] = -\frac{1}{2}\int d^2x \int_0^\infty \frac{ds}{s}\rho(\Lambda, s) \int \frac{d^2k}{(2\pi)^2} e^{-s(k^2+M^2)}. \tag{7.85}$$

The result is precisely the representation of η regularisation in generalised Schwinger proper time. While it is known that a regulator $\eta(k^2/\Lambda^2)$ can be represented in Schwinger proper time form under certain limits—particularly $M^2/\Lambda^2 \to 0$ as emphasised in an earlier chapter—our aim here is to highlight the deeper structural unification of these schemes. Namely, that the same divergent trace can be regularised consistently via $\rho(s)$ in proper time, $\eta(\lambda^2)$ in spectral space, or $\eta(k^2)$ in momentum space, all preserving gauge invariance and yielding the same chiral anomaly in the Schwinger model.

Indeed, from the spectral analysis the generalised Schwinger proper time regulator $\rho(\Lambda, s)$ can be related directly to the η regulators in the momentum space representation [1] via a modified or weighted Laplace transform. (If $M^2 \ll \Lambda^2$, then the Laplace equivalence is only approximate). This draws further connection between Fujikawa's spectral cut-off, generalised Schwinger proper time, and η regularisation of loop integrals. One furthermore sees that the eigenvalue sum discussed in Section 7.2.1 can be written in the form $\sum_n^\infty \langle n|\gamma_5|n\rangle \eta(\Lambda, \lambda_n)$. For the choice of $\eta(\Lambda, \lambda_n)$ equivalent to Fujikawa's regulator, we obtain the same asymptotic value of the regularised sum in (7.30). And since η is equivalent to ρ in their respective representations, we once again observe a unifying picture of how the same divergent series is regularised with the requirements of maintaining gauge invariance, with the resulting spectral asymmetry in the regularised summation ultimately leading to the chiral anomaly.

Of course it is trivial to observe that one can regularise the same quantity in three different ways; but it would be interesting to study more deeply the possible

equivalence of these different albeit related regularisation prescriptions. Furthermore, it is intriguing to ask whether there is a mapping between corresponding regulators, and whether any such mapping might be insightful in terms of a general theory of regularisation.

7.5 Conclusions

In this chapter, we have explored the relationship between Fujikawa's path integral formulation of the chiral anomaly and the more general idea of the regularised functional measure. Beginning with a review of Fujikawa's formalism, we emphasised the central role played by the divergent sum over eigenmodes of the Dirac operator in the computation of the anomaly. This sum, appearing in the functional Jacobian transformation under chiral symmetry, is inherently ill-defined, requiring an appropriate regularisation scheme to extract meaningful physical results.

To understand the mathematical structure of this divergence, we analysed the spectral summation in detail, demonstrating that the sum over eigenmodes takes the form of a generalised alternating series. Using smoothed asymptotics, we rigorously computed the weighted sum over eigenmodes, making use of the smoothed summation method associated with η regularisation to ensure well-defined convergence. This analysis not only reinforced the argument of regularising the functional trace in the sense of how it is defined, but also highlighted the interplay between spectral asymmetry and anomaly generation.

Building upon this foundation, we then formally defined a generalised framework for defining the regularised measure, incorporating an operator-valued regulating function that naturally captures Fujikawa's original expression for the chiral anomaly from the outset. This framework provided a rigorous justification for the regularisation of the path integral measure and allowed us to recover the standard anomaly result through a properly regulated spectral trace. We further derived the $\iota_E(\Lambda)$ function, encoding the response of the regularised measure to changes in the cut-off parameter.

Our findings demonstrate that the regularisation of the path integral measure is not an ad hoc procedure, but rather an intrinsic feature in an approach to QFT that ensures well-defined anomaly computations. The identification of the Mellin transforms

as a key feature of anomaly regularisation further suggests deeper connections to the theory renormalisation and asymptotic analysis. However, a suitable definition of "super enhanced" regulators ensuring the vanishing of the Mellin transforms remains an important task for the future.

In the final part of this work, we commented on conceptual connections to the Schwinger proper-time formalism, reviewing the direct correspondence between the regularised measure, proper-time cut-off methods, and η regularisation for loop integrals. We discussed in what way Fujikawa's exponential suppression of high-energy eigenmodes corresponds, in the proper-time framework, to an integration cut-off in the short-time region. By introducing a smooth proper-time regulator function $\rho(\Lambda, s)$, we surveyed how the regularisation of the path integral measure translates into a well-defined regularisation prescription within the heat kernel expansion. Finally, by transforming the regulated proper-time integral into momentum space, we made explicit the equivalence between η regularisation and proper-time regularisation, establishing that both methods belong to a broader class of gauge-invariant regulators for anomaly calculations that includes Fujikawa's regularisation scheme. This reveals the edges of a unified picture between different representations including Fujikawa's choice of cut-off, spectral regularisation, and the more general framework of η regularisation.

Future directions include extending these techniques to higher-dimensional anomalies, investigating their implications in topological field theories, and applying the regularised functional measure to gauge-invariant computations in curved spacetime. Of particular interest is the prospect of exploring the non-perturbative renormalisation of QED within this framework. Additionally, our framework may offer a new viewpoint on the multiplicative anomaly, a subtle feature arising in the regularisation of operator products, typically computed using standard zeta or heat-kernel techniques. Further investigation could clarify whether η regularisation, in conjunction with the functional measure, provides a consistent scheme for regularising products of elliptic operators in curved backgrounds.

More broadly, a full generalisation of our framework to curved spacetime naturally raises several questions related to UV completion. As discussed in Section 7.3.2, if the theory is interpreted as an effective field theory, or if background fields such as the metric are eventually promoted to dynamical variables, then previous simplifications

no longer apply. In such cases, divergences that appear to be removable in fixed-background settings—such as constant contributions to the regularised measure—can become field-dependent, thereby affecting the relative weighting of configurations in the full path integral. This suggests that a fully regulator-independent measure, as encoded in the vanishing of all relevant Mellin transforms, may be essential for a consistent UV completion.

An open question remains as to whether residual divergences, even when reduced to constant factors for fixed backgrounds, can still signal a breakdown of convergence when background fields are allowed to vary. If effective descriptions yield partition functions that differ only by constant prefactors for each fixed background configuration, does this guarantee a convergent path integral when all configurations are summed over? Or might such prefactors encode non-trivial global obstructions to UV completion?

Finally, we note recent developments by Branchina et al. [162, 163], which investigate a proper treatment of the functional measure in the context of the cosmological constant problem. Exploring potential connections between their approach and the regularised measure formalism proposed here may yield new insights. Similarly, Bilal and Ferrari [187] applies smooth spectral cut-off methods to link zeta-function regularisation with spectral flow techniques in curved spacetime, offering another possible point of contact. We also note that, from the perspective of our generalised framework, it would be valuable to understand whether these connections can be understood in a scheme-independent way similar to the discussion in Chapter 6.

PART III

String theory, analytic number theory, and UV completion

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8

Introduction to string theory

Physics is mathematical not because we know so much about the physical world, but because we know so little: it is only its mathematical properties that we can discover.

— Bertrand Russell, An outline of philosophy (1927)

So far, throughout this thesis, we have taken a bottom-up approach to a theory of regularisation. We have primarily asked if there are connections between the regularisation of divergent series and divergent QFT integrals, and whether there are clever ways to regularise divergent quantities that eliminate the divergences altogether. The universal features of logarithmic divergences, in contrast to power law divergences, were found to be strong hints at some deeper structure. When we regularised an infinite series that displayed power-law divergence, we saw how it was the finite terms that were instead universal, yielding the same results obtained via analytic continuation. How much further can we go? If the ultimate goal throughout this work is to use η regularisation to gain a window into how nature protects us from infinity, to what extent might we extend this analysis to draw connections with the structures that also govern, from the top-down, the UV finiteness of string theory amplitudes?

More pointedly, having previously introduced *enhanced regulators* to eliminate divergences appearing in number theory, and seeing the role they play for divergent integrals in QFT, we now want to ask: can we connect the methods we have so far developed, via number theory and QFT, to the softening of scattering amplitudes at high energies in a fundamental microscopic theory of nature? Is there such a thing

as stringy enhanced regulators cast down from perturbative string amplitudes to the low-energy field theory? If so, do they connect in any way to the large class of η regulators?

For the remainder of this work, in Part III, our goal is to in the very least begin exploring such questions—in particular, the possibility of deriving string theory induced regulators in UV divergent field theory integrals. In pursuit of such questions, the first step will be to identify and draw on a framework that enables detailed study of field theory amplitudes as directly descendent from perturbative string theory. In other words, we want to systematically study the field theory limit of string theory amplitudes, and in a way that allows us to carefully keep track of the full spectral sum and the underlying modular structure responsible for UV finiteness.

From a field theory viewpoint, it has already been noted that the link between η regularisation and Schwinger proper time integrals is intriguing, not least because of the well known connection between the field theory limit of string theory amplitudes and the worldline formulation of QFT. This shall serve as our starting point, and, restricting only to one-loop amplitudes, we are particularly motivated to search for any possible regulators that exhibit some remnant of modular invariance—what we will refer to as "modular shadows". We would also like to deepen this analysis and say something about the nature of QFT directly in the low-energy limit of string theory.

Whilst string field theory may offer an insightful perspective on the subject (see e.g., [64, 65, 188, 189]), given our interest is in computing string amplitudes and directly taking their field theory limit, following [66] first quantisation appears to be the most efficient approach. For reasons that will be made clear in Chapter 10, the recent tropical amplitudes programme, as developed by Tourkine [66], will serve as a key framework for this purpose.

The strength of the tropical amplitude approach to studying the field theory limit, is that it enables us to rigorously study the degeneration of stringy geometry, particularly by tracing the degeneration of the string worldsheet into worldlines, where the integration over the moduli spaces of Riemann surfaces becomes an integration over Schwinger-like proper time [189]. By studying the limit where strings collapse to point-like objects, the tropical formalism furthermore enables us to map the degeneration of one-loop string amplitudes directly to their corresponding Feynman graphs in QFT.

The tropical amplitudes framework thus makes it possible to carefully keep track of the key mechanisms underlying the UV finiteness of string amplitudes, weighting them against the emergence of UV divergences in the descendent low-energy field theory, which generally appear from regions where a loop in the worldline diagram collapses to a point.

Before arriving at this analysis, however, it will be useful in this chapter to briefly review perturbative bosonic and superstring theory. As this is not a complete review, the purpose is to introduce essential concepts. Then, in the following chapter, we build on these concepts to emphasise the modular structure of string theory amplitudes, detailing important aspects of the partition function as well as modular invariance as related to UV/IR mixing. We will then conclude with an introductory discussion on the field theory limit.

In this review chapter, we primarily follow the textbook by Polchinski [10, 11] as well as the textbook by Green, Schwarz, and Witten [190, 191]. Additionally, we also utilise the following secondary and supporting texts emphasising the study of the modular structure of string amplitudes in [43] and [192].

8.1 Overview

String theory is a theory of fundamental interactions. As the leading theory of quantum gravity, it offers significant insight into the unification of the standard model of particle physics with general relativity.

A key feature of string theory—what makes it a *stringy* theory of quantum gravity—is how zero-dimensional fundamental point particles are replaced by one-dimensional extended objects modelled as strings. Connected strings may have two different topologies: a *closed string*, or circle, or an *open string*, or an interval. Open strings with two endpoints can therefore be distinguished from closed strings that form a loop. Since the string is a generalisation of point particle theory (i.e., objects which trace a one-dimensional worldline), due to time evolution a one-dimensional closed string sweeps out a two-dimensional surface or topology of a cylinder. An open string sweeps out a rectangle. Such surfaces are known as the string worldsheet, represented in Figure 8.1. A key distinguishing feature is that, in open string theories, the worldsheet

can have boundaries. As a result, open string theories always also contain closed strings.

The string worldsheet can be orientable or non-orientable. For worldsheets which are non-orientable, the theory is simply called unoriented. In our discussion, however, we will be interested entirely in orientable theories.

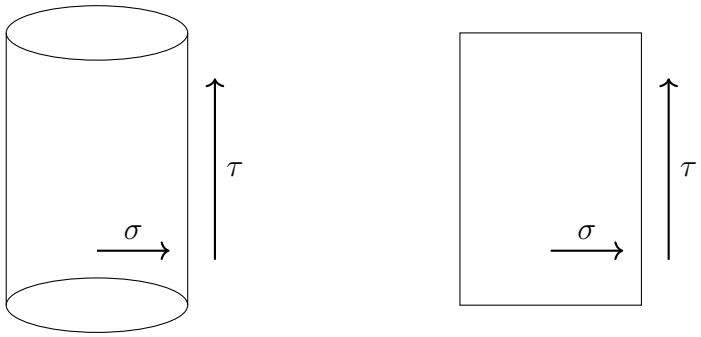


Figure 8.1: Worldsheets swept out by a closed string (left) and an open string (right).

The size of strings is set by the Planck length ℓ_P. The Planck length is approximately [43]

$$\ell_P = \sqrt{G_N \hbar / c^3} \approx 10^{-33} cm, \tag{8.1}$$

with G_N Newton's gravitational constant, $\hbar$ Planck's constant, and c the speed of light. All distance scales that are presently experimentally accessible are magnitudes larger than ℓ_P. Importantly, as we will discuss in a later chapter, at very long distance scales, which, equivalently, correspond to low-energy scales of the string, the string behaves effectively as a point particle. This is known as the field theory limit.

8.2 Bosonic strings

String theory is formulated in terms of a two-dimensional conformal field theory (CFT) on the worldsheet, with the embedding fields

$$X^\mu(\sigma^a), \quad \mu = 0, \ldots, D-1, \quad a = 0, 1, \tag{8.2}$$

mapping the worldsheet into the D-dimensional spacetime $X : \Sigma \mapsto \mathbb{R}^D$, with Σ the two-dimensional worldsheet. We denote the worldsheet coordinates as $\sigma^a = (\sigma^0, \sigma^1) = (\tau, \sigma)$, and spacetime indices with Greek letters (μ, ν). The worldsheet metric has Lorentzian signature $(-,+)$, and the spacetime metric is mostly-plus $(-,+,\ldots,+)$.

The classical dynamics of the string is governed by the first principle Nambu-Goto action, proportional to the area of the worldsheet

$$S_{\text{NG}} = -T \int d\tau d\sigma \, \sqrt{-\det h_{ab}}, \tag{8.3}$$

where

$$h_{ab} = \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X(\tau, \sigma)), \tag{8.4}$$

with $\partial_a = \partial/\partial\sigma^a$. We denote h to be the induced metric on the worldsheet (i.e., the pull-back under X of the spacetime metric $G_{\mu\nu}$) and T is the string tension. Adopting natural units $\hbar = c = 1$ unless otherwise specified in this thesis, we define

$$T = \frac{1}{2\pi\alpha'} \tag{8.5}$$

setting the fundamental length scale of the theory with $\alpha' = \ell_s^2$.

The Nambu-Goto action describes bosonic fields on the string worldsheet. It is analogous to the classical relativistic point particle action, which means it shares the same problem of containing derivatives in the square root. This makes quantisation difficult. This can be remedied by introducing an independent worldsheet metric $\gamma_{ab}(\tau, \sigma)$ with Lorentzian signature $(-,+)$. Doing so leads to the Polyakov action given as

$$S_{\text{Polyakov}} = -\frac{1}{4\pi\alpha'} \int d\tau d\sigma \, \sqrt{-\gamma}\, \gamma^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}, \tag{8.6}$$

with $\gamma = \det \gamma_{ab}$. The Polyakov action possesses both local and global symmetries. It is classically invariant under worldsheet and target space diffeomorphisms, as well as under Weyl transformations, with invariance under the Poincaré group assuming the target space metric is Minkowski. These symmetries are summarised as follows:

1. D-dimensional invariance under the Poincaré group:

$$X'^{\mu}(\tau,\sigma) = \Lambda^{\mu}_{\nu} X^{\nu}(\tau,\sigma) + a^{\mu}$$

$$\gamma'_{ab}(\tau,\sigma) = \gamma_{ab}(\tau,\sigma)$$

2. Worldsheet diffeomorphism invariance:

$$X'^{\mu}(\tau',\sigma') = X^{\mu}(\tau,\sigma)$$

$$\frac{\partial \sigma'^{c}}{\partial \sigma^{a}} \frac{\partial \sigma'^{d}}{\partial \sigma^{b}} \gamma'_{cd}(\tau',\sigma') = \gamma_{ab}(\tau,\sigma)$$

3. Worldsheet Weyl invariance:

$$X'^{\mu}(\tau,\sigma) = X^{\mu}(\tau,\sigma)$$

$$\gamma'_{ab}(\tau,\sigma) = \exp(2\omega(\tau,\sigma))\gamma_{ab}(\tau,\sigma),$$

where Λ^{μ}_{ν} is a D-dimensional Lorentz transformation matrix in target space, a^{μ} is a constant translation vector, and ω the standard conformal factor, reflecting the conformal symmetry of the theory. Moreover, by promoting the worldsheet metric to an independent field $\gamma_{ab}(\tau,\sigma)$, Weyl invariance becomes manifest (and an additional set of constraints are derived from the equations of motion). This is a local rescaling symmetry of the worldsheet metric.

The symmetries of the Polyakov action can be fixed by going to the conformal gauge, where $\gamma_{ab} = e^{2\omega(\tau,\sigma)}\delta_{ab}$ with δ_{ab} a flat metric of Lorentzian signature.

As a point of emphasis, X^{μ} can take on the interpretation as either spacetime coordinates or dynamical fields on the worldsheet. This means that when considering Lorentz transformations, for example, while local spacetime symmetries, they are in fact global symmetries of the worldsheet. In other words, the Poincaré transformations are global symmetries; but reparameterisations and Weyl transformations are local symmetries, with the latter used when making a decision on a gauge.

After Wick rotating $\tau \to -i\tau$, and by introducing complex coordinates on the Euclidean worldsheet $z = \sigma + i\tau$ and $\bar{z} = \sigma - i\tau$, with $X(z,\bar{z})$, and derivatives defined

as

$$\partial = \frac{1}{2}(\partial_\sigma - i\partial_\tau), \qquad \bar{\partial} = \frac{1}{2}(\partial_\sigma + i\partial_\tau), \tag{8.7}$$

the Polyakov action (8.6) for the Euclidean worldsheet becomes

$$S_{\text{Polyakov}} = \frac{1}{2\pi\alpha'} \int d^2z \, \partial X^\mu \bar{\partial} X_\mu, \tag{8.8}$$

with $d^2z = dzd\bar{z}$ and where $\gamma_{ab} \to \delta_{ab}$ denoting transformation to a flat space metric. Upon fixing the gauge by setting the worldsheet metric to be flat (i.e., the conformal gauge), the theory becomes a two-dimensional Euclidean CFT. The classical equations of motion reduce to the wave equation

$$\partial\bar{\partial} X^\mu = 0, \tag{8.9}$$

which implies that the solutions decompose into holomorphic and antiholomorphic parts by standard complex analysis, since any function annihilated by $\partial\bar{\partial}$ is harmonic. The most general solution to (8.9) is given by a linear combination of left- and right-moving modes $X^\mu(z, \bar{z}) = X^\mu_L(z) + X^\mu_R(\bar{z})$.

For the closed string, where $\sigma \in [0, 2\pi]$ and periodic boundary conditions are imposed, the general solution to the equations of motion is

$$X^\mu(z, \bar{z}) = x^\mu - i\frac{\alpha'}{2} p^\mu \ln \mid z \mid^2 + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \left(\frac{1}{n} \alpha^\mu_n z^{-n} + \frac{1}{n} \tilde{\alpha}^\mu_n \bar{z}^{-n} \right), \tag{8.10}$$

where α^μ_n and $\tilde{\alpha}^\mu_n$ correspond to the left- and right-moving modes respectively. They satisfy

$$[\alpha^\mu_m, \alpha^\nu_n] = m\delta_{m+n,0}\eta^{\mu\nu}, \quad [\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n] = m\delta_{m+n,0}\eta^{\mu\nu}, \tag{8.11}$$

with the Virasoro generators split into holomorphic and anti-holomorphic parts

$$L_0 = \frac{\alpha'}{4} p^2 + \sum_{n=1}^{\infty} \alpha^\mu_{-n} \alpha_{n,\mu}, \quad \tilde{L}_0 = \frac{\alpha'}{4} p^2 + \sum_{n=1}^{\infty} \tilde{\alpha}^\mu_{-n} \tilde{\alpha}_{n,\mu}, \tag{8.12}$$

and the mass-shell condition becomes $L_0 + \tilde{L}_0 - 2A = \alpha' M^2$. The normal ordering constant $A = -\frac{D-2}{24}$ appears in both sectors, arising from the infinite sum $\sum_{n=1}^{\infty} n$. Traditionally, zeta-function regularisation or a form of Gaussian cut-off is used. In

light of the findings in Chapter 4, one can of course also use η regularisation, with enhanced η regulators confirming the unique value of minus one-twelfth. However, before exploring the regularisation of this sum and its full implications, we turn to a summary of the open string case.

8.2.1 Open bosonic strings

For open strings, Neumann boundary conditions require

$$\partial_\sigma X^\mu|_{\text{boundary}} = 0 \quad \Rightarrow \quad \partial X^\mu(z) = \bar{\partial} X^\mu(\bar{z}) \text{ on the boundary } (z = \bar{z}). \tag{8.13}$$

This identification leads to the doubling trick, where one reflects the holomorphic field to define the full solution $X^\mu(z, \bar{z}) = X^\mu(z) + X^\mu(\bar{z})$. Importantly, as a hint of what is to come when considering the modular structure of the string, this construction uses the upper-half complex plane for $\text{Im}(z) > 0$, where the open string worldsheet lives, and the reflection allows us to define the theory holomorphically on the full complex plane. As a result, the open string can be described entirely by a single holomorphic field defined on the full complex plane, subject to this reflection symmetry. The holomorphic part $X^\mu(z)$ admits a Laurent expansion

$$X^\mu(z) = x^\mu - i\alpha' p^\mu \ln z + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n} \alpha_n^\mu z^{-n}, \tag{8.14}$$

which is the complex analogue of the Fourier mode expansion in real coordinates. The oscillators α_n^μ satisfy the standard commutation relations

$$[\alpha_m^\mu, \alpha_n^\nu] = m\eta^{\mu\nu}\delta_{m+n,0}, \quad [x^\mu, p^\nu] = i\eta^{\mu\nu}. \tag{8.15}$$

The energy-momentum tensor is given by $T(z) = -\frac{1}{\alpha'} : \partial X^\mu \partial X_\mu(z) :$, with Laurent modes $L_n = \frac{1}{2}\sum_m \alpha_{n-m}^\mu \alpha_{m,\mu}$ and where the colons denote normal ordering. In particular, the Virasoro generator L_0 governs the mass spectrum

$$L_0 = \alpha' p^2 + \sum_{n=1}^{\infty} \alpha_{-n}^\mu \alpha_{n,\mu}. \tag{8.16}$$

The sum of naturals and a σ-invariant regulator

Normal ordering introduces a constant shift due to the vacuum fluctuations of the infinite tower of oscillator modes. In the quantum theory, each non-zero oscillator mode α_n^μ behaves as an independent harmonic oscillator with zero-point energy $\frac{1}{2}n$. To see this, note that from (8.16), since the $\sum_{n=1}^{\infty} \alpha_{-n}^\mu \alpha_{n,\mu}$ is normal ordered, the vacuum expectation values of L_0 contains a contribution from the normal ordering constant. In string theory, famously, this is precisely where the infinite (and divergent) sum over natural numbers arises such that

$$\langle 0| \sum_{n=1}^{\infty} \alpha_{-n}^\mu \alpha_{n,\mu} |0\rangle = \sum_{n=1}^{\infty} n \cdot \delta_\mu^\mu = D \cdot \sum_{n=1}^{\infty} n,$$

in which the sum describes the contribution from a single transverse direction. Let us simply consider

$$\sum_{n=1}^{\infty} n. \tag{8.17}$$

This formally divergent expression encodes the collective vacuum energy of the infinite set of physical oscillators and plays a central role in determining the mass spectrum of the open string. We recognise this divergent sum from Chapter 4. As mentioned a short time ago, often one will use zeta function regularisation with $\zeta(-1) = -\frac{1}{12}$. But from previous discussions on η regularisation, it is interesting to instead consider a smooth cut-off.

A natural choice for a smooth regulator is $e^{-\epsilon \gamma_{\sigma\sigma}^{-1/2} |k_\sigma|}$, as Polchinski shows [10], with $k_\sigma = n\pi/\ell$. The factor $\gamma_{\sigma\sigma}^{-1/2}$ is included to ensure invariance under sigma parametrisation. The issue with this smooth regulator is that it produces a cut-off dependent term proportional to the length of the string. Weyl invariance requires the cut-off dependent term is cancelled, which can be done by including a counter-term in the action proportional to $\int d^2\sigma (-\gamma)^{-1/2}$. Alternatively, it is interesting to ask whether it is consistent to use the enhanced regulator of order one $\eta_{[1]}(x) = e^{-x} \cos x$ as we did previously to obtain (4.11) in Chapter 4. The advantage of η regularisation is that it completely kills the divergence, and thus in principle eliminates the need for the inclusion of a counter-term.

The important point is that the regulator must be a scalar function on the worldsheet;

otherwise, the regularisation breaks diffeomorphism invariance or Weyl invariance, and the conformal anomaly is not under control.

Let $x_n = \epsilon \cdot \frac{n\pi}{\ell\sqrt{\gamma_{\sigma\sigma}}}$ be a dimensionless scalar under worldsheet reparametrisations. We then define the regularised sum as

$$S(\epsilon) := \sum_{n=1}^{\infty} n\, \eta_{[1]}(x_n). \tag{8.18}$$

Since $\eta_{[1]}(x)$ is a scalar function of x_n, with both $\sqrt{\gamma_{\sigma\sigma}}$ and k_σ transforming appropriately under reparametrisations, the entire sum is manifestly sigma-invariant. To evaluate the regularised sum, we then write

$$S(\epsilon) := \sum_{n=1}^{\infty} n \cdot \eta_{[1]}\left(\epsilon \cdot \frac{n\pi/\ell}{\sqrt{\gamma_{\sigma\sigma}}}\right) = \sum_{n=1}^{\infty} n \cdot e^{-x_n} \cos(x_n), \tag{8.19}$$

which has a well-defined limit as $\epsilon \to 0$. We find

$$\lim_{\epsilon\to 0} S(\epsilon) = -\frac{1}{12} \tag{8.20}$$

giving the correct final expression for the Virasoro anomaly shift

$$L_0 = \alpha' p^2 + \sum_{n=1}^{\infty} \alpha^{\mu}_{-n}\alpha_{n,\mu} - \frac{D}{24}. \tag{8.21}$$

Thus, the enhanced regulator $\eta_{[1]}(x) = e^{-x}\cos x$ provides a smooth, convergent, and *sigma-invariant regularisation* of the divergent oscillator sum $\sum_{n=1}^{\infty} n$. By expressing the argument x_n in terms of the dimensionless scalar combination $k_\sigma/\sqrt{\gamma_{\sigma\sigma}}$, the regularisation preserves reparametrisation invariance of the worldsheet, and in the limit $\epsilon \to 0$ correctly reproduces the normal ordering constant of $-1/12$ for each transverse direction. Remarkably, the η regularisation damps the divergence so effectively that it eliminates the need for a counter-term to restore Weyl invariance, offering an elegant alternative.

Using this regularisation, the zero-point energy and infinite sum over natural numbers appear as the normal ordering constant in the Virasoro algebra. In the complex coordinate formulation, this divergence is interpreted as a conformal anomaly, reflecting the central charge of the theory.

8.2.2 Closed string

For closed strings, the worldsheet is topologically a cylinder, and the classical embedding fields $X^\mu(\tau,\sigma)$ satisfy periodic boundary conditions

$$X^\mu(\tau,\sigma+2\pi) = X^\mu(\tau,\sigma). \tag{8.22}$$

The left- and right-moving components admit Laurent expansions:

$$X^\mu(z,\bar{z}) = x^\mu - i\frac{\alpha'}{2}p^\mu \ln(z\bar{z}) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\left(\frac{1}{n}\alpha_n^\mu z^{-n} + \frac{1}{n}\tilde{\alpha}_n^\mu \bar{z}^{-n}\right), \tag{8.23}$$

where α_n^μ and $\tilde{\alpha}_n^\mu$ denote the left- and right-moving oscillators, respectively. The canonical quantisation procedure promotes the mode coefficients to operators with commutation relations

$$[\alpha_m^\mu, \alpha_n^\nu] = m\delta_{m+n,0}\eta^{\mu\nu}, \quad [\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu] = m\delta_{m+n,0}\eta^{\mu\nu}, \quad [x^\mu, p^\nu] = i\eta^{\mu\nu}. \tag{8.24}$$

The holomorphic and antiholomorphic components of the energy-momentum tensor are

$$T(z) = -\frac{1}{\alpha'} : \partial X^\mu \partial X_\mu :, \qquad \bar{T}(\bar{z}) = -\frac{1}{\alpha'} : \bar{\partial} X^\mu \bar{\partial} X_\mu :, \tag{8.25}$$

with Laurent mode expansions

$$T(z) = \sum_n \frac{L_n}{z^{n+2}}, \qquad \bar{T}(\bar{z}) = \sum_n \frac{\tilde{L}_n}{\bar{z}^{n+2}}. \tag{8.26}$$

The Virasoro generators are expressed in terms of the oscillators as

$$L_n = \frac{1}{2}\sum_m : \alpha_{n-m}^\mu \alpha_{m,\mu} :, \qquad \tilde{L}_n = \frac{1}{2}\sum_m : \tilde{\alpha}_{n-m}^\mu \tilde{\alpha}_{m,\mu} :, \tag{8.27}$$

and in particular the zero-mode components are

$$L_0 = \frac{\alpha'}{4}p^2 + \sum_{n=1}^{\infty} \alpha_{-n}^\mu \alpha_{n,\mu}, \qquad \tilde{L}_0 = \frac{\alpha'}{4}p^2 + \sum_{n=1}^{\infty} \tilde{\alpha}_{-n}^\mu \tilde{\alpha}_{n,\mu}. \tag{8.28}$$

Similar as the previous example of the open string, normal ordering introduces a

constant shift due to the zero-point energy of the infinite tower of oscillator modes. The important difference is that physical closed string states must satisfy the level-matching condition

$$L_0 - \tilde{L}_0 = 0, \tag{8.29}$$

along with the mass-shell condition

$$L_0 + \tilde{L}_0 - 2A = \alpha' M^2, \tag{8.30}$$

where A is the constant arising from the zero-point energy. Thus, the mass spectrum of the closed bosonic string is determined by the left- and right-moving oscillator numbers and the zero-point energy. The normal ordering constant, which arises from the vacuum fluctuations of the oscillators, plays a central role in fixing the critical dimension and ensuring consistency of the quantum theory. The conformal anomaly, encoded in the nonzero trace of the energy-momentum tensor, is proportional to the central charge $c = D$ of the worldsheet theory, and vanishes only in the critical dimension $D = 26$.

8.3 Scattering amplitudes

As discussed in [10], the computation of string amplitudes involves summing over all possible worldsheets with specified vertex operator insertions that encode initial and final asymptotic string states. This framework provides a natural generalisation of perturbative QFT, resulting in an S-matrix element—or scattering amplitude—that describes interactions of incoming and outgoing strings. Asymptotic states are represented by vertex operators inserted at selected points on the worldsheet, exploiting conformal symmetry, which greatly simplifies the construction and computation of amplitudes. Instead of treating external states through explicit boundaries on the worldsheet, vertex operators encapsulate all relevant physical information. Crucially, this formalism leverages the state-operator correspondence, establishing a direct one-to-one mapping between asymptotic string states and local operator insertions on the worldsheet.

In this sense, the basic idea in constructing string scattering amplitudes is that

they are computed as path integrals over two-dimensional worldsheet geometries *after* defining the appropriate worldsheet CFT correlators.

For bosonic strings, scattering amplitudes arise as sums over inequivalent Riemann surfaces of fixed topology, weighted by the worldsheet Polyakov action. Once the initial and final data of the incoming and outgoing string scattering states are fixed, the only remaining topological invariant is the genus $g \geq 0$, which counts the number of handles on the surface and determines the loop order. The genus g should not be confused with the string coupling g_s, which determines the relative weight of each topology in the perturbative expansion. The total perturbative string amplitude for a fixed scattering process can be expressed schematically as the following genus expansion in powers of g_s

$$\mathcal{A} = \sum_{g=0}^{\infty} g_s^{2g-2} \mathcal{A}^{(g)} + \text{non-perturbative corrections.} \tag{8.31}$$

This expansion is illustrated in Figure 8.3, where the sum over topologies plays a role analogous to the sum over Feynman diagrams in perturbative quantum field theory. For any fixed string theory, one must sum over all compact surfaces connecting asymptotic data. This structure emerges naturally from the coupling of the dilaton to worldsheet curvature.

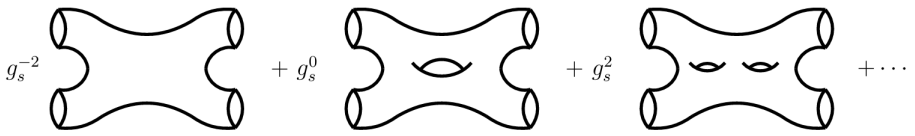


Figure 8.2: Sum over topologies for oriented closed and open string four-point amplitudes, courtesy of [43].

8.3.1 Path integral, CFT correlators, and gauge redundancies

From the Polyakov action (8.6), we have seen that string theory is constructed to be invariant under both diffeomorphisms and Weyl rescalings of the worldsheet, as well as diffeomorphisms of the target spacetime. As discussed in [43], in terms of the space of maps, a crucial feature is how the Polyakov approach may be formulated in terms of

continuous maps, with $X\colon \Sigma \to M$ a map from the two-dimensional worldsheet Σ to the target space M with spacetime metric G. The pullback of G under X defines the induced metric $X^*(G)$ on the worldsheet.[1] In the Polyakov formulation, an intrinsic metric γ_{ab} on Σ is introduced independently of X, and the path integral is defined over both X and γ. Since physical amplitudes must be independent of coordinate choices on Σ and M, the theory possesses diffeomorphism invariance under $\mathrm{Diff}(\Sigma)$ and $\mathrm{Diff}(M)$.

Feynman's prescription for computing quantum amplitudes translates to a path integral over both the space of maps $X \in \mathrm{Map}(\Sigma, M)$ and the space of metrics $\mathrm{Met}(\Sigma)$. The functional integration measures $\mathcal{D}\gamma$ and $\mathcal{D}X$ are formally constructed from L^2 norms on the respective tangent spaces in a way that respects the symmetries of the theory. A partial path integral over $\mathrm{Map}(\Sigma, M)$ defines the effective action at fixed worldsheet metric [43]

$$e^{-W_G[\gamma]} = \int_{\mathrm{Map}(\Sigma,M)} \mathcal{D}X\, e^{-S_{\mathrm{Polyakov}}[X,\gamma]}, \tag{8.32}$$

in which, heuristically, we may think of $-W_G[\gamma]$ as the generating functional (like a 2D effective action). This two-dimensional QFT defined by the action $-S_{\mathrm{Polyakov}}[X, \gamma]$ for fixed intrinsic metric γ and target space metric G is more specifically known as a two-dimensional non-linear sigma model. While such a model generally involves an infinite number of couplings via the functional dependence of $G_{\mu\nu}(X)$, in string theory it is special: the two-dimensional theory is renormalisable in the usual sense for two-dimensional field theories, and conformal invariance of the quantum theory constrains the beta functions. Requiring the absence of conformal anomalies leads to equations of motion for $G_{\mu\nu}$, effectively selecting consistent target space backgrounds. This is a distinctive feature of string theory.

Following the discussion in [43], the genus g contribution to the string amplitude can initially be represented as a formal integral over the space of worldsheet metrics γ on a surface Σ of fixed genus:

$$\mathcal{A}^{(g)} \sim \int_{\mathrm{Met}_g(\Sigma)} \mathcal{D}\gamma\, e^{-W_G[\gamma]}. \tag{8.33}$$

[1] To ground this statement in Polchinski's notation, note that $X^*(G)_{ab} = G_{\mu\nu}(X)\partial_a X^\mu \partial_b X^\nu$.

This integral is strictly formal and highlights an essential complication: it is ill-defined due to invariance of both the measure and the integrand under diffeomorphisms Diff(Σ) and Weyl transformations. Consequently, there is an over-counting of physically equivalent configurations, as any two configurations (X, γ) and (X', γ') related by local Diff $\times$ Weyl transformations describe the same physical state. Thus, the space Met(Σ) contains infinitely many redundant configurations connected by these gauge symmetries. To address this issue, a central step is to systematically remove or factor out the extreme gauge redundancy, which is a topic we will return to in a moment.

More formally, to find any correlation function, we define

$$S_{\text{bosonic}}[X, \gamma] = S_{\text{Poly}}[X, \gamma] + \lambda\chi, \tag{8.34}$$

where χ denotes the Euler characteristic of the worldsheet. To construct scattering amplitudes, it is essential to integrate over all possible worldsheet geometries, thus including variations in both metrics and topologies. However, since the Polyakov action does not include topological data, genus-dependent weighting must be introduced in the path integral, capturing the topological data and the genus expansion of the perturbative string. Hence, a topological term is included in the definition of (8.34) to ensure the path integral is sensitive to the worldsheet topology. This can be seen in how, upon constructing a scattering amplitude, the inclusion of the topological term $\lambda\chi$ allows the path integral to distinguish between different worldsheet topologies, and thereby generate the string coupling expansion. Each genus g surface contributes with a weight g_s^{2g-2}, reflecting the topology-dependence encoded by the Euler characteristic. It also serves to define suitable boundary conditions, because to calculate the path integral we need to access on-shell asymptotic states as its boundary.

The Euler characteristic can be derived from topological variables $\chi = 2 - 2g - b - c$ of the worldsheet, with g the genus, b denoting the number of boundaries, and c the number of cross-caps. It can be computed using a topological action constructed from the Gauss-Bonnet formula

$$\chi = \frac{1}{4\pi}\int_\Sigma d^2\sigma\sqrt{\det\gamma}\lambda(X)R + \frac{1}{2\pi}\int_{\partial\Sigma} ds\, k, \tag{8.35}$$

with λ the dilaton, R the curvature scalar of the intrinsic metric γ_{ab}, and $\partial\Sigma$ the

boundary of the worldsheet with boundary measure ds. We also denote k is the geodesic curvature such that $k = \pm t^a n^b \nabla_a t^b$, with t^a the unit vector tangent to the boundary and n_b an outward point vector orthogonal to t^a. This action has the same symmetries as the Polyakov action, with the dilaton defining the coupling constant $g_s \sim e^{\lambda}$.

With the definition (8.34), the path integral for an arbitrary field Φ can be written in terms of the correlation function

$$\langle \Phi(\sigma^1, \sigma^2)\Phi(\tilde{\sigma}^1, \tilde{\sigma}^2)\rangle \sim \int \frac{\mathcal{D}X\mathcal{D}\gamma}{\text{vol}(\text{Diff} \times \text{Weyl})} \Phi(\sigma^1, \sigma^2)\Phi(\tilde{\sigma}^1, \tilde{\sigma}^2) e^{-S_{\text{bosonic}}}, \quad (8.36)$$

with $\sim$ denoting that normalisation and ghost determinants are ignored at this stage. A proper treatment involves BRST quantisation and ghost contributions, to be dealt with in the next steps. As we'll also see, once the gauge redundancy is factored out, the path integral with vertex operator insertions defines a conformal correlator.

In string perturbation theory, one of the main objects of interest is the expectation value of the path integral of a product of local operators. For a closed string, these operators are inserted in the bulk of the worldsheet, while for an open string they are inserted on the boundary. In both cases, the local operator insertions may also be described as fields; however, in the context of CFTs, the notion of a field carries a different meaning than, for instance, the definition of a field in perturbative QFT. This is due to the following subtle difference: in CFTs, fields are more precisely understood as operators classified by transformations under the conformal group, which is conceptually distinct from fields in QFT, which typically refer to dynamical variables entering a Lagrangian (and their composites). Thus, in the present case, a field may be viewed generally as some local expression, which may be the generic field ϕ that enters the path integral in QFT or some composite operators $e^{i\phi}$ and derivatives $\partial^n \phi$. These are all different types of fields or local operators in the CFT dictionary. Consequently, the path integral over X translates into a CFT correlation function involving the insertion of vertex operators.

To compute scattering amplitudes, we therefore consider vertex operators $V_i(z_i, \bar{z}_i)$ inserted at points $z_i \in \Sigma$ to represent the emission and absorption of asymptotic

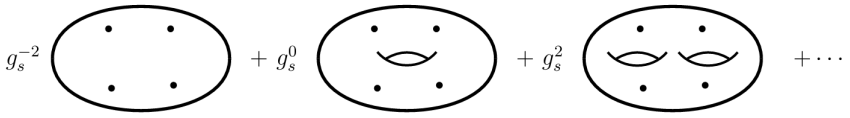


Figure 8.3: A schematic representation of the perturbative expansion in the string coupling constant g_s in terms of vertex operator insertions on the worldsheet, courtesy of [43].

string states, where z and $\bar{z}$ denote complex coordinates as defined in a previous section. In general, at any genus, the state-operator correspondence of two-dimensional CFT allows us to replace the asymptotic states that define boundary conditions with appropriate local vertex operators. As a simple example, the lowest energy state of bosonic string theory is known to be given by the tachyon. In terms of the operator insertions, this corresponds to

$$V(k,z) =: e^{ik\cdot X} :, \tag{8.37}$$

in which k_μ is the momentum of the tachyon satisfying $k^2 = \frac{4}{\alpha'}$. Thus, for n asymptotic closed string states, a general scattering amplitude can be written as

$$\mathcal{A}(1,\ldots,n) \sim \int \frac{\mathcal{D}X\mathcal{D}\gamma}{\text{vol}(\text{Diff}\times\text{Weyl})} e^{-S_{\text{bosonic}}} \prod_{i=1}^{n} \int d^2 z_i \sqrt{\det\gamma} V(k_i, z_i). \tag{8.38}$$

There are two distinct parts defining the structure of this amplitude. If we did not include any interactions, and thus there were no vertex operator insertions, then at one-loop for example all that would remain is the vacuum amplitude of the torus multiplied against an ill-defined measure of integration. This vacuum amplitude corresponds to the genus one string partition function, which is an example that we will cover in detail in a later chapter, albeit in a more complicated setting. If, on the other hand, we maintain the scattering of string states, then we must still deal with the ill-defined measure in addition to the CFT correlator such that

$$\int \mathcal{D}X\, e^{-S_{\text{bosonic}}} \prod_{i=1}^{n} V(k_i, z_i) \sim \left\langle \prod_{i=1}^{n} V(k_i, z_i) \right\rangle_\gamma . \tag{8.39}$$

Here we denote $\left\langle \prod_{i=1}^{n} V(k_i, z_i) \right\rangle_\gamma$ as the CFT correlator at fixed metric γ. At small coupling the perturbative expansion over different genera is given by

$$\mathcal{A}(k_1, \ldots, k_n) \sim \sum_{g=0}^{\infty} g_s^{-\chi} \int \frac{\mathcal{D}\gamma_g}{\text{vol}(\text{Diff} \times \text{Weyl})} \prod_{i=1}^{n} \int d^2 z_i \sqrt{\det \gamma} \left\langle \prod_{i=1}^{n} V_i(k_i, z_i) \right\rangle_{\gamma_g}, \tag{8.40}$$

with γ_g the metric at the fixed genus g.

With this general structure for a string scattering amplitude, we now may finally turn our attention back to the problem of the excess gauge freedom as motivated at the outset of this section. This is where we observe the first glimpse of the natural emergence of the modular structure of string theory amplitudes. To resolve the issue of over-counting, one can factor out the gauge redundancy by

$$\mathcal{M}_g = \frac{\text{Met}(\Sigma)}{\text{Diff}(\Sigma) \times \text{Weyl}}, \tag{8.41}$$

directly yielding the moduli space of genus g Riemann surfaces. This marks the first appearance of the deeply important modular structure underlying string perturbation theory, as elaborated in the next section. The Faddeev-Popov procedure implements this reduction, introducing a ghost determinant that may be represented by a path integral over anticommuting ghost fields b and c with action $S_{\text{gh}}[b, c, \gamma]$. These anticommuting ghost fields arise from the quantisation of the gauge-fixed worldsheet theory, with conformal weights $(2, 0)$ and $(-1, 0)$, respectively, in the holomorphic sector. Gauge fixing then consists of selecting a representative metric $\hat{\gamma}$ within each conformal class, reducing the functional integral to

$$\mathcal{A}_g^{(n)}(k_1, \ldots, k_n) = \sum_{h=0}^{\infty} g_s^{-\chi + n_c + \frac{1}{2} n_0} \int_{\mathcal{M}_{g,n}} \frac{\mathcal{D}\gamma_h}{\text{vol}(\text{Diff} \times \text{Weyl})} \langle \text{ghosts} \prod_{i=1}^{n} \mathcal{V}_i^{(g)}(k_i, z_i) \rangle_{\gamma_g}, \tag{8.42}$$

for a general genus g, n-point amplitude. Here $\mathcal{M}_{g,n}$ denotes the moduli space of conformal structures for n vertex operator insertions, n_0 and n_c are the number of open and closed vertex operators, and $k_1, \ldots, k_n$ denote momenta, polarisations, and other data of the asymptotic states. Without going into detail about the ghost fields, which will be dropped for simplicity, we note that in the complete amplitude

they are present. We also denote $\mathcal{V}_i\left(k_i, z_i\right)$ as the genus g vertex operator of the ith external state, with insertion at z_i. The correlator is computed in the worldsheet CFT defined on the genus-g Riemann surface Σ_g. In the context of this thesis, physical vertex operators typically correspond to BRST-closed states, ensuring decoupling of unphysical modes and consistency with spacetime gauge symmetries.

The moduli space $\mathcal{M}_{g,n}$ is the canonical definition for genus g Riemann surfaces (strictly, smooth conformal structures up to diffeomorphism). We will discuss the definition of the fully modular invariant integrals in a later section. Additionally, the formal structure in (8.42) provides the foundation for the detailed study of modular properties of one-loop amplitudes (specialising to $g = 1$), which will be a key focus of the following chapters. Modular invariance of the integrand is a crucial requirement, especially at one-loop, as it ensures finiteness and consistency.

8.3.2 Green function

The conformal Killing group $\mathrm{SL}(2, \mathbb{C})$ can be used to fix the positions of three vertex operators, leaving an integral over the remaining $n - 3$ positions. Thus, from (8.42), at arbitrary genus g, once the gauge has been fixed through the insertion of ghost fields, the path integral reduces to an integration over all gauge-inequivalent metrics. Eliminating these gauge redundancies effectively transforms the integral over metrics into an integral over the moduli space. Schematically, this is expressed as follows:

$$\int \frac{\mathcal{D}\gamma}{\mathrm{vol}(\mathrm{Diff} \times \mathrm{Weyl})} \to \int_{\mathcal{M}_{g,n}} d\mu_{g,n}, \tag{8.43}$$

where $d\mu_{g,n}$ denotes the modular invariant measure. In special cases like $g = 0$, the moduli space $\mathcal{M}_{0,n}$ corresponds to the space of inequivalent configurations of n punctures on the Riemann sphere modulo conformal transformations. There is residual gauge freedom from Möbius transformations corresponding to the conformal Killing group $\mathrm{SL}(2, \mathbb{C})$, even after conformal gauge fixing. Thus, in this case, one divides by the conformal Killing group, which corresponds to fixing positions of three vertex operators to eliminate the extra redundancy. Doing so—and to derive the correct tree-level amplitude—the corresponding CFT correlator is first derived using Wick's theorem and the two-point function

$$\langle X^\mu(z_1,\bar{z}_1)X^\nu(z_2,\bar{z}_2)\rangle = -\frac{\alpha'}{2}\log|z_1-z_2|^2\eta^{\mu\nu}. \tag{8.44}$$

This is the free field propagator on the complex plane (tree-level). For worldsheets containing cycles, we must ensure that the correlator does not exhibit monodromies when transported around these cycles. To facilitate global generalisations, it is more convenient to employ the Green function, which satisfies

$$\partial\bar{\partial}G(z_1,\bar{z}_1;z_2,\bar{z}_2) = -\pi\delta^{(2)}(z_1-z_2,\bar{z}_1-\bar{z}_2) + \text{extra terms}, \tag{2.15}$$

in the conformal gauge, where the Laplacian becomes $\partial\bar{\partial}$. We will consider such scalar Green functions in more detail in the next chapter, particularly in relation to the theory of automorphic forms. For now, we note that for genus $g \geq 0$ the extra terms are needed because on compact Riemann surfaces without boundary, the Laplacian has a zero mode, and one must define the Green function such that the integral over the surface vanishes

$$\int_\Sigma d^2z\,\sqrt{g}\,G(z,z') = 0. \tag{8.45}$$

Hence, the condition leads to extra terms that are modular dependent. At tree-level, the Green function is simply given by a logarithmic expression. However, as we will shortly discuss, additional terms must be included at the one-loop level. By using the Green function, we obtain a correlator that is sufficiently general to apply at higher genus as well

$$\langle X^\mu(z_1,\bar{z}_1)X^\nu(z_2,\bar{z}_2)\rangle = -\alpha' G(z_1,\bar{z}_1;z_2,\bar{z}_2)\eta^{\mu\nu}.$$

Koba-Nielsen factor

From the condition (8.45), since the Green function must be defined modulo the zero mode, it is assumed G is free of zero mode contributions. It is convention that the zero mode is investigated separately. A well-known result is that, given tachyonic vertex operators, the correlator can be expressed in terms of the Green function as follows

$$\left\langle \prod_{i=1}^{n} : e^{ik_i \cdot X(z_i, \bar{z}_i)} : \right\rangle \sim \delta^{(D)} \left(\sum_{i=1}^{n} k_i \right) \exp \left(\sum_{1 \leq i < j \leq n} -\frac{\alpha'}{2} k_i \cdot k_j G(z_i, \bar{z}_i; z_j, \bar{z}_j) \right), \tag{8.46}$$

which is derived via Wick's theorem using the propagator

$$\langle X^\mu(z_i) X^\nu(z_j) \rangle = -\alpha' G(z_i, z_j) \eta^{\mu\nu} \tag{8.47}$$

and the standard exponential operator identity $\langle : e^A :: e^B : \rangle = e^{\langle AB \rangle}$, with contractions between different insertions giving rise to pairwise $k_i \cdot k_j$ factors. The prefactor delta function ensures momentum conservation, and arises from integrating over the zero mode of X^μ. The exponential is referred to as the Koba–Nielsen factor. For n bosonic states we denote it by $\mathcal{G}_n$ and write

$$\mathcal{G}_n(z_i, z_j) = \exp \left(\sum_{1 \leq i < j \leq n} -\frac{\alpha'}{2} k_i \cdot k_j G(z_i, z_j) \right), \tag{8.48}$$

and similarly for $\bar{z}$.

At genus zero, as mentioned a moment ago, we have $G(z_i, z_j) = \log |z_i - z_j|^2$ and the Koba–Nielsen factor reduces to the standard product $\prod_{i<j} |z_i - z_j|^{-\alpha' k_i \cdot k_j}$. This factor encodes the momentum dependence of the amplitude and plays a central role in the explicit evaluation of tree-level string amplitudes such as the Veneziano and Virasoro–Shapiro amplitudes.

At higher genus, on the other hand, the Green function generalises from (8.46), but now depends on the complex structure and moduli of the worldsheet, and enters more elaborate expressions for the integrand of the moduli space integral. For instance, as elaborated in the next section, the correlator of normal-ordered exponential vertex operators is expressed in terms of a modular form. One sees that this factor captures the non-local interaction among vertex operator insertions and depends on the positions z_i as well as the modular parameter τ.

8.3.3 One-loop

From the discussion in Section 8.3.1, we saw from the view of the stringy path integral how the integration over the moduli space is motivated. At one-loop, the modular structure of string scattering amplitudes becomes much more explicit. This is because, for genus $g = 1$, the worldsheet takes the topology of a torus. After gauge fixing, the integration over worldsheet geometries reduces to an integration over the moduli space of flat tori, characterised by a single complex structure parameter $\tau \in \mathbb{H}$, where, as we'll discuss in detail in the next chapter, $\mathbb{H}$ denotes the upper complex half-plane.

The amplitude at genus one with n vertex operator insertions is given by

$$\mathcal{A}^{(1)}(k_1, \ldots, k_n) \sim \int_{\mathcal{M}_{1,n}} d\mu_{1,n} \left\langle \prod_{i=1}^{n} V(k_i, z_i) \right\rangle, \tag{8.49}$$

where $\mathcal{M}_{1,n}$ is the moduli space of tori with n marked points and $d\mu_{1,n}$ denotes the full measure on this space, including the integration over the modular parameter τ and the insertion points z_i. The correlator includes the contribution from both the matter and ghost sectors and is evaluated using the torus Green function. The structure of the amplitude closely follows that of tree-level, with modifications required to accommodate the toroidal geometry and modular invariance.

To evaluate the correlation function of vertex operators, we again use Wick's theorem and the Green function $G(z, \bar{z}; \tau)$, which on the torus satisfies

$$\partial\bar{\partial} G(z, \bar{z}; \tau) = -\pi\delta^{(2)}(z - \bar{z}) + \frac{\pi}{\operatorname{Im}\tau}. \tag{8.50}$$

This equation reflects the removal of the constant zero mode on a compact surface. The Green function must similarly satisfy the condition

$$\int d^2z\, G(z, z'; \tau) = 0, \tag{8.51}$$

ensuring that G is orthogonal to the zero mode. The explicit form of the Green function can be written in terms of the odd Jacobi theta function $\vartheta_1(z|\tau)$ as

$$G(z, \bar{z}; \tau) = -\log \left| \frac{\vartheta_1(z - \bar{z}|\tau)}{\vartheta_1'(0|\tau)} \right|^2 + \frac{2\pi}{\operatorname{Im}\tau} \operatorname{Im}(z - \bar{z})^2. \tag{8.52}$$

This form is manifestly modular invariant (up to additive constants) and is globally well-defined on the torus. Furthermore, the Green function must be doubly periodic, and ϑ provides the correct elliptic function structure. It reduces to the standard logarithmic form in the degeneration limit $\operatorname{Im}\tau \to \infty$.

8.3.4 Superstrings

Many of the key structures of bosonic string theory can be extended to the supersymmetric setting, with a number of important modifications. In this section, we present a concise summary of superstring theory without delving into technical derivations.

Bosonic string theory, while mathematically consistent in 26 dimensions, suffers from two major physical deficiencies: the presence of a tachyon in the spectrum and the absence of spacetime fermions. Both issues are addressed by extending the worldsheet theory to include two-dimensional supersymmetry, resulting in a superconformal field theory with local fermionic symmetry. The resulting theory—superstring theory—is anomaly-free and consistent only in ten-dimensional spacetime.

To incorporate spacetime fermions and eliminate the tachyonic ground state, one supersymmetrises the Polyakov action (8.6), leading to a worldsheet theory with local $\mathcal{N} = (1,1)$ supersymmetry in Euclidean signature. After gauge fixing the local symmetries, the action takes the form

$$S[X,\psi,\bar{\psi}] = \frac{1}{2\pi}\int d^2z \left(\frac{2}{\alpha'}\partial X^\mu \bar{\partial} X_\mu + \psi^\mu \bar{\partial}\psi_\mu + \bar{\psi}^\mu \partial \bar{\psi}_\mu\right), \tag{2.12}$$

with anticommuting worldsheet spinors $\psi^\mu(z)$ and $\bar{\psi}^\mu(\bar{z})$ taking values in the target space directions, and $\mu = 0,\ldots,9$. This action describes a two-dimensional superconformal field theory with separate left- and right-moving supersymmetries generated by holomorphic and antiholomorphic super-Virasoro algebras.

To define the theory on higher-genus surfaces such as the cylinder or torus, one must impose consistent boundary conditions on the worldsheet fermions. There are two inequivalent choices for the spin structure on the fermionic fields:

- **Neveu–Schwarz (NS)**: $\psi^\mu(e^{2\pi i}z) = -\psi^\mu(z)$, leading to half-integer moding. These states correspond to spacetime bosons.
- **Ramond (R)**: $\psi^\mu(e^{2\pi i}z) = \psi^\mu(z)$, yielding integer moding. The zero modes of

ψ^μ generate a Clifford algebra, and the corresponding ground states transform as spinors under the spacetime Lorentz group.

The formalism outlined in the following, in which worldsheet supersymmetry is manifest while spacetime supersymmetry arises only after a GSO projection (more on this in a moment), is known as the *Ramond–Neveu–Schwarz* (RNS) formalism. The quantisation of the RNS theory requires gauge fixing both bosonic and fermionic symmetries. In addition to the Faddeev–Popov ghosts (b, c) associated with diffeomorphism and Weyl invariance, one introduces bosonic superghosts (β, γ) to fix local worldsheet supersymmetry. The ghost action in conformal gauge is

$$S_{\text{superghosts}}[\beta, \gamma] = \frac{1}{\pi} \int d^2z \left(\beta \bar{\partial} \gamma + \bar{\beta} \partial \bar{\gamma} \right), \tag{2.13}$$

where the holomorphic fields β and γ have conformal weights $(\frac{3}{2}, 0)$ and $(-\frac{1}{2}, 0)$, respectively, and similarly for the antiholomorphic sector.

However, even after quantisation, the spectrum remains tachyonic and lacks spacetime supersymmetry. Modular invariance of the one-loop partition function imposes stringent constraints on the theory, necessitating a projection of the Hilbert space known as the *Gliozzi–Scherk–Olive* (GSO) projection. The GSO projection eliminates the tachyon in the NS sector and selects a subset of R-sector states with definite chirality, thereby ensuring both modular invariance and spacetime supersymmetry. For example, in the left-moving Ramond sector, the ground state is a ten-dimensional Majorana–Weyl spinor, and the GSO projection imposes a definite chirality condition. Consistent projection of left- and right-moving sectors leads to theories with $(\mathcal{N}_L, \mathcal{N}_R) = (1, 1)$ or $(2, 0)$ supersymmetry in ten dimensions.

Thus, the requirement of modular invariance not only ensures the consistency of the quantum theory, but also plays a central role in determining the physical spectrum. In particular, the pairing of spin structures between left- and right-movers constrains the allowed sectors and enforces the elimination of unphysical states.

Alternative formulations include the *Green–Schwarz* formalism [193], in which spacetime supersymmetry is manifest but worldsheet supersymmetry is hidden and quantisation is only consistent in light-cone gauge, and the *pure spinor* formalism [194], which preserves both manifest spacetime supersymmetry and Lorentz invariance, at the expense of a more abstract BRST structure. All three approaches are believed to

be equivalent at the quantum level and yield the same physical spectrum and S-matrix elements.

Depending on the choice of left- and right-moving sectors and the implementation of the GSO projection, one obtains the five consistent ten-dimensional superstring theories: *Type I*, *Type IIA*, *Type IIB*, *Heterotic SO(32)*, and *Heterotic* $E_8 \times E_8$. These theories differ in the amount of spacetime supersymmetry, the structure of their gauge groups, and their spectra of perturbative and non-perturbative objects such as D-branes. They are related to one another via various string dualities and are conjectured to arise as different limits of a more fundamental eleven-dimensional theory, known as *M-theory*, which is one candidate thought to represent the complete non-perturbative formulation of string theory.

8.3.5 A note on the path integral measure

To conclude this chapter, we observe that many of the key insights from Chapter 7 regarding the functional measure and its non-invariance under symmetry transformations extend naturally to the Polyakov path integral for string theory. While standard treatments of bosonic string theory often assume classical Weyl invariance and proceed through gauge fixing and operator quantisation, they tend to sidestep or defer the subtle issues that arise in a rigorous treatment of the quantum measure. As shown in the careful analysis by Fujikawa and Suzuki [48], the regularised path integral measure is not invariant under Weyl transformations. This gives rise to the Weyl anomaly and the associated Liouville dynamics. The demand that the Weyl anomaly vanishes leads to constraints on the theory (like the critical dimension). Their analysis provides a clear, rigorous explanation of how the path integral measure itself plays a role in determining the anomaly structure of the theory.

Furthermore, a modest calculation shows that, under a Weyl transformation of the worldsheet metric, the path integral measure acquires a non-trivial Jacobian

$$\mathcal{D}X^\mu \, \mathcal{D}\gamma_{ab} \longrightarrow \mathcal{D}X^\mu \, \mathcal{D}\gamma \, e^{S_L[\omega]},$$

where $\omega(\tau, \sigma)$ becomes the dynamical Liouville field, and $S_L[\omega]$ is the Liouville action given by

$$S_L[\omega] = \frac{25-D}{48\pi}\int d^2\sigma\,\sqrt{-\hat{\gamma}}\left(\frac{1}{2}\hat{\gamma}^{ab}\partial_a\omega\partial_b\omega + \omega\hat{R}\right), \tag{8.53}$$

with $\hat{\gamma}_{ab}$ a fiducial metric in the conformal class of γ_{ab}, and $\hat{R}$ its associated scalar curvature.

The Liouville action arises as a quantum correction due to the Weyl anomaly. It encodes the failure of the path integral measure to be Weyl invariant. The coefficient $\frac{25-D}{48\pi}$ reflects the conformal anomaly due to the deviation from the critical dimension $D = 26$. The Liouville field ω, originally a gauge degree of freedom, becomes a *dynamical field* when $D \neq 26$. In non-critical string theory, it must be included to ensure quantum consistency.

In summary, the full conformal anomaly is encoded in the central charge condition

$$c_{\text{matter}} + c_{\text{ghost}} + c_{\text{Liouville}} = 0, \tag{8.54}$$

where $c_{\text{matter}} = D$, $c_{\text{ghost}} = -26$, and $c_{\text{Liouville}} = 26 - D$, ensuring that the total central charge vanishes. This condition gives the well-known result that the critical dimension of bosonic string theory is $D = 26$. The necessity of regularising the path integral measure, and the emergence of Liouville dynamics, demonstrates that quantum consistency imposes constraints on the worldsheet geometry, subtly breaking classical background independence.

Even in the critical case where the Liouville mode decouples, the background metric $\hat{\gamma}_{\mu\nu}$ retains topological data of the worldsheet (such as its Euler characteristic). Thus, topological and geometric structure is deeply interwoven into the string path integral via the anomaly structure.

Although a deeper analysis of the functional measure in the stringy path integral extends beyond the scope of this thesis, we note the possible application of the η regularised measure defined in Chapter 7. Interestingly, given Fujikawa and Suzuki's path integral treatment shows that the quantum consistency of string theory requires a non-trivial interplay between gauge fixing, functional determinants, and conformal geometry, it would be interesting to explore these observations in full generality of the regularised measure. It is possible that such an investigation would lead to deeper insights into a generalised theory of regularisation, as well as into how η regularisation

might be related to essential properties of perturbative string theory. Furthermore, we note that calculating the beta functions via the regularised measure could potentially reveal insights into the conformal anomaly, and potentially also to the UV completion of the perturbative string.

9

Modular structure of string amplitudes: A brief tour through analytic number theory

Mathematics, rightly viewed, possesses not only truth, but supreme beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show. The true spirit of delight, the exaltation, the sense of being more than Man, which is the touchstone of highest excellence, is to be found in mathematics as surely as in poetry.

— ***Bertrand Russell, Mysticism and Logic (1910)***

In the last chapter we built up the general structure of string amplitudes from the Polyakov action. Up to the point of deriving the Green function, we glimpsed how the modular structure arises from quotienting out the excess gauge freedom. In this chapter, we discuss in more detail the general modular structure of genus one string amplitudes, before briefly discussing the field theory limit and tying together different aspects of our review.

In this review chapter, we continue to follow [10, 11] as well as [43], which explicitly covers the topic of modular forms and strings. From a mathematics perspective, this chapter also follows standard derivations in analytic number theory (see e.g.,

[31, 32, 195–198]) as well as applications of analytic number theory in string theory (see e.g., [10, 43, 199, 200]).

9.1 Tori, modularity, and modular forms for $\mathbf{SL}(2,\mathbb{Z})$

To build up and understand the moduli space of one-loop string worldsheets in a detailed way, we begin with a more general perspective from the theory of Riemann surfaces [192, 201, 202]. The *Uniformisation Theorem* classifies all simply connected Riemann surfaces as conformally equivalent to one of three canonical domains [192]: the Riemann sphere $\widehat{\mathbb{C}}$, the complex plane $\mathbb{C}$, or the unit disk $\mathbb{D}$. Any compact Riemann surface can thus be realised as a quotient of its universal cover by a discrete group of automorphisms, such as a Fuchsian or Kleinian group. The parameters distinguishing inequivalent complex structures on a fixed topological surface are known as *moduli*.

In the case of a genus-one Riemann surface—the torus—the universal cover is $\mathbb{C}$, and the surface itself may be constructed as a quotient by a rank-2 lattice. Let $\Lambda \subset \mathbb{C}$ be a lattice generated by two $\mathbb{R}$-linearly independent complex numbers $\omega_1, \omega_2 \in \mathbb{C}$:

$$\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2, \qquad \omega_1, \omega_2 \in \mathbb{C}, \quad \mathrm{Im}(\omega_2/\omega_1) > 0. \tag{9.1}$$

The torus is then defined as the complex manifold

$$T^2 = \mathbb{C}/\Lambda,$$

with the identification on the complex coordinate z given by

$$z \sim z + m\omega_1 + n\omega_2, \qquad m, n \in \mathbb{Z}. \tag{9.2}$$

This construction also arises naturally in the theory of elliptic functions. A function $f(z)$ is said to be *elliptic* with respect to a lattice Λ if it is meromorphic on $\mathbb{C}$ and doubly periodic:

$$f(z+\omega_1) = f(z+\omega_2) = f(z).$$

Thus, elliptic functions are functions on the torus $T^2 = \mathbb{C}/\Lambda$. Their analytic structure is deeply tied to the geometry of T^2, with elliptic functions and curves providing an

important set of tools in the study of amplitudes from a mathematical perspective. Later on, we will consider for example Jacobi ϑ functions, which are a class of elliptic functions common in the string amplitude dictionary.

9.1.1 Conformal structure and modular parameter

Given a Riemannian metric γ on a two-dimensional surface Σ, we may perform a Weyl rescaling $\gamma \mapsto e^{\omega}\gamma$ to obtain a conformally equivalent metric. For Riemann surfaces, the uniformisation theorem ensures that any conformal class contains a constant curvature representative. On the torus, this representative is flat, and thus we may choose global coordinates $z \in \mathbb{C}$ such that the metric takes the form $ds^2 = |dz|^2$. All conformal structure information is then encoded in the identification lattice Λ.

However, different choices of basis for the lattice—say, (ω_1, ω_2) and (ω_1', ω_2')—may yield conformally equivalent tori. Two lattices define the same complex torus if they are related by a $\mathrm{GL}^+(2,\mathbb{R})$ transformation, i.e. an invertible linear map with positive determinant

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = A \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}, \qquad A \in \mathrm{GL}^+(2,\mathbb{R}). \tag{9.3}$$

Since we are only interested in the conformal class, without loss of generality we may fix $\omega_1 = 1$, so that $\tau = \omega_2/\omega_1 \in \mathbb{H}$ parametrises the complex structure of the torus. The condition previously given $\mathrm{Im}(\tau) > 0$ ensures a non-degenerate orientation. The conformal structure is then determined up to the modular group $\mathrm{SL}(2,\mathbb{Z})$, as we mod out by integer linear transformations of the lattice basis. That is to say, the residual ambiguity in choosing $\tau \in \mathbb{H}$ corresponds to the action of $\mathrm{SL}(2,\mathbb{Z})$, which identifies physically equivalent tori. Thus, two tori $T^2 = \mathbb{C}/\Lambda$ and $T^2 = \mathbb{C}/\Lambda'$ are conformally equivalent if their lattices are related by a $\mathrm{GL}(2,\mathbb{Z})$ transformation. After normalising $\omega_1 = 1$, this action becomes the modular group $\mathrm{SL}(2,\mathbb{Z})$ acting on $\tau \in \mathbb{H}$, identifying moduli under $\tau \sim \frac{a\tau+b}{c\tau+d}$. Hence, up to rescaling and rotation, any torus may be represented as

$$T^2_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}), \qquad \tau \in \mathbb{H}. \tag{9.4}$$

The identifications on the complex coordinate z become

$$z \sim z + m + n\tau, \qquad m, n \in \mathbb{Z}. \tag{9.5}$$

The parameter $\tau \in \mathbb{H}$ is known as the *modular parameter* or complex structure modulus. It captures the conformal shape of the torus while discarding its overall size, which is irrelevant in conformally invariant theories such as string theory. Hence, in the conformal gauge on a genus-one worldsheet, we observed in (8.41) that the Polyakov path integral localises to an integral over moduli space, with the residual gauge freedom parametrised by τ. Drawing deeper connection with this discussion, this reduction in the path integral measure in fact corresponds to integrating over the modular parameter τ in the fundamental domain of $\mathrm{SL}(2,\mathbb{Z})$ acting on $\mathbb{H}$. Modular invariance arises precisely as the large diffeomorphisms preserving the conformal class.

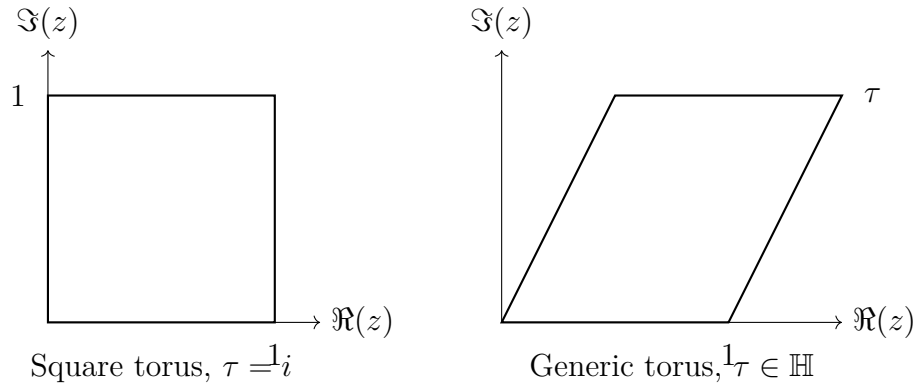


Figure 9.1: Left: Unit square corresponding to the square torus ($\tau = i$). Right: Fundamental parallelogram corresponding to a general lattice $\mathbb{Z} + \tau\mathbb{Z}$ with $\tau \in \mathbb{H}$.

A particularly symmetric case is the square torus, corresponding to $\tau = i$ and $\Lambda = \mathbb{Z} + i\mathbb{Z}$. More generally, the upper half-plane $\mathbb{H}$ parametrises the Teichmüller space of the torus—that is, the space of complex structures up to diffeomorphisms isotopic to the identity. The moduli space of inequivalent conformal structures is then given by the quotient $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}$. Thus, τ is a complex coordinate on the *Teichmüller space* of the torus, and physically equivalent tori correspond to points in $\mathbb{H}$ related by the action of the modular group.

An alternative way to arrive at the Teichmüller space of the torus is directly from the string worldsheet. Upon the use of diffeomorphisms and Weyl transformations to transform the worldsheet metric to a locally flat metric on the torus, one finds that there is a global restriction. In other words, globally

$$\gamma = \mid d\sigma^1 + \tau d\sigma^2 \mid^2 = (d\sigma^1)^2 + \mid \tau \mid^2 (d\sigma^2)^2 + 2\tau_1 d\sigma^1 d\sigma^2 \tag{9.6}$$

which is flat only when $\tau = i$. In this expression, the metric remains invariant under complex conjugation of τ, and becomes degenerate when τ is real. Therefore, we restrict to the region $\tau_2 > 0$.

Furthermore, the modular group $\mathrm{SL}(2,\mathbb{Z})$ acts on the basis of lattice vectors by integer-valued linear transformations

$$\begin{pmatrix} 1 \\ \tau \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ \tau \end{pmatrix} = \begin{pmatrix} a + b\tau \\ c + d\tau \end{pmatrix}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}).$$

This corresponds to a new basis $(1, \tau')$ with

$$\tau' = \frac{a\tau + b}{c\tau + d},$$

so that $(1, \tau)$ and $(1, \tau')$ generate the same lattice Λ. Thus, two complex tori are conformally equivalent if their modular parameters are related by an $\mathrm{SL}(2,\mathbb{Z})$ transformation. The moduli space of conformally inequivalent genus-one surfaces is the quotient

$$\mathcal{M}_1 = \mathbb{H}/\mathrm{SL}(2,\mathbb{Z}). \tag{9.7}$$

This quotient identifies all values of τ corresponding to the same complex torus. As we will discuss in the next section, the standard fundamental domain for this action captures one representative for each equivalence class.

9.1.2 Modular symmetries and equivalence of complex structures

Having introduced the torus T^2 as the quotient $\mathbb{C}/\Lambda$ by a rank-2 lattice, we now explore the symmetries that relate different but conformally equivalent choices of $\tau \in \mathbb{H}$. In doing so, we shall now also give precise definitions of several key concepts.

Although a torus admits uncountably many complex structures, conformal invari-

ance ensures that only the complex structure—encoded by the modular parameter $\tau \in \mathbb{H}$—is physically relevant. In the parameterisation (9.5), we have seen that conformal transformations are rescalings and rotations such that the complex structure is uniquely specified by the modular parameter

$$\tau = \tau_1 + i\tau_2,\ \tau_1, \tau_2 \in \mathbb{R}, \tag{9.8}$$

which remains unchanged under Weyl rescalings of the flat torus metric. As reviewed in Section 8.3.1, the path integral over γ becomes an integral over $\tau \in \mathbb{H}$

$$\mathbb{H} = \{\tau = \tau_1 + i\tau_2 \in \mathbb{C} \mid \tau_2 > 0\}, \tag{9.9}$$

which is a two-dimensional real manifold and serves as the domain for the action of the modular group $\Gamma = \mathrm{SL}(2, \mathbb{Z})$. This space naturally parametrises complex structures on tori.

The full isometry group preserving the structure on $\mathbb{H}$ is $\mathrm{PSL}(2, \mathbb{R})$. This is the group of orientation-preserving isometries of the hyperbolic plane $\mathbb{H}$, equipped with the Poincaré metric

$$ds^2 = \frac{d\tau_1^2 + d\tau_2^2}{\tau_2^2}, \quad \text{for } \tau = \tau_1 + i\tau_2 \in \mathbb{H}. \tag{9.10}$$

The full isometry group acts by Möbius transformations

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}, \quad \text{where } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2, \mathbb{R}) \tag{9.11}$$

modulo scalar multiples of $\pm I$, so we consider the projective group $\mathrm{PSL}(2, \mathbb{R}) = \mathrm{SL}(2, \mathbb{R})/\{\pm I\}$. This is the group of all orientation-preserving isometries of $\mathbb{H}$.

Given the complex modulus $\tau = \tau_1 + i\tau_2$, the modular group $\mathrm{SL}(2, \mathbb{Z})$, which is a subgroup of $\mathrm{SL}(2, \mathbb{R})$, is the group of 2×2 integer matrices with unit determinant:[1]

[1]For future reference, it is worth noting that we can define subgroups of the full modular group as

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (mod N) \right\}, \tag{9.12}$$

which is called the principal congruence subgroup of level N. For example, the case $N = 1$ we have

$$\mathrm{SL}(2,\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{Z}, ad-bc=1 \right\}. \tag{9.13}$$

Let $M \in \mathrm{SL}(2,\mathbb{Z})$.[2] Each such M acts on $\mathbb{H}$ via Möbius transformations

$$\tau \to M\cdot\tau = \tau' = \frac{a\tau+b}{c\tau+d}, \text{ where } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{9.14}$$

for all a,b,c,d with determinant $ad-bc=1$. Note that

$$d\tau' = \frac{d\tau}{(c\tau+d)^2}, \qquad \tau_2' = \frac{\tau_2}{|c\tau+d|^2} \Rightarrow Im(M\cdot\tau) = \frac{\tau_2}{|c\tau+d|^2}. \tag{9.15}$$

The modular group $\mathrm{SL}(2,\mathbb{Z})$ is generated by two elements, S and T transformations, which take the form

$$T:\ \tau' = \tau+1,\ \ S:\ \tau' = -\frac{1}{\tau}, \tag{9.16}$$

with

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{9.17}$$

These elements satisfy three fundamental relations

$$S^2 = (ST)^3 = (TS)^3 = -I. \tag{9.18}$$

Here S inverts the modulus and exchanges the two lattice generators, while T increments τ by one and corresponds to Dehn twists around one cycle of the torus. These define the structure of the modular group $\mathrm{PSL}(2,\mathbb{Z}) = \mathrm{SL}(2,\mathbb{Z})/\{\pm I\}$, which plays an essential role in string theory due to its invariance properties. In string theory, it is common to work with $\mathrm{SL}(2,\mathbb{Z})$, and only quotient by $\{\pm I\}$ in number theory for simplicity in function theory. Physically, signs can sometimes carry meaning (e.g., spin structures).

As we will discuss in a later chapter, the modular invariance of closed string

$\Gamma(1) = \mathrm{SL}(2,\mathbb{Z})$, or simply denoted Γ.

[2] Conventionally, it is standard in the literature to denote γ as the transformation matrix. However, here we use non-standard notation M to avoid confusion with the worldsheet metric γ_{ab}.

amplitudes at one-loop—their invariance under these precise transformations—is a powerful constraint and is directly tied to the absence of UV divergences and the analytic behaviour of amplitudes. Moreover, modular forms—holomorphic functions on $\mathbb{H}$ that transform under $\mathrm{SL}(2,\mathbb{Z})$ in specific ways—arise naturally in the string path integral as part of the integrand due to this underlying modular structure of the string.

Haar measure:

Given $\mathbb{H}$ can be equipped with the Poincaré metric (9.10), it is easily seen that we may define the area element

$$d\mu = \frac{d\tau_1 d\tau_2}{\tau_2^2}. \tag{9.19}$$

This is also known as the Haar measure for the Lie group $\mathrm{PSL}(2,\mathbb{R})$ acting on $\mathbb{H}$ by Möbius transformations (as well as its arithmetic subgroups like $\mathrm{SL}(2,\mathbb{Z})$). One-loop genus $g = 1$ string amplitudes are naturally defined in terms of the Haar measure $\int_{\mathcal{F}} d\mu$, in which the integration is specialised to the fundamental domain of the modular group $\mathrm{SL}(2,\mathbb{Z})$. We hereby refer to the integration measure as (9.19) unless otherwise specified.

9.1.3 UV finiteness

There are many robust arguments for why superstring theory is UV finite at each order [147–150], and there are reasons to believe why it should remain UV finite at higher loop order [10, 147–150]. Rigorous proof in terms of computing actual integrals over moduli space has been established up to two-loop order, with calculations from three-loops up to five-loops studied to different extents. For higher genus and for all orders, in terms of explicit calculation, technical issues including combinatorial complexity and rigorous definition of the measure at arbitrary genus (e.g., for the pure spinor moduli space) remain an open challenge.

In order to comment on UV finiteness at one-loop, we first need to say a word about the modular regularisation natural to closed string theory amplitudes. Tying together the various threads from the first quantised approach reviewed in the previous

chapter, with a brief review of the underlying modular structure of string amplitudes in the current chapter, the single most essential fact is that: at each genus, the contribution to the amplitude arises from a single connected worldsheet topology. This is in complete contrast to the proliferation of Feynman diagrams in perturbative QFT. The result is a compact and symmetric representation of scattering amplitudes in a theory of quantum gravity [10].

In addition to string integrands being based in conformal field theory techniques, with rich symmetry structure to constrain the correlators, a central ingredient in the UV behaviour of string theory is the integration over the moduli spaces of Riemann surfaces. Replacing the analogous momentum integrals of QFT, the smearing out of interactions can be viewed from how the string possesses an internal length scale $\sqrt{\alpha'}$. Crucially, instead of string theory being a theory of an infinite tower of interacting point particle states, which would not display UV completion (see discussion in e.g., [10, 62, 63]), the minimal length of string theory restricts the integration to regions absent of UV divergences.

At one-loop (i.e., genus one), this can be seen very simply in how integration is restricted to the fundamental domain of the modular group $\mathrm{SL}(2,\mathbb{Z})$ acting on the torus. Modular invariance explicitly shows the removal of the UV region of integration that would correspond to short distance divergences in field theory. Thus, the minimum length scale combined with modular structure provides perturbatively a naturally regulated and UV-complete framework.

The fundamental domain $\mathcal{F} \subset \mathbb{H}$ defined on the upper-half plane is given by

$$\mathcal{F} = \{\tau \in \mathbb{H} \,|\, -\frac{1}{2} < \tau_1 \leq \frac{1}{2}, \, |\tau| > 1\}, \tag{9.20}$$

with boundary identifications $\tau \sim \tau + 1$ and $\tau \sim -1/\tau$. The points $\tau = i$ and $\tau = e^{2\pi i/3}$ are fixed under elements of the modular group and correspond to elliptic points, while, as we will discuss in the next chapter, $\tau \to i\infty$ is a cusp.

The fundamental domain $\mathcal{F}$ is the standard representative of the quotient $\mathbb{H}/\mathrm{SL}(2,\mathbb{Z})$. Moreover, the fundamental domain $\mathcal{F}$ is a set of representatives for the orbits of $\mathrm{SL}(2,\mathbb{Z})$ acting on $\mathbb{H}$. Every point $\tau \in \mathbb{H}$ is modular equivalent to a unique point in $\overline{\mathcal{F}}$, the closure of $\mathcal{F}$, up to boundary identifications. This means that every point in $\mathbb{H}$ can be mapped into $\mathcal{F}$ via a transformation in $\mathrm{SL}(2,\mathbb{Z})$, and no two points in $\mathcal{F}$ are equivalent

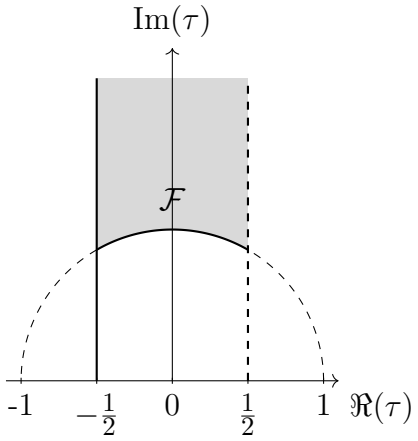


Figure 9.2: The fundamental domain $\mathcal{F}$ of $\mathrm{SL}(2,\mathbb{Z})$ in the upper half-plane.

under the modular group.

In number theory, this quotient space can be viewed as the moduli space of elliptic curves. In hyperbolic geometry, $\mathcal{F}$ is a hyperbolic surface with a single cusp at $i\infty$. Despite the presence of this cusp, and the subsequent fact that the fundamental domain is not compact, its volume is known to be finite

$$\mathrm{Vol}(\mathcal{F}) = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} = \frac{\pi}{3}. \tag{9.21}$$

9.1.4 Concluding comments

We have discussed how the reluctance of string theory to travel deep into the UV is a direct manifestation of the smoothing or smearing effect of the string length scale as seen in a study of modular invariance of the worldsheet. This was first observed in string theory by Shapiro [203], with a more modern treatment found in [10].

These key features of one-loop string amplitudes will serve an important role in the analysis of the next chapter, where we focus on the field theory limit of string amplitudes. In particular, we shall be interested in studying the degeneration of the torus, which, in contrast to the above, is associated with the infrared (IR) region of moduli space. We will see that this region corresponds to the $\alpha' \to 0$ limit, and

we will be especially interested in keeping track of stringy effects closer to the UV region of the theory via the subtle and profound feature of UV/IR mixing of the string spectrum.

Unlike in QFT, where UV and IR behaviours are largely decoupled, in string theory short-distance and long-distance physics are mixed. This mixing arises naturally in the first quantised formalism [10], since, at one-loop, integrals over moduli space are invariant under modular transformations (9.16) mapping UV and IR regions.

10

UV completion in the tropics

There is no royal road to geometry.

— Euclid, according to Proclus

In the introductory chapters to Part III, we briefly discussed how string theory renders amplitudes UV finite through two intertwined mechanisms: minimum length scale and modular invariance. It was mentioned that these properties enforce a form of UV/IR mixing. What was not detailed was how this UV/IR mixing is evidenced in the infinite tower of massive string states. Additionally, we also motivated the study of the field theory limit of string theory amplitudes. Tying together these ideas, and many others from previous chapters of this thesis, we now investigate the question of how UV divergences in QFT might be tamed through some regularising mechanism directly descendent from string theory. In particular, this chapter investigates the emergence of such possible mechanisms in the limit $\alpha' \to 0$, where the Riemann surface of the worldsheet degenerates to Feynman-like graphs.

To proceed, we first offer a general introduction, surveying various known methods for extracting field theory amplitudes from string theory, highlighting the respective strengths and limitations of each approach. In Section 10.2, we then review the exciting and actively developing tropical amplitudes framework as introduced by Tourkine [66], which makes rigorous the connection between $\alpha' \to 0$ limit of string theory (correspondingly, the geometric limit $\tau_2 \to \infty$) and the recent field of mathematics called *tropical geometry*. Emphasising a physics perspective, we focus primarily on the remarkable and recent observation that string theory directly induces a renormalisation

prescription for UV divergent field theory amplitudes.

Building on this discussion, the bulk of the present chapter proceeds from Section 10.3, which considers the non-trivial case of the one-loop closed string partition function. We show that its tropicalisation gives rise to the appropriate renormalisation structure, displaying how string theory can make UV finite the corresponding vacuum energy in the $\alpha' \to 0$ limit.

Finally, in Section 10.4, we use these insights to examine the challenges in deriving classes of string theory induced regulator functions. Complimenting the bottom-up approach examined in Part I and Part II of this thesis, we offer several remarks as to how multiplicative regulators directly descendent from string theory might appear in the *partial tropical limit*, including possible geometric interpretations in terms of dressed or smeared propagators, and also compare against insights from spectral and modular integral approaches (among others). Outlining many significant technical and mathematical hurdles, the discussion throughout this section describes both ongoing research in the background of this thesis, as well as lays the foundation for future directions.

10.1 Introduction

The relationship between perturbative string theory and quantum field theory—particularly regarding their respective ultraviolet behaviours and structure of scattering amplitudes—has been a central theme in quantum gravity research for decades. A long-standing hope, dating back to the early development of string theory, is that the UV finiteness of string amplitudes may point to a broader role for quantum gravity as a universal regulator in the field theory limit, $\alpha' \to 0$, where α' is the Regge slope, related to the string length via $\alpha' = \ell_s^2$.[1]

The process of extracting field theory amplitudes at genus one, and of studying the different limiting behaviours of the string, is already well understood thanks to the pioneering work of Bern and Kosower [204–208]. Their formalism originated from the

[1]Throughout this work we will use "field theory limit", "point particle limit," "infinite tension limit," and "tropical limit" interchangeably. Depending on context, this is often characterised by the limit $\tau_2 \to \infty$, but for consistency with later discussions, we will primarily define it as the limit $\alpha' \to 0$.

observation that gauge theory amplitudes could be extracted directly from one-loop string amplitudes for massless vector states in the $\alpha' \to 0$ limit. Originally developed in the context of heterotic string theory, they employed a first quantised approach to derive effective Feynman rules from a string-theoretic origin, producing master formulae for one-loop n-gluon amplitudes in gauge theory. By capturing the structure of QFT amplitudes without resorting directly to Feynman diagrams, the Bern-Kosower formalism has offered deep insights—particularly into how worldsheet moduli map to Schwinger proper time, and how string correlators reduce to worldline Green functions for point particles.

From the viewpoint of wanting to study string theory's possible regularising effects, a central limitation of the Bern–Kosower approach has to do with how it was constructed to preserve only the massless sector. In their framework, taking the $\alpha' \to 0$ limit causes the string length to shrink, rendering the massive string modes (with mass $M \sim 1/\alpha'$) extremely heavy. These contributions are exponentially suppressed and, as a result, are traditionally considered to decouple. The resulting amplitude therefore retains only the massless sector—an outcome consistent with the goal of reproducing effective field theory, but incompatible with the desire to study the full effects of massive mode contributions.

A similar issue is present in most other traditional methods (see, e.g., [67, 209–213] and the review in [66]). In particular, the Schottky parametrisation approach [212], the off-shell continuation framework [210], the sewing formalism [211], and the modular invariant spectral method [214]—each with its own advantages and limitations—generally decompose the string spectrum and discard the massive string contributions in one way or another. From the standpoint of deriving string theory induced regularisation, this presents a significant problem. Breaking modular invariance and truncating the infinite tower of massive modes erases the very string-theoretic data that, presumably, would be responsible for regularising UV divergences in field-theoretic integrals. As a result, these methods fall short in addressing key questions about the structure of UV completion in field theory as derived from string theory.

One particularly important example—as it relates to a more general discussion in the following section—the Schottky parametrisation method [212] is grounded in the representation of (super)string worldsheets via Schottky groups. A genus g

Riemann surface with boundaries is described in terms of projective transformations $S_i \in \mathrm{PSL}(2,\mathbb{R})$, each characterised by a pair of fixed points (ξ_i, η_i) and a complex multiplier $t_i \in (0,1)$. The index $i = 1, \ldots, g$ such that at one-loop there is only one t. The degeneration limit relevant for the field theory corresponds to the region $t_i \to 0$, where the handles of the surface collapse, giving rise to a corresponding Feynman graph topology. Crucially, the parametrisation of Schwinger proper time T_i in this formalism is given by

$$\log t_i = -\frac{T_i}{\alpha'},$$

so that the limit $\alpha' \to 0$ with fixed T_i corresponds to $t_i \to 0$. In this regime, the string integrand admits a power series expansion in t_i, with the leading term ($\mathcal{O}(1)$) encoding the propagation of massless modes and the higher powers suppressed by exponentials of the form $e^{-nT_i/\alpha'}$, corresponding to massive string excitations of mass $m_n^2 \sim n/\alpha'$ with n an integer. One can see quite clearly that, as $\alpha' \to 0$, the massive states are suppressed and decouple as common in most approaches.

From the perspective of these frameworks, this treatment of the tower of massive states and their residual effects appears natural. It can be shown, after all, that the correct field theory description is obtained. In more general settings, however, a deeper perspective on UV finiteness and the subtle role of massive string excitations can be observed in which the naive field theory limit no longer appears entirely satisfactory. For instance, in Zwiebach's formulation of string field theory (see [188], and also [64, 189] for review), using a geometric decomposition based on minimal area metrics and the geometric concept of *stubs*, it is demonstrated that UV finiteness can be attributed to the structure of the interaction vertices, each of which carries an exponential suppression factor. While only massless modes propagate along the internal lines of the Feynman-like diagrams, the "stubs" act as regulated truncations of the infinite sum over massive modes. These effects do not vanish but instead become localised on the vertices, appearing as *weighted insertions* and can be furthermore described in terms of exponentially dressed, or smoothly weighted, propagators. This is one example that provides insight into how string-theoretic corrections from the massive sector may directly influence field-theoretic amplitudes.

As noted in [66], however, Zwiebach's construction was not intended for practical implementation of the field theory limit $\alpha' \to 0$ as desired in this thesis. Instead,

the goal in taking a string field theory approach is much more aligned with the study of non-perturbative physics. While formal representations of string field theory amplitudes are given in terms of Feynman graphs, and although the $\alpha' \to 0$ limit might be useful for constructing field-theoretic amplitudes in a similar way that one would construct them from field theoretic Lagrangians [66], the overarching framework is designed with quite different objectives in mind.

In contrast, our interest here lies in deriving regularised field theory graphs directly from perturbative string amplitudes. The aim, in a sense, is to generalise the Bern–Kosower framework in a way that retains the contributions of the massive modes and gives their effects geometrical description. Achieving this requires defining the field theory limit such that the full modular sum over the infinite tower of string states is not prematurely decomposed, and the massive modes do not simply decouple. For reasons discussed in Section 10.4 this is a tall task.

As an intermediary first step, recently a unifying picture of the degeneration of worldsheet geometries to field theory diagrams has emerged through the mathematical development of tropical geometry [215]. Its application to string amplitudes was developed in the influential works of Tourkine [66] and others [216–218]. The tropical amplitudes programme offers a powerful set of tools for rigorously studying the correspondent map between string theory amplitudes and their field theory counterparts. Its key strength lies in enabling a controlled transition from extended string configurations to point-like limits, collapsing the worldsheet into tropical graphs whose edge lengths are parameterised by Schwinger proper time T. Crucially, in this process, the tropical framework allows one to neatly track the contributions of massive string modes. A remarkable result is that, in cases where the corresponding field theory amplitude exhibits a primary UV divergence, the massive modes are found to not naively decouple in the $\alpha' \to 0$ limit. Instead, similar to the picture offered by string field theory, they localise as vertex weights in the tropical graphs. These weights correspond to counter-terms, representing corrections arising from residual "stringy effects," which subtract the UV divergence and render the field theory amplitude finite.

Elegant in its formulation, standard tropicalisation relies on decomposing the moduli space $\mathcal{M}_{g,n}$ as a conceptual prerequisite. This allows for a detailed study of

worldsheet degeneration with a clear distinction between UV and IR contributions. However, the cost of this decomposition is the loss of UV/IR mixing inherent in the full modular integrand and spectral sum. Although much more sophisticated than other methods, this procedure still yields only an approximate shadow of the full modular structure of the string amplitude. While this is sufficient to demonstrate the rather astounding emergence of string theory induced renormalisation in the low-energy theory, the more ambitious goal of deriving string theory induced regulator functions requires significant generalisation of the tropical limit to something akin to a *global tropical limit*. As we will discuss, from the present viewpoint of tropical geometry, such a requirement is highly non-trivial, presenting significant technical and mathematical challenges.

10.2 Tropical amplitudes: a review

Tropical geometry is primarily motivated by the observation that there exists a deep analogy between classical Riemann surfaces and the geometry of metric graphs [219]. Furthermore, it can be viewed as a combinatorial tool to study degenerations of Riemann surfaces in algebraic geometry [215]. In this section we emphasise a physics point of view, and review the tropical amplitudes formalism strictly following the discussion by Tourkine [66] and references therein.

Recall from the last chapter that smooth Riemann surfaces of genus g with n punctures (i.e., external insertions) span a moduli space $\mathcal{M}_{g,n}$ up to conformal equivalence. The moduli space classifies all such complex structures up to isomorphism, with complex dimension $3g-3+n$. This space is not compact (i.e., not bounded and closed), since Riemann surfaces of different genus can degenerate. That is to say, the moduli space $\mathcal{M}_{g,n}$ admits boundary points corresponding to nodal curves formed when homotopy cycles—closed loops on the surface that cannot be continuously shrunk to a point—on the surface "pinch". Geometrically, as a Riemann surface degenerates, this corresponds to how certain geometric features "collapse"—the cycle shrinks to a point forming a nodal surface, which includes double points (i.e., two different points on the surface). This process gives rise to what are defined as nodal curves—a surface that is Riemannian everywhere but has singularities at the pinched points.

Physically, since nodal curves represent important boundary points of moduli space, they allow us to compactify $\mathcal{M}_{g,n}$ by adding these degeneration limits. This is known as Deligne-Mumford compactification producing $\bar{\mathcal{M}}_{g,n}$, which *is* a compact space. These nodal configurations are essential in string perturbation theory, since they correspond to physical processes like loop formation or string splitting. They also allow for a clean definition of integration over the moduli space, since without compactification, integrals might diverge or be ill-defined near degenerate regions. Most importantly, for our purposes, they provide a natural and rigorous language for mapping string theory to quantum field theory, in which degenerations lead to graphs resembling standard Feynman diagrams.

10.2.1 Degenerations

In tropical geometry, degenerations are generally classified in two types:

- separating degenerations, where the pinched cycle splits the surface into two disconnected components;
- or non-separating degenerations, where the pinched cycle results in a single surface with a new node, reducing the genus by one.

This process is illustrated in Figure 10.1:



Figure 10.1: An example of separating degeneration (a) and a non-separating degeneration (b) of a Riemann surface. Dashes represent double points. Borrowed from Tourkine [66].

These degenerations induce a stratification on $\bar{\mathcal{M}}_{g,n}$. This stratification arises by grouping curves according to their combinatorial degeneration types. Each stratum is defined by a dual graph, which captures the combinatorics of the degeneration:

- vertices represent components of the curve (with weights equal to their genus),
- edges represent nodes (components that are glued together),
- legs represent marked points (punctures).

To phrase it differently, the boundary of $\bar{\mathcal{M}}_{g,n}$ is stratified using dual graphs, with each component becoming a vertex labelled with its genus, each node corresponding to an edge, and each puncture becoming a leg attached to a vertex. This process is illustrated in Figure 10.2. The dual graph encodes both the the topology of the generated surface and the combinators of its boundary divisor. We will return to these important ideas pertaining to the decomposition of moduli space in a moment.

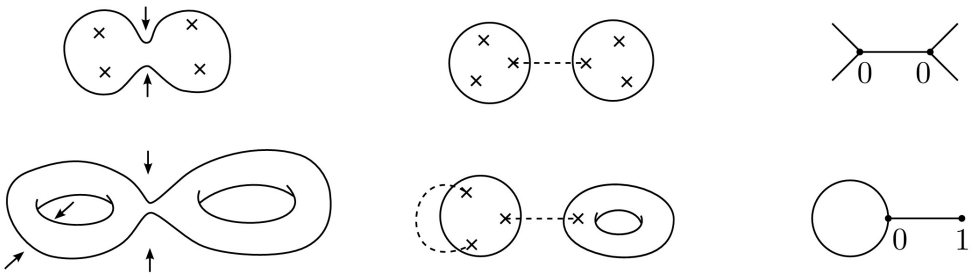


Figure 10.2: Degenerating Riemann surfaces (left), the associated nodal curves (center), and their dual graphs (right). These illustrate the connection between moduli space boundaries and Feynman-like diagrams. Borrowed from Tourkine [66].

There are different classifications of curves in the sense that, for curves that degenerate more, they belong to deeper strata. The most superficial stratum, which represents the top layer, is the interior of the moduli space. As we will see, in the context of string amplitudes, this is an important region representing the space of non-degenerate (smooth) curves.

10.2.2 General amplitude structure

Before discussing further mathematical technicalities, let us ground the discussion in physics. A generic closed string amplitude at genus g with n punctures is expressed as

$$\mathcal{A}^{(g,n)} = \int_{\mathcal{M}_{g,n}} d\mu_{\text{bos}}\, F_{g,n}, \tag{10.1}$$

where $F_{g,n} = \mathcal{W}_{g,n} \exp(Q_{g,n})$ with $\mathcal{W}_{g,n}$ containing kinematical and process specific data, and $\exp(Q_{g,n})$ the universal Koba-Nielsen factor

$$Q_{g,n} = \sum_{1\leq i<j\leq n} k_l k_j G(z_i, z_j).$$

The Green function $G(z_i, z_j)$ on the worldsheet is given by

$$G(z_i, z_j) = -\frac{1}{2}\log\left(|E(z_1, z_2)|^2\right) + \pi \mathrm{Im}\left(\int_{z_2} z_1 \omega_I\right)(\mathrm{Im}\Omega^{-1})^{IJ}\mathrm{Im}\left(\int_{z_2}^{z_1}\omega_J\right), \quad (10.2)$$

constructed using the prime form E and period matrix Ω. The integration in (10.1) is over the moduli space of Riemann surfaces $\mathcal{M}_{g,n}$, with $d\mu_{\mathrm{bos}}$ the standard modular invariant bosonic integration measure. As shown in [66], in the $\alpha' \to 0$ limit this integral localises onto $\mathcal{M}^{\mathrm{trop}}_{g,n}$, yielding the corresponding generic tropical amplitude

$$\lim_{\alpha'\to 0} A^{(g,n)}_{\alpha'} = \int_{\mathcal{M}^{\mathrm{trop}}_{g,n}} d\mu_{\mathrm{trop}}\, F^{\mathrm{trop}}_{g,n}. \quad (10.3)$$

In tropical geometry, the tropical measure is defined generally as

$$d\mu_{\mathrm{trop}} = (2\pi)^{d/2-|E(G)|}\prod_{i\in E(G)} d\ell_i \,\frac{1}{(\det K)^{d/2}}, \quad (10.4)$$

where K is the tropical period matrix built from edge lengths ℓ_i. Importantly, one can show the derivation of the tropical analogue of the Green function on the worldsheet, which, as $\alpha' \to 0$, becomes the worldline Green function

$$G^{\mathrm{trop}}(Z_i, Z_j) = -\frac{1}{2}E^{\mathrm{trop}}(Z_i, Z_j) - \frac{1}{2}\left(\int_{Z_i}^{Z_j}\omega^{\mathrm{trop}}_I\right)(K^{-1})_{IJ}\left(\int_{Z_i}^{Z_j}\omega^{\mathrm{trop}}_J\right), \quad (10.5)$$

with the tropical prime form E^{trop} identified as the distance along a graph. This formalism neatly recovers Feynman integrals in the tropical limit. The integrand's structure,

$$F^{\mathrm{trop}}_{g,n} = \mathcal{W}_{g,n}\,\exp\left(\sum_{i<j} k_i\cdot k_j\, G^{\mathrm{trop}}(Z_i, Z_j)\right), \quad (10.6)$$

exhibits a clear relation to the first and second Symanzik polynomials in quantum field theory. The technical achievement of this framework follows from how one can define the correspondence mapping the general structure of string theory amplitudes to field theory amplitudes

$$\mathcal{A}^{\text{string}}_{g,n}(k_1,\dots,k_n) = \int_{\mathcal{M}_{g,n}} [d\mu]^{3g-3+n}_{\mathbb{C}}\, \mathcal{W}_{g,n}\, e^{\mathcal{Q}_{g,n}}$$

$$\xrightarrow{\alpha'\to 0} \int_{\mathcal{M}^{\text{trop}}_{g,n}} [d\mu]^{3g-3+n}_{\text{Trop}}\, \mathcal{W}_{g,n}\, e^{\mathcal{Q}_{g,n}} = \mathcal{A}^{\text{Field theory}}_{g,n}(k_1,\dots,k_n).$$

As string amplitudes can be rigorously observed to descend to a tropical version, in general one obtains a generic structure of the integrand in terms of first and second Symanzik polynomials. To give a concrete example, we will review the known tropicalisation of the genus one, four-graviton Type II amplitude in Section 10.2.5.

10.2.3 Decomposition of moduli space

The extraction of the field theory limit from closed string amplitudes relies crucially on a systematic decomposition of the moduli space of Riemann surfaces $\mathcal{M}_{g,n}$. This decomposition isolates the divergent regions of moduli space responsible for Feynman diagram singularities and enables a piecewise tropicalisation, mapping each region to a corresponding domain in the tropical moduli space $\mathcal{M}^{\text{trop}}_{g,n}$.

The decomposition of the moduli space is conceptually essential, as it is fundamentally based on the behaviour of the integrand near various degenerating loci of the Riemann surfaces. Since string amplitudes contain singularities that arise at degeneration loci, tropicalisation entails: 1) the identification of the regions of moduli space associated with particular degenerations, 2) introduction of local coordinates around these degenerations, and 3) scaling of these coordinates to extract the tropical limit.

In general, the strategy to decompose the integration domain of a string amplitude into various regions or cells follows the so-called Kontsevich-Penner (KP) decomposition for decorated Teichmüller spaces of punctured surfaces [220, 221]. The idea [222] is to divide $\mathcal{M}_{g,n}$ into a disjoint union of cells. In the centre of this disjoint union resides a nodal curve of $\overline{\mathcal{M}_{g,n}}$ along with a dual graph labelling the cell. This means that one obtains disjoint domains as given by

$$\mathcal{M}_{g,n} = \bigsqcup_{G} D_G \sqcup D_0, \tag{10.7}$$

with $\bigsqcup$ denoting a disjoint union, and with each domain D_G centred on a region where the geometry of the surface is approximated by a stable nodal curve with dual graph G (i.e., a combinatorial type of degeneration). In each cell or domain D_G, one can define local coordinates t on $\mathcal{M}_{g,n}$ to parametrise the degenerating regions of the given Riemann surface. These degenerative domains are not uniquely defined; their boundaries can be shifted provided they do not absorb neighbouring singularities. On the other hand, D_0 is a compact analytic domain that captures the non-degenerate, finite-region contribution. Thus, intuitively, domains D_G correspond to field theory Feynman diagrams with finite proper times, while D_0 encodes ultraviolet counter-terms through its contraction to vertices of nonzero weight.

In practical terms, this decomposition amounts to the ability to split the moduli space $\mathcal{M}_{g,n}$ into distinct regions: degenerative, non-analytic regions and non-degenerative, analytic regions.

In the degenerative region, in the neighbourhood of each domain $\mathcal{D}_G$, the surface develops a number of narrow necks or long tubes (one for each internal edge of the dual graph). These are locally modelled by the equation:

$$xy = t$$

with $t \in \mathbb{C}$, $|t| < 1$, a complex deformation parameter. As $t \to 0$, the neck becomes infinitely thin and a node forms. The key idea is that near the boundaries of $\mathcal{M}_{g,n}$, where the Riemann surface degenerates, one can use local plumbing coordinates t_j to describe the long tubes or narrow necks forming in the surface, with $j = 1, \ldots, N$ labelling the labelling the set of local coordinates. These parameters relate to edge lengths T_j of tropical graphs via the scaling

$$T_j = -\alpha' \log |t_j|, \tag{10.8}$$

so that fixed Schwinger proper times correspond to

$$t_j = \exp(i(2\pi\phi + iT_j/\alpha')), \quad |t_j| \to 0, \quad \phi \in [0, 2\pi[\tag{10.9}$$

as $\alpha' \to 0$. This defines a family of degenerating Riemann surfaces $\Sigma_{\alpha'}$, whose

geometry converges to a tropical graph Γ with edge lengths $\ell_j = T_j$.

The analytic domain is compact, defined by removing all neighbourhoods around any singularities.[2] Thus, inside the analytic domain, we shall treat the τ_1 integration assuming no singularities such that the $\alpha' \to 0$ limit can be taken *inside* the integral. Typically, the Symanzik polynomial $\exp(\mathcal{Q}_{g,n})$ contains a factor of α', and in the tropical limit $\exp(\mathcal{Q}_{g,n}) \to 1$ in such a way that a dual graph of the analytic domain is a single vertex of weight g. These are the graphs precisely responsible for the presence of counter-terms to UV divergences in the corresponding field theory.

The tropical limit involves identifying a loop of finite proper time length T. To achieve this, one defines the tropical scaling as $T = 2\pi\alpha'\tau_2$ so that

$$q = e^{-2\pi\tau_2} \quad \longrightarrow \quad q = e^{-T/\alpha'}. \tag{10.10}$$

The scaling (10.10) effectively tropicalises the moduli space by converting complex degenerations into piecewise-linear graphs with finite edge lengths. In other words, as $\alpha' \to 0$, we scale the imaginary part of the modulus $\tau_2 \to \infty$ proportionally as T/α'. Clearly, this gives a finite size of the tropical loop (i.e., the field theory loop) as the worldsheet degenerates

$$\text{finite loop length } T \quad \Longleftrightarrow \quad \tau_2 = \frac{T}{\alpha'}. \tag{10.11}$$

10.2.4 Emergence of counter-terms

A key feature of the tropical approach to string amplitudes is its ability to not only reproduce the integrand structure of perturbative field theory, but also to encode renormalisation data—in the form of counter-terms and contact interactions—directly from the geometry of degenerating Riemann surfaces. This is quite an astounding success, with string theory found to induce a renormalisation for UV divergent field theory integrals. Furthermore, this higher renormalisation manifests as a precise

[2]This nomenclature is slightly deceiving, since this region can also contain non-analytic terms; however, in Tourkine's approach, one defines the analytic region in such a way that ensures it is compact. Furthermore, since the decomposition into domains D_G is not necessarily unique; it is important to note that it is defined up to how the singularities are partitioned. Nonetheless, any such partition provides a well-defined tropical limit as shown in [66].

structure in the tropical moduli space with precise geometric consequences.

As previously mentioned, contributions from compact regions of moduli space—i.e., the analytic region far from degenerations—are responsible for the emergence of this renormalisation structure, corresponding to the production of Schwinger worldline graphs that have vertices of nonzero weight. These weighted vertices do not arise from the direct degeneration of cycles (which yield finite-length edges), but instead from degenerations where full subregions of moduli space are contracted to a point. This situation typically occurs when:

- the moduli corresponding to a loop in a Riemann surface are completely integrated out (e.g., $\tau_2 < \epsilon$ for a torus, with ϵ a cut-off to be discussed in the next section);
- or when the Riemann surface degenerates to a configuration with collapsed handles.

Such configurations are mapped to tropical graphs where a vertex of weight w represents a localised contribution of genus w. Physically, these correspond to UV counter-terms: string-theoretic corrections that cancel divergences arising in the field theory limit.

Tourkine [66] formalises this structure in the central conjecture of his paper, which serves as a field theory renormalisation theorem:

Conjecture: The $\alpha' \to 0$ limit of a closed string amplitude reduces to an integral over the tropical moduli space

$$\int_{\mathcal{M}_{g,n}} d\mu_{\text{bos}} F_{g,n} = \int_{\mathcal{M}_{g,n}^{\text{trop}}} d\mu_{\text{trop}} F_{g,n}^{\text{trop}} + \mathcal{O}(\alpha'),$$

where the integral on the right is over all combinatorially distinct tropical graphs with genus g and n legs, and the tropical integrand includes both:

- the field-theoretic propagating contributions from pure trivalent graphs,
- and the string-induced local counter-terms, encoded by weighted tropical vertices.

This result implies that string theory defines a canonical renormalisation scheme for field theory: all divergent graphs in the tropical representation are regularised by finite (and calculable) contributions from compact regions of moduli space. No additional counter-terms or subtraction procedures are required beyond what string

theory prescribes. In practice, this correspondence requires:

- a decomposition of the moduli space such that each region maps cleanly to a class of field theory graphs;
- a renormalised field theory amplitude expressed entirely in tropical (worldline) variables

$$A^{(g,n)}_{\text{trop}} = \int_{\mathcal{M}^{\text{trop}}_{g,n}} d\mu_{\text{trop}} F^{\text{trop}}_{g,n},$$

  with counter-terms geometrically embedded.

This integral structure gives interesting geometric foundation to field theory renormalisation and supports the broader idea that string theory resums and regularises quantum field theory divergences in a natural, background-independent way.

10.2.5 Canonical renormalisation example: genus one, four-graviton Type II superstring amplitude

The canonical example of string theory induced renormalisation given by Tourkine [66], comes from examining the field theory limit of the one-loop four-graviton amplitude in type II superstring theory. In this setting, a genus-one Riemann surface is a complex torus $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$, parameterised by a complex structure modulus τ in the Siegel upper half-plane

$$\mathcal{H}_1 = \{\tau \in \mathbb{C} \mid \text{Im}(\tau) > 0\}.$$

The integration domain for τ is typically restricted to an $\text{SL}(2,\mathbb{Z})$ fundamental domain. For a four-point amplitude, the moduli space $\mathcal{M}_{1,4}$ includes τ and three puncture positions $\zeta_i \in \mathbb{T}$, with the fourth fixed by modular invariance. Upon decomposing the moduli space, a cut-off ϵ is implemented with $\tau_2 \geq \epsilon$ representing the non-analytic region containing the degeneration, and $\tau_2 < \epsilon$ the compact analytic region.

The one-loop four-graviton amplitude in type II superstring theory in 10D is:

$$\mathcal{A}^{(1,4)}_{\alpha'} = \int_{\mathcal{M}_{1,4}} d\mu_{\text{bos}} F_{1,4}, \tag{10.12}$$

where the integrand $F_{1,4}$ is given by:

$$F_{1,4} = (2\pi)^8 R^4 \exp\left(\alpha' \sum_{i<j} k_i \cdot k_j \, G(\zeta_i, \zeta_j)\right). \tag{10.13}$$

Here, R^4 encodes the kinematic tensor $t_8 t_8 R^4$, and $G(\zeta_i, \zeta_j)$ is the genus-one Green function on the torus.

Supersymmetry ensures that the integrand has no singularities from vertex operator collisions, eliminating diagrams with external trees attached to the loop. Thus, the amplitude is dominated by configurations where punctures remain well-separated. The tropical limit arises when $\alpha' \to 0$, converting worldsheet moduli space integrals into integrals over tropical graphs (combinatorial one-loop graphs). The relevant propagator limit becomes

$$\alpha' G(\zeta_i^{\alpha'}, \zeta_j^{\alpha'}) \to G_{\text{trop}}(X_i, X_j) = \frac{1}{2}\left(-|X_i - X_j| + \frac{(X_i - X_j)^2}{T}\right), \tag{10.14}$$

which corresponds to the Schwinger worldline propagator on a circle of length T, the tropical modulus replacing τ_2. Substituting this into the integrand yields

$$F_{1,4} \to (2\pi)^8 R^4 \exp\left(\sum_{i<j} k_i \cdot k_j \, G_{\text{trop}}(X_i, X_j)\right), \tag{10.15}$$

and the measure transforms as

$$d\mu_{\text{bos}} \to d\mu_{\text{trop}} = 2\pi\alpha' \frac{dT}{T^5} \prod_{i=1}^{3} dX_i. \tag{10.16}$$

Hence, the full one-loop amplitude becomes

$$\mathcal{A}^{(1,4)} \to \int dT \, \frac{1}{T^5} \int \prod_{i=1}^{3} dX_i \, \exp\left(\sum_{i<j} k_i \cdot k_j \, G_{\text{trop}}(X_i, X_j)\right), \tag{10.17}$$

which matches the standard field-theoretic Schwinger parameter representation of the one-loop four-graviton amplitude.

From (10.16), Tourkine shows the integration domain over proper times and vertex

insertions is specified for each channel, such as

$$D_{(ijk)} = \left\{ T \in [\alpha'\epsilon, \infty), X_i, X_j, X_k \in [0,T], 0 < X_i < X_j < X_k < T \right\}, \tag{10.18}$$

with this domain capturing the proper ordering of insertions along the worldline circle. The key observation follows from how the tropical exponent is written using the second Symanzik polynomial $\mathcal{Q}_{1,4}$, constructed from the pairwise differences $(X_i - X_j)$. The first Symanzik polynomial is simply T. Collecting all permutations (i.e., channels), the leading term of the tropical one-loop amplitude becomes

$$\mathcal{A}^{(1,4)}_{+,\alpha'}(L) = \sum_{\sigma} \int_{D_\sigma} d\mu_{\text{trop}} F_{1,4}, \tag{10.19}$$

which explicitly evaluates to

$$\begin{aligned} \mathcal{A}^{(1,4)}_{+,\alpha'}(\epsilon) = \alpha'(2\pi)^9 R^4 \Bigg(& \int_{2\pi\alpha'\epsilon}^{\infty} \frac{dT}{T^2} \int_0^T \frac{dX_1}{T} \int_0^{X_1} \frac{dX_3}{T} \int_0^{X_3} \frac{dX_2}{T} e^{\mathcal{Q}_{1,4}} \\ & + 2 \text{ permutations} \Bigg). \end{aligned} \tag{10.20}$$

The UV divergence arises from the region where the Schwinger parameter T goes to zero, corresponding to the worldline loop becoming short. This is cut off at $T > 2\pi\alpha'\epsilon$, with ϵ dimensionless. In 10 dimensions, the leading UV divergence is therefore easily seen to take the form

$$\mathcal{A}^{(1,4)}_{+,\alpha',\text{leading div.}} = \alpha'(2\pi)^9 R^4 \left(\frac{1}{2\pi\alpha'\epsilon} \right), \tag{10.21}$$

originating from the deepest (maximally non-analytic) stratum of moduli space. It corresponds to a tropical graph with a loop of vanishing length $T \to 0$.

Crucially, the contribution from the analytic domain, where no UV divergence exists, can be evaluated by setting $\alpha' \to 0$ inside the integral

$$\mathcal{A}^{(1,4)}_{\alpha'}(\epsilon) \to \mathcal{A}^{(1,4)}(\epsilon) = (2\pi)^8 R^4 \int_{\mathcal{F}_\epsilon} \frac{d^2\tau}{(\tau_2)^2} + \mathcal{O}(\alpha'). \tag{10.22}$$

Explicit integration gives

$$\mathcal{A}^{(1,4)}(L) = (2\pi)^9 R^4 \left(\frac{1}{6} - \frac{1}{2\pi\epsilon}\right) + \mathcal{O}(\alpha'), \tag{10.23}$$

with $\frac{1}{6}$ described as the analytic term, and $-\frac{1}{2\pi\epsilon}$ the counter-term needed to cancel the UV divergence in the field theory integral. Upon summing over the contributions from the decomposed amplitude, this incredible result shows how two distinct contributions render the field theory UV finite.

Tropicalisation can therefore be seen to amount to a loop with finite length T, corresponding to standard Feynman graphs, and a pinched graph at $T = 0$ corresponding to a single-vertex tropical graph of weight one, representing a local counter-term. The total field theory amplitude is thus expressed as an integral over the full tropical moduli space $\mathcal{M}_{1,4}^{\text{trop}}$, and the UV divergence at $T \to 0$ is cancelled by adding this local graph.

10.3 Genus one torus vacuum amplitude

An interesting example not found in the literature[3] is the tropicalisation of the torus vacuum amplitude for the non-supersymmetric, non-tachyonic superstring [69, 214, 223]. Since there exists a large class of non-supersymmetric superstring theories that are tachyon-free—i.e., do not exhibit any vacuum instabilities in the tropical, or, IR region [69, 223]—they serve as a very good setting to study tropicalisation since the behaviour of the string spectrum is exposed in the limit $\alpha' \to 0$. This is also a setting of interest for discussion in the following section on string theory induced regulators.

The general genus one torus vacuum amplitude can be written as

$$\mathcal{A}_{\text{Torus}} = \int_{\mathcal{F}} d\mu \, \mathcal{Z}_{\text{String}} \tag{10.24}$$

with

$$\mathcal{Z}_{\text{string}}(\tau, \bar{\tau}) = \text{Str}_{\mathcal{H}\times\bar{\mathcal{H}}} q^{L_0 - E_0} \bar{q}^{\bar{L}_0 - \bar{E}_0}. \tag{10.25}$$

Since there are no operator insertions, the amplitude is not represented in terms of the Green function. Instead, the integrand is given by the full modular invariant partition function. The moduli space $\mathcal{M}_{1,0}$ is simply represented by the fundamental

[3] To the best knowledge of the author at the time of writing

domain $\mathcal{F}$ for the modular group $\mathrm{SL}(2,\mathbb{Z})$. Here $d\mu$ is the modular invariant measure $d^2\tau/\tau_2^2$. The supertrace is defined over the full stringy Hilbert space $\mathcal{H}\times\bar{\mathcal{H}}$, with L_0 and $\bar{L}_0$ the Virasoro operators, and E_0 and $\bar{E}_0$ represent vacuum energy [69, 214, 223].

The full one-loop torus partition function (10.25) is an automorphic function of moderate decay [214] at the cusp, where the torus degenerates. To study the tropicalisation of this amplitude, we want to massage the integrand into a more suggestive form. Given $q = e^{2\pi i\tau}$, $\bar{q} = e^{-2\pi i\bar{\tau}}$ encoding the worldsheet modulus τ, with $\tau = \tau_1 + i\tau_2$ and $\bar{\tau} = \tau_1 - i\tau_2$, we can recast the partition function as

$$\mathcal{Z}_{\text{string}} = \mathrm{Str}_{\mathcal{H}\times\bar{\mathcal{H}}}\left(e^{2\pi i(L_0-\bar{L}_0)\tau_1}e^{-2\pi[(L_0-E_0)+(\bar{L}_0-\bar{E}_0)]\tau_2}\right), \tag{10.26}$$

highlighting the phases and moduli. The τ_1 exponent is purely a phase, encoding the left- and right-moving modes. Furthermore, integration over τ_1 enforces the level-matching condition $L_0 - \bar{L}_0 = 0$ by projecting onto terms with net zero phase. This means that only diagonal terms survive in physical amplitudes. As will be seen later, the τ_2 integration corresponds to the mass weighting. The vacuum energy E_0 is chosen (or projected) such that $M^2 \geq 0$, ensuring the theory is free of tachyons.

For simplicity, let $m = L_0 - E_0$ and $n = \bar{L}_0 - \bar{E}_0$. For the τ_1 phase-dependent exponent, note $L_0 - \bar{L}_0 = (m + E_0) - (n + \bar{E}_0) = m - n + (E_0 - \bar{E}_0)$, and if $E_0 = \bar{E}_0$ as is usually the case in critical string theory, then the τ_1 exponent simplifies to $L_0 - \bar{L}_0 = m - n$. Writing the supertrace as a double sum over non-negative integer valued excitation levels, the partition function can be recast as

$$\mathcal{Z}_{\text{string}} = \tau_2^{1-d/2}\sum_{m,n} a_{m,n}e^{-2\pi(m+n)\tau_2}e^{2\pi i(m-n)\tau_1}, \tag{10.27}$$

where the prefactor $\tau_2^{1-d/2}$ arises from the zero-mode momentum integration and the path integral. From the supertrace $\mathrm{Str}_{\mathcal{H}\times\bar{\mathcal{H}}}$ we denote $a_{m,n}$ as the weight that counts the net number of states at level m, n, with $a_{m,n} = d_B(m) - d_F(n)$. If the supertrace is exact, then $d_B(m) = d_F(n)$ such that $a_{m,n} = 0$, since with unbroken supersymmetry there is exact boson-fermion cancellation at each level. In studying the tropical limit, this case of exact supersymmetry is uninteresting. If supersymmetry is broken, however, then there is an imbalance, and the tropical limit becomes meaningful. (Note the

identity if all states are at level n such that $a_{nn} = (-1)^F g_n$ with $g_n = d_B(n) - d_F(n)$). The full amplitude can thus be written as

$$\mathcal{A}_{\text{Torus}} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \tau_2^{1-d/2} \sum_{m,n} a_{m,n} e^{-2\pi(m+n)\tau_2} e^{2\pi i(m-n)\tau_1} \tag{10.28}$$

representing a class of vacuum amplitudes that includes all unitary non-critical strings, critical Type II strings, as well as the phenomenologically interesting case of $d > 2$ critical heterotic strings [69, 223].

The insightful detail is in the asymptotics and the complementary limits where $\alpha' \to 0$ and $\tau_2 \to \infty$ in the tropical scaling $T = 2\pi\alpha'\tau_2$. In (10.28), it is clear that the τ_2 exponential controls the asymptotic behaviour. In the region of moduli space where $\tau_2 \sim 1/\alpha'$, holding $T = 2\pi\alpha'\tau_2$ fixed as $\alpha' \to 0$, corresponds to the geometric limit where the torus degenerates into a long, thin cylinder. When $m = n = 0$, this region is where the massless states dominate, as the τ_2 exponential becomes suppressed and the τ_1 integration trivially integrates to one. This just leaves

$$\mathcal{Z}^{\text{IR}}_{\text{string}} \sim \tau_2^{1-d/2} \sum_{m=n=0} a_{0,0}, \tag{10.29}$$

for $a_{0,0} = d_B(0) - d_F(0)$. Here, $a_{0,0}$ is the net number of massless bosonic minus fermionic degrees of freedom. As mentioned, if the theory is supersymmetric, this vanishes; otherwise, $a_{0,0}$ is a constant and contributes to the dominant IR behaviour. Importantly, it can be seen that only the massless modes propagate. This is the region where the string amplitude descends to a field theory integral with a worldline loop of length $T = \alpha'\tau_2$. Furthermore, in this region of moduli space with $m, n \geq 1$, asymptotically the dominant contribution comes from the smallest values of m and n. This means that, in the sum over the infinite tower of string states, the smallest eigenvalues dominate while there is natural exponential damping of other terms for increasingly large values of m and n.

Now comes the subtle point. In contrast, in the region of moduli space where $\tau_2 \sim \mathcal{O}(1)$ is fixed, the exponential suppression of massive string modes is no longer effective. Since typical massive string excitations generically satisfy $M^2 \sim n/\alpha'$ with the integer level excitation $n \geq 1$, the exponential $e^{-\pi\alpha' M^2 \tau_2} \sim e^{-\pi n \tau_2}$. Thus, for

$\tau_2 \sim \mathcal{O}(1)$, even the massive states are not strongly suppressed. Since the massive states do not decouple, the full tower contributes. This is the region crucial for UV consistency and is protected by modular invariance.

To inspect this behaviour in the full tropicalisation of the torus vacuum amplitude, the moduli space is decomposed similar to the previous example in Section 10.2.5. The fundamental domain $\mathcal{F}$ is split into two distinct regions

$$\mathcal{F} = \mathcal{F}_{\text{non-analytic}} \cup \mathcal{F}_{\text{analytic}} \tag{10.30}$$

with the non-analytic region $\mathcal{F}_{\text{non-analytic}} = \{\tau \in \mathcal{F} | \tau_2 > \epsilon\}$ containing the cusp at $\tau_2 \to \infty$ where $q \to 0$ and the torus degenerates.

To define the split we once again impose a hard cut-off ϵ (since $\mathcal{F}$ is curved, one can think of this as a horizontal cut-off), acting as the lower bound of the integration for the degenerative region such that $\tau_2 > \epsilon$.[4]. A common strategy is to keep ϵ fixed while $\alpha' \to 0$ in order to probe the degenerating boundary. Thus, for any $T \geq 2\pi\alpha'\epsilon$ we now have the standard family of complex tori in $\mathcal{F}_{\text{non-analytic}}(\epsilon)$ [66]. The analytic region $\mathcal{F} \setminus \mathcal{F}_{\text{non-analytic}}$ with $\mathcal{F}_{\text{analytic}} = \{\tau \in \mathcal{F}, \tau_2 \leq \epsilon\}$ is once again compact and represents the interior of the moduli space far from degeneration.

Thus, explicitly, the original torus vacuum amplitude can be written as

$$\mathcal{A}_{\text{Torus}}(\epsilon) = \mathcal{A}_{\text{non-analytic}}(\epsilon) + \mathcal{A}_{\text{analytic}}(\epsilon), \tag{10.31}$$

where

$$\mathcal{A}_{\text{non-analytic}} = \int_{\tau_2 \geq \epsilon} \frac{d^2\tau}{\tau_2^2} \tau_2^{1-d/2} \sum_{m,n} a_{m,n} e^{-2\pi(m+n)\tau_2} e^{2\pi i(m-n)\tau_1} \tag{10.32}$$

and

$$\mathcal{A}_{\text{analytic}} = \int_{\tau_2 < \epsilon} \frac{d^2\tau}{\tau_2^2} \tau_2^{1-d/2} \sum_{m,n} a_{m,n} e^{-2\pi(m+n)\tau_2} e^{2\pi i(m-n)\tau_1}. \tag{10.33}$$

We first consider the integral over the non-analytic region. Here the massless modes

[4]We note, as related to more general discussion in the following section, it is possible to use other cut-offs, such a smooth partition of unity. However, further investigation into possible generalisations along these lines extends beyond the remit of this work

dominate. The τ_1 integration is trivial and imposes level-matching $m = n$, giving a delta function constraint equal to one. In terms of the string mass, this means $M^2 = \frac{4}{\alpha'}m = \frac{4}{\alpha'}n$. The sum becomes $\sum_n a_{n,n} e^{-2\pi n \tau_2}$, with each level n corresponding to a string state with mass level squared such that $\frac{\alpha'}{4}M^2 = n \implies M^2 = \frac{4n}{\alpha'}$. Thus, the amplitude can be written as

$$\mathcal{A}_{\text{non-analytic}} = \int_{\tau_2 \geq \epsilon}^{\infty} \frac{d^2\tau}{\tau_2^2}\, \tau_2^{1-d/2} \sum_{M^2} a(M^2)\, e^{-\pi\tau_2\alpha' M^2}, \tag{10.34}$$

where the degeneracy $a(M^2)$ is now indexed by the mass level. Making a change of variables to introduce the tropical scaling $T = 2\pi\alpha'\tau_2$, it is easily found that in the limit $\alpha' \to 0$ the integral takes the form

$$\mathcal{A}_{\text{non-analytic}} = a(0)(2\pi\alpha')^{d/2} \int_{2\pi\alpha'\epsilon}^{\infty} dT\, T^{-1-d/2}. \tag{10.35}$$

As anticipated, this is just a Schwinger worldline vacuum loop integral over a single loop of length T. Explicitly, it is just a one-loop vacuum bubble diagram for a massless scalar particle in d dimensions, which is clearly UV divergent when $d \geq 2$ and $a_0 \neq 0$. For all $d > 2$ this integral evaluates to

$$\mathcal{A}_{\text{non-analytic}}\,|_{\text{leading div.}} = a(0)(2\pi\alpha'\epsilon)^{-d/2}\frac{1}{d/2}, \tag{10.36}$$

in which we find the leading power-behaved divergence in ϵ. For $d = 2$, the integral instead becomes logarithmically divergent. We therefore see how massless field theory emerges directly from the sum over all string states, giving rise to an effective tropical (field-theoretic) vacuum diagram.

Next, we consider the analytic region integration where τ_2 is $\mathcal{O}(1)$. As mentioned, the integration region is compact; however, unlike other cases for genus one amplitudes, we cannot immediately assume the integrand is analytic due to the presence of the sum over the infinite tower of string states. The main subtlety therefore is that we have analyticity of the region but not necessarily of the integrand, due to the degeneracy of the sum. Furthermore, as the sum over the infinite tower of massive states can exhibit Hagedorn growth (exponential growth of states at large mass), the degeneracy

factor asymptotically behaves like $a(M^2) \sim e^{C\sqrt{\alpha'}M}$ with C a positive constant [224]. It is therefore not immediately obvious that one can take the $\alpha' \to 0$ limit inside the integral, since one must ensure the sum converges.

Thankfully, for our present purposes, it is enough to note that since we assume there are no physical and unphysical tachyonic states, the fundamental domain cuts off the potential divergence in the degeneracy of states [225].[5] Assuming the appropriate rate of convergence and the satisfaction of Fubini-Tonelli theorem, τ_1 integration once again becomes trivial and simply gives one. We can take the $\alpha' \to 0$ limit inside the integral. For fixed M^2, the integration gives

$$Z_{\text{analytic}} = a(M^2) \cdot \frac{1}{d/2} \left(\left(\frac{\sqrt{3}}{2} \right)^{-d/2} - \epsilon^{-d/2} \right) + \mathcal{O}(\alpha'), \tag{10.37}$$

where we observe at leading order in α' the expected renormalisation structure in the $-1/\epsilon^{d/2}$ counter-term. The other term $(\sqrt{3}/2)^{-d/2}$ represents the analytic part of the amplitude. The appearance of $-1/\epsilon^{d/2}$ suggests the presence of a worldline graph with a vertex of nonzero weight [66], consistent with observations in the tropical formalism for one-loop amplitudes. If, instead, we do not fix M^2 and treat the full sum over the infinite tower of string states—for instance, by transforming to the Mellin representation and analysing the poles—it is possible to also study subleading corrections in α'. But in the full tower of states, with α' fixed, there is non-trivial contribution by the massive states, and the leading power-law divergence in ϵ no longer matches that from the non-analytic region. This observation raises a number of interesting questions, and serves as motivation for more detailed future study.

10.4 String theory induced regulators

The possibility that string theory might induce a novel form of renormalisation is deeply motivated by physical considerations, as explored in Parts I and II of this thesis. From a bottom-up perspective, however, the idea of deriving regulator functions directly from string theory might appear overly optimistic. Unlike string amplitudes, conventional

[5] Otherwise we would just need to formally treat the sum using a smooth cut-off or zeta-function regularisation.

quantum field theory amplitudes—expressed through Schwinger parameter integrals—lack an intrinsic modular structure. In this framework, short proper times in the Schwinger parametrisation correspond to high-energy modes and yield divergences that require explicit regularisation, often in an *ad hoc* way. This contrast highlights the conceptual gap between standard field theory and the modular data intrinsic to quantum gravity.

Yet it is precisely this contrast that makes the emergence of string theory induced renormalisation in the tropical formulation so remarkable. Far from being a mere technical curiosity, it offers a concrete example of how quantum gravitational effects might render low-energy theories UV finite. Moreover, the combinatorial structure of tropical degenerations appears to provide a new lens through which to reinterpret the regularising role of the string worldsheet. These insights support the tantalising possibility that a class of regulator functions—descended from string theory—could systematise UV finiteness in field-theoretic integrals, potentially making contact with a generalised theory of regularisation via the η framework. We conclude this chapter with a number of remarks exploring this idea and report on its many ongoing technical challenges.

10.4.1 The geometry of modular shadows

As a topic of ongoing research, it is interesting that, from a top-down viewpoint, there appears to be an emerging sense that the lack of modular structure in field theory amplitudes might be interpreted analytically and geometrically in how low-energy theories exist in a regime stripped of modular data [213, 226–232]. Indeed, in considering some of the key research questions and motivations posed in the introduction of this thesis, a very natural idea [151–153] in effort to connect the regularising effects of string theory with the bottom-up approach of η regularisation, is simply to imagine η style regulators that are modular invariant! Defining by hand any Wilsonian η regulator satisfying the usual properties of a Schwartz-like function, except now defined as a modular function $\eta(\Lambda, \tau, \bar{\tau})$, would seem like an obvious place to start. However, the issue is in deriving such multiplicative regulator functions directly from string theory, as opposed to defining them by hand.

Working within the tropical amplitudes framework, one explanation about the

absence of modularity in the low-energy theory may be viewed along the following lines: namely how, as the worldsheet geometry degenerates, it in a sense becomes flattened, corresponding with the geometry of field theory graphs. This flattened, modularly incomplete structure of tropical graphs of course captures only the massless portion of the string spectrum [66, 204, 233, 234], and, it could be argued, is absent of other important structural data. This viewpoint is akin to the modern view that such purely field theory amplitudes derived from string theory appears as a "shadow" of the full modular object of the string integrand [227, 228].

Moreover, from the mathematical study of modular forms, this notion of a *modular shadow* [235]—a non-modular function that can be "completed" to restore modularity—closely aligns with modern ideas concerning the modular completion of field theory amplitudes [151, 152, 236]. In the context of Zwegers' work on mock modular forms [39, 237], a shadow refers to the incomplete or non-holomorphic part of a modular object—essentially an absence of modular data—with modularity restored through the addition of a compensating term. Analogously, as discussed in [235] and references therein, what if the field theory integrand is simply a modular shadow—an object that has lost modular data? In contrast to the tropical approach, the modular integral approach to studying the field theory limit, which utilises the Rankin-Selberg-Zagier (RSZ) method, offers several clues evidencing this idea [213, 227–232].

A particularly instructive example in recent years arises from the spectral analysis of the string [214, 238]. Using the Rankin–Selberg–Zagier method [200, 214], one can precisely track how the modular structure of the amplitude decomposes—thereby making the modular shadow concept manifest.[6] Indeed, using the RSZ method in the spectral analysis of the genus one torus amplitude (10.28), one sees quite explicitly

[6]A helpful analogy is to view the modular structure of the one-loop string amplitude—such as the modular-invariant partition function—as a three-dimensional sculpture. The field theory integrand then corresponds to the shadow this sculpture casts when taking the limit $\alpha' \to 0$. The shadow is a real but flattened projection of the full object: incomplete, lacking modular symmetry, and missing essential data, unless one knows how to reconstruct the original form. Thus, in the string-theoretic view of emergent field theory amplitudes, the modular shadow represents a partial, IR-limited projection of the full modular structure. Recovering the complete data requires either (1) a modular completion that restores the symmetry, or (2) a reformulation of the low-energy theory that retains information about the massive modes.

how the partition function decomposes to take the form [214]

$$a_0(\tau_2) \sim \frac{3}{\pi}\Omega_{\text{Torus}} + d(0)\Psi(1-\frac{d}{2})\tau_2^{d/2} + \sum_{\zeta^*(\rho)=0} C_\rho \tau_2^{1-\rho/2}, \tag{10.38}$$

where, in the asymptotic tail of the torus partition function, the massless and low-lying part of the spectrum is given by the first term. UV corrections and oscillatory terms—essentially what are residual corrections, which can also be studied by analysing the poles and the residues of the full modular integrand—are representative of the contributions from massive fluctuations, captured by the last two terms. Although they are effectively decoupled (via additive decomposition), we see that they are also *always present*.

These sorts of cuspidal expansions (i.e., expansions around the degeneration region of the genus one surface), along with Mass-Selberg relations and other techniques, aid in understanding the joint behaviour between non-compact regions and compact regions of moduli space. In the full modular integrals, these contributions are not independent, connected via modular symmetry. The goal of modularly completing field theory amplitudes is then to ask whether one can piece back together the contributions from the two regions.

A more direct way to motivate this idea, without having to deal with non-trivial spectral sums over Riemann zeroes, is to instead directly consider the asymptotic form of the torus partition function (10.28). Deep in the IR region of moduli space, where the torus degenerates, it is possible to package the integrand in the following way [214, 227]

$$\mathcal{Z}_{\text{Torus}}(\tau_2)\,|_{\text{Asymp}} = \tau_2^{1-d/2}\left[a_{0,0} + \sum_{n\geq 1} a_{n,n} e^{-2\pi n\tau_2}\right], \tag{10.39}$$

which, as can be seen, is naturally decomposed into massless and massive contributions.[7] Naively, it is possible to recast the integrand so that in the partial tropical

[7]This decomposition would, in fact, appear *universal* in the context of genus one amplitudes in the low-energy limit, and is one reason why the decomposition of moduli space in the tropical programme is fundamentally consistent.

limit [239] one gets

$$\mathcal{A}_{\text{Torus}} = \int_0^\infty \frac{dT}{T^2} F_{\text{trop}}(T) \cdot H(\tau_2; \alpha'), \tag{10.40}$$

where $F_{\text{trop}}(T)$ is the standard field theory part including $a_{0,0}$, and $H(\tau_2; \alpha') := \left[1 + \frac{1}{a_{0,0}} \sum_{(m,n)\neq(0,0)} a_{m,n} e^{-2\pi\tau_2(m+n)}\right]$. In addition to this not being an especially nice regulator, its tropicalisation is problematic for reasons discussed in a later section. In any case, treated formally, this type of general integrand structure shows how one may naively begin to probe how "stringy effects" might begin to explicitly appear in the low-energy theory as a regulator function.

Recently, along similar lines, Abel and Dondi [62] proposed a framework for UV-completing particle theories based on the worldline formalism. A key feature of this work is the introduction of a worldline inversion symmetry, $T \to 1/T$ (with T the proper time), serving as a form of modular completion that mimics the regulatory role of modular invariance in string amplitudes. Their modified worldline theory successfully reproduced the UV/IR mixing behaviour seen in the string spectrum under a modular S-transformation. Related work by Abel, Dienes, and others (see e.g., [69, 151–153, 223, 224]) similarly explores various mechanisms for modular completion, offering new perspectives on how modular symmetry may be implemented or mirrored in field-theoretic settings.

In all of these approaches (and others), the hope of realising *rigorously* the concept of string theory induced regulators has proven elusive. But the search has been continuously stimulated by progress in diverse areas, including the development of new mathematical structures in the automorphic form program [240–243] and the study of multiple zeta values [244, 245]. Furthermore, related progress in the study of modular graph functions [246–249] suggest it might be possible to encapsulate the intricate structures in the full modular invariant partition function in a way that gives further insight to the derivation of string theory induced regulators. It is also poignant to note ongoing research in the area of iterated Eisenstein integrals [250] as potential sources of insight.

These developments intersect with the exciting development of the tropical amplitudes programme. From the viewpoint of tropical geometry, given how tropical

graphs can be interpreted as Feynman graphs in the Schwinger proper time representation, directly descended from degenerations of the worldsheet, with loops allowed to degenerate into vertices—it would seem natural that the massive fluctuations of the string, or its residual effects, might somehow weight the tropical integration in a more direct way given the appropriate generalisations. The question now changes from how to create familiar field theory graphs from out of closed Riemann surfaces [66], to how to create *regularised* field theory graphs from out of degenerate Riemann surfaces and their full modular structures.

10.4.2 Main challenges from a tropical viewpoint

From a tropical viewpoint, a solution to this challenge is not obvious. As we have seen, distinct limits are defined to strictly correspond to different physical regimes, in order to consistently degenerate the genus one geometries to piecewise-linear tropical graphs [66]. This conceptual approach is a requirement, due to the fact that a *global tropical limit* remains highly non-trivial, based fundamentally on the behaviour of the integrand near various degeneration loci and the need to identify these regions in terms of Schottky data (as reviewed in Section 10.2).

Given a string theory integral, once it is split up according to the KP decomposition, it is simply no longer possible to probe the emergence of a class of multiplicative regulators. Quite simply, any possible string theory induced regulator would need to act multiplicatively in the field theory integral. Presumably, such a function would scale the tropical integrand or measure, similar to key insights obtained in the bottom-up approach of η regularisation for field theory loop integrals. As a matter of proposition, it would effectively act like a form factor. This means one either has to:

1. preserve the analytic region's effect as a function within the tropical integrand—most naturally as a geometric correction which acts as a weight in the tropical graphs or in the tropical measure;
2. or re-encode this data into the tropical integrand as a correction, which is analogous to the modular integral approach that effectively aims to establish a form of modular completion.

Both strategies have proven to present their own significant complications in the background of this thesis. In the standard tropical framework, to recover an intrinsic

regulator, let alone a *tropical regulator*, would likely again require some notion of modular completion. There are examples, such as in tropical Gromov-Witten theory, and in non-Archimedean geometry, where certain tropical corrections can appear [251–253]. But it is unclear how these methods may formally apply.

Furthermore, the reliance on taking $\alpha' \to 0$, which suppresses the massive modes, means the tropical limit also requires generalisation. Whilst there are ways to modify and generalise the tropical scaling so that the massive tower of states can be probed and preserved more effectively (such as double scaling limits, or by treating α' as fixed but small), this requires further mathematical development along the lines of what becomes a *partial tropical limit*.

Another technical hurdle has to do with the reliance on moving $\alpha' \to 0$ inside the integral. For this to be rigorous, one has to ensure the integration region is compact and the integrand is fully analytic, which is not obvious in a global treatment of the integrand. This raises yet another technical hurdle: string integrands involve entire spectral sums that do not have a natural interpretation in the purely tropical context. Additionally, as with other approaches (e.g., modular integral approaches), we have to deal with the mathematical problem of needing to compute the full string spectrum, which, in the simplest settings, becomes as much a deeply number theoretic problem as a physical problem.

10.4.3 Probing regularising structures

Since tropical geometry is a recent and actively developing area of mathematical research, application in the domain of string theory amplitudes still faces several notable restrictions and open questions that call for further development. Given some of the technical challenges stated, and others to be discussed in this final section, we can at least formally, and in many instances *heuristically*, start to probe possible pathways forward.

In addition to all the possible research pathways to probe connections between η regularisation and the UV finiteness of string theory proposed throughout this thesis, a particularly foundational approach is motivated by observations found in non-Archimedean and Berkovich inspired geometries [254]. Instead of the standard tropicalisation whereby $\mathcal{M}_{g,n} \to \mathcal{M}_{g,n}^{\text{trop}}$, one considers [255], at genus one, an analyt-

ical enhancement such that $\mathcal{M}_{1,n}^{\text{hybrid}} := \mathcal{M}_{1,n}^{\text{trop}} \times \mathcal{F}$, where $\mathcal{F}$ encodes the spectral data. It is not presently obvious how one would rigorously define such a hybrid tropicalisation map, which schematically implies $\mathcal{M}_{g,n} \to \text{trop(graph)} \times \text{spectral data}$; but the idea is at least motivated physically in the following sense.

Consider, once again, the full torus amplitude (10.28). Without decomposing the moduli space, we make a change of variables and introduce *globally* the tropical parametrisation $T = 2\pi\alpha'\tau_2$. Under global tropical parametrisation in the context of the *partial tropical limit*, we thus at least formally define a generalised decorated tropical graph as a metric graph where each edge e of length T_e carries non-trivial weight given by the function $H_e(T;\alpha') := \sum_m a_e(M)e^{-M^2T}$, with $a_e(M)$ a spectral coefficient determined by the string spectrum. Such a partial tropicalised integrand then takes form

$$\mathcal{A}_{\text{Torus}}(\alpha') \,|_{\text{hybrid}} = (2\pi\alpha')^{d/2} \int_{T=2\pi\alpha'}^{\infty} \frac{dT}{T^{d/2}} H(T;\alpha'), \tag{10.41}$$

with only a subset of the moduli of the geometry fully tropicalised in the proper sense. In other words, the moduli space and the integration measure are allowed to fully tropicalise, as can be seen; but non-degenerate structure, given by a non-tropical analytic sum, is retained in the form $H(T;\alpha')$.

From the perspective of tropical geometry, it is not clear how to interpret the integrand in any sort of rigorous way. Containing both tropical and non-tropical data, naively it represents a hybrid or intermediate object—neither tropical nor purely string-theoretic—akin to something like a generalised *decorated tropical graph* in the form of a graph effectively thickened or smeared by the full inclusion of the stringy corrections. Since the algebraic data is enriched, and although the combinatorial interpretation of edge decorations is unclear, it is perhaps best viewed as an integrand in the sense of Berkovich [256] and Thuillier [257], where tropical and analytic structures may coexist.

Although speculative, and not rigorous, it can at least be heuristically commented that, if a string theory induced class of regulators were to be derived in this fashion, there is a developing body of research on hybrid objects defined on a hybrid spaces [256–258] that might prove insightful. Indeed, although formulated in a different context, these works have been developed explicitly to bridge tropical and complex

analytic geometries along various lines. In particular, a recent paper by Abramovich [259] presents sophisticated studies of the links between $\mathcal{M}_{g,n}$ and compactified moduli space $\overline{\mathcal{M}_{g,n}}$ using Berkovich analytic spaces, which would be interesting to explore more deeply in the context of string amplitudes. Additionally, the work by Abramovich–Caporaso–Payne [251], Kontsevich–Soibelman [254], and Gross–Siebert [258] has demonstrated that degenerations of algebraic curves can be encoded as maps into spaces that simultaneously carry tropical and complex structure. It would be interesting to investigate more deeply whether these approaches, including if not especially hybrid spaces, can be be given a wider application in studying the partial tropicalisations of string amplitudes under global parametrisation.

Another possible pathway forward, taking directly from η regularisation and motivated by some observations in non-Archimedean geometry and elsewhere (see e.g., [260–263]), seeks to define a weighted tropical measure that encodes stringy corrections. For instance, recall that the tropicalisation of a genus g Riemann surface (or more generally a punctured curve) produces a metric graph Γ. In a more general setting, consider the set of edges $E(\Gamma)$ that carry length parameters $T_e \in \mathbb{R}_{>0}$. These are the tropical analogues of the Schwinger parameters in field theory. The "measure" on the moduli of tropical curves may be schematically written as

$$\int_{M_{g,n}^{\mathrm{trop}}} \Big(\prod_{e\in E(\Gamma)} dT_e \Big)\, H\big(\{T_e\};\alpha'\big)\,,$$

with divergences arising when one or more $T_e \to 0$. Again, this is purely formal; but it is motivated in some sense by constructions in non-Archimedean geometry. The idea is to explore the implications (and geometric motivations) for the damping of the measure schematically written

$$\prod_{e\in E(\Gamma)} R(T_e)\,,$$

where $R\colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ satisfies $R(T) \xrightarrow[T\to\infty]{} 1$ and vanishes for $T \to 0$ thus regularising the UV. However, as at present this is entirely heuristic, it is unclear whether any sort of regularised tropical measure can be made rigorous. The larger hope is that a truly "string encoded" tropical moduli space—i.e., one that is not just the half-line $T \in [0,\infty)$ with measure $dT/T^{d/2}$—but the full tropical compactification

of $\mathcal{M}_{1,n}$ carrying all the extra analytic data, might be possible.

Lastly, a more obvious approach follows from the observation that a form of tropical regularisation may appear as a natural consequence of inducing a cut-off on the lower bound $T > 2\pi\alpha'$ of the T integration, which scales with α'. This is similar to the regularising of Kutasov and Seiberg [264]. It is well-known [265] that, upon extending the integration from the fundamental domain to the domain where $\tau_2 \to 0$, this can be understood as a gauge divergence in the volume of integration $\mathcal{S}_\epsilon = \left\{\tau \in \mathbb{C} : \mathrm{Re}\,\tau \in [-1/2, 1/2] \wedge \mathrm{Im}\,\tau \in [\epsilon^{-1}, +\infty[\right\}$, with $\epsilon \gg 1$. Furthermore, Kutasov and Seiberg show the relationship between regularised and unregularised volumes can be established [265]

$$\frac{1}{\mathrm{vol}_{\mathrm{PSL}_2(\mathbb{Z})}\mathcal{F}}\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}Z(\tau,\bar{\tau}) \overset{\epsilon\to\infty}{\simeq} \frac{1}{\mathrm{vol}_{\mathrm{PSL}_2(\mathbb{Z})}\mathcal{S}_\epsilon}\int_{\mathcal{S}_\epsilon}\frac{d^2\tau}{\tau_2^2}Z(\tau,\bar{\tau}), \tag{10.42}$$

with scaling parameter ϵ. It can be found that $1/(\mathrm{vol}_{\mathrm{PSL}_2(\mathbb{Z})}\mathcal{F}) = \pi/3$, while for the cut-off dependent volume $1/(\mathrm{vol}_{\mathrm{PSL}_2(\mathbb{Z})}\mathcal{S}_\epsilon) = \epsilon$. The key idea is to define a function $f(\tau_2)$ dependent only on the net degeneracy of physical states

$$f(\tau_2) = \int_{-1/2}^{1/2} d\tau_1\, Z(\tau_1,\tau_2) = \tau_2^{1-D/2}\sum_n a_{nn}e^{-2\pi n\tau_2}, \tag{10.43}$$

so that, using the Kutasov-Seiberg identity, one can write [265]

$$\lim_{\epsilon\to\infty}\left[\frac{1}{\mathrm{vol}_{\mathrm{PSL}_2(\mathbb{Z})}\mathcal{S}_\epsilon}\int_{\mathcal{S}_\epsilon}\frac{d^2\tau}{\tau_2^2}Z(\tau_1,\tau_2)\right] = \lim_{\epsilon\to\infty}\frac{1}{\epsilon}\int_{\epsilon^{-1}}^{\infty}\frac{d\tau_2}{\tau_2^2}f(\tau_2), \tag{10.44}$$

with $\tau_2 = \epsilon^{-1}$ cutting off the UV divergence in the lower bound. Tropicalisation of the torus amplitude can be formulated in a way that is found to achieve a much more physically motivated version of such a cut-off. It raises the question of whether an α' dependent cut-off can be generalised—for instance, by keep α' small but fixed, or introducing double scaling limits, to further probe how a tropical induced regularisation might be generalised.

While the proposal to define string theory induced regulators from within the tropical framework is a topic of ongoing research, and thus is presented here as speculative and heuristic, the proposed pathways forward can nevertheless be seen

to resonate with a rich network of ideas. Possible connections to Berkovich skeleta, logarithmic structures, and tropical limits of string amplitudes suggest that a more refined analytic structure on tropical moduli spaces—potentially encoding the full string-theoretic data—would be incredibly exciting. Of course, much work remains to be done, not least in formulating and justifying rigorously more concrete examples in which classes of regulator functions might directly descend from string theory. Despite this, however, the observations and remarks made throughout this concluding chapter hopefully point to a potentially fruitful research direction at the interface of number theory, quantum field theory, algebraic geometry, and quantum gravity. In effort to formalise a foundational theory of regularisation across scales, from low-energy field theory to high-energy regimes, what's clear is that a potentially unifying view seems to constantly tantalise and tease.

11

Conclusion and outlook: Taming infinity, charting the finite

The Road goes ever on and on. Down from the door where it began. Now far ahead the Road has gone. And I must follow, if I can.

Pursuing it with eager feet. Until it joins some larger way. Where many paths and errands meet. And whither then? I cannot say.

— **Frodo Baggins, *The Fellowship of the Ring***

This thesis has sought to explore the role and resolution of infinity in the fundamental description of nature. By tracing a path from number theory to quantum field theory and string theory, we have attempted to unify seemingly disparate mathematical insights under a single philosophical and technical framework grounded in the interplay between divergent expressions and their regularisation.

The central object of investigation has been the development and application of η regularisation, a novel technique motivated by number-theoretic ideas and the smoothed asymptotics framework of Tao. Starting from the treatment of divergent series—especially those historically explored by Ramanujan—we rigorously formulated η regularisation as a tool capable of extracting finite, physically meaningful values from formally divergent expressions.

Bridging number theory and physics

In **Part I**, we built a foundation by revisiting the philosophical and historical underpinnings of infinity in mathematics. Through the lens of number theory, and specifically via the Riemann zeta function and Ramanujan summation, we re-examined how the infinite can yield finite outcomes. This set the stage for a more formal approach to divergent series, culminating in a precise treatment using smoothed asymptotics. Here, enhanced regulators were introduced as a means of eliminating power-law divergences entirely—revealing that the finite parts are in some sense universal, echoing the structure of renormalisable theories in physics.

A developing theory of regularisation

In **Part II**, we extended these methods to quantum field theory. The application of η regularisation to loop integrals and the associated preservation of gauge invariance marked a significant development. Not only did η-inspired regularisation offer an intuitive and physically transparent regularisation scheme, but it also unified the treatment of divergences in both summation and integration frameworks. We showed explicitly how η-inspired regulators could preserve the Ward identities of gauge theories—suggesting that such techniques are not merely formal, but may align with deeper symmetries of nature.

Moreover, our work in generalising η regularisation and solving gauge consistency conditions revealed a broader family of regularisation schemes, including many known cases such as dimensional regularisation. The formalism naturally extended to the path integral measure, where our reinterpretation of the Fujikawa method uncovered interesting connections between spectral sums, anomalies, and index theory.

From the infinite to the UV complete

In **Part III**, we turned to string theory, the leading candidate for a UV complete theory of quantum gravity. After reviewing the structure of string amplitudes and their intrinsic modular symmetries, we explored how the modular invariance of the

string worldsheet naturally regulates ultraviolet divergences—echoing, naively, the smoothing effects of η-type regulators in field theory. The tropical geometry framework provided a particularly powerful language in this regard, as we demonstrated how string theory amplitudes degenerate into field theory diagrams with induced renormalisation structures. This perspective strengthened the motivation that, albeit difficult to rigorously prove, there appears to be elegant physical mechanisms: string theory not only resolves divergences via renormalisation, but it seems possible that it should also be able endow field theory with a natural regularisation prescription via its degeneration limits. This bottom-up and top-down consistency strengthens the interpretation that η regularisation—and the number-theoretic ideas behind it—may reflect more than mathematical convenience. They may hint at a deep structure in the continuum limits of quantum gravity.

Outlook

In additional to the future directions already proposed in the many chapters of this thesis, several other promising directions emerge from this work:

1. **Foundations of regularisation**: The η framework opens the door to a mathematically rigorous, physically motivated theory of regularisation. Future work could further develop this in terms of a variety of ways, including various extensions on the level of number theoretic sums as well as in terms of connections to Tauberian theorems, quasiasymptotics, and Stieltjes integrals, to name a few. Another future direction is to understand this theory of regularisation in categorical or geometric language, potentially bridging even more tightly with modern developments in homotopy theory or derived geometry.
2. **Modular shadows and regulator geometry**: The idea of "string-induced regulators" invites a deeper study of modular structures and their geometric interpretation. The concept of modular shadows remains rich with unexplored potential in both perturbative and non-perturbative regimes. It is especially important to study more deeply string theory derived regulator functions in terms of general modular integrals, not just in terms of finite volumes but also as fundamental corrections to the measure.

3. **Non-perturbative generalisations**: Resurgence theory, trans-series, and the inclusion of instanton sectors may extend η regularisation to fully non-perturbative settings, offering new tools to study phenomena like confinement, duality, or black hole microstates.
4. **Algebraic and tropical approaches to UV finiteness**: The tropical limit and associated decomposition of moduli space suggest novel ways to formulate and understand renormalisation in algebraic and combinatorial terms. It would be interesting to understand better how string theory induced regularisation, if indeed possible, fits within the broader picture of positive geometry and its many recent developments. This could enhance computational techniques for string amplitudes and inspire new UV-complete toy models.
5. **Implications for quantum gravity**: If UV finiteness and regularisation are linked at a fundamental level to modularity, then it is conceivable that new constraints—perhaps number-theoretic in origin—could help refine or classify consistent degenerations from quantum gravity theories to field theories.

In closing, this thesis has argued that the infinite, rather than being a pathological artefact to be discarded, may be an essential scaffold in the architecture of physical law. The reconciliation of the infinite and the finite—across number theory, quantum fields, and strings—is not just a technical necessity but a profound theme that speaks to the unity of mathematics and physics. Through the introduction and exploration of η regularisation, we have attempted to navigate this confluence, offering a new perspective on the taming of divergences and the search for a deeper mathematical order underlying the cosmos.

A

Vacuum polarisation and generalised ward identities

Following [103], the contribution of N_f equal mass fermion loops to the gauge boson self-energy is given by

$$\Pi^{(f)ab}_{\mu\nu}(p) = 4N_f \mathrm{Tr}(T^a T^b)(ig)^2 \int \frac{d^4k}{(2\pi)^4} \frac{\mathrm{Tr}[\gamma_\mu(\not{k}+m)\gamma_\nu(\not{k}+\not{p}+m)]}{(k^2+m^2-i\epsilon)((k+p)^2+m^2-i\epsilon)}. \quad \text{(A.1)}$$

After expressing the colour charge factor as $\mathrm{Tr}(T^a T^b) = C_2\delta^{ab}$ for some constant C_2 and computing the Dirac algebra, we use Feynman parametrisation to write the remaining integrals in terms of one-fold ILIs as clearly laid out in [103]. Doing so gives

$$\begin{aligned}\Pi^{(f)ab}_{\mu\nu}(p) = -4N_f g^2 C_2 \delta^{ab} \int_0^1 dx \, \Big[& 2I_{2\mu\nu}(\mathcal{M}) \\ & - g_{\mu\nu} I_2(\mathcal{M}) + 2x(1-x)(p^2 g_{\mu\nu} - p_\mu p_\nu) I_2(\mathcal{M}) \Big],\end{aligned} \quad \text{(A.2)}$$

where the mass factor $\mathcal{M} = m^2 - x(1-x)p^2$ in the ILIs includes a contribution from the fermion mass m, the external momentum, p_μ and the Feynman parameter, x. The derivation of this formula requires several comments. Here and in [103], the formula is presented in terms of unregulated integrals. However, these integrals are divergent. Strictly speaking, the chain of manipulations can only be done for convergent integrals, which in our case would correspond to a particular form of η regularisation [1]. In future work we will explore whether or not this choice can

[1] We thank Killian Moehling for pointing this out in detail.

generalise some of our conclusions. Further, one might also worry that the regulator breaks the invariance under translations of loop momenta. However, as we explain in more detail in Chapter 6, η regulators can be rendered translation invariant by redefining the origin of momentum space.

Taking the regularised form of (A.2), we have that

$$\Pi^{(f)ab}_{\mu\nu}(p)|_{\text{regularised}} \propto \int_0^1 dx \left[2I_{2\mu\nu}(\mathcal{M})|_{\text{regularised}} - g_{\mu\nu} I_2(\mathcal{M})|_{\text{regularised}} + 2x(1-x)(p^2 g_{\mu\nu} - p_\mu p_\nu) I_0(\mathcal{M})|_{\text{regularised}} \right]. \quad \text{(A.3)}$$

As stated in the main text, we now require that this expression satisfies the generalised Ward identity (6.15) asymptotically,

$$p^\mu \Pi^{(f)ab}_{\mu\nu} = \Pi^{(f)ab}_{\mu\nu} p^\nu = 0. \quad \text{(A.4)}$$

We immediately see that the logarithmically divergent terms on the second line of (A.3) vanish automatically when contracted with the external momenta and do not affect the generalised Ward identity (6.15). However, this is not the case with the two quadratically divergent terms on the first line of (A.3). These must cancel asymptotically in order to preserve gauge invariance, giving the condition (5.29) for $\alpha = -1$,

$$I_2^{\mu\nu}(\mathcal{M})|_{\text{regularised}} \sim \frac{1}{2} g^{\mu\nu} I_2(\mathcal{M})|_{\text{regularised}} \quad \text{(A.5)}$$

Note that the full set of conditions given by (5.29) and (5.30) for $\alpha = -1, 0, 1$ are obtained from the remaining generalised Ward identities coming from gauge field contributions to the vacuum polarisation tensor and from higher point correlation functions.

B

Undoing the Feynman parametrisation with η factors

In Chapter 6, we introduced our η regulators after performing a range of manipulations on the integrals, most notably, Feynman parametrisations. Such manipulations are not valid for divergent integrals so one is implicitly assuming that some form of regularisation is always present and that this will result in the proposed η regularisation of the ILIs at the end of the calculation. We shall now investigate how the regularisation scheme might look prior to Feynman parametrisation for certain integrals.

To this end, we consider a contribution to a correlation function coming from a regularised scalar ILI

$$\mathcal{A} = (n-1)! \int_x \int_k \frac{1}{(k^2 + \mathcal{M}^2)^n} \eta\left(\frac{k}{\mu}, \epsilon\right). \tag{B.1}$$

Here $\int_x = \int_0^1 dx_1 \cdots \int_0^1 dx_n \delta(\sum_i x_i - 1)$ denotes the integration over n Feynman parameters, x_i, subject to the constraint $\sum_i x_i = 1$. Furthermore, $\mathcal{M}^2 = \sum_i x_i(p_i^2 + m_i^2) - (\sum_i x_i p_i)^2$ is a function of these parameters, the particle masses, m_i and external momenta, p_i. $\int_k$ denotes the integration over the loop momentum, which could be in any number of dimensions.

To undo the Feynman parametrisation, it is convenient to rewrite the integral in

Schwinger form. To do so, we first use momentum routing to shift the loop momentum

$$\mathcal{A} = (n-1)! \int_x \int_k \frac{1}{((k+\sum_i x_i p_i)^2 + M^2)^n} \eta\left(\frac{|k+\sum_i x_i p_i|}{\mu}, \epsilon\right) \tag{B.2}$$

$$= (n-1)! \int_x \int_k \frac{1}{(\sum_i x_i \Delta_i)^n} \eta\left(\frac{|k+\sum_i x_i p_i|}{\mu}, \epsilon\right) \tag{B.3}$$

where $\Delta_i = (k+p_i)^2 + m_i^2$, before integrating in a Schwinger parameter, s

$$\mathcal{A} = \int_0^\infty ds\ s^{n-1} \int_x \int_k e^{-s(\sum_i x_i \Delta_i)} \eta\left(\frac{|k+\sum_i x_i p_i|}{\mu}, \epsilon\right). \tag{B.4}$$

We now introduce a set of n variables $s_i = s x_i$ allowing us to combine the integration of the Schwinger parameter and the Feynman parameters,

$$\mathcal{A} = \int_0^\infty ds_1 \cdots \int_0^\infty ds_n \int_k e^{-(\sum_i s_i \Delta_i)} \eta\left(\frac{|k+\frac{\sum_i s_i p_i}{\sum_i s_i}|}{\mu}, \epsilon\right). \tag{B.5}$$

Finally, we absorb the Δ_i into a redefinition of the $s_i \to s_i/\Delta_i$ so that we have

$$\mathcal{A} = \int_k \frac{1}{\Delta_1 \cdots \Delta_n} \tilde{\eta}, \tag{B.6}$$

where

$$\tilde{\eta} = \int_0^\infty ds_1 e^{-s_1} \cdots \int_0^\infty ds_n e^{-s_n} \eta\left(\frac{\left|k+\frac{\sum_i s_i p_i/\Delta_i}{\sum_i s_i/\Delta_i}\right|}{\mu}, \epsilon\right). \tag{B.7}$$

This is the form of regularisation factor prior to Feynman parametrisation. Note that $\tilde{\eta} \to 1$ and $\eta \to 1$, as of course it should.

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