

Equations over Groups

Noha Eljamel, MSc.

Thesis submitted to The University of Nottingham
for the degree of Doctor of Philosophy

December 2017

Contents

1	Introduction	4
1.1	Definitions and previous results	4
1.2	Method of proof	7
1.3	Statement of results	12
2	Case I:	20
2.1	Case I (a1)	20
2.2	Case I (a2)	25
2.3	Case I (b1)	34
2.4	Case I (b2)	39
2.5	Case I (b3)	42
3	Case II	45
3.1	Case II (a1)	45
3.2	Case II (a2)	58
3.3	Case II (a3)	67
4	Case III	71
4.1	Case III(a1)	71
4.2	Case III(a2)	74
4.2.1	Figures with a vertex label $\overline{214}$	75
4.2.2	Figures with a vertex label $\overline{241}$	127
5	Proof of Theorem 8	172
	Bibliography	175

Abstract

If G is a none trivial group and t is an element distinct from G then $r(t) = g_1 t^{l_1} \dots g_k t^{l_k} = 1$, $k \geq 1$, $g \in G \setminus \{1\}$, $l_i \in \mathbb{Z} \setminus \{0\}$ is said to be an equation over G . There has been much research aimed to investigate solvability of such equations over groups and these researches adopted two main approaches. The first considers the properties of G . The other direction which we are following here is to put restrictions on $r(t)$. The results obtained in this direction was of length restriction at first. More recent more results have been proved and the concept of isolated t -exponent has been introduced and used to study a generalised form of equations of unlimited length. In this study we investigate $r(t)$ which has the generalized form $w_1 t^{l_1} w_2 t^{l_2} w_3 t^{l_3} w_4 t^{l_4}$ where $w_i = g_{i,1} t^{m_{i,1}} g_{i,2} \dots t^{m_{i,k_i}-1} g_{i,k_i}$ with $g_{i,k} \in G \setminus \{1\}$, $k_i \geq 1$, $m_{i,j} \neq 0$.

In Chapter 1 we introduce the concept of equations over groups and we give some of the known results the geometric method of proof is explained. We also state our main theorem and the main lemma which will be proved in the following chapters and some technical lemmas are proved.

In Chapter 2, Chapter 3, and Chapter 4 the Cases I, II, III are discussed and the distribution is shown. In Chapter 5 the main lemma is proved and the proof of the main theorem is completed.

Acknowledgements

First praise is to Allah, the Almighty, on whom ultimately we depend for sustenance and guidance.

Secondly, I would like to express my deep gratitude to my supervisor Dr Martin Edjvet for his expert advice, patient guidance, encouragement and useful critiques of this research work. Without him nothing of this work would be possible.

I also owe my very great appreciation to my Family: my dear husband Abdula-dem and my precious kids Amna, Salem, Hadi and Abdulmonaem, thank you for your support, love, patience and care; my dear parents whose love, prayer and guidance are always with me; and my supportive sisters, brothers and friends.

My thanks extend to all my teachers specially in The University of Nottingham and The University of Misurata. I would also like to thank my office mates for their warm company.

Chapter 1

Introduction

1.1 Definitions and previous results

A polynomial equation of degree n over a field K is always solvable in a suitable extension K' of degree at most n over K [1]. There has been much research aimed at finding a corresponding result for groups. In order to put the problem we are studying in context we need to give some definitions.

Definition 1. *Let G be a non trivial group and let t be an element distinct from G . Consider $r(t) = g_1 t^{l_1} \dots g_k t^{l_k}$, $k \geq 1$, $g \in G \setminus \{1\}$, $l_i \in \mathbb{Z} \setminus \{0\}$, an element in the free product $G * \langle t \rangle$. Then $r(t) = 1$ is said to be an equation over G .*

We say that this equation has a solution over G if there is an embedding ϕ from G into a group H and an element $h \in H$ such that $\phi(g_1)h^{l_1} \dots \phi(g_k)h^{l_k} = 1$ in H . We define the length of $r(t)$ to be $|l_1| + \dots + |l_k|$ and the exponent sum as $l_1 + l_2 + \dots + l_k$.

An equation is called singular if its exponent sum is equal to zero, otherwise it is called non-singular.

There has been two approaches to this problem. The first considers the properties of G . Research in this direction include the following:

Theorem 1. [12] *Any equation of length at most 6 over a torsion-free group has a solution.*

Theorem 2. [13] *Any equation over a locally indicable group G has a solution over G .*

Locally indicable group is a group with each of its non-trivial finitely generated subgroups admits an epimorphism into the infinite cyclic group. Note that any locally indicable group is torsion-free. One of the main conjectures in this direction is the following:

Conjecture 1. *Any equation over a torsion-free group G has a solution over G .*

The other direction is to put restrictions on $r(t)$ such as the following theorems.

Theorem 3. [1] *If the length of an equation is equal to its exponent sum of t then the equation has a solution.*

Theorem 4. [3] *Any non-singular equation of length at most 5 has a solution.*

The following conjecture is the main conjecture in this direction:

Conjecture 2. *Any non-singular equation over any group G has a solution over G .*

And in this direction results were obtained only for $r(t)$ for limited length. More recent other results have been found. The next theorem follows from results that were proven in ([1],[6]-[9]).

Theorem 5. *let $r(t) = g_1 t^{l_1} \dots g_k t^{l_k} \in G^* \langle t \rangle$, $k \geq 1$, $l_i \in \mathbb{Z} \setminus \{0\}$ where :*

- (i) $|g_i| > 2$, ($1 \leq i \leq k$), where $|g_i|$ refers to the order of g_i in G
- (ii) $l_1 + l_2 + \dots + l_k \neq 0$ (i.e the equation $r(t) = 1$ is non-singular)
- (iii) $|l_i| \neq |l_j|$ for $i \neq j$

Then $r(t) = 1$ has a solution over G .

The next two theorems are proved in [10]

Theorem 6. *Let G be a group and let $r(t) = w_1 t^{l_1} \dots w_k t^{l_k}$ where $w_i = g_{i,1} t^{m_{i,1}} g_{i,2} \dots t^{m_{i,k_i-1}} g_{i,k_i}$ with $g_{i,k} \in G \setminus \{1\}$, $k_i \geq 1$, $m_{i,j} \neq 0$, such that the next condition holds:*

(*) *for $1 \leq i \leq k$, $|l_i|$ is distinct from $|l_j|$ when $i \neq j$ and is distinct from $m_{u,v}$ for any u, v .*

Then the equation $r(t) = 1$ has a solution over G if $k \geq 6$.

Lemma 1. *If $\forall i, \exists j \neq i$ such that $w_i = w_j^{\pm 1}$, then it can be assumed that $r(t)$ has one of the following forms (modulo cyclic permutation and inversion):*

$$\begin{array}{ll}
 A1 & wt^{l_1}wt^{l_2}wt^{l_3}wt^{l_4}wt^{l_5} \\
 A2 & wt^{l_1}wt^{l_2}wt^{l_3}wt^{l_4}w^{-1}t^{l_5} \\
 A3 & wt^{l_1}wt^{l_2}wt^{l_3}w^{-1}t^{l_4}w^{-1}t^{l_5} \\
 A4 & wt^{l_1}wt^{l_2}w^{-1}t^{l_3}wt^{l_4}w^{-1}t^{l_5} \\
 \\
 B1(a) & wt^{l_1}wt^{l_2}wt^{l_3}vt^{l_4}vt^{l_5} \\
 B1(b) & wt^{l_1}wt^{l_2}wt^{l_3}vt^{l_4}v^{-1}t^{l_5} \\
 B2(a) & wt^{l_1}wt^{l_2}w^{-1}t^{l_3}vt^{l_4}vt^{l_5} \\
 B2(b) & wt^{l_1}wt^{l_2}w^{-1}t^{l_3}vt^{l_4}v^{-1}t^{l_5} \\
 B3(a) & wt^{l_1}w^{-1}t^{l_2}wt^{l_3}vt^{l_4}vt^{l_5} \\
 B3(b) & wt^{l_1}w^{-1}t^{l_2}wt^{l_3}vt^{l_4}v^{-1}t^{l_5} \\
 B4(a) & wt^{l_1}wt^{l_2}vt^{l_3}wt^{l_4}vt^{l_5} \\
 B4(b) & wt^{l_1}wt^{l_2}vt^{l_3}wt^{l_4}v^{-1}t^{l_5} \\
 B5(a) & wt^{l_1}wt^{l_2}vt^{l_3}w^{-1}t^{l_4}vt^{l_5} \\
 B5(b) & wt^{l_1}wt^{l_2}vt^{l_3}w^{-1}t^{l_4}v^{-1}t^{l_5} \\
 B6(a) & wt^{l_1}w^{-1}t^{l_2}vt^{l_3}wt^{l_4}vt^{l_5} \\
 B6(b) & wt^{l_1}w^{-1}t^{l_2}vt^{l_3}wt^{l_4}v^{-1}t^{l_5}
 \end{array}$$

Definition 2. *A subword of the word $w = g_1t^{m_1}.....t^{m_{k-1}}g_k, w \in G^* < t >$, $k \geq 1$ is a word of the form $g_pt^{m_p}g_{p+1}t^{m_{p+1}}.....t^{m_{p+r-1}}g_{p+r}$ where $p \in \{1, \dots, k\}$ and $r \in \{0, \dots, k-p\}$. A subword is a proper if $(p, r) \neq (1, k-p)$, and an initial if $p = 1$, and terminal if $r = p - k$.*

Theorem 7. *Let G be a group and let $r(t) = w_1t^{l_1}.....w_kt^{l_k} (k \geq 5)$ where $w_i = g_{i,1}t^{m_{i,1}}g_{i,2}.....t^{m_{i,k_i-1}}g_{i,k_i}$ with $g_{i,k} \in G \setminus \{1\}$, $k_i \geq 1$, $m_{i,j} \neq 0$. Assume that condition (*) holds and in addition, the following conditions holds:*

(**) *No w_i is a conjugate of an element of G of order 2.*

(***) *No w_i is proper initial or end subword of any $w_i^{\pm 1}$ for $i \neq j$.*

Then the following statements are true:

(I) *let $r(t)$ be given by one of the 16 forms listed in the previous lemma. Then $r(t) = 1$ has a solution over G if one of the following holds:*

(1) *$r(t)$ is of the form A1 or A4.*

(2) *$r(t)$ is of the form A2 and none of the following sets of relations holds:*

(i) $l_1 = l_2 + l_4, l_2 = l_3 + l_5$ and $l_3 = l_1 + l_5$,

(ii) $l_1 = l_3 + l_4, l_2 = l_1 + l_4$ and $l_3 = l_2 + l_5$,

(iii) $l_3 = l_2 + l_4 = l_1 + l_5$,

(iv) $l_1 = l_2 + l_5 = l_3 + l_4$.

(3) *$r(t)$ is of the form A3 and neither of the following sets of relation holds:*

(i) $l_2 = l_1 + l_3$ and $l_2 + l_4 + l_5 = 0$

(ii) $l_1 = l_2 + l_5$ and $l_1 + l_3 + l_4 = 0$

(4) *$r(t)$ is of the form B1-B6.*

(II) If $r(t)$ is not one of the 16 forms listed in the previous lemma, i.e. $\exists i$ such that $w_i \neq w_j^{\pm 1}$ for all $i \neq j$, then $r(t) = 1$ has a solution over G .

This has been used in [10] to improve Theorem 5 when $k \geq 5$ and we get the following corollary:

Corollary 1. Let $r(t) = g_1 t^{l_1} g_2 t^{l_2} g_3 t^{l_3} g_4 t^{l_4} g_5 t^{l_5} \in G^* \langle t \rangle$, $l_i \in \mathbb{Z} \setminus \{0\}$ where:

(i) $|g_i| > 2$, ($1 \leq i \leq 5$).

(ii) $|l_i| \neq |l_j|$ for $i \neq j$.

Then $r(t) = 1$ has a solution over G .

In our work we studied $r(t)$ in the form described in Theorem 7 when $k = 4$ under some similar conditions, in particular we assume condition (*) in Theorem 6 holds. We will state our main results after the next section.

1.2 Method of proof

The proof method applied here is a geometric approach and it is an analogue to the one used in [10] which is originally was used in [7]. We will use van Kampen diagrams with some amendment. A van Kampen diagram over any group, say $G = \langle X \mid R \rangle$, is a finite connected planar graph whose edges are directed and labelled by elements of X in such away that each face is a disk whose boundary label belongs to R . The boundary of the whole graph is itself a relation in G [11]. The converse is also true [2], if $w = 1$ in $G = \langle X \mid R \rangle$ then there exist a van Kampen diagram whose boundary label is w . It is this fact we will use.

In order to show that $r(t) = 1$ has a solution over G it is enough to show that the map $G \rightarrow H = \langle G, t \mid r(t) \rangle$ given by $g \mapsto g$, $\forall g \in G$ is injective. The method of our proof uses a contradiction argument. We assume that $r(t) = 1$ has no solution over G , this means that the map is not injective, which means by [2] that there is a free product diagram K over H whose boundary is a simple closed curve with an element $g_0 \in G \setminus \{1\}$ as it's label.

The diagram corresponding to $r(t) = w_1 t^{l_1} w_2 t^{l_2} w_3 t^{l_3} w_4 t^{l_4}$ will have two different type of regions. The first type of region is the one whose boundary label is some cyclic permutation of $r(t)^{\pm 1}$ and they are called $r(t)$ -regions. The second type are G -regions whose boundary label is a word in G which yield the identity.

We will amend the diagram in a certain way such that all the regions are given by Figure 1.2(i). The first amendment is to contract each maximal t -segment to a point and we label the corresponding corner by l_i or $m_{u,v}$ as appropriate. Secondly we place a new vertex in the interior of each G -region, including the infinite external region of K . Then we create new edges between the newly added vertex and each vertex of the region and we delete all the old edges which forms the boundary of the region. Finally we label the new vertex with element of G that corresponds to the deleted edge, see Figure 1.5.

Following those steps we obtain a tessellation D of the 2-sphere, whose regions

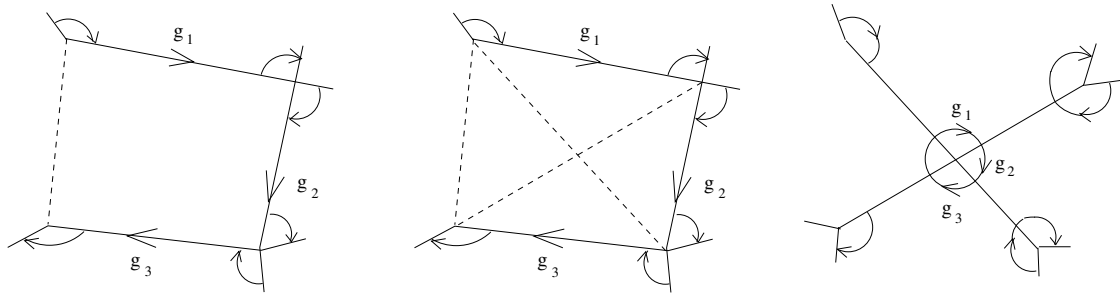


Figure 1.1: Amendment to G -regions

have corners labelled with some cyclic permutation of $r(t)^{\pm 1}$, reading around the region from any vertex.

We will refer by t -vertices to vertices which are labelled with powers of t , while g -vertices is used to refer to vertices labelled with elements of G . By convention, i is written in place of l_i at the corners of the diagrams, and $\bar{i}$ for $-l_i$. Other powers of t which are not one of l_i , $1 \leq i \leq 4$ will be denoted by letters such as k or k' and will be referred to as τ -powers.

Given a region Δ , we will refer to the four vertices with corner labels l_1, l_2, l_3, l_4 within Δ with v_1, v_2, v_3, v_4 respectively. The segment $(i, i+1)$ is called an edge and we say that the edge $(i, i+1)$ splits if at least one of the g - or t -vertices on that edge ($i, i+1$ are not included) has degree ≥ 3 , and we will refer to that vertex by u_{i+1} subscript $\text{mod } 4$. When we say that $d(u_{i+1}) = 2$ then we mean that there is no split on the edge $(i, i+1)$. For simplicity we will draw the figures as in Figure 1.2(ii). We will label the line segment which gives us the chain of g -vertices and t -vertices corresponding to w_i simply by w_i .

The sum of the corner labels of a t -vertex is 0 since $\langle t \mid \rangle$ is one of the free

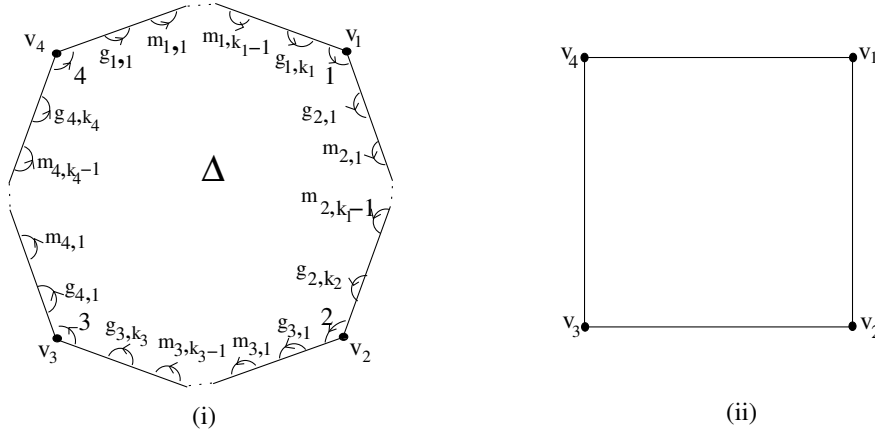


Figure 1.2: amendment

factors and the product of the directed corner labels at the g -vertices must be the identity in G except for the vertex labeled by g_0 . The degree of a vertex $\mathbf{v}$ is defined to be the number of edges incident at $\mathbf{v}$. It will be denoted by $d(\mathbf{v})$ and the label of $\mathbf{v}$ will be denoted by $l(\mathbf{v})$. While the degree of a region Δ is the number of its vertices with degree greater than 2 and it will be denoted by $d(\Delta)$. Let $\mathbf{v}_0$ be the vertex of the amended G -region whose label is g_0 . A boundary region is a region which has $\mathbf{v}_0$ as one of its vertices, otherwise it is called interior. We will refer to a boundary region by Δ_0 .

We may assume the following:

- (1) D is minimal with respect to the number of regions;
- (2) subject to (1) the number of interior vertices of D of degree 2 is maximal.

This is essential in proving the following lemma which is analogous to one proved in [9].

Lemma 2. (i) $i\bar{i}$ and $\bar{i}i$ can not be a sublabel of any vertex label in D .

(ii) $d(\mathbf{v}_i) > 2$ for $1 \leq i \leq 4$.

(iii) $d(\Delta) \geq 4$ for any region Δ of D .

Proof. (i) If $i\bar{i}$ or $\bar{i}i$ is a sublabel of any vertex label in D then the diagram is not reduced and it would be possible to cancel two regions which contradict the assumption of the minimality of D .

(ii) This follows from (i) along with the assumption that $|l_i| \neq |l_j|$ when $i \neq j$.

(iii) This follows from (ii) and the fact there are four of $\mathbf{v}_i$. $\square$

The curvature of a region Δ in D is given by $c(\Delta) = c(d_1, d_2, \dots, d_m) = (2 - m)\pi + 2\pi \sum_{i=1}^m \frac{1}{d_i}$ where $m = d(\Delta)$ and the d_i are the degree of vertices, $1 \leq i \leq m$. The total curvature of D is the sum of the curvatures of each region and it is denoted by $c(D)$. It can be shown that $c(D) = 4\pi$ as follows: Let V = the number of vertices in D , E = the number of edges and F = the number of regions. Then:

$$\begin{aligned} c(D) = \sum_{\Delta \in D} c(\Delta) &= \sum_{\Delta \in D} [(2 - d(\Delta))\pi + 2\pi(\frac{1}{d_1} + \frac{1}{d_2} + \dots + \frac{1}{d_m})] \\ &= 2\pi \sum_{\Delta \in D} (\frac{1}{d_1} + \frac{1}{d_2} + \dots + \frac{1}{d_m}) + 2\pi \sum_{\Delta \in D} -\pi \sum_{\Delta \in D} d(\Delta) \\ &= 2\pi V + 2\pi F - 2\pi E = 2\pi(V - E + F) = 4\pi \end{aligned}$$

This means that D must have regions of positive curvature. For each positively curved region Δ of D , we distribute $c(\Delta)$ to a neighbouring region $\hat{\Delta}$ in a systematic way. We subtract curvature from positive regions and add it to some neighbouring negative regions and we let $c^*(\hat{\Delta}) = c(\hat{\Delta})$ plus all positive curvature $\hat{\Delta}$ receives minus all the positive curvature $\hat{\Delta}$ distribute. This process is a purely numerical reassignment and the diagram itself remains unchanged. Then $4\pi \leq \sum[c^*(\hat{\Delta})] + \sum[c^*(\Delta_0)]$ where the first sum is taking over interior regions $\hat{\Delta}$ and the second sum is taking over boundary regions Δ_0 . When this is done, we prove our main lemma:

Lemma 3. *If $\hat{\Delta}$ is an interior region of D then $c^*(\hat{\Delta}) \leq 0$.*

After proving this lemma it is only remains to prove that $\sum c^*(\Delta_0) < 4\pi$ and we only need to prove that $c^*(\Delta_0) < \frac{4\pi}{k_0}$, $k_0 = d(\mathbf{v}_0)$ for each boundary region and this complete the proof since then $c(D) \leq k_0(c^*(\Delta_0)) < k_0(\frac{4\pi}{k_0}) = 4\pi$. The proof of this lemma will follow in the next chapters.

Remarks 1. (i) For a region of degree = 4:

$$\begin{aligned} c(3, 3, 3, 3) &= \frac{2\pi}{3}, \\ c(3, 3, 3, 4) &= \frac{\pi}{2}, \\ c(3, 3, 4, 4) &= \frac{\pi}{3}, \\ c(3, 4, 4, 4) &= \frac{\pi}{6}, \end{aligned}$$

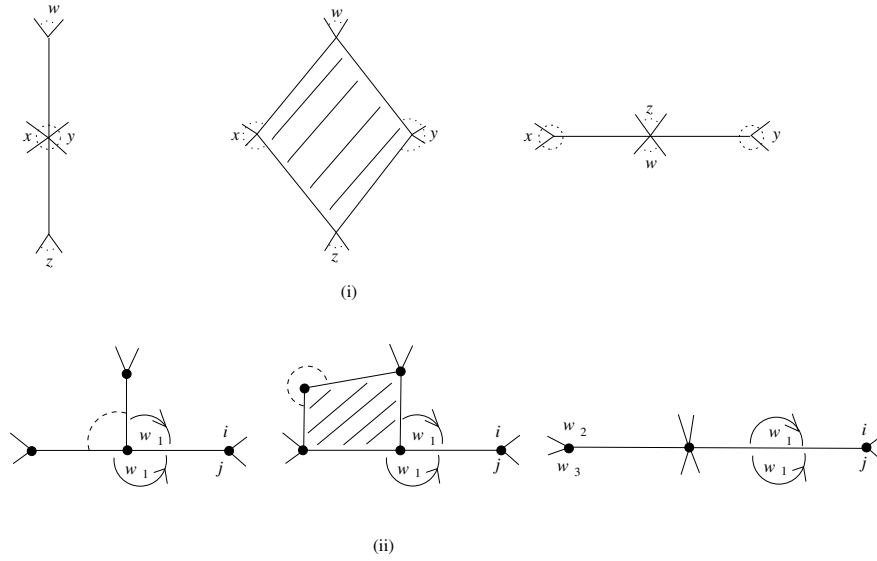


Figure 1.3: Bridge move

$$c(4, 4, 4, 4) = 0.$$

(ii) For a region of degree = 5:

$$c(3, 3, 3, 3, 3) = \frac{\pi}{3},$$

$$c(3, 3, 3, 3, 4) = \frac{\pi}{6},$$

$$c(3, 3, 3, 3, 5) = \frac{\pi}{15},$$

$$c(3, 3, 3, 3, 6) = 0.$$

(iii) Any region of degree ≥ 6 has curvature ≤ 0 .

Definition 3. [6] Let v be some vertex of D and suppose it is labeled xy and we know that $x = 1$. Then we can change D by a bridge move as shown in Figure 1.3

Lemma 4. [10] If $w_i = w_j$, $i \neq j$, no vertex label of D can have a proper sublabel $w_i w_j^{-1}$. Also if $w_i = w_j^{-1}$, $i \neq j$, no vertex label of D can have a proper sublabel $w_i w_j$.

Proof. If such label is allowed then a bridge move as in Figure 1.3(i) can be performed to increase the number of degree 2 vertices without changing the number of regions, which is a contradiction. An example is given in Figure 1.3(ii). $\square$

1.3 Statement of results

Before we state our results we need the following definition:

Definition 4. Let $w = g_1 t^{k_1} \dots t^{m_{k-1}} g_k, w \in G^* < t >$. Then w is strongly cyclically reduced if $g_1 \neq g_k^{-1}$. If either $g_1 \neq g_k^{-1}$ or $g_1 = g_k^{-1}$ but $t_{m_1} \neq t_{m_{k-1}}^{-1}$ then we say that w is cyclically reduced. Note that strongly cyclically reduced implies cyclically reduced but the opposite is not always true.

In our work we studied the equation $r(t) = w_1 t^{l_1} \dots w_k t^{l_k}$ where $w_i = g_{i,1} t^{m_{i,1}} g_{i,2} \dots t^{m_{i,k_i-1}} g_{i,k_i}$ with $g_{i,k} \in G \setminus \{1\}$, $k_i \geq 1$, $m_{i,j} \neq 0$ when $k = 4$. The following lemma is important in our main theorem:

Lemma 5. Let $r(t) = w_1 t^{l_1} w_2 t^{l_2} w_3 t^{l_3} w_4 t^{l_4}$. If $w_i \in \{w^{\pm 1}, v^{\pm 1}\}$, $w, v \in G^* < t >$ $1 \leq i \leq 4$ then it can be assumed that $r(t)$ has one of the following forms (modulo cyclic permutation and inversion):

Case(I):

$$\begin{array}{ll} (a1) & w t^{l_1} w t^{l_2} v t^{l_3} v t^{l_4} \\ (a2) & w t^{l_1} w t^{l_2} v t^{l_3} v^{-1} t^{l_4}, \\ (a3) & w t^{l_1} w^{-1} t^{l_2} v t^{l_3} v^{-1} t^{l_4} \end{array} \quad \begin{array}{ll} (b1) & w t^{l_1} v t^{l_2} w t^{l_3} v t^{l_4} \\ (b2) & w t^{l_1} v t^{l_2} w t^{l_3} v^{-1} t^{l_4} \\ (b3) & w t^{l_1} v t^{l_2} w^{-1} t^{l_3} v^{-1} t^{l_4} \end{array}$$

Case(II):

$$\begin{array}{ll} (a1) & w t^{l_1} w t^{l_2} w t^{l_3} v t^{l_4} \\ (a2) & w t^{l_1} w t^{l_2} w^{-1} t^{l_3} v t^{l_4} \\ (a3) & w t^{l_1} w^{-1} t^{l_2} w t^{l_3} v t^{l_4} \end{array}$$

Case(III):

$$\begin{array}{ll} (a1) & w t^{l_1} w t^{l_2} w t^{l_3} w t^{l_4} \\ (a2) & w t^{l_1} w t^{l_2} w t^{l_3} w^{-1} t^{l_4} \\ (a3) & w t^{l_1} w t^{l_2} w^{-1} t^{l_3} w^{-1} t^{l_4} \\ (a4) & w t^{l_1} w^{-1} t^{l_2} w t^{l_3} w^{-1} t^{l_4} \end{array}$$

Proof. Clearly it can be assumed that the numbers of $w^{\pm 1}$ is at least the same of $v^{\pm 1}$ in w_i 's. Since $w_i \in \{w^{\pm 1}, v^{\pm 1}\}$, $1 \leq i \leq 4$ then we either have two of the w_i 's are in $\{w, w^{-1}\}$ and the other two are in $\{v, v^{-1}\}$; or exactly three of the w_i 's are in $\{w, w^{-1}\}$; or $\{w_1, w_2, w_3, w_4\} \subset \{w, w^{-1}\}$. Any other form

that is not in the list can be rewritten using cyclic permutation and inversion to be one. For example if $r(t) = w^{-1}t^{l_1}wt^{l_2}wt^{l_3}vt^{l_4}$ then taking the inverse yields $r(t) = t^{-l_4}v^{-1}t^{-l_3}w^{-1}t^{-l_2}w^{-1}t^{-l_1}w$ and a cyclic permutation gives $r(t) = w^{-1}t^{-l_2}w^{-1}t^{-l_1}wt^{-l_4}v^{-1}t^{-l_3}$ and replacing w with w^{-1} and v with v^{-1} we have $wt^{-l_2}wt^{-l_1}w^{-1}t^{-l_4}vt^{-l_3}$ which with relabelling is of the form (a2) in Case II. $\square$

The main theorem that we prove is the following:

Theorem 8. *Let $r(t) = w_1t^{l_1}w_2t^{l_2}w_3t^{l_3}w_4t^{l_4}$ where $w_i = g_it^{m_{i,1}}\dots t^{m_{i,k_i-1}}g_it^{m_{i,k_i}}$ and assume that the following condition holds:*

(A1) *for $1 \leq i \leq k$, $|l_i|$ is distinct from $|l_j|$ when $i \neq j$ and is distinct from $m_{u,v}$ for any u, v .*

(A2) *Each w_i is cyclically reduced as a word in the free product.*

(A3) *No w_i is a proper subword of w_j for $i \neq j$.*

Then the following statements are true:

(I) *Let $r(t)$ be given by one of the 13 forms listed in Lemma 5 (modulo cyclic permutation and inversion). Then $r(t) = 1$ has a solution over G if one of the following holds:*

(1) *$r(t)$ is of the forms (a1) or (a3) or of the forms (b1)-(b3) in Case I.*

(2) *$r(t)$ is of the form (a2) in Case I and the following does not hold:*

(*) *$l_1 = l_2 + l_4$ and $l_1 = \pm ml_3$, $m \in \mathbb{N}$ and $m \geq 2$.*

(3) *$r(t)$ is of the forms (a1)-(a3) in Case II and $w^{\pm 1}v^2 \neq 1$ and $v^{\pm 1}w^2 \neq 1$.*

(4) *$r(t)$ is of the forms (a3) or (a4) in Case III.*

(5) *$r(t)$ is of the form (a1) in Case III and the following does not hold:*

(**) *$w = g_1t^{m_1}g_2t^{m_2}\dots g_{n-1}t^{m_{n-1}}g_1^{-1}$, $n > 2$ is even, $g_{\frac{n}{2}}^2 = 1$, $g_p = g_{n-p}^{-1}$, $1 \leq p < n$ and $m_p = m_{n-p-1}^{-1}$, $1 \leq p < n-1$.*

(6) *$r(t)$ is of the form (a2) in Case III and the condition (**) in (5) does not hold and none of the following sets of relations holds:*

(*) *$l_1 = l_2 + l_4$ and $l_4 = ml_3$, $m \in \mathbb{N}$ and $m \geq 2$.*

(**) *$l_2 = l_1 + l_3$ and $l_3 = ml_4$, $m \in \mathbb{N}$ and $m \geq 2$.*

(***) *$l_2 = l_1 + l_4$, $l_3 = m_1l_4$, $m_1 \in \{2, 3\}$ or $l_4 = m_2l_3$, $m_2 \in \{2, -3\}$.*

(****) *$l_1 = l_2 + l_3$ and $l_4 = m_1l_3$, $m_1 \in \{2, 3\}$ or $l_3 = m_2l_4$, $m_2 \in \{2, -3\}$.*

(II) *If $r(t)$ is not one of the forms listed in Lemma 5 then $r(t)$ has a solution over G .*

Remark 1. • Note that $r(t)$ has a solution under the restriction $6(*)$ if and only if it has a solution under the restriction $6(**)$. Similarly $r(t)$ has a solution under the restriction $6(***)$ if and only if it has a solution under the restriction $6(****)$. The symmetry is proved later in Lemma 6.

- In Theorem 8(II) since $r(t)$ is not one of the forms listed in Lemma 5 then $\exists i, j, i \neq j, 1 \leq i, j \leq 4$ such that $w_i \neq w_p^{\pm 1}, p \neq i, 1 \leq p \leq 4$ and $w_j \neq w_p^{\pm 1}, p \neq j$. This implies that when constructing D there will be no region of degree < 6 since neither w_i or w_j can be a piece. Thus $c(D) < 4\pi$.

Remark 2. In the light of Theorem 8, if $r(t)$ is given by one of the forms listed in Lemma 5 and provided that the conditions (A1), (A2), and (A3) given in Theorem 8 hold then the question of whether or not the equation $r(t) = 1$ has a solution remains open in the following cases:

- $r(t) = wt^{l_1}wt^{l_2}vt^{l_3}v^{-1}t^{l_4}, l_1 = l_2 + l_4$ and $l_1 = \pm ml_3, m \in \mathbb{N}, m \geq 2$.
- $r(t) = wt^{l_1}wt^{l_2}wt^{l_3}vt^{l_4}$, or $r(t) = wt^{l_1}wt^{l_2}w^{-1}t^{l_3}vt^{l_4}$, or $r(t) = wt^{l_1}w^{-1}t^{l_2}wt^{l_3}vt^{l_4}$ when $w^2v^{\pm 1} = 1$ or $v^2w^{\pm 1} = 1$.
- $r(t) = wt^{l_1}wt^{l_2}wt^{l_3}wt^{l_4}$ when $w = g_1t^{m_1}g_2t^{m_2}....g_{n-1}t^{m_{n-1}}g_1^{-1}, n > 2$ is even, $g_{\frac{n}{2}}^2 = 1, g_p = g_{n-p}^{-1}, 1 \leq p < n$ and $m_p = m_{n-p-1}^{-1}, 1 \leq p < n - 1$.
- $r(t) = wt^{l_1}wt^{l_2}wt^{l_3}w^{-1}t^{l_4}$ with one of the following holds:
 - $w = g_1t^{m_1}g_2t^{m_2}....g_{n-1}t^{m_{n-1}}g_1^{-1}, n > 2$ is even, $g_{\frac{n}{2}}^2 = 1, g_p = g_{n-p}^{-1}, 1 \leq p < n$ and $m_p = m_{n-p-1}^{-1}, 1 \leq p < n - 1$.
 - $l_1 = l_2 + l_4$ and $l_4 = ml_3, m \in \mathbb{N}$ and $m \geq 2$.
 - $l_2 = l_1 + l_3$ and $l_3 = ml_4, m \in \mathbb{N}$ and $m \geq 2$.
 - $l_2 = l_1 + l_4, l_3 = m_1l_4, m_1 \in \{2, 3\}$ or $l_4 = m_2l_3, m_2 \in \{2, -3\}$.
 - $l_1 = l_2 + l_3$ and $l_4 = m_1l_3, m_1 \in \{2, 3\}$ or $l_3 = m_2l_4, m_2 \in \{2, -3\}$.

In order to prove this theorem we need the following lemmas.

Lemma 6. The following relations are symmetric:

- Each of the pairs $l_1 = l_4 + l_2$ and $l_2 = l_1 + l_3$; and $l_2 = l_1 + l_4$ and $l_1 = l_2 + l_3$ when $r(t) = wt^{l_1}wt^{l_2}wt^{l_3}w^{-1}t^{l_4}$.
- Each of the pairs $l_1 = l_3 + l_4$ and $l_2 = l_3 + l_4$; $l_1 = l_4 + l_2$ and $l_2 = l_1 + l_3$;

- $l_2 = 2l_1$ and $l_1 = 2l_2$; and $l_2 = l_1 + l_4$ and $l_1 = l_2 + l_3$ when $r(t) = wt^{l_1}wt^{l_2}wt^{l_3}vt^{l_4}$.
- (iii) Each of the pairs $l_1 = l_2 + l_4$ and $l_3 = l_2 + l_4$; and $l_2 = l_1 + l_3$ and $l_4 = l_1 + l_3$ when $r(t) = wt^{l_1}wt^{l_2}vt^{l_3}vt^{l_4}$.
- (iv) $l_1 = 2l_3$, $l_2 = 2l_4$, $l_3 = 2l_1$ and $l_4 = 2l_2$ when $r(t) = wt^{l_1}vt^{l_2}wt^{l_3}vt^{l_4}$.
- (v) $l_3 = l_1 + l_4$, $l_1 = l_2 + l_3$, $l_4 = l_2 + l_3$ and $l_2 = l_1 + l_4$ when $r(t) = wt^{l_1}vt^{l_2}wt^{l_3}v^{-1}t^{l_4}$.
- (vi) $l_1 + l_2 + l_4 = 0$, $l_1 + l_2 + l_3 = 0$, $l_2 + l_3 + l_4 = 0$ and $l_1 + l_3 + l_4 = 0$ when $r(t) = wt^{l_1}vt^{l_2}w^{-1}t^{l_3}v^{-1}t^{l_4}$.

Proof. (i) Let $r(t) = wt^{l_1}wt^{l_2}wt^{l_3}w^{-1}t^{l_4}$ with $l_1 = l_4 + l_2$. Then without any loss we can replace w with w^{-1} and $r(t) = w^{-1}t^{l_1}w^{-1}t^{l_2}w^{-1}t^{l_3}wt^{l_4}$ taking the inverse gives $r(t) = t^{-l_4}w^{-1}t^{-l_3}wt^{-l_2}wt^{-l_1}w$ and cyclic permutation gives $r(t) = wt^{-l_2}wt^{-l_1}wt^{-l_4}w^{-1}t^{-l_3}$ which can be rewritten as $r(t) = wt^{l'_1}wt^{l'_2}wt^{l'_3}w^{-1}t^{l'_4}$ and in which $l'_2 = l'_1 + l'_3$ is equivalent to $l_1 = l_2 + l_4$ and $l'_1 = l'_2 + l'_3$ is equivalent to $l_2 = l_1 + l_4$.

(ii) Let $r(t) = wt^{l_1}wt^{l_2}wt^{l_3}vt^{l_4}$. Then writing w^{-1} in place of w and v^{-1} in place of v gives $r(t) = w^{-1}t^{l_1}w^{-1}t^{l_2}w^{-1}t^{l_3}v^{-1}t^{l_4}$. Taking the inverse yields $r(t) = t^{-l_4}vt^{-l_3}wt^{-l_2}wt^{-l_1}w$ and cyclically permutation this we has $r(t) = wt^{-l_2}wt^{-l_1}wt^{-l_4}vt^{-l_3}$ which can be rewritten as $r(t) = wt^{l'_1}wt^{l'_2}wt^{l'_3}vt^{l'_4}$ and in which all the pairs are equivalent.

(iii) Let $r(t) = wt^{l_1}wt^{l_2}vt^{l_3}vt^{l_4}$. Then without any loss we can replace w 's with v 's and this yields $r(t) = vt^{l_1}vt^{l_2}wt^{l_3}wt^{l_4}$. By cyclic permutation this gives $r(t) = wt^{l_3}wt^{l_4}vt^{l_1}vt^{l_2}$ which we rewrite as $r(t) = wt^{l'_1}wt^{l'_2}vt^{l'_3}vt^{l'_4}$ and in which the given relations are equivalent.

(iv) Let $r(t) = wt^{l_1}vt^{l_2}wt^{l_3}vt^{l_4}$. Then the pairs are symmetric by the following: first rewrite w 's as v 's and vice versa and we get $r(t) = vt^{l_1}wt^{l_2}vt^{l_3}wt^{l_4}$. Then a cyclic permutation gives $r(t) = wt^{l_2}vt^{l_3}wt^{l_4}vt^{l_1}$ in which the pairs $l_1 = 2l_3$ with $l_2 = 2l_4$, $l_2 = 2l_4$ with $l_3 = 2l_1$, $l_3 = 2l_1$ with $l_4 = 2l_2$, and $l_4 = 2l_2$ with $l_1 = 2l_3$.

(v) Let $r(t) = wt^{l_1}vt^{l_2}wt^{l_3}v^{-1}t^{l_4}$. Then without any loss we can write v^{-1} for v and this gives $r(t) = wt^{l_1}v^{-1}t^{l_2}wt^{l_3}vt^{l_4}$. By cyclic permutation we have $r(t) = wt^{l_3}vt^{l_4}wt^{l_1}v^{-1}t^{l_2}$ in which the pairs $l_3 = l_1 + l_4$ and $l_1 = l_2 + l_3$; and $l_4 = l_2 + l_3$ and $l_2 = l_1 + l_4$ are equivalent. Now consider $r(t) = wt^{l_1}vt^{l_2}wt^{l_3}v^{-1}t^{l_4}$. Then taking the inverse gives $r(t) = t^{-l_4}vt^{-l_3}w^{-1}t^{-l_2}v^{-1}t^{-l_1}w^{-1}$ and cyclic permutation yields $r(t) = w^{-1}t^{-l_2}v^{-1}t^{-l_1}w^{-1}t^{-l_4}vt^{-l_3}$. Now replacing v with v^{-1} and w with w^{-1} yields $r(t) = wt^{-l_2}vt^{-l_1}wt^{-l_4}v^{-1}t^{-l_3}$ in which the pairs $l_3 = l_1 + l_4$ and $l_4 = l_2 + l_3$; and $l_1 = l_2 + l_3$ and $l_2 = l_1 + l_4$ are equivalent.

(vi) Let $r(t) = wt^{l_1}vt^{l_2}w^{-1}t^{l_3}v^{-1}t^{l_4}$. Then the pairs $l_1+l_2+l_4 = 0$ and $l_2+l_3+l_4 = 0$; and $l_1 + l_2 + l_3 = 0$ and $l_1 + l_3 + l_4 = 0$ can be seen symmetric by the following: first we replace the w 's and the v 's by their inverses and we get $r(t) = w^{-1}t^{l_1}v^{-1}t^{l_2}wt^{l_3}vt^{l_4}$. Then a cyclic permutation gives the desired equation $r(t) = wt^{l_3}vt^{l_4}w^{-1}t^{l_1}v^{-1}t^{l_2}$. While to see the symmetry in the pairs $l_1 + l_2 + l_4 = 0$ and $l_1 + l_2 + l_3 = 0$; and $l_2 + l_3 + l_4 = 0$ and $l_1 + l_3 + l_4 = 0$ we first take the inverse and this gives $r(t) = t^{-l_4}vt^{-l_3}wt^{-l_2}v^{-1}t^{-l_1}w^{-1}$. Then a cyclic permutation gives $r(t) = wt^{-l_2}v^{-1}t^{-l_1}w^{-1}t^{-l_4}vt^{-l_3}$ and replacing v with v^{-1} yields the desired equation $r(t) = wt^{-l_2}vt^{-l_1}w^{-1}t^{-l_4}v^{-1}t^{-l_3}$. $\square$

The following Lemma is important when considering possibilities of labellings of positive regions in D .

Lemma 7. *The following cannot be a interior vertex label in D :*

- (i) w^p or v^p , $p \geq 1$.
- (ii) w^2w_1 or v^2v_1 where w_1 and v_1 are subwords of $w^{\pm 1}$ and $v^{\pm 1}$ respectively.

Proof. (i) If $p = 1$ then $w = 1$ a contradiction to assumption. If $p \geq 2$ then the statement is true because w is a cyclically reduced word by condition (A2) and so $w^p \neq 1$. In particular w^2 and v^2 can never be a labelling in D .

(ii) $w^2w_1 \neq 1$ when $w_1 = w^{\pm 1}$ and this follows by (i). If w_1 is a proper subword of $w^{\pm 1}$ and if $w \neq gt^mg^{-1}$ then $w^2w_1 = 1$ implies that $2(l(w)) = l(w_1) < l(w)$ which is a contradiction, where $l(w)$ is the free product length, since there is no cancelation in the t terms. While if $w = gt^mg^{-1}$ then $w^2 = gt^{2m}g^{-1} = w_1^{-1}$ which is a contradiction since $w_1 \in \{g^{\pm 1}, g^{\pm 1}t^{\pm \epsilon}\}$. Similarly v^2v_1 can not be a vertex label in D . $\square$

It is possible however to have vv_1v_2 where v_1 and v_2 are proper subwords of $v^{\pm 1}$ and ww_1w_2 where w_1 and w_2 are proper subwords of $w^{\pm 1}$

Lemma 8. *Let $w = g_1t^{m_1}.....t^{m_{n-1}}g_n$, $w \neq gt^{m_1}g^{-1}$ be a cyclically reduced word and assume that the edge (i, j) has the labelling shown in Figure 1.4(i) where w_1 is a subword of w . If i, j and j' are in $\{l_1, l_2, l_3, l_4\}$ then there must be a split on the edge (i, j) and the split occurs in the last g - or t -vertex before the right-hand end of the edge (i, j) .*

Proof. since i, j and j' are in $\{l_i, 1 \leq i \leq 4\}$ then there is a split since $w^2 \neq 1$ and $w_1 \neq w^{-1}$. Assume that the split does not occur in the last g - or t -vertex before

the end of the edge and assume that the segment from the split u to the end of the edge is labeled by θ . Note that since $j' \in \{l_1, l_2, l_3, l_4\}$ then $w = w_1 w'_1 = \theta^{-1} w''_1$ and since $j \in \{l_1, l_2, l_3, l_4\}$ then $w = w_2 \theta$, where w'_1, w''_1, w_2 are subwords of w . So $w = \theta^{-1} w''_1 = w_2 \theta$ is not cyclically reduced, a contradiction. $\square$

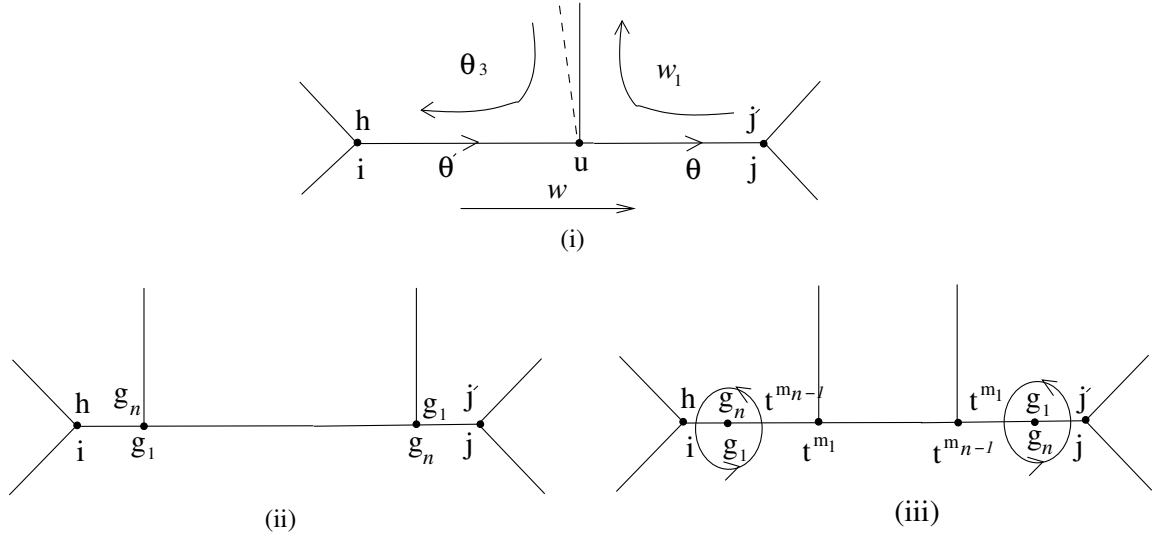


Figure 1.4: Lemma8

Lemma 9. Let w_1, θ and w be as in Lemma 8 and let $i, j, j' \in \{l_i, 1 \leq i \leq 4\}$ in Figure 1.4(i). If θ_3 in Figure 1.4(i) is also a subword of w , or a subword of w^{-1} provided that $h \notin \{l_1, l_2, l_3, l_4\}$, then either there is another split on the edge (i, j) as shown in Figure 1.4(ii)-(iii) or w has one of the following forms:

- (i) $w = (g_1 t^{m_1})^\beta g_1^{-1}$, $\beta \geq 1$, $g_1^2 = 1$.
- (ii) $w = g_1 t^{m_1} g_2 t^{m_2} \dots g_{n-1} t^{m_{n-1}} g_1^{-1}$, $n > 2$ is even, $g_{\frac{n}{2}}^2 = 1$, $g_p = g_{n-p}^{-1}$, $1 \leq p < n$ and $m_p = m_{n-p-1}^{-1}$, $1 \leq p < n-1$

Proof. Let θ_3 be a subword of the cyclically reduced element w . If $h \in \{l_1, l_2, l_3, l_4\}$ and $w \neq g_1 t^{m_1} g_1^{-1}$ then by the previous lemma there is another split on the last g - or t -vertex as shown in Figure 1.4(ii)-(iii). Now assume h is a τ -power. First let u be a g -vertex or a t -vertex that does not involve l_i and let θ' be the label of the segment from the left end of the edge to u . If the only split occurs in the last g -vertex then $l(\theta_3) \geq l(\theta') + 1 = l(g_1 t^{m_1} \dots t^{m_{n-1}}) + 1 = l(w) + 1$ which is a contradiction since $l(\theta_3)$ is a subword of w . If the only split occurs

in the last t -vertex then $l(\theta_3) \geq l(\theta') + 2 = l(g_1 t^{m_1} \dots g_{m_{n-1}}) + 2 = l(w) + 1$ which is again a contradiction. Now let u be a t -vertex that involves i' where $i' \in \{l_1, l_2, l_3, l_4\}$. Then it can be assumed that u is given by Figure 1.5(i) otherwise the above argument still applies. Then $w = g_1 t^{m_1} g_2 t^{m_2} \dots g_{n-1} t^{m_{n-1}} g_1^{-1}$ and the possible orientations are given in Figures 1.5(ii)-(iii).

In Figure 1.5(ii) $g_1 = g_2 = g_3 = \dots g_{n-1} = g_1^{-1}$ and $m_1 = m_2 = \dots = m_{n-1}$ and

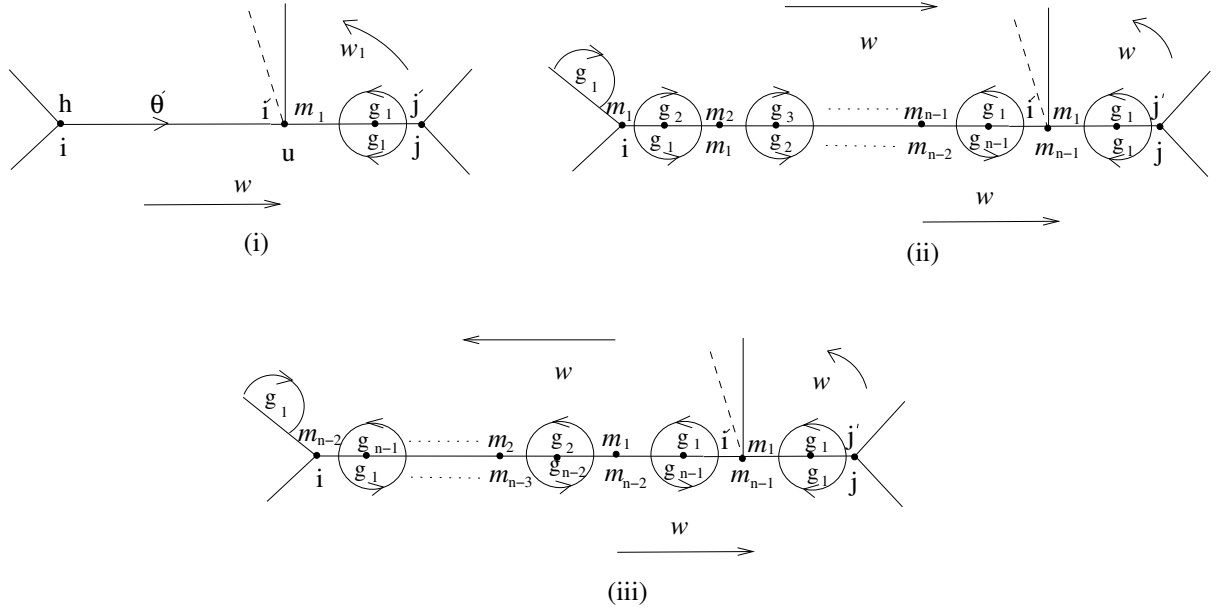


Figure 1.5: Lemma 9

so $w = (g_1 t^{m_1})^\beta g_1^{-1}$, $\beta \geq 1$ and $g_1^2 = 1$. In Figure 1.5(iii) $g_1 = g_{n-1}^{-1}$, $g_2 = g_{n-2}^{-1}$, ... and $m_{n-2} = m_1^{-1}$, $m_{n-3} = m_2^{-1}$ If n is odd then this yields that $m_{\frac{n+1}{2}} = -m_{\frac{n+1}{2}}$ which is a contradiction. If n is even and $n = 2$ then $w = g_1 t^{m_1} g_1^{-1}$ with $g_1^2 = 1$ which is the form in (i) of this Lemma. If n is even and $n > 2$ then this yields that $g_{\frac{n}{2}} = g_{\frac{n}{2}}^{-1}$ and $g_p = g_{n-p}^{-1}$, $1 \leq p < n$ and $m_p = m_{n-p-1}^{-1}$, $1 \leq p < n - 1$ which is the second form in the statement of the lemma. $\square$

For example $w = g_1 t^{m_1} g_2 t^{m_2} g_3 t^{-m_2} g_2^{-1} t^{-m_1} g_1^{-1} t^{m_5} g_1^{-1}$, $g_3^2 = 1$ is a cyclically reduced word in the form (ii) described in Lemma 9, see Figure 1.6.

Remark 3. Let w be a cyclically reduced word that is not one of the forms (i)-(ii) in Lemma 9 and $i, j \in \{l_1, l_2, l_3, l_4\}$ with orientation as shown in Figure 1.7 in which only $w^{\pm 1}$ is involved in the labelling of u . If u is the only split occurring

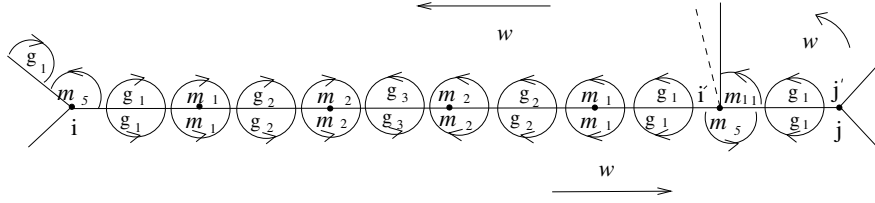


Figure 1.6: Example

on the edge (i, j) then both h and h' must be τ -powers since otherwise there is another split by Lemma 9.

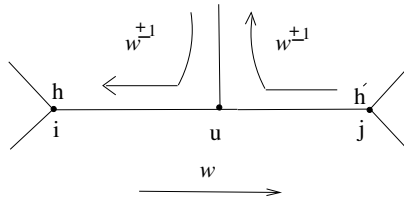


Figure 1.7: Exactly one split

Lemma 10. If $r(t) = w_1 t^{l_1} \dots t^{l_{k-1}} w_{k-1} t^{l_k}$, $k \geq 1$ and $w_i = c_i t^{p_i} c_i^{-1}$, $1 \leq i \leq k-1$ where $c_i = g_{i,1} t^{m_{i,1}} \dots g_{i,q}$, $q \geq 1$, that is w_i is not strongly cyclically reduced with its core a power of t . Then $r(t)$ has a solution over G .

Proof. The proof is immediate as $t = 1$ implies that $w_i = c_i c_i^{-1} = 1$ and so it is a solution for such equation. In particular if $w = (g_1 t^{m_1})^\beta g_1^{-1}$ and $\beta = 1$ then $t = 1$ is a solution for case III for $r(t) = w_1 t^{l_1} w_2 t^{l_2} w_3 t^{l_3} w_4 t^{l_4}$ in our main theorem. $\square$

Lemma 11. If $w = (g_1 t^{m_1})^\beta g_1^{-1}$, $g_1^2 = 1$ and $\beta \geq 1$ then $t = 1$ is a solution for case III of $r(t) = w_1 t^{l_1} w_2 t^{l_2} w_3 t^{l_3} w_4 t^{l_4}$ in our main theorem.

Proof. If $\beta = 1$ then $t = 1$ is a solution as in the above lemma; and $t = 1$ is a solution for (a3) and (a4). But $t = 1$ is a solution for (a1) as this implies $r(1) = w w w w = ((g_1)^\beta g_1^{-1})^4 = (g_1)^{4\beta-4} = ((g_1)^2)^{2\beta-2} = 1$. Moreover $t = 1$ is a solution for (a2) as this implies $r(1) = w w w w^{-1} = w^2 = ((g_1)^\beta g_1^{-1})^2 = (g_1)^{2\beta-2} = ((g_1)^2)^{\beta-1} = 1$. $\square$

In the next chapters we will prove our main Theorem. In Chapter 2 we study case I, in Chapter 3 we study case II and in Chapter 4 we study case III.

Chapter 2

Case I:

Up to cyclic permutation and inversion $r(t)$ is one of the following subcases:

- | | |
|---|---|
| (a1) $wt^{l_1}wt^{l_2}vt^{l_3}vt^{l_4}$ | (b1) $wt^{l_1}vt^{l_2}wt^{l_3}vt^{l_4}$ |
| (a2) $wt^{l_1}wt^{l_2}vt^{l_3}v^{-1}t^{l_4}$, | (b2) $wt^{l_1}vt^{l_2}wt^{l_3}v^{-1}t^{l_4}$ |
| (a3) $wt^{l_1}w^{-1}t^{l_2}vt^{l_3}v^{-1}t^{l_4}$ | (b3) $wt^{l_1}vt^{l_2}w^{-1}t^{l_3}v^{-1}t^{l_4}$ |

Observe that $t = 1$ is a solution for case I(a3). We consider the rest case by case as follows.

2.1 Case I (a1)

First assume that $d(u_i) = 2$, $1 \leq i \leq 4$. Figure 2.1 shows the only possible labelling in this case.

Then checking shows that $d(v_i) = d(v_j) = 3$, $i \neq j$ yields a contradiction to

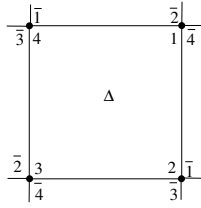


Figure 2.1: All possible labellings case I (a1).

(A3) (it is sometimes referred to by labelling contradiction), so at most we have $d(v_i) = 3$ for one of $1 \leq i \leq 4$. If $d(v_1) = 3$ then $l(v_1) = \bar{2}1\bar{4}$, see Figure 2.2(i). If $d(v_2) = 3$ then $l(v_2) = \bar{1}2\bar{3}$, see Figure 2.2(ii). If $d(v_3) = 3$ then $l(v_3) = \bar{4}3\bar{2}$, see

Figure 2.2(iii). If $d(\mathbf{v}_4) = 3$ then $l(\mathbf{v}_4) = \bar{3}4\bar{1}$, see Figure 2.2(iv). Note that when $d(\mathbf{v}_i) = 3$, $1 \leq i \leq 4$ then this yields a split as shown.

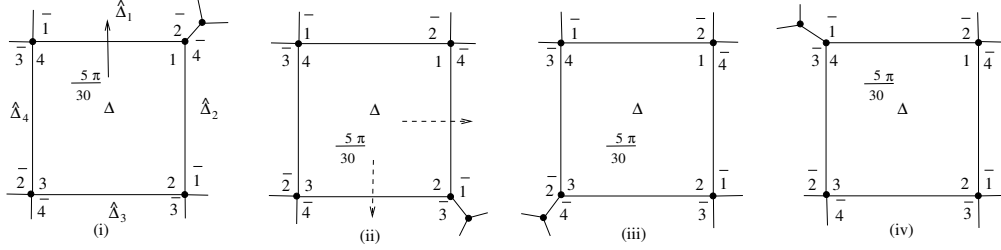


Figure 2.2: $d(\mathbf{v}_1) \geq 4$.

Now assume that $d(u_i) \geq 3$ for exactly one $1 \leq i \leq 4$. Let $d(u_1) \geq 3$ then either $d(\mathbf{v}_2) \geq 4$ or $d(\mathbf{v}_3) \geq 4$. If $d(u_1) = d(\mathbf{v}_1) = d(\mathbf{v}_3) = d(\mathbf{v}_4) = 3$ then $l(\mathbf{v}_1) \in \bar{1}4\{1, 2, \bar{3}, \bar{4}, k\}$ and $l(\mathbf{v}_3) = l(\mathbf{v}_4) = \bar{4}3\bar{2}$, see Figure 2.3(i). If $d(u_1) = d(\mathbf{v}_1) = d(\mathbf{v}_2) = d(\mathbf{v}_4) = 3$ then $l(\mathbf{v}_2) = \bar{1}2\bar{3}$ and either $l(\mathbf{v}_1) = \bar{1}4k$, $l(\mathbf{v}_4) \in \bar{3}4\{1, 4, \bar{2}, \bar{3}, k'\}$; or $l(\mathbf{v}_1) = \bar{1}4\bar{1}$, $l(\mathbf{v}_4) \in \bar{3}4\{1, 4, \bar{2}, k'\}$; or $l(\mathbf{v}_1) = \bar{1}4\bar{2}$, $l(\mathbf{v}_4) \in \bar{3}4\{1, 4, \bar{3}, k'\}$; or $l(\mathbf{v}_1) = \bar{1}4\bar{3}$, $l(\mathbf{v}_4) = \bar{3}4\{4, \bar{2}, \bar{3}, k'\}$; or $l(\mathbf{v}_1) = \bar{1}4\bar{4}$, $l(\mathbf{v}_4) \in \bar{3}4\{1, \bar{2}, \bar{3}, k'\}$ and each choice of $l(\mathbf{v}_4)$ yields a split, see Figure 2.3(ii).

Let $d(u_2) \geq 3$ then either $d(\mathbf{v}_3) \geq 4$ or $d(\mathbf{v}_4) \geq 4$. If $d(u_2) = d(\mathbf{v}_1) = d(\mathbf{v}_2) = d(\mathbf{v}_4) = 3$ then $l(\mathbf{v}_4) = \bar{4}\bar{1}\bar{3}$ and either $l(\mathbf{v}_1) = \bar{1}2\bar{1}$ and $l(\mathbf{v}_2) \in \bar{2}\bar{3}\{1, 2, \bar{4}, k'\}$; or $l(\mathbf{v}_1) \in \bar{1}2\{4, \bar{3}\}$ and $l(\mathbf{v}_2) \in \bar{2}\bar{3}\{2, \bar{3}, k'\}$; or $l(\mathbf{v}_1) = \bar{1}2\bar{2}$ and $l(\mathbf{v}_2) \in \bar{2}\bar{3}\{1, \bar{3}, \bar{4}, k'\}$; or $l(\mathbf{v}_1) = \bar{1}2k$ and $l(\mathbf{v}_2) \in \bar{2}\bar{3}\{1, 2, \bar{3}, \bar{4}, k'\}$, see Figure 2.3(iii). If $d(u_2) = d(\mathbf{v}_1) = d(\mathbf{v}_2) = d(\mathbf{v}_3) = 3$ then $l(\mathbf{v}_1) \in \bar{2}\bar{1}\{1, 4, \bar{2}, \bar{3}, k\}$ and $l(\mathbf{v}_2) = l(\mathbf{v}_3) = \bar{2}\bar{3}4$, see Figure 2.3(iv).

Let $d(u_3) \geq 3$ then either $d(\mathbf{v}_1) \geq 4$ or $d(\mathbf{v}_4) \geq 4$. If $d(u_3) = d(\mathbf{v}_2) = d(\mathbf{v}_3) = d(\mathbf{v}_4) = 3$ then $l(\mathbf{v}_4) = \bar{4}\bar{1}\bar{3}$ and either $l(\mathbf{v}_2) = \bar{1}2k$ and $l(\mathbf{v}_3) \in \bar{3}\bar{2}\{3, 4, \bar{1}, \bar{2}, k'\}$; or $l(\mathbf{v}_2) = \bar{1}2\bar{2}$ and $l(\mathbf{v}_3) \in \bar{3}\bar{2}\{3, 4, \bar{1}, k'\}$; or $l(\mathbf{v}_2) = \bar{1}2\bar{3}$ and $l(\mathbf{v}_3) \in \bar{3}\bar{2}\{3, 4, \bar{2}, k'\}$; or $l(\mathbf{v}_2) = \bar{1}2\bar{4}$ and $l(\mathbf{v}_3) \in \bar{3}\bar{2}\{3, \bar{1}, \bar{2}, k'\}$; or $l(\mathbf{v}_2) = \bar{1}2\bar{1}$ and $l(\mathbf{v}_3) \in \bar{3}\bar{2}\{4, \bar{1}, \bar{2}, k'\}$, see Figure 2.3(v). If $d(u_3) = d(\mathbf{v}_1) = d(\mathbf{v}_2) = d(\mathbf{v}_3) = 3$ then $l(\mathbf{v}_1) = l(\mathbf{v}_2) = \bar{1}24$ and $l(\mathbf{v}_3) \in \bar{3}\bar{2}\{3, 4, \bar{1}, \bar{2}, k'\}$, see Figure 2.3(vi).

Finally let $d(u_4) \geq 3$ then either $d(\mathbf{v}_1) \geq 4$ or $d(\mathbf{v}_2) \geq 4$. If $d(u_4) = d(\mathbf{v}_1) = d(\mathbf{v}_3) = d(\mathbf{v}_4) = 3$ then $l(\mathbf{v}_1) = \bar{2}1\bar{4}$ and $l(\mathbf{v}_4) = \bar{4}\bar{1}2$ and $l(\mathbf{v}_3) \in \bar{4}3\{2, 3, \bar{1}, \bar{4}, k'\}$, see Figure 2.3(vii). If $d(u_4) = d(\mathbf{v}_2) = d(\mathbf{v}_3) = d(\mathbf{v}_4) = 3$ then $l(\mathbf{v}_2) = \bar{1}2\bar{3}$ and either $l(\mathbf{v}_3) = \bar{4}33$ and $l(\mathbf{v}_4) \in \bar{4}\bar{1}\{4, \bar{2}, k'\}$; or $l(\mathbf{v}_3) = \bar{4}32$ and $l(\mathbf{v}_4) \in$

$4\bar{1}\{3, 4, \bar{1}, k'\}$; or $l(v_3) = \bar{4}3\bar{1}$ and $l(v_4) \in 4\bar{1}\{3, 4, \bar{1}, \bar{2}, k'\}$; or $l(v_3) = \bar{4}3\bar{4}$ and $l(v_4) \in 4\bar{1}\{3, \bar{1}, \bar{2}, k'\}$; or $l(v_3) = \bar{4}3k$ and $l(v_4) \in 4\bar{1}\{3, 4, \bar{1}, \bar{2}, k'\}$ and each choice of $l(v_4)$ yields a split, see Figure 2.3(viii).

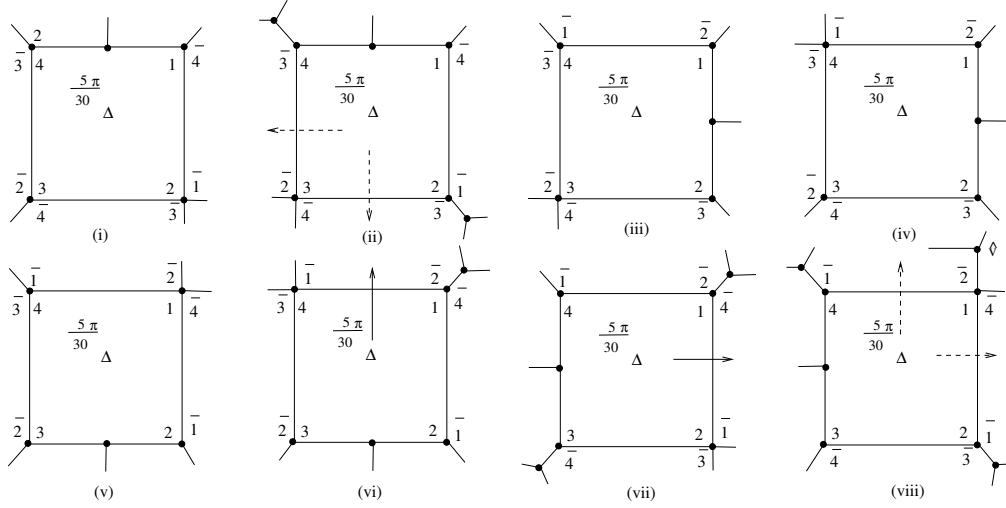


Figure 2.3: $d(u_i) \geq 3$.

Note that Figures with $l(v_1) = \bar{2}1\bar{4}$ are symmetric to Figures with $l(v_3) = \bar{4}3\bar{2}$ and Figures with $l(v_2) = \bar{1}2\bar{3}$ are symmetric to Figures with $l(v_3) = \bar{3}4\bar{1}$ by Lemma 6(iii). Moreover none of these labellings can coincide and each positive region has exactly one of them. Thus it is enough to consider regions with $l(v_1) = \bar{2}1\bar{4}$ and $l(v_2) = \bar{1}2\bar{3}$ and the argument applies to other regions by symmetry. The distribution is shown case by case.

Δ is given by Figure 2.2(i):

Here $d(v_4) \in \{4, 5\}$ since otherwise $c(\Delta) \leq c(3, 4, 4, 6) = 0$. Then checking the labelling shows that $l(v_4)$ forces the edge $(4, 1)$ in $\hat{\Delta}_1$ to split otherwise $l(v_4)$ will have the sublabel $\bar{3}4\bar{1}2$ which contradicts $l(v_1) = \bar{2}1\bar{4}$. Thus $c(\hat{\Delta}_1) \leq c(3, 3, 3, 3, 3, 4) = \frac{-\pi}{6}$. Add $c(\Delta) \leq c(3, 4, 4, 4) = \frac{\pi}{6}$ to $c(\hat{\Delta}_1)$.

Δ is given by Figure 2.2(ii):

Let $d(v_3) = 4$. Then this forces the edge $(4, 1)$ in $\hat{\Delta}_3$ to split otherwise $l(v_3) = \bar{4}3\bar{2}1$ which yields a contradiction with $l(v_2) = \bar{1}2\bar{3}$ and hence $c(\hat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 4) = \frac{-\pi}{6}$. Add $c(\Delta) \leq c(3, 4, 4, 4) = \frac{\pi}{6}$ to $c(\hat{\Delta}_3)$. Now let $d(v_3) \geq 5$.

Then $d(v_1) = 4$ and $c(\Delta) \leq c(3, 4, 4, 5) = \frac{\pi}{15}$ and this forces the edge $(3, 4)$ in $\widehat{\Delta}_2$ to split otherwise $l(v_1) = \bar{2}1\bar{4}3$ which yields a labelling contradiction. Thus $c(\widehat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 4) = \frac{-\pi}{6}$. Add $c(\Delta) \leq \frac{\pi}{15}$ to $c(\widehat{\Delta}_2)$.

Δ is given by Figure 2.3(ii):

If $d(v_3) = 4$ then this forces the edge $(4, 1)$ in $\widehat{\Delta}_3$ to split otherwise there is a labelling contradiction and we add $c(\Delta) \leq c(3, 3, 3, 3, 4) = \frac{\pi}{6}$ to $c(\widehat{\Delta}_3) \leq c(3, 3, 3, 3, 4) = \frac{-\pi}{6}$. If $d(v_3) = 5$ then this also forces the edge $(4, 1)$ in $\widehat{\Delta}_3$ to split except for $l(v_3) = \bar{4}3\bar{2}41$ and $c(\Delta) \leq c(3, 3, 3, 3, 5) = \frac{\pi}{15}$. In the former case we add $c(\Delta)$ to $c(\widehat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$. In the latter case $l(v_3) = \bar{4}3\bar{2}41$ forces the edge $(1, 2)$ in $\widehat{\Delta}_4$ to split and we add $c(\Delta)$ to $c(\widehat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$.

Δ is given by Figure 2.3(vi):

Here $d(v_4) \in \{4, 5\}$ forces the edge $(4, 1)$ in $\widehat{\Delta}_1$ to split otherwise $l(v_4)$ yields a labelling contradiction. Thus $c(\widehat{\Delta}_1) \leq c(3, 3, 3, 3, 3, 4) = \frac{-\pi}{6}$. Add $c(\Delta) \leq c(3, 4, 4, 4) = \frac{\pi}{6}$ to $c(\widehat{\Delta}_1)$.

Δ is given by Figure 2.3(vii):

Here $d(v_2) \in \{4, 5\}$ forces the edge $(1, 2)$ in $\widehat{\Delta}_2$ to split otherwise $l(v_2)$ yields a labelling contradiction. Thus $c(\widehat{\Delta}_1) \leq c(3, 3, 3, 3, 3, 4) = \frac{-\pi}{6}$. Add $c(\Delta) \leq c(3, 4, 4, 4) = \frac{\pi}{6}$ to $c(\widehat{\Delta}_1)$.

Δ is given by Figure 2.3(viii):

Let $d(v_1) = 4$. Then this forces the edge $(4, 1)$ in $\widehat{\Delta}_2$ to split otherwise there is a labelling contradiction and we add $c(\Delta) \leq c(3, 3, 3, 3, 4) = \frac{\pi}{6}$ to $c(\widehat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 4) = \frac{-\pi}{6}$. Now let $d(v_4) = 5$. Then this also forces the edge $(4, 1)$ in $\widehat{\Delta}_3$ to split except for $l(v_4) = \bar{2}1\bar{4}34$ and $c(\Delta) \leq c(3, 3, 3, 3, 5) = \frac{\pi}{15}$. In the former case we add $c(\Delta)$ to $c(\widehat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$. In the latter case $l(v_4) = \bar{2}1\bar{4}34$ forces $d(\widehat{v}) \geq 4$ and we add $c(\Delta)$ to $c(\widehat{\Delta}_1) \leq c(3, 3, 3, 4, 5) = \frac{-\pi}{10}$.

Assume now that $\widehat{\Delta} \in \{\widehat{\Delta}_1, \widehat{\Delta}_2, \widehat{\Delta}_3, \widehat{\Delta}_4\}$. If $\widehat{\Delta}$ is receiving from exactly one region then we see from above that $c^*(\widehat{\Delta}) \leq 0$. Moreover checking all possibilities shows that (up to symmetry) only two regions may send their positive curvature to the same $\widehat{\Delta}$, namely Δ to $\widehat{\Delta}_4$ in Figure 2.3(ii) with Δ to $\widehat{\Delta}_2$ in Figure 2.3(viii) may coincide. Other regions cannot be placed on the same region as

for example Δ to $\hat{\Delta}_1$ in Figure 2.2(i) cannot coincide with Δ to $\hat{\Delta}_1$ in Figure 2.3(vi) as they use the same edge, it also cannot coincide with Δ to $\hat{\Delta}_2$ in Figure 2.3(vii) as in the last mentioned figure the edge (1,2) splits. However when the two regions coincide $\hat{\Delta}$ is forced to have at least 2 vertices of degree ≥ 4 and 2 edges that have at least one split and receives at most $\frac{\pi}{6}$ from each region and thus $c^*(\hat{\Delta}) \leq c(3, 3, 3, 3, 4, 4) + 2(\frac{\pi}{6}) = 0$.

2.2 Case I (a2)

Figure 2.4 shows the possible labellings in this case.

First assume that $l_1 - l_2 - l_4 \neq 0$. If $d(u_i) = 2$ for $1 \leq i \leq 4$ then $c(\Delta) \leq$

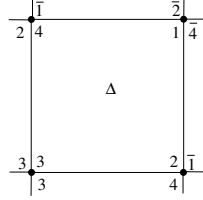


Figure 2.4: All possible labellings case I (a2).

$c(4, 4, 4, 4) = 0$. Now assume that exactly one $d(u_i) > 2$. If $d(u_1) > 2$ then $d(v_2) \geq 4$ and $d(v_3) \geq 4$; If $d(u_2) > 2$ then $d(v_3) \geq 4$ and $d(v_4) \geq 4$; If $d(u_3) > 2$ then $d(v_4) \geq 4$ and $d(v_1) \geq 4$; If $d(u_4) > 2$ then $d(v_1) \geq 4$ and $d(v_2) \geq 4$. In each case, $c(\Delta) \leq c(3, 3, 3, 4, 4) = 0$.

Next assume that $l_1 - l_2 - l_4 = 0$. In this case we do further amendment to D by deleting all vertices of degree 2 labelled by $(vv^{-1})^{\pm 1}$ and all edges incident to such vertices, see Figure 2.5(i). This introduces the following corner labels: $(24)^{\pm 1}$ and $(33\dots 3)^{\pm 1}$ (m times for some $m \geq 2$), which will be denoted by $\tilde{3}^{\pm 1}$. Applying these changes a new type of region Δ has been introduced. Figure 2.5(ii) shows an example of such regions. Observe that since $\tilde{3}^{\pm 1}$ cannot be a vertex label then for a maximal m , $d(y_2) > 2$ and $d(y_4) > 2$. Observe also that Δ has $4m + 4$ vertices. We further assume that $l_1 \neq \pm ml_3$, $m \in \mathbb{N}, m \geq 2$.

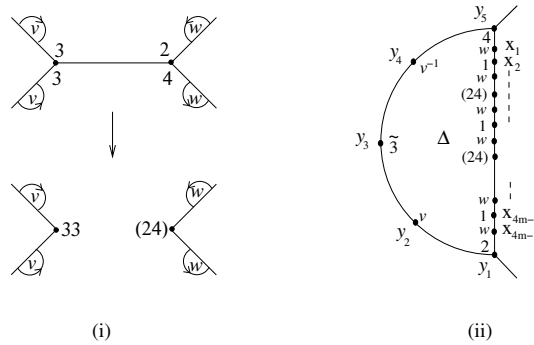


Figure 2.5

Lemma 12. (i) No vertex of (the amended) D has a sublabel 24.

(ii) No vertex of D has a sublabel $4\bar{1}2$.

(iii) If $l_2 = \pm ml_3$, $m \in \mathbb{N}$, then $l_4 \neq \pm m'l_3$, $m' \in \mathbb{N}$.

(iv) $d(y_2) \geq 3$ and $d(y_4) \geq 3$ for a maximal m .

Proof. (i) Every sublabel of the form 24 is changed by the amendment in Figure 2.5(i) and this transforms 24 to (24) .

(ii) If D has a sublabel $4\bar{1}2$ then using a bridge move if necessary, it can be assumed that $4\bar{1}2$ is a vertex label in D in which case the previous part of this lemma applies.

(iii) Letting $l_2 = \pm ml_3$, $m \in \mathbb{N}$ and $l_4 = \pm m'l_3$, $m' \in \mathbb{N}$ implies $l_1 = l_2 + l_4 = \pm m''l_3$ which is a contradiction to assumption.

(iv) $d(y_2) \geq 3$ and $d(y_4) \geq 3$ by the maximality of m . $\square$

Remark 4. Observe that some labelling cannot occur since they yield the exceptional case $l_1 = \pm ml_3$, $m \in \mathbb{N}$, $m \geq 2$. For example $1\tilde{3}^\pm, (24)\tilde{3}^\pm$. So from here on contradiction means either a labelling contradiction or a contradiction to the assumption $l_1 \neq \pm ml_3$.

Let Δ be a region in the amended diagram and assume $\tilde{3}$ is maximum. Assume now that $d(x_1) \geq 3$. If $d(x_2) = 2$ then $l(x_2) = 1\overline{(24)}$ otherwise there is a contradiction. But $l(x_2) = 1\overline{(24)}$ implies that $l(x_1)$ involves ww^{-1} as a proper subword which is a contradiction and so $d(x_2) \geq 3$, see Figure 2.6(i).

Similarly assuming $d(x_{4m-1}) \geq 3$ with $d(x_{4m-2}) = 2$ forces $l(x_{4m-2}) = 1\overline{(24)}$ otherwise there is a contradiction. but this forces $l(x_{4m-1})$ to have ww^{-1} as a proper subword which is a contradiction and so $d(x_{4m-2}) \geq 3$, see Figure 2.6(ii).

Assume now that $d(x_i) \geq 3$, $i \equiv 2 \pmod{4}$ for any $2 \leq i \leq 4m - 2$. If in addition



Figure 2.6: $d(x_i) \geq 3$, $i \in \{1, 4m - 1\}$.

$d(x_{i-1}) = 2$ then either $d(x_i) \geq 4$ as in Figure 2.7(i) or $l(x_i) = \bar{2}1\tilde{3}^\pm$ which in turn

forces $d(x_{i+1}) \geq 3$, see Figure 2.7(ii). While if $d(x_{i+1}) = 2$ then either $d(x_i) \geq 4$ as in Figure 2.7(i) or $l(x_i) = 1\bar{4}\tilde{3}^\pm$ which in turn forces $d(x_{i-1}) \geq 3$, see Figure 2.7(iii).

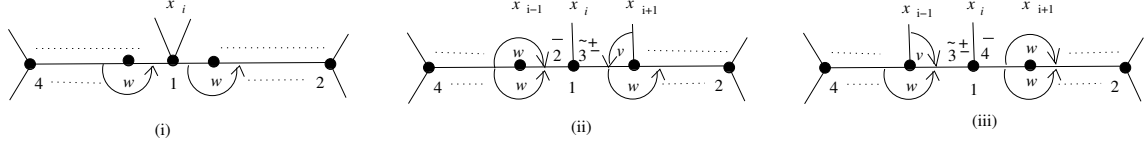


Figure 2.7: $d(x_i) \geq 3$, $i \equiv 2 \pmod{4}$.

Assume now that $m \geq 2$ and $d(x_i) \geq 3$, $i \equiv 0 \pmod{4}$ for any $4 \leq i \leq 4m - 4$. If in addition $d(x_{i-1}) = 2$ then $l(x_i) = \bar{1}(24)\omega$ which forces $d(x_i) \geq 4$ as in Figure 2.8(i). While if $d(x_{i+1}) = 2$ then either $d(x_i) \geq 4$ as in Figure 2.7(i) or $l(x_i) = \tilde{3}^\pm(24)\bar{2}$ which in turn forces $d(x_{i-1}) \geq 3$, see Figure 2.8(ii).



Figure 2.8: $d(x_i) \geq 3$, $i \equiv 0 \pmod{4}$.

Finally assume that $m \geq 2$ and $d(x_i) \geq 3$, $i \equiv 1 \pmod{2}$ for any $3 \leq i \leq 4m - 3$. Then $d(x_{i-1}) \geq 3$ and $d(x_{i+1}) \geq 3$ since each of $1\bar{3}^\pm$ and $(24)\tilde{3}^\pm$ yields a contradiction to the assumption and $1\overline{(24)}$ and $(24)\bar{1}$ forces the proper sublabel ww^{-1} which is a contradiction, see Figure 2.9.

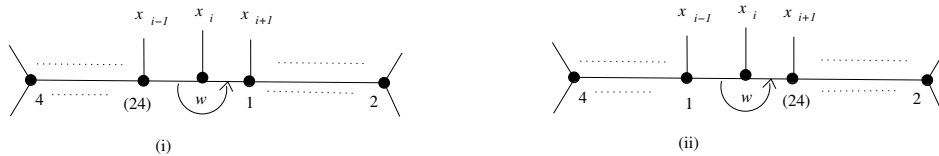


Figure 2.9: $d(x_i) \geq 3$, $i \equiv 1 \pmod{2}$.

Lemma 13. (i) $d(x_2) = 2$ forces $d(x_1) = 2$ and either $d(y_5) \geq 4$ or $l(y_5) = 4\bar{1}\tilde{3}^\pm$.
(ii) $d(x_{4m-1}) = 2$ forces either $d(y_1) \geq 4$ or $l(y_1) = \bar{1}2\tilde{3}^\pm$.
(iii) $d(x_2) = d(x_{4m-1}) = 2$ with $d(y_5) = d(y_1) = 3$ yields a contradiction.

Proof. (i) Letting $d(x_2) = 2$ forces $l(x_2) = 1(\overline{24})$ which in turn forces $d(x_1) = 2$ and this implies $l(y_5) = 4\bar{1}\omega$. If in addition $d(y_5) = 3$ then $l(y_5) = 4\bar{1}\tilde{3}^\pm$ otherwise there is a labelling contradiction.

(ii) Letting $d(x_{4m-1}) = 2$ forces $l(y_1) = \bar{1}2\omega$. If in addition $d(y_1) = 3$ then $l(y_1) = \bar{1}2\tilde{3}^\pm$ otherwise there is a labelling contradiction.

(iii) $d(x_2) = d(x_{4m-1}) = 2$ with $d(y_5) = d(y_1) = 3$ forces $l(y_5) = 4\bar{1}\tilde{3}^\pm$ and $l(y_1) = \bar{1}2\tilde{3}^\pm$ by (i)-(ii) in this lemma. These labelling however implies that $l_2 = \pm ml_3$, $m \in \mathbb{N}$ and $l_4 = \pm m'l_3$, $m' \in \mathbb{N}$ which is a contradiction by Lemma 12(iii). $\square$

Assume now that Δ is a region in the amended diagram with $\tilde{3}$ is maximum, $m \geq 2$ and assume $c(\Delta) > 0$. Then $d(y_2) \geq 3$ and $d(y_4) \geq 3$. We proceed on a case-by-case basis: $d(y_1) = d(y_5) = 2$; $d(y_1) = 2$ and $d(y_5) \geq 3$; $d(y_1) \geq 3$, $d(y_5) = 2$; $d(y_1) \geq 3$ and $d(y_5) \geq 3$.

$d(y_1) = d(y_5) = 2$:

Note that $d(y_1) = 2$ implies $l(y_1) = 2\tilde{3}^\pm$ and $d(y_5) = 2$ implies $l(y_5) = 4\tilde{3}^\pm$ which yield a contradiction by Lemma 12(iii).

$d(y_1) = 2$ and $d(y_5) \geq 3$:

Then $d(y_1) = 2$ implies $l(y_1) = 2\tilde{3}^\pm$ which forces $d(x_{4m-1}) \geq 3$ which in turn forces $d(x_{4m-2}) \geq 3$ as in Figure 2.6(ii) and hence $d(\Delta) \geq 5$. Then $d(x_i) = 2$, $1 \leq i \leq 4m - 3$, $d(y_3) = 2$. Note that in this case $d(x_{4m-2}) = 3$ forces $l(x_{4m-2}) = \bar{2}1\tilde{3}^\pm$ which with $l(y_1)$ yields a contradiction by Lemma 12(iii) and so $d(x_{4m-2}) \geq 4$. This forces $d(y_5) = 3$ which in turn implies $l(y_5) = 4\bar{1}\tilde{3}^\pm$ by Lemma 13(i) and $c(\Delta) \leq c(3, 3, 3, 3, 4) = \frac{\pi}{6}$, see Figure 2.10(i).

$d(y_1) \geq 3$ and $d(y_5) = 2$:

Then $d(y_5) = 2$ implies $l(y_5) = 4\tilde{3}^\pm$ which forces $d(x_1) \geq 3$ which in turn forces $d(x_2) \geq 3$ as in Figure 2.6(i). Hence $d(\Delta) \geq 5$. This implies that $d(x_i) = 2$, $3 \leq i \leq 4m - 3$, $d(y_3) = 2$. Moreover if $d(x_2) = 3$ then $l(x_2) = 4\bar{1}\tilde{3}^\pm$ which

with $l(y_5)$ yields a contradiction by Lemma 12(iii) and so $d(x_2) \geq 4$. This implies $d(y_1) = 3$ and then $l(y_1) = \bar{1}2\bar{3}^\pm$ by Lemma 13(ii), see Figure 2.10(ii).

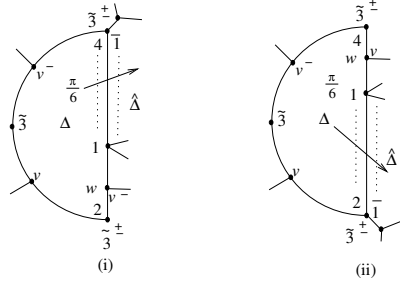


Figure 2.10: $d(y_1) = 2$ or $d(y_5) = 2$.

$d(y_1) \geq 3$ and $d(y_5) \geq 3$:

Then $d(\Delta) \geq 4$. First let $d(x_i) = 2$, $1 \leq i \leq 4m - 1$. Then $d(y_1) = d(y_5) = 3$ yields a contradiction by Lemma 13(iii). Thus Δ has at least one vertex of degree ≥ 4 and $c(\Delta) \leq c(3, 3, 3, 4) = \frac{\pi}{2}$, see Figures 2.11(i)-(iii). Now let $d(x_i) \geq 3$ for some $1 \leq i \leq 4m - 2$. Then either this forces at least one more split or x_i is as in one of Figures 2.7(i) or 2.8(i). In the former case the further split implies $d(\Delta) \geq 6$. In the latter case $d(x_i) \geq 4$ and letting $d(y_1) = d(y_5) = 3$ yields a contradiction by Lemma 13(iii) and so $c(\Delta) \leq c(3, 3, 3, 4, 4) = 0$.

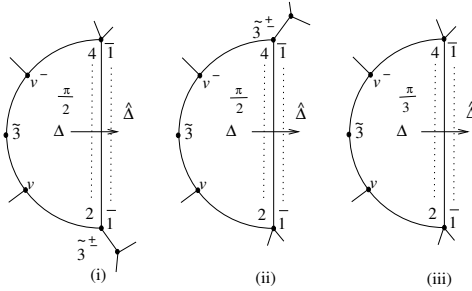


Figure 2.11: $d(y_1) \geq 3$ and $d(y_5) \geq 3$.

Assume next that $m = 1$. Then $d(y_2) \geq 3$, $d(y_4) \geq 3$ and $d(y_3) \geq 3$. By Lemma 12(iii) $d(y_1) = d(y_5) = 2$ yields a contradiction. Moreover by the above discussion letting only $d(y_1) = 2$ or only $d(y_5) = 2$ forces at least 2 more splits and this implies $d(\Delta) \geq 6$. Thus $d(y_1) \geq 3$ and $d(y_5) \geq 3$ and this implies

$d(\Delta) \geq 5$ and $d(x_i) = 2$, $1 \leq i \leq 4m - 1$. Moreover letting $d(y_1) = d(y_5) = 3$ yields a contradiction by Lemma 12(iii). Thus $d(y_5) \geq 4$ or $d(y_1) \geq 4$ and $c(\Delta) \leq c(3, 3, 3, 3, 4) = \frac{\pi}{6}$, see Figures 2.12(i)-(ii).

This complete the description of positive regions in this case and we show the

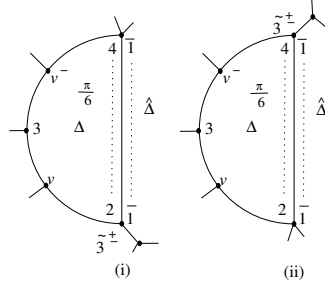


Figure 2.12: $m = 1$.

distribution case by case.

Δ is given by Figure 2.10(i):

Note that $l(y_5) = l(\hat{x}_i) = 4\bar{1}\tilde{3}^\pm$ forces $d(\hat{x}_{i-1}) \geq 3$. Note further that $d(\hat{y}_5) \geq 3$ otherwise there is a contradiction with $l(y_1)$ by Lemma 12(iii). Thus $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 3, 4) = \frac{-\pi}{6}$. Add $c(\Delta) \leq c(3, 3, 3, 3, 4) = \frac{\pi}{6}$ to $c(\hat{\Delta})$.

Δ is given by Figure 2.10(ii):

Note that $l(y_1) = l(\hat{x}_j) = \bar{1}2\tilde{3}^\pm$ forces $d(\hat{x}_{j+1}) \geq 3$. Note further that $d(\hat{y}_1) \geq 3$ otherwise there is a contradiction with $l(y_5)$ by Lemma 12(iii). Thus $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 3, 4) = \frac{-\pi}{6}$. Add $c(\Delta) \leq c(3, 3, 3, 3, 4) = \frac{\pi}{6}$ to $c(\hat{\Delta})$.

Δ is given by Figure 2.11(i):

Note that $l(y_1) = l(\hat{x}_j) = \bar{1}2\tilde{3}^\pm$ forces $d(\hat{x}_{j+1}) \geq 3$. Note further that $d(\hat{y}_1) \geq 3$ otherwise there is a contradiction by Lemma 12(iii). Moreover either $d(\hat{y}_5) \geq 3$ or $d(\hat{x}_1) \geq 3$ and we have $d(\hat{y}_2) \geq 3$ and $d(\hat{y}_4) \geq 3$. Thus $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 3, 3, 4) = \frac{-\pi}{2}$. Add $c(\Delta) \leq c(3, 3, 3, 4) = \frac{\pi}{2}$ to $c(\hat{\Delta})$.

Δ is given by Figure 2.11(ii):

Note that $l(y_5) = l(\hat{x}_i) = 4\bar{1}\tilde{3}^\pm$ forces $d(\hat{x}_{i-1}) \geq 3$. Note further that $d(\hat{y}_5) \geq 3$ otherwise there is a contradiction by Lemma 12(iii). Moreover either $d(\hat{y}_1) \geq 3$ or $d(\hat{x}_{4\hat{m}-1}) \geq 3$ and we have $d(\hat{y}_2) \geq 3$ and $d(\hat{y}_4) \geq 3$. Thus

$c(\hat{\Delta}) \leq c(3, 3, 3, 3, 3, 3, 4) = \frac{-\pi}{2}$. Add $c(\Delta) \leq c(3, 3, 3, 4) = \frac{\pi}{2}$ to $c(\hat{\Delta})$.

Δ is given by Figure 2.11(iii):

Then $d(\hat{\Delta}) \geq 6$ since $d(\hat{y}_2) \geq 3$, $d(\hat{y}_4) \geq 3$, at least one of $d(\hat{y}_5)$ or $d(\hat{x}_1)$ is ≥ 3 and at least one of $d(\hat{y}_1)$ or $d(\hat{x}_{4\hat{m}-1})$ is ≥ 3 . Thus $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 4, 4) = \frac{-\pi}{3}$. Add $c(\Delta) \leq c(3, 3, 4, 4) = \frac{\pi}{3}$ to $c(\hat{\Delta})$.

Δ is given by Figure 2.12(i)-(ii):

The same argument used in Figure 2.11(i)(ii) applies here and we add $c(\Delta) \leq c(3, 3, 3, 3, 4) = \frac{\pi}{6}$ to $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 3, 3, 4) = \frac{-\pi}{2}$.

If $\hat{\Delta}$ is receiving from exactly one region then by the above discussion $c^*(\Delta) \leq 0$. Now assume that $\hat{\Delta}$ receives positive curvature from k neighbouring regions Δ_i , as shown in Figure 2.13(i), where $k \geq 2$ arranged in an anti-clockwise way around $\partial\hat{\Delta}$.

Looking at Figures 2.10-2.12 we can categorise the regions in Types 1-3 according

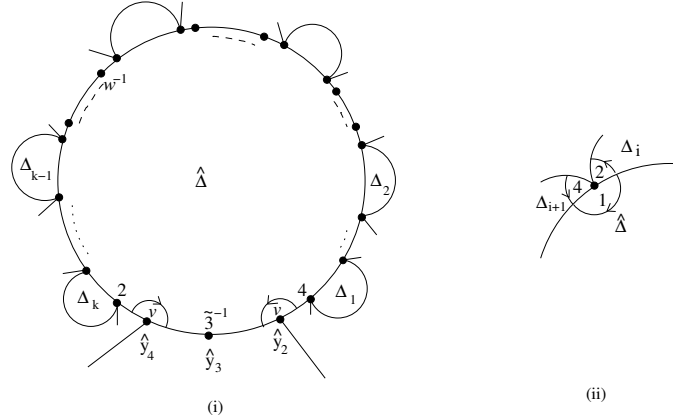


Figure 2.13

to their intersection points with $\hat{\Delta}$, see Figure 2.14. Type 1a occurs in Figure 2.11(iii); Type 1b occurs in Figure 2.11(ii) and Figure 2.12(ii); Type 1c occurs in Figure 2.11(i) and Figure 2.12(i); Type 2a and Type 2b occur in Figure 2.10(ii) according to the labelling of $l(x_2)$; and Type 3a and Type 3b occur in Figure 2.10(i) according to the labelling of $l(x_{4m-2})$.

If $\Delta \in \{\Delta_2, \dots, \Delta_{k-1}\}$ then Δ is one of the following types: Type 1, Type 2a, Type 3a, see Figure 2.14. While if $\Delta \in \{\Delta_1, \Delta_k\}$ then Δ is one of the types:

Type 2b, Type 3b. Note that successive Δ_i, Δ_{i+1} cannot be incident at the same

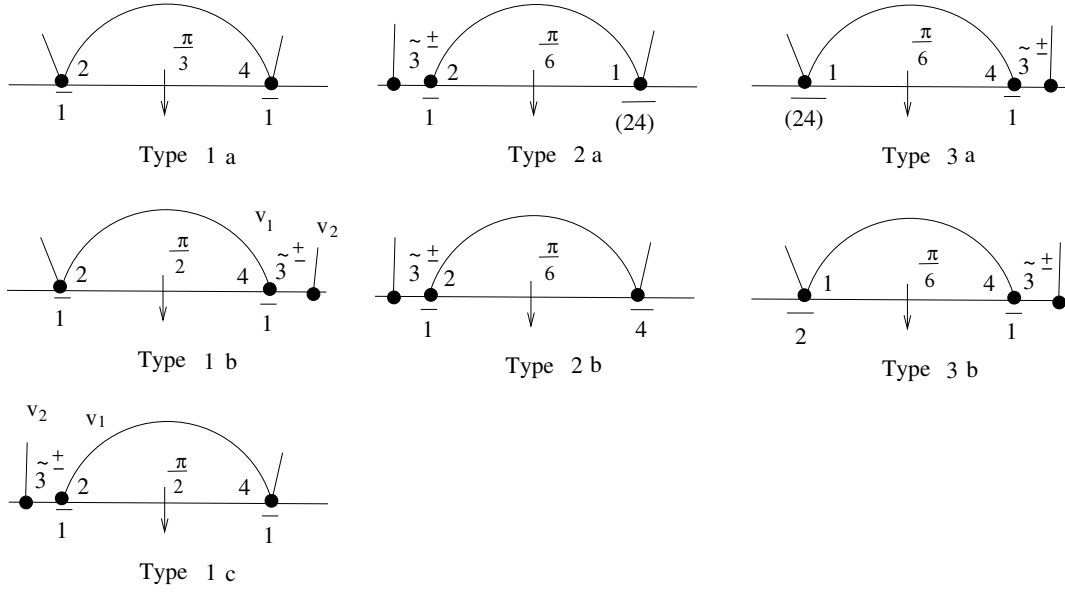


Figure 2.14

vertex of $\hat{\Delta}$. That is because in a Type 1 region when $d(y_1)$ or $d(y_5)$ is of degree 3 then the labelling forces at least one w -split and when the degree ≥ 4 then this forces $2\bar{1}4$ to be a sublabel of a vertex in $\hat{\Delta}$, see Figure 2.13(ii). Moreover Type 1b and Type 1c can never coincide since this yields a contradiction by Lemma 13 (iii). Also Type 1 can not be incident at the same vertex of $\hat{\Delta}$ with regions of Type 2 or Type 3 since the last mentioned types force the end point vertex to be of degree 3 with one further split. Moreover Type 2 or Type 3 regions can never coincide since there is a labelling contradiction by Lemma 13 (iii).

Note further that Type 2 or Type 3 regions send $\leq \frac{\pi}{6}$ and Type 1a regions send $\leq \frac{\pi}{3}$ while Type 1b and Type 1c regions send $\leq \frac{\pi}{2}$. However, for Type 1b and Type 1c regions the contribution of v_1 and v_2 when $d(v_1) = 3$ is $\leq 2(\frac{2\pi}{3})$ with a further $\frac{\pi}{2}$ which gives a total $\frac{11\pi}{6}$ while for Type 1a regions $d(v_1) \geq 4$ and this allows $d(v_2) = 2$ and the contribution is $\leq \frac{2\pi}{4} + \frac{2\pi}{2} + \frac{\pi}{3} = \frac{11\pi}{6}$ and so it is enough to consider Type 1a with $c(\Delta_i) \leq \frac{\pi}{3}$ and that the two end point are of degree ≥ 4 . This estimation works as well for regions of Type 2 or Type 3 as each contribute with one vertex of degree ≥ 4 and two vertices of degree ≥ 3 . So as an estimation we can assume that Δ_i sends $\leq \frac{\pi}{3}$ and that each one of Δ_i forces 2 vertices of degree ≥ 4 . Thus for $k \geq 1$, $\hat{\Delta}$ contains at least $2k + 2$ vertices of

degree > 2 at least $2k$ are of degree ≥ 4 and it receives at most $k(\frac{\pi}{3})$. So, $c^*(\hat{\Delta}) \leq (2 - (2k + 2))\pi + 2(\frac{2\pi}{3}) + 2k(\frac{2\pi}{4}) + (k)(\frac{\pi}{3})$ which implies that $c^*(\hat{\Delta}) \leq 0$ when $k \geq 2$.

Observe in this case that when Δ_i , $1 \leq i < k$ is sending positive curvature to $\hat{\Delta}$ then there is no curvature sent across the left-hand adjacent edge and the most sent by Δ_i as estimated above is $\leq \frac{\pi}{3}$. Note further that Δ_k sends at most $\frac{\pi}{6}$. Thus, for later use, the average curvature sent to $\hat{\Delta}$ can be assumed to be $\frac{\pi}{6}$ per edge.

2.3 Case I (b1)

First assume that $d(u_i) = 2$, $1 \leq i \leq 4$. Then Figure 2.15 shows all possible labellings in this case. Note that letting $d(v_1) = d(v_3) = 3$ yields a labelling

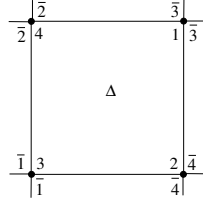


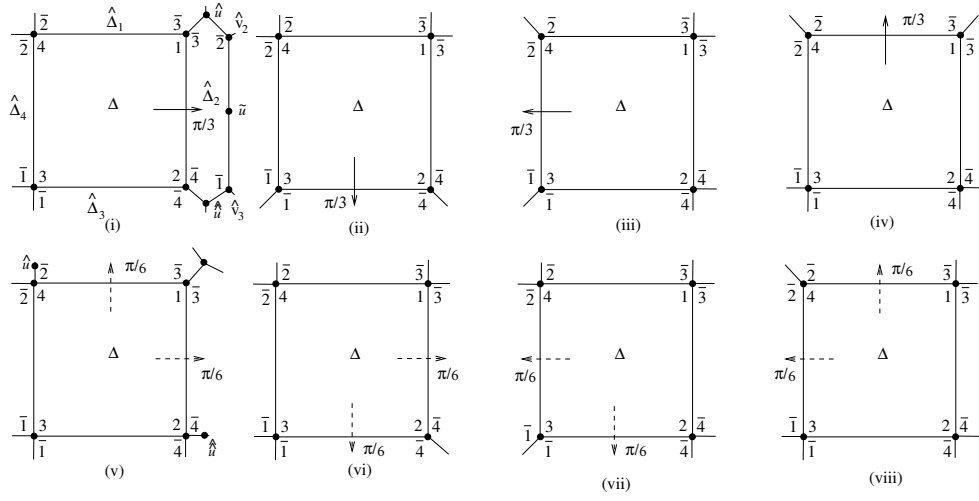
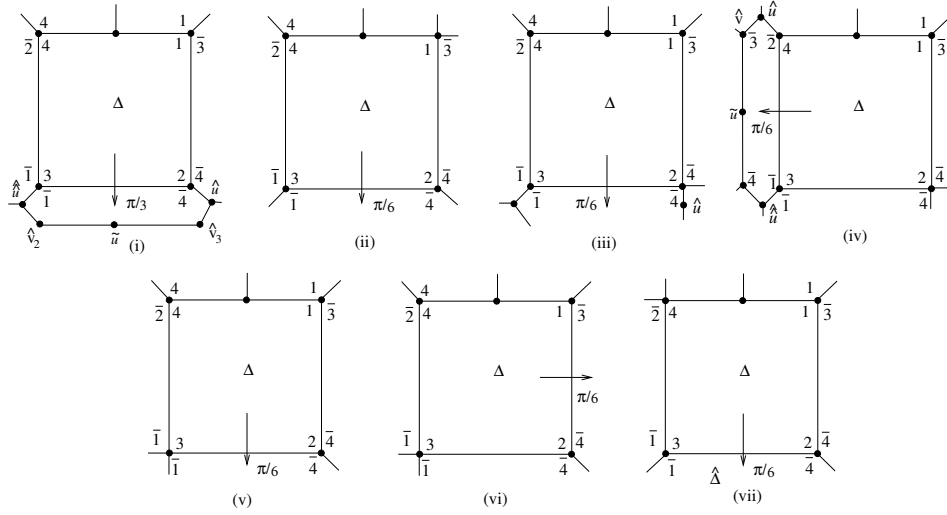
Figure 2.15: All possible labellings case I (b1).

contradiction and $d(v_2) = d(v_4) = 3$ also yields a labelling contradiction, so at most two of the vertices have degree 3. First let $d(v_1) = d(v_2) = 3$. Then $l(v_1) = \bar{3}1\bar{3}$, $l(v_2) = \bar{4}2\bar{4}$ and Δ is given by Figure 2.16(i). Now let $d(v_2) = d(v_3) = 3$. Then $l(v_2) = \bar{4}2\bar{4}$, $l(v_3) = \bar{1}3\bar{1}$, see Figure 2.16(ii). Assume now that $d(v_3) = d(v_4) = 3$. Then $l(v_3) = \bar{1}3\bar{1}$, $l(v_4) = \bar{2}4\bar{2}$, see Figure 2.16(iii). Finally let $d(v_4) = d(v_1) = 2$. Then $l(v_4) = \bar{2}4\bar{2}$, $l(v_1) = \bar{3}1\bar{3}$, see 2.16(iv).

Next let only one $d(v_i) = 3$, $1 \leq i \leq 4$. First let $d(v_1) = 3$. Then $l(v_1) = \bar{3}1\bar{3}$, see Figure 2.16(v). If $d(v_2) = 3$ then $l(v_2) = \bar{4}2\bar{4}$, see Figure 2.16(vi). If $d(v_3) = 3$ then $l(v_3) = \bar{1}3\bar{1}$, see Figure 2.16(vii). If $d(v_4) = 3$ then $l(v_4) = \bar{2}4\bar{2}$, see Figure 2.16(viii).

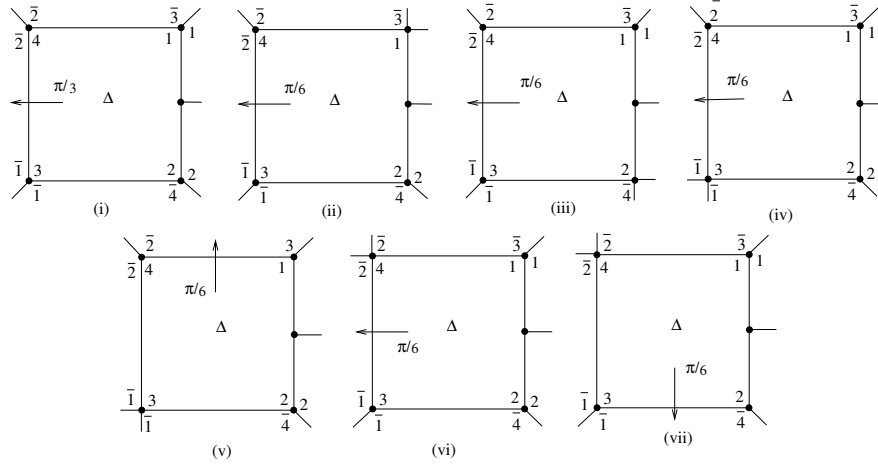
Now assume that $d(u_i) \geq 3$ for exactly one $1 \leq i \leq 4$. First let $d(u_1) \geq 3$. If $d(v_i) = 3$ for all $1 \leq i \leq 4$ then this forces $l(v_1) = l(v_3) = 11\bar{3}$ and $l(v_2) = l(v_4) = \bar{4}2\bar{4}$, see Figure 2.17(i). If $d(v_1) \geq 4$ then $l(v_2) = l(v_4) = \bar{4}2\bar{4}$ and $l(v_3) = \bar{1}3\bar{1}$, see Figure 2.17(ii). If $d(v_2) \geq 4$ then $l(v_1) = l(v_3) = 11\bar{3}$ and $l(v_4) \in \bar{2}4\{1, 3, 4, \bar{1}, \bar{3}\}$, see Figure 2.17(iii)-(iv). If $d(v_3) \geq 4$ then $l(v_2) = l(v_4) = \bar{4}2\bar{4}$ and $l(v_1) \in \bar{1}3\{1, 2, 4, \bar{2}, \bar{4}\}$, see Figure 2.17(v)-(vi). If $d(v_4) \geq 4$ then $l(v_1) = l(v_3) = \bar{1}3\bar{1}$ and $l(v_2) = \bar{4}2\bar{4}$, see Figure 2.17(vii).

Now let $d(u_2) \geq 3$. If $d(v_i) = 3$ for all $1 \leq i \leq 4$ then this forces $l(v_1) = l(v_3) = 11\bar{3}$ and $l(v_2) = l(v_4) = \bar{2}4\bar{2}$, see Figure 2.18(i). If $d(v_1) \geq 4$ then $l(v_2) = l(v_4) = \bar{2}4\bar{2}$ and $l(v_3) = \bar{1}3\bar{1}$, see Figure 2.18(ii). If $d(v_2) \geq 4$ then $l(v_1) = l(v_3) = 11\bar{3}$ and $l(v_4) = \bar{2}4\bar{2}$, see Figure 2.18(iii). If $d(v_3) \geq 4$ then $l(v_2) = l(v_4) = \bar{2}4\bar{2}$ and $l(v_1) \in \bar{3}1\{1, 2, 4, \bar{2}, \bar{4}\}$, see Figure 2.18(iv)-(v). If $d(v_4) \geq 4$ then

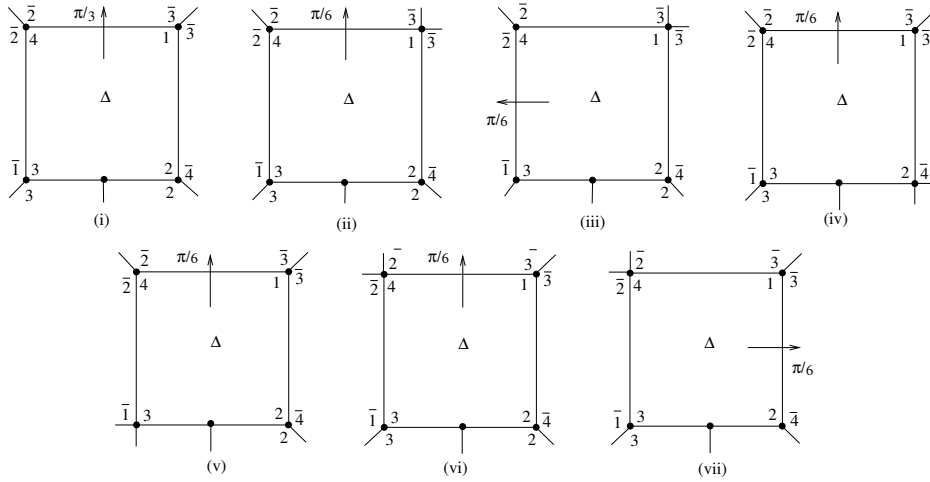
Figure 2.16: $d(u_i) = 2$, $1 \leq i \leq 4$ Figure 2.17: $d(u_1) > 2$

$l(\mathbf{v}_1) = l(\mathbf{v}_3) = \bar{1}3\bar{1}$ and $l(\mathbf{v}_2) \in \bar{2}4\{1, 2, 3, \bar{1}, \bar{3}\}$, see Figure 2.18(vi)-(vii).

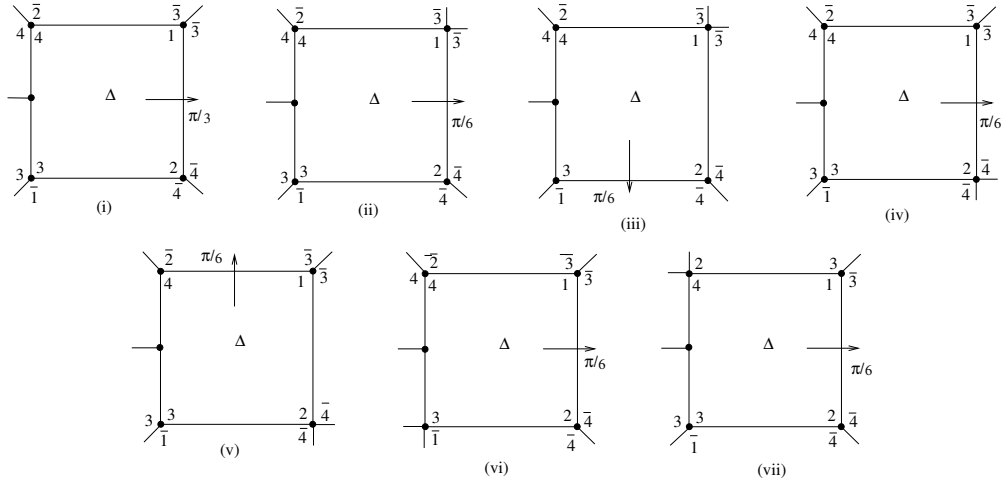
Now let $d(u_3) \geq 3$. If $d(\mathbf{v}_i) = 3$ for all $1 \leq i \leq 4$ then this forces $l(\mathbf{v}_1) = l(\mathbf{v}_3) = \bar{3}1\bar{3}$ and $l(\mathbf{v}_2) = l(\mathbf{v}_4) = \bar{4}2\bar{2}$, see Figure 2.19(i). If $d(\mathbf{v}_1) \geq 4$ then $l(\mathbf{v}_2) = l(\mathbf{v}_4) = \bar{4}2\bar{2}$ and $l(\mathbf{v}_3) \in 3\bar{1}\{2, 3, 4, \bar{2}, \bar{4}\}$, see Figure 2.19(ii)-(iii). If $d(\mathbf{v}_2) \geq 4$ then $l(\mathbf{v}_1) = l(\mathbf{v}_3) = \bar{3}1\bar{3}$ and $l(\mathbf{v}_4) = \bar{2}4\bar{2}$, see Figure 2.19(iv). If $d(\mathbf{v}_3) \geq 4$ then $l(\mathbf{v}_2) = l(\mathbf{v}_4) = \bar{4}2\bar{2}$ and $l(\mathbf{v}_1) = \bar{3}1\bar{3}$, see Figure 2.19(v). If $d(\mathbf{v}_4) \geq 4$ then

Figure 2.18: $d(u_2) > 2$

$l(v_1) = l(v_3) = \bar{3}1\bar{3}$ and $l(v_2) \in \bar{4}2\{1, 2, 3, \bar{1}, \bar{3}\}$, see Figure 2.19(vi)-(vii).

Figure 2.19: $d(u_3) > 2$

Now let $d(u_4) \geq 3$. If $d(v_i) = 3$ for all $1 \leq i \leq 4$ then this forces $l(v_1) = l(v_3) = \bar{3}1\bar{3}$ and $l(v_2) = l(v_4) = \bar{4}2\bar{4}$, see Figure 2.20(i). If $d(v_1) \geq 4$ then $l(v_2) = l(v_4) = \bar{4}2\bar{4}$ and $l(v_3) \in \bar{1}3\{2, 3, 4, \bar{2}, \bar{4}\}$, see Figure 2.20(ii)-(iii). If $d(v_2) \geq 4$ then $l(v_1) = l(v_3) = \bar{3}1\bar{3}$ and $l(v_4) \in \bar{4}2\{1, 3, 4, \bar{1}, \bar{3}\}$, see Figure 2.20(iv)-(v). If $d(v_3) \geq 4$ then $l(v_2) = l(v_4) = \bar{4}2\bar{4}$ and $l(v_1) = \bar{3}1\bar{3}$, see Figure 2.20(vi). If $d(v_4) \geq 4$ then $l(v_1) = l(v_3) = \bar{3}1\bar{3}$ and $l(v_2) = \bar{4}2\bar{4}$, see Figure 2.20(vii).

Figure 2.20: $d(u_4) > 2$

Note that there is symmetry by Lemma 6(iv) when $l(v_i) \in \{\bar{3}1\bar{3}, \bar{4}2\bar{4}, \bar{1}3\bar{1}, \bar{2}4\bar{2}\}$. In particular Figure 2.16(i) is symmetric to Figures 2.16(ii)-(iv); Figure 2.16(v) is symmetric to Figures 2.16(vi)-(viii); Figure 2.17(i) is symmetric to Figure 2.18(i), Figure 2.19(i) and Figure 2.20(i); Figure 2.17(ii) is symmetric to Figure 2.17(vii), Figures 2.18(ii)-(iii), Figure 2.19(iv)-(v) and Figure 2.20(vi)-(vii); Figure 2.17(iii) is symmetric to Figure 2.17(v), Figures 2.18(iv) and 2.18(vi), Figures 2.19(ii) and 2.19(vi) and Figures 2.20(ii) and 2.20(ii)(iv); Figure 2.17(iv) is symmetric to Figure 2.17(vi), Figures 2.18(v) and 2.18(vii), Figures 2.19(iii) and 2.19(vii) and Figures 2.20(iii) and 2.20(iii)(v). Thus it is enough to consider one of each symmetric figures and the other figures are done by symmetric argument.

Δ is given by Figure 2.16(i):

Here $l(v_1) = \bar{3}1\bar{3}$ which forces $d(\hat{u}) \geq 3$ and $l(v_2) = \bar{4}2\bar{4}$ forces $l(\hat{u}) \geq 3$. If in addition $d(\hat{u}) = 2$ then $d(\hat{v}_2) = 3$ implies that $l(\hat{v}_2) = \bar{2}4\bar{4}$ but the sublabel $4\bar{2}$ would force $w^{-1}w$ as a proper sublabel of $l(\hat{u})$ which is a contradiction. Moreover $d(\hat{v}_1) = 3$ implies $l(\hat{v}_1) = \bar{1}3\bar{3}$ but the sublabel $\bar{1}3$ would force ww^{-1} as a proper sublabel of $l(\hat{u})$ which is a contradiction. Thus $c(\hat{\Delta}_2) \leq c(3, 3, 3, 3, 4, 4) = -\frac{\pi}{3}$ and we add $c(\Delta) \leq c(3, 3, 4, 4) = \frac{\pi}{3}$ to $c(\hat{\Delta}_2)$.

Δ is given by Figure 2.16(v):

In this case at least one of $d(v_2)$ or $d(v_4) = 4$. Let $d(v_2) = 4$. Then checking

all possible labellings shows that $d(\widehat{u}) \geq 3$ and so $c(\widehat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 4) = -\frac{\pi}{6}$. Add $c(\Delta) \leq c(3, 4, 4, 4) = \frac{\pi}{6}$ to $c(\widehat{\Delta}_2)$. Otherwise $d(v_4) = 4$ and checking shows that $d(\widehat{u}) \geq 3$ and hence $c(\widehat{\Delta}_1) \leq c(3, 3, 3, 3, 3, 4) = -\frac{\pi}{6}$. Add $c(\Delta) \leq c(3, 4, 4, 4) = \frac{\pi}{15}$ to $c(\widehat{\Delta}_1)$.

Δ is given by Figure 2.17(i)-(ii):

Similarly to the argument used for Figure 2.16(i) $l(v_2) = \overline{424}$ forces $d(\widehat{u}) \geq 3$ and $l(v_3) = \overline{131}$ forces $l(\widehat{u}) \geq 3$. If in addition $d(\widehat{u}) = 2$ then $d(\widehat{v}_3) = 3$ implies that $l(\widehat{v}_3) = \overline{311}$ but the sublabel $\overline{13}$ would force $v^{-1}v$ as a proper sublabel of $l(\widehat{u})$ which is a contradiction. Moreover $d(\widehat{v}_2) = 3$ implies $l(\widehat{v}_1) = \overline{244}$ but the sublabel $\overline{24}$ would force vv^{-1} as a proper sublabel of $l(\widehat{u})$ which is a contradiction. Thus $c(\widehat{\Delta}_3) \leq c(3, 3, 3, 3, 4, 4) = \frac{\pi}{3}$ and we add $c(\Delta) \leq c(3, 3, 4, 4) = \frac{\pi}{3}$ to $c(\widehat{\Delta}_2)$.

Δ is given by Figure 2.17(iii):

In Figure 2.17(iii) $d(v_2) \in \{4, 5\}$ and checking all possible labelling shows that $d(\widehat{u}) \geq 3$ and so $c(\widehat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 4) = \frac{\pi}{6}$. Add $c(\Delta)$ to $c(\widehat{\Delta}_3)$.

Δ is given by Figure 2.17(iv):

Here $l(v_3)$ forces $d(\widehat{u}) \geq 3$ and $l(v_4) \in \overline{24}\{1, 3, \overline{1}, \overline{3}\}$ each forces $d(\widehat{u}) \geq 3$. Moreover if $d(\widehat{u}) = 2$ then $d(\widehat{v}_3) = 3$ forces $l(\widehat{v}_3) = \overline{311}$ and again the sublabel $\overline{13}$ would force ww^{-1} as a proper sublabel of $l(\widehat{u})$ which is a contradiction. So $c(\widehat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 4) = \frac{\pi}{6}$ and we add $c(\Delta) \leq \frac{\pi}{6}$ to $c(\widehat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 4) = \frac{\pi}{6}$.

Now assume $\widehat{\Delta} \in \{\widehat{\Delta}_1, \widehat{\Delta}_2, \widehat{\Delta}_3, \widehat{\Delta}_4\}$. If $\widehat{\Delta}$ is receiving from exactly one region then by the above discussion $c^*(\widehat{\Delta}) \leq 0$. If $\widehat{\Delta}$ is receiving from two or more regions then checking Figures 2.17- 2.20 shows that at most 2 positive regions can send their curvature to the same $\widehat{\Delta}$ and the only possibilities for such regions are: $\widehat{\Delta}_1$ in Figure 2.16(v) with $\widehat{\Delta}_3$ in Figures 2.16(vi); $\widehat{\Delta}_2$ in Figure 2.16(v) with $\widehat{\Delta}_4$ in Figures 2.16(viii). In these cases the two positive regions send at most $2(\frac{\pi}{6})$ and force two vertices of degree ≥ 4 at least two edges to have one split and so $c(\widehat{\Delta}) \leq \frac{\pi}{3}$, and hence $c^*(\widehat{\Delta}) \leq c(3, 3, 3, 3, 4, 4) + 2(\frac{\pi}{6}) = 0$.

2.4 Case I (b2)

First assume that $d(u_i) = 2$. Figure 2.21 shows all possible labellings in this case.

Then $d(v_i) = d(v_j) = 3$, $1 \leq i, j \leq 4$, $i \neq j$ yields a contradiction. Thus if Δ is a

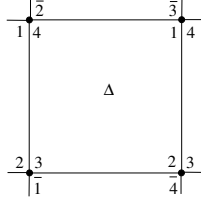


Figure 2.21: All possible labellings case I (b2).

positive region then at most one of v_i has degree 3. If $d(v_1) = 3$ then $l(v_1) = \bar{3}14$, see Figure 2.22(i). If $d(v_2) = 3$ then $l(v_2) = 32\bar{4}$, see Figure 2.22(ii). If $d(v_3) = 3$ then $l(v_3) = \bar{1}32$, see Figure 2.22(iii). If $d(v_4) = 3$ then $l(v_4) = 4\bar{2}1$, see Figure 2.22(iv).

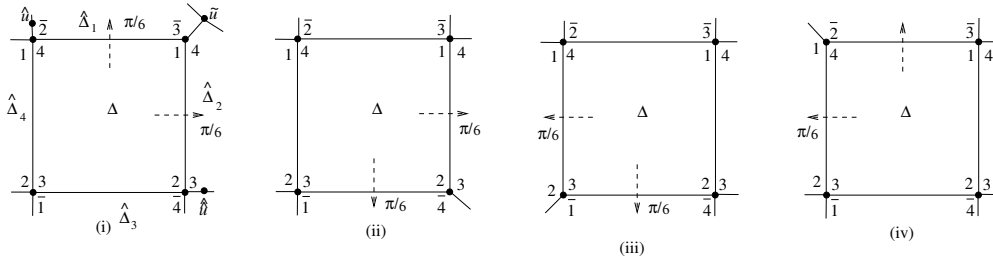


Figure 2.22: $d(u_i) = 2$.

Now assume that exactly one $d(u_i) \geq 3$. First assume that $d(u_1) \geq 3$. Then $d(u_1) = d(v_1) = d(v_4) = 3$ and exactly one of $d(v_2)$ and $d(v_3)$ is 4. If $d(v_2) = 3$ then $l(v_2) = 32\bar{4}$, see Figure 2.23(i). If $d(v_3) = 3$ then $l(v_3) = \bar{1}32$, see Figure 2.23(ii). Moreover $l(v_1) = l(v_4) = 14\{1, 2, 3, 4, k\}$. Now let $d(u_2) \geq 3$. Then $d(u_2) = d(v_1) = d(v_2) = 3$ and exactly one of $d(v_4)$ and $d(v_3)$ is ≥ 4 . If $d(v_3) = 3$ then $l(v_3) = \bar{1}32$, $l(v_1) = \bar{3}1\bar{2}$ and $l(v_2) \in 2\bar{4}\{1, 2, \bar{3}, \bar{4}, k\}$, see Figure 2.23(iii). If $d(v_4) = 3$ then $l(v_4) = 14\bar{2}$, $l(v_2) = 2\bar{4}\bar{1}$ and $l(v_1) \in \bar{3}1\{1, 2, \bar{3}, \bar{4}, k\}$, see Figure 2.23(iv). Now let $d(u_3) \geq 3$. Then $d(u_3) = d(v_2) = d(v_3) = 3$ and exactly one of $d(v_4)$ and $d(v_1)$ is ≥ 4 . If $d(v_4) = 3$ then $l(v_4) = 14\bar{2}$, see Figure 2.23(v).

If $d(v_1) = 3$ then $l(v_1) = \bar{3}14$, see Figure 2.23(vi). Moreover $l(v_2) = l(v_3) = 32\{1, 2, 3, 4, \bar{1}, k\}$. Finally let $d(u_4) \geq 3$. Then $d(u_4) = d(v_3) = d(v_4) = 3$ and exactly one of $d(v_1)$ and $d(v_2)$ is ≥ 4 . If $d(v_2) = 3$ then $l(v_2) = 32\bar{4}$, $l(v_4) = 4\bar{2}\bar{3}$ and $l(v_3) \in \bar{1}3\{3, 4, \bar{1}, \bar{2}, k\}$, see Figure 2.23(vii). If $d(v_1) = 3$ then $l(v_1) = \bar{3}14$, $l(v_3) = \bar{1}3\bar{4}$ and $l(v_4) \in 4\bar{2}\{3, 4, \bar{1}, \bar{2}, k\}$, see Figure 2.23(viii).

Recall that $l(v_i) \in \{\bar{3}14, \bar{1}23, \bar{4}32, \bar{2}14\}$ are all symmetric by Lemma6(v). Note

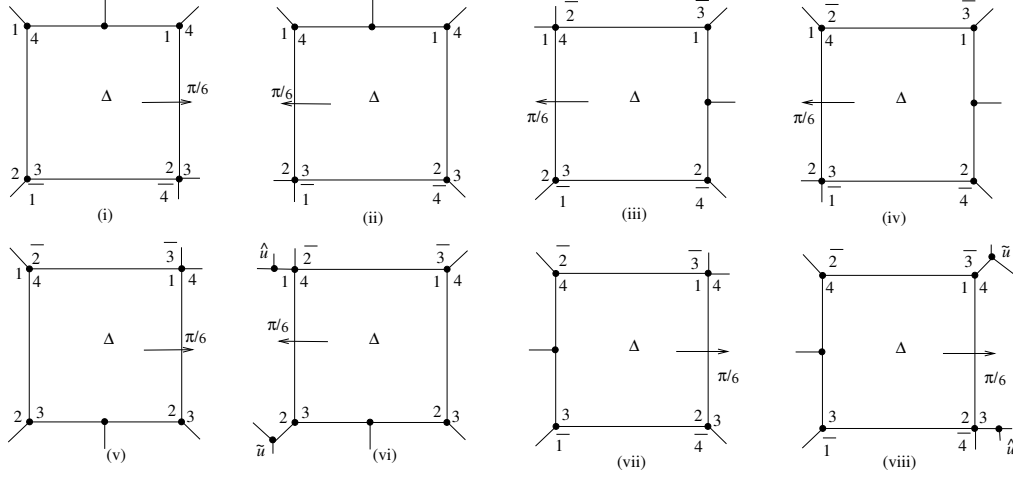


Figure 2.23: $d(u_i) > 3$ for exactly one i .

further that none of these labelling can coincide and that in each positive region there is exactly one of them. Thus it is enough to do the distribution when $l(v_1) = \bar{3}14$ and a symmetric argument applies for the other regions.

Δ is given by Figure 2.22(i):

Note that $d(\tilde{u}) \geq 3$ and at least one of $d(v_2)$ and $d(v_4)$ is $= 4$. If $d(v_2) = 4$ then $d(\hat{u}) \geq 3$ since $d(\hat{u}) = 2$ implies $l(v_2) = 32\bar{4}\bar{1}$ which yields a contradiction with $l(v_1) = \bar{3}14$ and so $d(\hat{\Delta}_2) \geq 6$. Add $c(\Delta)$ to $c(\hat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 4) = -\frac{\pi}{6}$. Otherwise $d(v_4) = 4$ and this forces $d(\hat{u}) \geq 3$ since otherwise $l(v_4) = 14\bar{2}\bar{3}$ which yields a contradiction with $l(v_1) = \bar{3}14$ and so $d(\hat{\Delta}_1) \geq 6$. Add $c(\Delta) \leq c(3, 4, 4, 5) = \frac{\pi}{15}$ to $c(\hat{\Delta}_1) \leq c(3, 3, 3, 3, 3, 4) = -\frac{\pi}{6}$.

Δ is given by Figure 2.23(vi):

Note here that all possible labelling for v_3 forces $d(\tilde{u}) \geq 3$. Moreover checking the possible labels for v_4 shows that when $d(v_4) = 4$ then either $d(\hat{u}) \geq 3$ or $l(v_4) =$

$\bar{3}14\bar{2}$ which contradict $l(v_1)$ and that when $d(v_4) = 5$ then either $d(\hat{u}) \geq 3$ or $l(v_4) = \bar{3}14\bar{2}\omega$ which also contradicts $l(v_1)$. Thus $c(\hat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 4) = -\frac{\pi}{6}$ and we add $c(\Delta) \leq c(3, 3, 3, 3, 4) = \frac{\pi}{6}$ to $c(\hat{\Delta}_4)$.

Δ is given by Figure 2.23(viii):

Note here that $l(v_1)$ forces $d(\hat{u}) \geq 3$. Moreover checking the possible labels for v_2 shows that $l(v_4)$ forces $d(\hat{u}) \geq 3$ when $d(v_4) \in \{4, 5\}$ otherwise there is a contradiction. Thus $c(\hat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 4) = -\frac{\pi}{6}$ and we add $c(\Delta) \leq c(3, 3, 3, 3, 4) = \frac{\pi}{6}$ to $c(\hat{\Delta}_2)$.

Assume now that $\hat{\Delta} \in \{\hat{\Delta}_1, \hat{\Delta}_2, \hat{\Delta}_3, \hat{\Delta}_4\}$ in Figures 2.22 and 2.23. If $\hat{\Delta}$ receives from exactly one positive region then by above discussion $c^*(\hat{\Delta}) \leq 0$. Now assume $\hat{\Delta}$ receives from two or more regions. Then checking shows that $\hat{\Delta}$ receives at most from 2 positive region a maximum of $\frac{\pi}{6}$ each and that at least two of its edges have at least one split each and each of the positive regions forces at least one vertex of degree ≥ 4 and these regions (up to symmetry) are, $\hat{\Delta} = \hat{\Delta}_2$ in Figure 2.22(i) with $\hat{\Delta} = \hat{\Delta}_4$ in Figure 2.23(vi); and $\hat{\Delta} = \hat{\Delta}_4$ in Figure 2.23(vi) with $\hat{\Delta} = \hat{\Delta}_2$ in Figure 2.23(viii). However this implies that $c^*(\hat{\Delta}) \leq c(3, 3, 3, 3, 4, 4) + 2(\frac{\pi}{6}) = 0$.

2.5 Case I (b3)

Figure 2.24 shows all possible labellings in this case.

First assume that $d(u_i) = 2$. Then $d(v_i) = d(v_j) = 3$, $1 \leq i, j \leq 4$, $i \neq j$ yields a

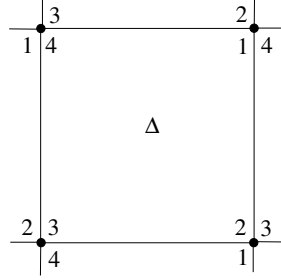


Figure 2.24: All possible labellings case I (b3).

contradiction. Thus if Δ is a positive region then at most one of v_i has degree 3. If $d(v_1) = 3$ then $l(v_1) = 214$, see Figure 2.25(i). If $d(v_2) = 3$ then $l(v_2) = 321$, see Figure 2.25(ii). If $d(v_3) = 3$ then $l(v_3) = 432$, see Figure 2.25(iii). If $d(v_4) = 3$ then $l(v_4) = 431$, see Figure 2.25(iv).

Now assume that exactly one $d(u_i) \geq 3$. First assume that $d(u_1) \geq 3$. Then

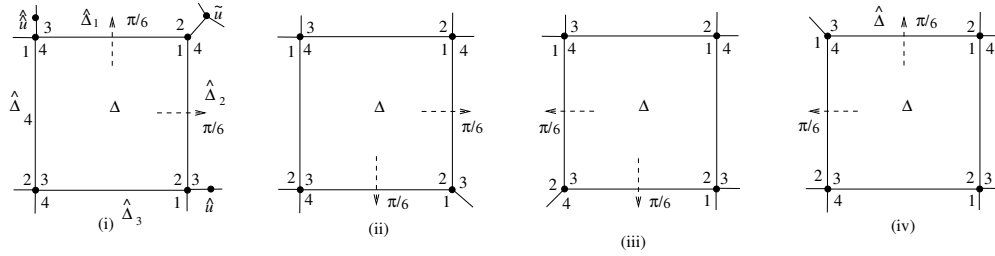
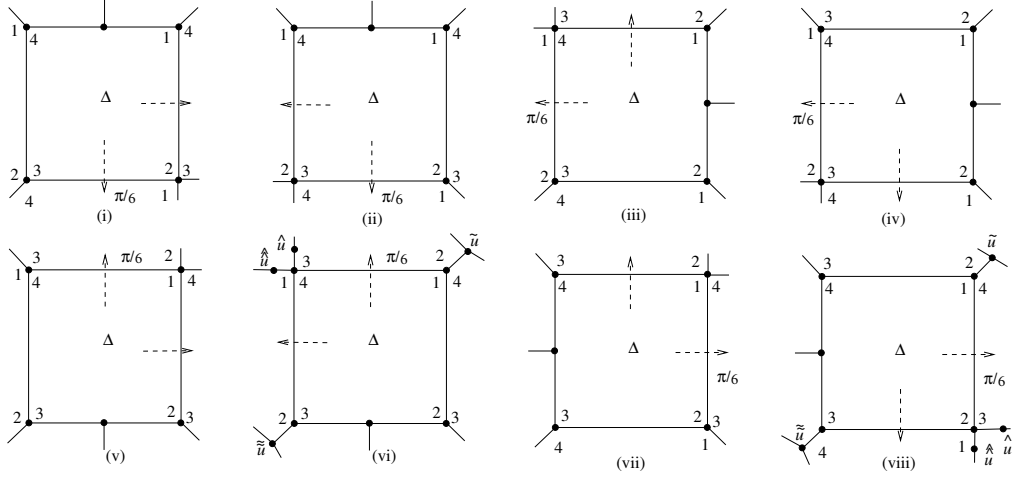


Figure 2.25: $d(u_i) = 2$.

$d(u_1) = d(v_1) = d(v_4) = 3$ and exactly one of $d(v_2)$ and $d(v_3)$ is 4. If $d(v_2) = 3$ then $l(v_2) = 321$, see Figure 2.26. If $d(v_3) = 3$ then $l(v_3) = 432$, see Figure 2.26(ii). Moreover $l(v_1) = l(v_4) = 14\{1, 4, \bar{2}, \bar{3}, k\}$. Now let $d(u_2) \geq 3$. Then $d(u_2) = d(v_1) = d(v_2) = 3$ and exactly one of $d(v_4)$ and $d(v_3)$ is ≥ 4 . If $d(v_3) = 3$ then $l(v_3) = 432$, see Figure 2.26(iii). If $d(v_4) = 3$ then $l(v_4) = 431$, see Figure 2.26(iv). Moreover $l(v_1) = l(v_2) = 21\{1, 2, \bar{3}, \bar{4}, k\}$. Now let $d(u_3) \geq 3$. Then $d(u_3) = d(v_2) = d(v_3) = 3$ and exactly one of $d(v_4)$ and $d(v_1)$ is ≥ 4 . If $d(v_4) = 3$

then $l(v_4) = 431$, see Figure 2.26(v). If $d(v_1) = 3$ then $l(v_1) = 214$, see Figure 2.26(vi). Moreover $l(v_2) = l(v_3) = 32\{2, 3, 4, \bar{1}, \bar{2}, k\}$. Finally let $d(u_4) \geq 3$. Then $d(u_4) = d(v_3) = d(v_4) = 3$ and exactly one of $d(v_1)$ and $d(v_2)$ is ≥ 4 . If $d(v_2) = 3$ then $l(v_2) = 321$, see Figure 2.26(vii). If $d(v_1) = 3$ then $l(v_1) = 214$, see Figure 2.26(viii). $l(v_2) = l(v_3) = 43\{2, 3, \bar{1}, \bar{4}, k\}$.

Figure 2.26: $d(u_i) > 3$.

Observe that the labellings $l(v_1) = 214$, $l(v_2) = 321$, $l(v_3) = 432$ and $l(v_4) = 431$ are symmetric by Lemma 6(vi) and in each positive region we have exactly one of these labellings and no two of these labelling can occur together. Thus it is enough to show the distribution when $d(v_1) = 214$ and a symmetric distribution work for the other labels.

Δ is given by Figure 2.25(i):

Note that $l(v_1)$ forces $d(\tilde{u}) \geq 3$. In addition if $d(v_2) = 4$ then $d(\hat{u}) \geq 3$ unless $l(v_2) = 3214$ which however contradicts $l(v_1) = 214$. Thus $d(v_2) = 4$ implies that $c(\hat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 4) = -\frac{\pi}{6}$ and we add $c(\Delta) \leq c(3, 4, 4, 4) = \frac{\pi}{6}$ to $c(\hat{\Delta}_2)$. If $d(v_2) \geq 5$ then $d(v_4) = 4$ and this forces $d(\hat{u}) \geq 3$ unless $l(v_4) = 1432$ which however contradict $l(v_1) = 214$. Thus $c(\hat{\Delta}_1) \leq c(3, 3, 3, 3, 3, 4) = -\frac{\pi}{6}$ and we add $c(\Delta) \leq c(3, 4, 4, 5) = \frac{\pi}{15}$ to $c(\hat{\Delta}_1)$.

Δ is given by Figure 2.26(vi):

Then the labellings force $d(\tilde{u}) \geq 3$ and $d(\tilde{\tilde{u}}) \geq 3$. If $d(\mathbf{v}_4) = 4$ then $d(\hat{u}) \geq 3$ unless $l(\mathbf{v}_4) = 1432$ which however contradicts $l(\mathbf{v}_1) = 214$. Thus $d(\mathbf{v}_4) = 4$ implies that $c(\hat{\Delta}_1) \leq c(3, 3, 3, 3, 3, 4) = -\frac{\pi}{6}$ and we add $c(\Delta) \leq c(3, 3, 3, 3, 4) = \frac{\pi}{6}$ to $c(\hat{\Delta}_1)$. If $d(\mathbf{v}_4) = 5$ then $c(\Delta) \leq c(3, 3, 3, 3, 5) = \frac{\pi}{15}$ and either $d(\hat{u}) \geq 3$ or $l(\mathbf{v}_4) = 1432\bar{3}$ which forces $d(\hat{\tilde{u}}) \geq 3$. In the former case we add $c(\Delta)$ to $c(\hat{\Delta}_1) \leq c(3, 3, 3, 3, 3, 5) = -\frac{8\pi}{30}$. In the latter case we add $c(\Delta)$ to $c(\hat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 5) = -\frac{8\pi}{30}$.

Δ is given by Figure 2.26(viii):

Then the labellings force $d(\tilde{u}) \geq 3$ and $d(\tilde{\tilde{u}}) \geq 3$. If $d(\mathbf{v}_2) = 4$ then $d(\hat{u}) \geq 3$ unless $l(\mathbf{v}_2) = 1432$ which however contradict $l(\mathbf{v}_1) = 214$. Thus $d(\mathbf{v}_2) = 4$ implies that $c(\hat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 4) = -\frac{\pi}{6}$ and we add $c(\Delta) \leq c(3, 3, 3, 3, 4) = \frac{\pi}{6}$ to $c(\hat{\Delta}_2)$. If $d(\mathbf{v}_2) = 5$ then $c(\Delta) \leq c(3, 3, 3, 3, 5) = \frac{\pi}{15}$ and either $d(\hat{u}) \geq 3$ or $l(\mathbf{v}_2) = 4321\bar{3}$ which forces $d(\hat{\tilde{u}}) \geq 3$. In the former case we add $c(\Delta)$ to $c(\hat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 5) = -\frac{8\pi}{30}$. In the latter case we add $c(\Delta)$ to $c(\hat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 5) = -\frac{8\pi}{30}$.

Assume now that $\hat{\Delta} \in \{\hat{\Delta}_1, \hat{\Delta}_2, \hat{\Delta}_3, \hat{\Delta}_4\}$ in Figures 2.25 and 2.26. If $\hat{\Delta}$ receives from exactly one positive region then by above discussion $c^*(\hat{\Delta}) \leq 0$. Now assume $\hat{\Delta}$ receives from two or more regions. Then checking Figures 2.25-2.26 shows that only two regions can send their curvature to $\hat{\Delta}$ at the same time and this forces $\hat{\Delta}$ to have at least 2 edges that split and the most received is $2(\frac{\pi}{6})$ and each region forces at least one vertex of degree ≥ 4 . These regions (up to symmetry) are $\hat{\Delta}_1$ in Figure 2.25(i) with $\hat{\Delta}_3$ in Figure 2.26(viii); $\hat{\Delta}_2$ in Figure 2.26(i) with $\hat{\Delta}_4$ in Figure 2.26(vi); $\hat{\Delta}_1$ in Figure 2.26(vi) with $\hat{\Delta}_3$ in Figure 2.26(viii); and $\hat{\Delta}_4$ in Figure 2.26(vi) with $\hat{\Delta}_2$ in Figure 2.26(viii). However $c^*(\hat{\Delta}) \leq c(3, 3, 3, 3, 4, 4) + 2(\frac{\pi}{6}) = 0$.

Chapter 3

Case II

Up to cyclic permutation and inversion $r(t)$ is one of the following subcases:

- (a1) $wt^{l_1}wt^{l_2}wt^{l_3}vt^{l_4}$
- (a2) $wt^{l_1}wt^{l_2}w^{-1}t^{l_3}vt^{l_4}$
- (a3) $wt^{l_1}w^{-1}t^{l_2}wt^{l_3}vt^{l_4}$

We assume further that $w^{\pm 1}v^2$ and $v^{\pm 1}w^2$ can not be a vertex labelling in D . Let Δ be a region of positive curvature corresponding to one of these subcases. Then Δ is given by Figure 3.1(i). This is because v is not a piece so the (3,4) edge must split.

3.1 Case II (a1)

Figure 3.1(ii) shows all possible labellings in this case.

Assume first that $d(v_i) = 3$, $1 \leq i \leq 4$. Then $l(v_1) \in \{\bar{3}1\bar{4}, \bar{2}1\bar{2}, \bar{3}1\bar{2}\}$ otherwise there is a labelling contradiction at v_2 . Let $l(v_1) = \bar{3}1\bar{4}$. Then $l(v_2) \in \bar{1}2\{\bar{1}, \bar{4}\}$. If $l(v_1) = \bar{3}1\bar{4}$ and $l(v_2) = \bar{1}2\bar{1}$ then either $l(v_3) = \bar{2}3k$ and $l(v_4) \in 4\bar{2}\{\bar{2}, \bar{3}, k\}$; or $l(v_3) = \bar{2}3\bar{4}$ and $l(v_4) = 4\bar{2}k$; or $l(v_3) = \bar{2}3\bar{2}$ and $l(v_4) = 4\bar{2}k$, see Figure 3.2(i). If $l(v_1) = \bar{3}1\bar{4}$ and $l(v_2) = \bar{1}2\bar{4}$ then $l(v_3) = \bar{1}34$ and $l(v_4) = 4\bar{2}1$, see Figure 3.2(ii). Let $l(v_1) = \bar{2}1\bar{2}$. Then $l(v_2) = \bar{3}2\bar{4}$ and either $l(v_3) \in \bar{1}3\{\bar{4}, \bar{1}\}$ and $l(v_4) = 4\bar{1}k$; or $l(v_3) = \bar{1}3k$ and $l(v_4) \in 4\bar{1}\{\bar{1}, \bar{3}, k\}$, see Figure 3.2(iii). Let $l(v_1) = \bar{3}1\bar{2}$. Then $l(v_2) = \bar{3}2\bar{4}$, $l(v_3) = \bar{1}3\bar{2}$ and $l(v_4) = 4\bar{2}3$, see Figure 3.2(iv).

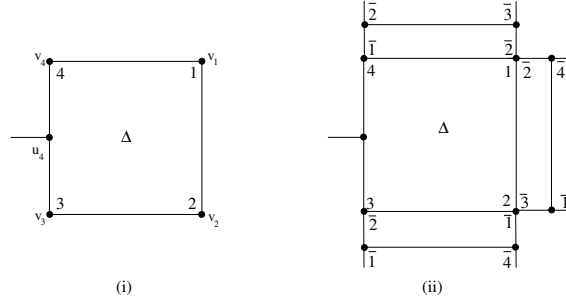
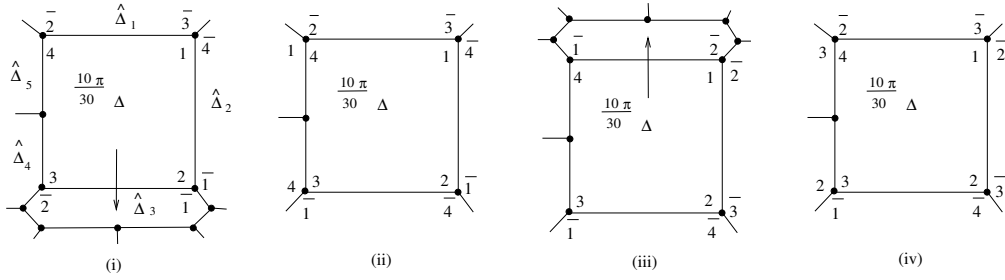
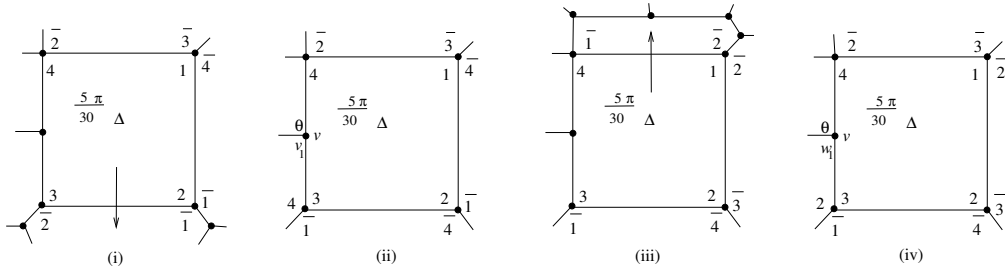


Figure 3.1: Positive region of case II

Figure 3.2: $d(v_i) = 3, 1 \leq i \leq 4$

Assume now that $d(v_i) \geq 4$, for exactly one of $1 \leq i \leq 4$. If $d(v_1) = d(v_2) = d(v_3) = 3$ then either $l(v_1) = \bar{3}1\bar{4}$, $l(v_2) = \bar{1}2\bar{1}$, and $l(v_3) \in \bar{2}3\{2, \bar{4}, k\}$, see Figure 3.3(i); or $l(v_1) = \bar{3}1\bar{4}$, $l(v_2) = \bar{1}2\bar{4}$, and $l(v_3) = \bar{1}3\bar{4}$, see Figure 3.3(ii); or $l(v_1) = \bar{2}1\bar{2}$, $l(v_2) = \bar{3}2\bar{4}$, and $l(v_3) \in \bar{1}3\{\bar{4}, \bar{1}, k\}$, see Figure 3.3(iii); or $l(v_1) = \bar{3}1\bar{2}$, $l(v_2) = \bar{3}2\bar{4}$, and $l(v_3) = \bar{1}3\bar{2}$, see Figure 3.3(iv).

If $d(v_1) = d(v_2) = d(v_4) = 3$ then either $l(v_1) = \bar{3}1\bar{4}$, $l(v_2) = \bar{1}2\bar{1}$, and

Figure 3.3: $d(v_4) \geq 4$.

$l(v_4) \in 4\bar{2}\{\bar{2}, \bar{3}, k\}$, see Figure 3.4(i); or $l(v_1) = \bar{3}1\bar{4}$, $l(v_2) = \bar{1}2\bar{4}$, and $l(v_4) = \bar{4}2\bar{1}$, see Figure 3.4(ii); or $l(v_1) = \bar{2}1\bar{2}$, $l(v_2) = \bar{3}2\bar{4}$, and $l(v_4) \in 4\bar{1}\{\bar{1}, \bar{3}, k\}$, see Figure

3.4(iii); or $l(v_1) = \bar{3}1\bar{2}$, $l(v_2) = \bar{3}2\bar{4}$, and $l(v_4) = 4\bar{2}3$, see Figure 3.4(iv).

If $d(v_1) = d(v_3) = d(v_4) = 3$ then either $l(v_1) = \bar{2}1\bar{4}$, $l(v_3) \in \{\bar{1}3\{\bar{1}, \bar{2}, 3, \bar{4}\}$

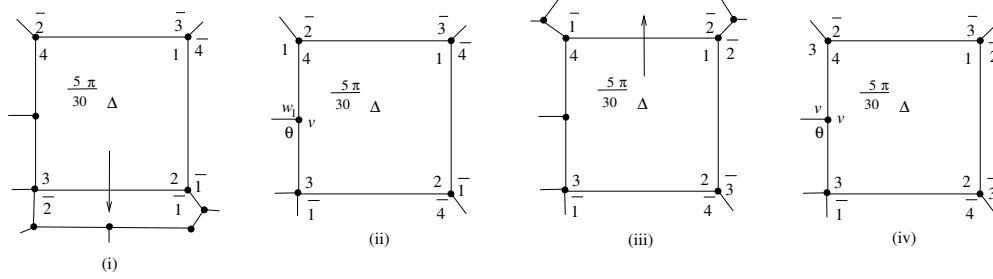
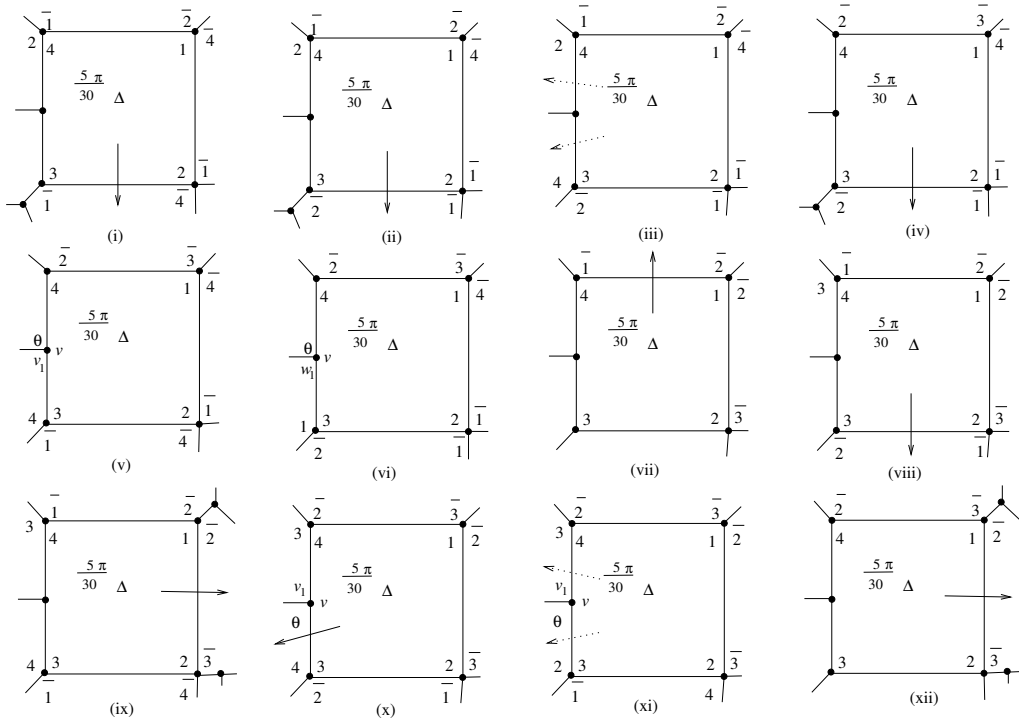


Figure 3.4: $d(v_3) \geq 4$.

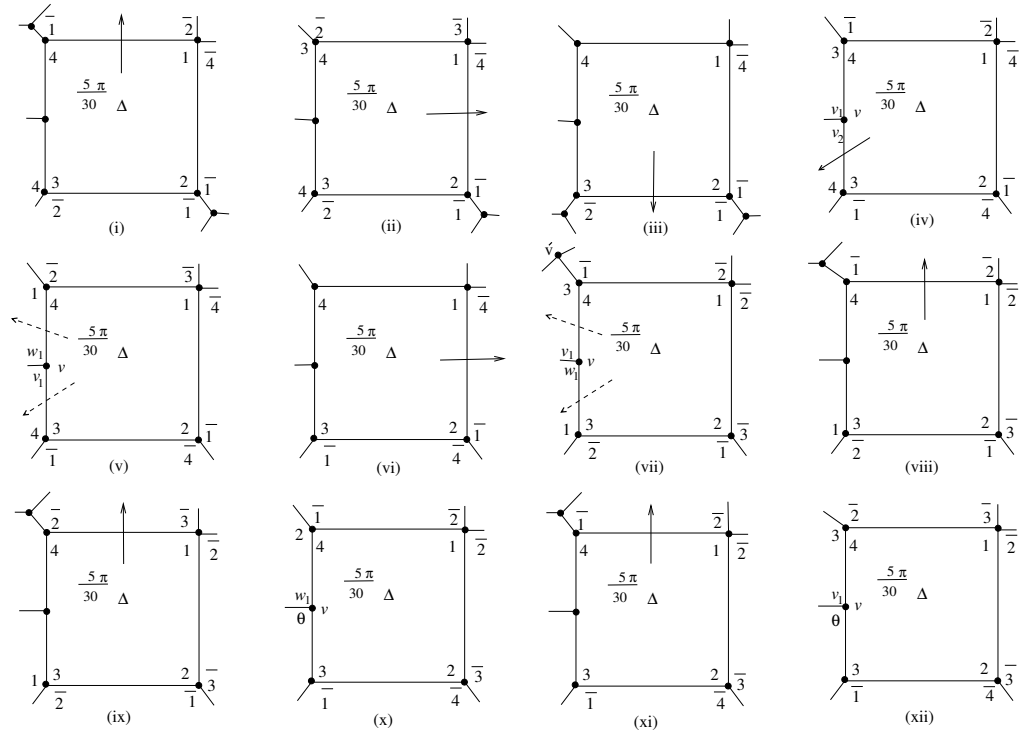
, $k\}$, $\bar{2}3\{\bar{1}, \bar{2}, 3, 4, k\}$, and $l(v_4) = 4\bar{1}2$, see Figure 3.5(i)-(iii); or $l(v_1) = \bar{3}1\bar{4}$, $l(v_3) \in \{\bar{2}3k, \bar{1}34\}$, and $l(v_4) \in 4\bar{2}\{1, 4, \bar{2}, \bar{3}, k\}$, see Figure 3.5(iv) and (v); or $l(v_1) = \bar{3}1\bar{4}$, $l(v_3) = \bar{2}3\bar{4}$, and $l(v_4) \in 4\bar{2}\{4, \bar{2}, \bar{3}, k\}$, see Figure 3.5(iv); or $l(v_1) = \bar{3}1\bar{4}$, $l(v_3) = \bar{2}3\bar{2}$, and $l(v_4) \in 4\bar{2}\{1, 4, \bar{3}, k\}$, see Figure 3.5(iv); or $l(v_1) = \bar{3}1\bar{4}$, $l(v_3) = \bar{2}33$, and $l(v_4) \in 4\bar{2}\{1, \bar{2}, \bar{3}, k\}$, see Figure 3.5(iv); or $l(v_1) = \bar{3}1\bar{4}$, $l(v_3) = \bar{2}31$, and $l(v_4) \in 4\bar{2}\{4, \bar{2}, k\}$, see Figure 3.5(vi); or $l(v_1) = \bar{2}1\bar{2}$, $l(v_3) \in \{\bar{2}3k, \bar{2}33\}$, and $l(v_4) \in 4\bar{1}\{\bar{1}, \bar{2}, \bar{3}, k, 3\}$, see Figure 3.5(vii)-(viii); or $l(v_1) = \bar{2}1\bar{2}$, $l(v_3) = \bar{2}3\bar{4}$, and $l(v_4) \in 4\bar{1}\{\bar{1}, \bar{2}, k, 3\}$, see Figure 3.5(vii)-(viii); or $l(v_1) = \bar{2}1\bar{2}$, $l(v_3) = \bar{2}34$, and $l(v_4) \in 4\bar{1}\{\bar{1}, \bar{3}, k\}$, see Figure 3.5(vii); or $l(v_1) = \bar{2}1\bar{2}$, $l(v_3) \in \bar{2}3\bar{1}$, and $l(v_4) \in 4\bar{1}\{\bar{1}, \bar{3}, k\}$, see Figure 3.5(vii); or $l(v_1) = \bar{2}1\bar{2}$, $l(v_3) \in \bar{1}3k$, and $l(v_4) \in 4\bar{1}\{\bar{1}, \bar{2}, k\}$, see Figure 3.5(vii); or $l(v_1) = \bar{2}1\bar{2}$, $l(v_3) = \bar{1}3\bar{4}$, and $l(v_4) \in 4\bar{1}\{\bar{1}, \bar{2}, k\}$, see Figure 3.5(vii); or $l(v_1) = \bar{2}1\bar{2}$, $l(v_3) = \bar{1}3\bar{1}$, and $l(v_4) \in 4\bar{1}\{\bar{2}, \bar{3}, k\}$, see Figure 3.5(vii); or $l(v_1) = \bar{2}1\bar{2}$, $l(v_3) = \bar{1}3\bar{2}$, and $l(v_4) \in 4\bar{1}\{\bar{1}, \bar{3}, k\}$, see Figure 3.5(vii); or $l(v_1) = \bar{2}1\bar{2}$, $l(v_3) = \bar{1}34$, and $l(v_4) = 4\bar{1}3$, see Figure 3.5(ix); or $l(v_1) = \bar{3}1\bar{2}$, $l(v_3) = \bar{2}34$, and $l(v_4) \in 4\bar{2}3$, see Figure 3.5(x); or $l(v_1) = \bar{3}1\bar{2}$, $l(v_3) = \bar{1}32$, and $l(v_4) \in 4\bar{2}\{3, 4, \bar{1}, \bar{2}, k\}$, see Figure 3.5(xi)-(xii); or $l(v_1) = \bar{3}1\bar{2}$, $l(v_3) = \bar{2}3k$, and $l(v_4) \in 4\bar{2}\{4, \bar{1}, \bar{2}, k\}$; or $l(v_1) = \bar{3}1\bar{2}$, $l(v_3) = \bar{2}3\bar{2}$, and $l(v_4) \in 4\bar{2}\{4, \bar{1}, k\}$; or $l(v_1) = \bar{3}1\bar{2}$, $l(v_3) = \bar{2}33$, and $l(v_4) \in 4\bar{2}\{\bar{1}, \bar{2}, k\}$; or $l(v_1) = \bar{3}1\bar{2}$, $l(v_3) = \bar{2}3\bar{4}$, and $l(v_4) \in 4\bar{2}\{\bar{2}, 4, k\}$, see Figure 3.5(xii).

If $d(v_2) = d(v_3) = d(v_4) = 3$ then either $l(v_2) = \bar{1}2\bar{1}$, $l(v_3) = \bar{2}34$, and $l(v_4) \in \{4\bar{1}\{4, \bar{3}, k\}, 4\bar{2}3\}$, see Figure 3.6(i)-(ii); or $l(v_2) = \bar{1}2\bar{1}$, $l(v_3) = \bar{2}3k$, and $l(v_4) \in \{4\bar{1}\{3, 4, \bar{2}, \bar{3}, k\}, 4\bar{2}\{\bar{1}, \bar{2}, \bar{3}, k\}\}$; or $l(v_2) = \bar{1}2\bar{1}$, $l(v_3) = \bar{2}3\bar{4}$, and $l(v_4) \in \{4\bar{1}\{3, 4, \bar{2}, k\}, 4\bar{2}\{\bar{1}, \bar{2}, k\}\}$; or $l(v_2) = \bar{1}2\bar{1}$, $l(v_3) = \bar{2}3\bar{1}$, and

Figure 3.5: $d(v_2) \geq 4$.

$l(v_4) \in \{4\bar{1}\{4, \bar{3}, k\}, 4\bar{2}\{2, \bar{3}, k\}\}$; or $l(v_2) = \bar{1}2\bar{1}$, $l(v_3) = \bar{2}3\bar{2}$, and $l(v_4) \in \{4\bar{1}\{3, 4, \bar{2}, \bar{3}, k\}, 4\bar{2}\{1, \bar{3}, k\}\}$, see Figure 3.6(iii); or $l(v_2) = \bar{1}2\bar{4}$, $l(v_3) = \bar{1}3\bar{4}$, and $l(v_4) \in \{4\bar{1}3, 4\bar{2}1\}$, see Figure 3.6(iv)-(v); or $l(v_2) = \bar{1}2\bar{4}$, $l(v_3) = \bar{1}3\bar{k}$, and $l(v_4) \in \{4\bar{1}\{4, \bar{1}, \bar{3}, k\}, 4\bar{2}1\}$, see Figure 3.6(vi); or $l(v_2) = \bar{1}2\bar{4}$, $l(v_3) = \bar{1}3\bar{1}$, and $l(v_4) \in \{4\bar{1}\{4, \bar{1}, k\}, 4\bar{2}1\}$; or $l(v_2) = \bar{1}2\bar{4}$, $l(v_3) = \bar{1}3\bar{3}$, and $l(v_4) \in \{4\bar{1}\{1, \bar{3}, k\}, 4\bar{2}1\}$, see Figure 3.6(vi); or $l(v_2) = \bar{3}2\bar{1}$, $l(v_3) = \bar{2}3\bar{1}$, and $l(v_4) \in \{4\bar{1}\{3, 4, \bar{1}, \bar{2}, k\}, 4\bar{2}\{4, \bar{1}, \bar{2}, \bar{3}, k\}\}$, see Figure 3.6(vii)-(ix); or $l(v_2) = \bar{3}2\bar{4}$, $l(v_3) = \bar{1}3\bar{k}$, and $l(v_4) \in \{4\bar{1}\{2, 4, \bar{1}, \bar{3}, k\}, 4\bar{2}3\}$, see Figure 3.6(x)-(xii); or $l(v_2) = \bar{3}2\bar{4}$, $l(v_3) = \bar{1}3\bar{4}$, and $l(v_4) \in \{4\bar{1}\{3, 4, \bar{1}, k\}, 4\bar{2}3\}$, see Figure 3.6(x)-(xii); or $l(v_2) = \bar{3}2\bar{4}$, $l(v_3) = \bar{1}3\bar{1}$, and $l(v_4) \in \{4\bar{1}\{2, 4, \bar{3}, k\}, 4\bar{2}3\}$, see Figure 3.6(x)-(xii); or $l(v_2) = \bar{3}2\bar{4}$, $l(v_3) = \bar{1}3\bar{2}$, and $l(v_4) \in \{4\bar{1}\{4, \bar{1}, k\}, 4\bar{2}3\}$, see Figure 3.6(xi)-(xii); or $l(v_2) = \bar{3}2\bar{4}$, $l(v_3) = \bar{1}3\bar{3}$, and $l(v_4) \in \{4\bar{1}\{2, \bar{1}, \bar{3}, k\}, 4\bar{2}3\}$, see Figure 3.6(x)-(xii).

Note that by Lemma 6(ii) the labels $\bar{3}1\bar{4}$ and $\bar{4}2\bar{3}$, $\bar{2}1\bar{4}$ and $\bar{3}2\bar{1}$, $\bar{1}2\bar{1}$ and

Figure 3.6: $d(v_1) \geq 4$.

$\bar{2}1\bar{2}$, and $\bar{1}2\bar{4}$ and $\bar{3}1\bar{2}$ are symmetric. Note that every positive region has at least one of these labels. Checking shows that, by the above named symmetries, Figures 3.2(i)-(ii) are symmetric to Figures 3.2(iii)-(iv), Figures 3.3(i),(ii), (iii) and (iv) are symmetric to Figures 3.4(iii),(iv),(i) and (ii) respectively and Figures 3.5(i),(ii),(iii),(iv),(v),(vi),(vii),(viii),(ix),(x),(xi) and (xii) are symmetric to Figures 3.6(ix),(viii),(vii), (xi),(xii),(x),(iii),(i),(ii),(iv),(v) and (vi) respectively. Note further that the label $\bar{3}1\bar{4}$ contradicts the labels $\bar{4}2\bar{3}$ and $\bar{2}1\bar{4}$ and the label $\bar{3}2\bar{4}$ contradicts the labels $\bar{4}1\bar{3}$ and $\bar{1}2\bar{3}$. So from the first two pairs of symmetries it is enough to consider the regions with labels $\bar{3}1\bar{4}$ and the regions with labelling $\bar{3}2\bar{1}$ and the same argument applies to the other labellings. Thus we only need to show the distribution in Figures 3.2(i)-(ii), Figures 3.3(i)-(ii) and Figures 3.4(i)-(ii). Also we need to show the distribution for Figure 3.5 when the labelling $\bar{4}2\bar{3}$ does not occur and we will do the distribution for Figure 3.6 when $\bar{4}1\bar{3}$ is a vertex labelling; other regions in these figures are symmetric to one of the considered regions. Thus we consider all Figure 3.5 except for 3.5(iii), (x) and (xi) which are symmetric to Figures 3.6(vii), (iv) and (v). In what follows we will show the

distribution for the above named figures one by one.

Δ is given by Figure 3.2(i):

$c(\Delta) \leq c(3, 3, 3, 3, 3) = \frac{10\pi}{30}$ and $c(\hat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$ since $l(v_2)$ and $l(v_3)$ forces at least one split each and the edge $(3, 4)$ must split as well. Add $c(\Delta)$ to $c(\hat{\Delta}_3)$.

Δ is given by Figure 3.2(ii):

Either $d(u_4) \geq 4$ or $l(u_4) = v_1 v w_1$ where at least one of v_1 or w_1 is a proper subword. If $d(u_4) \geq 4$ then $c(\Delta) \leq c(3, 3, 3, 3, 4) = \frac{5\pi}{30}$ and $c(\hat{\Delta}_5) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$. Add $c(\Delta)$ to $c(\hat{\Delta}_5)$, see Figure 3.7(i) [we note here that once all the distribution of curvature has been completed, checking all possibilities shows that Δ' never send positive curvature to $\hat{\Delta}_5$]. Now Let $d(u_4) = 3$. If w_1 is a proper subword of w then $c(\hat{\Delta}_5) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$ and again add $c(\Delta)$ to $c(\hat{\Delta}_5)$, see Figure 3.7(ii). If $w_1 = w$ then v_1 is a proper subword of v and this implies $d(\hat{\Delta}_4) \geq 6$, see Figure 3.7(iii). Moreover either $d(\hat{v}) \geq 4$ or the edge $(1, 2)$ in $\hat{\Delta}_4$ splits. If in addition $d(\hat{v}) \geq 4$ then $c(\hat{\Delta}_4) \leq c(3, 3, 3, 3, 4, 4) = \frac{-10\pi}{30}$ and we add $c(\Delta)$ to $c(\hat{\Delta}_4)$, see Figure 3.7(iii). If $d(\hat{v}) = 3$ then checking the possible labelling shows that the edge $(2, 3)$ in $\hat{\Delta}_5$ must split and $c(\hat{\Delta}_5) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$. Add $c(\Delta)$ to $c(\hat{\Delta}_5)$, see Figure 3.7(iv).

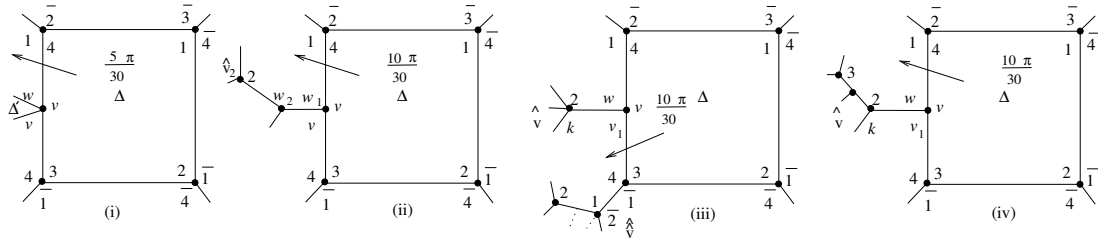


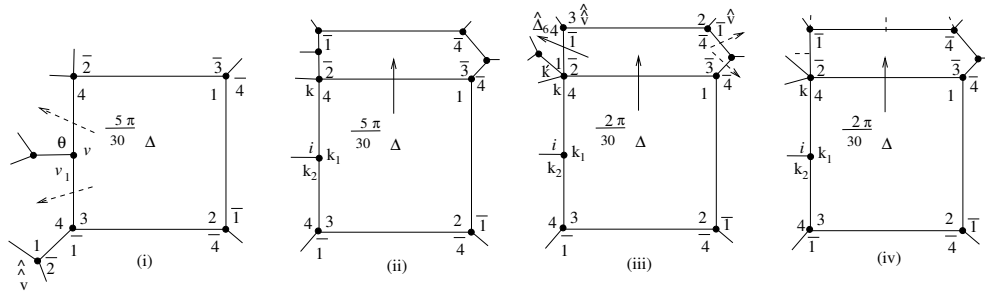
Figure 3.7: $l(v_1) = l(v_3) = \bar{3}1\bar{4}$ and $l(v_2) = l(v_4) = \bar{1}2\bar{4}$.

Δ is given by Figure 3.3(i):

Add $c(\Delta) \leq c(3, 3, 3, 3, 4) = \frac{\pi}{6}$ to $c(\hat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{\pi}{3}$ as in Figure 3.2(i).

Δ is given by Figure 3.3(ii):

First assume $l(u_4)$ does not involve l_i , $1 \leq i \leq 4$. If θ is a proper subword of $w^{\pm 1}$ or a proper subword of v then $d(\widehat{\Delta}_5) \geq 6$ and so $c(\widehat{\Delta}_5) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$. Add $c(\Delta)$ to $c(\widehat{\Delta}_5)$ in Figure 3.8(i). Otherwise, that is $\theta = w^{\pm 1}$ or $\theta = v$, v_1 is a proper subword of v . Moreover either $d(\widehat{v}) \geq 4$ or the edge $(1, 2)$ in $\widehat{\Delta}_4$ splits and we add $c(\Delta)$ to $c(\widehat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 4) = \frac{-\pi}{6}$, see Figure 3.8(i). Now assume $l(u_4)$ involve l_i , $1 \leq i \leq 4$. Then $l(u_4) = k_1 k_2 i^{\pm}$ and $l(v_4)$ involves a τ -power since otherwise there is a contradiction to (A3). If $d(v_4) = 4$ then checking possible labellings show that this forces the edge $(1, 2)$ in $\widehat{\Delta}_1$ to split and $c(\widehat{\Delta}_1) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$ and we add $c(\Delta)$ to $c(\widehat{\Delta}_1)$, see Figure 3.8(ii). If $d(v_4) = 5$ then $c(\Delta) \leq c(3, 3, 3, 3, 5) = \frac{2\pi}{30}$. Moreover if neither $(4, 1)$ or $(1, 2)$ in $\widehat{\Delta}_1$ splits and all other vertices are of degree 3 then $l(\widehat{v}) = 2\bar{4}\bar{1}$ and $l(\widehat{v}) = \bar{1}34$ and this in turn forces $l(v_4) = 4\bar{2}1k'k$ which forces $d(\widehat{\Delta}_6) \geq 6$. In this case we add $c(\Delta)$ to $c(\widehat{\Delta}_6) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$, see Figure 3.8(iii) [we note here that once all the distribution of curvature has been completed, checking shows that $\widehat{\Delta}_1$ in Figure 3.5(iii) never sends positive curvature to $\widehat{\Delta}_6$]. Otherwise, that is there is at least one split or at least one vertex of degree ≥ 4 , we add $c(\Delta) \leq \frac{2\pi}{30}$ to $c(\widehat{\Delta}_1) \leq c(3, 3, 3, 4, 5) = \frac{-3\pi}{30}$, see Figure 3.8(iv).

Figure 3.8: $d(v_4) \geq 4$.

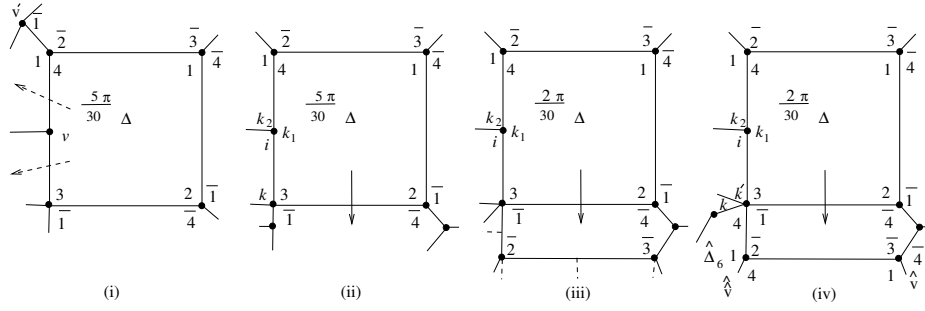
Δ is given by Figure 3.4(i):

Add $c(\Delta) \leq c(3, 3, 3, 3, 4) = \frac{\pi}{6}$ to $c(\widehat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 4) = \frac{-\pi}{6}$.

Δ is given by Figure 3.4(ii):

If w_1 is a proper subword of $w^{\pm 1}$ then $c(\widehat{\Delta}_5) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$ and add $c(\Delta)$ to $c(\widehat{\Delta}_5)$ in Figure 3.9(i). Otherwise, that is $w_1 = w^{\pm 1}$, if $l(u_4)$ does not involve l_i , $1 \leq i \leq 4$ then θ is either a proper subword of $w^{\pm 1}$ or a proper subword

of v and $c(\widehat{\Delta}_5) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$. Add $c(\Delta)$ to $c(\widehat{\Delta}_4)$, see Figure 3.9(i). If $l(u_4) = k_2 k_1 i^{\pm}$ then (A3) implies $l(v_3)$ involves a τ -power and letting $d(v_3) = 4$ forces a split on $\widehat{\Delta}_3$ and $c(\widehat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$ and we add $c(\Delta)$ to $c(\widehat{\Delta}_3)$, see Figure 3.9(ii). If $d(v_3) = 5$ then $c(\Delta) \leq c(3, 3, 3, 3, 3, 5) = \frac{2\pi}{30}$. If $\widehat{\Delta}_3$ has at least one further split or another vertex of degree ≥ 4 then add $c(\Delta) \leq \frac{2\pi}{30}$ to $c(\widehat{\Delta}_3) \leq c(3, 3, 3, 4, 5) = \frac{-3\pi}{30}$, see Figure 3.9(iii). Otherwise $\widehat{\Delta}_3$ is given by Figure 3.9(iv) in which $l(\widehat{v}) = 4\bar{3}1$ and $l(\widehat{v}) = \bar{2}41$ and this in turn forces $l(v_4) = 4\bar{1}3k'k$ and forces $d(\widehat{\Delta}_6) \geq 7$. Add $c(\Delta)$ to $c(\widehat{\Delta}_6) \leq c(3, 3, 3, 3, 3, 3, 5) = \frac{-18\pi}{30}$. We note here once all the distribution of curvature has been completed, checking shows that in this case $\widehat{\Delta}_1$ in Figure 3.6(iv) never sends positive curvature to $\widehat{\Delta}_6$.

Figure 3.9: $d(v_3) \geq 4$.

Δ is given by Figures 3.5(i)-(ii), 3.5(iv) and 3.5(viii):

Then $c(\widehat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$. Add $c(\Delta)$ to $c(\widehat{\Delta}_3)$.

Δ is given by Figure 3.5(vii):

Add $c(\Delta)$ to $c(\widehat{\Delta}_1) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$ since $l(v_1)$ and $l(v_4)$ forces at least one split each on $\widehat{\Delta}_1$.

Δ is given by Figure 3.5(v):

Then $l(v_4) \in 4\bar{2}\{1, 4, \bar{2}, \bar{3}, k\}$. Let $l(v_4) \in 4\bar{2}\{1, 4, \bar{2}, \bar{3}\}$. If θ is a proper subword of $w^{\pm 1}$, that is $l(v_4) \in 4\bar{2}\{1, 4, \bar{2}\}$, then add $c(\Delta) \leq \frac{5\pi}{30}$ to $c(\widehat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$; otherwise if $\theta = w^{\pm 1}$ or θ is a proper subword of v then $d(\widehat{\Delta}_5) \geq 6$ and we distribute the curvature to $d(\widehat{\Delta}_4)$ as follows: if $l(v_4) \in 4\bar{2}\{1, 4, \bar{2}\}$ then either $d(\widehat{v}) \geq 4$ or $l(\widehat{v})$ forces at least one split the edge $(1, 2)$ or $\widehat{\Delta}_4$ and so $c(\widehat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$; or if $l(v_4) = 4\bar{2}\bar{3}$ then either

$d(\widehat{v}) \geq 4$ or $d(\widetilde{v}) \geq 4$ or at least one of the edges $(1, 2)$ or $(2, 3)$ splits in $\widehat{\Delta}_4$ and so $c(\widehat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$. Add $c(\Delta)$ to $c(\widehat{\Delta}_4)$ as shown. If $l(v_4) = 4\bar{2}k$ then note that in $\widehat{\Delta}_1$ either $d(v'') \geq 4$ or the edge $(4, 1)$ splits or $l(v'') \in \{1\bar{4}\bar{3}, 2\bar{4}k\}$. In the first two cases add $c(\Delta)$ to $c(\widehat{\Delta}_1) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$, see Figure 3.10(ii). In the last case if $d(\widehat{u}) \geq 4$ then add $c(\Delta)$ to $c(\widehat{\Delta}_1) \leq -\frac{\pi}{6}$ so assume otherwise; if $l(v'') = 1\bar{4}\bar{3}$, see Figure 3.10(iii), then this implies that at least 2 of v_2, v_3 and v_4 are proper subwords of v^{-1} . If v_2 is not a proper subword then v_3 is and we add $c(\Delta)$ to $c(\widehat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$ (which, by checking the possibilities once all the distribution of curvature has been completed, never receive positive curvature from $\widehat{\Delta}_1$ nor $\widehat{\Delta}_7$). If v_2 is a proper subword then $d(\widehat{\Delta}_1) \geq 7$ and we either have v_3 is a proper subword and we again add $c(\Delta)$ to $c(\widehat{\Delta}_2)$ (which, by checking the possibilities once all the distribution of curvature has been completed, never receive positive curvature from $\widehat{\Delta}_1$ nor $\widehat{\Delta}_7$ but $\widehat{\Delta}_7$ may compensate to $\widehat{\Delta}_1$) or $v_3 = v^{-1}$ and we add $c(\Delta)$ to $c(\widehat{\Delta}_1) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$. Observe here that once all the distribution of curvature has been completed, checking shows that if $\widehat{\Delta}_2$ is positive then it is the regions in Figures 3.6(iii) and it never send positive curvature to $\widehat{\Delta}_1$, see Figure 3.10(iii). If $l(v'') = 2\bar{4}k$ then this implies $l(\widetilde{v}) = 1\bar{3}\omega$ which in turn implies that $d(\widetilde{v}) \geq 4$ otherwise this forces ww^{-1} in $l(v')$ which is a contradiction and we add $c(\Delta)$ to $c(\widehat{\Delta}_1) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$, see Figure 3.10(iv).

Δ is given by Figure 3.5(vi):

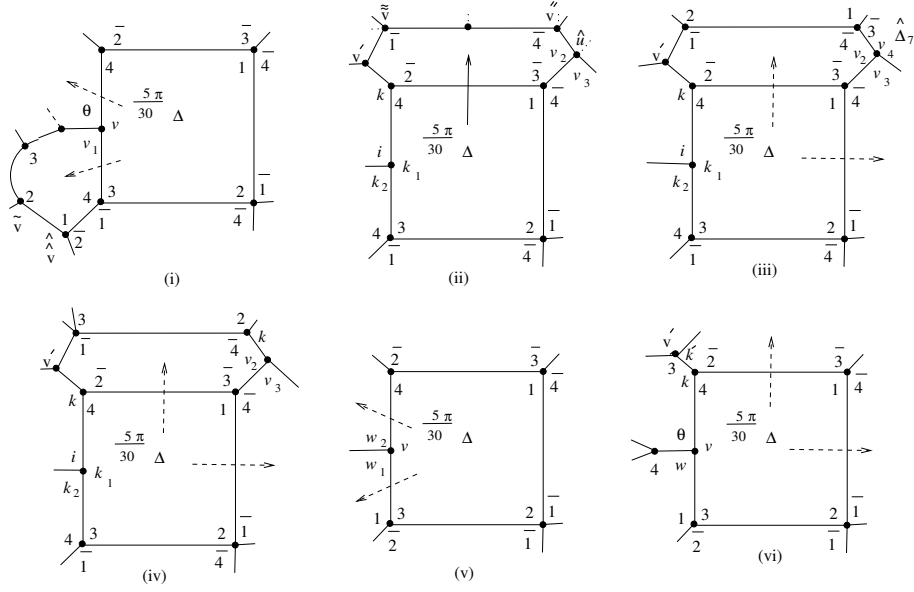
If w_1 is a proper subword of w then we add $c(\Delta) \leq \frac{5\pi}{30}$ to $c(\widehat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$. Otherwise, that is $w_1 = w$, if $l(v_4) \in 4\bar{2}\{2, 4\}$ then θ is a proper subword of $w^{\pm 1}$ and $l(u_4) = w_1vw_2$ and we add $c(\Delta) \leq \frac{5\pi}{30}$ to $c(\widehat{\Delta}_5) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$, see Figure 3.10(v). While if $l(v_4) = 4\bar{2}k$ then using the same argument as for Figure 3.10(ii)-(iv) we add $c(\Delta) \leq \frac{5\pi}{30}$ to $c(\widehat{\Delta}_1)$ or $c(\widehat{\Delta}_2)$, see Figure 3.10(vi).

Δ is given by Figures 3.5 (ix) and 3.5(xii):

Then $c(\widehat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$ and we add $c(\Delta)$ to $c(\widehat{\Delta}_3)$.

Δ is given by Figure 3.6(iv):

Then $l(u_4) = vv_1v_2$ where both v_1 and v_2 are proper subwords of v . Moreover similarly to Figure 3.8(i) either $d(\widehat{v}) \geq 4$ or the edge $(1, 2)$ in $\widehat{\Delta}_4$ splits and we


 Figure 3.10: $d(v_2) \geq 4$.

add $c(\Delta) \leq \frac{5\pi}{30}$ to $c(\widehat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$.

Δ is given by Figure 3.6(v):

Add $c(\Delta) \leq \frac{5\pi}{30}$ to $c(\widehat{\Delta}_4)$ or $c(\widehat{\Delta}_5)$ as in Figure 3.7(ii)-(iv) for region in Figure 3.2(ii).

Δ is given by 3.6(vii):

In Figure if w_1 is a proper subword of w then add $c(\Delta) \leq \frac{5\pi}{30}$ to $c(\widehat{\Delta}_4) \leq \frac{-10\pi}{30}$ in Figure 3.6(vii). Otherwise v_1 is a proper subword of v and either $d(v') \geq 4$ or $l(v') = 24\omega$ which forces the edge $(1, 2)$ in $\widehat{\Delta}_5$ to split and so $c(\widehat{\Delta}_5) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$. Add $c(\Delta) \leq \frac{5\pi}{30}$ to $c(\widehat{\Delta}_5)$ in Figure 3.6(vii).

Now assume that $\widehat{\Delta} \in \{\widehat{\Delta}_1, \widehat{\Delta}_2, \widehat{\Delta}_3, \widehat{\Delta}_4, \widehat{\Delta}_5, \widehat{\Delta}_6\}$. If $\widehat{\Delta}$ receives from exactly one region then by the above dissolution $c^*(\widehat{\Delta}) \leq 0$. Now assume $\widehat{\Delta}$ receives from two or more regions. First consider the regions that send curvature only through whole edges, so $\widehat{\Delta} \in \{\widehat{\Delta}_1, \widehat{\Delta}_2, \widehat{\Delta}_3, \widehat{\Delta}_6\}$ and we denote such regions by Type A. Then checking all regions of this type (up to symmetry) shows that $\widehat{\Delta}$ receives from at most two. If these regions occur at adjacent edges then the shared vertex must be of degree ≥ 5 and each region sends at most $\frac{2\pi}{30}$ and at least two edges

in $\hat{\Delta}$ have splits and hence $c^*(\hat{\Delta}) \leq c(3, 3, 3, 3, 3, 5) + 2(\frac{2\pi}{30}) < 0$. These regions are given by $\hat{\Delta} = \hat{\Delta}_3$ in Figure 3.9(iii) with $\hat{\Delta} = \hat{\Delta}_3$ in Figure 3.5(iv), $\hat{\Delta} = \hat{\Delta}_2$ in Figures 3.10(iii), (iv) or (vi) with $\hat{\Delta} = \hat{\Delta}_3$ Figure 3.5(iv) or (viii). If they occur at non-adjacent edges and this forces one of the following: either two vertices of degree ≥ 5 and each region sends at most $\frac{2\pi}{30}$ and at least one edge in $\hat{\Delta}$ has a split and hence $c^*(\hat{\Delta}) \leq c(3, 3, 3, 5, 5) + 2(\frac{2\pi}{30}) < 0$, this case is given by regions in Figure 3.9(iii) with Figures 3.8(iv); or they force two vertices of degree ≥ 4 and each region sends at most $\frac{5\pi}{30}$ and at least two edges in $\hat{\Delta}$ have splits and hence $c^*(\hat{\Delta}) \leq c(3, 3, 3, 3, 4, 4) + 2(\frac{5\pi}{30}) = 0$. This case is given by $\hat{\Delta} = \hat{\Delta}_3$ in Figures 3.9(ii) or (iv) with $\hat{\Delta} = \hat{\Delta}_1$ in Figures 3.8(ii) or (iv); $\hat{\Delta} = \hat{\Delta}_3$ in Figure 3.9(iii) with $\hat{\Delta} = \hat{\Delta}_1$ in Figure 3.8(ii); $\hat{\Delta} = \hat{\Delta}_1$ in Figure 3.8(ii) with $\hat{\Delta} = \hat{\Delta}_6$ in Figure 3.8(iii) or in Figure 3.9(iv); $\hat{\Delta} = \hat{\Delta}_6$ in Figure 3.8(iii) with $\hat{\Delta} = \hat{\Delta}_1$ in Figure 3.8(iv); $\hat{\Delta} = \hat{\Delta}_1$ in Figure 3.8(iv) with $\hat{\Delta} = \hat{\Delta}_6$ in Figure 3.9(iv) or in Figure 3.9(iv); $\hat{\Delta} = \hat{\Delta}_2$ in Figures 3.10(iii), (iv) or (v) with $\hat{\Delta} = \hat{\Delta}_2$ 3.5(ix); $\hat{\Delta} = \hat{\Delta}_1$ in Figure 3.10(iv) with $\hat{\Delta} = \hat{\Delta}_2$ in Figure 3.10(iv); $\hat{\Delta} = \hat{\Delta}_1$ in Figure 3.10(vi) with $\hat{\Delta} = \hat{\Delta}_2$ in Figure 3.10(vi).

Now consider regions that send curvature through a part of an edge so $\hat{\Delta} \in \{\hat{\Delta}_4, \hat{\Delta}_5\}$, we indicate them as Type B. Then such a region sends $\leq \frac{5\pi}{30}$ unless it is Δ in Figures 3.7 which send $\leq \frac{10\pi}{30}$. Checking Figures 3.2- 3.10 (up to symmetry) shows that $\hat{\Delta}_5$ can never receive positive curvature from Δ and $\hat{\Delta}_4$ at the same time; and similarly $\hat{\Delta}_4$ can never receive positive curvature from Δ and $\hat{\Delta}_5$ at the same time (this explains the zero in Figure 3.11). Moreover, two Type B regions are adjacent on the same edge only if the shared vertex is of degree ≥ 5 . That is only possible when a τ -power is involved in the labelling of $\hat{\Delta}_5$ or $\hat{\Delta}_4$ in Figures 3.8 or 3.9 respectively. Note further that if a Type B region is adjacent to a positive region of Type A then the shared vertex must be of degree ≥ 5 except for regions in Figure 3.8(iii) with Figures 3.7(ii), 3.6(iv) or 3.6(v). Moreover when compensating to $\hat{\Delta}_4$ or $\hat{\Delta}_5$ then this either forces at least one further split on $\hat{\Delta}$ as in Figure 3.7(iv) for example or it forces at vertex of degree ≥ 4 as in Figure 3.7(i) for instance. Thus if $\hat{\Delta}$ is receiving from at least one Type A region such that $c^*(\hat{\Delta}) \leq 0$ and we place regions, say Δ_i , of Type B on the edges of $\hat{\Delta}$ such that one region occurs per edge then each one of Δ_i either contributes by $\leq \frac{\pi}{3} + (-1 + \frac{2\pi}{3}) = 0$ when it forces a further split or by $\leq \frac{\pi}{6} + (\frac{\pi}{2} - \frac{2\pi}{3}) = 0$ when it forces a vertex degree to be ≥ 4 so we can ignore such occurrence since it

does not increase $c^*(\widehat{\Delta})$. If Δ_i are occurring more than once per edge then there are two cases; either no τ -powers are involved in the labelling of Δ_i as for Δ in Figure 3.2(ii) for example or there is possibly a τ -power as for Δ in Figure 3.3(ii) when $\widehat{\Delta} = \widehat{\Delta}_5$. In the former case checking all such regions shows that the only possibilities are regions in Figure 3.6(vii) with Figures 3.5(v) or 3.5(vi) where they can occur in the same time on the edge $(4, 1)$ in which case $\widehat{\Delta} = \widehat{\Delta}_5$ in Figure 3.5(v) or Figure 3.5(vi) with $\widehat{\Delta} = \widehat{\Delta}_4$ in Figure 3.6(vii) when $l(\widehat{v}_4) = 44\bar{2}$; or the edge $(3, 4)$ in which case $\widehat{\Delta} = \widehat{\Delta}_5$ in Figure 3.6(vii) with $\widehat{\Delta} = \widehat{\Delta}_5$ in Figure 3.5(v). They, however, send at most $\frac{5\pi}{30}$ each and force at least one further split and so the most they contribute to $c^*(\widehat{\Delta}) \leq \frac{-\pi}{3} + 2(\frac{5\pi}{30}) = 0$ and hence we can ignore such occurrence, see Figure 3.11(i). In the latter case when τ -powers may occur regions may occur more than twice per edge. Let (i, j) be an edge in $\widehat{\Delta}$ on which k , $k \geq 3$ regions are placed. Let (i, j) be an edge in $\widehat{\Delta}$ on which k , $k \geq 3$ regions are placed. Then there are two possible setting as shown in Figure 3.11(ii)-(iii). In Figure 3.11(ii) note that when Δ_i and Δ_{i+1} are both positive and sending their curvature to $\widehat{\Delta}$ then the shared t-vertex is of degree ≥ 5 and each of Δ_i and Δ_{i+1} is sending $\leq \frac{2\pi}{30}$ while in Figure 3.11(iii) if Δ_i is sending positive curvature to $\widehat{\Delta}$ then it sends $\leq \frac{5\pi}{30}$ and forcing a vertex of degree ≥ 4 while Δ_{i+1} is sending no positive curvature to $\widehat{\Delta}$. Thus for Δ_i , $3 \leq i \leq k-2$ regions placed on the edge (i, j) , two consecutive regions either send $\leq 2(\frac{2\pi}{30})$ and force a vertex of degree ≥ 5 so the net contribution is $\leq \frac{4\pi}{30} + (-1 + \frac{2}{5})\pi < 0$; or send $\leq \frac{5\pi}{30}$ and force a vertex of degree ≥ 4 so the net contribution is $\leq \frac{5\pi}{30} + (-1 + \frac{2}{4})\pi = 0$. Therefore we can ignore the regions $\Delta_3, \Delta_4, \dots, \Delta_{k-2}$ and the curvature they may send as this will not increase $c^*(\widehat{\Delta})$. This leaves the case when $k = 3$. In this case the situation is as in Figure 3.11(i) and for $c(\Delta_1) + c(\Delta_k) \leq \frac{10\pi}{30}$ and the same argument used above works here and we can again ignore such occurrence as it does not increase $c^*(\widehat{\Delta})$. So let $\frac{10\pi}{30} < c(\Delta_1) + c(\Delta_k) \leq \frac{15\pi}{30}$. This implies that Δ_1 or Δ_k is sending $\frac{10\pi}{30}$ and hence is one of the Figures 3.7(ii)-(iii), note that Figure 3.7(iv) send $\leq \frac{\pi}{3}$ but its the only Type B region which can be placed on $(1, 2)$. In Figure 3.7(ii) checking shows that $\widehat{\Delta} = \widehat{\Delta}_5$ does not receive from $\widehat{\Delta}_1$ nor $\widehat{\Delta}_4$. If $\widehat{\Delta}$ is receiving any positive curvature through the half edge $(w_2, 2)$ from one region then this forces $d(\widehat{v}_2) \geq 4$ and the most received is $\leq \frac{5\pi}{30}$ and the net contribution to $c^*(\widehat{\Delta})$ is ≤ 0 . If in addition $\widehat{\Delta}$ is receiving any positive curvature through the edge $(2, 3)$ then the only possible region is Δ in Figure 3.3(ii) but this forces $d(\widehat{v}_2) \geq 5$ and the most received is $\leq 2(\frac{2\pi}{30})$ and so the net contribution

is $\leq (\frac{2}{5} - \frac{2}{3}) + 2(\frac{2\pi}{30}) < 0$.

In Figure 3.7(iii) if $\hat{\Delta} = \hat{\Delta}_4$ receives through the half edge $(3, k)$ then $d(\hat{v}_3) \leq 4$ and the net contribution through the edge $(3, 4)$ is $\leq (-1 + \frac{2}{3})\pi + (\frac{1}{2} - \frac{2}{3})\pi + \frac{\pi}{3} + \frac{\pi}{6} = 0$. If in addition $\hat{\Delta}$ is receiving through the edge $(2, 3)$ then this forces $d(\hat{v}_3) \leq 5$ and the most received across the two regions adjacent to $\hat{v}_3$ is $2(\frac{2\pi}{30})$ and so the net contribution is $\leq (\frac{2}{5} - \frac{2}{3}) + 2(\frac{2\pi}{30}) < 0$. Moreover if $\hat{\Delta}_3$ is sending positive curvature to $\hat{\Delta} = \hat{\Delta}_4$ then $\hat{\Delta}$ plays the role of $\hat{\Delta}_6$ in Figure 3.8(iii) and the most it send is $\frac{2\pi}{30}$ and it forces $d(\hat{v}_1) \geq 5$. Also if $\hat{\Delta}$ is receiving through the edge $(1, 2)$ then the most it send is $\frac{2\pi}{30}$ and so the net contribution is $\leq (\frac{2}{5} - \frac{2}{3}) + 2(\frac{2\pi}{30}) < 0$.

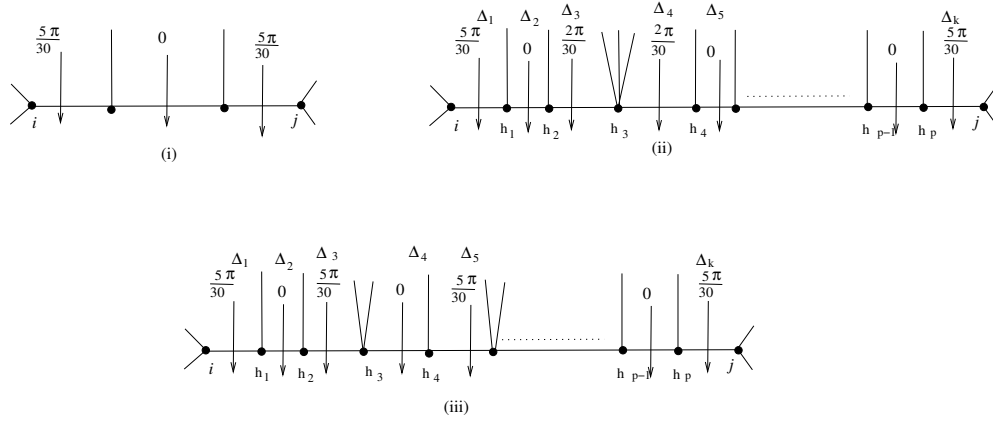


Figure 3.11

3.2 Case II (a2)

In this case $r(t) = wt^{l_1}wt^{l_2}w^{-1}t^{l_3}vt^{l_4}$ and Figure 3.12 shows all possible labellings for $d(\Delta) = 5$.

Assume first that $d(v_i) = 3$, $1 \leq i \leq 4$. Then $d(v_1) = d(v_2) = 3$ forces

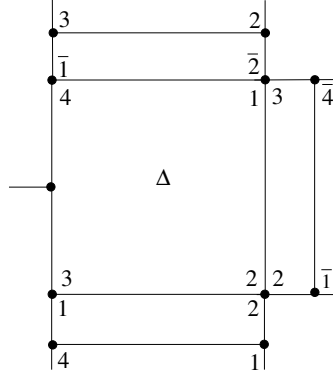


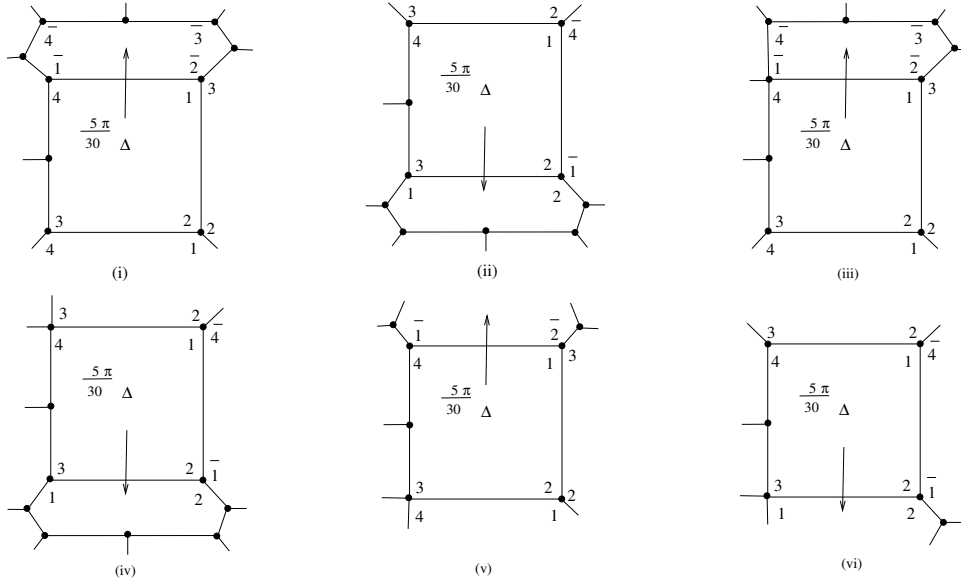
Figure 3.12: All possible labelling.

$l(v_1) \in \{\bar{2}13, 21\bar{4}\}$. If $l(v_1) = \bar{2}13$ then $l(v_2) = 221$ and either $l(v_3) \in 43\{2, k\}$ and $l(v_4) \in 4\bar{1}\{\bar{1}, k\}$; or $l(v_3) \in 43\{4, 3\}$ and $l(v_4) = 4\bar{1}k$; or $l(v_3) \in 34\bar{1}$ and $l(v_4) = 4\bar{1}3$ so $l(v_4)$ forces a split on $\hat{\Delta}_1$, see Figure 3.13(i). If $l(v_1) = 21\bar{4}$ then $l(v_2) = \bar{1}22$ and either $l(v_3) = 13k$ and $l(v_4) \in 43\{2, 3, 4, k\}$; or $l(v_3) = 134$ and $l(v_4) = 431$; or $l(v_3) = 131$ and $l(v_4) = 43\{2, 3, k\}$ so $l(v_3)$ forces a split on $\hat{\Delta}_3$, see Figure 3.13(ii).

Now assume that exactly one vertex is of degree ≥ 4 . If $d(v_1) = d(v_2) = d(v_3) = 3$ then either $l(v_1) = \bar{2}13$, $l(v_2) = 221$ and $l(v_3) \in 43\{2, 3, 4, \bar{1}, k\}$ forcing a split on $\hat{\Delta}_3$, see Figure 3.13(iii); or $l(v_1) = 21\bar{4}$, $l(v_2) = \bar{1}22$ and $l(v_3) \in 13\{1, 4, k\}$, see Figure 3.13(iv).

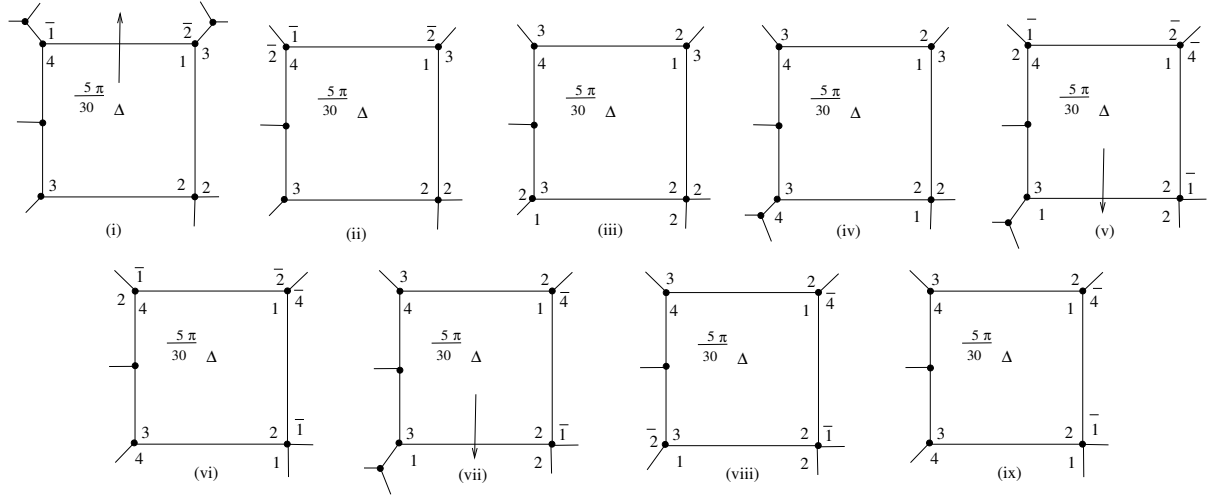
If $d(v_1) = d(v_2) = d(v_4) = 3$ then either $l(v_1) = \bar{2}13$, $l(v_2) = 221$ and $l(v_4) \in 4\bar{1}\{\bar{1}, 3, k\}$ forcing a split on $\hat{\Delta}_1$, see Figure 3.13(v); or $l(v_1) = 21\bar{4}$, $l(v_2) = \bar{1}22$ and $l(v_4) \in 43\{1, 2, 3, 4, k\}$, see Figure 3.13(vi).

If $d(v_1) = d(v_3) = d(v_4) = 3$ then $l(v_1) \in \{\bar{2}13, 213, \bar{2}1\bar{4}, 21\bar{4}\}$. Let $l(v_1) = \bar{2}13$, see Figure 3.14(i)-(ii). Then either $l(v_3) = 13\bar{2}$ and $l(v_4) \in 4\bar{1}\{\bar{1}, 3, 4, k, \bar{2}\}$; or $l(v_3) = 43k$ and $l(v_4) \in 4\bar{1}\{\bar{1}, 4, k, \bar{2}\}$; or $l(v_3) = 434$ and $l(v_4) \in 4\bar{1}\{\bar{1}, k, \bar{2}\}$; or

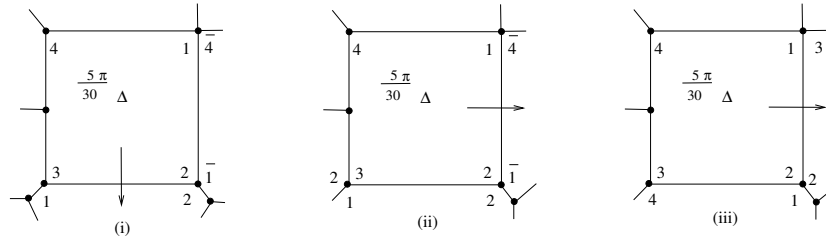
Figure 3.13: $d(v_i) = 3$, $1 \leq i \leq 4$.

$l(v_3) = 43\bar{1}$ and $l(v_4) \in 4\bar{1}3$; or $l(v_3) = 433$ and $l(v_4) \in 4\bar{1}\{4, k, \bar{2}\}$; or $l(v_3) = 432$ and $l(v_4) \in 4\bar{1}\{\bar{1}, 4, k\}$. Let $l(v_1) = 213$, see Figure 3.14(iii)-(iv). Then either $l(v_3) = 132$ and $l(v_4) \in 43\{\bar{1}, \bar{2}, 3, 4, k\}$, see Figures 3.14(iii); or $l(v_3) = l(v_4) \in 43\{\bar{1}, \bar{2}, 3, 4, k\}$, see Figure 3.14(iv). Let $l(v_1) = \bar{2}1\bar{4}$, see Figures 3.14(v)-(vii). Then $l(v_3) \in \{13\{3, 4, 2, k\}, 43\{\bar{2}, 1, 3, 4, k\}\}$ and $l(v_4) = 4\bar{1}2$. Let $l(v_1) = 21\bar{4}$, see Figure 3.14(viii)-(x). Then either $l(v_3) = 13k$ and $l(v_4) \in 43\{2, 3, 4, k\}$; or $l(v_3) = l(v_4) = 431$; or $l(v_3) = 133$ and $l(v_4) \in 43\{2, 4, k\}$; or $l(v_3) = 131$ and $l(v_4) \in 43\{2, 3, k\}$, see Figure 3.14(viii); or $l(v_3) = 13\bar{2}$ and $l(v_4) \in 43\{3, 4, k\}$, see Figure 3.14(ix); or $l(v_3) = l(v_4) \in 43\{1, 2, 3, 4, k\}$, see Figure 3.14(x).

If $d(v_2) = d(v_3) = d(v_4) = 3$ then $l(v_2) \in \{\bar{1}22, 221\}$. Let $l(v_2) = \bar{1}22$, see Figure 3.15(i)-(ii). Then either $l(v_3) = 13k$ and $l(v_4) \in \{4\bar{1}\{\bar{1}, \bar{2}, \bar{3}, 3, k\}, 43\{2, 3, 4, \bar{1}, \bar{2}, k\}\}$; or $l(v_3) = 134$ and $l(v_4) \in \{4\bar{1}\{\bar{1}, \bar{2}, k\}, 431\}$; $l(v_3) = 13\bar{4}$ and $l(v_4) \in \{4\bar{1}\bar{3}, 43\{2, 3, 4, \bar{2}, k\}\}$; $l(v_3) = 131$ and $l(v_4) \in \{4\bar{1}\{3, \bar{2}, k\}, 43\{2, 3, \bar{1}, \bar{2}, k\}\}$, see Figure 3.15(i); $l(v_3) = 132$ and $l(v_4) \in \{4\bar{1}\{3, \bar{1}, k\}, 43\{3, 4, \bar{1}, \bar{2}, k\}\}$, see Figure 3.15(ii). Let $l(v_2) = 221$, see Figure 3.15(iii). Then either $l(v_3) = 43k$ and $l(v_4) \in \{4\bar{1}\{\bar{1}, \bar{3}, 2, k\}, 43k\}$; or $l(v_3) = 434$ and $l(v_4) \in \{4\bar{1}\{\bar{1}, \bar{3}, 2, k\}, 434\}$; or $l(v_3) = 43\bar{1}$ and $l(v_4) \in \{4\bar{1}3, 43\bar{1}\}$; or $l(v_3) = 433$ and $l(v_4) \in \{4\bar{1}\{\bar{1}, 2, \bar{3}, k\}, 433\}$; or $l(v_3) = 431$ and $l(v_4) \in$


 Figure 3.14: $d(v_2) \geq 4$.

$\{4\bar{1}\{2, 1, k\}, 431\}$; or $l(v_3) = 432$ and $l(v_4) \in \{4\bar{1}\{\bar{1}, \bar{3}, k\}, 432\}$; or $l(v_3) = 43\bar{2}$ and $l(v_4) \in \{4\bar{1}\{2, \bar{3}, k\}, 43\bar{2}\}$.


 Figure 3.15: $d(v_1) \geq 4$.

This complete the description of positive regions and we will proceed by showing the distribution for the positive regions case by case.

Δ is given by Figures 3.13(i), 3.13(v):

Then $d(\hat{\Delta}_1) \geq 7$ and $c(\hat{\Delta}_1) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$ and we add $c(\Delta)$ to $c(\hat{\Delta}_1)$.

Δ is given by Figures 3.13(ii), 3.13(iv):

Similarly to above $d(\hat{\Delta}_3) \geq 7$ and $c(\hat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$ and we add $c(\Delta)$ to $c(\hat{\Delta}_3)$.

Δ is given by Figure 3.13(iii):

We have $d(\widehat{\Delta}_1) \geq 6$ with $d(v_4) \geq 4$ and we add $c(\Delta) \leq \frac{5\pi}{30}$ to $c(\widehat{\Delta}_1) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$.

Δ is given by Figure 3.13(vi):

We have $d(\widehat{\Delta}_3) \geq 6$ with $d(v_2) \geq 4$ and we add $c(\Delta) \leq \frac{5\pi}{30}$ to $c(\widehat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$.

Δ is given by Figure 3.14(i):

As in Figure 3.13(i) we add $c(\Delta)$ to $c(\widehat{\Delta}_1) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$.

Δ is given by Figure 3.14(ii):

In this case $l(v_3) \in \{13\bar{2}, 43\{3, 4, k\}\}$. If $l(v_3) \in \{13\bar{2}, 433\}$ then $l(u_4) = vw_1w_2$ and at least one of w_1 and w_2 is a proper subword of $w^{\pm 1}$. If w_1 is a proper subword of w^{-1} then add $c(\Delta) \leq \frac{5\pi}{30}$ to $c(\widehat{\Delta}_5) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$ otherwise w_2 is a proper subword of $w^{\pm 1}$ and add $c(\Delta) \leq \frac{5\pi}{30}$ to $c(\widehat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$, see Figure 3.16(i). If $l(v_3) \in 43\{4, k\}$ then checking all possible labellings for v_2 when $d(v_2) \in \{4, 5\}$ shows that at least the edge $(1, 2)$ in $\widehat{\Delta}_2$ splits or the edge $(1, 2)$ in $\widehat{\Delta}_3$ splits. In the former case we add $c(\Delta) \leq \frac{5\pi}{30}$ to $c(\widehat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$ and in the latter case we add $c(\Delta) \leq \frac{5\pi}{30}$ to $c(\widehat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$, see Figure 3.16(ii).

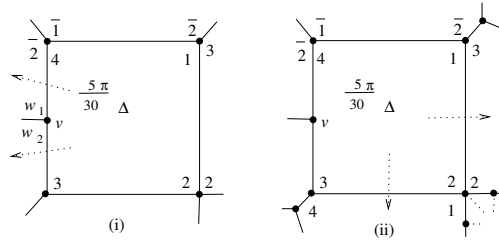
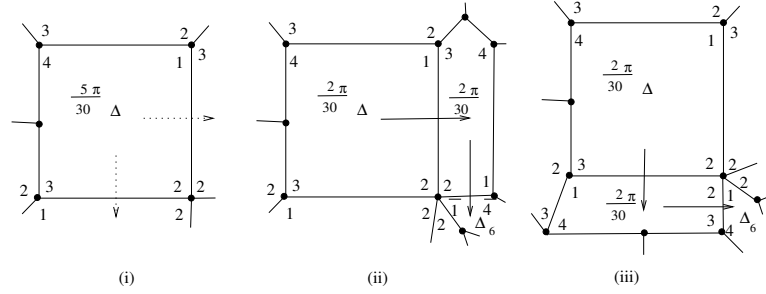


Figure 3.16: $d(v_2) \geq 4$.

Δ is given by Figure 3.14(iii):

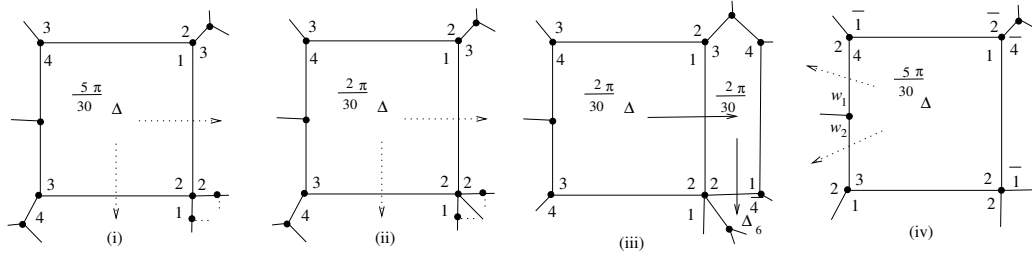
Then $l(v_4) \in 43\{3, \bar{1}, \bar{2}, 4, k\}$. First let $d(v_2) = 4$. Then this implies $c(\Delta) \leq \frac{5\pi}{30}$ and $l(v_2) \in 222\{1, \bar{1}, 3, \bar{3}, 4, \bar{4}, k\}$ and this except for $l(v_2) = 222\bar{1}$, forces $d(\widehat{\Delta}_2) \geq 6$ with at least one vertex of degree ≥ 4 and we add $c(\Delta)$ to $c(\widehat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$. While if $l(v_2) = 222\bar{1}$ then this forces $d(\widehat{\Delta}_3) \geq 6$ with at

least one vertex of degree ≥ 4 and we add $c(\Delta)$ to $c(\hat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$, see Figure 3.17(i). Now let $d(v_2) = 5$. Then $c(\Delta) \leq \frac{2\pi}{30}$ and $l(v_2)$ forces one more split on at least one of $\hat{\Delta}_2$ or $\hat{\Delta}_3$ except for $l(v_2) \in \{2222\bar{1}, 22212\}$. If $l(v_2)$ forces one more split on $\hat{\Delta}_2$ then add $c(\Delta)$ to $c(\hat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$; if not and $l(v_2)$ forces one more split on $\hat{\Delta}_3$ then add $c(\Delta)$ to $c(\hat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$; otherwise if $l(v_2) = 2222\bar{1}$ then this forces $d(\Delta_6) \geq 6$ and we add $c(\Delta) + c(\hat{\Delta}_2) \leq \frac{4\pi}{30}$ to $c(\hat{\Delta}_6) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$, see Figure 3.17(ii) and if $l(v_2) = 22212$ then this forces $d(\Delta_6) \geq 7$ and we add $c(\Delta) + c(\hat{\Delta}_3) \leq \frac{4\pi}{30}$ to $c(\hat{\Delta}_7) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$, see Figure 3.17(iii).

Figure 3.17: $d(v_2) \geq 4$.

Δ is given by Figure 3.14(iv):

Then $l(v_3) = l(v_4) \in 43\{3, 4, \bar{1}, \bar{2}, k\}$. First let $d(v_2) = 4$. Then $c(\Delta) \leq \frac{5\pi}{30}$ and $l(v_3) = l(v_4) \in 43\{3, 4, \bar{1}, k\}$ implies $l(v_2) \in 221\{2, \bar{3}, 4, k\}$ and $l(v_3) = l(v_4) = 43\bar{2}$ implies $l(v_2) \in 221\{2, \bar{3}, \bar{4}, 4, k\}$ and this except for $l(v_2) = 2212$, forces $d(\hat{\Delta}_2) \geq 6$ with at least one vertex of degree ≥ 4 and we add $c(\Delta)$ to $c(\hat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$. While if $l(v_2) = 2212$ then this forces $d(\hat{\Delta}_3) \geq 6$ with at least one vertex of degree ≥ 4 and we add $c(\Delta)$ to $c(\hat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$, see Figure 3.18(i)-(ii). Now let $d(v_2) = 5$. Then this implies $c(\Delta) \leq \frac{2\pi}{30}$ and $l(v_2)$ forces one more split on at least one of $\hat{\Delta}_2$ or $\hat{\Delta}_3$ except for $l(v_2) \in 2214\{\bar{1}, 2\}$ which is only possible when $l(v_3) = l(v_4) \in 43\{4, \bar{1}, k\}$ otherwise there is a labelling contradiction. If $l(v_2)$ forces one more split on $\hat{\Delta}_2$ then add $c(\Delta)$ to $c(\hat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$; if not and $l(v_2)$ forces one more split on $\hat{\Delta}_3$ then add $c(\Delta)$ to $c(\hat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$; otherwise $l(v_2) \in 2214\{\bar{1}, 2\}$ and this forces $d(\Delta_6) \geq 6$ and we add $c(\Delta) + c(\hat{\Delta}_2) \leq \frac{4\pi}{30}$ to $c(\hat{\Delta}_6) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$, see Figure 3.18(iii).

Figure 3.18: $d(v_2) \geq 4$.

Δ is given by Figures 3.14(v) and 3.14(viii):

As in Figure 3.13(vi) $l(v_3)$ forces at least one split on $\hat{\Delta}_3$ and we add $c(\Delta) \leq \frac{5\pi}{30}$ to $c(\hat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$.

Δ is given by Figures 3.14(vi):

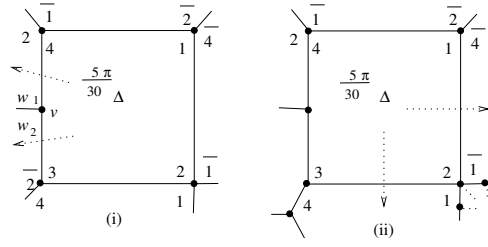
In this case we at least have w_1 be a proper subword of w^{-1} or w_2 be a proper subword of w . In the former case we add $c(\Delta) \leq \frac{5\pi}{30}$ to $c(\hat{\Delta}_5) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$ and in the latter case we add $c(\Delta) \leq \frac{5\pi}{30}$ to $c(\hat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$, see Figure 3.18(iv).

Δ is given by Figure 3.14(vii):

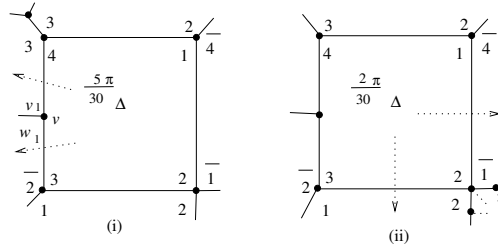
In this case $l(v_3) \in 43\{\bar{2}, 1, 3, 4, k\}$. If $l(v_3) = 43\bar{2}$ then $l(u_4) = vw_1w_2$, see Figure 3.19(i). If w_1 is a proper subword of w^{-1} then we add $c(\Delta) \leq \frac{5\pi}{30}$ to $c(\hat{\Delta}_5) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$ otherwise w_2 is a proper subword of w^{-1} and so $c(\hat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$ we add $c(\Delta) \leq \frac{5\pi}{30}$ to $c(\hat{\Delta}_4)$. If $l(v_3) \neq 43\bar{2}$ then checking all the possible labelling when $d(v_2) \in \{4, 5\}$ implies $l(v_2)$ forces one more split on at least one of $\hat{\Delta}_2$ or $\hat{\Delta}_3$. If $l(v_2)$ forces one more split on $\hat{\Delta}_2$ then add $c(\Delta)$ to $c(\hat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$; if not then $l(v_2)$ forces one more split on $\hat{\Delta}_3$ then add $c(\Delta)$ to $c(\hat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$, see Figure 3.19(ii).

Δ is given by Figure 3.14(ix):

Then $l(v_4) \in 43\{3, 4, k\}$. If $l(v_4) = 433$ then $l(u_4) = vv_1w_1$, see Figure 3.20(i). If w_1 is a proper subword of w then we add $c(\Delta) \leq \frac{5\pi}{30}$ to $c(\hat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$ otherwise v_1 is a proper subword of v . Also $l(v_4)$ forces one more split on $\hat{\Delta}_5$ and so $c(\hat{\Delta}_5) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$.

Figure 3.19: $d(v_2) \geq 4$.

we add $c(\Delta) \leq \frac{5\pi}{30}$ to $c(\hat{\Delta}_5)$. If $l(v_3) \in 43\{4, k\}$ then checking all the possible labelling when $d(v_2) \in \{4, 5\}$ implies $l(v_2)$ forces one more split on at least one of $\hat{\Delta}_2$ or $\hat{\Delta}_3$. If $l(v_2)$ forces one more split on $\hat{\Delta}_2$ then add $c(\Delta)$ to $c(\hat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$; if not then $l(v_2)$ forces one more split on $\hat{\Delta}_3$ then add $c(\Delta)$ to $c(\hat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$, see Figure 3.20(ii).

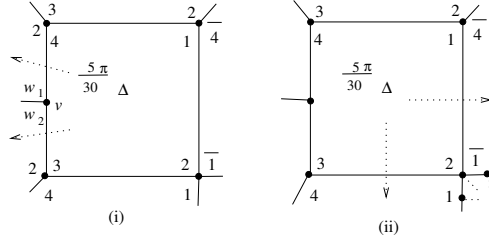
Figure 3.20: $d(v_2) \geq 4$.

Δ is given by Figure 3.14(x):

Then $l(v_3) = l(v_4) \in 43\{3, 4, 1, 2, k\}$. If $l(v_3) = l(v_4) = 432$ then $l(u_4) = vw_1w_2$, see Figure 3.21(i). If w_1 is a proper subword of w^{-1} then add $c(\Delta) \leq \frac{5\pi}{30}$ to $c(\hat{\Delta}_5) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$ otherwise w_2 is a proper subword of w and we add $c(\Delta) \leq \frac{5\pi}{30}$ to $c(\hat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$. If $l(v_3) = l(v_4) \neq 432$ then checking all the possible labelling when $d(v_2) \in \{4, 5\}$ implies $l(v_2)$ forces one more split on at least one of $\hat{\Delta}_2$ or $\hat{\Delta}_3$. If $l(v_2)$ forces one more split on $\hat{\Delta}_2$ then add $c(\Delta)$ to $c(\hat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$; if not then $l(v_2)$ forces one more split on $\hat{\Delta}_3$ then add $c(\Delta)$ to $c(\hat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$, see Figure 3.21(ii).

Δ is given by Figure 3.15(i):

Then $l(v_2)$ and $l(v_3)$ forces at least one split on $\hat{\Delta}_3$ each and so $d(\hat{\Delta}_3) \geq 7$ and

Figure 3.21: $d(v_2) \geq 4$.

$$c(\hat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}. \text{ Add } c(\Delta) \text{ to } c(\hat{\Delta}_3).$$

Δ is given by Figures 3.15(ii)-(iii):

Then $l(v_2)$ forces at least one split on $\hat{\Delta}_2$ and so $d(\hat{\Delta}_2) \geq 6$ with one vertex of degree ≥ 4 . add $c(\Delta) \leq \frac{5\pi}{30}$ to $c(\hat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$.

Now assume that $\hat{\Delta} \in \{\hat{\Delta}_1, \hat{\Delta}_2, \hat{\Delta}_3, \hat{\Delta}_4, \hat{\Delta}_5, \hat{\Delta}_6, \hat{\Delta}_7\}$. If $\hat{\Delta}$ receives from exactly one region then as we see from above $c^*(\hat{\Delta}) \leq 0$. Now assume $\hat{\Delta}$ receives from two or more regions. First consider regions that compensate through a whole edge, say Type A. Then $\hat{\Delta} \in \{\hat{\Delta}_1, \hat{\Delta}_2, \hat{\Delta}_3, \hat{\Delta}_6, \hat{\Delta}_7\}$ and checking all Type A regions shows that $\hat{\Delta}$ receives from at most two Type A regions namely; $\hat{\Delta} = \hat{\Delta}_2$ in Figure 3.15(ii) with $\hat{\Delta} = \hat{\Delta}_2$ in Figure 3.17(ii) or in Figure 3.18(i)-(ii); $\hat{\Delta} = \hat{\Delta}_2$ in Figure 3.16(ii) with $\hat{\Delta} = \hat{\Delta}_3$ in Figure 3.16(ii); $\hat{\Delta} = \hat{\Delta}_2$ in Figure 3.16(ii) with $\hat{\Delta} = \hat{\Delta}_2$ in Figure 3.20(ii); $\hat{\Delta} = \hat{\Delta}_2$ in Figure 3.16(ii) with $\hat{\Delta} = \hat{\Delta}_2$ or $\hat{\Delta} = \hat{\Delta}_3$ in Figure 3.21(ii); $\hat{\Delta} = \hat{\Delta}_2$ in Figure 3.17(i)-(ii) with $\hat{\Delta} = \hat{\Delta}_6$ in Figure 3.17(iii) or with $\hat{\Delta} = \hat{\Delta}_3$ in Figure 3.18(i)-(ii) or with $\hat{\Delta} = \hat{\Delta}_6$ in Figure 3.18(iii) or with $\hat{\Delta} = \hat{\Delta}_2$ or $\hat{\Delta} = \hat{\Delta}_3$ in Figure 3.19(ii); $\hat{\Delta} = \hat{\Delta}_6$ in Figure 3.17(iii) with $\hat{\Delta} = \hat{\Delta}_2$ in Figure 3.18(i)-(ii); $\hat{\Delta} = \hat{\Delta}_7$ in Figure 3.17(iii) with $\hat{\Delta} = \hat{\Delta}_6$ in Figure 3.18(iii); $\hat{\Delta} = \hat{\Delta}_2$ in Figure 3.18(i)-(ii) with $\hat{\Delta} = \hat{\Delta}_3$ in Figure 3.18(i)-(ii); $\hat{\Delta} = \hat{\Delta}_2$ in Figure 3.18(i)-(ii) with $\hat{\Delta} = \hat{\Delta}_2$ or $\hat{\Delta} = \hat{\Delta}_3$ in Figure 3.19(ii); and $\hat{\Delta} = \hat{\Delta}_2$ or $\hat{\Delta} = \hat{\Delta}_3$ in Figure 3.19(ii) with $\hat{\Delta} = \hat{\Delta}_2$ in Figure 3.15(iii). In each case the region $\hat{\Delta}$ is forced to have at least 2 vertices of degree ≥ 4 with 2 edges split and receives at most $\frac{5\pi}{30}$ from each region and hence $c^*(\Delta) \leq c(3, 3, 3, 3, 4, 4) + 2(\frac{5\pi}{30}) = 0$.

Now consider the regions that compensate through a part of an edge, Type B regions. Then $\hat{\Delta} \in \{\hat{\Delta}_4, \hat{\Delta}_5\}$ and the most $\hat{\Delta}$ can receive from one region of Type B is $\frac{5\pi}{30}$. Moreover checking all regions of Type B shows that $\hat{\Delta}_4$ can never receive

positive curvature from Δ and $\widehat{\Delta}_5$ at the same time and that $\widehat{\Delta}_5$ can never receive positive curvature from Δ and $\widehat{\Delta}_4$ at the same time. In addition checking shows that at most 2 region of this Type B can be placed together on the same edge namely regions : $\widehat{\Delta}_4$ in Figure 3.16(i) when $l(v_3) = 13\bar{2}$ with $\widehat{\Delta}_4$ in Figure 3.16(i) when $l(v_3) = 433$; and $\widehat{\Delta}_4$ in Figure 3.16(i) when $l(v_3) = 433$ with $\widehat{\Delta}_4$ in Figure 3.20(i) ; can be placed on the edge $(2, 3)$. However placing this Type B region at the edge (i, j) on $\widehat{\Delta}$ yields the situation given by Figure 3.11(i). Assuming that $c^*(\Delta) \leq 0$, then Type B regions contribute at most by $\leq 2(\frac{5\pi}{30}) + (-1 + \frac{2\pi}{3}) = 0$ and so we can ignore such occurrence since it does not increase $c^*(\widehat{\Delta})$.

3.3 Case II (a3)

In this case $r(t) = wt^{l_1}w^{-1}t^{l_2}wt^{l_3}vt^{l_4}$ and Figure 3.22 shows all possible labellings for $d(\Delta) = 5$.

First assume that $d(v_i) = 3$ for $1 \leq i \leq 4$. Then $d(v_1) = 3$ implies

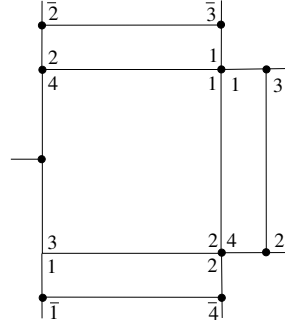


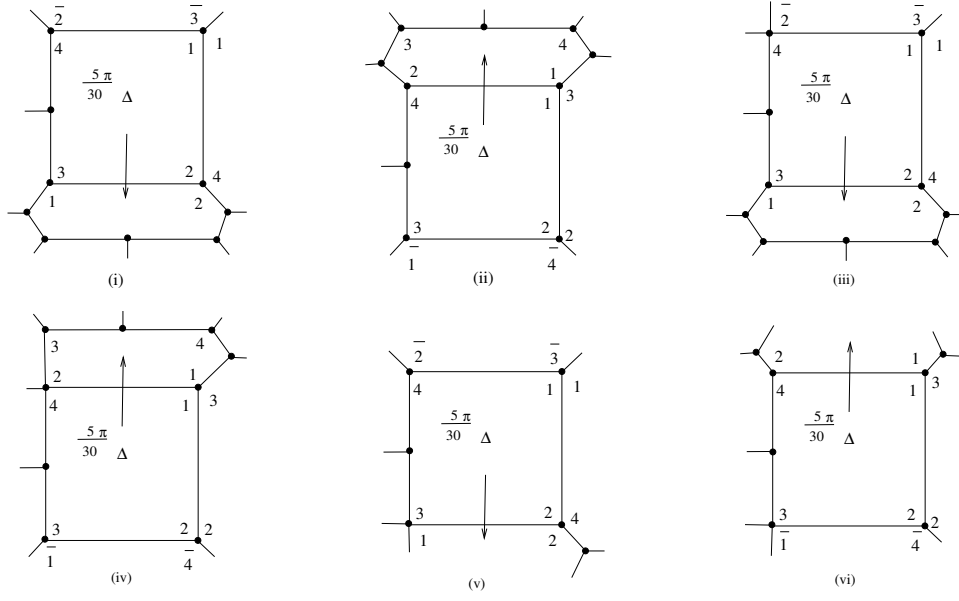
Figure 3.22: All possible labelling.

$l(v_1) \in \{\bar{3}11, 113\}$. Let $l(v_1) = \bar{3}11$. Then this forces $l(v_2) = 422$ and either $l(v_3) \in 13\{2, \bar{2}, 4, \bar{4}\}$ and $l(v_4) = 4\bar{2}k'$; or $l(v_3) = 13k$ and $l(v_4) \in 4\bar{2}\{1, \bar{1}, 3, \bar{3}, k'\}$, see Figure 3.23(i). Let $l(v_1) = 113$. Then this forces $l(v_2) = 22\bar{4}$ and either $l(v_3) \in \bar{1}3\{2, \bar{2}, 4, \bar{4}\}$ and $l(v_4) = 42k'$; or $l(v_3) = \bar{1}3k$ and $l(v_4) \in 42\{1, \bar{1}, 3, \bar{3}, k'\}$, see Figure 3.23(ii).

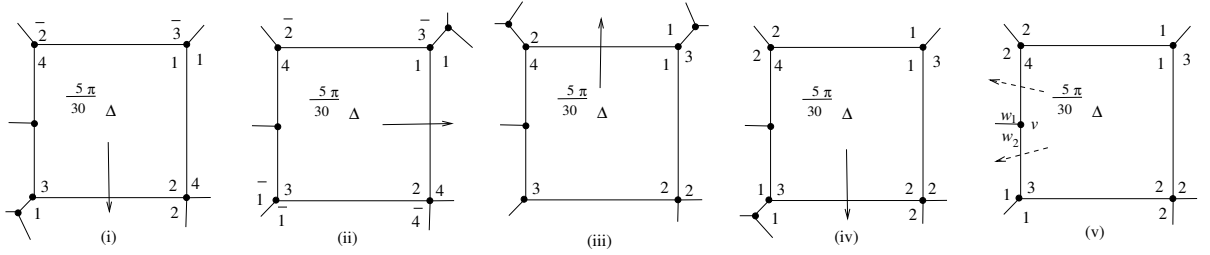
Now assume that exactly one vertex is of degree ≥ 4 . If $d(v_1) = d(v_2) = d(v_3) = 3$ then either $l(v_1) = \bar{3}11$, $l(v_2) = 422$ and $l(v_3) \in 13\{2, \bar{2}, 4, \bar{4}, k\}$, see Figure 3.23(iii); or $l(v_1) = 113$, $l(v_2) = 22\bar{4}$ and $l(v_3) \in \bar{1}3\{2, \bar{2}, 4, \bar{4}, k\}$, see Figure 3.23(iv).

If $d(v_1) = d(v_2) = d(v_4) = 3$ then either $l(v_1) = \bar{3}11$, $l(v_2) = 422$ and $l(v_4) \in 4\bar{2}\{1, \bar{1}, 3, \bar{3}, k'\}$, see Figure 3.23(v); or $l(v_1) = 113$, $l(v_2) = 22\bar{4}$ and $l(v_4) \in 42\{1, \bar{1}, 3, \bar{3}, k'\}$, see Figure 3.23(vi).

If $d(v_1) = d(v_3) = d(v_4) = 3$ then again $l(v_1) \in \{\bar{3}11, 113\}$. If $l(v_1) = \bar{3}11$ then either $l(v_3) = 13\{\bar{2}, 4\}$ and $l(v_4) \in 4\bar{2}\{\bar{1}, \bar{2}, \bar{3}, 4, k'\}$; or $l(v_3) \in 13\{2, \bar{4}\}$ and $l(v_4) \in 4\bar{2}\{1, \bar{2}, 3, 4, k'\}$; or $l(v_3) = 13k$ and $l(v_4) \in 4\bar{2}\{1, \bar{1}, \bar{2}, 3, \bar{3}, 4, k'\}$, see Figure 3.24(i); or $l(v_3) = \bar{1}3\bar{1}$ and $l(v_4) \in 4\bar{2}\{1, \bar{1}, \bar{2}, 3, \bar{3}, 4, k'\}$, see Figure 3.24(ii); or $l(v_3) = \bar{1}3\{2, 4\}$ and $l(v_4) \in 42\{1, \bar{3}, 4, k', 2\}$; or $l(v_3) \in \bar{1}3\{\bar{2}, \bar{4}\}$ and $l(v_4) \in 42\{\bar{1}, 3, 4, k', 2\}$; or $l(v_3) = \bar{1}3k$ and $l(v_4) \in 42\{1, \bar{1}, 3, \bar{3}, 4, k', 2\}$, see Figures 3.24(iii) and 3.24(iv); or $l(v_3) = 131$ and $l(v_4) \in 42\{1, \bar{1}, 3, \bar{3}, 4, k', 2\}$, see Figures 3.24(iii)-(v).

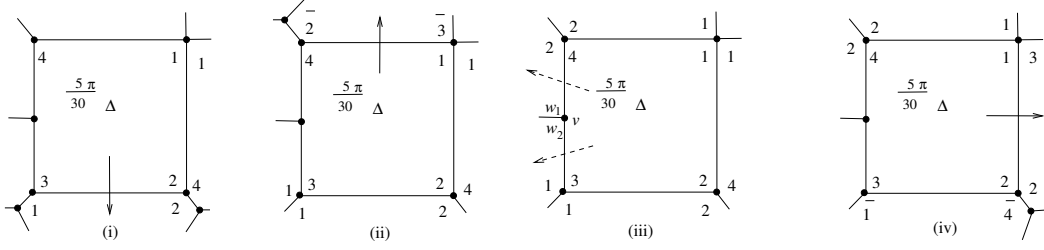
Figure 3.23: $d(v_1) = d(v_2) = 3$

If $d(v_2) = d(v_3) = d(v_4) = 3$ then $l(v_2) \in \{422, 22\bar{4}\}$. If $l(v_2) = 422$

Figure 3.24: $d(v_2) \geq 4$.

then either $l(v_3) \in 13\{2, \bar{4}\}$ and $l(v_4) \in \{4\bar{2}\{1, 3, k'\}, 422\}$; or $l(v_3) = 13\{\bar{2}, 4\}$ and $l(v_4) \in \{4\bar{2}\{\bar{1}, \bar{3}, k'\}, 422\}$, see Figure 3.25(i); or $l(v_3) = 13\{3, k, 1\}$ and $l(v_4) \in \{4\bar{2}\{1, \bar{1}, 3, \bar{3}, k'\}, 422\}$, see Figures 3.25(i)-(iii). If $l(v_2) = 22\bar{4}$ then either $l(v_3) \in \bar{1}3\{\bar{2}, \bar{4}\}$ and $l(v_4) \in \{4\bar{2}\bar{2}, 42\{\bar{1}, 3, k'\}\}$; or $l(v_3) = \bar{1}3\{2, 4\}$ and $l(v_4) \in \{4\bar{2}\bar{2}, 42\{1, \bar{3}, k'\}\}$; or $l(v_3) \in \bar{1}3\{3, \bar{1}, k\}$ and $l(v_4) \in \{4\bar{2}\bar{2}, 42\{1, \bar{1}, 3, \bar{3}, k'\}\}$, see Figure 3.25(iv).

In Figures 3.23(i), 3.23(iii) and 3.25(i) the labelling forces $d(\hat{\Delta}_3) \geq 7$ and we add $c(\Delta) \leq \frac{\pi}{3}$ to $c(\hat{\Delta}_3) \leq \frac{-\pi}{3}$. While in Figures 3.23(ii), 3.23(vi) and 3.24(iii) the


 Figure 3.25: $d(v_1) \geq 4$.

labelling forces $d(\hat{\Delta}_1) \geq 7$ and we add $c(\Delta) \leq \frac{\pi}{3}$ to $c(\hat{\Delta}_1) \leq \frac{-\pi}{3}$. Also in Figures 3.23(iv) and 3.25(ii) the labelling forces $d(\hat{\Delta}_1) \geq 6$ with at least one vertex of degree ≥ 4 and we add $c(\Delta) \leq \frac{\pi}{6}$ to $c(\hat{\Delta}_1) \leq \frac{-\pi}{6}$. Whereas in Figures 3.23(v), 3.24(i) and 3.24(iv) the labelling forces $d(\hat{\Delta}_3) \geq 6$ with at least one vertex of degree ≥ 4 and we add $c(\Delta) \leq \frac{\pi}{6}$ to $c(\hat{\Delta}_3) \leq \frac{-\pi}{6}$. Similarly in Figures 3.24(ii) and 3.25(iv) the labelling forces $d(\hat{\Delta}_2) \geq 6$ with at least one vertex of degree ≥ 4 and we add $c(\Delta) \leq \frac{\pi}{6}$ to $c(\hat{\Delta}_2) \leq \frac{-\pi}{6}$. Finally in Figures 3.24(v) and 3.25(iii) $l(u_4) = vw_1w_2$ and either w_1 is a proper subword of w and we add $c(\Delta) \leq \frac{\pi}{6}$ to $c(\hat{\Delta}_5) \leq \frac{-\pi}{6}$. Otherwise w_2 is a proper subword of w and we add $c(\Delta) \leq \frac{\pi}{6}$ to $c(\hat{\Delta}_4) \leq \frac{-\pi}{6}$.

Now assume that $\hat{\Delta} \in \{\hat{\Delta}_1, \hat{\Delta}_2, \hat{\Delta}_3, \hat{\Delta}_4, \hat{\Delta}_5\}$. If $\hat{\Delta}$ receives from exactly one region then as we see from above $c^*(\hat{\Delta}) \leq 0$. Now assume $\hat{\Delta}$ receives from two or more regions. First consider regions that compensate through a whole edge, say Type A. Then $\hat{\Delta} \in \{\hat{\Delta}_1, \hat{\Delta}_2, \hat{\Delta}_3\}$ and checking all regions of Type A shows that $\hat{\Delta}$ receives from at most 2 regions of this type namely; $\hat{\Delta}_2$ in Figure 3.24(ii) with $\hat{\Delta}_2$ in Figure 3.25(iv). However the region $\hat{\Delta}$ must have at least 2 vertices of degree ≥ 4 with 2 edges split and receives at most $\frac{5\pi}{30}$ from each region and hence $c^*(\Delta) \leq c(3, 3, 3, 3, 4, 4) + 2(\frac{5\pi}{30}) = 0$.

Now consider the regions that compensate through a part of an edge, Type B. Then $\hat{\Delta} \in \{\hat{\Delta}_3, \hat{\Delta}_4\}$ and the most can be received from one Type B region is $\frac{5\pi}{30}$. Moreover checking regions of Type B shows that $\hat{\Delta}_4$ can never receive positive curvature from Δ and $\hat{\Delta}_5$ at the same time and that $\hat{\Delta}_5$ can never receive positive curvature from Δ and $\hat{\Delta}_4$ at the same time. Moreover we never have more than one Type B region sending positive curvature through the same edge and so the net contribution is $\leq \frac{5\pi}{30} + (1 - \frac{2\pi}{3}) < 0$ and so we can ignore such occurrence

since it does not increase $c^*(\Delta)$.

Chapter 4

Case III

In this chapter we study the Case(III) and up to cyclic permutation and inversion $r(t)$ is one of the following subcases:

- (a1) $wt^{l_1}wt^{l_2}wt^{l_3}wt^{l_4}$
- (a2) $wt^{l_1}wt^{l_2}wt^{l_3}w^{-1}t^{l_4}$
- (a3) $wt^{l_1}wt^{l_2}w^{-1}t^{l_3}w^{-1}t^{l_4}$
- (a4) $wt^{l_1}w^{-1}t^{l_2}wt^{l_3}w^{-1}t^{l_4}$

Note that $t = 1$ is a solution for (a3) and (a4) so we only have (a1) and (a2) to consider. In addition if $w = (gt^k)^\beta g^{-1}$, $\beta \geq 1$ then $t = 1$ is a solution for the subcases (a1) and (a2) by Lemma 11. Thus in the following we assume $w \neq (gt^k)^\beta g^{-1}$. We further assume that $w \neq g_1 t^{k_1} g_2 t^{k_2} \dots g_{k-1} t^{k_{n-1}} g_1^{-1}$, $n > 2$ is even, $g_{\frac{n}{2}}^2 = 1$, $g_p = g_{n-p}^{-1}$, $1 \leq p < n$ and $k_p = -k_{n-p-1}$, $1 \leq p < n-1$ as by Lemma 9 a second split on the edge might not occur if w is of this form, however this is essential in the proof.

4.1 Case III(a1)

Figure 4.1(i) shows all possible labellings in this case. Note that if i and j are both from $\{l_1, l_2, l_3, l_4\}$ and both are in the same direction, then the vertex sublabel $(\vec{i} \vec{j})$ forces 2 splits on each of the edges $(i-1, i)$ and $(j, j+1)$ by Lemma 9. An illustration is given in Figure 4.1(ii). First assume that $d(u_i) = 2$ for $1 \leq i \leq 4$. Then the possible positive regions (up to cyclic permutation and inversion) and the distribution of curvature is shown in Figures 4.2(i)-(v). Now assume that $d(u_i) \geq 3$ for exactly one $1 \leq i \leq 4$ as shown in Figure 4.2(vi)-(vii). Then the

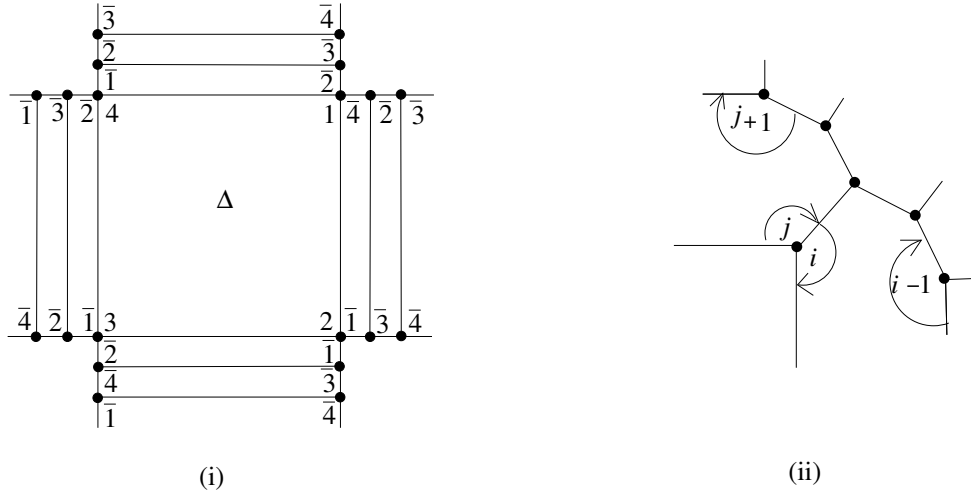


Figure 4.1: All possible labellings case III (a1).

labels k' and k must be τ -powers, otherwise there are at least two splits on the edge by Lemma9 and $c(\Delta) \leq 0$. Figures 4.2(vi)-(vii) show the possible positive regions and the distribution in this case.

Let $\hat{\Delta} \in \{\hat{\Delta}_1, \hat{\Delta}_2, \hat{\Delta}_3, \hat{\Delta}_4\}$. If $\hat{\Delta}$ is given by Figures 4.2(i),(ii),(iii) or (vi) then $d(\hat{\Delta}) \geq 8$ and at least two of its edges have two splits. Therefore $\hat{\Delta}$ receives positive curvature from at most two regions and so $c^*(\hat{\Delta}) \leq c(3, 3, 3, 3, 3, 3, 3, 3) + \frac{2\pi}{3} = 0$. Assume otherwise and that $\hat{\Delta}$ is given by Figure 4.2(iv) or 4.2(v) or $\hat{\Delta} = \hat{\Delta}_3$ in Figure 4.2(vii). Then $\hat{\Delta}$ receives from at most two positive region. If $d(\hat{\Delta}) \geq 7$ then $c^*(\hat{\Delta}) \leq c(3, 3, 3, 3, 3, 3, 4) + 2(\frac{\pi}{12}) < 0$; or if $d(\hat{\Delta}) = 6$ then either $\hat{\Delta}$ receives from exactly one region and $c^*(\hat{\Delta}) \leq c(3, 3, 3, 3, 3, 4) + \frac{\pi}{12} < 0$ or $\hat{\Delta}$ is forced to have at least two vertices of degree ≥ 4 and $c^*(\hat{\Delta}) \leq c(3, 3, 3, 3, 4, 4) + 2(\frac{\pi}{12}) < 0$. Finally if $\hat{\Delta} = \hat{\Delta}_2$ in Figure 4.2(vii) then $d(\hat{\Delta}) \geq 7$ and it receives positive curvature from at most two regions, so $c^*(\hat{\Delta}) \leq c(3, 3, 3, 3, 3, 3, 3) + 2(\frac{\pi}{12}) < 0$.

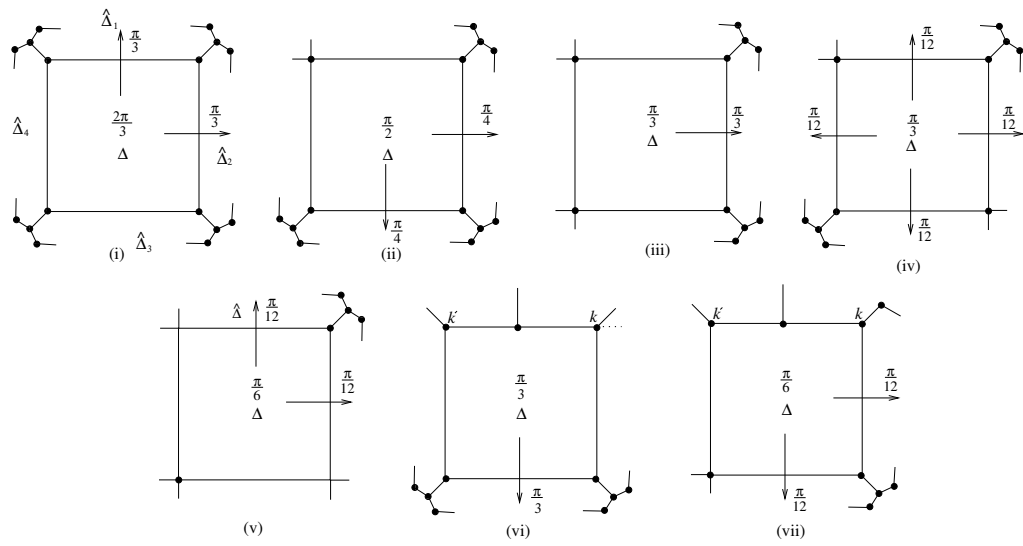


Figure 4.2: All possible positive regions caseI (a1)

4.2 Case III(a2)

Figure 4.3(i) shows all possible labellings in this case.

Consider the sublabel (ij) as shown in Figure 4.3(ii). Checking shows that if

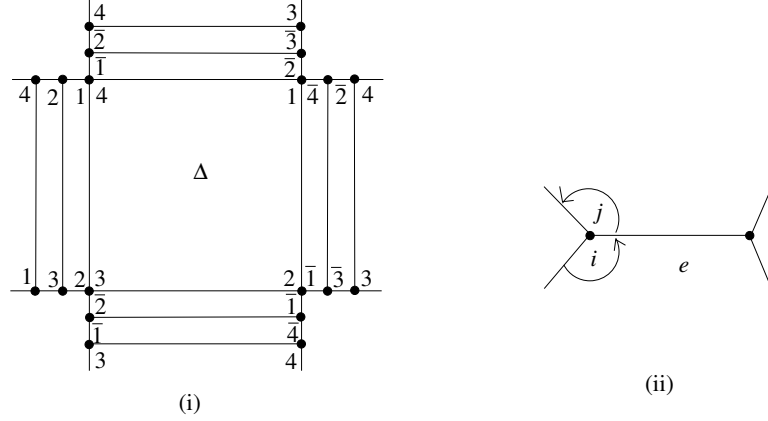


Figure 4.3: All possible labellings case III (a2).

$(ij) \in S = \{11, 12, 13, 1\bar{3}, 21, 22, 23, 2\bar{3}, 41, 42, 43, 4\bar{3}, \bar{4}1, \bar{4}2, \bar{4}3, \bar{4}\bar{3}, \bar{1}4, \bar{1}\bar{1}, \bar{1}\bar{2}, \bar{1}\bar{4}, \bar{2}\bar{4}, \bar{2}\bar{1}, \bar{2}\bar{2}, \bar{2}\bar{4}, \bar{3}\bar{4}, \bar{3}\bar{1}, \bar{3}\bar{2}, \bar{3}\bar{4}, \bar{3}\bar{4}, \bar{3}\bar{1}, \bar{3}\bar{2}, \bar{3}\bar{4}\}$ then the corner edge e of Figure 4.3(ii) has at least two splits by Lemma 9.

First assume that $(ij) \in S$ occurs as a corner label of each vertex of degree 3 in Δ that does not involve τ -power. Then the possible positive regions are the same as in case (a1) and the distribution is done in the same way in Figure 4.2. Now assume otherwise, that is if $d(v_p) = 3$ and $l(v_p) = pij$ then $(ij) \notin S$, for some $1 \leq p \leq 4$ and $d(u_i) = 2$ for $1 \leq i \leq 4$. If $p = 1$ then $d(v_1) = 3$ implies $l(v_1) \in \{\bar{2}1\bar{4}, \bar{2}14, \bar{3}1\bar{2}, \bar{3}1\bar{2}\}$; if $p = 2$ then $d(v_2) = 3$ implies $l(v_2) \in \{\bar{1}2\bar{4}, \bar{1}24, \bar{3}2\bar{1}, \bar{3}2\bar{1}\}$; if $p = 3$ then $d(v_3) = 3$ implies $l(v_3) \in \{\bar{1}3\bar{2}, \bar{2}3\bar{1}\}$; and if $p = 4$ then $d(v_4) = 3$ implies $l(v_4) \in \{14\bar{2}, 24\bar{1}\}$.

Assume that $d(v_p) = 3$ for two values of $1 \leq p \leq 4$. If $d(v_1) = d(v_2) = 3$ then either $l(v_1) = \bar{2}1\bar{4}$ and $l(v_2) = \bar{1}24$, or $l(v_1) = \bar{2}3\bar{1}$ and $l(v_2) = \bar{1}\bar{3}\bar{2}$; if $d(v_2) = d(v_3) = 3$ then $l(v_2) = \bar{3}2\bar{1}$ and $l(v_3) = \bar{2}3\bar{1}$; if $d(v_3) = d(v_4) = 3$ then there is a contradiction; and if $d(v_1) = d(v_4) = 3$ then $l(v_1) = \bar{2}1\bar{4}$ and $l(v_4) = 4\bar{1}\bar{2}$; if $d(v_1) = d(v_3) = 3$ then either $l(v_1) = \bar{3}1\bar{2}$ and $l(v_2) = \bar{1}3\bar{2}$, or $l(v_1) = l(v_2) = 31\bar{2}$; and if $d(v_2) = d(v_4) = 3$ then either $l(v_1) = \bar{1}2\bar{4}$ and $l(v_4) = 14\bar{2}$, or $l(v_2) = l(v_4) = 24\bar{1}$.

Now assume that the above holds for three values of $1 \leq p \leq 4$. Then there are two possibilities. Either $d(v_1) = d(v_2) = d(v_4) = 3$ in which case $l(v_1) = \bar{2}1\bar{4}$ and $l(v_2) = l(v_4) = \bar{1}24$; or $d(v_1) = d(v_2) = d(v_3) = 3$ in which case $l(v_1) = l(v_3) = \bar{2}31$ and $l(v_2) = \bar{1}\bar{3}2$. It follows that if $d(v_p) = 3, 1 \leq p \leq 4$ then there is a contradiction

Note that in each region exactly one of $\in \{\bar{2}1\bar{4}, \bar{2}14, \bar{3}1\bar{2}, 31\bar{2}\}$ can occur, maybe more than once. Note further that regions with vertex label $\bar{2}1\bar{4}$ are symmetric to regions with vertex label $31\bar{2}$; and regions with vertex label $\bar{2}14$ are symmetric to regions with vertex label $\bar{3}1\bar{2}$ by Lemma 6(i). It therefore suffices to consider only the cases $l_2 + l_4 - l_1 = 0$ and $l_1 + l_4 - l_2 = 0$

4.2.1 Figures with a vertex label $\bar{2}1\bar{4}$

In this subsection we assume that $l_2 + l_4 - l_1 = 0$ and that $l_4 \neq \pm ml_3, m \in \mathbb{N}, m \geq 2$. In this case we further amend D by deleting all vertices of degree 2 labelled by ww^{-1} with corner sublabels 24 and 33 as in Figure 4.4(i). This introduces the following corner labels: $(24)^{\pm 1}$ and $(33\dots 3)^{\pm 1}$ (m times for some $m \geq 1$), which will be denoted by $\tilde{3}^{\pm}$. Applying these changes creates a new type of region Δ . Figure 4.4(ii) shows an example of such a region. Observe that Δ has $4m + 4$ vertices. Observe further that $l_4 \neq \pm ml_3$ implies $d(y_5) \geq 3$.

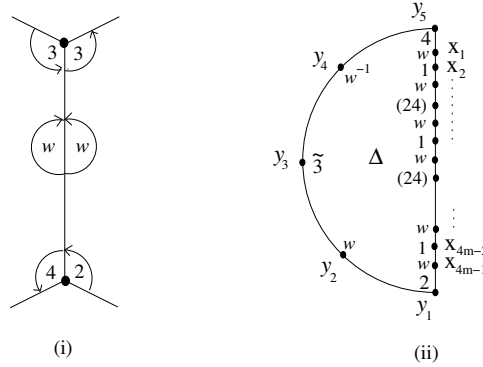


Figure 4.4

Lemma 14. (i) No vertex of (the amended) D has a sublabel 24.
(ii) No vertex of D has a sublabel $4\bar{1}2$.

Proof. (i) Every sublabel of the form 24 is changed by the amendment in Figure 4.4(i) and this transforms 24 to (24) .

(ii) If D has a sublabel $4\bar{1}2$ then using a bridge move if necessary, it can be assumed that $4\bar{1}2$ is a vertex label in D in which case the previous part of this lemma applies. □

Let Δ be a region in the amended diagram and assume $\tilde{3}$ is maximum. Assume now that $d(x_1) \geq 3$, see Figure 4.4(ii). Then (by Lemma 9) either there is another split on the edge (y_5x_2) as in Figure 4.5(i) or each of $l(y_5)$ and $l(x_2)$ involves a τ -power as in Figure 4.5(ii). If there are two splits on the edge (y_5x_2) then $d(x_2) \geq 3$ since otherwise $l(x_2) \in 1\{\tilde{3}^\pm, \overline{(24)}\}$ which yields a contradiction since it forces ww^{-1} to be a proper sublabel of x'_i in Figure 4.5(i). Note that $d(x_2) \geq 3$ in Figure 4.5(ii) since it involves a τ -power.

Now assume that $d(x_{4m-1}) \geq 3$. Then similarly to the above either there is

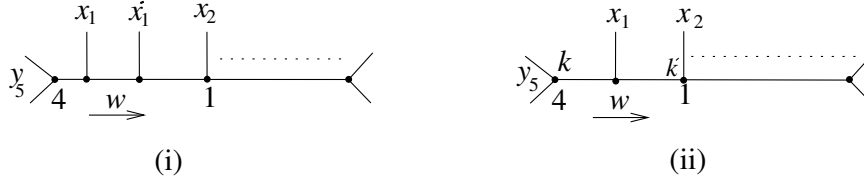


Figure 4.5: when $d(x_1) \geq 3$

another split on the edge (x_{4m-2}, y_1) which in turn forces either $d(x_{4m-2}) \geq 3$ as in Figure 4.6(i) or $l(x_{4m-2}) = 1\tilde{3}^\pm$ as in Figure 4.6(ii) in which $l(x_{4m-2}) = 1\tilde{3}^\pm$ forces $d(x_{4m-3}) = 2$ and so $l(x_{4m-3}) = ww^{-1}$ which forces $d(x_{4m-4}) \geq 3$ when $m \geq 2$; or both $l(y_1)$ and $l(x_{4m-2})$ involve a τ -power which forces $d(x_{4m-2}) \geq 3$ and $d(y_1) \geq 3$, see Figure 4.6(iii).

Now consider x_i for $i \equiv 1 \pmod{2}$, $2 \leq i \leq 4m - 2$ and $m \geq 2$. Then $d(x_i) \geq 3$ implies either both of $l(x_{i-1})$ and $l(x_{i+1})$ involve τ -powers and so both of the vertices are of degree ≥ 3 or there are two splits on the edge $(x_{i-1}x_{i+1})$. In the latter case $d(x_{i+1}) \geq 3$ otherwise it forces ww^{-1} to be a proper sublabel of x'_j , $\{x_j, x'_j\} = \{x_i, x'_i\}$, see Figure 4.7.

Now let $d(x_i) \geq 3$ for some $i \equiv 2 \pmod{4}$ and assume $d(x_{i-1}) = d(x_{i+1}) = 2$ as shown in Figure 4.8. First assume $m \geq 2$. Let $i \neq 2$ and $i \neq 4m - 2$. Then either $d(x_i) \geq 4$ as in Figure 4.8(i) or $l(x_i) = \tilde{3}^\pm 1\{4, \bar{4}\}$ which in turn forces $d(x_{i-2}) \geq 3$

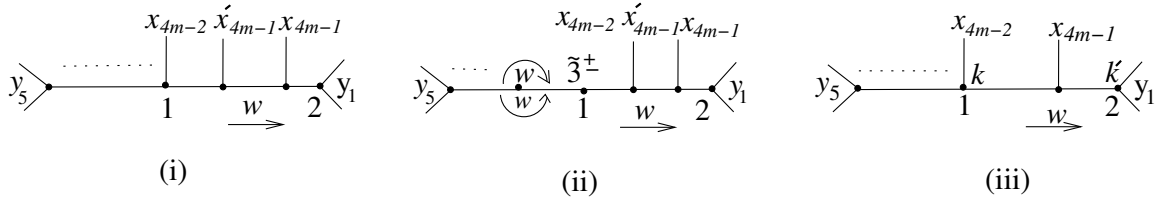

 Figure 4.6: when $d(x_{4m-1}) \geq 3$

 Figure 4.7: $d(x_i) \geq 3$ where $i \equiv 1 \pmod{2}$

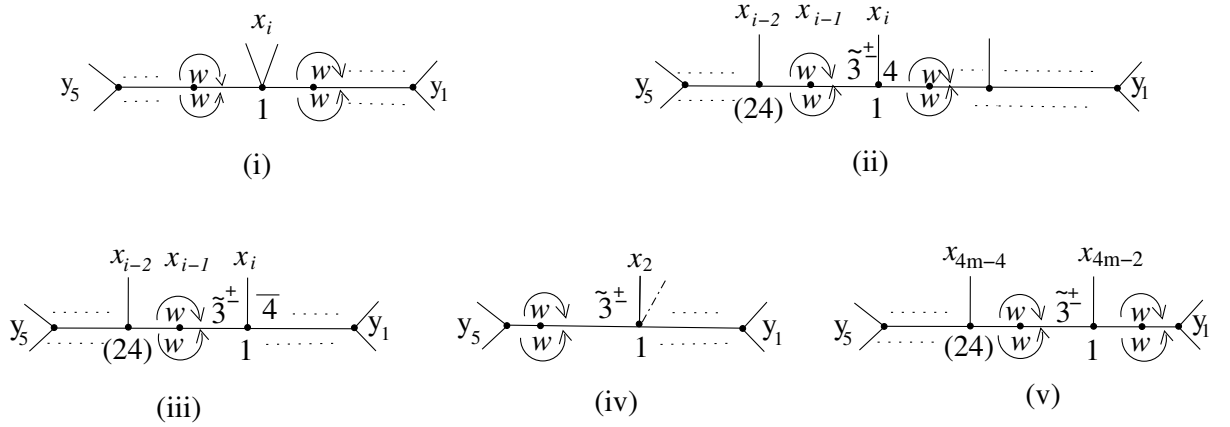
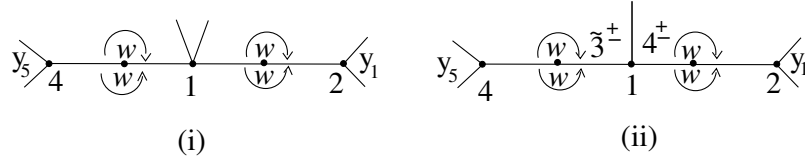
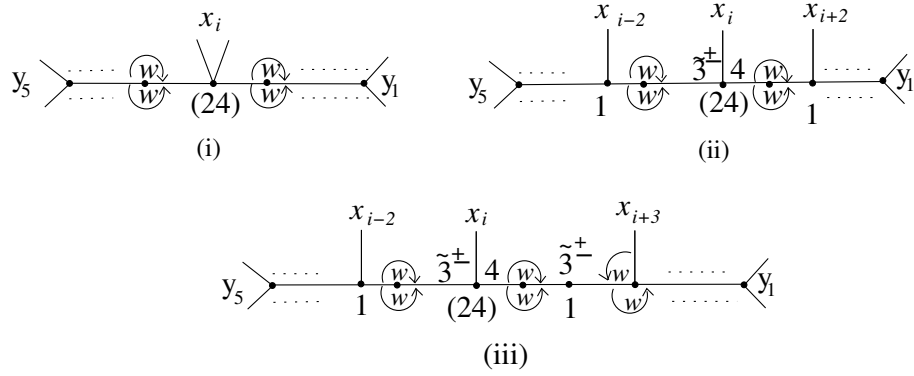
as shown in Figure Figure 4.8(ii)-(iii). Now let $i = 2$. Then again either $d(x_2) \geq 4$ or $l(x_2) \in \tilde{3}^\pm 1\{4, \bar{4}\}$ which forces $l(y_5) = 4\{\bar{2}, 4\}\omega$. Observe that for $i \neq 4m - 2$ $l(x_i) = \tilde{3}^\pm 14$ also forces either $d(x_{i+2}) \geq 3$ or $l(x_{i+2}) = (24)\tilde{3}$ and so $d(x_{i+3}) \geq 3$, that is, there is at least one split between x_{i+1} and y_1 and that is illustrated in Figure 4.8(ii). Now let $i = 4m - 2$. Then either $d(x_i) \geq 4$ or $l(x_i) \in \tilde{3}^\pm 1\{4, \bar{4}\}$ which again forces $d(x_{4m-4}) \geq 3$, see Figures 4.8(v). Assume now that $m = 1$. Then either $d(x_1) \geq 4$ or $l(x_1) = \tilde{3}^\pm 1\{4, \bar{4}\}$, see Figure 4.9.

Assume now that $d(x_i) \geq 3$ where $i \equiv 0 \pmod{4}$. Then $d(x_{i-1}) = d(x_{i+1}) = 2$ forces either $d(x_i) \geq 4$ as in Figure 4.10(i) or $l(x_i) = \tilde{3}^\pm (24)4$ which in turn forces $d(x_{i-2}) \geq 3$ and either $d(x_{i+2}) \geq 3$ as in Figure 4.10(ii) or $d(x_{i+3}) \geq 3$ as in Figure 4.10(iii).

Remark 5. Observe that some labelling cannot occur since they yield the exceptional case $l_4 = \pm ml_3$, $m \in \mathbb{N}$, $m \geq 2$. For example $1\bar{2}\tilde{3}^\pm, 4\tilde{3}^\pm, 41\tilde{3}^\pm\bar{1}$. So from here on contradiction means either a labelling contradiction or a contradiction to the assumption $l_4 \neq \pm ml_3$.

Lemma 15. Let Δ be the region in Figure 4.4 and assume $\tilde{3}$ is maximum. Then

- (i) if $l_2 = \pm ml_3$, $m \in \mathbb{N}$, then $l_1 \neq \pm m'l_3$, $m' \in \mathbb{N}$.
- (ii) if $d(y_2) = d(y_4) = 2$ then $d(y_3) \geq 4$.

Figure 4.8: $d(x_i) \geq 3$ where $i \equiv 2 \pmod{4}$, $m \geq 2$ Figure 4.9: $d(x_i) \geq 3$ where $i \equiv 2 \pmod{4}$, $m = 1$ Figure 4.10: $d(x_i) \geq 3$ where $i \equiv 0 \pmod{4}$

(iii) if $d(y_4) = d(x_1) = d(x_2) = d(x_3) = 2$ then $d(y_5) \geq 4$. Moreover if $d(x_1) = d(x_2) = d(x_3) = 2$ and $d(y_5) = 3$ then there are at least two splits on the edge (y_3, y_5) and $l(y_5) = 4\bar{1}3^\pm$.

(iv) if $d(x_{4m-3}) = d(x_{4m-2}) = d(x_{4m-1}) = 2$ then $d(y_1) \geq 4$.

(v) Let $m \geq 2$ and let $d(x_i) \geq 3$ for $i = 1$ only or any $3 \leq i \leq 4m - 2$ only; if in

addition $d(y_5) = 3$ then there is at least one further split on the edge (y_5, x_{i+1}) or two splits on the edge (y_3, y_5) .

(vi) Let $m \geq 2$ and let $d(x_{4m-2}) = 3$. Then this forces at least one split on segment (y_5, x_{4m-2}) . Moreover if there is exactly one split then it is x_{4m-4} and $d(x_{4m-4}) \geq 4$ and either $d(y_5) \geq 4$ or there are at least 2 splits on the edge (y_3, y_5) .

Proof. (i) Letting $l_2 = \pm ml_3$, $m \in \mathbb{N}$ and $l_1 = \pm m'l_3$, $m' \in \mathbb{N}$ implies $l_4 = l_1 - l_2 = \pm m''l_3$ which is a contradiction.

(ii) $d(y_2) = d(y_4) = 2$ then $d(y_3) \neq 2$ and $d(y_3) = 3$ forces $l(y_3) \in \{\bar{1}\bar{2}, 2\bar{1}, 2(\bar{2}\bar{4}), (\bar{2}\bar{4})\bar{2}\}\bar{3}$ which implies $l_4 = \pm ml_3$ which is a contradiction.

(iii) If $d(y_4) = d(x_1) = d(x_2) = d(x_3) = 2$ then $l(x_2) = \bar{1}(\bar{2}\bar{4})$ and so $d(y_5) = 3$ implies $l(y_5) = \{4, (\bar{2}\bar{4}), 1\}4\bar{1}$ and each yields a labelling contradiction. If $d(x_1) = d(x_2) = d(x_3) = 2$ and $d(y_5) = 3$ and there is only one split on the edge (y_3, y_5) then $l(y_5) = 4\bar{1}k$ which is a contradiction and the only possible labelling is $l(y_5) = 4\bar{1}\bar{3}^\pm$.

(iv) If $d(x_{4m-3}) = d(x_{4m-2}) = d(x_{4m-1}) = 2$ then $l(x_{4m-2}) = \bar{1}(\bar{2}\bar{4})$. If in addition $d(y_1) = 3$ then either $d(y_2) = 2$ and this implies $l(y_1) = \bar{1}\bar{2}\{\bar{4}, (\bar{2}\bar{4}), \bar{1}\}$ or $d(y_2) \geq 3$ and this implies $l(y_1) = \bar{1}\bar{2}\{1, 2, (\bar{2}\bar{4}), \bar{3}^{\pm 1}, k\}$ and each yields a contradiction.

(v) Let $m \geq 2$ and assume $d(y_5) = 3$. If $d(x_i) \geq 3$, $i \in \{1, 3\}$ then as in Figures 4.5 and Figure 4.7 this forces at least one more split on the edge (y_5, x_{i+1}) . If $d(x_i) \geq 3$ for some $i \geq 4$ only then $d(x_1) = d(x_2) = d(x_3) = 2$ and by (iii) of this Lemma this forces 2 splits on the edge (y_3, y_5) .

(vi) Let $m \geq 2$ and let $d(x_{4m-2}) = 3$. If $\underline{d(x_{4m-3}) = 2}$ then $l(x_{4m-2}) \in \bar{3}^{\pm 1}\{4, \bar{4}\}$ and this in turn forces $d(x_{4m-4}) \geq 3$ as in Figure 4.8(v). Moreover if $d(x_{4m-5}) = 2$ then either $d(x_{4m-4}) \geq 4$ as in Figure 4.10(i) or $l(x_{4m-4}) = \bar{3}^\pm(\bar{2}\bar{4})4$ as in Figure 4.10(ii) which in turn forces a further split at x_{4m-6} . In addition when $d(x_{4m-4}) \geq 4$ then either there is at least one more split on the edge (x_{4m-4}, y_5) or 2 splits on the edge (y_3, y_5) or $d(y_5) \geq 4$ by Lemma 15(iii). If $d(x_{4m-5}) \geq 3$ then as in Figure 4.7 this forces at least 2 splits on the edge (x_{4m-4}, y_5) . If $\underline{d(x_{4m-3}) \geq 3}$ then as in Figure 4.7 this forces either there are at least 2 splits on the edge (x_{4m-2}, x_{4m-4}) or one split with $d(x_{4m-4}) \geq 3$. $\square$

Now assume that Δ is a positive region in the amended diagram and that $m \geq 2$. Recall that $d(y_5) \geq 3$. We proceed on a case-by-case basis: $d(y_1) = d(y_3) = 2$; $d(y_1) = 2$, $d(y_3) \geq 3$; $d(y_1) \geq 3$, $d(y_2) = d(y_4) = 2$; $d(y_1) \geq 3$, $d(y_2) \geq 3$, $d(y_4) = 2$; $d(y_1) \geq 3$, $d(y_2) = 2$, $d(y_4) \geq 3$; $d(y_1) \geq 3$, $d(y_2) \geq 3$, $d(y_4) \geq 3$.

$d(y_1) = d(y_3) = 2$:

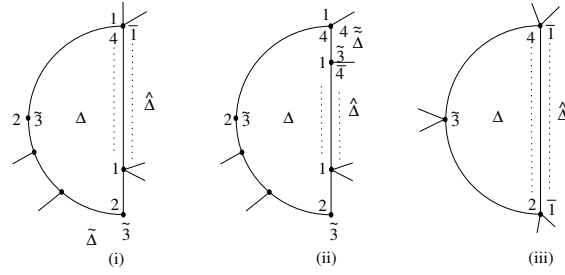
Letting $d(y_1) = 2$ implies $l(y_1) = 2\tilde{3}^\pm$. This forces 2 splits on the edge (y_1, y_3) by Lemma 9 and forces $l(x_{4m-2})$ to involve one of the sublabels 14 or $1\bar{2}$ which in turn forces $d(x_{4m-2}) \geq 3$. Assume $d(y_4) = 2$. Then $l(y_3) \in \tilde{3}\{1, (24), 2\}$ and by Lemma 15(i) $l(y_3) \in \tilde{3}\{1, (24)\}$ with $l(y_1) = 2\tilde{3}^\pm$ yields a contradiction. Thus, $l(y_1) = l(y_3) = \tilde{23}$. If $d(x_{4m-2}) = 3$ then by Lemma 15(vi) this forces either $c(\Delta) \leq c(3, 3, 3, 4, 4) = 0$ or $d(\Delta) \geq 6$. Thus $d(x_{4m-2}) = 3$ implies $c(\Delta) \leq 0$. So let $d(x_{4m-2}) \geq 4$. If $d(x_i) = 2$ for all $1 \leq i \leq 4m - 3$ then $d(y_5) \geq 4$ and $c(\Delta) \leq c(3, 3, 4, 4) = \frac{\pi}{3}$, see Figure 4.11(i). If $d(x_i) \geq 3$ for $i = 1$ or any $3 \leq i \leq 4m - 3$ then $c(\Delta) \leq 0$ by Lemma 15(iii). If $d(x_2) \geq 4$ then again $c(\Delta) \leq 0$, so let $d(x_2) = 3$. This forces $l(x_2) = 14\tilde{3}$ and $d(y_5) = 3$ which forces $l(y_5) = 441$ and $c(\Delta) \leq c(3, 3, 3, 3, 4) = \frac{\pi}{6}$, see Figure 4.11(ii). While if $d(y_4) \geq 3$ then it must be one split on the edge (y_3, y_5) and $d(x_i) = 2$ for all $1 \leq i \leq 4x - 3$ otherwise $d(\Delta) \geq 6$. This however forces $d(y_5) \geq 4$ and $d(x_{4m-2}) \geq 4$ by Lemma 15(vi) and hence $c(\Delta) \leq c(3, 3, 3, 4, 4) = 0$.

$d(y_1) = 2, d(y_3) \geq 3$:

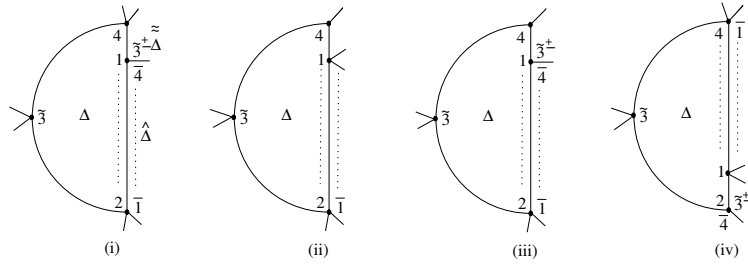
If $d(y_3) \geq 3$ then from the above case we see that $d(\Delta) \geq 5$ and either $d(\Delta) \geq 6$ or $d(x_{4m-2}) \geq 4$ and $d(y_5) \geq 4$ so $c(\Delta) \leq c(3, 3, 3, 4, 4) = 0$ by Lemma 15(vi).

$d(y_1) \geq 3, d(y_2) = d(y_4) = 2$:

Letting $d(y_2) = d(y_4) = 2$ forces $d(y_3) \geq 4$ by Lemma 15(ii). If in addition $d(x_i) = 2$ for all $1 \leq i \leq 4m - 1$ then $d(y_1) \geq 4$ and $d(y_5) \geq 4$ by Lemma 15(iii)-(iv). Thus $c(\Delta) \leq c(4, 4, 4) = \frac{\pi}{2}$, see Figure 4.11(iii). Now let $d(x_i) \geq 3$ for some $i \in \{1, 4m - 1\}$ or any $3 \leq i \leq 4m - 3$. If $i \equiv 1 \pmod{2}$ then this forces at least one further split as in Figures 4.5, 4.6, 4.7 and at least one of $d(y_1)$ or $d(y_5)$ is of degree ≥ 4 by Lemma 15(iii)-(iv) and so $c(\Delta) \leq c(3, 3, 3, 4, 4) = 0$. If $i \equiv 0 \pmod{2}$ then this forces either $d(x_i) \geq 4$ as in Figures 4.8(i) and 4.10(i) with $d(y_1) \geq 4$ and $d(y_5) \geq 4$, by Lemma 15(iii)-(iv), which implies $c(\Delta) \leq c(4, 4, 4, 4) = 0$; or it forces at least one further split with $d(y_3) \geq 4$ and at least one of $d(y_1)$ and $d(y_5)$ is of degree ≥ 4 , by Lemma 15(iii)-(iv), and $c(\Delta) \leq c(3, 3, 3, 4, 4) = 0$. If only $d(x_2) \geq 3$ then $d(y_1) \geq 4$ by Lemma 15(iv) and at most one of $d(y_1)$ and $d(x_2)$ can be of degree ≥ 4 and $c(\Delta) \leq c(3, 4, 4, 4) = \frac{\pi}{6}$ or $c(\Delta) \leq c(3, 3, 4, 4) = \frac{\pi}{3}$, see Figures 4.12(i)-(iii). Suppose that only $d(x_{4m-2}) \geq 3$.


 Figure 4.11: $m \geq 2$

If $\underline{d(x_{4m-2}) = 3}$ then this forces either $d(\Delta) \geq 6$ or $d(\Delta) = 5$ with at least $d(y_3) \geq 4$ and $d(y_1) \geq 4$ and so $c(\Delta) \leq 0$ by Lemma 15 (vi). If $\underline{d(x_{4m-2}) \geq 4}$ then $d(y_1) \geq 4$ by Lemma 15 (vi) and this forces $d(y_1) = 3$ which implies $l(y_1) = \tilde{3}^\pm 2\bar{4}$ and $c(\Delta) \leq c(3, 4, 4, 4) = \frac{\pi}{6}$, see Figure 4.12(iv).


 Figure 4.12: $m \geq 2$ and only $d(y_3) \geq 3$.

$\mathbf{d}(y_1) \geq 3, \mathbf{d}(y_2) \geq 3, \mathbf{d}(y_4) = 2$:

Note that if $l(y_1)$ or $l(y_3)$ only involves l_i , $1 \leq i \leq 4$ then there are at least 2 splits on the edge (y_1, y_3) by Lemma 9. Note further that when there is exactly one split then both of $l(y_1)$ and $l(y_3)$ involve a τ -power by corollary following Lemma 14 and this would force $d(y_3) \geq 3$ since otherwise $l(y_3) = \tilde{3}k$ and this forces $d(y_4) \geq 3$ which is a contradiction. Thus $d(\Delta) \geq 4$. First let $d(x_i) = 2$ for all $1 \leq i \leq 4m - 1$. Then $d(y_1) \geq 4$ by Lemma 15(iv) and $d(y_5) \geq 4$ by Lemma 15(iii). If there are 2 splits on the edge (y_1, y_3) then this forces $d(y_3) = 2$ otherwise $c(\Delta) \leq c(3, 3, 3, 4, 4) = 0$, see Figure 4.13(i). If there is only one split on the edge (y_1, y_3) then $d(y_3) \geq 3$ and $c(\Delta) \leq c(3, 3, 4, 4) = \frac{\pi}{3}$, see Figure 4.13(ii). Next let $d(x_i) \geq 3$ for some $1 \leq i \leq 4m - 1$ so $d(\Delta) = 5$. Then the only possible split resulting in a positive region is x_2 since $i \equiv 1 \pmod{4}$ always forces

at least one further split as in Figures 4.5, 4.6 and 4.7. Also $i \equiv 2 \pmod{4}$, $i \neq 2$ forces at least one further split as in Figures 4.8(ii) and 4.10(ii) or $d(x_i) \geq 4$ as in Figures 4.8(i) and 4.10(i) with at least one of $d(y_1) \geq 4$ and $d(y_5) \geq 4$ by Lemma 15(iii)-(iv) and so $c(\Delta) \leq c(3, 3, 3, 4, 4) = 0$. If $i = 2$ then the possibilities are shown in Figure 4.13(iii)-(iv).

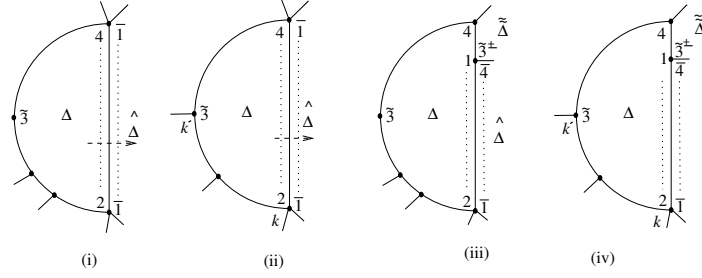


Figure 4.13: $m \geq 2$, $d(y_4) = 2$ and $d(y_2) \geq 3$.

$d(y_1) \geq 3$, $d(y_2) = 2$, $d(y_4) \geq 3$:

As in above case we either have 2 splits on the edge (y_3, y_5) or we have exactly one split and $d(y_3) \geq 3$. Let $d(x_i) = 2$, $1 \leq i \leq 4m - 1$. Then this forces $d(y_1) \geq 4$ by Lemma 15(iv) and either $l(y_5) = 4\bar{1}\bar{3}^\pm$ (and this forces at least 2 splits on the edge (y_3, y_5)) and we have the two cases $d(y_3) \geq 3$ or $d(y_3) = 2$; or $d(y_5) \geq 4$ in which case we only have $d(y_3) \geq 3$ if there is only one split on the edge (y_3, y_5) , Figures 4.14(i)-(iv) shows the positive regions in this case. Note that in Figure 4.14(ii) $d(y_2) = 2$ forces $l(y_3) \in \bar{3}\{\bar{1}, \bar{2}, \overline{(24)}\}$. However $l(y_5) = 4\bar{1}\bar{3}^\pm = \bar{2}\bar{3}^\pm$ and all the labelling, except $l(y_3) = \bar{3}\bar{2}$ and $l(y_5) = 4\bar{1}\bar{3}$, yield a contradiction by Lemma 15(i). Now let $d(x_i) \geq 3$ for some $1 \leq i \leq 4m - 1$. This implies $d(\Delta) = 5$. Then at most only $d(x_2) \geq 3$ or only $d(x_{4m-2}) \geq 4$, otherwise we have $d(\Delta) \geq 6$ or $d(x_i) \geq 4$ as in Figures 4.8(i) and 4.10(i) and $d(y_1) \geq 4$ by Lemma 15(iv), so $c(\Delta) \leq c(3, 3, 3, 4, 4) = 0$. Let $d(x_2) \geq 3$. Then $d(y_1) \geq 4$ by Lemma 15(iv), see Figures 4.14(v)-(vi). Note that in Figure 4.14(v) similarly to Figure 4.14(ii) $l(y_3) = \bar{3}\bar{2}$ and $l(x_2) = 1\bar{4}\bar{3}^-$ otherwise there is a contradiction by Lemma 15(i). Moreover $l(x_2) = 1\bar{4}\bar{3}^{-1}$ forces $l(y_5) = 4\bar{2}\bar{3}^\pm$ which yields a contradiction to assumption $l_4 \neq ml_3$. Thus in Figure 4.14(v) $c(\Delta) \leq 0$. Finally let $d(x_{4m-2}) \geq 3$. Then $d(x_{4m-2}) = 3$ forces either $d(\Delta) \geq 6$ or $c(\Delta) \leq c(3, 3, 3, 4, 4) = 0$ as in Lemma 15(vi). If $d(x_{4m-2}) \geq 4$ then $d(y_5) = d(y_1) = 3$ which implies $l(y_5) = 4\bar{1}\bar{3}^\pm = \bar{2}\bar{3}^\pm$ and hence there are 2 splits on the edge (y_3, y_5) and $d(y_3) = 2$ since

otherwise $d(\Delta) \geq 6$. Also $l(y_1) = 2\bar{4}3^\pm$ which implies $l(y_3) = \bar{3}1$. This however yields a contradiction to our assumption.

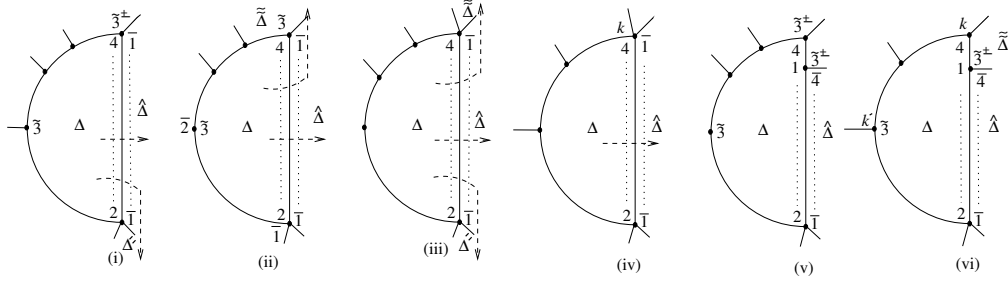


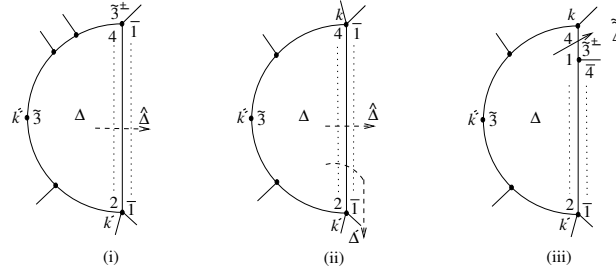
Figure 4.14: $m \geq 2$, $d(y_2) = 2$ and $d(y_4) \geq 3$.

$d(y_1) \geq 3$, $d(y_2) \geq 3$, $d(y_4) \geq 3$:

If there are 2 splits in each of the edges (y_1, y_3) and (y_3, y_5) then $d(\Delta) \geq 6$. If there are 2 splits on the edge (y_1, y_3) and only one split on the edge (y_3, y_5) then $d(\Delta) = 5$ and $d(x_i) = 2$ for all $1 \leq i \leq 4m - 1$ which implies $d(y_1) \geq 4$ by Lemma 15(iv) and $d(y_5) \geq 4$ by Lemma 15(iii) and hence $c(\Delta) \leq c(3, 3, 3, 4, 4) = 0$. If there is exactly one split on the edge (y_1, y_3) and 2 splits on the edge (y_3, y_5) then $d(\Delta) = 5$ and $d(x_i) = 2$ for all $1 \leq i \leq 4m - 1$ which implies $d(y_1) \geq 4$ by Lemma 15(iv). Moreover $d(y_5) = 3$ implies $l(y_5) = 4\bar{1}3^\pm$ and $d(y_3) = 2$ implies $l(y_3) = \bar{3}k''$ and $c(\Delta) \leq c(3, 3, 3, 3, 4) = \frac{\pi}{6}$, see Figure 4.15(i).

If there is exactly one split on both edges (y_1, y_3) and (y_3, y_5) then $d(\Delta) \geq 4$. If $d(y_3) \geq 3$ then $d(x_i) = 2$ for all $1 \leq i \leq 4m - 1$ which implies $d(y_1) \geq 4$ by Lemma 15(iv) and $d(y_5) \geq 4$ by Lemma 15(iii) and hence $c(\Delta) \leq c(3, 3, 3, 4, 4) = 0$. If $d(y_3) = 2$ then $l(y_3) = \bar{3}k''$ and letting $d(x_i) = 2$ for all $3 \leq i \leq 4m - 3$ implies $d(y_1) \geq 4$ by Lemma 15(iv) and $d(y_5) \geq 4$ by Lemma 15(iii) and hence $c(\Delta) \leq c(3, 3, 4, 4) = \frac{\pi}{3}$, see Figure 4.15(ii). Now let $d(x_i) \geq 3$ for some $1 \leq i \leq 4m - 1$. Then $d(\Delta) = 5$. If $i \neq 2$ then $d(x_i) \geq 4$ and either $d(y_1) \geq 4$ by Lemma 15(iv) or $d(y_5) \geq 4$ by Lemma 15(iii) and hence $c(\Delta) \leq c(3, 3, 3, 4, 4) = 0$. If $i = 2$ then $d(y_1) \geq 4$ by Lemma 15(iv). This implies $d(x_2) = d(y_5) = 3$ and so $l(x_2) = 1\bar{4}3^\pm$, $l(y_5) \in \{44k, 4\bar{2}k\}$ and $c(\Delta) \leq c(3, 3, 3, 3, 4) = \frac{\pi}{6}$, see Figure 4.15(iii).

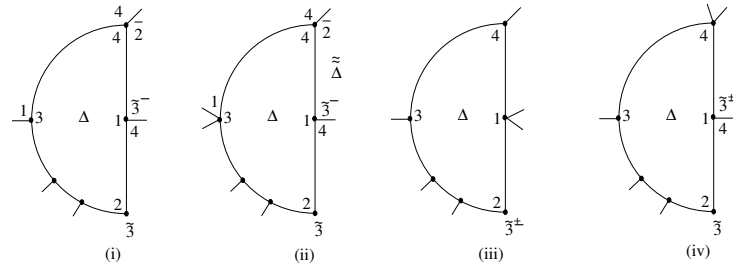
This complete the description of the positive regions when $m \geq 2$. Assume now that Δ is a positive region in the amended diagram and that $m = 1$ and recall


 Figure 4.15: $m \geq 2$, $d(y_2) \geq 2$ and $d(y_4) \geq 3$.

that $d(y_5) \geq 3$. Then $d(y_3) \geq 3$ since $m = 1$ and we proceed on a case-by-case basis: $d(y_1) = 2$; $d(y_1) \geq 3$, $d(y_2) = d(y_4) = 2$; $d(y_1) \geq 3$, $d(y_2) \geq 3$, $d(y_4) = 2$; $d(y_1) \geq 3$, $d(y_2) = 2$, $d(y_4) \geq 3$, $d(y_1) \geq 3$, $d(y_2) \geq 3$, $d(y_4) \geq 3$.

$d(y_1) = 2$:

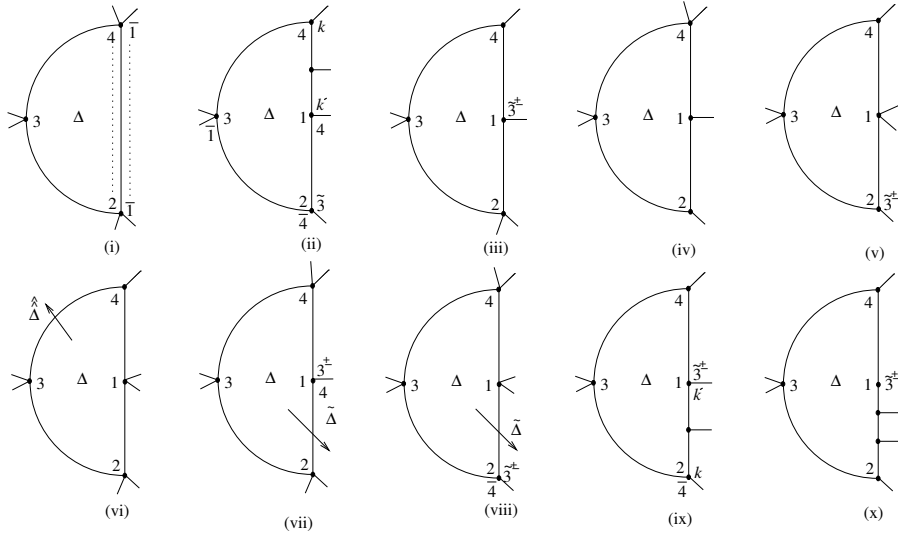
This implies $l(y_1) = 2\tilde{3}^\pm$ which forces two splits on the edge (y_1y_3) by Lemma 9 and forces $l(x_2)$ to involve one of the sublabels 14 or $1\bar{2}$ which in turn forces $d(x_2) \geq 3$. Hence $d(\Delta) \geq 5$. This forces $d(y_4) = d(x_1) = 2$, otherwise $c(\Delta) \leq 0$. The positive regions in this case are shown in Figure 4.16. Observe that $d(x_2) = 3$ forces $l(x_2) \in \{14\omega, 1\bar{2}\omega\}$ which forces $l(x_2) = 14\tilde{3}^\pm$ since $1\bar{2}\tilde{3}^\pm = \bar{4}\tilde{3}^\pm$ a contradiction to assumption. Observe further that $l(x_2) = 14\tilde{3}^\pm$ implies $l(y_1) = 2\tilde{3}$. In addition note that in Figures 4.16(i)-(ii) $d(y_5) = d(x_2) = 3$ and if $l(x_2) = 14\tilde{3}$ then this implies $l(y_5) \in \{441, 44(24)\}$ which yields a contradiction to the assumption. While if $l(x_2) = 14\tilde{3}^-$ then $l(y_5) = 4\bar{2}4$.


 Figure 4.16: $m = 1$

$d(y_1) \geq 3$, $d(y_2) = d(y_4) = 2$:

This forces $d(y_3) \geq 4$ by Lemma 15(ii). If $d(x_i) = 2$, $1 \leq i \leq 3$ then $d(y_1)$ and

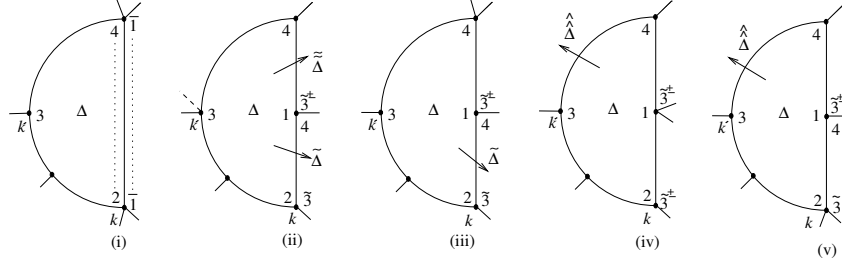
$d(y_5)$ are both ≥ 4 by Lemma 15(iii)-(iv), see Figure 4.17(i). Now let $d(x_1) \geq 3$. If there are 2 splits on the edge (y_5, x_2) then $d(x_2) \geq 3$ as in Figure 4.5(i) and $d(\Delta) \geq 6$. If there is only one split then both of $l(y_5)$ and $l(x_2)$ involve τ -powers, $d(x_3) = 2$ and $d(y_1) = d(x_2) = d(y_5) = 3$. Checking all possible labellings shows that $l(y_1) = 2\bar{4}\bar{3}^\pm$. This implies $l(x_2) \in \{14k, 1\bar{2}k\}$. However $l(x_2) = 1\bar{2}k = \bar{4}k$ is a contradiction to (A1). Thus $l(x_2) = 14k$, $l(y_1) = 2\bar{4}\bar{3}$ and $c(\Delta) \leq c(3, 3, 3, 3, 4) = \frac{\pi}{6}$, see Figure 4.17(ii). Now let $d(x_1) = 2$ and $d(x_2) \geq 3$. Assume $d(x_3) = 2$. Then checking shows that at least one of $d(y_1)$, $d(x_2)$ and $d(y_5)$ is ≥ 4 otherwise there is a labelling contradiction, see Figures 4.17(iii)-(viii). If $d(x_3) \geq 3$ then only one split can occur and both of $l(y_1)$ and $d(x_2)$ involve τ -powers and $d(y_1) = d(x_2) = d(y_5) = 3$ which forces $l(y_1) \in \{2\bar{4}k, 2\bar{1}k, 2(\bar{24})k\}$ and all, except $l(y_1) = 2\bar{4}k$, yield a contradiction to (A1), see Figure 4.17(ix). Finally if $d(x_1) = d(x_2) = 2$ and $d(x_3) \geq 3$ then $l(x_2) = \bar{1}3^\pm$ which forces at least 2 splits on the edge (x_2, y_1) and $l(y_1) \in 2\bar{4}\{k, 2, \bar{4}\}$, see Figure 4.17(x).

Figure 4.17: $m = 1$

$d(y_1) \geq 3, d(y_2) \geq 3, d(y_4) = 2$:

If there are 2 splits on the edge (y_1, y_3) then $d(x_i) = 2, 1 \leq i \leq 3$ which implies $d(y_1) \geq 4$ by Lemma 15(iv) and $d(y_5) \geq 4$ by Lemma 15(iii) and $c(\Delta) \leq c(3, 3, 3, 4, 4) = 0$ so assume there is only one split. Then both of $l(y_1)$ and $l(y_3)$ involve τ -powers. If $d(x_i) = 2, 1 \leq i \leq 3$ then $d(y_1) \geq 4$ by Lemma

15(iv) and $d(y_5) \geq 4$ by Lemma 15(iii), see Figure 4.18(i). If $d(x_1)$ or $d(x_3)$ is ≥ 3 then this forces at least one more split and $d(\Delta) \geq 6$. If $d(x_2) \geq 3$ then at most one of the vertices is of degree ≥ 4 , see Figure 4.18(ii)-(v).

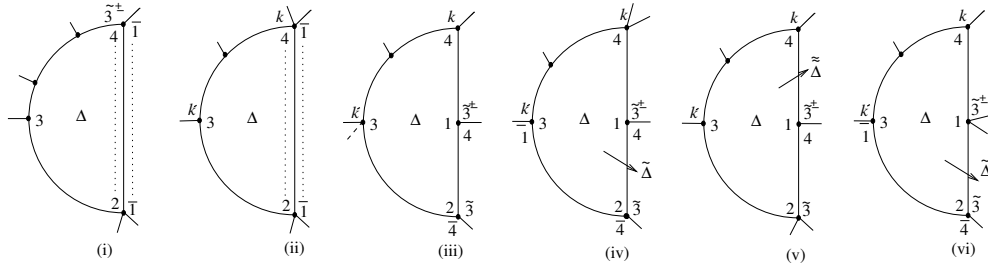

 Figure 4.18: $m = 1$

$d(y_1) \geq 3, d(y_2) = 2, d(y_4) \geq 3$:

A similar argument to the above yields the possibilities shown in Figure 4.19.

$d(y_1) \geq 3, d(y_2) \geq 3, d(y_4) \geq 3$:

Then $d(\Delta) \geq 5$. This forces that there is exactly one split one each of the edges (y_1, y_3) and (y_3, y_5) and that $d(x_i) = 2, 1 \leq i \leq 3$. This however forces both $d(y_1)$ and $d(y_5)$ to be ≥ 4 by Lemma 15 (iii)-(iv) and so $c(\Delta) \leq c(3, 3, 3, 4, 4) = 0$.


 Figure 4.19: $m = 1$

This complete the description of the positive regions in this case. Therefore if $c(\Delta) > 0$ then Δ is given by Figures 4.11- 4.19. We treat each Δ in turn. The distribution is as follows.

Notation:

The vertices in $\hat{\Delta}$ or $\tilde{\Delta}$ corresponding to the vertices y_i, x_j in Δ will be denoted

$\widehat{y}_i, \widehat{x}_j$ or $\widetilde{y}_i, \widetilde{x}_j$.

Note:

When adding $c(\Delta)$ to more than one region then we divide it equally on the regions unless it was stated otherwise.

Δ is given by Figure 4.11(i):

First let $d(y_5) = 4$. Then $l(y_5) = 14\overline{1}4$ and it forces at least 2 splits on $\widehat{\Delta}$ (by Lemma 9), see Figure 4.20. Moreover letting $d(x_{4m-2}) = 4$ forces $l(x_{4m-2}) \in \overline{2}14\{1, \overline{2}, \widetilde{3}^\pm, (24), k\}$. Add $c(\Delta) \leq c(3, 3, 4, 4) = \frac{\pi}{3}$ to $c(\widehat{\Delta}) \leq c(3, 3, 3, 3, 4, 4) = \frac{-\pi}{3}$, see Figure 4.20. (Note that $c(\Delta_1) \leq 0$ and it sends no positive curvature to $\widehat{\Delta}$ nor $\widetilde{\Delta}$). If $d(x_{4m-2}) \geq 5$ then $c(\widehat{\Delta}) \leq c(3, 3, 3, 3, 4, 5) = \frac{-16\pi}{30}$. Add

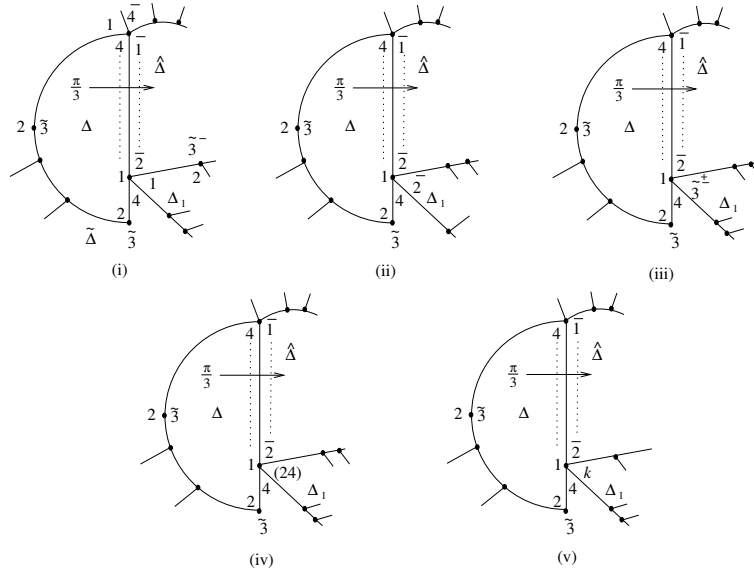


Figure 4.20: $m \geq 2$, $d(y_1) = 2$.

$c(\Delta) \leq c(3, 3, 4, 5) = \frac{7\pi}{30}$ to $c(\widehat{\Delta})$, see Figure 4.21(i)-(ii).

Now let $d(y_5) \geq 5$ and $d(x_{4m-2}) = 4$. Then $c(\Delta) \leq c(3, 3, 4, 5) = \frac{7\pi}{30}$. If $l(x_{4m-2}) \in \overline{2}14\{1, (24), \widetilde{3}^\pm\}$ then these labelling forces at least 2 splits on $\widetilde{\Delta}$ of Figure 4.11(i) (by Lemma 9). Also, $l(y_1)$ forces at least 2 more splits on $\widetilde{\Delta}$ (by Lemma 15(iii)) forcing $d(\widetilde{\Delta}) \geq 7$. Add $c(\Delta)$ to $c(\widetilde{\Delta}) \leq \frac{-\pi}{3}$ as in Figure 4.21(iii). If $l(x_{4m-2}) = \overline{2}14k$ then this forces at least one split on $\widetilde{\Delta}$ with $d(\widetilde{x}_2) \geq 3$ and one split on $\widehat{\Delta}$. Moreover $\widetilde{m} \geq 2$ and either $d(\widetilde{y}_1) \geq 3$ or $l(\widetilde{y}_1) = 2\widetilde{3}$ which forces $d(\widetilde{x}_{4\widetilde{m}-2}) \geq 3$. In addition either $d(\widetilde{x}_3) \geq 3$ or $d(\widetilde{x}_2) \geq 4$. Thus $c(\widehat{\Delta}) \leq c(3, 3, 3, 4, 5) = \frac{-3\pi}{30}$ and $c(\widetilde{\Delta}) \leq c(3, 3, 3, 3, 4, 4) = \frac{-\pi}{3}$. Add $c(\Delta)$ to $c(\widehat{\Delta})$

and $c(\tilde{\Delta})$ as shown in Figure 4.21(iv). Now let $l(x_{4m-2}) = \bar{2}14\bar{2}$. If $d(\hat{y}_3) \geq 3$ then

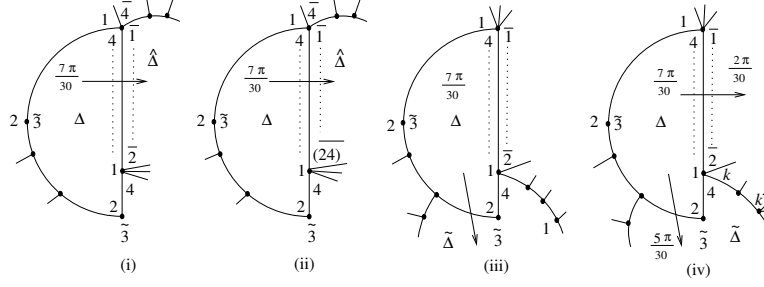


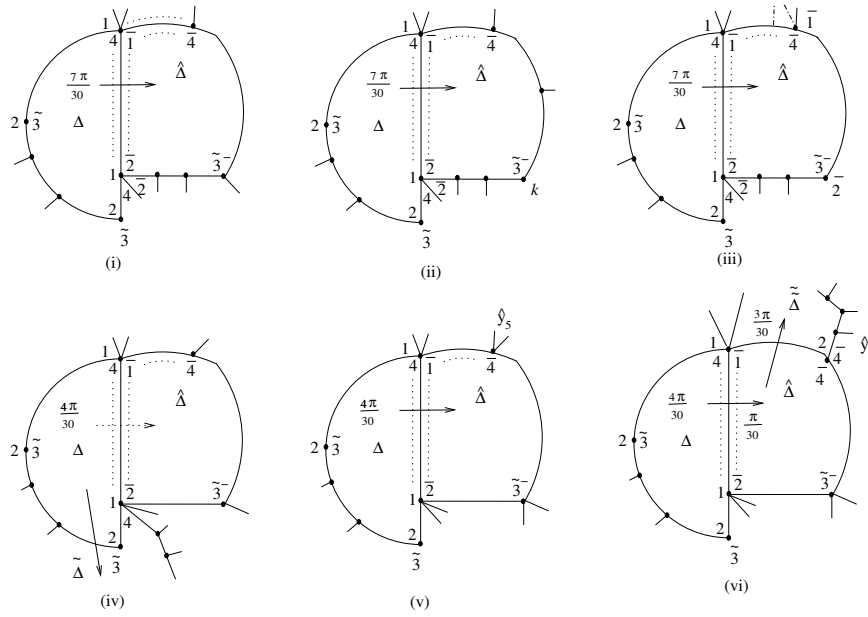
Figure 4.21: $m \geq 2$, $d(y_1) = 2$.

$c(\hat{\Delta}) \leq c(3, 3, 3, 3, 4, 5) = \frac{-13\pi}{30}$. Add $c(\Delta)$ to $c(\hat{\Delta})$ as in Figure 4.22(i). Now let $d(\hat{y}_3) = 2$. Then $l(\hat{y}_3) \in \tilde{3}^- \{k, \bar{2}\}$. If $l(\hat{y}_3) = \tilde{3}^- k$ then $d(\hat{y}_4) \geq 3$ and $c(\hat{\Delta}) \leq \frac{-13\pi}{30}$ as in Figure 4.22(ii). Add $c(\Delta)$ to $c(\hat{\Delta})$. Otherwise if $l(\hat{y}_3) = \bar{2}\tilde{3}^-$ then this forces either $d(\hat{x}_1) \geq 3$ or $d(\hat{y}_5) \geq 4$. Thus $c(\hat{\Delta}) \leq c(3, 3, 4, 4, 5) = \frac{-8\pi}{30}$. Add $c(\Delta)$ to $c(\hat{\Delta})$, see Figure 4.22(iii).

Finally let $d(y_5) \geq 5$ and $d(x_{4m-2}) \geq 5$. Then $c(\Delta) \leq c(3, 3, 5, 5) = \frac{4\pi}{30}$ and we add $c(\Delta)$ to $c(\hat{\Delta})$ when $d(\hat{\Delta}) \geq 5$ in which case $c(\hat{\Delta}) \leq c(3, 3, 3, 5, 5) = \frac{-6\pi}{30}$. Now let $d(\hat{\Delta}) = 4$. Then $\hat{\Delta}$ is as in Figures 4.22(iv)-(v) and $d(\hat{y}_3) \geq 4$ (by Lemma 15 (ii)). If $l(x_{4m-2})$ forces at least 2 splits on $\tilde{\Delta}$ then add $c(\Delta)$ to $c(\tilde{\Delta})$ as in Figure 4.22(iv). Otherwise if $d(x_{4m-2}) = 5$ then $l(x_{4m-2}) \in \{\bar{2}14\bar{1}4, \bar{2}14\bar{2}4, \bar{2}1441, \bar{2}144(24)\}$. If $l(x_{4m-2}) = \bar{2}14\bar{1}4$ then either $d(\hat{y}_5) \geq 4$ or $l(\hat{y}_5) = 2\bar{4}$. In the former case add $c(\Delta)$ to $c(\hat{\Delta}) \leq c(4, 4, 5, 5) = \frac{-6\pi}{30}$. In the latter case $l(\hat{y}_5)$ forces at least 2 splits on $\tilde{\Delta}$ and forces $l(\tilde{x}_{4m-2}) \geq 3$. Hence $c(\tilde{\Delta}) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$. Add $\frac{\pi}{30}$ of $c(\Delta)$ to $c(\tilde{\Delta}) \leq c(3, 4, 5, 5) = \frac{-\pi}{30}$ and $\frac{3\pi}{30}$ of $c(\Delta)$ to $c(\tilde{\Delta})$, see Figures 4.22(v)-(vi). Now let $l(x_{4m-2}) \in \{\bar{2}14\bar{2}4, \bar{2}1441, \bar{2}144(24)\}$. Then $d(\hat{y}_5) \geq 4$ and add $c(\Delta)$ to $c(\hat{\Delta}) \leq \frac{-6\pi}{30}$ as in Figure 4.22(v). If $d(x_{4m-2}) \geq 6$ then $c(\Delta) \leq c(3, 3, 5, 6) = \frac{2\pi}{30}$ and $c(\hat{\Delta}) \leq c(3, 4, 5, 6) = \frac{-3\pi}{30}$ and we add $c(\Delta)$ to $c(\hat{\Delta})$.

Δ is given by Figure 4.11(ii):

In this case note that $l(x_2) = 1\bar{4}\tilde{3}$ forces 2 splits on each of $\hat{\Delta}$ and $\tilde{\Delta}$. Note further that $l(y_5)$ forces at least 3 splits on $\tilde{\Delta}$ as in Figure 4.5(i). Add $c(\Delta) \leq c(3, 3, 3, 3, 4) = \frac{5\pi}{30}$ to $c(\tilde{\Delta}) \leq c(3, 3, 3, 3, 3, 3, 3, 3) = \frac{-2\pi}{3}$, see Figure 4.23(i).

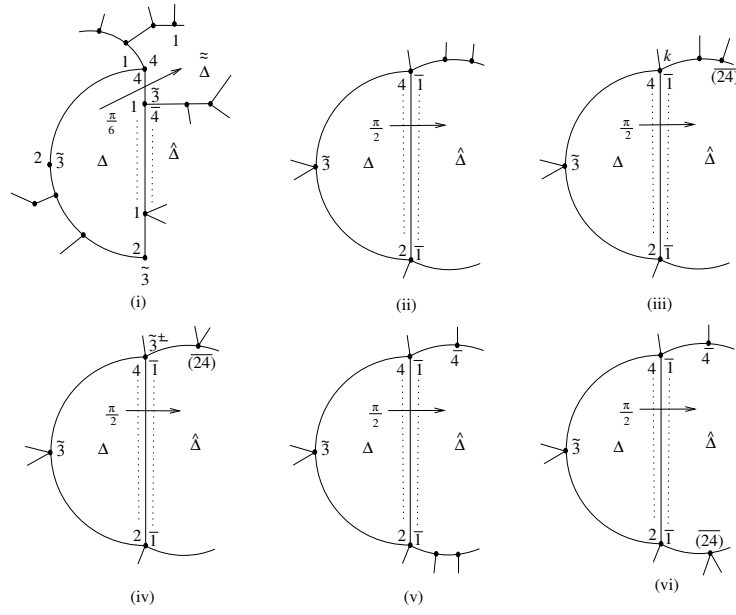
Figure 4.22: $m \geq 2$, $d(y_1) = 2$.

Δ is given by Figure 4.11(iii):

In this case $c(\Delta) \leq c(4, 4, 4) = \frac{\pi}{2}$. Moreover letting $d(y_5) = 4$ forces $l(y_5) \in \{14\bar{1}\bar{4}, (24)4\bar{1}\bar{4}, 44\bar{1}\{\bar{2}, \bar{3}^\pm, 4, k\}\}$ while letting $d(y_1) = 4$ forces $l(y_1) \in \{\bar{1}2\bar{1}\{\bar{1}, \bar{3}^\pm, \bar{4}, \overline{(24)}\}, k\}, \bar{1}2\bar{4}\{\bar{1}, 2, \bar{3}^\pm, \overline{(24)}\}, k\}, \bar{1}2\overline{(24)}\{\bar{1}, \bar{3}^\pm, \overline{(24)}\}, \bar{4}\}\}$. First let $d(y_5) = 4$ and assume $l(y_5) \in \{14\bar{1}\bar{4}, (24)4\bar{1}\bar{4}, 44\bar{1}\{\bar{2}, 4\}\}$. Then $l(y_5)$ forces at least 2 splits on $\hat{\Delta}$. We also have at least 3 vertices of degree ≥ 3 on the edges $(\hat{y}_1, \hat{y}_3), (\hat{y}_3, \hat{y}_5)$ and so $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 3, 4, 4) = \frac{-2\pi}{3}$. Add $c(\Delta)$ to $c(\hat{\Delta})$, see Figure 4.23(ii).

Now let $d(y_5) = 4$ and $l(y_5) = l(\hat{x}_i) \in 44\bar{1}\{k, \bar{3}^\pm\}$. Then we have 2 cases: $\hat{x}_i \neq \hat{x}_2$ and $\hat{x}_i = \hat{x}_2$. If $\hat{x}_i \neq \hat{x}_2$ then labelling forces at least two splits or one split of degree ≥ 4 and $c(\hat{\Delta}) \leq c(3, 3, 3, 4, 4, 4) = \frac{-\pi}{2}$. Add $c(\Delta)$ to $c(\hat{\Delta})$, see Figure 4.23(iii)-(iv). Now let $\hat{x}_i = \hat{x}_2$ then we have 2 cases $d(y_1) = 4$ and $d(y_1) \geq 5$.

If $d(y_1) = 4$ then letting $l(y_1) \in \{\bar{1}2\bar{1}\{\bar{1}, \bar{3}^\pm, \overline{(24)}\}, \bar{1}2\bar{4}\{\bar{1}, \bar{3}^\pm, \overline{(24)}\}, \bar{1}2\overline{(24)}\{\bar{1}, \bar{3}^\pm, \overline{(24)}\}\}$ forces at least 2 splits on $\hat{\Delta}$ and again $d(\hat{\Delta}) \geq 7$. If $l(y_1) = l(\hat{x}_j) \in \{\bar{1}2\bar{1}\{\bar{4}, k\}, \bar{1}2\bar{4}\{2, k\}, \bar{1}2\overline{(24)}\{\bar{4}, k\}\}$ then there are 2 cases $\hat{x}_j \neq \hat{x}_{4\hat{m}-2}$ and $\hat{x}_j = \hat{x}_{4\hat{m}-2}$. If $\hat{x}_j \neq \hat{x}_{4\hat{m}-2}$ then this forces at least two splits or one split of degree ≥ 4 and $c(\hat{\Delta}) \leq c(3, 3, 3, 4, 4, 4) = \frac{-\pi}{2}$. Add $c(\Delta)$ to $c(\hat{\Delta})$, see Figure 4.23(v)-(vi). Now let $\hat{x}_j = \hat{x}_{4\hat{m}-2}$. Then $\hat{m} = m + 1$ and we proceed by studying


 Figure 4.23: $m \geq 2$

each labelling of y_5 in turn.

First let $l(y_5) = 44\bar{1}k$. If $l(y_1)$ involves a τ -power then $d(\hat{\Delta}) \geq 7$. Add $c(\Delta)$ to $c(\hat{\Delta})$, see Figure 4.24(i). If $l(y_1) = \bar{1}2\bar{1}\bar{4}$ then this forces 2 splits on Δ' and 3 on Δ'' and either $l(\hat{y}_1) = 2\bar{3}^-$ or $d(\hat{y}_1) \geq 3$. If $d(\hat{y}_1) = 2$ then there are at least 2 splits on each of $\hat{\Delta}$ and Δ'' (by Lemma 9) and so $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 4, 4) = \frac{-\pi}{3}$, $c(\Delta') \leq c(3, 3, 3, 3, 3, 4) = \frac{-\pi}{6}$ and $c(\Delta'') \leq c(3, 3, 3, 3, 3, 4) = \frac{-\pi}{6}$, see Figure 4.24(ii). Now let $d(\hat{y}_1) \geq 3$. Then again $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 4, 4) = \frac{-\pi}{3}$, $c(\Delta') \leq c(3, 3, 3, 3, 3, 4) = \frac{-\pi}{6}$ and $c(\Delta'') \leq c(3, 3, 3, 3, 3, 4) = \frac{-\pi}{6}$, see Figure 4.24(iii). Add $c(\Delta)$ to $c(\hat{\Delta}) + c(\Delta') + c(\Delta'')$ as shown in Figures 4.24(i)-(iii). Next assume $l(y_1) = \bar{1}2\bar{4}2$. Then again $c(\hat{\Delta}) \leq \frac{-\pi}{3}$ and $l(y_1)$ forces 2 splits on Δ' and 3 on Δ'' . Moreover if $m' = 1$ then $d(y'_3) \geq 3$ and $d(\Delta') \geq 6$ and if $m' \geq 2$ then either $d(y'_3) \geq 3$ and $d(\Delta') \geq 6$ or $d(y'_3) = 2$ and this implies $l(y'_3) \in \bar{3}^-\{k, 2\}$. If $l(y'_3) = \bar{3}^-k$ then $d(y'_4) \geq 3$ and so $d(\Delta') \geq 6$ and if $l(y'_3) = \bar{3}^-2$ then $l(y'_1) = \bar{1}2\omega$ which either forces a split on the segment (y'_1, y_3) or $d(y'_1) \geq 4$ and hence $c(\Delta') \leq c(3, 3, 4, 4, 4) = \frac{-\pi}{6}$. Also $c(\Delta'') \leq c(3, 3, 3, 3, 3, 4) = \frac{-\pi}{6}$. Add $c(\Delta)$ to $c(\hat{\Delta}) + c(\Delta') + c(\Delta'')$, see Figure 4.24(iv)-(v). Similarly letting $l(y_1) = \bar{1}2\bar{4}2\bar{4}$ forces $c(\hat{\Delta}) \leq \frac{-\pi}{3}$ and $l(y_1)$ forces 2 splits on Δ' (by Lemma 9) and 3 on Δ'' (as in Figure 4.6). Hence $c(\Delta') \leq \frac{-\pi}{6}$ and $c(\Delta'') \leq \frac{-\pi}{6}$. Add $c(\Delta)$ to $c(\hat{\Delta}) + c(\Delta') + c(\Delta'')$, see Figure 4.24(vi)-(vii).

Next let $l(y_5) = 44\bar{1}3^\pm$. Note that $l(y_5)$ forces at least 2 splits on $\widehat{\widehat{\Delta}}$ and at least

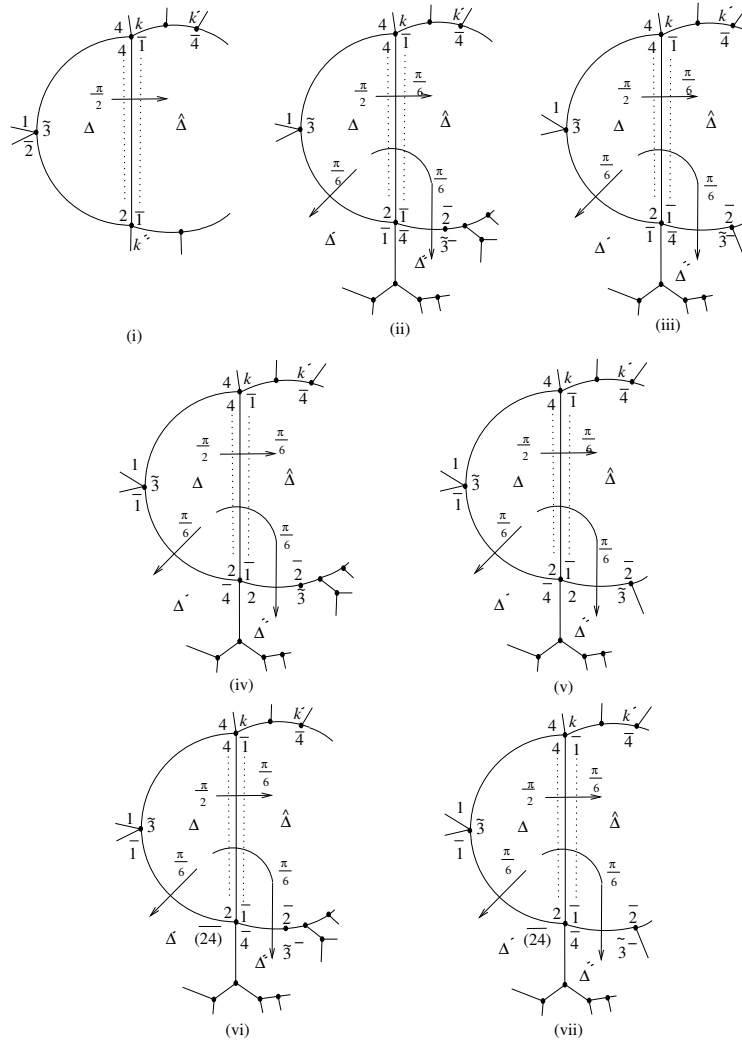


Figure 4.24: $m \geq 2$

2 splits on $\widehat{\widehat{\Delta}}$ and hence $c(\widehat{\widehat{\Delta}}) \leq \frac{-\pi}{6}$. If $l(y_1)$ involves a τ -power then this forces at least one split on $\widehat{\Delta}$ and $c(\widehat{\Delta}) \leq \frac{-\pi}{2}$. Add $c(\Delta)$ to $c(\widehat{\Delta}) + c(\widehat{\widehat{\Delta}})$ as shown in Figure 4.25(i). Now let $l(y_1) \in \{\bar{1}2\bar{1}4, \bar{1}2\bar{4}2, \bar{1}2(\bar{2}4)4\}$. If $d(\widehat{y}_5) \geq 4$ or $d(\widehat{y}_4) \geq 3$ then $c(\widehat{\Delta}) \leq \frac{-\pi}{6}$. Add $c(\Delta)$ to $c(\widehat{\Delta}) + c(\widehat{\widehat{\Delta}}) + c(\Delta')$. If $d(\widehat{y}_5) = 3$ then $c(\widehat{\Delta}) \leq 0$. Add $c(\Delta)$ to $c(\widehat{\Delta}) + c(\Delta') + c(\Delta'')$, see Figures 4.25(ii)-(iii).

Now let $d(y_1) \geq 5$. Then $c(\Delta) \leq c(4, 4, 5) = \frac{12\pi}{30}$. Moreover, letting $l(y_5) = 44\bar{1}k$ forces $c(\widehat{\Delta}) \leq c(3, 3, 3, 3, 4, 5) = \frac{-13\pi}{30}$. Add $c(\Delta)$ to $c(\widehat{\Delta})$, see Figure 4.25(iv). Now let $l(y_5) = 44\bar{1}3^\pm$. If $d(\widehat{y}_5) \geq 4$ then $c(\widehat{\Delta}) \leq c(3, 3, 4, 4, 5) = \frac{-8\pi}{30}$. Add $\frac{4\pi}{30}$

of $c(\Delta)$ to $c(\hat{\Delta})$ and add the remaining $\frac{8\pi}{30}$ to $c(\hat{\Delta})$, see Figure 4.25(v). Now let $d(\hat{y}_5) = 3$. Then $l(\hat{y}_5) \in \{\bar{4}\bar{4}\{1, \tilde{3}^\pm, \overline{(24)}, k\}, 2\bar{4}\{\bar{4}, \tilde{3}^\pm, k\}\}$ each forces at least 2 splits on $\tilde{\Delta}$. If $l(\hat{y}_3) \in \{\bar{4}\bar{4}\{\tilde{3}^\pm, k\}, 2\bar{4}\{\tilde{3}^\pm, k\}\}$ then the labelling forces $d(\hat{y}_4) \geq 3$ so $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 4, 5) = \frac{-13\pi}{30}$. Add $\frac{4\pi}{30}$ of $c(\Delta)$ to $c(\hat{\Delta})$ and add the remaining $\frac{8\pi}{30}$ to $c(\hat{\Delta})$, see Figure 4.25(vi). Otherwise if $l(\hat{y}_3) \in \{\bar{4}\bar{4}\bar{1}, \bar{4}\bar{4}\overline{(24)}, 22\bar{4}\bar{4}\}$ and hence $c(\tilde{\Delta}) \leq c(3, 3, 3, 3, 3, 3, 4) = \frac{-\pi}{2}$. Also $c(\hat{\Delta}) \leq c(3, 3, 3, 4, 5) = \frac{-3\pi}{30}$. Add $\frac{4\pi}{30}$ of $c(\Delta)$ to $c(\hat{\Delta})$ and add $\frac{5\pi}{30}$ to $c(\hat{\Delta})$ and add the remaining $\frac{4\pi}{30}$ to $c(\tilde{\Delta})$, see Figure 4.25(vii).

Now let $d(y_5) = d(\hat{x}_i) \geq 5$ and $d(y_1) = d(\hat{x}_j) = 4$. Then $c(\Delta) \leq \frac{12\pi}{30}$. If

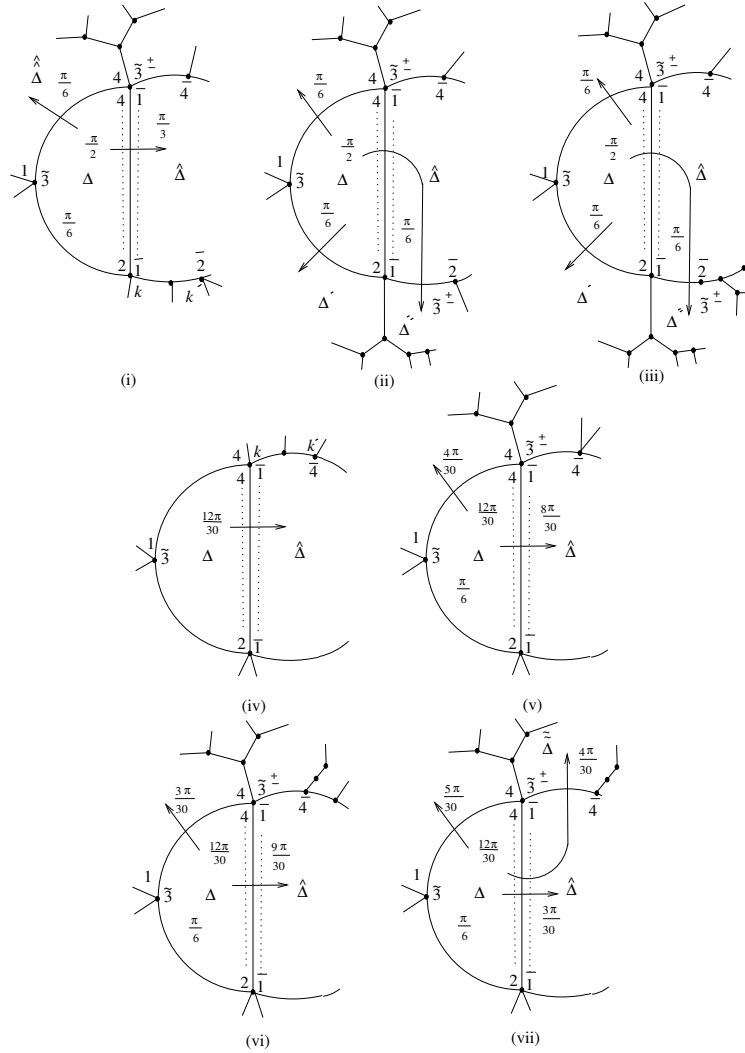


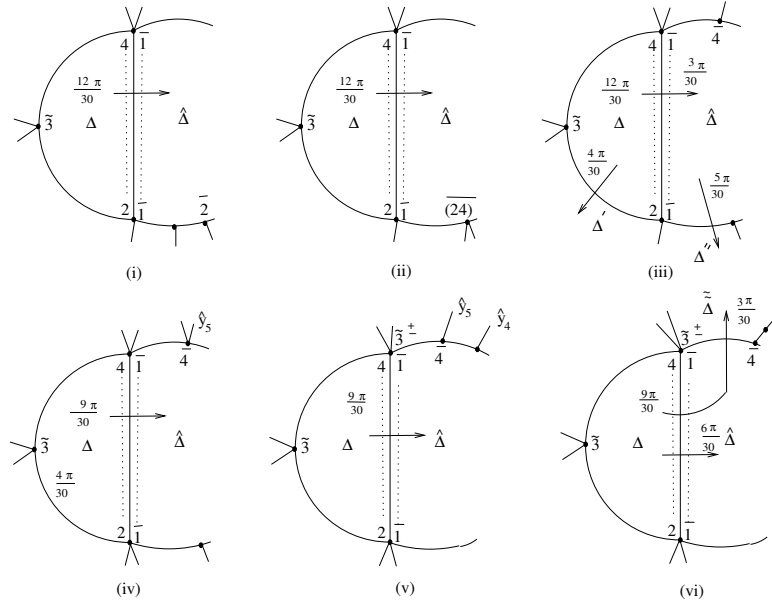
Figure 4.25: $m \geq 2$.

$l(y_1) \in \{\bar{1}2\bar{1}\{\bar{1}, \tilde{3}^\pm, \overline{(24)}, k\}, \bar{1}2\bar{4}\{\bar{1}, \tilde{3}^\pm, \overline{(24)}, k\}, \bar{1}2\overline{(24)}\{\bar{1}, \tilde{3}^\pm, \overline{(24)}, k\}\}$ or $l(y_1) = l(\hat{x}_j) \in \{\bar{1}2\bar{1}\bar{4}, \bar{1}2\bar{4}2, \bar{1}2\overline{(24)}\bar{4}\}$ with $\hat{x}_j \neq \hat{x}_{4\hat{m}-2}$ then this forces at least one split on $\hat{\Delta}$ and so $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 4, 5) = \frac{-16\pi}{30}$. Add $c(\Delta)$ to $c(\hat{\Delta})$, see Figure 4.26(i)-(ii). If $l(y_1) = l(\hat{x}_j) \in \{\bar{1}2\bar{1}\bar{4}, \bar{1}2\bar{4}2, \bar{1}2\overline{(24)}\bar{4}\}$ and $\hat{x}_j = \hat{x}_{4\hat{m}-2}$ then add $\frac{3\pi}{30}$ of $c(\Delta)$ to $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 4, 5) = \frac{-3\pi}{30}$ and add $\frac{4\pi}{30}$ to $c(\Delta')$ and add the remaining $\frac{3\pi}{30}$ to $c(\Delta'')$, see Figure 4.26(iii).

Now let $d(y_5) = d(\hat{x}_i) \geq 5$ and $d(y_1) = d(\hat{x}_j) \geq 5$. Then $c(\Delta) \leq \frac{9\pi}{30}$. If $\hat{x}_i \neq \hat{x}_2$ then either $d(\Delta) \geq 6$ or $d(\hat{y}_5) \geq 4$. Thus, $c(\hat{\Delta}) \leq c(3, 3, 4, 5, 5) = \frac{-11\pi}{30}$. Add $c(\Delta)$ to $c(\hat{\Delta})$. Now let $\hat{x}_i = \hat{x}_2$. If $d(\hat{y}_5) \geq 4$ then add $c(\Delta)$ to $c(\hat{\Delta}) \leq c(3, 3, 4, 5, 5) = \frac{-11\pi}{30}$, see Figure 4.26(iv). If $d(\hat{y}_5) = 3$ then either $d(\hat{y}_4) \geq 3$ or $l(\hat{y}_3) \in \{\bar{4}\bar{4}\bar{1}, \bar{4}\bar{4}2\}$. In the former case $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 5, 5) = \frac{-16\pi}{30}$. Add $c(\Delta)$ to $c(\hat{\Delta})$, see Figure 4.26(v). In the latter case letting $l(\hat{y}_5) = \bar{4}\bar{4}\{\bar{1}, \overline{(24)}\}$ forces at least 3 splits on $\tilde{\Delta}$ and hence $c(\tilde{\Delta}) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$ and $c(\hat{\Delta}) \leq c(3, 3, 3, 5, 5) = \frac{-6\pi}{30}$. Add $\frac{5\pi}{30}$ of $c(\Delta)$ to $c(\hat{\Delta})$ and add the remaining $\frac{4\pi}{30}$ to $c(\tilde{\Delta})$, see Figure 4.26(vi). Whereas letting $l(\hat{y}_5) = \bar{4}\bar{4}2$ forces at least 2 splits on $\tilde{\Delta}$. If in addition $d(\hat{y}_5) \geq 4$; or if $d(\tilde{x}_{4\tilde{m}-2}) \geq 3$; or if $d(\tilde{x}_{4\tilde{m}-2}) = 2$ with $\tilde{m} \geq 2$ then $c(\tilde{\Delta}) \leq c(3, 3, 3, 4, 5) = \frac{-3\pi}{30}$ and add $\frac{6\pi}{30}$ of $c(\Delta)$ to $c(\hat{\Delta})$ and add the remaining $\frac{3\pi}{30}$ to $c(\tilde{\Delta})$ as in Figure 4.26(vi). Otherwise $l(\tilde{x}_{4\tilde{m}-2}) = 1\bar{3}^-$, $l(\hat{y}_5) = 4\bar{4}2$ and $\tilde{m} = 1$. Hence $\tilde{\Delta}$ is as Δ in Figure 4.45(iii) and we describe the distribution later.

Δ is given by Figure 4.12(i):

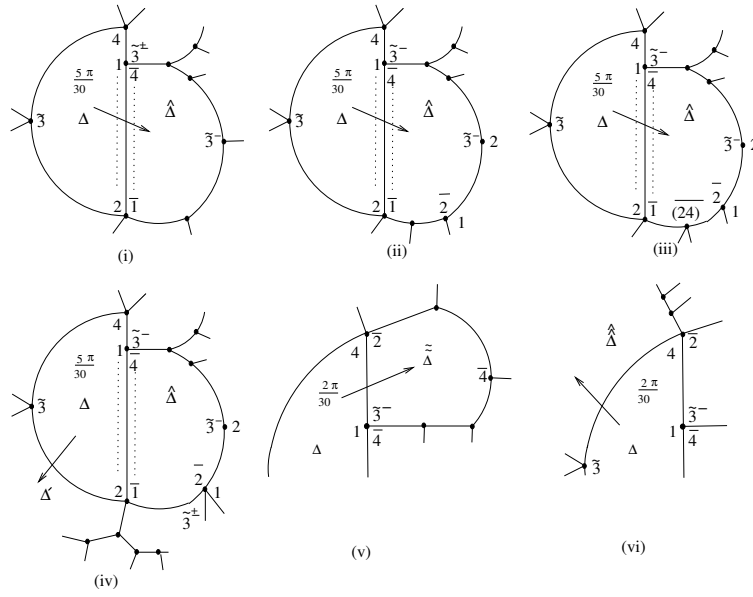
Note that $l(x_2)$ forces at least 2 splits on each of $\hat{\Delta}$ and $\tilde{\Delta}$, see Figure 4.27. First let $d(\hat{y}_3) \geq 3$. Then $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 3, 4) = \frac{-\pi}{6}$. Add $c(\Delta) \leq c(3, 4, 4, 4) = \frac{\pi}{6}$ to $c(\hat{\Delta})$, See Figures 4.27(i). Next let $d(\hat{y}_3) = 2$. Then $l(\hat{y}_3) \in \tilde{3}^-\{2, \bar{2}\}$. However, $l(\hat{y}_3) = \bar{2}\tilde{3}^-$ yields a contradiction as it forces ww^{-1} on $l(\hat{y}_4)$. Thus $l(\hat{y}_3) = 2\tilde{3}^-$. If in addition $d(y_1) = 4$ then $l(y_1) = l(\hat{x}_j) \in \{\bar{1}2\bar{1}\{\bar{1}, \tilde{3}^\pm, \overline{(24)}, k\}, \bar{1}2\bar{4}\{\bar{1}, 2, \tilde{3}^\pm, \overline{(24)}, k\}, \bar{1}2\overline{(24)}\{\bar{1}, \tilde{3}^\pm, \overline{(24)}, \bar{4}\}\}$. If $l(y_1) \in \{\bar{1}2\bar{1}\{\bar{1}, \tilde{3}^\pm, \overline{(24)}, k\}, \bar{1}2\bar{4}\{\bar{1}, \tilde{3}^\pm, \overline{(24)}, k\}, \bar{1}2\overline{(24)}\{\bar{1}, \tilde{3}^\pm, \overline{(24)}\}\}$. Then the labelling forces at least one split on $\tilde{\Delta}$ and $c(\tilde{\Delta}) \leq c(3, 3, 3, 3, 3, 4) = \frac{-\pi}{6}$. Add $c(\Delta)$ to $c(\tilde{\Delta})$, see Figures 4.27(ii). Otherwise if $l(y_1) = l(\hat{x}_i) \in \{\bar{1}2\bar{1}\bar{4}, \bar{1}2\bar{4}2, \bar{1}2\overline{(24)}\bar{4}\}$ with $\hat{x}_i \neq \hat{x}_{4\hat{m}-2}$ then this also forces at least one split and add $c(\Delta)$ to $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 4, 4) = \frac{-\pi}{3}$, see Figures 4.27(iii). Whereas letting $\hat{x}_i = \hat{x}_{4\hat{m}-2}$ implies $\hat{m} = m$, $d(\hat{y}_1) \geq 4$, $c(\hat{\Delta}) \leq 0$, $c(\Delta') \leq \frac{-\pi}{6}$. Add $c(\Delta)$ to $c(\Delta')$, see Figures 4.27(iv). Next assume $\underline{d(y_1) \geq 5}$. Then $d(y_5) = 4$ and $c(\Delta) \leq c(3, 4, 4, 5) = \frac{2\pi}{30}$. Moreover,

Figure 4.26: $m \geq 2$.

$d(y_5) = 4$ forces $l(y_5) \in \{14\bar{2}\{1, \bar{2}, \tilde{3}^\pm, k\}, (24)4\bar{2}\{1, \tilde{3}^\pm, k\}, 44\bar{2}\{\bar{1}, \tilde{3}^\pm, \overline{(24)}, k, 4\}\}$. All these labellings except for $l(y_5) \in \{14\bar{2}1, (24)4\bar{2}1\}$ forces at least one split on $\tilde{\Delta}$ and $c(\tilde{\Delta}) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$. Add $c(\Delta)$ to $c(\tilde{\Delta})$, see Figure 4.27(v). If $l(y_5) \in \{14\bar{2}1, (24)4\bar{2}1\}$ then these labellings forces at least 2 splits on $\hat{\Delta}$ and $c(\hat{\Delta}) \leq \frac{-5\pi}{30}$. Add $c(\Delta)$ to $c(\hat{\Delta})$, see Figure 4.27(vi).

Δ is given by Figure 4.12(ii):

In this case $d(y_5) = 3$ implies $l(y_5) \in \{441, 44(24), 4\bar{2}4\}$. If $l(y_5) \in \{441, 44(24)\}$ then these labelling forces at least 2 splits on $\hat{\Delta}$ and $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$, see Figure 4.28(i)-(ii). Add $c(\Delta) \leq c(3, 4, 4, 4) = \frac{\pi}{6}$ to $c(\hat{\Delta})$. Now let $l(y_5) = 4\bar{2}4$. Then this forces as well at least 2 splits on $\hat{\Delta}$. If in addition $d(\hat{y}_3) \geq 3$ then $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$ and add $c(\Delta)$ to $c(\hat{\Delta})$, see Figure 4.28(iii). Now let $d(\hat{y}_3) = 2$. Then $\hat{m} \geq 2$ and $l(\hat{y}_3) \in \tilde{3}\{k, \bar{1}, \bar{2}, \overline{(24)}\}$. If $l(\hat{y}_3) = \tilde{3}k$ then $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$ and add $c(\Delta)$ to $c(\hat{\Delta})$, see Figure 4.28(iv). If $l(\hat{y}_3) \in \tilde{3}\{\bar{1}, \bar{2}, \overline{(24)}\}$ then either there is another split on $\hat{\Delta}$ or $l(y_3)$ involves one of $\{\overline{(24)}, \bar{2}\}$ and $d(\hat{y}_1) \geq 4$. In the former case add $c(\Delta)$ to $c(\hat{\Delta}) \leq -\frac{\pi}{6}$, see Figure 4.28(v). In the latter case if $d(y_3) = 4$ then $l(y_3) \in \{\tilde{3}1\overline{(24)}\{\bar{1}, \bar{2}, \overline{(24)}\}, \tilde{3}1\bar{2}\{\bar{1}, \bar{2}, \overline{(24)}\}\}$ each forces at lest 2 splits on Δ' . In

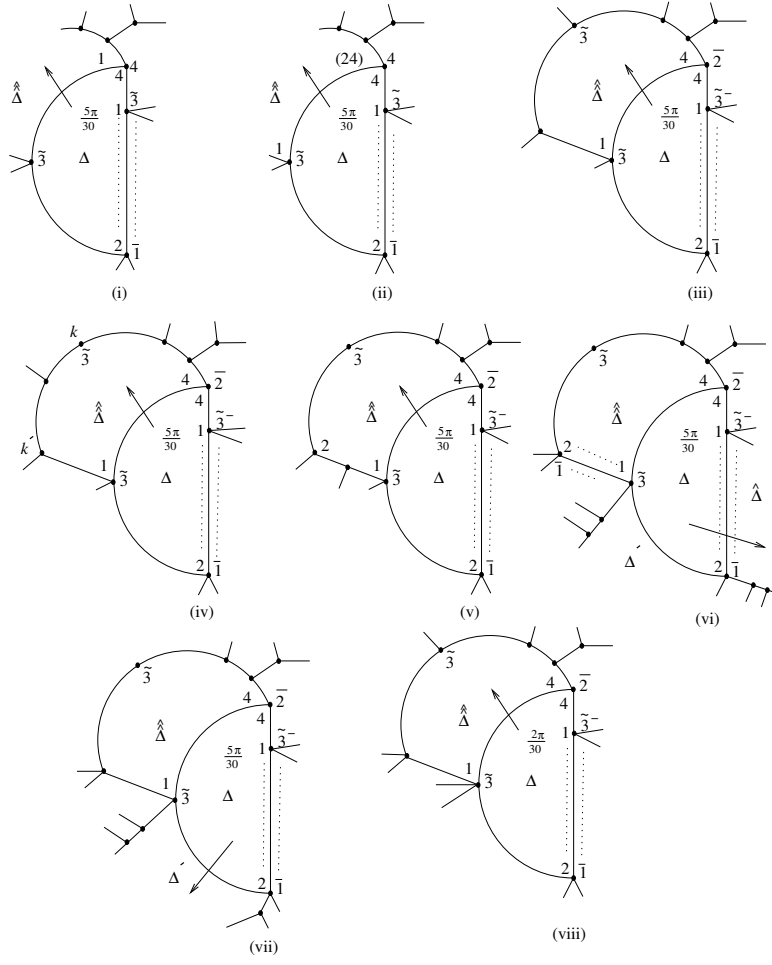
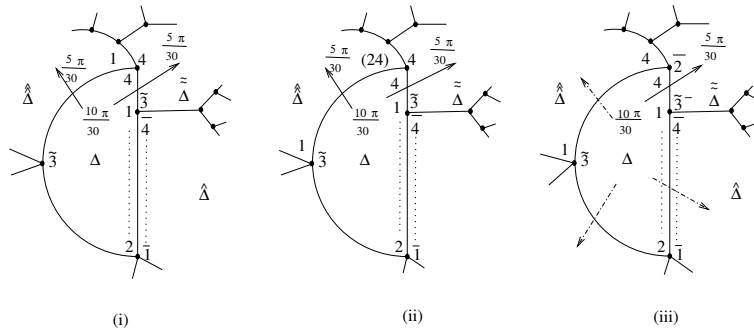

 Figure 4.27: $m \geq 2$.

addition $d(y_1) = 4$ forces $l(y_1) \in \{\bar{1}2\bar{1}\{\tilde{3}^\pm, k\}, \bar{1}2\bar{4}\{2, \tilde{3}^\pm\}, \bar{1}2\overline{(24)}\{\tilde{3}^\pm, k\}\}$. If $d(y_1) = 4$ and $l(y_1) \in \{\bar{1}2\bar{1}\tilde{3}^\pm, \bar{1}2\bar{4}\tilde{3}^\pm, \bar{1}2\overline{(24)}\tilde{3}^\pm\}$ then this labelling forces at least 2 split on $\hat{\Delta}$ and $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 4, 4) = \frac{-\pi}{3}$. Add $c(\Delta)$ to $c(\hat{\Delta})$, see Figure 4.28(vi). If $d(y_1) = 4$ and $l(y_1) \in \{\bar{1}2\bar{1}k, \bar{1}2\bar{4}2, \bar{1}2\overline{(24)}k\}$ then this labelling forces at least one more split on Δ' and $c(\Delta') \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$. Add $c(\Delta)$ to $c(\Delta')$, see Figure 4.28(vii). While if $d(y_1) \geq 5$ then $c(\Delta) \leq c(3, 4, 4, 5) = \frac{2\pi}{30}$ and $c(\Delta') \leq c(3, 3, 3, 4, 5) = \frac{-3\pi}{30}$. Add $c(\Delta)$ to $c(\Delta')$. Now let $d(y_3) \geq 5$ then $c(\Delta) \leq c(3, 4, 4, 5) = \frac{2\pi}{30}$ and $c(\hat{\Delta}) \leq c(3, 3, 3, 4, 5) = \frac{-3\pi}{30}$. Add $c(\Delta)$ to $c(\hat{\Delta})$, see Figure 4.28(viii).

Δ is given by Figure 4.12(iii):

Note that $l(x_2)$ forces at least 2 splits on $\tilde{\Delta}$. On the other hand $l(y_5) \in \{441, 44(24), 4\bar{2}4\}$ also forces 2 splits on $\tilde{\Delta}$. If $l(y_5) \in \{441, 44(24)\}$ then add $c(\Delta) \leq \frac{10\pi}{30}$ to $c(\hat{\Delta})$ and $c(\hat{\hat{\Delta}})$, see Figure 4.29(i)-(ii). If $l(y_5) = 4\bar{2}4$ then send $\frac{5\pi}{30}$ to $\tilde{\Delta}$ and using a similar argument to the one used in 4.28(iii)-(viii) send the remainder of $c(\Delta)$ to $\hat{\hat{\Delta}}$, $\hat{\Delta}$ or Δ' , see Figure 4.29(iii). In this case note that $1\bar{4}\tilde{3}^- = 2\tilde{3}^-$ and so if $d(\hat{y}_3) = 2$ then $l(\hat{y}_3) = \tilde{2}\tilde{3}^-$.

Δ is given by Figure 4.12(iv):


 Figure 4.28: $m \geq 2$.

 Figure 4.29: $m \geq 2$.

Then $c(\Delta) \leq c(3, 4, 4, 4) = \frac{5\pi}{30}$. Similarly to Figure 4.11(iii) letting $d(y_5) = 4$ forces $l(y_5) \in \{14\bar{1}\bar{4}, (24)4\bar{1}\bar{4}, 44\bar{1}\{2, \tilde{3}^\pm, 4, k\}\}$. If $l(y_5) \in \{14\bar{1}\bar{4}, (24)4\bar{1}\bar{4},$

$44\bar{1}\{\bar{2}, 4\}$ then these labelling forces at least 2 splits on $\hat{\Delta}$, see Figure 4.30(i), and so $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 4, 4) = \frac{-10\pi}{30}$. Add $c(\Delta)$ to $c(\hat{\Delta})$ as shown. Now let $l(y_5) = l(\hat{x}_i) = 44\bar{1}\tilde{3}^\pm$. If $\hat{x}_i \neq \hat{x}_2$ then $l(y_5)$ forces at least one split of degree ≥ 4 on $\hat{\Delta}$ and $c(\hat{\Delta}) \leq c(3, 3, 4, 4, 4) = \frac{-5\pi}{30}$, see Figure 4.30(ii). Add $c(\Delta)$ to $c(\hat{\Delta})$ as shown. Now let $\hat{x}_i = \hat{x}_2$. Then $l(y_5)$ forces at least 2 splits on $\hat{\Delta}$, see Figure 4.30(iii). If in addition $d(\hat{y}_3) \geq 3$ then $d(\hat{\Delta}) \geq 6$ while if $d(\hat{y}_3) = 2$ then $c(\hat{\Delta}) \leq c(3, 3, 4, 4, 4) = \frac{-5\pi}{30}$ as if $d(\hat{\Delta}) = 5$ then $d(\hat{y}_1) \geq 4$. Add $c(\Delta)$ to $c(\hat{\Delta})$, see Figures 4.30(iii)-(iv). Next let $l(y_5) = l(\hat{x}_i) = 44\bar{1}k$. If $\hat{x}_i \neq \hat{x}_2$ then $l(y_5)$ forces at least 2 split on $\hat{\Delta}$ and $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 4, 4) = \frac{-10\pi}{30}$, see Figure 4.30(v). Add $c(\Delta)$ to $c(\hat{\Delta})$ as shown. If $\hat{x}_i = \hat{x}_2$ then $l(y_5)$ forces at least one split with $d(\hat{y}_5) \geq 4$ or $d(\hat{y}_4) \geq 3$, see Figure 4.30(vi), and $c(\hat{\Delta}) \leq c(3, 3, 4, 4, 4) = \frac{-5\pi}{30}$. Add $c(\Delta)$ to $c(\hat{\Delta})$ as shown.

Now assume $d(y_5) \geq 5$. Then $d(y_3) = d(x_{4m-2}) = 4$ and $c(\Delta) \leq c(3, 4, 4, 5) = \frac{2\pi}{30}$.

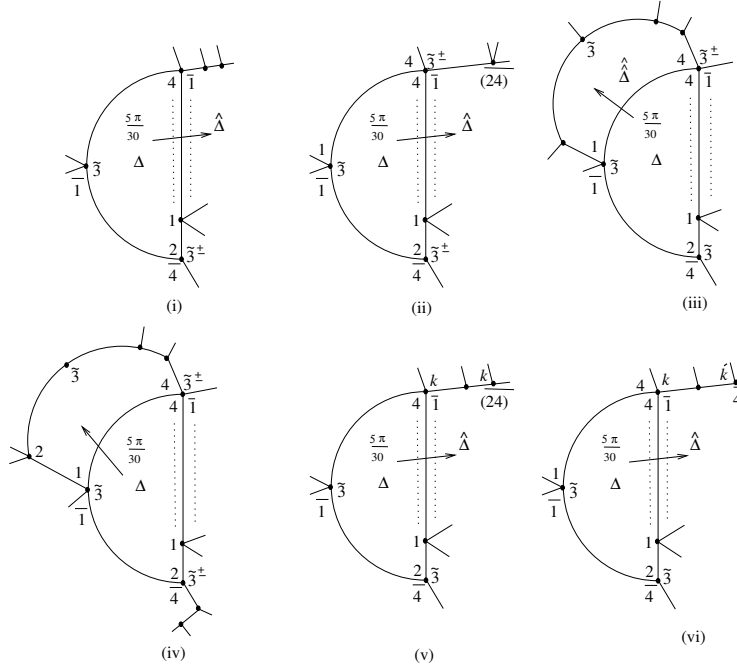
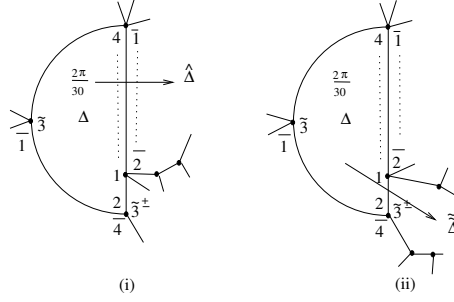


Figure 4.30: $m \geq 2$.

If $l(x_{4m-2}) = \overline{(24)}1\{\bar{2}, 4\}\omega$ then $d(x_{4m-2}) \geq 5$ and $c(\Delta) \leq 0$. This is because $l(x_{4m-2}) = \overline{(24)}1\tilde{2}\omega$ yields a contradiction with $l(y_1) = 2\bar{4}\tilde{3}^\pm$ and $l(x_{4m-2}) = \overline{(24)}14\omega$ yields a labelling contradiction. If $l(x_{4m-2}) = \bar{2}1\{\bar{2}, 4\}\omega$ then checking all possible labellings shows that $l(x_{4m-2}) \in \{\bar{2}1\bar{2}\{\bar{2}, \tilde{3}^\pm, k\}, \bar{2}14\{1, \tilde{3}^\pm, k\}\}$. Letting

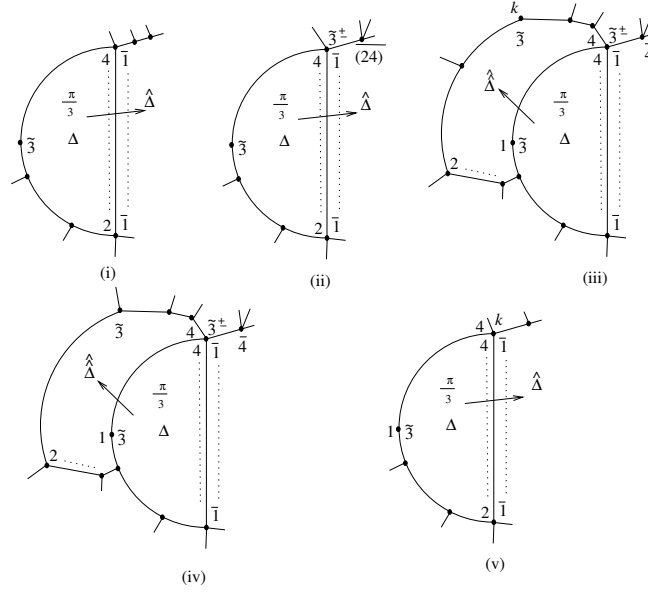
$l(x_{4m-2}) \in \{\bar{2}1\bar{2}\{\bar{2}, \tilde{3}^\pm\}, \bar{2}14\tilde{3}^\pm\}$ then this labelling forces at least 2 splits on $\hat{\Delta}$, see Figure 4.31(i), and $c(\hat{\Delta}) \leq c(3, 3, 3, 4, 5) = \frac{-3\pi}{30}$. Add $c(\Delta)$ to $c(\hat{\Delta})$ as shown. Whereas letting $l(x_{4m-2}) \in \{\bar{2}1\bar{2}k, \bar{2}14\{1, k\}\}$ forces at least 1 split on $\tilde{\Delta}$. Moreover $l(y_1) = \bar{2}4\tilde{3}^\pm$ forces 2 splits on $\tilde{\Delta}$ and hence $c(\tilde{\Delta}) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$. Add $c(\Delta)$ to $c(\tilde{\Delta})$, see Figure 4.31(ii).


 Figure 4.31: $m \geq 2$.

Δ is given by Figure 4.13(i):

In this case $c(\Delta) \leq c(3, 3, 4, 4) = \frac{\pi}{3}$ and $l(y_3) \in \tilde{3}\{1, 2, (24)\}$. Note that $\hat{m} \geq m+1$ and that either $d(\hat{y}_3) \geq 3$ or $l(\hat{y}_3) = \tilde{3}k$ otherwise there is a contradiction. Moreover as in Figure 4.11(iii) letting $d(y_5) = 4$ forces $l(y_5) \in \{14\bar{1}\bar{4}, (24)4\bar{1}\bar{4}, 44\bar{1}\{\bar{2}, \tilde{3}^\pm, 4, k\}\}$. If $l(y_5) \in \{14\bar{1}\bar{4}, (24)4\bar{1}\bar{4}, 44\bar{1}\{\bar{2}, 4\}\}$ then these labelling forces at least 2 splits on $\hat{\Delta}$, see Figure 4.32(i), and hence $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 3, 4, 4) = \frac{-2\pi}{3}$. Add $c(\Delta)$ to $c(\hat{\Delta})$ as shown. Now let $l(y_5) = l(\hat{x}_i) = 44\bar{1}\tilde{3}^\pm$. If $\hat{x}_i \neq \hat{x}_2$ then this forces at least one split of degree ≥ 4 , see Figure 4.32(ii), and $c(\hat{\Delta}) \leq c(3, 3, 3, 4, 4, 4) = \frac{-\pi}{2}$. Add $c(\Delta)$ to $c(\hat{\Delta})$ as shown. Now let $\hat{x}_i = \hat{x}_2$. Note that $l(y_5)$ forces at least 2 splits on $\hat{\Delta}$. Moreover, $l(y_5) = 44\bar{1}\tilde{3}^\pm$ forces $l(y_3) = \tilde{3}1$ which also forces 2 splits on $\hat{\Delta}$, see Figure 4.32(iii), and since either $d(\hat{y}_3) \geq 3$ or $l(\hat{y}_3) = \tilde{3}k$. Then $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 3, 3, 4) = \frac{-\pi}{2}$. Add $c(\Delta)$ to $c(\hat{\Delta})$, see Figure 4.32(iii)-(iv). Now let $l(y_5) = l(\hat{x}_i) = 44\bar{1}k$. Then $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 4, 4) = \frac{-\pi}{3}$. Add $c(\Delta)$ to $c(\hat{\Delta})$, see Figure 4.32(v).

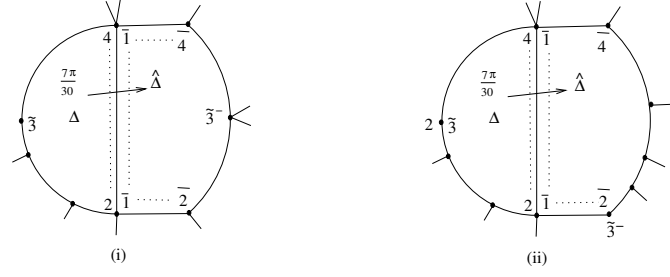
Next let $d(y_5) \geq 5$. If in addition $d(y_1) = d(\hat{x}_j) = 4$ then $c(\Delta) \leq c(3, 3, 4, 5) = \frac{7\pi}{30}$ and each of the possible labellings of y_1 forces at least $d(\hat{x}_{j+1}) \geq 3$ or $d(\hat{x}_{j+2}) \geq 3$ or $l(\hat{x}_{j+2}) = \bar{(24)}\tilde{3}^\pm$ and $d(\hat{x}_{j+3}) \geq 3$ except for $l(y_1) \in \{\bar{1}2\bar{2}\{\bar{2}, \bar{4}\}, \bar{1}2\tilde{3}^\pm\{\bar{2}, \bar{4}\}\}$ with $y_1 = \hat{x}_j = \hat{x}_{4\hat{m}-2}$ or $l(y_1) = \bar{1}2\tilde{3}^\pm(24)$ which in turn forces $l(y_3) = \tilde{3}2$. In the former case if y_1 forces a split then $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 4, 5) = \frac{-13\pi}{30}$. Add $c(\Delta)$ to


 Figure 4.32: $m \geq 2$

$c(\hat{\Delta})$ in Figure 4.13(i). Now assume $l(y_1) = l(\hat{x}_{4m-2}) \in \{\bar{1}22\{2, \bar{4}\}, \bar{1}2\bar{3}^\pm\{2, \bar{4}\}\}$. First let $l(y_3) \in \bar{3}\{1, (24)\}$, see Figure 4.33(i). Then $d(\hat{y}_1) \geq 3$ otherwise it is $2\bar{3}^\pm$ a contradiction. Moreover $d(\hat{y}_3) = 2$ forces both of $d(\hat{y}_4)$ and $d(\hat{y}_2)$ to be ≥ 3 and $d(\hat{y}_3) = 3$ forces at least one of $d(\hat{y}_4)$ and $d(\hat{y}_2)$ to be ≥ 3 . Thus $c(\hat{\Delta}) \leq c(3, 3, 4, 4, 5) = \frac{-8\pi}{30}$. Add $c(\Delta)$ to $c(\hat{\Delta})$, see Figure 4.33(i). Next let $l(y_3) = \bar{3}2$, see Figure 4.33(ii). If $d(\hat{y}_2) = 2$ then this implies $l(\hat{y}_2) = \bar{2}\bar{3}^-$ which forces at least 2 splits on $\hat{\Delta}$. Moreover at least one of $d(\hat{y}_3)$ and $d(\hat{y}_4)$ is ≥ 3 . Thus $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 4, 5) = \frac{-13\pi}{30}$. Add $c(\Delta)$ to $c(\hat{\Delta})$, see Figure 4.33(ii). If $d(\hat{y}_2) \geq 3$ then the same argument for Figure 4.33(i) applies here and again add $c(\Delta)$ to $c(\hat{\Delta})$. In the latter case, that is $l(y_1) = \bar{1}2\bar{3}^\pm(24)$, then $d(\hat{y}_1) \geq 3$. If $l(y_5)$ forces w-split or $d(\hat{y}_4) \geq 3$ or $d(\hat{y}_2) \geq 3$ then $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 4, 5) = \frac{-13\pi}{30}$. Add $c(\Delta)$ to $c(\hat{\Delta})$. If none of the above holds then $d(\hat{y}_3) \geq 4$ and $c(\hat{\Delta}) \leq c(3, 3, 4, 4, 5) = \frac{-8\pi}{30}$. Add $c(\Delta)$ to $c(\hat{\Delta})$. Now let $d(y_1) = d(\hat{x}_j) \geq 5$ then $c(\Delta) \leq c(3, 3, 5, 5) = \frac{4\pi}{30}$ and $c(\hat{\Delta}) \leq c(3, 3, 3, 5, 5) = \frac{-6\pi}{30}$. Add $c(\Delta)$ to $c(\hat{\Delta})$ in Figure 4.13(i).

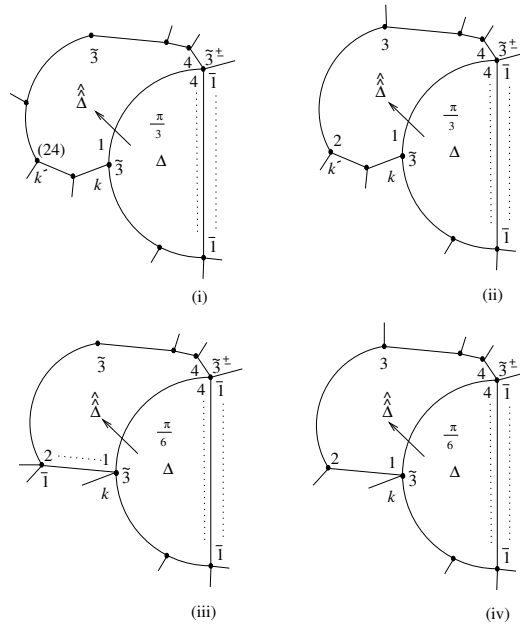
Δ is given by Figure 4.13(ii):

When $d(y_5) = 4$ then the labelling possibilities are as in Figure 4.13(i) and similarly we add $c(\Delta) \leq c(3, 3, 4, 4) = \frac{\pi}{3}$ to $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 3, 4, 4) = \frac{-2\pi}{3}$ if the

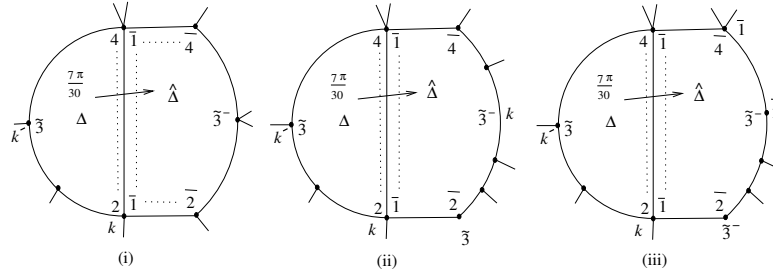

 Figure 4.33: $m \geq 2$

labelling forces at least 2 splits on $\hat{\Delta}$ in Figure 4.13(ii). Also when $l(y_5) = 44\bar{1}k$ then $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 4, 4) = \frac{-\pi}{3}$ and we add $c(\Delta)$ to $c(\hat{\Delta})$ in Figure 4.13(ii). If $l(y_5) = l(\hat{x}_i) = 44\bar{1}3^\pm$ then there are 2 cases: $d(y_3) = 3$ and $d(y_3) \geq 4$. If $d(y_3) = 3$ then $l(y_3) = \bar{3}1k$ which forces at least one split on $\hat{\Delta}$, see Figure 4.34(i). Moreover, if $\hat{m} \geq 2$ then $l(y_3)$ forces $d(\hat{x}_4) \geq 3$ and at least one of $d(\hat{y}_1)$, $d(\hat{y}_2)$ and $d(\hat{y}_3)$ is ≥ 3 . While $\hat{m} = 1$ forces $d(\hat{y}_1)$ and $d(\hat{y}_3)$ to be ≥ 3 . Thus add $c(\Delta)$ to $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 3, 3, 4) = \frac{-15\pi}{30}$, see Figure 4.34(i)-(ii). If $d(y_3) \geq 4$ then $c(\Delta) \leq c(3, 4, 4, 4) = \frac{5\pi}{30}$, see Figure 4.34(iii). In this case letting $\hat{m} \geq 2$ forces either $d(\hat{x}_i) \geq 3$ for $i \in \{4\hat{m} - 1, 4\hat{m} - 2\}$; or $d(\hat{y}_1) \geq 4$ and add $c(\Delta)$ to $c(\hat{\Delta}) \leq c(3, 3, 4, 4, 4) = \frac{-5\pi}{30}$, see Figure 4.34(iii). Whereas $\hat{m} = 1$ forces $d(\hat{y}_3) \geq 3$ and add $c(\Delta)$ to $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 4, 4) = \frac{-10\pi}{30}$, see Figure 4.34(iv).

Now let $d(y_5) \geq 5$. If $d(y_1) = 4$ then checking all possible labellings shows that $l(y_1)$ either forces at least one of $d(\hat{x}_{j+1})$ or $d(\hat{x}_{j+2})$ to be ≥ 3 or both $= 2$ and hence $l(\hat{x}_{j+2}) = \bar{(24)}\bar{3}^\pm$ and $d(\hat{x}_{j+3}) \geq 3$; or $l(y_1) \in \bar{1}2k\{2, \bar{4}\}$ with $y_1 = \hat{x}_j = \hat{x}_{4\hat{m}-2}$. In the former case $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 4, 5) = \frac{-13\pi}{30}$. Add $c(\Delta)$ to $c(\hat{\Delta})$ in Figure 4.13(ii). In the latter case first let $d(\hat{y}_1) \geq 3$. Then as in Figure 4.35(i) $c(\hat{\Delta}) \leq c(3, 3, 4, 4, 5) = \frac{-8\pi}{30}$. Add $c(\Delta)$ to $c(\hat{\Delta})$. Now let $d(\hat{y}_1) = 2$. Then $l(\hat{y}_1) = \bar{3}^\pm 2$ which forces at least 2 splits on $\hat{\Delta}$, see Figures 4.35(ii)-(iii). If in addition $d(\hat{y}_3) \geq 3$ then add $c(\Delta)$ to $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 4, 5) = \frac{-13\pi}{30}$ in Figure 4.13(ii). Now let $d(\hat{y}_3) = 2$. Then there are 2 cases: either $l(\hat{y}_3) = \bar{3}^-k$ or $l(\hat{y}_3) = l(\hat{y}_1) = \bar{3}^-2$. If $l(\hat{y}_3) = \bar{3}^-k$ then this forces at least 1 more split on $\hat{\Delta}$ and $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 4, 5) = \frac{-13\pi}{30}$. Add $c(\Delta)$ to $c(\hat{\Delta})$, see Figure 4.35(ii). Now let $l(\hat{y}_3) = l(\hat{y}_1) = \bar{2}3^-$ then this forces $l(y_1) = \bar{1}2k\bar{4}$. If $d(\hat{y}_5) \geq 4$ then $c(\hat{\Delta}) \leq c(3, 3, 4, 4, 5) = \frac{-8\pi}{30}$. Add $c(\Delta)$ to $c(\hat{\Delta})$, see Figure 4.35(iii). If $d(\hat{y}_5) = 3$ then either it forces at least one split on the edge $(\hat{y}_5, \hat{x}_i)$ or $l(\hat{y}_5) = \bar{4}\bar{4}\bar{1}$ which


 Figure 4.34: $m \geq 2$

however yields a contradiction with $l(y_1)$. Thus $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 4, 5) = \frac{-13\pi}{30}$. Add $c(\Delta)$ to $c(\hat{\Delta})$ in Figure 4.13(ii). If $d(y_1) \geq 5$ then $c(\Delta) \leq c(3, 3, 5, 5) = \frac{4\pi}{30}$ and $c(\hat{\Delta}) \leq c(3, 3, 3, 5, 5) = \frac{-6\pi}{30}$. Add $c(\Delta)$ to $c(\hat{\Delta})$ in Figure 4.13(ii).


 Figure 4.35: $m \geq 2$

Δ is given by Figure 4.13(iii):

Checking the possible labellings shows that as in Figure 4.11(ii) $c(\Delta) \leq \frac{\pi}{6}$, $l(x_2) = 1\bar{4}\tilde{3}$, $l(y_3) = \tilde{3}2$ and $l(y_5) = 441$. Add $c(\Delta)$ to $c(\tilde{\Delta}) \leq c(3, 3, 3, 3, 3, 3) = \frac{-\pi}{3}$ in Figure 4.13(iii).

Δ is given by Figure 4.13(iv):

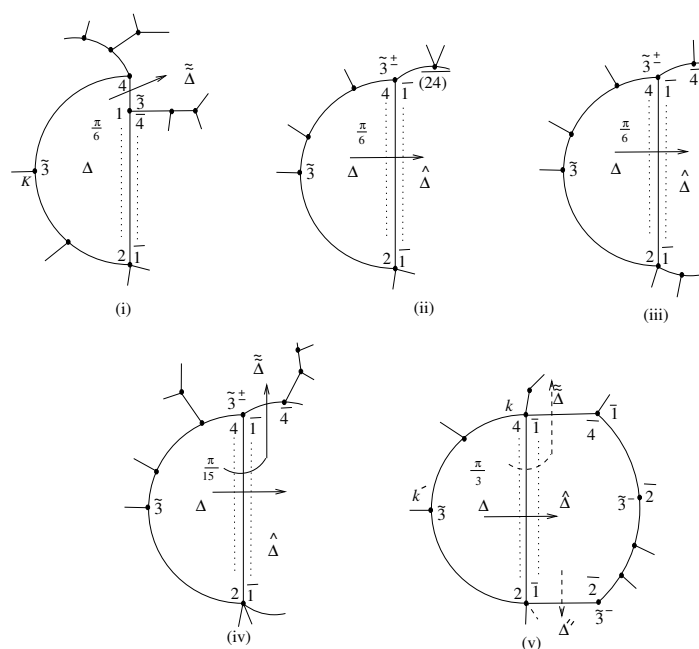
In Figure 4.13(iv) $l(y_5) \in \{441, 44(24), 4\bar{2}4\}$ and these labelling forces at least 2 splits on $\tilde{\Delta}$, see Figure 4.36(i). Moreover $l(x_2)$ forces more 2 splits on $\tilde{\Delta}$. Add $c(\Delta) \leq c(3, 3, 3, 3, 4) = \frac{5\pi}{30}$ to $c(\tilde{\Delta}) \leq c(3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$ as shown.

Δ is given by Figure 4.14(i):

Assume first that $y_5 = \hat{x}_i \neq \hat{x}_2$. Then $l(y_5)$ forces at least $d(\hat{x}_{i-1}) \geq 4$ and add $c(\Delta) \leq c(3, 3, 3, 3, 4) = \frac{5\pi}{30}$ to $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 4, 4) = \frac{-10\pi}{30}$, see Figure 4.36(ii). Assume now that $y_5 = \hat{x}_i = \hat{x}_2$. If $d(y_1) = 4$ then $l(y_1) \in \{\bar{1}2\bar{1}\{\bar{1}, \tilde{3}^\pm, \bar{4}, \overline{(24)}, k\}, \bar{1}24\{\bar{1}, 2, \tilde{3}^\pm, \overline{(24)}, k\}, \bar{1}2(\overline{24})\{\bar{1}, \tilde{3}^\pm, \overline{(24)}, \bar{4}\}\}$. If $l(y_1) \in \{\bar{1}2\bar{1}\{\bar{1}, \tilde{3}^\pm, \overline{(24)}, k\}, \bar{1}24\{\bar{1}, \tilde{3}^\pm, \overline{(24)}, k\}, \bar{1}2(\overline{24})\{\bar{1}, \tilde{3}^\pm, \overline{(24)}, k\}\}$ then this forces at least 1 split on $\hat{\Delta}$ and $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 3, 4) = \frac{-\pi}{6}$. Add $c(\Delta)$ to $c(\hat{\Delta})$, see Figure 4.36(iii). If $l(y_1) = l(\hat{x}_j) \in \{\bar{1}2\bar{1}\bar{4}, \bar{1}24\bar{2}, \bar{1}2(\overline{24})\bar{4}\}$ and $\hat{x}_j \neq \hat{x}_{4\hat{m}-2}$ then this forces at least one split of degree ≥ 4 and $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 3, 4) = \frac{-\pi}{6}$ and again add $c(\Delta)$ to $c(\hat{\Delta})$ in Figure 4.14(i). If $\hat{x}_j = \hat{x}_{4\hat{m}-2}$ then we use the same argument used in Figures 4.24(ii)-(vii) for Figure 4.11(iii) and add $c(\Delta)$ to $c(\Delta'') \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$ in Figure 4.14(i). Note that $c(\hat{\Delta}) \leq c(3, 3, 3, 4, 4) = 0$ since $d(\hat{\Delta}) \geq 5$ and letting $d(\hat{\Delta}) = 5$ forces $d(\hat{y}_3) \geq 4$. Now let $d(y_1) = 5$ then $c(\Delta) \leq \frac{2\pi}{30}$. If in addition $d(\hat{y}_5) \geq 4$ then $c(\hat{\Delta}) \leq c(3, 3, 3, 4, 5) = \frac{-3\pi}{30}$. Add $c(\Delta)$ to $c(\hat{\Delta})$ in Figure 4.14(i). If $d(\hat{y}_5) = 3$ and $d(\hat{y}_4) \geq 3$ then add $c(\Delta)$ to $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$ in Figure 4.14(i). If $d(\hat{y}_5) = 3$ and $d(\hat{y}_4) = 2$ then $l(\hat{y}_5) \in \{\bar{4}\bar{4}\bar{1}, 24\bar{1}\}$ which forces at least 2 splits on $\tilde{\Delta}$, see Figure 4.36(iv). Also, $l(y_5)$ forces 2 more splits on $\tilde{\Delta}$ and hence $c(\tilde{\Delta}) \leq c(3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$. The maximum sent to $\tilde{\Delta}$ is $c(\Delta) + c(\hat{\Delta}) \leq \frac{2\pi}{30} + \frac{2\pi}{30} = \frac{4\pi}{30}$, see Figure 4.36(iv).

Δ is given by Figure 4.14(ii):

First assume $d(y_1) = 4$. Then $l(y_1) = l(\hat{x}_j) \in \bar{1}2\bar{1}\{\bar{1}, \tilde{3}^\pm, \bar{4}, \overline{(24)}, k\}$. If $l(y_1) \in \bar{1}2\bar{1}\{\bar{1}, \tilde{3}^\pm, \overline{(24)}\}$ or $l(y_1) = \bar{1}2\bar{1}k$ with $\hat{x}_j \neq \hat{x}_{4\hat{m}-2}$ then this labelling forces at least 2 splits on $\hat{\Delta}$ and $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 3, 4) = \frac{-15\pi}{30}$, see Figure 4.37(i). Add $c(\Delta) \leq c(3, 3, 3, 4) = \frac{\pi}{2}$ to $c(\hat{\Delta})$ as shown. If $l(y_1) = \bar{1}2\bar{1}k$ with $\hat{x}_j = \hat{x}_{4\hat{m}-2}$ then either this forces 2 splits on $\hat{\Delta}$ and add $c(\Delta)$ to $c(\hat{\Delta})$ as above or this forces exactly one split and this forces $d(\hat{y}_1) \geq 3$, see Figure 4.37(ii)-(iii). If in addition $y_5 = \hat{x}_i \neq \hat{x}_2$ then this forces at least

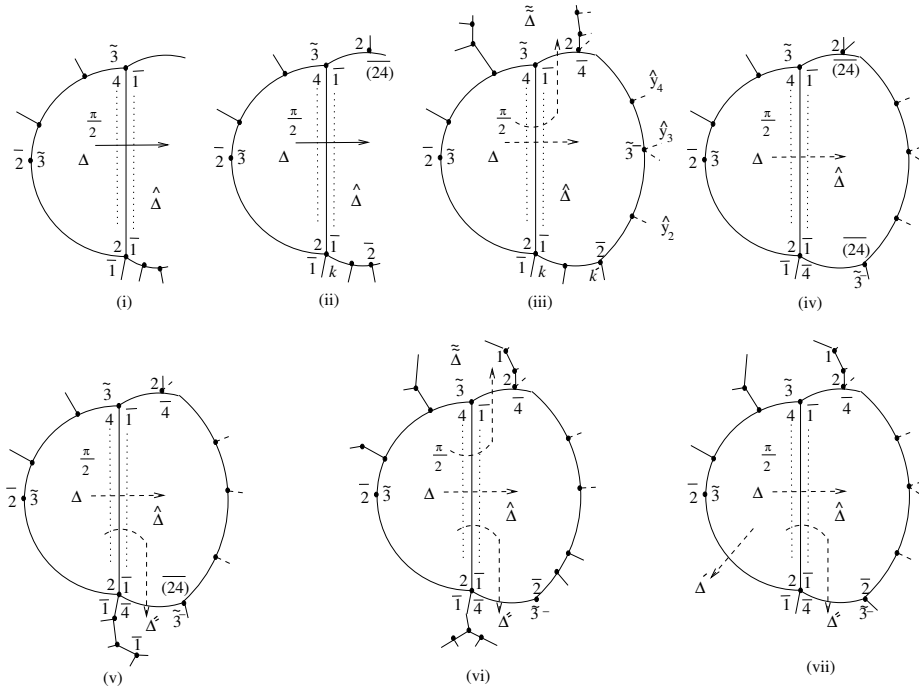
Figure 4.36: $m \geq 2$

one split on $\widehat{\Delta}$ and we add $c(\Delta)$ to $c(\widehat{\Delta}) \leq c(3, 3, 3, 3, 3, 3, 4) = \frac{-15\pi}{30}$, see Figure 4.37(ii). While if $y_5 = \widehat{x}_i = \widehat{x}_2$ then we have one of the following three cases for Figure 4.37(iii): either $d(\widehat{y}_2) = d(\widehat{y}_4) = 2$ and this forces $d(\widehat{y}_3) \geq 4$; or $d(\widehat{y}_3) = 3$ which forces at least one of $d(\widehat{y}_2)$ or $d(\widehat{y}_4)$ to be ≥ 3 ; or $d(\widehat{y}_3) = 2$ which forces $l(\widehat{y}_3) = \widetilde{3}^-k$, otherwise there is a contradiction with $l(y_3) = \widetilde{3}\overline{2}$, and this forces both of $d(\widehat{y}_2)$ and $d(\widehat{y}_4)$ to be ≥ 3 . In the first case if $d(\widehat{y}_5) \geq 4$ then $c(\widehat{\Delta}) \leq c(3, 3, 3, 4, 4, 4) = \frac{-15\pi}{30}$ and add $c(\Delta)$ to $c(\widehat{\Delta})$ in Figure 4.37(iii) whereas if $d(\widehat{y}_5) = 3$ then this forces at least 2 splits on $\widetilde{\widetilde{\Delta}}$ plus 2 other splits forced by $l(y_5)$ on $\widetilde{\widetilde{\Delta}}$ and so $c(\widetilde{\widetilde{\Delta}}) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$ and we add $\frac{\pi}{3}$ of $c(\Delta)$ to $c(\widetilde{\widetilde{\Delta}}) \leq c(3, 3, 3, 3, 4, 4) = \frac{-10\pi}{30}$ and we add $\frac{\pi}{6}$ to $c(\widetilde{\widetilde{\Delta}})$ in Figure 4.37(iii). In the second and third cases we add $c(\Delta)$ to $c(\widehat{\Delta}) \leq c(3, 3, 3, 3, 3, 3, 4) = \frac{-15\pi}{30}$ in Figure 4.37(iii).

Now let $l(y_1) = \overline{121}4$. Then there are 2 cases: $y_1 = \hat{x}_j \neq \hat{x}_{4\hat{m}-2}$, see Figures 4.37(iv)-(v), and $y_1 = \hat{x}_j = \hat{x}_{4\hat{m}-2}$, see Figures 4.37(vi)-(vii). If $y_1 = \hat{x}_j \neq \hat{x}_{4\hat{m}-2}$ then this forces at least one split on $\hat{\Delta}$ and either $y_5 = \hat{x}_i \neq \hat{x}_2$ and we add $c(\Delta)$ to $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 3, 3, 4) = \frac{-15\pi}{30}$, see Figure 4.37(iv); or $y_5 = \hat{x}_i = \hat{x}_2$ and then either $d(\hat{y}_4) \geq 3$ or $d(\hat{y}_5) \geq 4$ otherwise there is a labelling contradiction and we add $\frac{\pi}{3}$ of $c(\Delta)$ to $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 4, 4) = \frac{-10\pi}{30}$ and we add the remaining $\frac{\pi}{6}$ to

Δ'' as $l(y_1)$ forces at least 3 splits on Δ'' and so $c(\Delta'') \leq c(3, 3, 3, 3, 3, 4) = \frac{-\pi}{6}$, see Figure 4.37(v). If $y_1 = \hat{x}_j = \hat{x}_{4\hat{m}-2}$ then whether $d(\hat{y}_1) = 2$ or $d(\hat{y}_1) \geq 3$ we always have $c(\Delta'') \leq \frac{-\pi}{6}$ by the same argument in Figures 4.25(ii)-(iii) and we add $\frac{\pi}{6}$ of $c(\Delta)$ to $c(\Delta'')$ in Figure 4.37(v)-(vii). This leaves $\frac{\pi}{3}$ of $c(\Delta)$ to deal with. If $y_5 = \hat{x}_i \neq \hat{x}_2$ then add the remaining $\frac{\pi}{3}$ of $c(\Delta)$ to $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 4, 4) = \frac{-\pi}{3}$ so assume $y_5 = \hat{x}_i = \hat{x}_2$. If $d(\hat{y}_1) = 2$ then this forces at least 2 splits on $\hat{\Delta}$ and at least $d(\hat{y}_3) \geq 3$ or $d(\hat{y}_4) \geq 3$. If both $d(\hat{y}_3) \geq 3$ and $d(\hat{y}_4) \geq 3$, or if $d(\hat{y}_3) = 2$ and there are 2 splits on the edge $(\hat{y}_3, \hat{y}_5)$ then add the remainder $\frac{\pi}{3}$ of $c(\Delta)$ to $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 3, 3, 4) = \frac{-\pi}{2}$ in Figure 4.37(vi). If $d(\hat{y}_3) = 2$ and there is exactly one split on the edge $(\hat{y}_3, \hat{y}_5)$ then add the remaining $\frac{\pi}{3}$ of $c(\Delta)$ to $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 4, 4) = \frac{-\pi}{3}$ when $d(\hat{y}_5) \geq 4$; or add the remaining $\frac{\pi}{3}$ of $c(\Delta)$ to $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 3, 4) = \frac{-\pi}{6}$ and $c(\tilde{\Delta}) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-\pi}{3}$ when $d(\hat{y}_5) = 3$ since this forces $l(\hat{y}_5) = 2\bar{4}k$ which in turn forces at least 2 splits on $\tilde{\Delta}$. If $d(\hat{y}_3) \geq 3$ and $d(\hat{y}_4) = 2$ then checking the labelling shows that this forces $d(\hat{y}_3) \geq 4$ and $d(\hat{y}_5) \geq 4$ and we add the remaining $\frac{\pi}{3}$ of $c(\Delta)$ to $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 4, 4) = \frac{-\pi}{3}$ in Figure 4.37(vi). Now suppose $d(\hat{y}_1) \geq 3$. Then $d(\hat{\Delta}) \geq 5$. If $d(\hat{\Delta}) = 5$ then we have $d(\hat{y}_5) \geq 4$, $d(\hat{y}_2) = d(\hat{y}_4) = 2$, $d(\hat{y}_3) \geq 4$, $d(\hat{y}_1) \geq 4$ and so $c(\hat{\Delta}) \leq c(3, 4, 4, 4, 4) = \frac{-\pi}{3}$; and if $d(\hat{\Delta}) \geq 6$ then $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 3, 4) = \frac{-\pi}{6}$. We add $\frac{\pi}{6}$ of $c(\Delta)$ to $c(\hat{\Delta})$ in Figure 4.37(vii). This leaves $\frac{\pi}{6}$ of $c(\Delta)$ to deal with. But $l(y_1)$ and $l(y_3)$ force at least 2 splits each on Δ' and either $d(y'_3) \geq 3$ or $l(y'_3) = 2\bar{3}^-$ which forces $m' \geq 2$ and more 2 splits on Δ' . Thus $c(\Delta') \leq c(3, 3, 3, 3, 3, 3, 4) = \frac{-\pi}{2}$ and we add the remaining $\frac{\pi}{6}$ of $c(\Delta)$ to $c(\Delta')$ in Figure 4.37(vii).

Now assume $d(y_1) = d(\hat{x}_j) = 5$. Then $c(\Delta) \leq c(3, 3, 3, 5) = \frac{12\pi}{30}$. First let $d(\hat{x}_{j+1}) \geq 3$. Then $d(\hat{\Delta}) \geq 6$. If in addition $y_5 = \hat{x}_i \neq \hat{x}_2$ then add $c(\Delta)$ to $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 3, 3, 5) = \frac{-18\pi}{30}$, see Figure 4.38(i). If $y_5 = \hat{x}_i = \hat{x}_2$ then we have either at least vertices of degree ≥ 3 on the segment $(\hat{y}_1, \hat{y}_5)$ and we add $c(\Delta)$ to $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 3, 3, 5) = \frac{-18\pi}{30}$ in Figure 4.38(ii); or there are exactly three vertices of degree ≥ 3 on the segment $(\hat{y}_1, \hat{y}_5)$ and this forces $d(\hat{y}_2) = d(\hat{y}_4) = 2$, $d(\hat{y}_1) \geq 3$ and $d(\hat{y}_3) \geq 4$. Then we add $c(\Delta)$ to $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 4, 5) = \frac{-13\pi}{30}$, see Figure 4.38(ii). Now let $d(\hat{x}_{j+1}) = 2$. If $\hat{x}_{j+2} \neq \hat{y}_1$ then either $d(\hat{x}_{j+2}) \geq 3$ or $l(y_1) = (24)\bar{1}2\bar{1}4$, see Figures 4.38(iii)-(iv). In the former case as above add $c(\Delta)$ to $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 4, 5) = \frac{-13\pi}{30}$, see Figure 4.38(iii). In the latter case $l(y_1)$ forces at least 3 splits on Δ'' and $c(\Delta'') \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-\pi}{3}$ and we add $\frac{\pi}{6}$ of

Figure 4.37: $m \geq 2$

$c(\Delta)$ to $c(\Delta'')$ in Figure 4.38(iv). This leaves $\frac{7\pi}{30}$ of $c(\Delta)$ to deal with. But either there are at least 4 vertices of degree ≥ 3 on the segment $(\hat{y}_1, \hat{y}_5)$ and we add the remaining curvature $\leq \frac{7\pi}{30}$ to $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 5) = \frac{-8\pi}{30}$ in Figure 4.38(iv); or there are exactly 3 vertices of degree ≥ 3 on the segment $(\hat{y}_1, \hat{y}_5)$ and so $d(\hat{y}_3) \geq 4$ and we add the remaining curvature $\leq \frac{7\pi}{30}$ to $c(\hat{\Delta}) \leq c(3, 3, 4, 4, 5) = \frac{-8\pi}{30}$ when $y_5 = \hat{x}_i \neq \hat{x}_2$ or when $d(\hat{y}_5) \geq 4$, whereas we add $\frac{3\pi}{30}$ of the remaining curvature to $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 4, 5) = \frac{-3\pi}{30}$ and $\frac{4\pi}{30}$ to $c(\tilde{\Delta}) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-\pi}{3}$ when $y_5 = \hat{x}_i = \hat{x}_2$ with $d(\hat{y}_5) = 3$, see Figure 4.38(iv). If $\hat{x}_{j+2} = \hat{y}_1$ then there are 2 cases regarding $d(\hat{y}_1)$. First if $d(\hat{y}_1) = 2$ then $l(\hat{y}_1) = \bar{2}\bar{3}$ which forces 2 splits on $\hat{\Delta}$ and we have at least one of $d(\hat{y}_3) \geq 3$ or $d(\hat{y}_4) \geq 3$ and we add $c(\Delta) \leq \frac{12\pi}{30}$ to $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 4, 5) = \frac{-13\pi}{30}$ when $y_5 = \hat{x}_i \neq \hat{x}_2$ or when $d(\hat{y}_5) \geq 4$, whereas we add $\frac{7\pi}{30}$ of $c(\Delta)$ to $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$ and $\frac{5\pi}{30}$ to $c(\tilde{\Delta}) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-\pi}{3}$ when $y_5 = \hat{x}_i = \hat{x}_2$ with $d(\hat{y}_5) = 3$, see Figure 4.38(v). Now let $d(\hat{y}_1) \geq 3$. Then either there are at least 5 vertices of degree ≥ 3 on the segment $(\hat{y}_1, \hat{y}_5)$ and we add $c(\Delta)$ to $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 3, 3, 5) = \frac{-18\pi}{30}$; or there are exactly 4 vertices of degree ≥ 3 on the segment $(\hat{y}_1, \hat{y}_5)$ and we add $c(\Delta)$ to $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 4, 5) = \frac{-13\pi}{30}$ or $\frac{7\pi}{30}$ to $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$ and

$\frac{5\pi}{30}$ to $c(\tilde{\Delta}) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$ as above; or there are exactly 3 vertices of degree ≥ 3 on the segment $(\hat{y}_1, \hat{y}_5)$ and so $d(\hat{y}_3) \geq 4$. Checking the possible labelling shows that in the last case $l(y_1) \in \{2\bar{1}2\bar{1}\omega, 4\bar{1}2\bar{1}\omega, (24)\bar{1}2\bar{1}\omega\}$, $d(\hat{y}_1) \geq 4$ and we add $c(\Delta)$ to $c(\hat{\Delta}) \leq c(3, 4, 4, 4, 5) = \frac{-13\pi}{30}$ when $y_5 = \hat{x}_i \neq \hat{x}_2$ or when $d(\hat{y}_5) \geq 4$, whereas we add $\frac{7\pi}{30}$ of $c(\Delta)$ to $c(\hat{\Delta}) \leq c(3, 3, 4, 4, 5) = \frac{-8\pi}{30}$ and $\frac{5\pi}{30}$ to $c(\tilde{\Delta}) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-\pi}{3}$ when $y_5 = \hat{x}_i = \hat{x}_2$ with $d(\hat{y}_5) = 3$, see Figure 4.38(vi). Note that Δ'' in Figures 4.38(v)-(vi) never sends positive curvature to $\hat{\Delta}$.

Now assume $d(y_1) = d(\hat{x}_j) \geq 6$. Then $c(\Delta) \leq c(3, 3, 3, 6) = \frac{10\pi}{30}$. Similarly to the

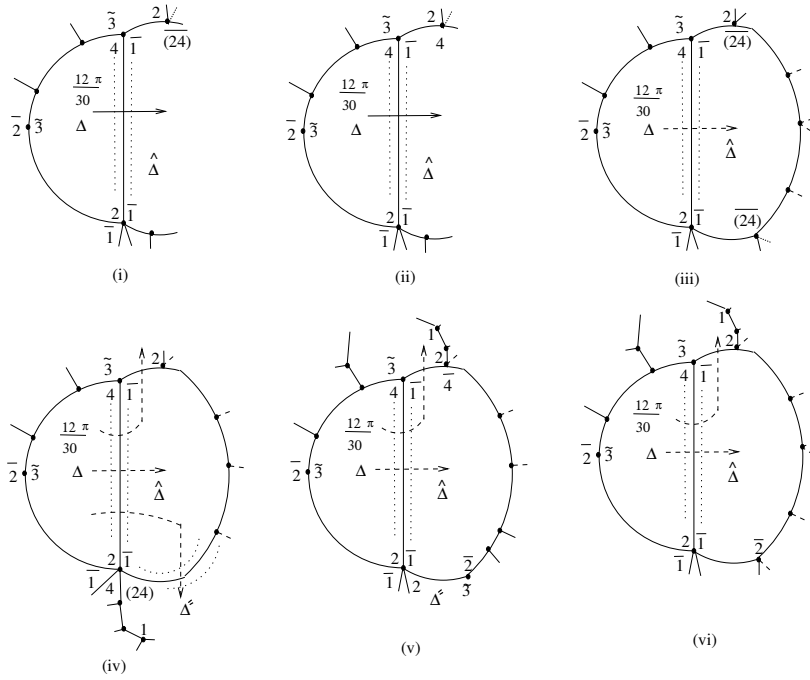


Figure 4.38: $m \geq 2$

above discussion we add $c(\Delta)$ to $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 3, 6) = \frac{-\pi}{3}$ when $y_5 = \hat{x}_i \neq \hat{x}_2$, or $d(\hat{x}_{j+1}) \geq 3$, or $3 \leq d(\hat{x}_{j+2}) \neq d(\hat{y}_1)$, or when there are at least 4 vertices of degree ≥ 3 on the segment $(\hat{y}_1, \hat{y}_5)$. If none of the above holds then $d(\hat{y}_3) \geq 4$ and we add $c(\Delta)$ to $c(\hat{\Delta}) \leq c(3, 3, 4, 4, 6) = \frac{-10\pi}{30}$ when $d(\hat{y}_5) \geq 4$. While we add $\frac{\pi}{6}$ of $c(\Delta)$ to $c(\hat{\Delta}) \leq c(3, 3, 3, 4, 6) = \frac{-5\pi}{30}$ and $\frac{\pi}{6}$ to $c(\tilde{\Delta}) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-\pi}{3}$ when $d(\hat{y}_5) = 3$.

Δ is given by Figure 4.14(iii):

Note that $l(y_3) \in \widetilde{3}\{\overline{1}, \overline{2}, \overline{(24)}\}$ and $\widehat{m} \geq m + 1$. If $d(\widehat{y}_3) = 2$ then $l(\widehat{y}_3) = \widetilde{3}^-k$ otherwise there is a contradiction. First let $d(y_1) = 4$ then $l(y_1) \in \{\overline{12}\overline{1}\{\overline{1}, \widetilde{3}^\pm, \overline{4}, \overline{(24)}\}, k\}, \overline{124}\{\overline{1}, \overline{2}, \widetilde{3}^\pm, \overline{(24)}\}, k\}, \overline{12(24)}\{\overline{1}, \widetilde{3}^\pm, \overline{(24)}, \overline{4}\}\}$. If $l(y_1) \in \{\overline{12}\overline{1}\{\overline{1}, \widetilde{3}^\pm, \overline{(24)}\}, k\}, \overline{124}\{\overline{1}, \widetilde{3}^\pm, \overline{(24)}\}, k\}, \overline{12(24)}\{\overline{1}, \widetilde{3}^\pm, \overline{(24)}\}\}$ then this forces at least one split and $c(\widehat{\Delta}) \leq c(3, 3, 3, 3, 4, 4) = \frac{-10\pi}{30}$, in fact $\widehat{\Delta}$ is given in Figure 4.23(v). Add $c(\Delta) \leq c(3, 3, 4, 4) = \frac{10\pi}{30}$ to $c(\widehat{\Delta})$ in Figure 4.14(iii) (with the understanding that $\widehat{\Delta}$ is as in Figure 4.23(v)). Next let $l(y_1) = l(\widehat{x}_j) \in \{\overline{12}\overline{1}\overline{4}, \overline{1242}, \overline{12(24)}\overline{4}\}$. If $\widehat{x}_j \neq \widehat{x}_{4\widehat{m}-2}$ then this also forces at least one split and add $c(\Delta)$ to $c(\widehat{\Delta}) \leq c(3, 3, 3, 3, 4, 4) = \frac{-10\pi}{30}$, in fact $\widehat{\Delta}$ is given in Figure 4.23(vi). Now let $\widehat{x}_j = \widehat{x}_{4\widehat{m}-2}$. If in addition $d(y_5) = 4$ then $l(y_5) = l(\widehat{x}_i) \in \{\overline{1}, \overline{2}, \overline{(24)}, \widetilde{3}^\pm, k\}4\overline{1}\omega$ which forces at least one more split on $\widehat{\Delta}$ except for $l(y_5) = l(\widehat{x}_i) \in \{\overline{1}, \overline{2}, \overline{(24)}, k\}4\overline{1}\widetilde{3}^\pm$ with $\widehat{x}_i = \widehat{x}_2$. In the former case add $c(\Delta)$ to $c(\widehat{\Delta}) \leq c(3, 3, 3, 3, 4, 4) = \frac{-10\pi}{30}$, $\widehat{\Delta}$ is given in Figures 4.23(ii)-(iv). In the latter case if $d(\widehat{y}_5) \geq 4$, $\widehat{\Delta}$ is given in Figure 4.25(v), or if $d(\widehat{y}_5) = 3$ with $d(\widehat{y}_4) \geq 3$, $\widehat{\Delta}$ is given in Figure 4.25(vi), then $c(\widehat{\Delta}) \leq c(3, 3, 4, 4, 4) = \frac{-5\pi}{30}$. Moreover $c(\Delta'') \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$, Δ'' is given in Figure 4.25(ii)-(iii). Add $c(\Delta)$ to $c(\widehat{\Delta}) + c(\Delta'')$ in Figure 4.14(iii). While if $d(\widehat{y}_5) = 3$ with $d(\widehat{y}_4) = 2$ then $c(\widehat{\Delta}) \leq 0$ and $c(\widetilde{\Delta}) \leq c(3, 3, 3, 3, 4, 4) = \frac{-10\pi}{30}$. Add $c(\Delta)$ to $c(\widetilde{\Delta}) + c(\Delta'')$, $\widetilde{\Delta}$ is given in Figure 4.25(viii). Now let $d(y_5) \geq 5$. Then $c(\Delta) \leq \frac{7\pi}{30}$, $c(\widehat{\Delta}) \leq \frac{-3\pi}{30}$ and $c(\Delta'') \leq \frac{-\pi}{6}$. Add $\frac{3\pi}{30}$ of $c(\Delta)$ to $c(\widehat{\Delta})$ and $\frac{4\pi}{30}$ to $c(\Delta'')$ in Figure 4.14(iii). Finally let $d(y_1) \geq 5$. First let $d(y_5) = 4$. If $l(y_5)$ forces at least one split on $\widehat{\Delta}$; or if $d(\widehat{y}_5) \geq 4$; or if $d(\widehat{y}_5) = 3$ with $d(\widehat{y}_4) \geq 3$ then add $c(\Delta) \leq \frac{7\pi}{30}$ to $c(\widehat{\Delta}) \leq c(3, 3, 4, 4, 5) = \frac{-8\pi}{30}$ in Figure 4.14(iii) otherwise add $\frac{3\pi}{30}$ of $c(\Delta)$ to $c(\widehat{\Delta}) \leq \frac{-3\pi}{30}$ and add the remaining $\frac{4\pi}{30}c(\widetilde{\Delta}) \leq \frac{-5\pi}{30}$ in Figure 4.14(iii). Now let $d(y_5) \geq 5$. Then add $c(\Delta) \leq \frac{4\pi}{30}$ to $c(\widehat{\Delta}) \leq \frac{-6\pi}{30}$ in Figure 4.14(iii).

Δ is given by Figure 4.14(iv):

First let $d(y_1) = 4$. Then $l(y_1)$ as for Figure 4.14(iii). If $l(y_1)$ forces at least one split then add $c(\Delta) \leq c(3, 3, 4, 4) = \frac{\pi}{3}$ to $c(\widehat{\Delta}) \leq c(3, 3, 3, 3, 4, 4) = \frac{-10\pi}{30}$ in Figure 4.14(iv). Otherwise $l(y_1) = l(\widehat{x}_j) \in \{\overline{12}\overline{1}\overline{4}, \overline{1242}, \overline{12(24)}\overline{4}\}$ with $\widehat{x}_j = \widehat{x}_{4\widehat{m}-2}$. If $d(y_5) = 4$ then $l(y_5) \in k4\overline{1}\{\overline{1}, \overline{2}, \widetilde{3}^\pm, \overline{4}, \overline{(24)}\}$ each forces at least a split on $\widehat{\Delta}$ except for $l(y_5) = l(\widehat{x}_j) = k4\overline{1}\widetilde{3}^\pm$. In the former case add $c(\Delta)$ to $c(\widehat{\Delta}) \leq c(3, 3, 3, 3, 4, 4) = \frac{-\pi}{3}$ in Figure 4.14(iv). In the latter case if $d(\widehat{y}_3) = 2$ then $l(\widehat{y}_3) \in \widetilde{3}^-\{1, \overline{1}, \overline{2}, \overline{2}, k\}$. If $l(\widehat{y}_3) \in \widetilde{3}^-\{2, 1, \overline{1}, k\}$ then this forces

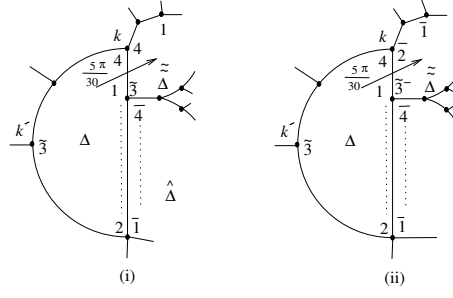
$d(\hat{\Delta}) \geq 6$ by a similar argument to the one used in Figure 4.35(ii) for Figure 4.13(iii) and add $c(\Delta) \leq c(3, 3, 4, 4) = \frac{\pi}{3}$ to $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 4, 4) = \frac{-10\pi}{30}$ in Figure 4.14(iv). If $l(\hat{y}_3) = \bar{2}\bar{3}^-$ then either $d(\hat{y}_1) \geq 3$ and again add $c(\Delta) \leq c(3, 3, 4, 4) = \frac{\pi}{3}$ to $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 4, 4) = \frac{-10\pi}{30}$; or $l(\hat{y}_3) = l(\hat{y}_1) = \bar{2}\bar{3}^-$ and $\hat{\Delta}$ is given in Figure 4.36(v). In the latter case we have $c(\Delta'') \leq \frac{-5\pi}{30}$, Δ'' is in fact given in Figure 4.25(iii). If $d(\hat{y}_3) \geq 4$ then $c(\hat{\Delta}) \leq c(3, 3, 4, 4, 4) = \frac{-\pi}{6}$ and we add $c(\Delta)$ to $c(\hat{\Delta}) + c(\Delta'')$ while if $d(\hat{y}_3) = 3$ then $l(\hat{y}_5) = \bar{4}\bar{1}\bar{4}$ and add $c(\Delta)$ to $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-\pi}{3}$ and $c(\Delta'')$. If $d(\hat{y}_3) \geq 3$ then $c(\hat{\Delta}) \leq c(3, 3, 4, 4, 4) = \frac{-\pi}{6}$ and again we use $c(\Delta'') \leq \frac{-5\pi}{30}$ of Figure 4.25(iii). Add $c(\Delta)$ to $c(\hat{\Delta})$ and $c(\Delta'')$. If $d(\hat{y}_5) \geq 5$ then $c(\Delta) \leq \frac{7\pi}{30}$ and if $d(\hat{\Delta}) \geq 6$ then add $c(\Delta)$ to $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$ while if $d(\hat{\Delta}) = 5$ then it is given by $\hat{\Delta}$ in Figures 4.33(i) or (ii). If it is given by 4.33(i) then add $c(\Delta)$ to $c(\hat{\Delta}) \leq c(3, 3, 4, 4, 5) = \frac{-\pi}{30}$ or if it is given by 4.33(ii) then add $c(\Delta)$ to $\hat{\Delta}$ or to $c(\hat{\Delta})$ and $c(\tilde{\Delta})$ according to $d(\hat{y}_5) \geq 4$ or $d(\hat{y}_5) = 3$. Now let $d(y_1) \geq 5$. Then $c(\Delta) \leq \frac{7\pi}{30}$. If $d(\hat{y}_3) = 2$ with $l(\hat{y}_3) \in \bar{3}^-\{2, 1, \bar{1}, k\}$ then as above add $c(\Delta)$ to $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 4, 4) = \frac{-10\pi}{30}$ in Figure 4.14(iv). If $l(\hat{y}_3) = \bar{2}\bar{3}^-$ then again either $d(\hat{y}_1) \geq 3$ and add $c(\Delta)$ to $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 4, 4) = \frac{-10\pi}{30}$; or $l(\hat{y}_3) = l(\hat{y}_1) = \bar{2}\bar{3}^-$ and then add $c(\Delta)$ to $c(\hat{\Delta}) \leq \frac{-8\pi}{30}$ when $d(\hat{y}_5) \geq 4$ and add $c(\Delta)$ to $c(\hat{\Delta}) \leq \frac{-3\pi}{30}$ and $c(\tilde{\Delta}) \leq \frac{-10\pi}{30}$ when $d(\hat{y}_5) = 3$, see Figure 4.36(v). If $d(\hat{y}_3) \geq 3$ then $c(\hat{\Delta}) = c(3, 3, 4, 4, 5) = \frac{-8\pi}{30}$. Add $c(\Delta)$ to $c(\hat{\Delta})$.

Δ is given by Figure 4.14(vi):

Then $c(\Delta) \leq c(3, 3, 3, 3, 4) = \frac{\pi}{6}$. If $l(x_1) = \bar{1}\bar{4}\bar{3}$ then this forces at least 2 splits on each of $\hat{\Delta}$ and $\tilde{\Delta}$, see Figure 4.39(i). Also this implies $l(y_5) = 44k$. Let $l(x_1) = \bar{1}\bar{4}\bar{3}^-$. Then this also forces at least 2 splits on each of $\hat{\Delta}$ and $\tilde{\Delta}$, see Figure 4.39(ii). Also this implies $l(y_5) = 4\bar{2}k$. In each case add $c(\Delta)$ to $c(\tilde{\Delta}) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-\pi}{3}$ as shown in Figure 4.39.

Δ is given by Figure 4.15(i):

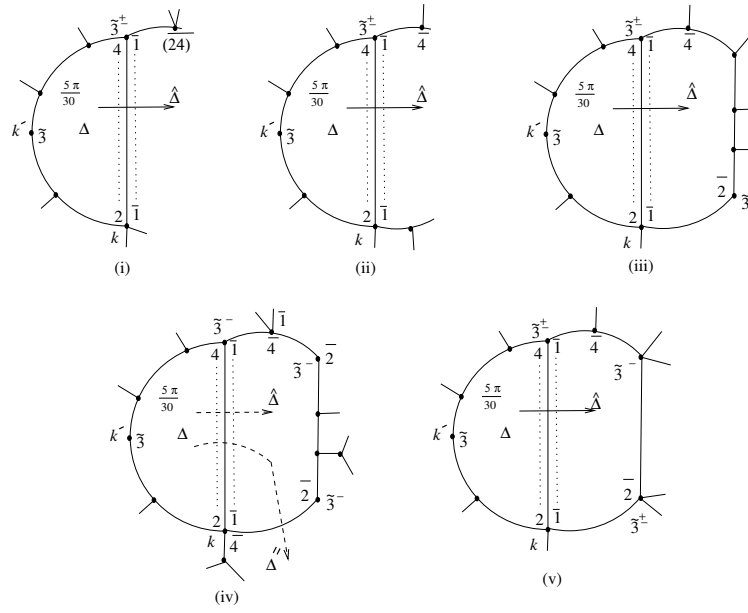
If $y_5 = \hat{x}_i \neq \hat{x}_2$ then $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$. Add $c(\Delta) \leq \frac{5\pi}{30}$ to $c(\hat{\Delta})$, see Figure 4.40(i). Now let $y_5 = \hat{x}_i = \hat{x}_2$. If $d(y_1) = 4$ then all the possible labelling forces at least one split on $\hat{\Delta}$ unless $y_1 = \hat{x}_j = \hat{x}_{4\bar{m}-2}$ and $l(y_1) \in \{\bar{1}2k\bar{4}, \bar{1}2k2\}$. In the former case $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$. Add $c(\Delta) \leq \frac{5\pi}{30}$ to $c(\hat{\Delta})$, see Figure 4.40(ii). In the latter case either $l(\hat{y}_1) = \bar{2}\bar{3}^\pm$ or $d(\hat{y}_1) \geq 3$. First let


 Figure 4.39: $m \geq 2$

$l(\hat{y}_1) = \bar{2}\bar{3}^\pm$. Then this forces at least 2 splits on $\hat{\Delta}$. If in addition $d(\hat{y}_3) \geq 3$ or $d(\hat{y}_4) \geq 3$ then $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$. Add $c(\Delta)$ to $c(\hat{\Delta})$, see Figure 4.40(iii). While if $d(\hat{y}_3) = d(\hat{y}_4) = 2$ then $l(\hat{y}_3) = \bar{2}\bar{3}^-$ which forces $l(y_1) = \bar{1}2k\bar{4}$ and $l(\hat{y}_5) = 4\bar{1}\bar{3}^-$ otherwise there is a labelling contradiction. If in addition $d(\hat{y}_5) = 3$ then $l(\hat{y}_5) = \bar{4}\bar{4}\bar{1}$ and this yields a contradiction with $l(y_1) = \bar{1}2k\bar{4}$. So $d(\hat{y}_5) \geq 4$ and $c(\hat{\Delta}) \leq 0$. Moreover $l(\hat{y}_1)$ forces at least 2 splits on Δ'' and $l(y_1)$ forces at least one split and hence $c(\Delta'') \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$. Add $c(\Delta)$ to $c(\Delta'')$, see Figure 4.40(iv). Now let $d(\hat{y}_1) \geq 3$. If in addition there are at least 2 more vertices on the edges $(\hat{y}_1, \hat{y}_3)$ and $(\hat{y}_3, \hat{y}_4)$ of degree ≥ 3 then $d(\hat{\Delta}) \geq 6$ and add $c(\Delta)$ to $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$ in Figure 4.15(i). Otherwise $d(\hat{y}_3)$ and $d(\hat{y}_1)$ are both ≥ 4 , see Figure 4.40(v), and so $c(\hat{\Delta}) \leq \frac{-5\pi}{30}$. Add $c(\Delta)$ to $c(\hat{\Delta})$ as shown. Now let $d(y_1) = 5$. Then $c(\Delta) \leq \frac{2\pi}{30}$. If $d(\hat{\Delta}) \geq 6$ then add $c(\Delta)$ to $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$. If $d(\hat{\Delta}) = 5$ then $\hat{\Delta}$ is as in Figure 4.40(iv) or Figure 4.40(v). If $\hat{\Delta}$ is as in Figure 4.40(iv) and $d(\hat{y}_5) \geq 4$ then add $c(\Delta) \leq \frac{2\pi}{30}$ to $c(\hat{\Delta}) \leq c(3, 3, 3, 4, 5) = \frac{-3\pi}{30}$. If $d(\hat{y}_5) = 3$ then $c(\tilde{\Delta}) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$, in fact $\tilde{\Delta}$ is given in 4.36(iv). Add $c(\Delta)$ to $c(\tilde{\Delta})$. If $\hat{\Delta}$ is as in Figure 4.40(v) then $c(\hat{\Delta}) \leq c(3, 3, 4, 4, 5) = \frac{-8\pi}{30}$. Add $c(\Delta)$ to $c(\hat{\Delta})$ as shown.

Δ is given by Figure 4.15(ii):

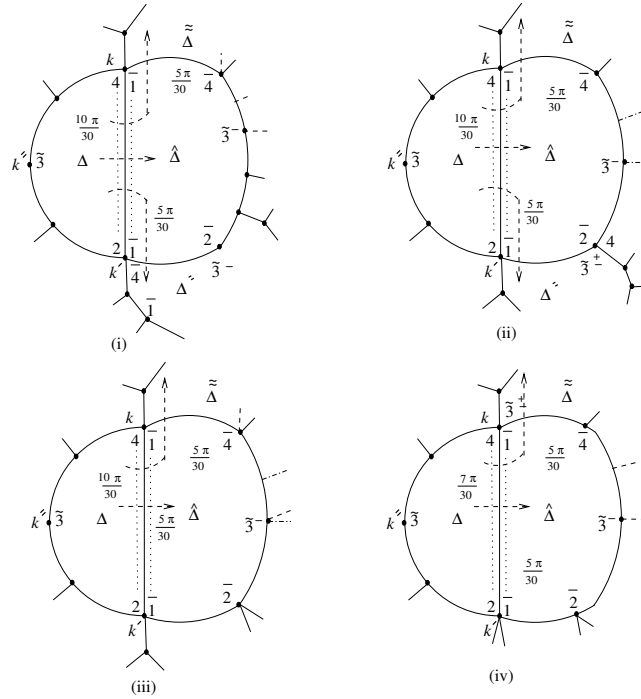
Letting $d(y_5) = 4$ forces $l(y_5) = l(\hat{x}_i) \in k4\bar{1}\{\bar{1}, \bar{2}, \bar{3}^\pm, 4, \hat{k}, \overline{(24)}\}$. If $l(y_5) \in k4\bar{1}\{\bar{1}, \bar{2}, 4, \hat{k}, \overline{(24)}\}$ then this labelling forces at least 1 split on $\hat{\Delta}$ and so $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 4, 4) = \frac{-10\pi}{30}$. Add $c(\Delta) \leq \frac{\pi}{3}$ to $c(\hat{\Delta})$ in Figure 4.15(ii). Now let $l(y_5) = l(\hat{x}_i) = k4\bar{1}\bar{3}^\pm$. If $\hat{x}_i \neq \hat{x}_2$ then the labelling forces at least one split of degree ≥ 4 and $c(\hat{\Delta}) \leq c(3, 3, 3, 4, 4, 4) = \frac{-\pi}{2}$. Add $c(\Delta)$ to $c(\hat{\Delta})$ in Figure 4.15(ii). Now let $\hat{x}_i = \hat{x}_2$. If in addition $d(y_1) = 4$ then $l(y_1) = l(\hat{x}_j) \in \bar{1}2k'\{\bar{1}, 2, \bar{3}^\pm, \bar{4}, \hat{k}, \overline{(24)}\}$


 Figure 4.40: $m \geq 2$

and this labelling except for $l(y_1) = l(\hat{x}_j) \in \bar{1}2k'\{2, \bar{4}\}$ with $\hat{x}_j = \hat{x}_{4\hat{m}-2}$ forces at least one split on $\hat{\Delta}$ and so $c(\hat{\Delta}) \leq \frac{-10\pi}{30}$. Add $c(\Delta)$ to $c(\hat{\Delta})$ in Figure 4.15(ii) when the labelling forces a split. If $l(y_1) = l(\hat{x}_j) \in \bar{1}2k'\{2, \bar{4}\}$ with $\hat{x}_j = \hat{x}_{4\hat{m}-2}$ then there are 2 possibilities for $d(\hat{y}_1)$, see Figures 4.41(i)-(ii). First let $d(\hat{y}_1) = 2$. If in addition $d(\hat{y}_3) \geq 3$ or $d(\hat{y}_4) \geq 3$ then $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 4, 4) = \frac{-\pi}{3}$. Add $c(\Delta)$ to $c(\hat{\Delta})$, see Figure 4.41(i). If $d(\hat{y}_3) = d(\hat{y}_4) = 2$ then $l(\hat{y}_3) = \bar{3}^-\bar{2}$. If $d(\hat{y}_5) \geq 4$ then $c(\hat{\Delta}) \leq c(3, 3, 4, 4, 4) = \frac{-5\pi}{30}$ and $c(\Delta'') \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$. Add $c(\Delta)$ to $c(\hat{\Delta}) + c(\Delta'')$, see Figure 4.41(i). If $d(\hat{y}_5) = 3$ then $c(\hat{\Delta}) \leq 0$ and $c(\tilde{\Delta}) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$. Also $c(\Delta'') \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$. Add $c(\Delta)$ to $c(\tilde{\Delta}) + c(\Delta'')$, see Figure 4.41(i). Now let $d(\hat{y}_1) \geq 3$. If $d(\hat{y}_1) = 3$ then $l(\hat{y}_1) = 4\bar{2}\bar{3}^\pm$ which in turn forces at least one of $d(\hat{y}_3)$ and $d(\hat{y}_4)$ to be ≥ 3 , see Figure 4.41(ii). It also forces at least 2 splits on Δ'' and $c(\Delta'') \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$. Using the previous argument depending on $d(\hat{y}_5)$ we add $c(\Delta)$ to $c(\hat{\Delta}) + c(\Delta'')$ if $d(\hat{y}_5) \geq 4$ or to $c(\hat{\Delta}) + c(\tilde{\Delta})$ if $d(\hat{y}_5) = 3$, see Figure 4.41(ii). If $d(\hat{y}_1) \geq 4$ then $c(\hat{\Delta}) \leq c(3, 3, 4, 4, 4) = \frac{-5\pi}{30}$ when $d(\hat{y}_5) = 3$ and add $c(\Delta)$ to $c(\hat{\Delta}) + c(\tilde{\Delta})$, see Figure 4.41(iii). Whereas $c(\hat{\Delta}) \leq c(3, 4, 4, 4, 4) = \frac{-10\pi}{30}$ when $d(\hat{y}_5) \geq 4$ and add $c(\Delta)$ to $c(\hat{\Delta})$ in Figure 4.41(iii). Now let $d(y_1) \geq 5$. If $d(\hat{\Delta}) \geq 6$ then add $c(\Delta)$ to $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 3, 4, 5) = \frac{-13\pi}{30}$ so let $d(\hat{\Delta}) = 5$. If $d(\hat{y}_5) \geq 4$ then add

$c(\Delta)$ to $c(\hat{\Delta}) \leq c(3, 4, 4, 4, 4) = \frac{-10\pi}{30}$ whereas if $d(\hat{y}_5) = 3$ then add $c(\Delta)$ to $c(\hat{\Delta}) \leq c(3, 3, 4, 4, 4) = \frac{-5\pi}{30}$ and $c(\tilde{\Delta}) \leq \frac{-10\pi}{30}$, see Figure 4.41(iv).

Finally let $d(y_5) \geq 5$. First let $d(y_1) = 4$. Then $c(\Delta) \leq \frac{7\pi}{30}$. If $d(\hat{\Delta}) \geq 6$ then add $c(\Delta)$ to $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 4, 5) = \frac{-13\pi}{30}$ in Figure 4.15(ii). If $d(\Delta) = 5$ then $\hat{\Delta}$ is as in one of the Figures 4.41(i)-(iii) and we add $c(\Delta)$ to $c(\hat{\Delta}) \leq c(3, 3, 4, 4, 5) = \frac{-8\pi}{30}$ in Figure 4.15(ii) when $d(\hat{y}_5) \geq 4$ or $d(\hat{y}_1) \geq 4$. Whereas we add $\frac{3\pi}{30}$ of $c(\Delta)$ to $c(\hat{\Delta}) \leq c(3, 3, 3, 4, 5) = \frac{-3\pi}{30}$ and $\frac{5\pi}{30}$ to $c(\Delta'') \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$ in Figure 4.15(ii) when $d(\hat{y}_1) = d(\hat{y}_5) = 3$. If $d(y_1) \geq 5$ then $c(\Delta) \leq \frac{4\pi}{30}$ and $c(\hat{\Delta}) \leq c(3, 3, 3, 5, 5) = \frac{-6\pi}{30}$. Add $c(\Delta)$ to $c(\hat{\Delta})$ in Figure 4.15(ii).

Figure 4.41: $m \geq 2$

Δ is given by Figure 4.15(iii):

In Figure 4.15(iii) similarly to Figure 4.39 $l(x_2)$ forces at least 2 splits on each of $\hat{\Delta}$ and $\tilde{\Delta}$ and $l(y_1)$ forces at least 2 splits on $\tilde{\Delta}$. Hence add $c(\Delta) \leq \frac{\pi}{6}$ to $c(\tilde{\Delta}) \leq \frac{-10\pi}{30}$ in Figure 4.15(iii).

Δ is given by Figure 4.16(i):

$l(x_1) = 14\tilde{3}^-$ and $l(y_5) = 4\tilde{2}4$. Checking all possible labellings shows that $l(y_3) =$

31k. As used before $l(x_1)$ and $l(y_5)$ forces splits on $\tilde{\Delta}$ and $\hat{\Delta}$. Add $c(\Delta) \leq \frac{10\pi}{30}$ to $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$ and $c(\tilde{\Delta}) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$, see Figure 4.42(i).

Δ is given by Figure 4.16(ii):

$l(x_1) = 14\tilde{3}^-$ and $l(y_5) = 4\bar{2}4$. Add $c(\Delta) \leq \frac{5\pi}{30}$ to $c(\tilde{\Delta}) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$ in Figure 4.16(ii).

Δ is given by Figure 4.16(iii):

$l(y_5) \in \{441, 442, 4\bar{2}4\}$ and each forces at least 3 splits on $\tilde{\Delta}$ and $c(\tilde{\Delta}) \leq c(3, 3, 3, 3, 3, 4) = \frac{-\pi}{6}$. Add $c(\Delta) \leq \frac{\pi}{6}$ to $\tilde{\Delta}$, see Figure 4.42(ii).

Δ is given by Figure 4.16(iv):

$l(x_2) = 14\tilde{3}^\pm$ forces at east 2 splits on $\tilde{\Delta}$ and $l(y_1)$ forces at least 2 more splits on $\tilde{\Delta}$. Thus $c(\tilde{\Delta}) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$. Add $c(\Delta) \leq \frac{5\pi}{30}$ to $c(\tilde{\Delta})$, see Figure 4.42(iii).

Δ is given by Figure 4.17(i):

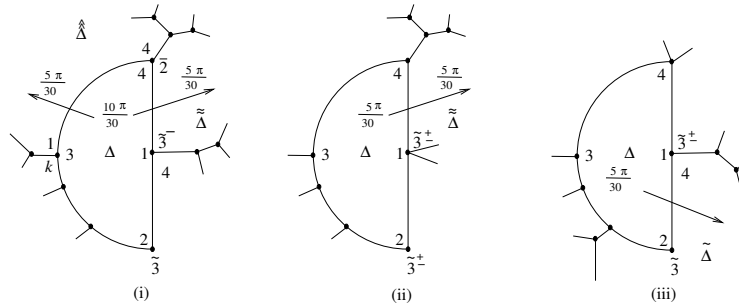


Figure 4.42: $m \geq 2$

The same argument used in Figure 4.11(iii) applies here.

Δ is given by Figure 4.17(ii):

Observe that $l(x_2) = 14k'$ forces at least one split on $\tilde{\Delta}$ and $d(\tilde{x}_2) \geq 3$, see Figure 4.43(i). Moreover $l(y_1)$ forces at least 2 splits on $\tilde{\Delta}$ and forces $d(\tilde{y}_1) \geq 3$ otherwise there is a contradiction. Add $c(\Delta)$ to $c(\tilde{\Delta}) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$ as shown.

Δ is given by Figure 4.17(iii):

$d(y_5) = d(x_2) = 3$ forces $l(x_2) = 14\tilde{3}^-$ and $l(y_5) = 4\bar{2}4$ and $c(\Delta) \leq \frac{10}{30}$. Note that as in Figure 4.42(i) this forces $c(\tilde{\Delta}) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$. On the other hand $l(x_2)$ forces at least 2 splits on $\tilde{\Delta}$ with $d(\tilde{x}_2) \geq 3$, see Figure 4.43(ii), and hence $c(\tilde{\Delta}) \leq c(3, 3, 3, 3, 3, 4) = \frac{-\pi}{6}$. Add $c(\Delta)$ to $c(\tilde{\Delta}) + c(\tilde{\Delta})$ as shown.

Δ is given by Figure 4.17(iv):

$d(y_1) = d(x_2) = 3$ implies $l(x_2) = 14\tilde{3}^\pm$ which forces at least 2 splits on $\tilde{\Delta}$ and $l(y_1) = 2\tilde{4}\tilde{3}$ which also forces at least 2 splits on $\tilde{\Delta}$, see Figure 4.43(iii). Thus $c(\tilde{\Delta}) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-20\pi}{30}$. Add $c(\Delta)$ to $c(\tilde{\Delta})$ as shown.

Δ is given by Figure 4.17(v):

$d(y_1) = d(y_5) = 3$ forces $l(y_5) \in 44\{1, (24)\}$ and $l(y_1) = 2\tilde{4}\tilde{3}$. First let $d(x_2) = 4$. Then checking all possible labelling shows that $l(x_2) \in \{14\tilde{3}2, 14\tilde{3}\bar{2}, 14\tilde{3}k\}$. If $l(x_2) \in \{14\tilde{3}2, 14\tilde{3}k\}$ then this forces at least 2 splits on $\tilde{\Delta}$, see Figures 4.43(iv)-(vi), and $c(\tilde{\Delta}) \leq c(3, 3, 3, 3, 3, 3, 4) = \frac{-15\pi}{30}$; or if $l(x_2) = 14\tilde{3}\bar{2}$ then this forces at least 2 splits on $\tilde{\Delta}$, see Figures 4.43(vii), and $c(\tilde{\Delta}) \leq c(3, 3, 3, 3, 3, 3, 4) = \frac{-15\pi}{30}$. On the other hand $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$ since $l(y_5)$ forces at least 2 splits on $\hat{\Delta}$. Add $c(\Delta)$ to $c(\hat{\Delta}) + c(\tilde{\Delta})$ or $c(\hat{\Delta}) + c(\tilde{\Delta})$ as shown in Figures 4.43(iv)-(vi). Now let $d(x_2) \geq 5$. Then $c(\Delta) \leq \frac{7\pi}{30}$. If $d(\tilde{x}_i) \geq 3$ for at least one $1 \leq i \leq 4\tilde{m} - 1$ then $c(\tilde{\Delta}) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$ while letting $d(\tilde{x}_i) = 2$ for all $1 \leq i \leq 4\tilde{m} - 1$ forces $d(\tilde{y}_1) \geq 4$ and $c(\tilde{\Delta}) \leq c(3, 3, 3, 4, 5) = \frac{-3\pi}{30}$. We also have $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$. Add $c(\Delta)$ to $c(\tilde{\Delta}) + c(\hat{\Delta})$, see Figure 4.43(viii).

Δ is given by Figure 4.17(vi):

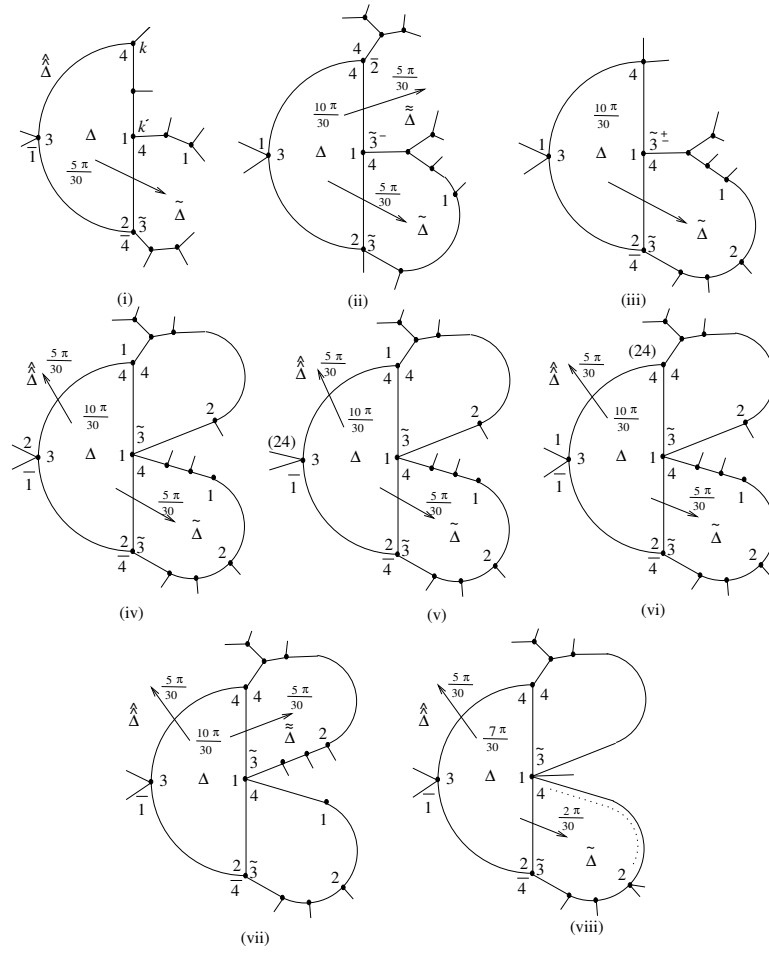
$c(\Delta) \leq \frac{5\pi}{30}$ and $l(y_5) \in \{441, 44(24), 4\bar{2}4\}$ forces at least 2 splits on $\hat{\Delta}$ and we add $c(\Delta)$ to $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$ in Figure 4.17(vi).

Δ is given by Figure 4.17(vii):

$l(x_2) = 14\tilde{3}^\pm$ forces at least 2 splits on $\tilde{\Delta}$ and this implies $d(\tilde{\Delta}) \geq 6$. Add $c(\Delta)$ to $c(\tilde{\Delta}) \leq \frac{-5\pi}{30}$ in Figure 4.17(vii).

Δ is given by Figure 4.17(viii):

$d(y_1) = 3$ implies $l(y_1) = 2\tilde{4}\tilde{3}$ which forces at least 2 splits on $\tilde{\Delta}$. If $d(x_2) = 4$ then

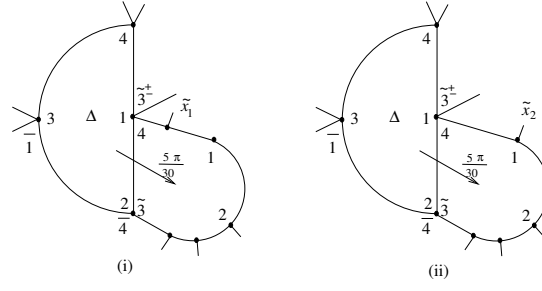
Figure 4.43: $m \geq 2$

$l(x_2) \in 14\tilde{3}\{1, 2, \bar{2}, 4, k, (24)\}$. Letting $l(x_2) \in 14\tilde{3}\{1, 2, k, (24)\}$ forces $d(\tilde{x}_1) \geq 3$, see Figure 4.44(i), and $l(x_2) \in 14\tilde{3}\{\bar{2}, 4\}$ forces $d(\tilde{x}_2) \geq 3$, see Figure 4.44(ii). Thus $c(\tilde{\Delta}) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$. Add $c(\Delta)$ to $c(\tilde{\Delta})$ as shown. Now let $d(x_2) \geq 5$. Then $c(\Delta) \leq \frac{2\pi}{30}$ and as in Figure 4.43(vii) $c(\tilde{\Delta}) \leq c(3, 3, 3, 4, 5) = \frac{-3\pi}{30}$. Add $c(\Delta)$ to $c(\tilde{\Delta})$ in Figure 4.17(viii).

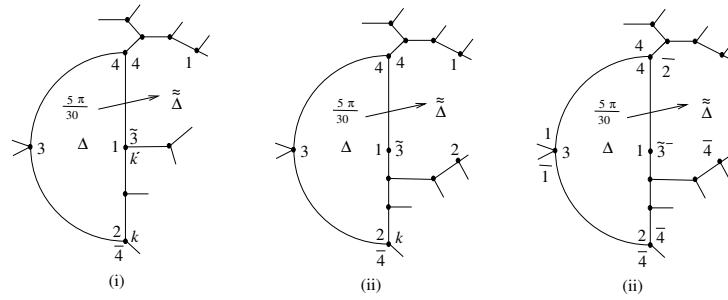
Δ is given by Figure 4.17(ix):

Note that $l(x_2) = \tilde{3}^-1k$ forces $l(y_5) = 4\bar{2}4$ which yields a contradiction with $l(y_1)$. Thus $l(x_2) = \tilde{3}1k$ which forces $l(y_5) \in 44\{1, (24)\}$, see Figure 4.45(i). This labelling forces $c(\tilde{\Delta}) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-15\pi}{30}$. Add $c(\Delta)$ to $c(\tilde{\Delta})$ as shown.

Δ is given by Figure 4.17(x):


 Figure 4.44: $m \geq 2$

Checking all possibilities shows that $l(y_1) \in 2\bar{4}\{k, 2, \bar{4}\}$. First let $l(y_1) = 2\bar{4}k$. Then $l(y_5) \in 44\{1, (24)\}$ and $l(x_2) = 1\bar{3}$. This labelling forces $c(\tilde{\Delta}) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-15\pi}{30}$. Add $c(\Delta)$ to $c(\tilde{\Delta})$, see Figure 4.45(ii). If $l(y_1) = 2\bar{4}2$ then $d(y_5) = 3$ yields a labelling contradiction. Now let $l(y_1) \in 2\bar{4}\bar{4}$. Then $l(y_1) = 4\bar{2}4$ and $l(x_2) = 1\bar{3}^-$ and each forces at least 2 splits on $\tilde{\Delta}$, see Figure 4.45(iii). Moreover $\tilde{m} \geq 2$ and either $d(\tilde{x}_{4\tilde{m}-2}) \geq 3$ or $l(\tilde{x}_{4\tilde{m}-2}) = 1\bar{3}^-$. In the latter case $l(\tilde{x}_{4\tilde{m}-2})$ forces $d(\tilde{x}_{4\tilde{m}-3}) \geq 3$. Thus $c(\tilde{\Delta}) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-15\pi}{30}$. Add $c(\Delta)$ to $c(\tilde{\Delta})$ in Figures 4.45(iii). Note that in this case Δ can receive extra $\frac{3\pi}{30}$ if Δ coincide with $\tilde{\Delta}$ in Figure 4.26(vi). However this forces $d(y_3) \geq 5$ and $c(\Delta) \leq \frac{2\pi}{30}$ and the sum is at most $\frac{5\pi}{30}$ and again we add the sum to $c(\tilde{\Delta})$, see Figure 4.45(iii).


 Figure 4.45: $m \geq 2$

Δ is given by Figure 4.18(i):

The same argument applied in Figure 4.13(ii) works here.

Δ is given by Figure 4.18(ii):

Then $d(y_1) = d(x_2) = d(y_5) = 3$ forces $l(x_2) = 14\bar{3}^-$ and $l(y_5) = 4\bar{2}4$. Then as in Figure 4.46(i) this labelling forces $c(\tilde{\Delta}) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$, $d(\tilde{y}_1) \geq 3$ and $c(\tilde{\Delta}) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$. Add $c(\Delta) \leq c(3, 3, 3, 3, 3) = \frac{10\pi}{30}$ to $c(\hat{\Delta}) + c(\tilde{\Delta})$ as shown.

Δ is given by Figure 4.18(iii):

$d(y_1) = d(x_2) = 3$ forces, as in Figure 4.46(i), $c(\tilde{\Delta}) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$. Add $c(\Delta) \leq c(3, 3, 3, 3, 4) = \frac{5\pi}{30}$ to $c(\tilde{\Delta})$ in Figure 4.18(iii).

Δ is given by Figures 4.18(iv)-(v):

Add $c(\Delta) \leq c(3, 3, 3, 3, 4) = \frac{5\pi}{30}$ to $c(\hat{\Delta}) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$ in Figures 4.18(iv)-(v) by a similar argument to the one used in Figures 4.43(iv)-(vi) for Figure 4.17(v).

Δ is given by Figures 4.19(i)-(ii):

The same argument used in Figure 4.14(i) and Figure 4.14(iv) respectively applies here.

Δ is given by Figure 4.19(iii):

Note that $d(y_1) = d(x_2) = d(y_5) = 3$ implies that $c(\tilde{\Delta}) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$ and $c(\tilde{\Delta}) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$ as $l(x_1)$ forces at least 2 splits on $\tilde{\Delta}$ and $\tilde{\Delta}$, $l(y_5)$ forces at least 2 splits on $\tilde{\Delta}$ and $l(y_5)$ forces at least 2 splits on $\tilde{\Delta}$. Add $c(\Delta)$ to $c(\hat{\Delta}) + c(\tilde{\Delta})$, see Figure 4.46(ii).

Δ is given by Figure 4.19(iv):

$c(\Delta) \leq \frac{5\pi}{30}$ and as in Figure 4.19(iii) $c(\tilde{\Delta}) \leq \frac{-10\pi}{30}$. Add $c(\Delta)$ to $c(\tilde{\Delta})$.

Δ is given by Figure 4.19(v):

Then $c(\Delta) \leq \frac{5\pi}{30}$ and $c(\tilde{\Delta}) \leq \frac{-10\pi}{30}$. Add $c(\Delta)$ to $c(\tilde{\Delta})$.

Δ is given by Figure 4.19(vi):

Assume first $d(x_1) = 4$. Then $l(x_1) \in \bar{3}^{\pm}14\{2, \bar{2}, 4, (24), k\}$ and each of these labelling forces at least one split on $\tilde{\Delta}$ as in Figure 4.44 for Figure 4.17(viii). Thus

Figure 4.46: $m \geq 2$

- a w -split on $\widehat{\Delta}$ forces at least one other split;
- a split on $\widehat{\Delta}$ on a vertex labeled by $\overline{1}$ or $\overline{(24)}$ either forces at least one other split or it is of degree ≥ 4 ;
- either $d(\widehat{y}_3) = 2$ and forces at least one split on each side of $\widehat{y}_3$ or $d(\widehat{y}_3) = 3$ and forces at least one split on one side of $\widehat{y}_3$ or $d(\widehat{y}_3) \geq 4$;
- either $d(\widehat{y}_5) \geq 4$ or $l(\widehat{y}_5) \in \{\overline{143}^{\pm}, \overline{24}\{4, k\}, 44\{1, (24), k\}, 2\overline{3}^{\pm}\overline{4}\}$ which forces at least one split on $\widehat{\Delta}$;
- either $d(\widehat{y}_1) \geq 4$ or it forces at least one split on $\widehat{\Delta}$;
- If $\widehat{\Delta}$, u , $\mathbf{v}$ and $\mathbf{v}'$ are as in Figure 4.47 in which $d(\mathbf{v}) > 2$ and if the edge $(\mathbf{v}, \mathbf{v}')$ is labeled by $w^{\pm 1}$ then we never sends curvature through the segment $(\mathbf{v}, u)$.

In what follows many observations follow from checking, that is, a check of Figures 4.11- 4.46 and we do this often without explicit mention. First consider

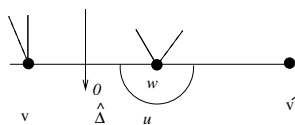
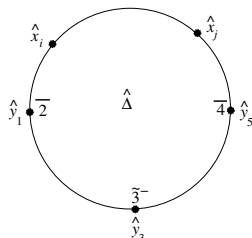


Figure 4.47

Figure 4.48: $\hat{\Delta}$

the regions that compensate through a segment $(\hat{x}_j, \hat{x}_i)$, $2 \leq j < i \leq 4\hat{m} - 2$. Then there are four types of these regions, see Figure 4.49 . Examples of these

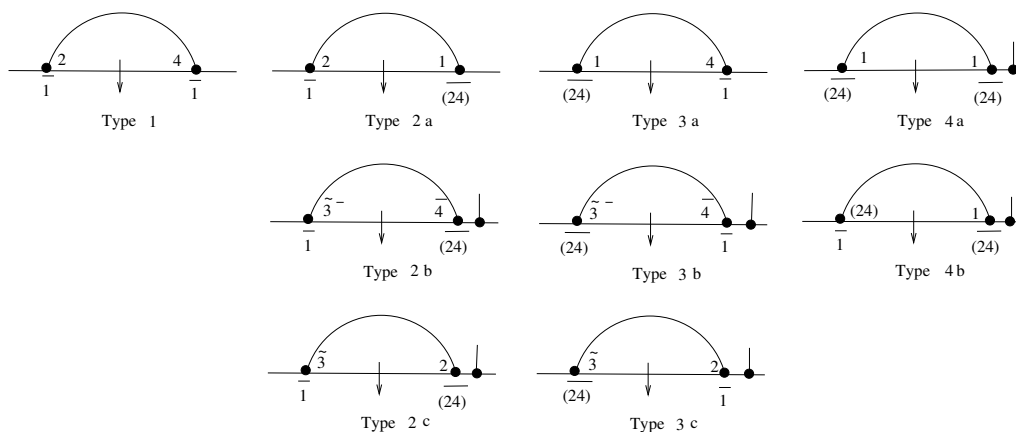


Figure 4.49: regions that compensate through a segment $(\hat{x}_i, \hat{x}_j)$

types are as follows.

- Type 1: $\hat{\Delta}$ in Figure 4.23(ii)-(vi) ;
- Type 2: $\hat{\Delta}$ in Figure 4.28(vi) when $l(x_2) = \widetilde{3}^{-1}\overline{(24)}\omega$ (Type 2a); or $\hat{\hat{\Delta}}$ in Figure 4.28(ii)(Type 2b); or Δ' in Figure 4.24 (vi)(Type 2c);
- Type 3: $\hat{\Delta}$ in Figure 4.30(i) when $l(x_2)$ involves the sublabel $\overline{(24)}1$ (Type 2a); or $\hat{\hat{\Delta}}$ in Figure 4.27(vi) when $d(y_5) = 14\overline{2}1$ and $d(y_3)$ involves the

sublabel $\tilde{3}(24)$ (Type 2b); or Δ' in Figure 4.24(iii) when $l(y_3) = \overline{(24)}\tilde{3}1\omega$ (Type 2c).

- Type 4: the only example is Δ'' in Figure 4.38(iv).

Note that a region of Type2 sends at most $\frac{\pi}{6}$ to $\hat{\Delta}$ while a region of Type3 sends at most $\frac{\pi}{6}$ to $\hat{\Delta}$ unless it is Δ in Figure 4.21(ii) where it is of Type 3a and sends $\leq \frac{7\pi}{30}$ but it forces a vertex of degree ≥ 5 . Furthermore a region of Type3 can never be adjacent to a region of Type1 since Type3 (b) and (c) regions force 2 splits on the adjacent edge as for example in in Figure 4.27(vi) and Figure 4.24(iii). A region of Type3 (a) can not be adjacent to a Type1 region since this forces $4\bar{1}2$ in the labelling of the shared vertex. In addition a region of Type2 can only be adjacent to a region of Type1 if it is Type2b or Type2c and either the shared vertex $\hat{x}_j$ is of degree ≥ 5 , see Figures 4.50(i)-(ii). Note that Δ_1 in Figures 4.50(i)-(ii) always forces at least 2 splits on the adjacent edge and that the maximum curvature it can send to $\hat{\Delta}$ is $\frac{5\pi}{30}$. Moreover a check shows that the maximum curvature Δ_2 may send to $\hat{\Delta}$ is $\frac{12\pi}{30}$ and it forces Δ_3 to send no positive curvature to $\hat{\Delta}$ because of the split $\hat{x}_{j+1}$. Note that Type2a and Type1 cannot be adjacent as this forces the sublabel $4\bar{1}2$ in the labelling of the shared vertex which is a contradiction. In addition a region of Type 4 sends at most $\frac{\pi}{6}$ to $\hat{\Delta}$ and it can never be adjacent to Type 3 regions as it forces at least two splits on the adjacent edge and it can be adjacent to Type 1 or Type 2 regions only if the shard vertex is of degree ≥ 5 .

On the other hand Type22b and Type2c forces splits on the adjacent edges and

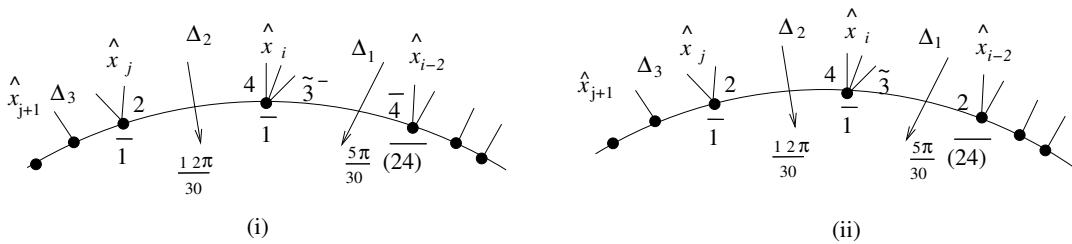


Figure 4.50: $\hat{\Delta}$ receives from more than one region

a check shows that regions of Type2 and Type3 can only be adjacent and send positive curvature to the same $\hat{\Delta}$ if Type2 follows Type3 as shown in Figure 4.51. Note that in Figure 4.51 Δ_3 always forces at least 2 w -splits on the next edge as shown. Moreover in Figures 4.51 (ii)-(vii) $\hat{\Delta}$ receives no positive curvature

Moreover two regions of Type1 can never share a vertex $\hat{x}_i$ as this forces $2\bar{1}4$ in

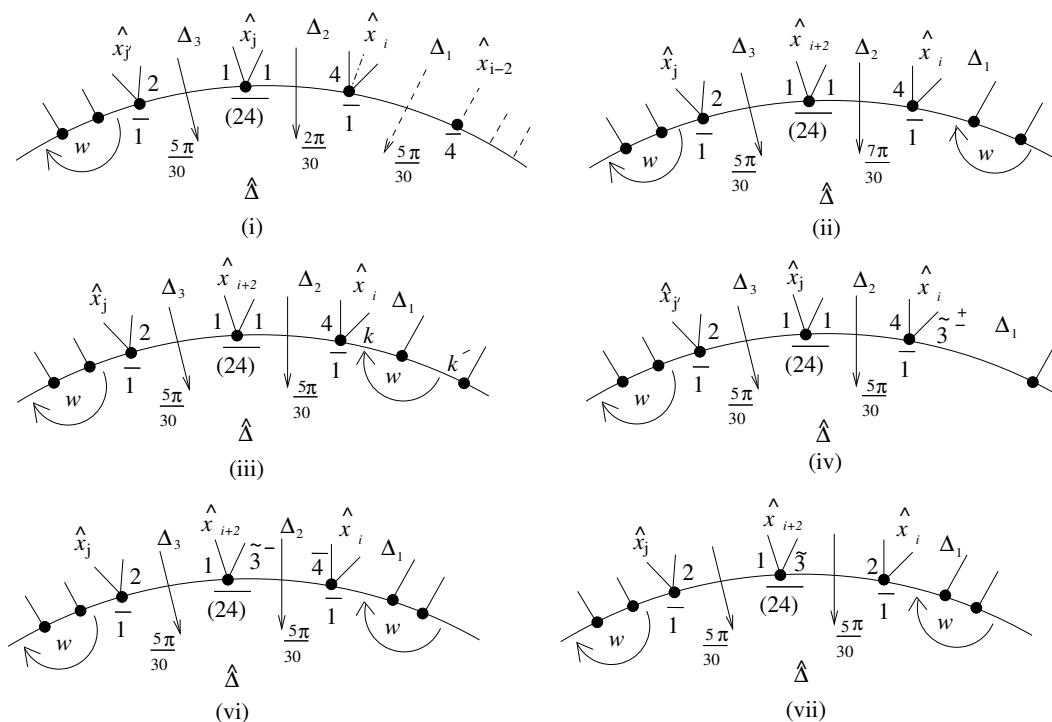


Figure 4.51: $\hat{\Delta}$ receives from more than one region

the labelling which is a contradiction, see Figure 4.52(i). In addition if Δ_i is a region of Type 1 then the most it sends to $\widehat{\Delta}$ is $\frac{\pi}{2}$ and between two consecutive regions either there is at least one split or at least $d(y_1) \geq 4$ in Δ_1 and $d(y_5) \geq 5$ in Δ_2 , see Figure 4.52(ii)-(v). Observe further that Δ^{**} will send no positive curvature to $\widehat{\Delta}$ since it would have to be of Type 3 which cannot be placed here as discussed before. Moreover Δ^* in Figure 4.52(iii) never sends positive curvature to $\widehat{\Delta}$ and this can be seen by the following. Note that we have $d(\widehat{x}_{j-1}) = 2$. If the sublabel involved is (24) then it has to be Type4b which can not be placed here so assume otherwise. If $d(y_5) = 3$ then $l(y_5) = 4\widetilde{13}^\pm$ which forces at least 2 splits on Δ^* and either $d(\Delta^*) \geq 6$ or $c(\Delta^*) \leq c(3, 3, 3, 4, 4) = 0$ as $d(\Delta^*) = 5$ forces both $d(y_1^*)$ and $d(y_5^*) \geq 4$ unless $l(y_5^*) = l(y_5) = 4\widetilde{13}$, see Figure 4.53(i). In this

case however Δ^* is Δ in Figure 4.14(i) and it sends the positive curvature to $\hat{\Delta}^*$ and Δ'' in Figure 4.53(i) and never sends any positive curvature to $\hat{\Delta}$. If $d(y_5) = 4$ then $l(y_5) = 4\bar{1}3^\pm\omega$ and again all possible labelling forces either 2 splits on Δ^* and $c(\Delta^*) \leq 0$ as above; or it forces exactly one split and $l(y_5) = 4\bar{1}3^\pm k$, see Figure 4.53(ii)-(iii), and this implies that either $c(\Delta^*) \leq c(3, 3, 3, 4, 4) = 0$ when there is another split; or both $d(y_1^*)$ and $d(y_5^*)$ are ≥ 4 , $c(\Delta^*) \leq c(3, 4, 4, 4) = \frac{\pi}{6}$ and Δ^* is given by Δ in Figure 4.13(ii) or in Figure 4.14(iv) or Figure 4.18(i) or Figure 4.19(ii). If Δ^* is Δ in Figure 4.13(ii) or Figure 4.18(i) then, as shown in Figure 4.53(ii), this forces $d(y_5^*) \geq 5$ and by checking the distribution done in Figure 4.35 we see that Δ^* sends the positive curvature to $\hat{\Delta}^*$ and no positive curvature to $\hat{\Delta}$, see Figure 4.53(ii). If Δ^* is Δ in Figure 4.14(iv) or Figure 4.19(ii) then, as shown in Figure 4.53(iii), this forces either $d(y_1^*) \geq 5$ or $l(y_1^*) = \bar{1}2(\bar{2}4)3^\pm$. If $d(y_1^*) = 4$ then we see from Figure 4.14(iv) or Figure 4.19(ii) that Δ^* sends the positive curvature to $\hat{\Delta}^*$, and if $d(y_1^*) \geq 5$ then it sends to $\hat{\Delta}^*$ or Δ' . In Figure 4.52(iv) checking shows that Δ^* never sends positive curvature to $\hat{\Delta}$. In Figure 4.52(v) $c(\Delta^*) \leq 0$ and it sends no positive curvature to $\hat{\Delta}$.

By the above discussion it can be seen that through three consecutive edges of

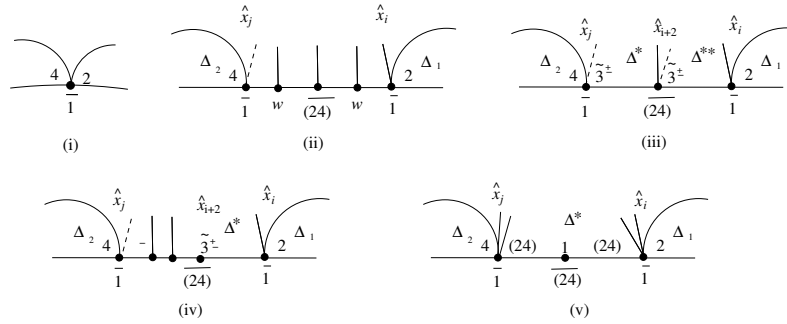


Figure 4.52: $\hat{\Delta}$ receives from more than one region

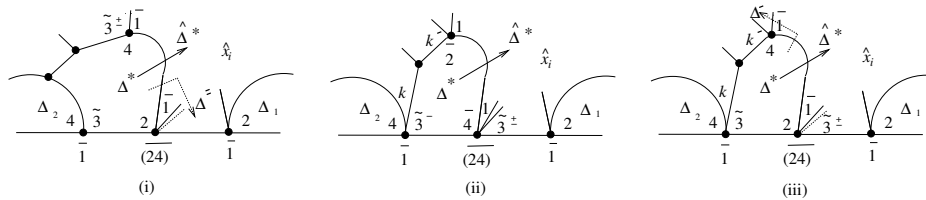


Figure 4.53: $c^*(\hat{\Delta})$ distribution

$\hat{\Delta}$ and from regions of Type1, Type2, and Type3 we have one of the following:

- three regions sending at most $\frac{\pi}{6}$ each;
- if not three then two adjacent regions of the three regions sending a sum of $< \frac{\pi}{2}$; or at most a sum of $\frac{17\pi}{30}$ as in Figure 4.50 but this forces $d(\hat{x}_i) \geq 5$;
- two but not adjacent regions of the three regions sending a sum of $< \frac{20\pi}{30}$ as in Figure 4.52 but forces at least one vertex of degree ≥ 5 ;
- only one of the regions is sending $\frac{\pi}{2}$ and the other two send no positive curvature to $\hat{\Delta}$.

Observe that only a region of Type 1 can send $> \frac{7\pi}{30}$ and that the possibilities are as shown in Figure 4.54. In Figures 4.54(i)-(ii) Δ_3 is sending up to $\frac{\pi}{2}$ but Δ_2 and Δ_4 sends no positive curvature to $\hat{\Delta}$. Moreover if Δ_1 is sending any positive curvature to $\hat{\Delta}$ then it is either of Type 2 or Type 4 and sends at most $\frac{\pi}{6}$. In addition if Δ_5 is sending positive curvature to $\hat{\Delta}$ then it is of Type 3 and either it is sending at most $\frac{\pi}{6}$ or it forces further splits and then Δ_6 will send no positive curvature to $\hat{\Delta}$, see Figure 4.54(ii). Note that in Figure 4.54(ii) Δ_7 is of Type 3 as well and it has the same possibilities as Δ_5 . If Δ_3 is given by Figure 4.54(iii), in which case it forces no splits on the next vertex and it is labelled by $1(24)$, then Δ_3 plays the role of Δ in Figure 4.14(i) when $l(y_1) = (24)\bar{1}2\bar{3}^\pm$ and send $\leq \frac{\pi}{3}$ to $\hat{\Delta}$, Δ_4 and Δ_2 send no positive curvature to $\hat{\Delta}$ and Δ_5 as in Figures 4.54 (i) and (ii). Note that this forces $d(\hat{v}) \geq 5$ so Δ_1 is on of the cases 4.54 (vii),(viii) or (ix).

If Δ_3 is forcing one vertex of degree ≥ 5 then as shown in Figures 4.54(iv)-(ix) Δ_3 is sending $\leq \frac{12\pi}{30}$ and Δ_4 is sending $\leq \frac{5\pi}{30}$ to $\hat{\Delta}$ only if it is given by Figures 4.54(vi)-(vii) where $d(\hat{v}) \geq 5$ while Δ_2 does not send any since in Figures 4.54(iv),(vi) and (ix) it would have to be of Type 3 to send positive curvature to $\hat{\Delta}$ but Type 3 regions cannot be adjacent to Type 1 regions and it would have to be of Type 1 in the other cases which is not possible. Moreover Δ_5 is as in Figures 4.54(i)-(ii) whereas Δ_1 in 4.54(iv),(vi) and (ix) is as in Figure 4.54(i) sending $\leq \frac{\pi}{6}$ while in (iii),(v),(vii),(viii) plays the role of Δ_3 in Figure 4.54 (vi),(vii),(viii) and (xi). If Δ_3 is forcing 2 vertices of degree ≥ 5 then the most it can send is $\frac{9\pi}{30}$ and it is either as in Figure 4.54(vi) or it forces no splits as in Figure 4.52(x) when $d(y_1) \geq 5$. Observe that Δ_2 is as in Figures 4.54(iii)-(iv) never sends positive

curvature to $\hat{\Delta}$ and that Δ_1 and Δ_5 are as in Figures 4.54(i)-(ii).

Checking shows that a region of Type1, Type2, and Type3 that sends cur-

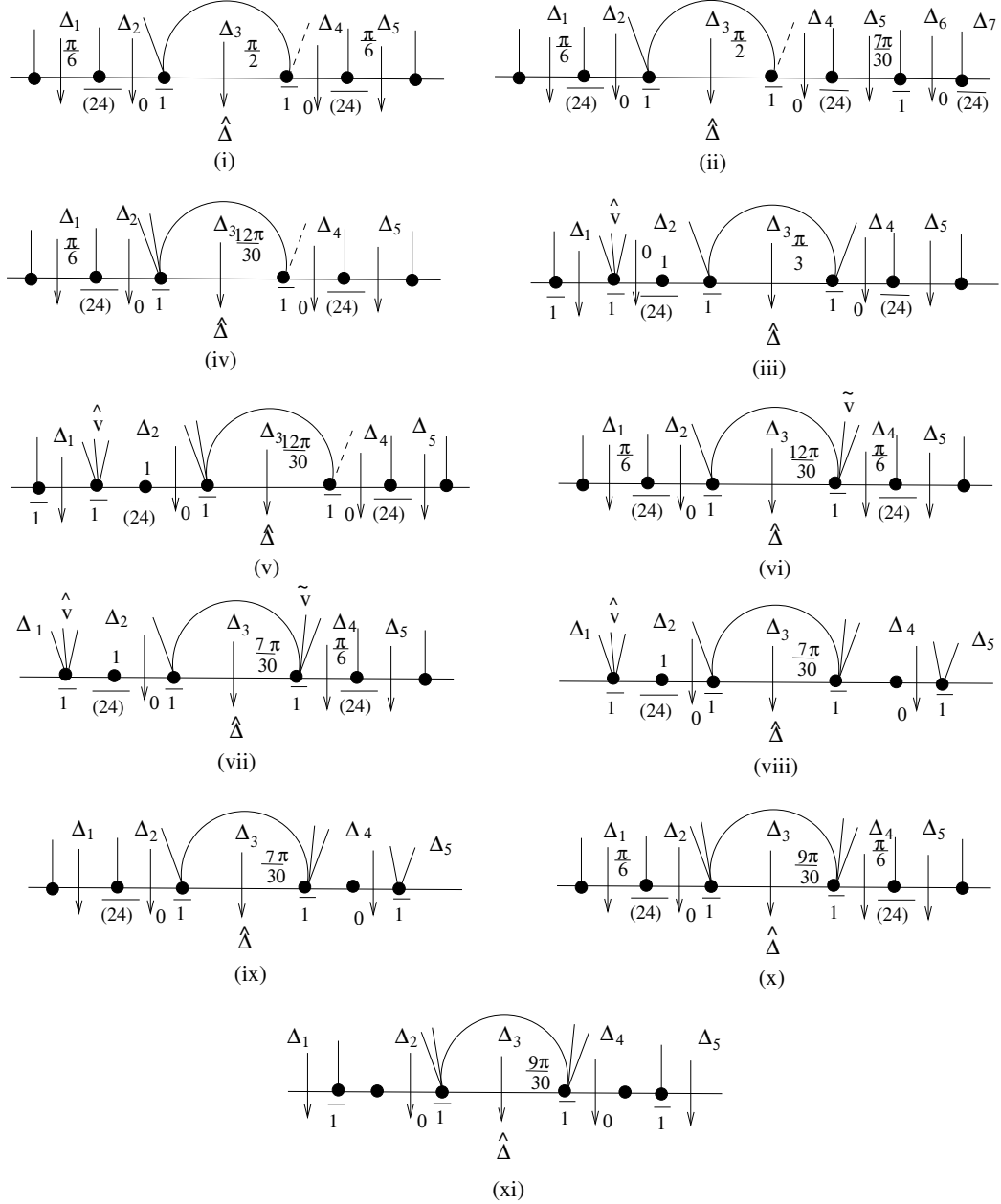


Figure 4.54: $\hat{\Delta}$ receives from more than one region

vature to $\hat{\Delta}$ forces the two end vertices to be of degree ≥ 4 except when $l(y_5) = 4\bar{1}\bar{3}^\pm$ as for example $\hat{\Delta}$ in Figures 4.14(i)-(ii) and $\hat{\Delta}$ in Figure 4.15(i); or when $l(y_5) \in 44\{1, (24)\}$ as for instance in $\hat{\Delta}$ in Figures 4.28(i)-(ii). Figure

4.55 shows these possibilities and the maximum curvature that can be sent in these cases. Note that in Figure 4.55(i) either $d(\widehat{v}) \geq 4$ or it forces a further split. Also in Figures 4.55(ii)-(iii) $l(y_5)$ forces at least 2 splits on the edge $(y_5, \widehat{v})$ and we can, for our estimation, ignore the splits and assume that $d(\widehat{v}) \geq 4$. We can assume in Figure 4.52(iv) as well that only one split of degree ≥ 4 in place of the two splits.

Thus when estimating $c^*(\widehat{\Delta})$ we can assume that half of the vertices forced by the regions of Type1, Type2, and Type3 are of degree ≥ 4 . We can also assume that the maximum curvature sent from regions of Type1, Type2, and Type3 is $\frac{\pi}{6}$ per edge otherwise the maximum is either $\frac{17\pi}{30}$ but a vertex of degree ≥ 5 is forced but this is less than our estimation.

Now consider the regions that compensate through the segment $(\widehat{y}_5, \widehat{x}_j)$,

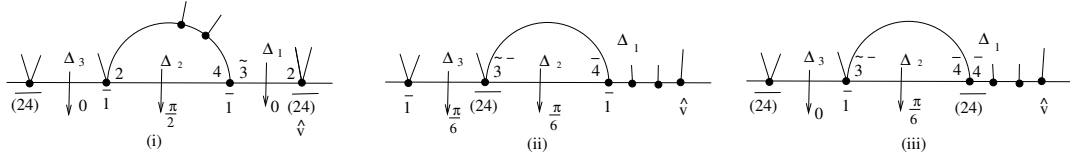
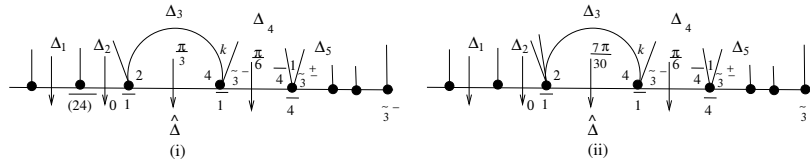


Figure 4.55: $\widehat{\Delta}$ receives from more than one region

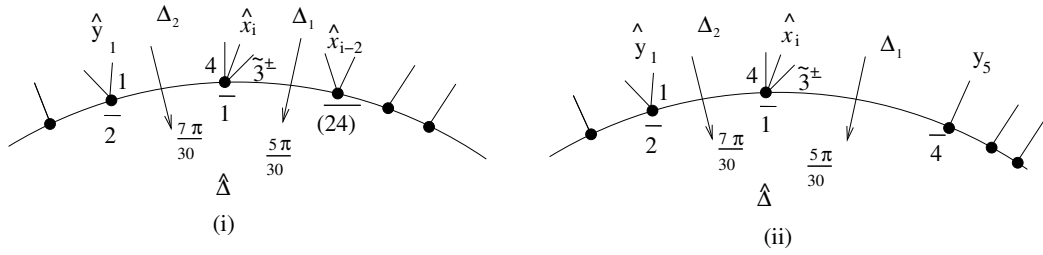
$2 \leq j \leq 4\widehat{m} - 2$ in Figure 4.48. The maximum curvature sent by such regions is $\frac{\pi}{3}$. If a region of this type is sending $> \frac{5\pi}{30}$ then it is only given by Figures 4.32(iii)-(iv) or 4.34(i)-(ii). In both cases Δ is forcing at least one w -split on the next edge $(\widehat{x}_j, \widehat{x}_{j+1})$ and it forces at least 2 w -splits on the edge $(\widehat{y}_3, \widehat{y}_5)$ so no positive curvature can be sent through these edges. If a region of this type is sending at most $\frac{5\pi}{30}$, as for example Figures 4.25(i)-(ii) and Figures 4.28(iii)-(v), then it is either as above or it has the same possibilities as a Type2 region in the way in which it is adjacent to a Type1 region or a Type2 region except when it is given by Figure 4.56 in which case Δ_4 plays the role of Δ in Figure 4.13(i) or Figure 4.18(i) with $l(y_5) = 44\bar{1}\bar{3}^\pm$ and Δ_3 plays the role of Δ in Figure 4.14(iv) or Figure 4.18(ii). However Δ_2 and Δ_3 never send any positive curvature to $\widehat{\Delta}$ and Δ_1 in Figure 4.56(i) send $\leq \frac{\pi}{6}$ while in Figure 4.56(ii) Δ_3 is forcing a vertex of degree ≥ 5 and send $\leq \frac{7\pi}{30}$ which is less than our estimation, see Figure 4.56. Moreover these type of regions either forces the two end vertices to be of degree ≥ 4 as in Figures 4.25(i) or one of the end vertices is of degree ≥ 4 while the other one is of ≥ 3 but it forces another split. So (when maximizing $c^*(\widehat{\Delta})$)

Figure 4.56: $\hat{\Delta}$ receives from more than one region

we can ignore the splits and assume this type of regions forces at least 2 vertices of degree ≥ 4 when sending positive curvature to $\hat{\Delta}$ and that the estimation with $\frac{\pi}{6}$ per edge works for this Type.

Now consider the regions that compensate through a segment $(\hat{x}_i, \hat{y}_1)$, $2 \leq i \leq 4\hat{m} - 2$. Then there are 2 possibilities for $d(\hat{y}_1)$. If $d(\hat{y}_1) = 2$ then Δ is the region in Figure 4.37(vii) with $l(y_1) = \tilde{3}2$. This however sends $\leq \frac{\pi}{6}$ and forces one of the end vertices to be of degree ≥ 4 , at least 1 split per edge on the adjacent edges and so no positive curvature can be sent through these edges to $\hat{\Delta}$. If $d(\hat{y}_1) \geq 3$ then $\hat{\Delta}$ is given by $\hat{\Delta}$ in Figures 4.22(i)-(iii) or Δ' in Figures 4.24(ii)-(iii) when $l(y_3)$ has the sublabel $\tilde{2}3$ or $\hat{\Delta}$ in Figure 4.43(iv) for instance. A region of this type that compensate to $\hat{\Delta}$ may send up to $\frac{\pi}{3}$. If Δ is of this type and it sends a positive curvature $> \frac{5\pi}{30}$ then Δ is one of the positive regions in Figure 4.20 or Figure 4.21(i) or Figures 4.22(i)-(iii). This implies that $\hat{\Delta}$ never receives positive curvature through the edge $(\hat{y}_1, \hat{y}_3)$ as it is split in case of Figures 4.20 and Figures 4.22(i)-(iii) and by checking in case of Figure 4.21(i). Also Δ forces at least 2 vertices of degree ≥ 4 . Moreover either it does not receive through the segment $(\hat{x}_{i-2}, \hat{x}_i)$ or $d(\hat{x}_i) \geq 5$. Note that the only positive region that can share $\hat{x}_i$ is a region of Type2, see Figure 4.57, and the most positive curvature sent to $\hat{\Delta}$ from Δ_1 and Δ_2 is $\frac{12\pi}{30}$. However it forces $d(\hat{x}_i) \geq 5$. If Δ is of this type and it sends a positive curvature $\leq \frac{5\pi}{30}$ then Δ either forces 2 vertices of degree ≥ 4 as in Figures 4.22(v) for example or forces one vertex of degree ≥ 4 and one vertex of degree 3 which in turn forces at least 2 splits on the adjacent edge as for example $\hat{\Delta}$ in Figures 4.29(i)-(ii) with $l(y_3)$ involves the sublabel $\tilde{3}2$. So as above when such a region is sending positive curvature to $\hat{\Delta}$ we can ignore the splits and assume this type of regions forces at least 2 vertices of degree ≥ 4 .

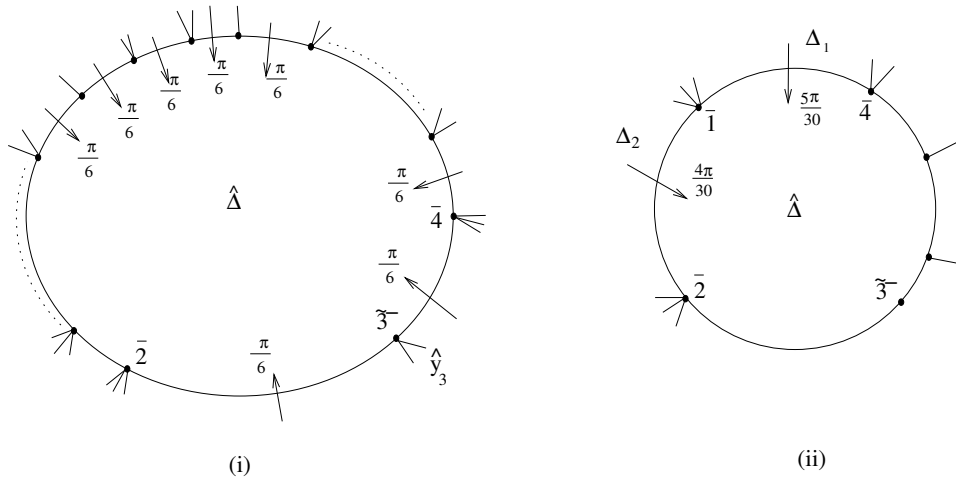
Now consider regions that compensate through the edges $(\hat{y}_1, \hat{y}_3)$ and $(\hat{y}_3, \hat{y}_5)$.

Figure 4.57: $\hat{\Delta}$ receives from more than one region

Let Δ be a region of this type that compensate to $\hat{\Delta}$. Checking shows that if $d(\hat{y}_3) = 2$ then $\hat{\Delta}$ receives through the segment $(\hat{y}_1, \hat{y}_5)$ only if $l(\hat{y}_3) \in \{2\tilde{3}, 1\tilde{3}^\pm\}$. If $l(\hat{y}_3) = 2\tilde{3}$ then, as $\tilde{\Delta}$ in Figures 4.21(iii)-(iv) for example, this forces at least 3 splits on $\hat{\Delta}$ and no positive curvature can be received through the segment $(\hat{y}_5, \hat{x}_j)$ and the most that can be received from Δ is $\frac{7\pi}{30}$ (note that when $d(\hat{y}_1) = 2$ then $l(\hat{y}_1) = \bar{2}\tilde{3}^-$ and there is no positive curvature can be sent across this). If $l(\hat{y}_3) = 1\tilde{3}^\pm$ then, as $\tilde{\Delta}$ in Figures 4.45(ii)-(iii), this forces at least 3 splits on $\hat{\Delta}$ and no positive curvature can be received through the edge $(\hat{x}_{4\hat{m}-2}, \hat{y}_1)$ or the edge $(\hat{x}_2, \hat{y}_5)$ and the most that can be received from Δ is $\frac{5\pi}{30}$. Now let $d(\hat{y}_3) = 3$ then at least one of the edges $(\hat{y}_1, \hat{y}_3)$ and $(\hat{y}_3, \hat{y}_5)$ has a split and hence $\hat{\Delta}$ receives through at most one of them. Let $\hat{\Delta}$ receives through the edge $(\hat{y}_1, \hat{y}_3)$, for instance as Δ'' in Figure 4.24(v), $\tilde{\Delta}$ in Figure 4.25(vii) when $l(y_5) = 44\bar{1}\tilde{3}$ and $l(\tilde{y}_5) = 24\omega$ or $\tilde{\Delta}$ in Figure 4.31(ii) when $l(y_1) = 24\tilde{3}^-$ and $l(\tilde{x}_{4\tilde{m}-2}) \in \bar{2}1\bar{2}\{\bar{2}, \tilde{3}^\pm\}$. Then the most curvature received is $\frac{5\pi}{30}$. Moreover checks shows that no positive curvature can be received through the edge $(\hat{x}_{4\hat{m}-2}, \hat{y}_1)$. Let $\hat{\Delta}$ receives through the edge $(\hat{y}_3, \hat{y}_5)$ as for example Δ'' in Figure 4.24(iii), $\tilde{\Delta}$ in Figure 4.25(vii) when $l(y_5) = 44\bar{1}\tilde{3}^-$ and $l(\tilde{y}_5) = \bar{4}4\omega$ or $\tilde{\Delta}$ in Figure 4.31(ii) when $l(y_1) = 24\tilde{3}$ and $l(\tilde{x}_{4\tilde{m}-2}) = \bar{2}14\tilde{3}^\pm$. Then the most curvature received is $\frac{5\pi}{30}$. Moreover no positive curvature can be received through the edge $(\hat{y}_5, \hat{x}_1)$. Now let $d(\hat{y}_3) \geq 4$. If $\hat{\Delta}$ receives through the edge $(\hat{y}_1, \hat{y}_3)$, for example $\tilde{\Delta}$ in Figure 4.42(ii) when $l(x_2)$ involves the sublabel $\tilde{3}\bar{1}$ and Δ'' in Figure 4.24(v) when $d(y_1'') \geq 4$, then the most curvature received is $\frac{5\pi}{30}$. Moreover when $d(\hat{y}_1) \in \{3, 4\}$ then no positive curvature can be received through the edge $(\hat{x}_{4\hat{m}-2}, \hat{y}_1)$. If $\hat{\Delta}$ receives through the edge $(\hat{y}_3, \hat{y}_5)$, for example $\tilde{\Delta}$ in Figure 4.25(vii) when $l(y_1) = 44\bar{1}\tilde{3}^-$ and Δ'' in Figure 4.24(iii) when $d(y_1'') \geq 4$, then the most curvature received is $\frac{5\pi}{30}$. Moreover when $d(\hat{y}_5) \in \{3, 4\}$ no positive curvature can be received through the edge $(\hat{y}_5, \hat{x}_1)$.

By the previous discussion we can see when $\hat{\Delta}$ receives cross the edges $(\hat{x}_{4\hat{m}-2}, \hat{y}_1)$, $(\hat{y}_1, \hat{y}_3)$, $(\hat{y}_3, \hat{y}_5)$ and $(\hat{y}_5, \hat{x}_2)$ then it is at most $\frac{\pi}{6}$ per edge and this will force at least 2 vertices of degree ≥ 5 , in particular $\hat{y}_2$ and $\hat{y}_5$, and the other vertices are of degree ≥ 4 .

Therefore if $\hat{\Delta}$ is a region that receives from N regions of Type1, Type2, Type3 and Type4 then this forces at least $\frac{N+1}{2}$ of the $N+1$ vertices are of degree ≥ 4 and the most received is $N(\frac{\pi}{6})$. If $\hat{\Delta}$ is receiving through the other four edges then the most received is $4(\frac{\pi}{6})$, and this forces at least 1 vertex of degree ≥ 4 , 2 vertices of degree ≥ 5 and increases the degree of at least one vertex, $\hat{x}_i$, to ≥ 5 , see Figure 4.58. Thus such a region will have $N+4$ vertices of which $\frac{N+1}{2}$ of degree ≥ 4 and 3 of degree ≥ 5 and receives at most $(N+4)(\frac{\pi}{6})$ and hence $c^*(\Delta) \leq [2 - (N+4)]\pi + (\frac{N+1}{2})\frac{2\pi}{3} + (\frac{N+1}{2})\frac{2\pi}{4} + 3(\frac{2\pi}{5}) + (N+4)\frac{\pi}{6}$ which implies that $c^*(\Delta) \leq 0$ for $N \geq 2$. Assume now that $N < 2$ that is $d(\hat{\Delta}) < 6$. Then the only case where 2 regions can compensate to the same $\hat{\Delta}$ is shown in Figure 4.58(ii) where Δ_1 is as Δ in Figure 4.22(iii), $c(\Delta_1) \leq \frac{5\pi}{30}$, Δ_2 is as Δ in Figure 4.34(iii), $c(\Delta_2) \leq \frac{4\pi}{30}$ and $c(\hat{\Delta}) \leq c(3, 3, 4, 5, 5) = \frac{-11\pi}{30}$. Thus $c^*(\Delta) < 0$.


 Figure 4.58: $m \geq 2$

4.2.2 Figures with a vertex label $\bar{2}41$

Assume now that $l_1 - l_2 + l_4 = 0$ with the assumption $l_3 \neq \pm m_1 l_4$, $m_1 \in \{2, 3\}$ and $l_4 \neq m_2 l_3$, $m_2 \in \{\pm 2, -3\}$. First assume $d(u_i) = 2$ for all $1 \leq i \leq 4$ and

let $d(v_i) = 3$ for all $1 \leq i \leq 4$. Then checking all the possible labelling shows that this yields a contradiction. Now assume exactly three of $d(v_i)$, $1 \leq i \leq 4$ are 3. First let $d(v_1) = d(v_2) = d(v_3) = 3$. Then letting $d(v_1) = 3$ implies $l(v_1) \in \{\bar{3}1\bar{4}, 31\bar{4}, \bar{2}14\}$. If $l(v_1) = \bar{3}1\bar{4}$ then this forces $l(v_2) = \bar{1}2\bar{4}$ but $d(v_3) = 3$ yields a contradiction; if $l(v_1) = 31\bar{4}$ then this forces $l(v_2) = \bar{1}2\bar{4}$ and $l(v_3) = \bar{1}33$; if $l(v_1) = \bar{2}14$ then this forces $l(v_2) = 324$ and $l(v_3) = 331$ however this yields a contradiction to the assumption $l_3 \neq 2l_4$. Now let $d(v_2) = d(v_3) = d(v_4) = 3$. Then checking the possible labellings shows that there are two possibilities; either $l(v_2) = \bar{1}2\bar{4}$, $l(v_3) = \bar{1}33$ and $l(v_4) = 244$; or $l(v_2) = \bar{3}24$, $l(v_3) = 332$ and $l(v_4) = 14\bar{2}$. Now let $d(v_3) = d(v_4) = d(v_1) = 3$. Then this forces $l(v_3) = 332$, $l(v_4) = 14\bar{2}$ and $l(v_1) = \bar{3}1\bar{4}$ otherwise there is a labelling contradiction. Now let $d(v_4) = d(v_1) = d(v_2) = 3$. Then checking the possible labelling shows that there are three possibilities; either $l(v_4) = 44\bar{1}$, $l(v_1) = \bar{2}14$ and $l(v_2) = 324$; or $l(v_4) = 14\bar{2}$, $l(v_1) = \bar{3}1\bar{4}$ and $l(v_2) = \bar{1}2\bar{4}$; or $l(v_4) = 244$, $l(v_1) = 31\bar{4}$ and $l(v_2) = \bar{1}2\bar{4}$. This yields the seen cases shown in Figure 4.59.

Now assume exactly two of $d(v_i)$, $1 \leq i \leq 4$ are 3. If $d(v_1) = d(v_2) = 3$ then

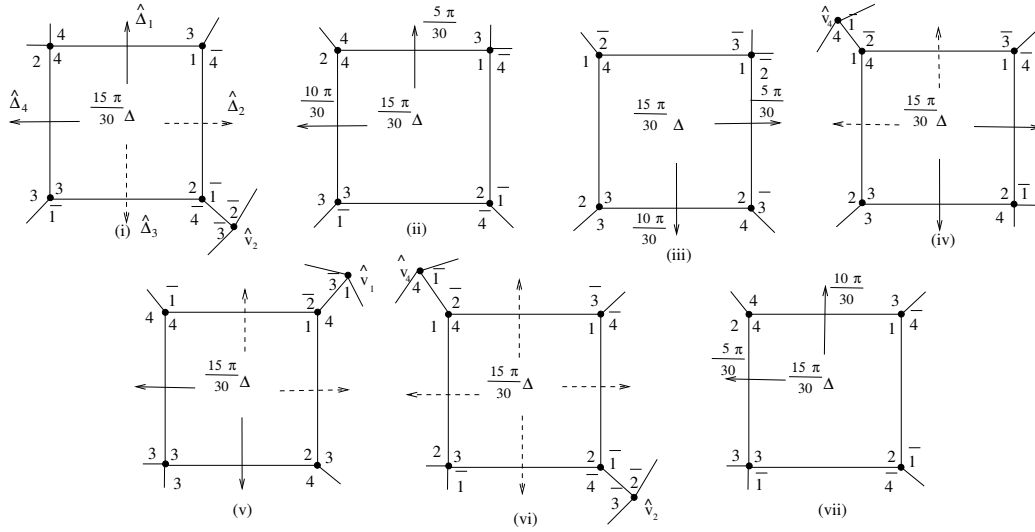
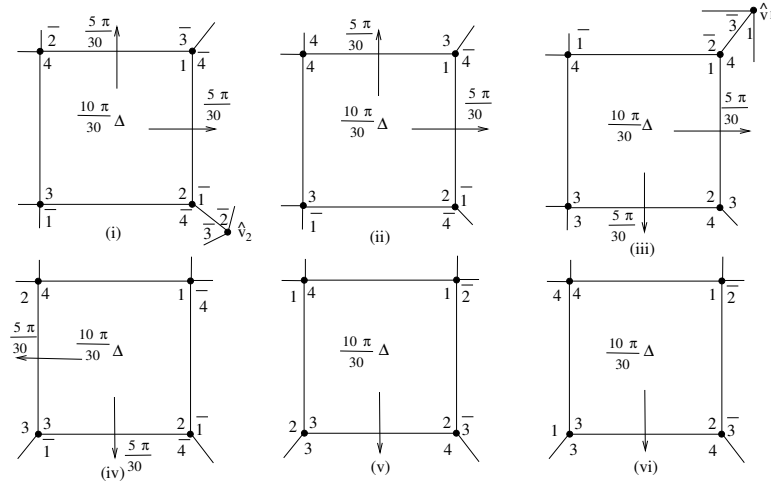
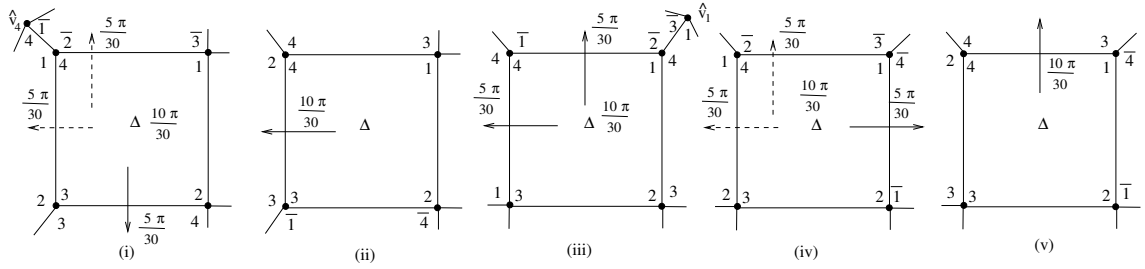


Figure 4.59: $d(u_i) = 2$ for all $1 \leq i \leq 4$

either $l(v_1) \in \{\bar{3}1\bar{4}, 31\bar{4}\}$ and $l(v_2) = \bar{1}2\bar{4}$; or $l(v_1) = \bar{2}14$ and $l(v_2) = 324$. If $d(v_2) = d(v_3) = 3$ then either $l(v_2) = \bar{1}2\bar{4}$ and $l(v_3) = \bar{1}33$; or $l(v_2) = \bar{3}24$ and $l(v_3) \in \{332, 331\}$, see Figure 4.60. If $d(v_3) = d(v_4) = 3$ then either $l(v_3) = 332$ and $l(v_4) = 14\bar{2}$; or $l(v_3) = \bar{1}33$ and $l(v_4) = 244$. If $d(v_1) = d(v_4) = 3$ then either


 Figure 4.60: $d(u_i) = 2$ for all $1 \leq i \leq 4$

$l(v_1) = \bar{2}14$ and $l(v_4) = 44\bar{1}$; or $l(v_1) = \bar{3}1\bar{4}$ and $l(v_4) = 14\bar{2}$; or $l(v_1) = 31\bar{4}$ and $l(v_4) = 244$, see Figure 4.61. If $d(v_1) = d(v_3) = 3$ or $d(v_2) = d(v_4) = 3$ then the


 Figure 4.61: $d(u_i) = 2$ for all $1 \leq i \leq 4$

possible labellings are given in Tables 4.1, 4.2 respectively, see Figure 4.62, 4.63.

Assume now that only one of $d(v_i)$, $1 \leq i \leq 4$ is 3. If $d(v_1) = 3$ then

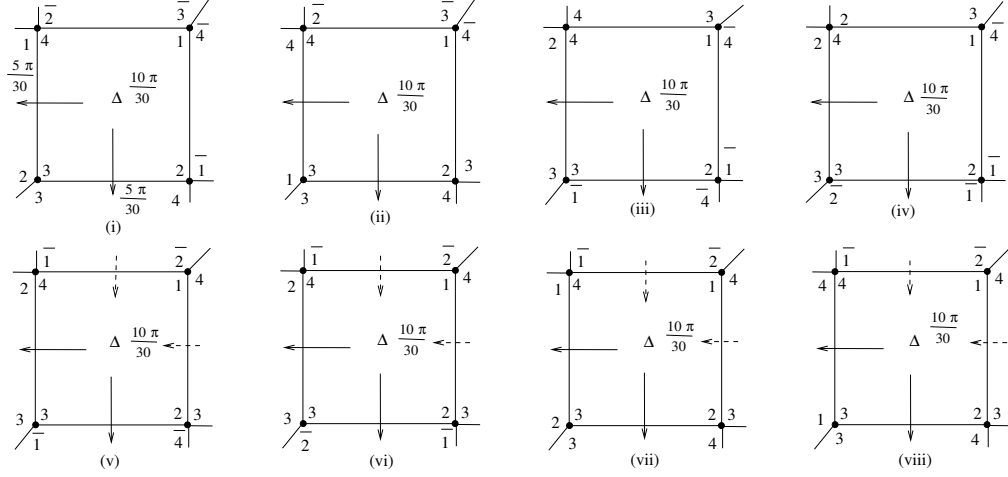
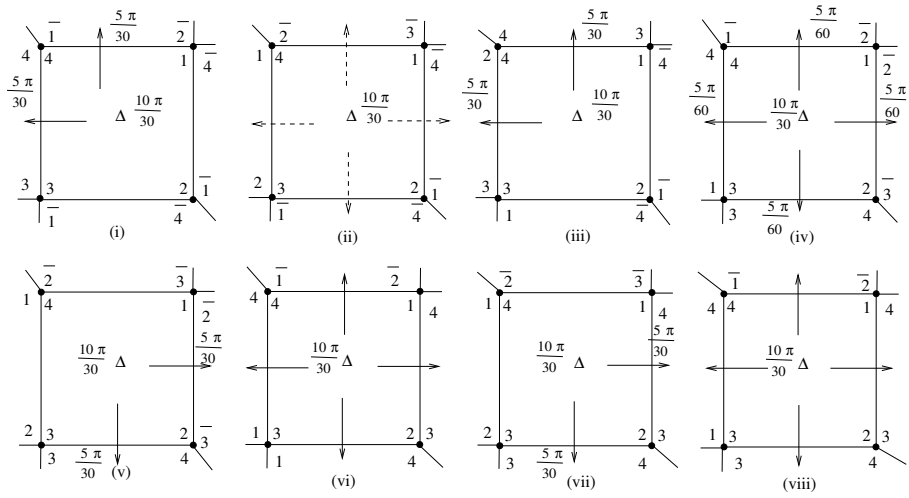
v_1	v_3	v_1	v_3	v_1	v_3	v_1	v_3
$\bar{3}1\bar{4}$	332	$\bar{3}1\bar{4}$	331	$\bar{3}1\bar{4}$	$\bar{1}33$	$\bar{3}1\bar{4}$	$\bar{2}33$
$\bar{2}14$	$\bar{1}33$	$\bar{2}14$	$\bar{2}33$	$\bar{2}14$	332	$\bar{2}14$	331

 Table 4.1: $d(v_1) = d(v_3) = 3$

$l(v_1) \in \{\bar{3}1\bar{4}, 31\bar{4}, \bar{2}14\}$; if $d(v_2) = 3$ then $l(v_2) \in \{\bar{1}2\bar{4}, \bar{3}24, 324\}$; if $d(v_3) = 3$ then $l(v_3) \in \{\bar{1}33, \bar{2}33, 332, 331\}$ and if $d(v_4) = 3$ then $l(v_4) \in \{14\bar{2}, 244, 44\bar{1}\}$.

Next assume that exactly one $d(u_i) \geq 3$. First let $d(u_1) \geq 3$. Then this forces

v_2	v_4	v_2	v_4	v_2	v_4	v_2	v_4
$\overline{124}$	$44\overline{1}$	$\overline{124}$	$14\overline{2}$	$\overline{124}$	244	$\overline{324}$	$44\overline{1}$
$\overline{324}$	$14\overline{2}$	$\overline{324}$	$44\overline{1}$	$\overline{324}$	$14\overline{2}$	$\overline{324}$	$44\overline{1}$

 Table 4.2: $d(v_1) = d(v_3) = 3$

 Figure 4.62: $d(u_i) = 2$ for all $1 \leq i \leq 4$

 Figure 4.63: $d(u_i) = 2$ for all $1 \leq i \leq 4$

$l(v_1)$ and $l(v_2)$ to involve τ -powers. Suppose $d(v_i) = 3$ for all $1 \leq i \leq 4$. Then checking all possible labellings shows that $l(v_1) = 14k$, $l(v_2) = \overline{124}$, $l(v_3) = \overline{133}$ and $l(v_4) = 24k'$. Suppose now that exactly one vertex of Δ is of degree ≥ 4 . If $d(v_1) \geq 4$ then either $l(v_2) = \overline{124}$, $l(v_3) = \overline{133}$ and $l(v_4) = 24k'$; or $l(v_2) = \overline{324}$,

$l(v_3) = 331$ and $l(v_4) = 44k'$. Whereas if $d(v_2) \geq 4$ then $l(v_1) = \bar{14}k$ and either $l(v_3) \in \{\bar{1}33, \bar{2}33\}$ and $l(v_4) = 24k'$; or $l(v_3) = 331$ and $l(v_4) = 44k'$. In addition if $d(v_3) \geq 4$ then $l(v_1) = \bar{14}k$, $l(v_2) = \bar{1}2\bar{4}$ and $l(v_4) \in \{24k', 44k'\}$ while if $d(v_4) \geq 4$ then $l(v_1) = \bar{14}k$, $l(v_2) = \bar{1}2\bar{4}$ and $l(v_3) = \bar{1}33$, see Figure 4.64.

Next let $d(u_2) \geq 3$. Then $l(v_1)$ and $l(v_2)$ involve τ -powers. Suppose $d(v_i) = 3$

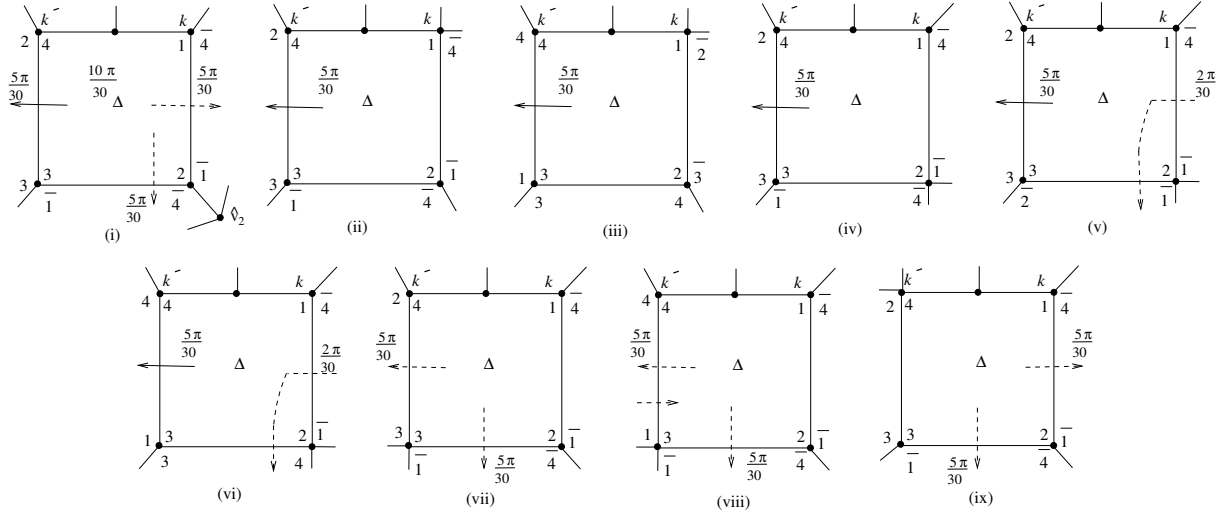
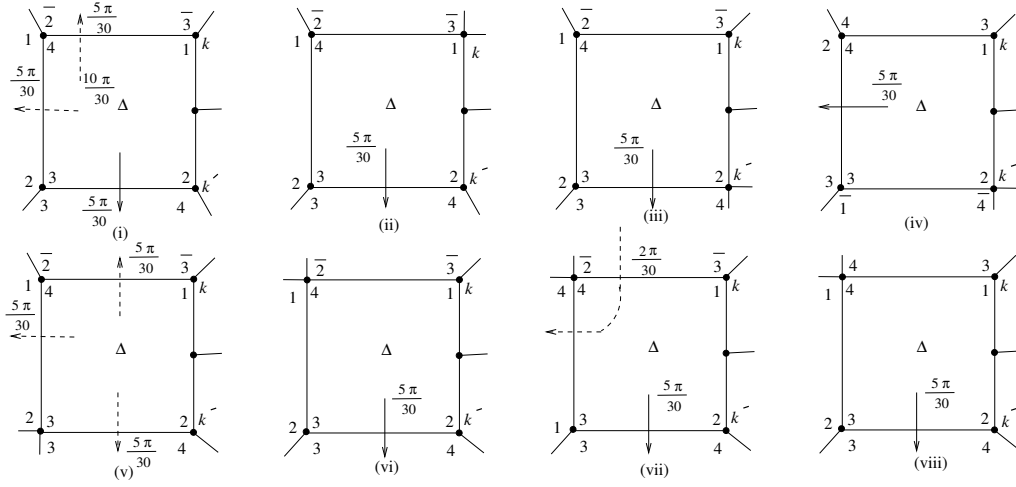


Figure 4.64: $d(u_1) \geq 3$

for all $1 \leq i \leq 4$. Then checking all the possible labellings shows that $l(v_1) = \bar{3}1k$, $l(v_2) = 24k'$, $l(v_3) = 332$ and $l(v_4) = 4\bar{2}1$. Suppose now that exactly one vertex of Δ is of degree ≥ 4 . If $d(v_1) \geq 4$ then $l(v_2) = 24k'$, $l(v_3) = 332$ and $l(v_4) = 4\bar{2}1$ while if $d(v_2) \geq 4$ then either $l(v_1) = \bar{3}1k$, $l(v_3) = 332$ and $l(v_4) = 4\bar{2}1$; or $l(v_1) = 31k$, $l(v_3) = \bar{1}33$ and $l(v_4) = 442$. Also, if $d(v_3) \geq 4$ then $l(v_1) = \bar{3}1k$, $l(v_2) = 24k'$ and $l(v_4) = 4\bar{2}1$ while if $d(v_4) \geq 4$ then either $l(v_1) = \bar{3}1k$, $l(v_2) = 24k'$ and $l(v_3) \in \{332, 331\}$; or $l(v_1) = 31k$, $l(v_2) = 24k'$ and $l(v_4) = 332$, see Figure 4.65.

Next let $d(u_3) \geq 3$. Then $l(v_2)$ and $l(v_3)$ involve τ -powers. If $d(v_i) = 3$ for all $1 \leq i \leq 4$ then $l(v_1) = \bar{2}14$, $l(v_2) = 32k$, $l(v_3) = 31k'$ and $l(v_4) = 44\bar{1}$. If $d(v_1) \geq 4$ then either $l(v_2) = \bar{3}2k$, $l(v_3) = 32k'$ and $l(v_4) = 14\bar{2}$; or $l(v_2) = \bar{3}2k$, $l(v_3) = 33k'$ and $l(v_4) = 244$; or $l(v_2) = \bar{3}2k$, $l(v_3) = 31k'$ and $l(v_4) = 44\bar{1}$; or $l(v_2) = 32k$, $l(v_3) = 32k'$ and $l(v_4) = 14\bar{2}$; or $l(v_2) = 32k$, $l(v_3) = 33k'$ and $l(v_4) = 244$; or $l(v_2) = 32k$, $l(v_3) = 31k'$ and $l(v_4) = 44\bar{1}$. If $d(v_2) \geq 4$ then either $l(v_1) = \bar{2}14$, $l(v_3) = 31k'$ and $l(v_4) = 44\bar{1}$; or $l(v_1) = \bar{3}1\bar{4}$, $l(v_3) = 32k'$ and $l(v_4) = 14\bar{2}$; or $l(v_1) = 31\bar{4}$, $l(v_3) = 33k'$ and $l(v_4) = 244$. If $d(v_3) \geq 4$ then

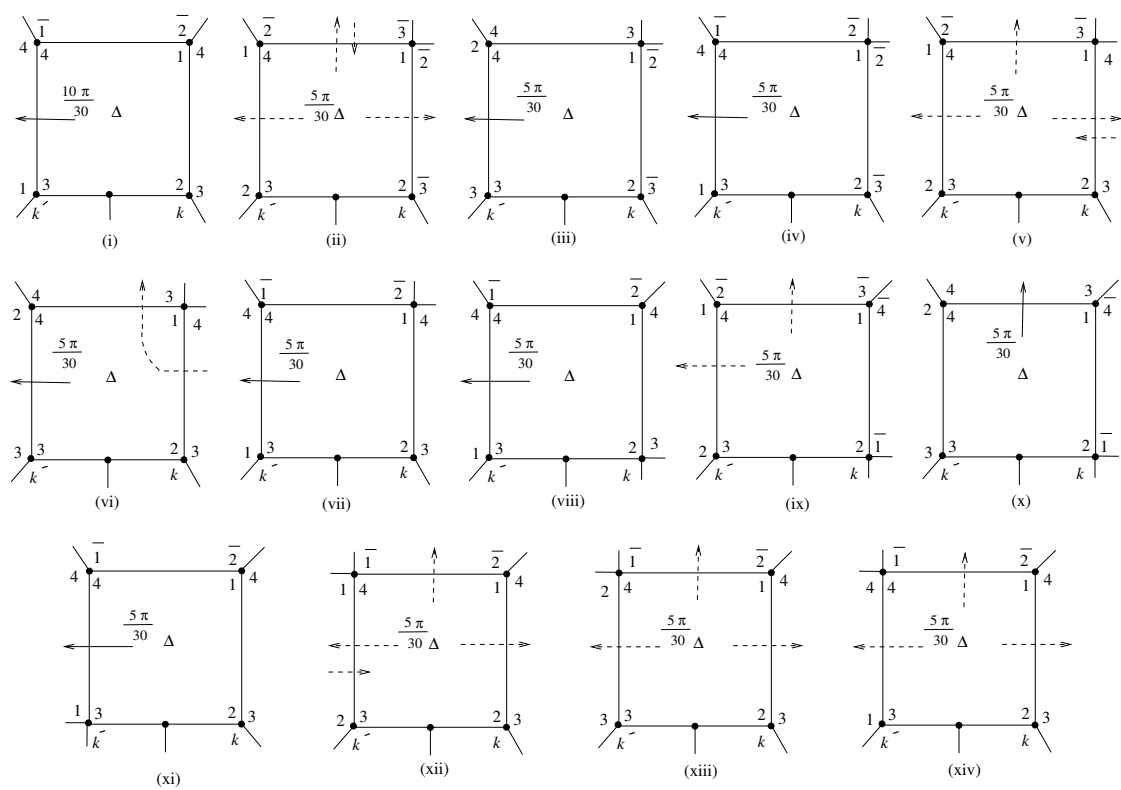
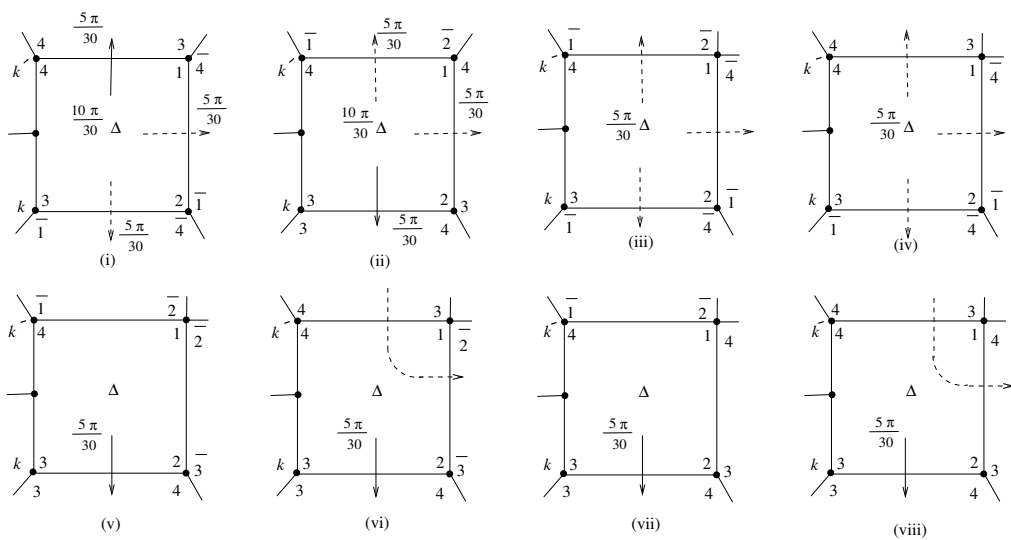
Figure 4.65: $d(u_2) \geq 3$

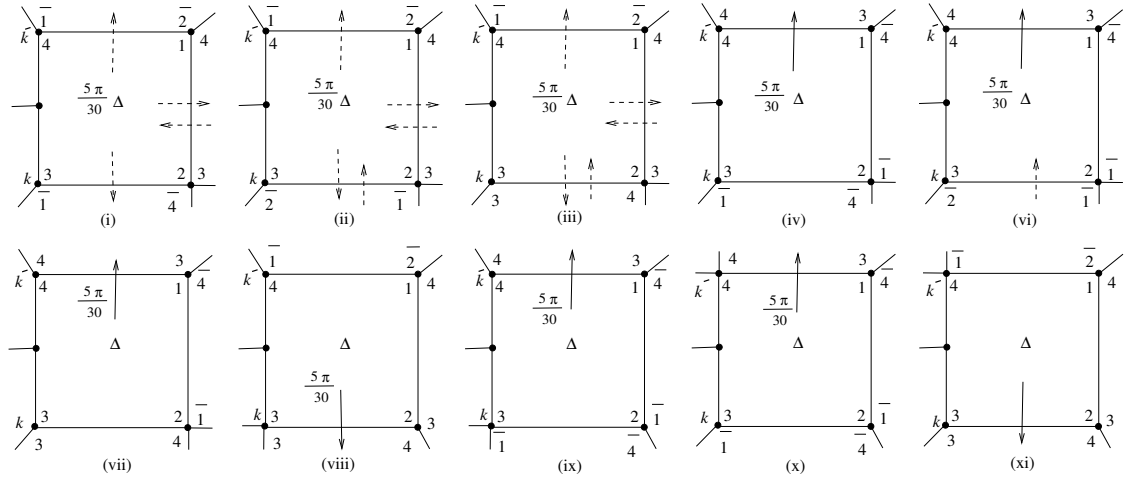
$l(v_1) = \bar{2}14$, $l(v_2) = 32k$ and $l(v_4) = 44\bar{1}$. While if $d(v_4) \geq 4$ then $l(v_1) = \bar{2}14$, $l(v_2) = 32k$ and $l(v_3) \in \{32k', 33k', 31k'\}$, see Figures 4.66.

Finally let $d(u_4) \geq 3$. Then $l(v_3)$ and $l(v_4)$ involve τ -powers and if $d(v_i) = 3$ for all $1 \leq i \leq 4$ then either $l(v_1) = 31\bar{4}$, $l(v_2) = \bar{1}2\bar{4}$, $l(v_3) = \bar{1}3k$ and $l(v_4) = 44k'$ or $l(v_1) = \bar{2}14$ and $l(v_2) = 324$, $l(v_3) = 33k$ and $l(v_4) = 4\bar{1}k'$. If $d(v_1) \geq 4$ then either $l(v_2) = \bar{1}2\bar{4}$, $l(v_3) = \bar{1}3k$ and $l(v_4) \in \{4\bar{1}k', 44k'\}$; or $l(v_2) \in \{\bar{3}24, 324\}$, $l(v_3) = 33k$ and $l(v_4) \in \{4\bar{1}k', 44k'\}$. If $d(v_2) \geq 4$ then either $l(v_1) = \bar{2}14$, $l(v_3) \in \{\bar{1}3k, \bar{2}3k, 33k\}$ and $l(v_4) = 4\bar{1}k'$; or $l(v_2) = 31\bar{4}$, $l(v_3) \in \{\bar{1}3k, \bar{2}3k, 33k\}$ and $l(v_4) = 44k'$. If $d(v_3) \geq 4$ then $l(v_1) = \bar{2}14$, $l(v_2) = 324$ and $l(v_4) = 4\bar{1}k'$ or $l(v_1) = 31\bar{4}$, $l(v_2) = \bar{1}2\bar{4}$ and $l(v_4) = 44k'$. While if $d(v_4) \geq 4$ then either $l(v_1) = 31\bar{4}$, $l(v_2) = \bar{1}2\bar{4}$ and $l(v_3) = \bar{1}3k'$ or $l(v_1) = \bar{2}14$, $l(v_2) = 324$ and $l(v_3) = 33k'$, see Figures 4.67 and 4.68.

This complete the description of positive regions in this case. Note that when $d(u_i) = 2$, $d(u_{i+1}) = 2$, $1 \leq i \leq 4$ then $d(v_i) = 3$ implies that $l(v_i)$ forces at least 2 splits on the adjacent edges as in Figure 4.1(ii) **except when** $l(v_i) = \bar{2}14$. We will use this fact in what follows often without explicit mention. Distribute $c(\Delta)$ as shown in Figures 4.59- 4.68. We consider each Δ case by case.

Δ is given by Figure 4.59(i):


 Figure 4.66: $d(u_3) \geq 3$

 Figure 4.67: $d(u_4) \geq 3$

Figure 4.68: $d(u_4) \geq 3$

Here $l(v_1)$ forces at least two splits on each of $\hat{\Delta}_1$ and $\hat{\Delta}_2$ and $l(v_3)$ forces at least two splits on each of $\hat{\Delta}_3$ and $\hat{\Delta}_4$ (this follows from the note made immediately above). Moreover letting $d(v_4) = 4$ forces $l(v_4) = 244k$ which in turn forces at least one more split on each of $\hat{\Delta}_1$ and $\hat{\Delta}_4$. This implies that $c(\hat{\Delta}_1) \leq c(3, 3, 3, 3, 3, 3, 4) = \frac{-15\pi}{30}$ and $c(\hat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 3, 4) = \frac{-15\pi}{30}$. Add $c(\Delta) \leq \frac{15\pi}{30}$ to $c(\hat{\Delta}_1)$ and $c(\hat{\Delta}_2)$. Now let $d(v_4) \geq 5$. Then $c(\Delta) \leq \frac{12\pi}{30}$, $c(\hat{\Delta}_1) \leq c(3, 3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$ and $c(\hat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$. Add a third of $c(\Delta)$ ($\leq \frac{4\pi}{30}$) to each of $c(\hat{\Delta}_1)$ and $c(\hat{\Delta}_4)$ and send the remaining $\leq \frac{4\pi}{30}$ to $\hat{\Delta}_2$ or $\hat{\Delta}_3$ as follows. If $d(\hat{v}_2) = 3$ then $l(\hat{v}_2) = 2\bar{3}k$ which in turn forces another split on $\hat{\Delta}_2$ and add the remaining to $c(\hat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$; if $d(\hat{v}_2) = 4$ then $l(\hat{v}_2)$ either forces at least one split on $\hat{\Delta}_2$ and add the remaining to $c(\hat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 3, 4) = \frac{-15\pi}{30}$ or it forces at least one split on $\hat{\Delta}_3$ and add the remainder to $c(\hat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 3, 4) = \frac{-15\pi}{30}$; otherwise $d(\hat{v}_2) \geq 5$ and add the remainder to $c(\hat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$. We note here that once all the distribution of curvature has been completed, checking all the possible labellings for $d(v_4) \in \{4, 5\}$ shows that $\hat{\Delta}_1$ and $\hat{\Delta}_4$ do not receive any positive curvature through their edges $(4, 1)$ and $(1, 2)$ respectively.

Δ is given by Figure 4.59(ii):

Letting $d(v_1) = 4$ forces $l(v_4) = 31\bar{4}k$ which in turn forces at least one more split on $\hat{\Delta}_1$. Moreover $c(\hat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-20\pi}{30}$ since each of $l(v_3)$ and $l(v_4)$

forces at least 2 splits on $\hat{\Delta}_4$. Add $\frac{10\pi}{30}$ of $c(\Delta)$ to $c(\hat{\Delta}_4)$ and add the remaining, which is $\leq \frac{5\pi}{30}$ to $c(\hat{\Delta}_1) \leq c(3, 3, 3, 3, 3, 3, 4) = \frac{-15\pi}{30}$ when $d(v_1) = 4$; and $\leq \frac{2\pi}{30}$ to $c(\hat{\Delta}_1) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$ when $d(v_1) \geq 5$.

Δ is given by Figure 4.59(iii):

Let $d(v_1) = 4$. Then this forces $l(v_4) = \bar{3}1\bar{2}k$ which in turn forces at least one more split on $\hat{\Delta}_2$. Moreover $c(\hat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-20\pi}{30}$ since each of $l(v_2)$ and $l(v_3)$ forces at least 2 splits on $\hat{\Delta}_3$. Add $\frac{10\pi}{30}$ of $c(\Delta)$ to $c(\hat{\Delta}_3)$ and add the remaining curvature, which is $\leq \frac{5\pi}{30}$ to $c(\hat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 3, 4) = \frac{-15\pi}{30}$ when $d(v_1) = 4$; and $\leq \frac{2\pi}{30}$ to $c(\hat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$ when $d(v_1) \geq 5$.

Δ is given by Figure 4.59(iv):

First let $d(v_2) = 4$. Then this forces $l(v_2) = \bar{1}24k$ which in turn forces at least one more split on each of $\hat{\Delta}_2$ and $\hat{\Delta}_3$. This implies that $c(\hat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 3, 4) = \frac{-15\pi}{30}$ and $c(\hat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 3, 4) = \frac{-15\pi}{30}$. Add $c(\Delta) \leq \frac{15\pi}{30}$ to $c(\hat{\Delta}_2)$ and $c(\hat{\Delta}_3)$. Now let $d(v_2) \geq 5$. Then $c(\Delta) \leq \frac{12\pi}{30}$, $c(\hat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$ and $c(\hat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$. Add a third of $c(\Delta)$ ($\leq \frac{4\pi}{30}$) to each of $c(\hat{\Delta}_2)$ and $c(\hat{\Delta}_3)$ and add the remainder $\leq \frac{4\pi}{30}$ to $c(\hat{\Delta}_1)$ or $c(\hat{\Delta}_4)$ as follows. If $l(\hat{v}_4) = 4\bar{1}3$ then it forces at least two splits on $\hat{\Delta}_4$ and we add the remaining curvature to $c(\hat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-20\pi}{30}$. If $d(\hat{v}_4) = 4$ then it is either forces at least one more split on $\hat{\Delta}_4$ and we add the remaining curvature to $c(\hat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 3, 4) = \frac{-15\pi}{30}$; or forces at least one more split on $\hat{\Delta}_1$ and we add the remaining curvature to $c(\hat{\Delta}_1) \leq c(3, 3, 3, 3, 3, 3, 4) = \frac{-15\pi}{30}$. Otherwise $d(\hat{v}_4) \geq 5$ and $c(\hat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$ and we add the remaining curvature to $c(\hat{\Delta}_4)$. We note here that once all the distribution of curvature has been completed, checking all the possible labellings for $d(v_2) \in \{4, 5\}$ shows that $\hat{\Delta}_2$ and $\hat{\Delta}_3$ do not receive positive curvature through their edges $(1, 2)$ and $(4, 1)$ respectively.

Δ is given by Figure 4.59(v):

If $d(v_3) = 4$ then this implies $l(v_3) = 331k$ which forces at least one more split on each of $\hat{\Delta}_3$ and $\hat{\Delta}_4$ and so $c(\hat{\Delta}_3) \leq \frac{-15\pi}{30}$ and $c(\hat{\Delta}_4) \leq \frac{-15\pi}{30}$. Add $c(\Delta) \leq \frac{15\pi}{30}$ to $c(\hat{\Delta}_3)$ and $c(\hat{\Delta}_4)$. If $d(v_3) \geq 5$ then $c(\Delta) \leq \frac{12\pi}{30}$, $c(\hat{\Delta}_3) \leq \frac{-8\pi}{30}$ and $c(\hat{\Delta}_4) \leq \frac{-8\pi}{30}$ and we add a third ($\leq \frac{4\pi}{30}$) of $c(\Delta)$ to each of $c(\hat{\Delta}_3)$ and $c(\hat{\Delta}_4)$ and add the remaining curvature ($\leq \frac{4\pi}{30}$) to $c(\hat{\Delta}_1)$ if $d(\hat{v}_1) = 3$ (which implies $l(\hat{v}_1) = \bar{3}1k$

and this forces at least one further split on $\widehat{\Delta}_1$) or if $d(\widehat{v}_1) = 4$ with at least one split on $\widehat{\Delta}_1$ or $d(\widehat{v}_1) \geq 5$; otherwise, that is $d(\widehat{v}_1) = 4$ with at least one split on $\widehat{\Delta}_2$, add the remaining curvature to $c(\widehat{\Delta}_2)$. We note here that once all the distribution of curvature has been completed, checking all the possible labellings for $d(v_2) \in \{4, 5\}$ shows that $\widehat{\Delta}_3$ and $\widehat{\Delta}_4$ do not receive positive curvature through their edges $(2, 3)$ and $(1, 2)$ respectively.

Δ is given by Figure 4.59(vi):

Let $d(v_3) = 4$. Then $l(v_3) = \overline{1}32\overline{1}$ otherwise there is a contradiction. Moreover letting $d(\widehat{v}_2) = 3$ forces $l(\widehat{v}_2) \in \overline{2}3\{\overline{2}, \overline{3}, k\}$ which in turn forces at least one split on $\widehat{\Delta}_2$ or $\widehat{\Delta}_3$, see Figures 4.69(i)-(iii); and letting $d(\widehat{v}_2) = 4$ forces either at least one split on $\widehat{\Delta}_2$ or on $\widehat{\Delta}_3$ or $l(\widehat{v}_2) \in \overline{2}3\{\overline{3}4, 11\}$, see Figures 4.69(iv)-(vi). In addition $d(\widehat{v}_4) = 3$ forces $l(\widehat{v}_4) = 4\overline{1}3$ and $d(\widehat{v}_4) = 4$ forces either at least one split on $\widehat{\Delta}_4$ or on $\widehat{\Delta}_1$ or $l(\widehat{v}_2) = 4\overline{1}22$, see Figures 4.69(vii)-(ix).

First assume $l(v_3) = \overline{1}32\overline{1}$, see Figure 4.70(i). Then this forces at least two

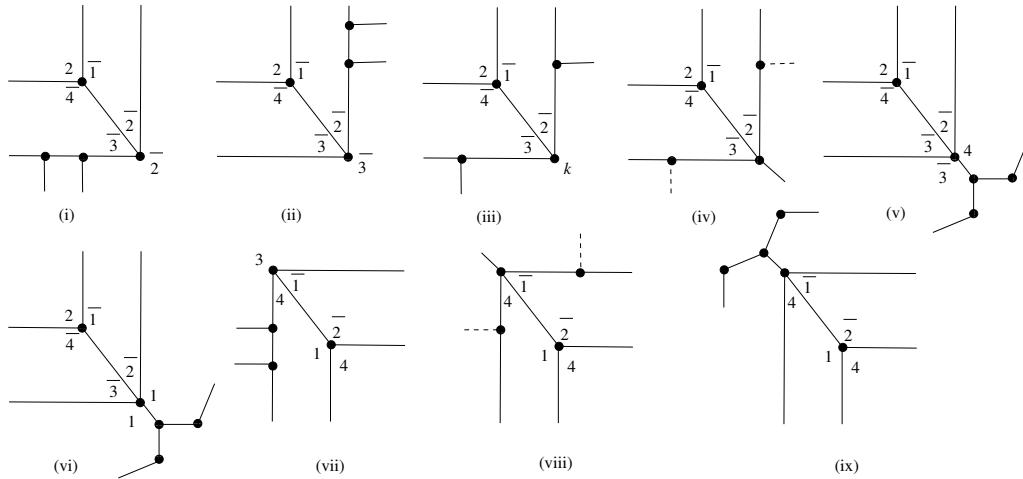


Figure 4.69: $\{d(\widehat{v}_2), d(\widehat{v}_4)\} \in \{3, 4\}$

splits on $\widehat{\Delta}_3$ and $c(\Delta) \leq c(3, 3, 3, 4) = \frac{15\pi}{30}$. If in addition $d(\widehat{v}_2) = 3$ then $c(\widehat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$ and $c(\widehat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$ and we add $c(\Delta) \leq \frac{15\pi}{30}$ to $c(\widehat{\Delta}_2)$ and $c(\widehat{\Delta}_3)$. Now let $d(\widehat{v}_2) = 4$. If this forces at least one split on $\widehat{\Delta}_2$ then $c(\widehat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 3, 4) = \frac{-15\pi}{30}$ and so add $c(\Delta)$ to $c(\widehat{\Delta}_2)$; otherwise if $d(\widehat{v}_2) = 4$ and forces at least one split on $\widehat{\Delta}_3$ then $c(\widehat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 4, 4) = \frac{-20\pi}{30}$ and add $c(\Delta)$ to $c(\widehat{\Delta}_3)$; while if $l(\widehat{v}_2) \in \overline{2}3\{\overline{3}4, 11\}$ then

$c(\hat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$ and $c(\hat{\Delta}_3) \leq c(3, 3, 3, 3, 4, 4) = \frac{-10\pi}{30}$ and we add $\frac{5\pi}{30}$ of $c(\Delta) \leq \frac{15\pi}{30}$ to $c(\hat{\Delta}_2)$ and we add the remaining $\frac{10\pi}{30}$ to $c(\hat{\Delta}_3)$. Now let $d(\hat{v}_2) \geq 5$. Then $c(\hat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$ and $c(\hat{\Delta}_3) \leq c(3, 3, 3, 3, 4, 5) = \frac{-13\pi}{30}$ we add $\frac{10\pi}{30}$ of $c(\Delta)$ to $c(\hat{\Delta}_3)$ and the remainder $\frac{5\pi}{30}$ to $c(\hat{\Delta}_2)$.

Now assume $d(v_3) = 5$. Then $c(\Delta) \leq c(3, 3, 3, 5) = \frac{12\pi}{30}$. Checking all possible labellings shows that $l(v_3)$ forces at least one split on $\hat{\Delta}_3$ or on $\hat{\Delta}_4$. If $l(v_3)$ forces at least 2 splits on $\hat{\Delta}_3$ then the distribution is exactly as above.

Now let $l(v_3)$ force exactly one split on $\hat{\Delta}_3$, see Figure 4.70(ii). Suppose $d(\hat{v}_2) = 3$. Then either $l(\hat{v}_2)$ forces at least 2 splits on $\hat{\Delta}_2$ and we add $c(\Delta)$ to $c(\hat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 3, 3, 3) = \frac{-20\pi}{30}$; or $l(\hat{v}_2) = \bar{2}\bar{3}k$ which forces $c(\hat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$ and $c(\hat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$ and we add $c(\Delta)$ to $c(\hat{\Delta}_2)$ and $c(\hat{\Delta}_3)$. Now suppose $d(\hat{v}_2) = 4$. Then as above $l(\hat{v}_2)$ either forces at least one split on $\hat{\Delta}_2$ and we add $c(\Delta)$ to $c(\hat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 3, 4) = \frac{-15\pi}{30}$; or it forces at least one split one $\hat{\Delta}_3$ and we add $c(\Delta)$ to $c(\hat{\Delta}_3) \leq c(3, 3, 3, 3, 4, 5) = \frac{-13\pi}{30}$; or $l(\hat{v}_2) \in \bar{2}\bar{3}\{\bar{3}4, 11\}$ and so $c(\hat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$ and $c(\hat{\Delta}_3) \leq c(3, 3, 3, 4, 5) = \frac{-3\pi}{30}$. Add $\frac{5\pi}{30}$ of $c(\Delta) \leq \frac{12\pi}{30}$ to $c(\hat{\Delta}_2)$ and $\frac{3\pi}{30}$ to $c(\hat{\Delta}_3)$ and we send the remaining curvature $\leq \frac{4\pi}{30}$ to $c(\hat{\Delta}_1)$ or $c(\hat{\Delta}_4)$ as follows. If $d(\hat{v}_4) = 3$ then $l(\hat{v}_4) = 4\bar{1}\bar{3}$ and $c(\hat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$ and we add the remaining $\leq \frac{4\pi}{30}$ of $c(\Delta)$ to $c(\hat{\Delta}_4)$. Whereas if $d(\hat{v}_4) = 4$ then $c(\hat{\Delta}_1) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$ and we add the remaining $\leq \frac{4\pi}{30}$ of $c(\Delta)$ to $c(\hat{\Delta}_1)$. If $d(\hat{v}_4) \geq 5$ then add the remaining $\leq \frac{4\pi}{30}$ to $c(\hat{\Delta}_1) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$. Next suppose $d(\hat{v}_2) \geq 5$. Then $c(\hat{\Delta}_2) \leq \frac{-8\pi}{30}$ and $c(\hat{\Delta}_3) \leq \frac{-6\pi}{30}$ and we add $\frac{5\pi}{30}$ of $c(\Delta)$ to $c(\hat{\Delta}_2)$ and $\frac{3\pi}{30}$ of $c(\Delta)$ to $c(\hat{\Delta}_3)$ and add the remainder to $c(\hat{\Delta}_1)$ or $c(\hat{\Delta}_4)$ as above. Now suppose $l(v_3)$ does not force any splits on $\hat{\Delta}_3$. Then it forces at least one split on $\hat{\Delta}_4$. First let $l(v_3)$ forces at least 2 split on $\hat{\Delta}_4$, see Figure 4.70(iii). Then we add $c(\Delta)$ to $c(\hat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 3, 5) = \frac{-18\pi}{30}$ when either $l(\hat{v}_4) = 4\bar{1}\bar{3}$ or $d(\hat{v}_4) = 4$ with at least one split on $\hat{\Delta}_4$ while we add $c(\Delta)$ to $c(\hat{\Delta}_1) \leq c(3, 3, 3, 3, 3, 3, 4) = \frac{-15\pi}{30}$ when $d(\hat{v}_4) = 4$ with at least one split on $\hat{\Delta}_1$. Otherwise either $l(\hat{v}_4) = 4\bar{1}\bar{2}\bar{2}$ and we add $c(\Delta)$ to $c(\hat{\Delta}_4) \leq c(3, 3, 3, 3, 4, 5) = \frac{-13\pi}{30}$ or $d(\hat{v}_4) \geq 5$ and we add $c(\Delta)$ to $c(\hat{\Delta}_4) \leq c(3, 3, 3, 3, 5, 5) = \frac{-16\pi}{30}$. Now let $l(v_3)$ force exactly one split on $\hat{\Delta}_4$, see Figure 4.70(iv). We use a similar argument to the above applied to $\hat{v}_4$, $\hat{\Delta}_4$ and $\hat{\Delta}_1$. Add $c(\Delta)$ to $c(\hat{\Delta}_4) \leq \frac{-13\pi}{30}$ when $d(\hat{v}_4) = 3$ and when $d(\hat{v}_4) = 4$ with at least one split on $\hat{\Delta}_4$; or add $c(\Delta)$ to $c(\hat{\Delta}_1) \leq \frac{-15\pi}{30}$ when $d(\hat{v}_4) = 4$ with at least one split on $\hat{\Delta}_1$; or add $\frac{5\pi}{30}$ of $c(\Delta)$

to $c(\hat{\Delta}_1) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$ and $\frac{3\pi}{30}$ to $c(\hat{\Delta}_4) \leq c(3, 3, 3, 4, 5) = \frac{-3\pi}{30}$ when $l(\hat{v}_4) = 4\bar{1}22$ and send the remaining $\frac{4\pi}{30}$ to $c(\hat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$ (note that $d(\hat{v}_2) \geq 4$ or it forces at least one split on $\hat{\Delta}_2$); or add $\frac{5\pi}{30}$ of $c(\Delta)$ to $c(\hat{\Delta}_1) \leq \frac{-8\pi}{30}$ and $\frac{3\pi}{30}$ of $c(\Delta)$ to $c(\hat{\Delta}_4) \leq \frac{-6\pi}{30}$ when $d(\hat{v}_4) \geq 5$ and add the remaining to $c(\hat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$.

Now assume $d(v_3) \geq 6$, see Figure 4.70(v). Then $c(\Delta) \leq c(3, 3, 3, 6) = \frac{10\pi}{30}$. Using the same above argument we add $\frac{5\pi}{30}$ of $c(\Delta)$ to $c(\hat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$ and add the remaining $\frac{5\pi}{30}$ either to $c(\hat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$ when $l(\hat{v}_4) = 34\bar{1}$; or to $c(\hat{\Delta}_1) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$ when $d(\hat{v}_4) \geq 4$.

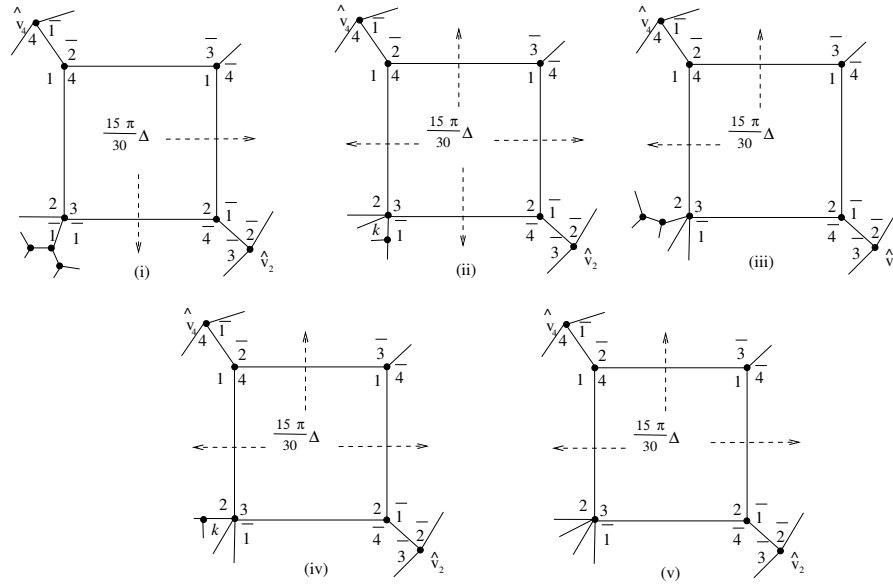


Figure 4.70: $l(v_2) = l(v_4) = \bar{1}2\bar{4}$

Δ is given by Figure 4.59(vii):

Add $\frac{10\pi}{30}$ of $c(\Delta)$ to $c(\hat{\Delta}_1) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-20\pi}{30}$. If $d(v_3) = 4$ then $l(v_3) = \bar{1}33k$ and add the remaining, which is $\frac{5\pi}{30}$ to $c(\hat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 3, 4) = \frac{-15\pi}{30}$; or if $d(v_3) \geq 5$ then add the remaining $\frac{2\pi}{30}$ to $c(\hat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$.

Δ is given by Figure 4.60(i)-(ii):

Note that $d(\hat{v}_2) = 3$ implies $l(\hat{v}_2) \in \bar{2}\bar{3}\{2, \bar{3}, k\}$ which forces $d(\hat{\Delta}_2) \geq 7$. Thus $c(\hat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$. Also $c(\hat{\Delta}_1) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$. If $d(v_4) = 4$ then add $\frac{5\pi}{30}$ of $c(\Delta) \leq \frac{10\pi}{30}$ to each of $c(\hat{\Delta}_2)$ and $c(\hat{\Delta}_1)$. If $d(v_4) \geq 5$ then add $\frac{5\pi}{30}$

of $c(\Delta) \leq \frac{7\pi}{30}$ to $c(\widehat{\Delta}_2)$ and the remaining $\frac{2\pi}{30}$ to $c(\widehat{\Delta}_1)$.

Δ is given by Figure 4.60(iii):

Letting $d(\widehat{v}_1) = 3$ forces $l(\widehat{v}_2) \in \overline{31}\{1, \overline{3}, k\}$ which forces $d(\widehat{\Delta}_2) \geq 7$ and so $c(\widehat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 4) = \frac{-\pi}{6}$, $c(\widehat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 4) = \frac{-\pi}{6}$. If $d(v_3) = 4$ then add $\frac{5\pi}{30}$ of $c(\Delta) \leq \frac{10\pi}{30}$ to each of $c(\widehat{\Delta}_2)$ and $c(\widehat{\Delta}_3)$. If $d(v_3) \geq 5$ then add $\frac{5\pi}{30}$ of $c(\Delta) \leq \frac{7\pi}{30}$ to $c(\widehat{\Delta}_2)$ and the remaining $\frac{2\pi}{30}$ to $c(\widehat{\Delta}_3)$.

Δ is given by Figure 4.60(iv):

Here $d(\widehat{v}_2) = 3$ implies $l(\widehat{v}_1) \in \overline{32}\{\overline{1}, \overline{2}, \overline{4}, k\}$ which forces $d(\widehat{\Delta}_3) \geq 7$ and so $c(\widehat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 4) = \frac{-\pi}{6}$, $c(\widehat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 4) = \frac{-\pi}{6}$. If $d(v_4) = 4$ then add $\frac{5\pi}{30}$ of $c(\Delta) \leq \frac{10\pi}{30}$ to each of $c(\widehat{\Delta}_3)$ and $c(\widehat{\Delta}_4)$. If $d(v_4) \geq 5$ then add $\frac{5\pi}{30}$ of $c(\Delta) \leq \frac{7\pi}{30}$ to $c(\widehat{\Delta}_3)$ and the remaining $\frac{2\pi}{30}$ to $c(\widehat{\Delta}_4)$.

Δ is given by Figure 4.60(v)-(vi):

Add $c(\Delta) \leq \frac{\pi}{3}$ to $c(\widehat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 3, 3, 3) = \frac{-2\pi}{3}$.

Δ is given by Figure 4.61(i):

Observe that $d(\widehat{v}_4) = 3$ forces $l(\widehat{v}_4) \in 4\overline{1}\{\overline{1}, 3, \overline{3}, 4, k\}$ and $d(\widehat{v}_4) = 4$ forces either at least one split on $\widehat{\Delta}_1$ or on $\widehat{\Delta}_4$ or $l(\widehat{v}_4) \in 4\overline{1}\{22, 34, \overline{34}\}$. First let $d(v_1) = 4$. Then $c(\Delta) \leq \frac{\pi}{3}$. If $d(\widehat{v}_4) \geq 4$ or $l(\widehat{v}_4) \in 4\overline{1}\{\overline{1}, 3, \overline{3}, k\}$ then $c(\widehat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$ while if $l(\widehat{v}_4) = 4\overline{1}4$ then $c(\widehat{\Delta}_1) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$. In the former case add $\frac{5\pi}{30}$ of $c(\Delta)$ to $c(\widehat{\Delta}_4)$ and in the latter case to $c(\widehat{\Delta}_1)$ and add the remaining curvature, which is $\leq \frac{5\pi}{30}$ when $d(v_2) = 4$ and $\leq \frac{2\pi}{30}$ when $d(v_2) \geq 5$, to $c(\widehat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$ since $l(v_3)$ forces at least 2 splits on $\widehat{\Delta}_3$. Now let $d(v_1) \geq 5$. Then $c(\Delta) \leq \frac{7\pi}{30}$ and by exactly the same above argument we add $\frac{3\pi}{30}$ of $c(\Delta)$ to $c(\widehat{\Delta}_1)$ or to $c(\widehat{\Delta}_4)$ and $\leq \frac{4\pi}{30}$ to $c(\widehat{\Delta}_3)$ when $d(v_2) = 4$ whereas we add $\leq \frac{2\pi}{30}$ of $c(\Delta) \leq \frac{4\pi}{30}$ to $c(\widehat{\Delta}_1)$ or $c(\widehat{\Delta}_4)$ and add the remaining $\frac{2\pi}{30}$ to $c(\widehat{\Delta}_3)$ when $d(v_2) \geq 5$.

Δ is given by Figure 4.61(ii):

Add $c(\Delta) \leq \frac{10\pi}{30}$ to $c(\widehat{\Delta}_4) \leq \frac{-20\pi}{30}$.

Δ is given by Figure 4.61(iii):

Observe that $d(\widehat{v}_1) = 3$ implies $l(\widehat{v}_1) \in \overline{31}\{1, 2, k\}$ which forces $d(\widehat{\Delta}_1) \geq 7$ and so

$c(\widehat{\Delta}_1) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$. Add $\frac{5\pi}{30}$ of $c(\Delta)$ to $c(\widehat{\Delta}_1)$ and add the remaining to $c(\widehat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$.

Δ is given by Figure 4.61(iv):

As used in Figure 4.59(vi), either $c(\widehat{\Delta}_1) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$ or $c(\widehat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$. Add $\frac{5\pi}{30}$ of $c(\Delta)$ to $c(\widehat{\Delta}_1)$ or $c(\widehat{\Delta}_4)$ and add the remaining to $c(\widehat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$.

Δ is given by Figure 4.61(v):

Add $c(\Delta) \leq \frac{\pi}{3}$ to $c(\widehat{\Delta}_1) \leq \frac{-2\pi}{3}$.

Δ is given by Figure 4.62:

Note that $l(v_3)$ forces at least 2 splits on each of $\widehat{\Delta}_3$ and $\widehat{\Delta}_4$. First let $d(v_4) = 4$ and $d(v_2) = 4$. Then add $\frac{5\pi}{30}$ of $c(\Delta) \leq \frac{10\pi}{30}$ to each of $c(\widehat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 4) = \frac{-\pi}{6}$ and $c(\widehat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 4) = \frac{-\pi}{6}$; or if $d(v_4) \geq 5$ and $d(v_2) = 4$ then add $\frac{5\pi}{30}$ of $c(\Delta) \leq \frac{7\pi}{30}$ to $c(\widehat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$ and add the remaining $\frac{2\pi}{30}$ to $c(\widehat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$; or if $d(v_4) = 4$ and $d(v_2) \geq 5$ then add $\frac{5\pi}{30}$ of $c(\Delta) \leq \frac{7\pi}{30}$ to $c(\widehat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 4) = \frac{-\pi}{6}$ and add the remaining $\frac{2\pi}{30}$ to $c(\widehat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$; or if $d(v_4) \geq 5$ and $d(v_2) \geq 5$ then add $\frac{2\pi}{30}$ of $c(\Delta) \leq \frac{4\pi}{30}$ to each of $c(\widehat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 8) = \frac{-8\pi}{30}$ and $c(\widehat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 8) = \frac{-8\pi}{30}$. We note here that once all curvature distributions has been completed, Δ in Figures 4.62(v)-(viii) may receive extra $\frac{\pi}{30}$ from $\widehat{\Delta}_1$ when $d(v_4) \geq 5$ and $d(v_2) \geq 5$ [if $\widehat{\Delta}_1$ is Δ in Figure 4.78(vi)] and in this case we send $\leq \frac{3\pi}{30}$ of the curvature sum to $c(\widehat{\Delta}_4)$ and we send the remaining curvature $\leq \frac{2\pi}{30}$ to $c(\widehat{\Delta}_3)$. Observe further that again in Figures 4.62(v)-(viii) when $d(v_2) \geq 5$ then Δ may receive extra $\frac{2\pi}{30}$ from $\widehat{\Delta}_2$ [as in Figure 4.85(i)] and in this case $l(v_2)$ forces splits on $\widehat{\Delta}_3$ and we send $\leq \frac{5\pi}{30}$ of the curvature sum $\leq \frac{9\pi}{30}$ to $c(\widehat{\Delta}_3)$ and send the remaining curvature to $c(\widehat{\Delta}_4)$. Note that the last mention cases cannot occur in the same time.

Δ is given by Figures 4.63(i) and 4.63(iii):

Add $c(\Delta) \leq \frac{10\pi}{30}$ to $c(\widehat{\Delta}_1) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$ and $c(\widehat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$ when $d(v_1) = d(v_3) = 4$; or add $\frac{5\pi}{30}$ of $c(\Delta) \leq \frac{7\pi}{30}$ to $c(\widehat{\Delta}_1) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$ and add the remaining $\frac{2\pi}{30}$ to $c(\widehat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$ when $d(v_1) = 4$ only ; or add $\frac{5\pi}{30}$ of $c(\Delta) \leq \frac{7\pi}{30}$ to $c(\widehat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$ and add the

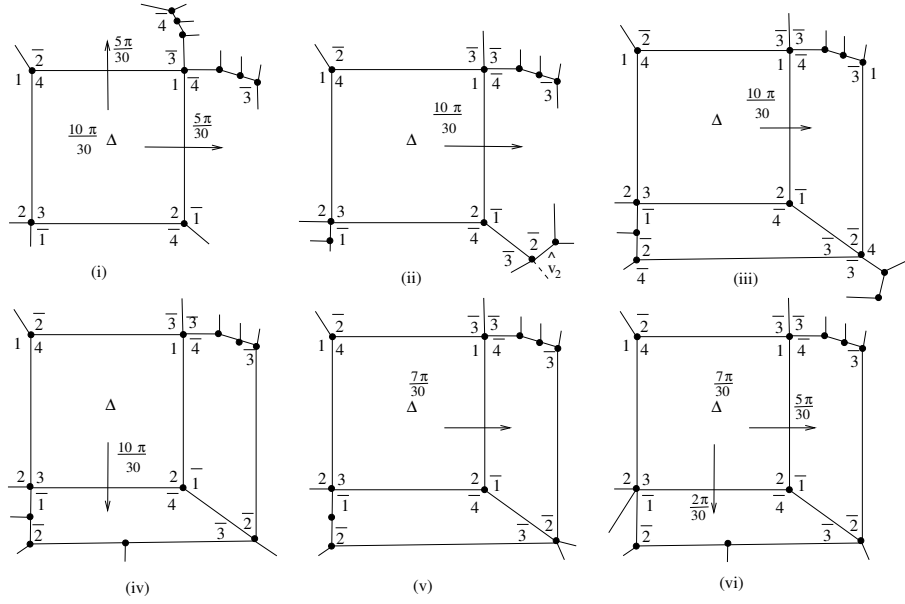
remaining $\frac{2\pi}{30}$ to $c(\hat{\Delta}_1) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$ when $d(v_3) = 4$ only; or add $c(\Delta) \leq \frac{4\pi}{30}$ to $c(\hat{\Delta}_1) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$ and $c(\hat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$ when both $d(v_1)$ and $d(v_3)$ are ≥ 5 .

Δ is given by Figure 4.63(ii):

First let $d(v_1) = 4$. Then $l(v_1) \in \{\bar{3}1\bar{4}1, \bar{3}1\bar{4}2, \bar{3}1\bar{4}\bar{3}, \bar{3}1\bar{4}\bar{4}, \bar{3}1\bar{4}k\}$ otherwise there is a contradiction. Assume $l(v_1) \in \bar{3}1\bar{4}\{1, 2\}$. Then this forces at least two splits on each of $\hat{\Delta}_1$ and $\hat{\Delta}_2$, so $c(\hat{\Delta}_1) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$ and $c(\hat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$. Add $c(\Delta) \leq \frac{\pi}{3}$ to $c(\hat{\Delta}_1)$ and $c(\hat{\Delta}_2)$, see Figure 4.71(i).

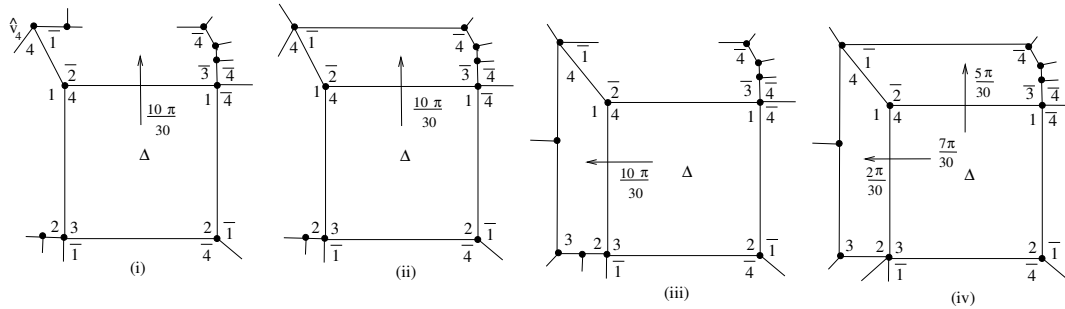
Now assume $l(v_1) = \bar{3}1\bar{4}\bar{3}$, see Figure 4.71(ii)-(vi). Then this forces at least two splits on $\hat{\Delta}_2$. Let $d(v_3) = 4$. Then $l(v_3) = \bar{1}32k$ and this in turn forces at least one split on $\hat{\Delta}_3$. If $d(\hat{v}_2) = 3$ then $l(\hat{v}_2) \in \bar{3}\bar{2}\{\bar{1}, \bar{2}, k\}$ and this in turn forces at least one split on each of $\hat{\Delta}_2$ and $\hat{\Delta}_3$, so $c(\hat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$. Add $c(\Delta) \leq \frac{\pi}{3}$ to $c(\hat{\Delta}_2)$, see Figure 4.71(ii). let $d(\hat{v}_2) = 4$. Then $l(\hat{v}_2)$ forces at least one split on $\hat{\Delta}_2$; or at least one split on $\hat{\Delta}_3$, otherwise $l(\hat{v}_2) = 4\bar{2}\bar{3}\bar{3}$. If $l(\hat{v}_2)$ forces at least one split on $\hat{\Delta}_2$ or if $l(\hat{v}_2) = 4\bar{2}\bar{3}\bar{3}$ then $c(\hat{\Delta}_2) \leq c(3, 3, 3, 3, 4, 4) = \frac{-10\pi}{30}$. Add $c(\Delta) \leq \frac{10\pi}{30}$ to $c(\hat{\Delta}_2)$, see Figure 4.71(ii)-(iii). If $l(\hat{v}_2)$ forces at least one split on $\hat{\Delta}_3$ then $c(\hat{\Delta}_3) \leq c(3, 3, 3, 3, 4, 4) = \frac{-10\pi}{30}$. Add $c(\Delta) \leq \frac{10\pi}{30}$ to $c(\hat{\Delta}_3)$, see Figure 4.71(iv). Now let $d(\hat{v}_2) \geq 5$. Then $c(\hat{\Delta}_2) \leq c(3, 3, 3, 3, 4, 5) = \frac{-16\pi}{30}$. Add $c(\Delta) \leq \frac{10\pi}{30}$ to $c(\hat{\Delta}_2)$, see Figure 4.71(v). Now let $d(v_3) \geq 5$. Then $c(\Delta) \leq \frac{7\pi}{30}$ and we have the same labelling possibilities for $d(\hat{v}_2)$ as above. Therefore we add $c(\Delta)$ to $c(\hat{\Delta}_2) \leq \frac{-10\pi}{30}$ when $l(\hat{v}_2)$ forces at least one split on $\hat{\Delta}_2$ or when $l(\hat{v}_2) = 4\bar{2}\bar{3}\bar{3}$; or when $d(\hat{v}_2) \geq 5$. Also when $d(\hat{v}_2) = 4$ and $l(\hat{v}_2)$ forces at least one split on $\hat{\Delta}_3$ we add $\frac{2\pi}{30}$ of $c(\Delta)$ to $c(\hat{\Delta}_3) \leq c(3, 3, 3, 4, 5) = \frac{-3\pi}{30}$ and we add the remaining $\frac{5\pi}{30}$ to $c(\hat{\Delta}_3) \leq c(3, 3, 3, 3, 4, 4) = \frac{-10\pi}{30}$, see Figure 4.71(vi).

Next assume $l(v_1) = \bar{3}1\bar{4}\bar{4}$, see Figure 4.72. Then this forces at least two splits on $\hat{\Delta}_1$. Let $d(v_3) = 4$. Then $l(v_3) = \bar{1}32k$ which forces at least one split on $\hat{\Delta}_3$. Let $d(\hat{v}_4) = 3$. Then $l(\hat{v}_2) = 4\bar{1}k$ and this forces at least one split on $\hat{\Delta}_1$, so $c(\hat{\Delta}_1) \leq c(3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$. Add $c(\Delta) \leq \frac{10\pi}{30}$ to $c(\hat{\Delta}_1)$, see Figure 4.72(i). Let $d(\hat{v}_4) = 4$. Then $l(\hat{v}_4)$ forces at least a split on $\hat{\Delta}_1$; or a split on $\hat{\Delta}_4$ otherwise $l(\hat{v}_4) \in 4\bar{1}\{22, 34\}$. If $l(\hat{v}_4)$ forces at least one split on $\hat{\Delta}_1$ or if $l(\hat{v}_4) \in 4\bar{1}\{22, 34\}$ then $c(\hat{\Delta}_1) \leq c(3, 3, 3, 3, 4, 4) = \frac{-10\pi}{30}$. Add $c(\Delta) \leq \frac{\pi}{3}$ to $c(\hat{\Delta}_1)$, see Figure 4.72(i)-(ii). If $l(\hat{v}_4)$ forces at least one split on $\hat{\Delta}_4$ then $c(\hat{\Delta}_4) \leq c(3, 3, 3, 3, 4, 4) = \frac{-10\pi}{30}$.

Figure 4.71: $l(v_2) = l(v_4) = \bar{2}14$.

Add $c(\Delta) \leq \frac{10\pi}{30}$ to $c(\hat{\Delta}_4)$, see Figure 4.72(iii). Now let $d(\hat{v}_4) \geq 5$. Then $c(\hat{\Delta}_1) \leq c(3, 3, 3, 3, 4, 5) = \frac{-16\pi}{30}$. Add $c(\Delta) \leq \frac{10\pi}{30}$ to $c(\hat{\Delta}_1)$. Now let $d(v_3) \geq 5$. Then $c(\Delta) \leq \frac{7\pi}{30}$ and we have the same labelling possibilities for $\hat{v}_4$ as above. Therefore add $c(\Delta)$ to $c(\hat{\Delta}_1) \leq \frac{-10\pi}{30}$ when $l(\hat{v}_4)$ forces at least one split on $\hat{\Delta}_1$; or when $l(\hat{v}_4) \in 4\bar{1}\{22, 34\}$; or when $d(\hat{v}_4) \geq 5$. When $d(\hat{v}_2) = 4$ and $l(\hat{v}_2)$ forces at least one split on $\hat{\Delta}_4$ we add $\frac{2\pi}{30}$ of $c(\Delta)$ to $c(\hat{\Delta}_4) \leq c(3, 3, 3, 4, 5) = \frac{-3\pi}{30}$ and we add the remaining curvature to $c(\hat{\Delta}_1) \leq c(3, 3, 3, 3, 4, 4) = \frac{-10\pi}{30}$, see Figure 4.72(iv).

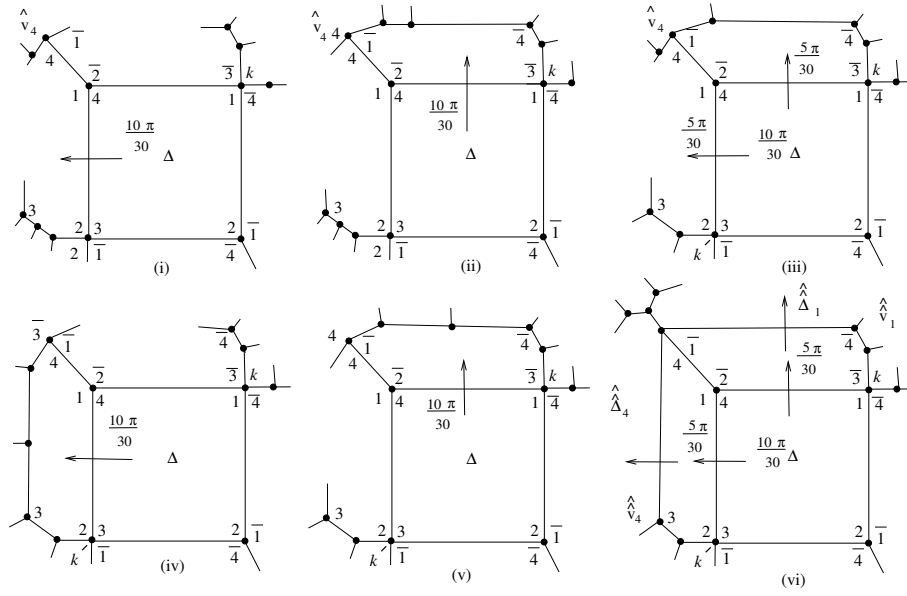
Finally when $d(v_1) = 4$ assume $l(v_1) = \bar{3}1\bar{4}k$, see Figures 4.73- 4.75. Then

Figure 4.72: $l(v_2) = l(v_4) = \bar{2}14$.

this forces at least one split on each of $\widehat{\Delta}_1$ and $\widehat{\Delta}_2$. Let $d(\mathbf{v}_3) = 4$. Then $l(\mathbf{v}_3) \in \overline{132}\{2, k'\}$ and $c(\Delta) = \frac{\pi}{3}$. First let $l(\mathbf{v}_3) = \overline{1322}$. Then this forces at least 2 splits on $\widehat{\Delta}_4$. Checking the possible labelling shows that $d(\widehat{\mathbf{v}}_4) = 3$ forces $l(\widehat{\mathbf{v}}_4) \in 4\overline{1}\{\overline{1}, 4, k\}$ and $d(\widehat{\mathbf{v}}_4) = 4$ forces at least one split on $\widehat{\Delta}_1$ or $\widehat{\Delta}_2$. If $l(\widehat{\mathbf{v}}_4) \in 4\overline{1}\{\overline{1}, k\}$ then this forces at least one split on $\widehat{\Delta}_4$ and $c(\widehat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 3, 4) = \frac{-15\pi}{30}$. Add $c(\Delta)$ to $c(\widehat{\Delta}_4)$, see Figure 4.73(i). If $l(\widehat{\mathbf{v}}_4) = 4\overline{1}4$ then this forces at least two splits on $\widehat{\Delta}_1$ and $c(\widehat{\Delta}_1) \leq c(3, 3, 3, 3, 3, 3, 4) = \frac{-15\pi}{30}$. Add $c(\Delta)$ to $c(\widehat{\Delta}_1)$, see Figure 4.73(ii). If $d(\widehat{\mathbf{v}}_4) = 4$ and it forces at least one split on $\widehat{\Delta}_1$ then $c(\widehat{\Delta}_1) \leq c(3, 3, 3, 3, 4, 4) = \frac{-10\pi}{30}$. Add $c(\Delta)$ to $c(\widehat{\Delta}_1)$. If $d(\widehat{\mathbf{v}}_4) = 4$ and it forces at least one split on $\widehat{\Delta}_4$ then $c(\widehat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 4, 4) = \frac{-20\pi}{30}$. Add $c(\Delta)$ to $c(\widehat{\Delta}_1)$. Finally if $d(\widehat{\mathbf{v}}_4) \geq 5$ then add $c(\Delta)$ to $c(\widehat{\Delta}_4) \leq c(3, 3, 3, 3, 4, 5) = \frac{-16\pi}{30}$.

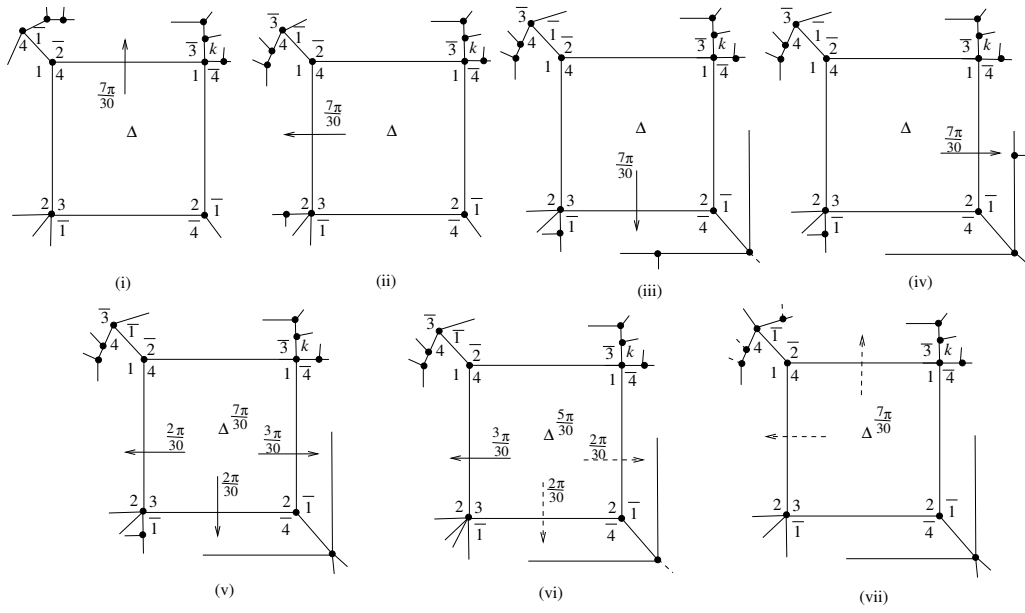
Next let $l(\mathbf{v}_3) = \overline{132k'}$, see Figure 4.73(iii)-(vi). Then this forces at least one split $\widehat{\Delta}_4$. If $d(\widehat{\mathbf{v}}_4) = 3$ then $l(\widehat{\mathbf{v}}_4) \in 4\overline{1}\{\overline{1}, \overline{3}, 4, k''\}$. Letting $l(\widehat{\mathbf{v}}_4) \in 4\overline{1}\{\overline{1}, k''\}$ forces at least one split on each of $\widehat{\Delta}_1$ and $\widehat{\Delta}_4$, so $c(\widehat{\Delta}_1) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$ and $c(\widehat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$. Add $c(\Delta)$ to $c(\widehat{\Delta}_3)$ and $c(\widehat{\Delta}_4)$, see Figure 4.73(iii). Whereas letting $l(\widehat{\mathbf{v}}_4) = 4\overline{1}\overline{3}$ forces at least two split on $\widehat{\Delta}_4$ and so $c(\widehat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 3, 4) = \frac{-15\pi}{30}$. Add $c(\Delta)$ to $c(\widehat{\Delta}_4)$, see Figure 4.73(iv). If $l(\widehat{\mathbf{v}}_4) = 4\overline{1}4$ then this forces at least two splits on $\widehat{\Delta}_1$ and so $c(\widehat{\Delta}_1) \leq c(3, 3, 3, 3, 3, 3, 4) = \frac{-15\pi}{30}$. Add $c(\Delta)$ to $c(\widehat{\Delta}_1)$, see Figure 4.73(v). Let $d(\widehat{\mathbf{v}}_4) = 4$. Then either it forces at least one split on $\widehat{\Delta}_4$ and $c(\widehat{\Delta}_4) \leq c(3, 3, 3, 3, 4, 4) = \frac{-10\pi}{30}$; or it forces at least one split on $\widehat{\Delta}_3$ and $c(\widehat{\Delta}_3) \leq c(3, 3, 3, 3, 4, 4) = \frac{-10\pi}{30}$; or $l(\widehat{\mathbf{v}}_4) \in 4\overline{1}\{22, \overline{3}4\}$. In the first case add $c(\Delta)$ to $c(\widehat{\Delta}_4)$. In the second case add $c(\Delta)$ to $c(\widehat{\Delta}_3)$. Otherwise, that is $l(\widehat{\mathbf{v}}_4) \in 4\overline{1}\{22, \overline{3}4\}$, if $d(\widehat{\mathbf{v}}_1) \geq 4$ or there are another split on the edge $(3, 4)$ in $\widehat{\Delta}_1$ in Figure 4.73(vi) then $c(\widehat{\Delta}_1) \leq c(3, 3, 4, 4, 4) = \frac{-5\pi}{30}$ and we add $\frac{1}{2}(c(\Delta)) \leq \frac{5\pi}{30}$ to $c(\widehat{\Delta}_1)$ whereas if $d(\widehat{\mathbf{v}}_1) = 3$ then $l(\widehat{\mathbf{v}}_1)$ involves a τ -power and it forces at least one split on $\widehat{\Delta}_1$, $c(\widehat{\Delta}_1) \leq 0$ and $l(\widehat{\mathbf{v}}_4)$ forces at least two splits on $\widehat{\Delta}_1$. This forces $c(\widehat{\Delta}_1) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$ and we add $\frac{1}{2}(c(\Delta)) \leq \frac{5\pi}{30}$ to $c(\widehat{\Delta}_1)$. The same above argument works with $\widehat{\mathbf{v}}_4$, $\widehat{\Delta}_4$ and $\widehat{\Delta}_4$. We add $\frac{5\pi}{30}$ of $c(\Delta)$ to $\widehat{\Delta}_4$ when $d(\widehat{\mathbf{v}}_4) \geq 4$ or when there is another split on the edge $(1, 2)$ in $\widehat{\Delta}_4$ and to $\widehat{\Delta}_4$ when $d(\widehat{\mathbf{v}}_4) = 3$, see Figure 4.73(vi).

Now let $d(\mathbf{v}_3) \geq 5$, see Figure 4.74. Then $c(\Delta) \leq \frac{7\pi}{30}$. First suppose $d(\widehat{\mathbf{v}}_4) = 3$. Then $l(\widehat{\mathbf{v}}_4) \in 4\overline{1}\{\overline{1}, 4, \overline{3}\}$. If $l(\widehat{\mathbf{v}}_4) \in 4\overline{1}\{\overline{1}, 4\}$ then this forces at least 2 splits on $\widehat{\Delta}_1$ and $c(\widehat{\Delta}_1) \leq c(3, 3, 3, 3, 3, 3, 4) = \frac{-15\pi}{30}$ and add $c(\Delta)$ to $c(\widehat{\Delta}_1)$, see Figure 4.74(i).

Figure 4.73: $l(v_2) = l(v_4) = \bar{2}14$.

Now let $l(\hat{v}_4) = 4\bar{1}\bar{3}$. Then this forces at least two splits on $\hat{\Delta}_4$. If in addition $d(v_3) = 5$ then checking all possible labellings shows that $l(v_3)$ forces at least one split on $\hat{\Delta}_4$ or on $\hat{\Delta}_3$. In the former case $c(\hat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 3, 5) = \frac{-18\pi}{30}$ and we add $c(\Delta)$ to $c(\hat{\Delta}_4)$, see Figure 4.74(ii). In the latter case if $d(\hat{v}_2) = 3$ then $l(\hat{v}_2) = \bar{2}\bar{3}\{\bar{2}, k\}$ which forces at least one split on $\hat{\Delta}_3$ and $c(\hat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$ so we add $c(\Delta)$ to $c(\hat{\Delta}_3)$, see Figure 4.74(iii); while if $d(\hat{v}_2) = 4$ then checking possible labelling shows that either $l(\hat{v}_2)$ forces at least one split on $\hat{\Delta}_3$ and again we add $c(\Delta)$ to $c(\hat{\Delta}_3) \leq c(3, 3, 3, 3, 4, 5) = \frac{-8\pi}{30}$; or it forces at least one split on $\hat{\Delta}_2$ and we add $c(\Delta)$ to $c(\hat{\Delta}_2) \leq c(3, 3, 3, 3, 4, 4) = \frac{-10\pi}{30}$, see Figure 4.74(iii)-(iv); or $d(\hat{v}_2) \geq 5$ then $c(\hat{\Delta}_2) \leq c(3, 3, 3, 4, 5) = \frac{-3\pi}{30}$ and $c(\hat{\Delta}_3) \leq c(3, 3, 3, 4, 5) = \frac{-3\pi}{30}$. Add $c(\Delta)$ to $c(\hat{\Delta}_2)$ and $c(\hat{\Delta}_3)$ and $c(\hat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$ as shown in Figure 4.74(v). Now assume $d(v_3) \geq 6$, see Figure 4.74(vi). Then $c(\Delta) \leq \frac{5\pi}{30}$ and we add $\frac{3\pi}{30}$ to $c(\hat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 6) = \frac{-10\pi}{30}$. We add the remaining $\frac{2\pi}{30}$ to $c(\hat{\Delta}_2)$ and $c(\hat{\Delta}_3)$ as follows. We add $\frac{2\pi}{30}$ to $c(\hat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$ when $d(\hat{v}_2) = 3$ (which forces $l(\hat{v}_2) \in \bar{2}\bar{3}\{\bar{2}, k\}$ which in turn forces at least one more split on $\hat{\Delta}_2$) or when $d(\hat{v}_2) = 4$ and it forces at least one split on $\hat{\Delta}_2$; otherwise either $l(\hat{v}_2)$ forces at least one split on $\hat{\Delta}_3$ and we add $\frac{2\pi}{30}$ to $c(\hat{\Delta}_3) \leq c(3, 3, 3, 4, 6) = \frac{-5\pi}{30}$ or $d(\hat{v}_2) \geq 5$ and we add $\frac{2\pi}{30}$ to $c(\hat{\Delta}_2) \leq c(3, 3, 3, 4, 5) = \frac{-3\pi}{30}$.

Now suppose $d(\hat{v}_4) \geq 4$, see Figure 4.74(vii). If the labelling forces at least

Figure 4.74: $l(\mathbf{v}_2) = l(\mathbf{v}_4) = \bar{2}14$.

one split on $\hat{\Delta}_1$ then $c(\hat{\Delta}_1) \leq \frac{-10\pi}{30}$ and we add $c(\Delta)$ to $c(\hat{\Delta}_1)$, so suppose otherwise. Let $d(\hat{\mathbf{v}}_4) = 4$. If the labelling forces two splits on $\hat{\Delta}_4$ then $c(\hat{\Delta}_4) \leq c(3, 3, 3, 3, 4, 5) = \frac{-13\pi}{30}$. Add $c(\Delta)$ to $c(\hat{\Delta}_4)$. Suppose now that the labelling forces exactly one split on $\hat{\Delta}_4$, see Figure 4.75(i), then $c(\hat{\Delta}_4) \leq c(3, 3, 3, 3, 4, 5) = \frac{-3\pi}{30}$ and $l(\hat{\mathbf{v}}_4) \in 4\bar{1}\{2k', \bar{3}k', 3k'\}$ which forces a split on $\hat{\Delta}_1$. Moreover either $d(\hat{\mathbf{v}}_1) \geq 4$ and $c(\hat{\Delta}_1) \leq c(3, 3, 3, 4, 4, 4) = \frac{-5\pi}{30}$ or $d(\hat{\mathbf{v}}_1) = 3$ and $l(\hat{\mathbf{v}}_1)$ involves a τ -power which forces a split on $\hat{\Delta}_1$ and so $c(\hat{\Delta}_1) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$. Add $\frac{2\pi}{30}$ of $c(\Delta)$ to $c(\hat{\Delta}_4)$ and add the remaining $\leq \frac{5\pi}{30}$ to $c(\hat{\Delta}_1)$ when $d(\hat{\mathbf{v}}_1) \geq 4$ or to $c(\hat{\Delta}_1)$ when $d(\hat{\mathbf{v}}_1) = 3$ as shown. This leaves $l(\hat{\mathbf{v}}_4) \in 4\bar{1}\{22, \bar{3}4\}$ and then by the same argument used in Figure 4.73(vi) we can assume that either $c(\hat{\Delta}_1) \leq c(3, 3, 3, 4, 4, 4) = \frac{-5\pi}{30}$ or $c(\hat{\Delta}_1) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$ and we send $\frac{5\pi}{30}$ to $c(\hat{\Delta}_1)$ or $c(\hat{\Delta}_1)$, see Figures 4.75(ii)-(iv). This leaves $\frac{2\pi}{30}$ to deal with. If $d(\hat{\mathbf{v}}_2) = 3$ then this implies $l(\hat{\mathbf{v}}_2) \in \bar{2}\bar{3}\{\bar{1}, \bar{2}, \bar{3}, k\}$ and each forces at least one split on $\hat{\Delta}_2$ and so $c(\hat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$. Add the remaining curvature $\frac{2\pi}{30}$ to $c(\hat{\Delta}_2)$, see Figure 4.75(ii). If $d(\hat{\mathbf{v}}_2) = 4$ and $l(\hat{\mathbf{v}}_2)$ forces at least one split on $\hat{\Delta}_2$ then again add the remaining curvature to $c(\hat{\Delta}_2)$ in Figure 4.75(ii). If $l(\hat{\mathbf{v}}_2)$ forces at least one split on $\hat{\Delta}_3$ then $c(\hat{\Delta}_3) \leq \frac{-3\pi}{30}$ and add the remaining curvature to $c(\hat{\Delta}_3)$, see Figure 4.75(iii). If $l(\hat{\mathbf{v}}_2) \in \bar{2}\bar{3}\{11, \bar{3}4, \bar{3}\bar{4}\}$ then each forces at least two splits

on $\hat{\Delta}_2$ and either we have $d(\hat{v}_2) \geq 4$ and $c(\hat{\Delta}_2) \leq c(3, 3, 4, 4, 4) = \frac{-5\pi}{30}$ or $l(\hat{v}_2)$ involves a τ -power and $c(\hat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$. Add the remaining $\frac{2\pi}{30}$ of $c(\Delta)$ to $c(\hat{\Delta}_2)$ or $c(\hat{\Delta}_2)$ as appropriate, see Figure 4.75(iv). If $d(\hat{v}_2) \geq 5$. Then $c(\hat{\Delta}_2) \leq c(3, 3, 3, 4, 5) = \frac{-3\pi}{30}$. Add the remaining $\frac{2\pi}{30}$ of $c(\Delta)$ to $c(\hat{\Delta}_2)$ as above. Finally letting $d(\hat{v}_4) \geq 5$ implies $c(\hat{\Delta}_1) \leq \frac{-3\pi}{30}$, see Figure 4.75(v), and we add $\frac{3\pi}{30}$ to $c(\hat{\Delta}_1)$. By the above argument add the remaining curvature to $c(\hat{\Delta}_2) \leq \frac{-5\pi}{30}$ or $c(\hat{\Delta}_2) \leq \frac{-5\pi}{30}$ when $d(\hat{v}_2) \in \{3, 4\}$. Otherwise add $\frac{3\pi}{30}$ to $c(\hat{\Delta}_2) \leq c(3, 3, 3, 4, 5) = \frac{-3\pi}{30}$. This leaves $\frac{\pi}{30}$. If $d(\hat{v}_4) \geq 4$ or if $\hat{\Delta}_3$ has extra split then $c(\hat{\Delta}_3) \leq c(3, 4, 5, 5) = \frac{-5\pi}{30}$ and add $\frac{\pi}{30}$ to $c(\hat{\Delta}_3)$. If $d(\hat{v}_4) = 3$ then $c(\hat{\Delta}_3) \leq c(3, 3, 5, 5) = \frac{4\pi}{30}$ and we distribute $c(\Delta) + c(\hat{\Delta}_3) \leq \frac{5\pi}{30}$ as in Figure 4.62(v)-(viii) in which Δ plays the role of $\hat{\Delta}_3$.

Next assume $d(v_1) \geq 5$, see Figures 4.76-4.81 . First let $d(v_3) = 4$. Then $l(v_3) \in$

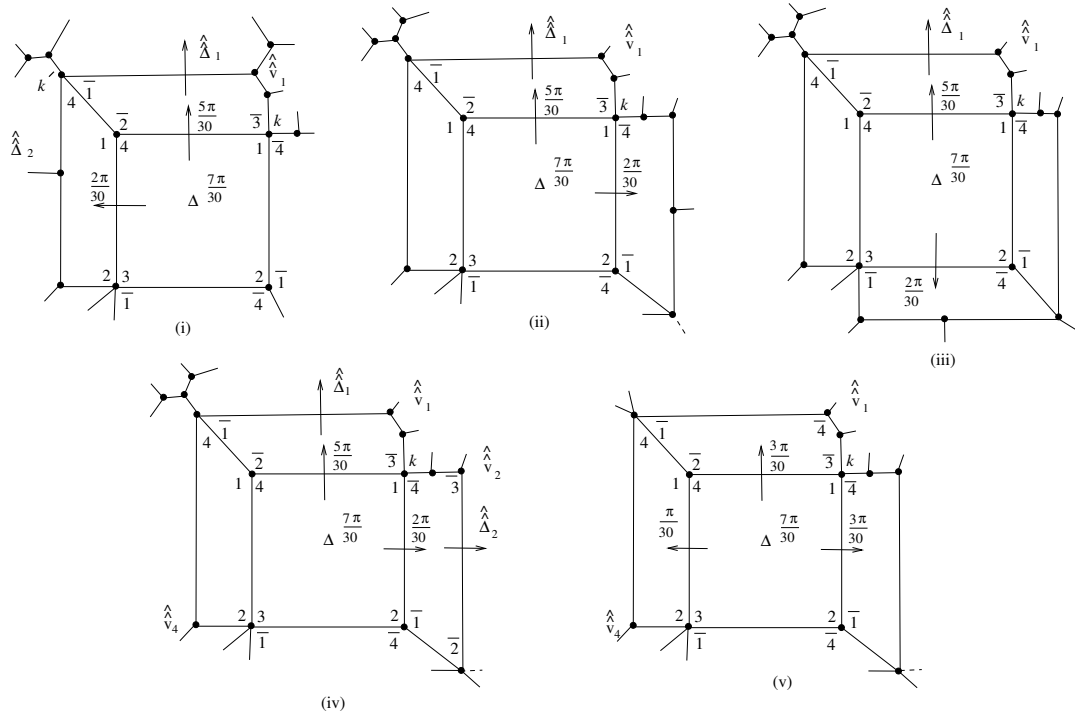


Figure 4.75: $l(v_2) = l(v_4) = \bar{2}14$.

$\bar{1}32\{2, \bar{1}, k\}$. First suppose $l(v_3) = \bar{1}322$, see Figures 4.76(i)-(iii). Then this forces at least two splits on $\hat{\Delta}_4$. If $d(\hat{v}_4) = 3$ then $l(\hat{v}_4) \in 4\bar{1}\{\bar{1}, 4, k\}$. We add $c(\Delta) \leq \frac{7\pi}{30}$ to $c(\hat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 3, 4) = \frac{-15\pi}{30}$ when $l(\hat{v}_4) \in 4\bar{1}\{\bar{1}, k\}$, see Figure 4.76(i); otherwise $l(\hat{v}_4)$ forces at least two splits on $\hat{\Delta}_1$, so $c(\hat{\Delta}_1) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$

and $c(\hat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$. Add $\frac{5\pi}{30}$ of $c(\Delta)$ to $c(\hat{\Delta}_4)$ and add the remaining $\frac{2\pi}{30}$ to $c(\hat{\Delta}_1)$, see Figure 4.76(ii). If $d(\hat{v}_4) \geq 4$ then add $c(\Delta)$ to $c(\hat{\Delta}_4) \leq c(3, 3, 3, 3, 4, 4) = \frac{-10\pi}{30}$, see Figure 4.76(iii).

Now suppose $\underline{l(v_3)} = \bar{1}32\bar{1}$, see Figures 4.76(iv)-(vi). Then this forces at least two splits on $\hat{\Delta}_3$. If $d(\hat{v}_2) = 3$ then $l(\hat{v}_2) \in \bar{2}\bar{3}\{\bar{2}, \bar{3}, k\}$ which all except $l(\hat{v}_2) = \bar{2}\bar{3}\bar{3}$, forces at least one split on $\hat{\Delta}_3$ and we add $c(\Delta) \leq \frac{7\pi}{30}$ to $c(\hat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 3, 4) = \frac{-15\pi}{30}$ when $l(\hat{v}_2) \neq \bar{2}\bar{3}\bar{3}$, see Figure 4.76(iv). If $l(\hat{v}_2) = \bar{2}\bar{3}\bar{3}$ then this forces at least two splits on $\hat{\Delta}_2$ and we add $c(\Delta)$ to $c(\hat{\Delta}_2)$ and $c(\hat{\Delta}_3)$ as in Figure 4.76(v). If $d(\hat{v}_2) \geq 4$ then add $c(\Delta)$ to $c(\hat{\Delta}_3) \leq c(3, 3, 3, 3, 4, 4) = \frac{-10\pi}{30}$, see Figure 4.76(vi).

Now suppose $\underline{l(v_3)} = \bar{1}32k$, see Figures 4.77-4.78. Then this forces at least one

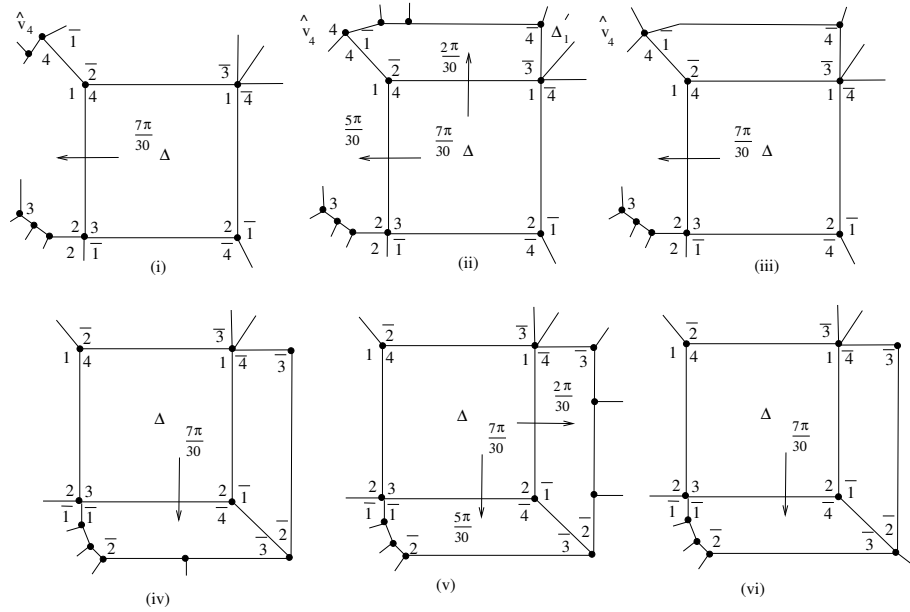


Figure 4.76: $l(v_2) = l(v_4) = \bar{2}14$.

split on each of $\hat{\Delta}_3$ and $\hat{\Delta}_4$. Let $d(\hat{v}_4) = 3$. This forces either two splits on $\hat{\Delta}_4$ so $c(\hat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 3, 4) = \frac{-15\pi}{30}$ and we add $c(\Delta)$ to $c(\hat{\Delta}_4)$ as in Figure 4.77(i); or forces one split on each of $\hat{\Delta}_4$ and $\hat{\Delta}_1$ so $c(\hat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$ and $c(\hat{\Delta}_1) \leq c(3, 3, 3, 4, 5) = \frac{-3\pi}{30}$ and we add $c(\Delta)$ to $c(\hat{\Delta}_4)$ and $c(\hat{\Delta}_1)$ as in Figure 4.77(ii); or $l(\hat{v}_4) = 4\bar{1}4$ and this forces two splits on $\hat{\Delta}_1$ so $c(\hat{\Delta}_1) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$, see Figures 4.77(iii). We add $\frac{5\pi}{30}$ of $c(\Delta)$ to $c(\hat{\Delta}_1)$ and we add the remaining curvature $\leq \frac{2\pi}{30}$ to $c(\hat{\Delta}_2)$ or $c(\hat{\Delta}_3)$ as follows: if $d(\hat{v}_2) = 3$

then checking shows that $l(\widehat{v}_2)$ forces at least one split on $\widehat{\Delta}_2$ and we add the remaining curvature to $c(\widehat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$. If $d(\widehat{v}_2) = 4$ then it either forces at least one split on $\widehat{\Delta}_3$ and we add the remaining curvature to $c(\widehat{\Delta}_3) \leq c(3, 3, 3, 3, 4, 4) = \frac{-10\pi}{30}$; or it forces at least one split on $\widehat{\Delta}_2$ and we add the remaining curvature to $c(\widehat{\Delta}_2) \leq c(3, 3, 3, 4, 5) = \frac{-5\pi}{30}$; or $l(\widehat{v}_2) \in \overline{2}3\{\overline{3}4, \overline{3}\overline{4}\}$ which forces at least 2 splits on $\widehat{\Delta}_3$ and either $d(\widehat{v}_3) \geq 4$ and we add the remaining to $c(\widehat{\Delta}_3) \leq c(3, 3, 4, 4, 4) = \frac{-5\pi}{30}$ or $d(\widehat{v}_3) = 3$ and so $l(\widehat{v}_3)$ involves a τ -power which would force a further split on $\widehat{\Delta}_3$ and add the remaining to $c(\widehat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$ as in 4.71(v). If $d(\widehat{v}_2) \geq 5$ then add the remaining to $c(\widehat{\Delta}_3) \leq c(3, 3, 3, 4, 5) = \frac{-3\pi}{30}$.

Now let $d(\widehat{v}_4) = 4$. Then this forces either $c(\widehat{\Delta}_4) \leq c(3, 3, 3, 3, 4, 4) = \frac{-10\pi}{30}$ as in Figure 4.77(iv); or forces $c(\widehat{\Delta}_1) \leq c(3, 3, 3, 3, 4, 5) = \frac{-13\pi}{30}$ as in Figure 4.77(v); or forces $c(\widehat{\Delta}_1) \leq c(3, 3, 3, 4, 5) = \frac{-3\pi}{30}$ and either $c(\widehat{\Delta}_4) \leq c(3, 3, 4, 4, 4) = \frac{-5\pi}{30}$ or $c(\widehat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$ and we distribute as in Figure 4.77(vi); or $l(\widehat{v}_4) \in 4\overline{1}\{22, \overline{3}4\}$ where we can compensate $\frac{5\pi}{30}$ to $c(\widehat{\Delta}_4) \leq c(3, 3, 4, 4, 4) = \frac{-5\pi}{30}$ or $c(\widehat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$ and we send the remaining to $\widehat{\Delta}_2$ or $\widehat{\Delta}_3$ as in Figure 4.77(iii), see Figure 4.78(i).

Finally if $d(\widehat{v}_4) \geq 5$ then $c(\widehat{\Delta}_4) \leq c(3, 3, 3, 4, 5) = \frac{-3\pi}{30}$. Add $\frac{2\pi}{30}$ to $c(\widehat{\Delta}_4)$

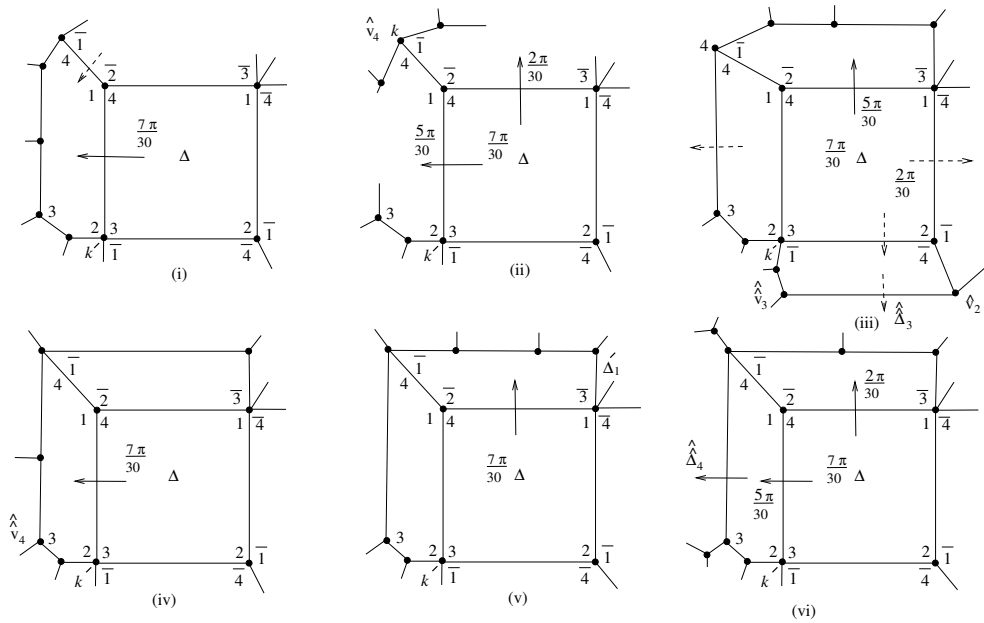


Figure 4.77: $l(v_2) = l(v_4) = \overline{2}14$.

and send the remaining $\frac{5\pi}{30}$ to $c(\hat{\Delta}_2)$, $c(\hat{\Delta}_3)$ or $c(\hat{\hat{\Delta}}_3)$ as in Figures 4.78(ii)-(vi). Note that in Figure 4.78(ii) $l(\hat{v}_2)$ forces at least one split on $\hat{\Delta}_3$ and that $d(\hat{v}_2) \in \{3, 4\}$ and so $c(\hat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$. Also in Figure 4.78(iii) $l(\hat{v}_2)$ forces at least two splits on $\hat{\Delta}_2$ and that $d(\hat{v}_2) \in \{3, 4\}$ and so $c(\hat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$. In Figure 4.78(iv) $d(\hat{v}_2) = 4$ and it forces only one split on $\hat{\Delta}_2$. Then either $d(\hat{v}_3) \geq 4$ and $c(\hat{\Delta}_3) \leq c(3, 3, 4, 4, 4) = \frac{-5\pi}{30}$ or $l(\hat{v}_3)$ involves a τ -power and $c(\hat{\hat{\Delta}}_3) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$. In Figure 4.78(v) similarly to above either $d(\hat{v}_3) \geq 4$ and $c(\hat{\Delta}_3) \leq c(3, 3, 4, 4, 4) = \frac{-5\pi}{30}$ or $l(\hat{v}_3)$ involves a τ -power and $c(\hat{\hat{\Delta}}_3) \leq c(3, 3, 3, 3, 3, 3, 4) = \frac{-51\pi}{30}$. In Figure 4.78(vi) $d(\hat{v}_2) \geq 5$ and either $d(\hat{v}_2) \geq 4$ and $c(\hat{\Delta}_2) \leq c(3, 4, 5, 5) = \frac{-\pi}{30}$; or $d(\hat{v}_2) = 3$ and $c(\hat{\Delta}_2) \leq c(3, 3, 5, 5) = \frac{4\pi}{30}$ so $c(\hat{\Delta}_2) + \frac{\pi}{30} \leq \frac{5\pi}{30}$ is distributed as in Figure 4.62(vii)-(x) in which Δ plays the role of $\hat{\Delta}_2$.

Now let $d(v_3) \geq 5$. Then $c(\Delta) \leq \frac{4\pi}{30}$. First assume $d(v_3) = 5$. Then we pro-

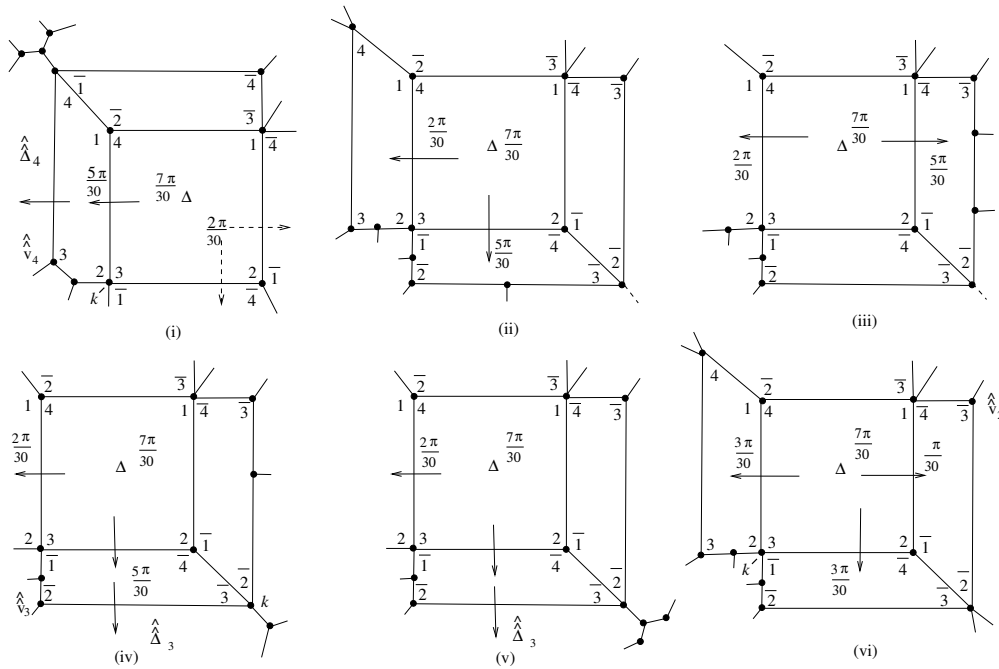


Figure 4.78: $l(v_2) = l(v_4) = \bar{2}14$.

ceed by considering the splits that $l(v_3)$ may force on $\hat{\Delta}_3$ or $\hat{\Delta}_4$. Suppose $l(v_3)$ forces two splits on $\hat{\Delta}_3$, see 4.79(i), then add $c(\Delta)$ to $c(\hat{\Delta}_3) \leq \frac{-8\pi}{30}$. Now suppose $l(v_3)$ forces only one split, see 4.79(ii)-(vi). If in addition $l(\hat{v}_2)$ forces at least one split on $\hat{\Delta}_3$ then again add $c(\Delta)$ to $c(\hat{\Delta}_3) \leq \frac{-8\pi}{30}$, see Figure 4.79(ii); If $l(\hat{v}_2)$

does not force any splits then either $l(\widehat{v}_2) = \overline{2}\overline{3}\overline{3}$ or $d(\widehat{v}_2) \geq 4$. In the former case $l(\widehat{v}_2)$ forces at least two splits on $\widehat{\Delta}_2$ and we add $c(\Delta)$ to $c(\widehat{\Delta}_2) \leq \frac{-8\pi}{30}$, see Figure 4.79(iii). In the latter case if $d(\widehat{v}_2) = 4$ then it either forces one split on $\widehat{\Delta}_2$ or $l(\widehat{v}_2) = \overline{2}\overline{3}\{11, \overline{3}4, \overline{3}\overline{4}\}$. In the first case add $c(\Delta)$ to $c(\widehat{\Delta}_3) \leq \frac{-3\pi}{30}$ and $c(\widehat{\Delta}_2) \leq \frac{-3\pi}{30}$ as in Figure 4.79(iv); in the second case either $d(\widehat{v}_3) \geq 4$ and add $c(\Delta)$ to $c(\widehat{\Delta}_3) \leq \frac{-5\pi}{30}$ or $d(\widehat{v}_3) = 3$ and add $c(\Delta)$ to $c(\widehat{\Delta}_3) \leq \frac{-10\pi}{30}$, see Figure 4.79(v). If $d(\widehat{v}_2) \geq 5$ then add $c(\Delta)$ to $c(\widehat{\Delta}_3) \leq c(3, 3, 3, 5, 5) = \frac{-6\pi}{30}$, see Figure 4.79(vi).

Next suppose that $l(v_3)$ forces no split on $\widehat{\Delta}_3$ but $l(\widehat{v}_2)$ forces at least two splits

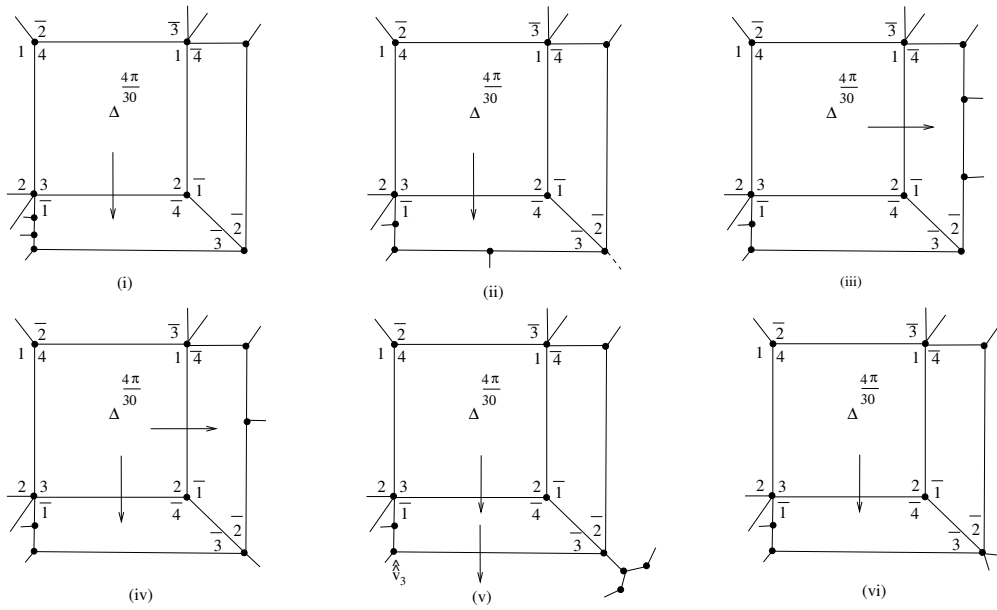
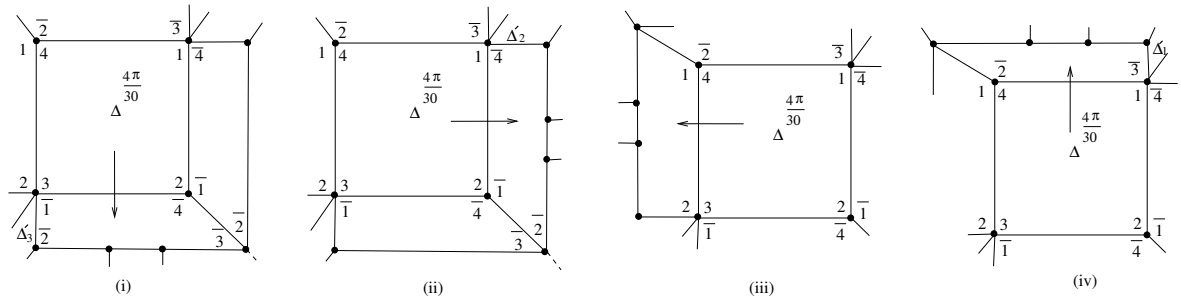


Figure 4.79: $l(v_2) = l(v_4) = \overline{2}14$.

on $\widehat{\Delta}_3$, see Figure 4.80(i). Then we add $c(\Delta)$ to $c(\widehat{\Delta}_3) \leq \frac{-8\pi}{30}$. Otherwise if $l(\widehat{v}_2)$ forces at least two splits on $\widehat{\Delta}_2$ then we add $c(\Delta)$ to $c(\widehat{\Delta}_2) \leq \frac{-8\pi}{30}$, see Figure 4.80(ii).

Assume now none of the above occur. If $l(v_3)$ forces splits on $\widehat{\Delta}_4$ then using a similar argument to the above we distribute the curvature to $c(\widehat{\Delta}_4)$, $c(\widehat{\Delta}_1)$ and $c(\widehat{\Delta}_4)$ similarly to Figures 4.76(i)-(iii) and Figure 4.77. If $l(\widehat{v}_4)$ forces at least two splits on $\widehat{\Delta}_4$, see Figure 4.80(iii), then add $c(\Delta)$ to $c(\widehat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 8) = \frac{-13\pi}{30}$. Otherwise if $l(\widehat{v}_4)$ forces at least two splits on $\widehat{\Delta}_1$, see Figure 4.80(iv), then we add $c(\Delta)$ to $c(\widehat{\Delta}_1)$.

If none of the above holds then $l(v_3)$ forces no splits on $\widehat{\Delta}_3$ nor $\widehat{\Delta}_4$ and this

Figure 4.80: $l(v_2) = l(v_4) = \bar{2}14$.

implies $l(v_3) \in \{\bar{1}32\bar{1}4, \bar{1}3242\}$ otherwise there is a contradiction. So assume now that $l(v_3) \in \{\bar{1}32\bar{1}4, \bar{1}3242\}$ and that neither $l(\hat{v}_4)$ nor $l(\hat{v}_2)$ forces two splits on $\hat{\Delta}_i$, $1 \leq i \leq 4$. If $d(v_1) = 5$ then a similar argument to the above can be used for $l(v_1)$ and we distribute the curvature to $c(\hat{\Delta}_1)$, $c(\hat{\Delta}_1)$, $c(\hat{\Delta}_2)$, $c(\hat{\Delta}_2)$ unless $l(v_1) \in S_0 = \{\bar{3}1\bar{4}\bar{1}\bar{2}, \bar{3}1\bar{4}\bar{2}\bar{1}, \bar{3}1\bar{4}\bar{2}\bar{2}, \bar{3}1\bar{4}\bar{4}\bar{3}\}$ where these are the only labels that forces no splits on $\hat{\Delta}_1$ nor on $\hat{\Delta}_2$. So let $l(v_1) \in S_0$ or $d(v_1) \geq 6$.

First suppose $l(v_3) = \bar{1}32\bar{1}4$. Then $d(v_1) \geq 6$ otherwise there is a contradiction; and $c(\Delta) \leq c(3, 3, 5, 6) = \frac{2\pi}{30}$. Observe that $l(v_3)$ forces at least two splits on Δ'_4 . First let $d(\hat{v}_4) = 3$. Then $l(\hat{v}_4) = 4\bar{1}k$ which forces at least one split on $\hat{\Delta}_4$. If in addition $d(\hat{v}_4) \geq 4$ then $c(\hat{\Delta}_4) \leq c(3, 3, 3, 4, 5) = \frac{-3\pi}{30}$ and so add $c(\Delta)$ to $c(\hat{\Delta}_4)$ while if $d(\hat{v}_4) = 3$ then add $c(\Delta) + c(\hat{\Delta}_4) \leq \frac{4\pi}{30}$ to $c(\Delta'_4)$, see Figure 4.81(i). Note that in this case $\hat{\Delta}_4$ plays the role of Δ in Figure 4.68(ii) and that $l(\hat{v}_4) = \bar{2}3k$ which forces a split on Δ'_4 so $c(\Delta'_4) \leq c(3, 3, 3, 3, 3, 5) = \frac{-18\pi}{30}$. Now let $d(\hat{v}_4) \geq 4$. Then either $l(\hat{v}_4)$ forces at least one split on $\hat{\Delta}_4$ and we add $c(\Delta)$ to $c(\hat{\Delta}_4) \leq \frac{-3\pi}{30}$; or $l(\hat{v}_4)$ forces no splits, see Figure 4.81(ii): in this case if $d(\hat{v}_4) = 3$ then $l(\hat{v}_4) = 33\bar{2}$ which yields with other labelling the exception case $l_3 = 3l_4$ so this is a contradiction and we let $d(\hat{v}_4) \geq 4$. Then $\hat{\Delta}_4$ is Δ in Figure 4.83(vii) and the sum of curvature is $\leq \frac{4\pi}{30}$. Add $\frac{2\pi}{30}$ of the sum to $c(\Delta'_4) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$ and distribute the remaining $\frac{2\pi}{30}$ as in Figure 4.83, see Figure 4.81(ii).

Now let $l(v_3) = \bar{1}3242$. Then either $d(v_1) \geq 6$ or $l(v_1) = \bar{3}1\bar{4}\bar{2}\bar{2}$. In the former case a similar argument to the above works here as $l(v_3)$ forces at least 2 splits on Δ'_4 . In the latter case $l(v_1)$ forces at least two splits on Δ'_2 and by similar argument we send $\frac{2\pi}{30}$ to $c(\hat{\Delta}_2)$ or to $c(\Delta'_2)$ and $c(\hat{\Delta}_2)$ if needed and add $\frac{2\pi}{30}$ to $c(\hat{\Delta}_4)$ or to $c(\Delta'_4)$ and $c(\hat{\Delta}_4)$ if needed, see Figures 4.81(iii)-(iv).

Now let $d(v_3) \geq 6$. Then $d(v_1) = 5$ and applying the same above argument we

send $c(\Delta) \leq \frac{2\pi}{30}$ to $c(\hat{\Delta}_2)$, see Figures 4.81(v)-(vi).

We note here once all the curvature distribution has been completed that $\hat{\Delta}_4$ in Figures 4.81(iii)-(iv) might receive positive curvature from $\hat{\Delta}_1$ if $d(\hat{v}_4) \geq 5$ and $\hat{\Delta}_1$ plays the role of Δ in Figures 4.81(iv) or (vi) in which case either the edge $(4, 3)$ splits in $\hat{\Delta}_4$ or $d(\hat{v}_4) \geq 4$ and so $c(\hat{\Delta}_4) \leq c(3, 4, 5, 5) = \frac{-\pi}{30}$. Then the curvature sum added to $c(\hat{\Delta}_4)$ is $\frac{1}{2}[c(\Delta) + c(\hat{\Delta}_1)] \leq \frac{4\pi}{30}$ and we add $c(\hat{\Delta}_4) + \frac{4\pi}{30} = \frac{3\pi}{30}$ to $c(\Delta'_4) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$. We note further that $\hat{\Delta}_3$ in Figure 4.81(iv) may play the role of Δ in Figure 4.81(iii)-(iv) when $d(\hat{v}_2) \geq 5$ and either the edge $(2, 3)$ splits in $\hat{\Delta}_2$ or $d(\hat{v}_1) \geq 4$ so $c(\hat{\Delta}_2) \leq c(3, 4, 5, 5) = \frac{-\pi}{30}$ and the curvature sum added to $c(\hat{\Delta}_2)$ is $\frac{1}{2}[c(\Delta) + c(\hat{\Delta}_3)] \leq \frac{4\pi}{30}$ and we add $c(\hat{\Delta}_4) + \frac{4\pi}{30} = \frac{3\pi}{30}$ to $c(\Delta'_2) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$.

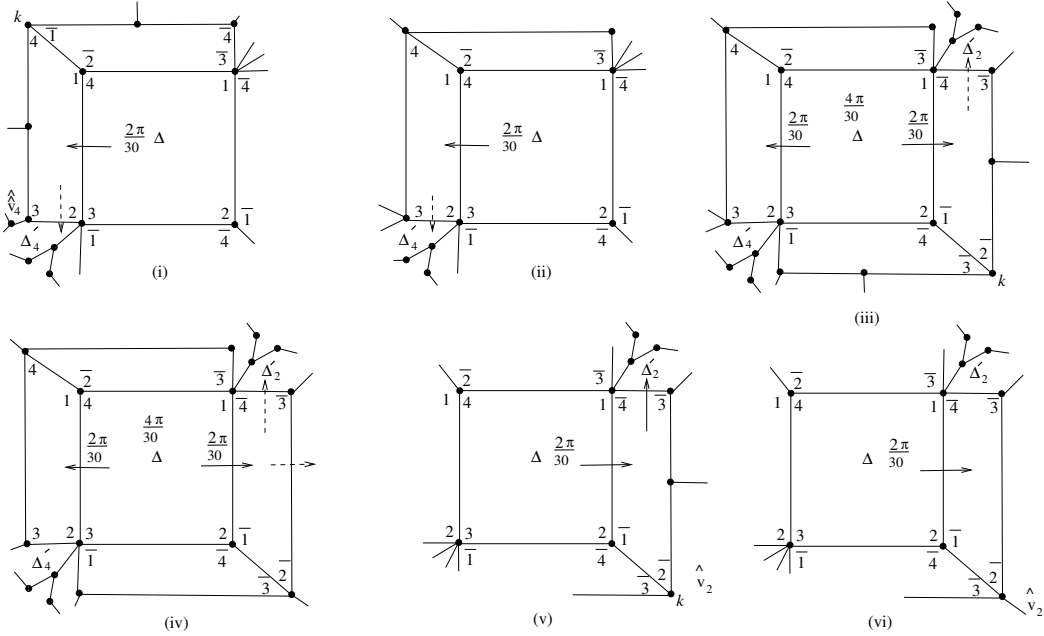


Figure 4.81: $l(v_2) = l(v_4) = \bar{2}14$.

Δ is given by Figures 4.63(iv),(vi) and (viii):

Add $\frac{1}{4}(c(\Delta)) \leq \frac{\pi}{12}$ to $c(\hat{\Delta}_i) \leq c(3, 3, 3, 3, 3, 4) = \frac{-\pi}{6}$, $1 \leq i \leq 4$.

Δ is given by Figures 4.63(v) and (vii):

Add $\frac{1}{2}(c(\Delta)) \leq \frac{\pi}{6}$ to each of $c(\hat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 4) = \frac{-\pi}{6}$ and $c(\hat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 4) = \frac{-\pi}{6}$.

Δ is a positive region with only one vertex of degree = 3:

If the labelling is not $\bar{2}14$ then we distribute as in Figure 4.2(v) unless $l(v_1) \in \{31\bar{4}, \bar{3}1\bar{4}\}$ where we add $c(\Delta) \leq \frac{5\pi}{30}$ to $c(\hat{\Delta}_1)$, see Figure 4.82. Observe that when $d(v_4) = 4$ then $l(v_4)$ involves a τ -power which forces at least one split on $\hat{\Delta}_1$ and so $c(\hat{\Delta}_1) \leq c(3, 3, 3, 3, 3, 3, 4) = \frac{-15\pi}{30}$ and when $d(v_4) \geq 5$ then $c(\Delta) \leq \frac{3\pi}{30}$ and $c(\hat{\Delta}_1) \leq \frac{-8\pi}{30}$.

If $l(v_i) = \bar{2}14$, $i \in \{1, 2, 4\}$ then the distribution is given in Figure 4.83 (in

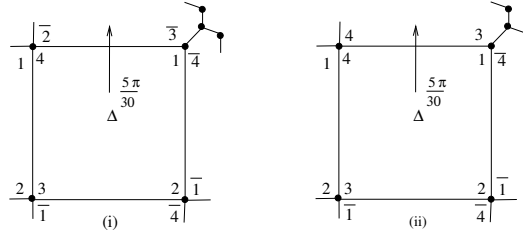


Figure 4.82: $l(v_1) \in \{31\bar{4}, \bar{3}1\bar{4}\}$.

which Δ is given up to anti-clockwise rotation). First assume $l(v_1) = \bar{2}14$. If $d(v_2) = 4$ then $l(v_2) \in \{32\bar{1}\{\bar{1}, 2, k\}, 324\{1, 2, 3, 4, k\}, 32\bar{4}\{1, 2, 3, \bar{4}, k\}\}$. If $l(v_2)$ does not involve a τ -power then it forces at least two splits either on $\hat{\Delta}_2$ or on $\hat{\Delta}_3$ and we add $c(\Delta) \leq \frac{5\pi}{30}$ to $c(\hat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$ or to $c(\hat{\Delta}_3) \leq c(3, 3, 3, 3, 4, 4) = \frac{-10\pi}{30}$, see Figures 4.83(i)-(ii). If $l(v_2)$ involves a τ -power then it forces at least one split on each of $\hat{\Delta}_2$ and $\hat{\Delta}_3$. If in addition $d(v_3) = 4$ and $l(v_3)$ forces at least one split on $\hat{\Delta}_3$ then we add $c(\Delta) \leq \frac{5\pi}{30}$ to $c(\hat{\Delta}_3) \leq c(3, 3, 3, 3, 4, 4) = \frac{-10\pi}{30}$, see Figure 4.83(ii); otherwise $l(v_3) \in S_1 = \{\bar{2}33\{4, \bar{4}\}, \bar{2}311, 3323, 333\{\bar{1}, \bar{2}\}, 3313, \bar{1}33\{4, \bar{4}\}, \bar{1}322\}$ each forces at least two splits on $\hat{\Delta}_4$ and we add $c(\Delta) \leq \frac{5\pi}{30}$ to $c(\hat{\Delta}_4) \leq c(3, 3, 3, 3, 4, 4) = \frac{-10\pi}{30}$, see Figure 4.83(iv). If $d(v_3) \geq 5$ then add $c(\Delta) \leq \frac{2\pi}{30}$ to $c(\hat{\Delta}_3) \leq c(3, 3, 3, 4, 5) = \frac{-3\pi}{30}$. If $d(v_2) \geq 5$ then $c(\Delta) \leq \frac{2\pi}{30}$ and $d(v_3) = d(v_4) = 4$. If in addition $l(v_3)$ forces at least one split on $\hat{\Delta}_3$ then add $c(\Delta)$ to $c(\hat{\Delta}_3) \leq c(3, 3, 3, 4, 5) = \frac{-3\pi}{30}$, see Figures 4.83(v) otherwise $l(v_3) \in S_1$ and add $c(\Delta)$ to $c(\hat{\Delta}_4) \leq c(3, 3, 3, 3, 4, 4) = \frac{-10\pi}{30}$, see Figure 4.83(vi). Note that when $d(v_2) \geq 5$ then Δ may receive extra $\frac{2\pi}{30}$ from $\hat{\Delta}_2$ and $l(v_2)$ in this case forces at least 2 splits on $\hat{\Delta}_2$ and we add the sum $\frac{4\pi}{30}$ to $c(\hat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 3, 5) = \frac{-15\pi}{30}$ when $l(v_3)$ forces at least one split on $\hat{\Delta}_3$ as in Figure 4.83(vi) or we add $\frac{2\pi}{30}$ to $c(\hat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$ and $\frac{2\pi}{30}$ to $c(\hat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$ otherwise, Figure 4.83(vii).

Next assume $l(v_2) = \bar{1}2\bar{4}$. Then a similar argument to the above works here with

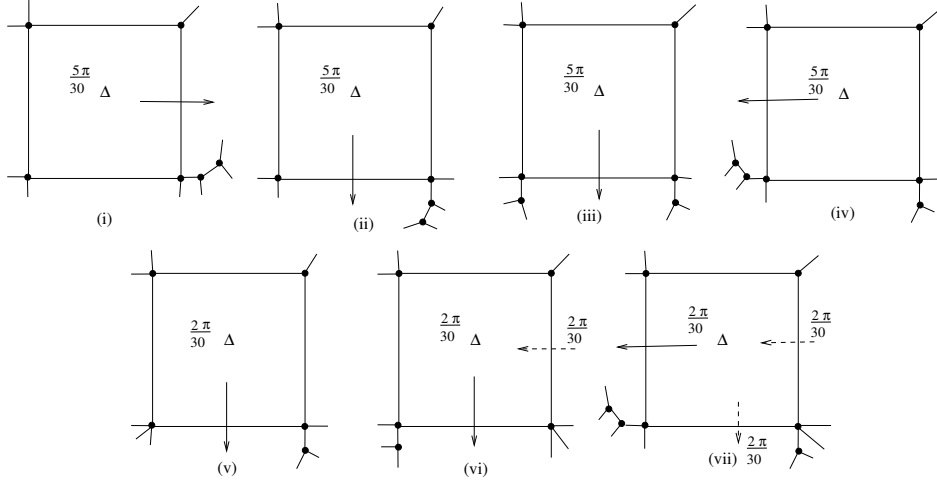


Figure 4.83: $l(v_i) = \bar{2}14$, $i \in \{1, 2, 4\}$.

$d(v_3) = 4$ which implies $l(v_3) \in \{\bar{1}31\bar{3}, \bar{1}32\{\bar{1}, 2, k\}, \bar{1}33\{\bar{2}, 3, 4, k\}\}$. If $l(v_3)$ does not involve a τ -power then we add $c(\Delta) \leq \frac{5\pi}{30}$ to $c(\hat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$ or to $c(\hat{\Delta}_4) \leq c(3, 3, 3, 3, 4, 4) = \frac{-10\pi}{30}$. If $l(v_2)$ involves a τ -power then we distribute to $c(\hat{\Delta}_4) \leq c(3, 3, 3, 3, 4, 4) = \frac{-10\pi}{30}$ when $d(v_4) = 4$ and $l(v_4)$ forces at least one split on $\hat{\Delta}_4$; or we add $c(\Delta) \leq \frac{5\pi}{30}$ to $c(\hat{\Delta}_1) \leq c(3, 3, 3, 3, 4, 4) = \frac{-10\pi}{30}$ when $l(v_4) \in S_2 = \{44\bar{1}4, 44\bar{2}4, 444\{1, 2\}, 24\bar{1}\bar{1}, 144\{3, \bar{3}\}, 244\{3, \bar{3}\}\}$ (each forces at least two splits on $\hat{\Delta}_1$); or $d(v_4) \geq 5$ and we add $c(\Delta) \leq \frac{2\pi}{30}$ to $c(\hat{\Delta}_3) \leq c(3, 3, 3, 4, 5) = \frac{-3\pi}{30}$. Now let $d(v_3) \geq 5$. Then this forces $c(\Delta) \leq \frac{2\pi}{30}$ and $d(v_1) = d(v_4) = 4$. If $l(v_4)$ forces at least one split on $\hat{\Delta}_4$ then add $c(\Delta)$ to $c(\hat{\Delta}_4) \leq c(3, 3, 3, 4, 5) = \frac{-3\pi}{30}$ otherwise $l(v_4) \in S_2$ and add $c(\Delta) \leq \frac{5\pi}{30}$ to $c(\hat{\Delta}_1) \leq c(3, 3, 3, 3, 4, 4) = \frac{-10\pi}{30}$.

Next let $l(v_4) = 14\bar{2}$. Then again a similar argument to the above works here with $d(v_1) = 4$ which implies $l(v_1) \in \{\bar{3}1\bar{2}\{1, \bar{2}, k\}, \bar{3}14\{1, 2, \bar{3}, 4, k\}, \bar{3}1\bar{4}\{1, 2, \bar{3}, \bar{4}, k\}\}$. If $l(v_1)$ does not involve a τ -power then we add $c(\Delta) \leq \frac{5\pi}{30}$ to $c(\hat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$ or to $c(\hat{\Delta}_4) \leq c(3, 3, 3, 3, 4, 4) = \frac{-10\pi}{30}$. If $l(v_1)$ involves a τ -power and $d(v_2) = 4$ such that $l(v_2)$ forces at least one split on $\hat{\Delta}_2$ then add $c(\Delta)$ to $c(\hat{\Delta}_1) \leq c(3, 3, 3, 3, 4, 4) = \frac{-10\pi}{30}$; otherwise either $l(v_2) \in S_3 = \{\bar{1}2\bar{1}4, \bar{1}24\bar{2}, \bar{3}2\bar{1}\bar{1}, \bar{3}24\bar{3}, \bar{3}2\bar{4}\bar{3}, 32\bar{1}\bar{1}, 324\bar{3}, 324\bar{3}\}$ each forces at least two splits on $\hat{\Delta}_3$ and we add $c(\Delta) \leq \frac{5\pi}{30}$ to $c(\hat{\Delta}_3) \leq c(3, 3, 3, 3, 4, 4) = \frac{-10\pi}{30}$; or $d(v_2) \geq 5$ and we add $c(\Delta) \leq \frac{2\pi}{30}$ to $c(\hat{\Delta}_2) \leq c(3, 3, 3, 4, 5) = \frac{-3\pi}{30}$. Now let $d(v_1) \geq 5$. Then this forces $c(\Delta) \leq \frac{2\pi}{30}$ and $d(v_2) = d(v_3) = 4$. If $l(v_2)$ forces at least one split on $\hat{\Delta}_2$

then we add $c(\Delta) \leq \frac{2\pi}{30}$ to $c(\hat{\Delta}_2) \leq c(3, 3, 3, 4, 5) = \frac{-3\pi}{30}$. Otherwise $l(v_2) \in S_3$ and we add $c(\Delta) \leq \frac{5\pi}{30}$ to $c(\hat{\Delta}_3) \leq c(3, 3, 3, 3, 4, 4) = \frac{-10\pi}{30}$.

This completes the cases when $d(u_i) = 2$, $1 \leq i \leq 4$.

Δ is given by Figures 4.64(i):

In Figure 4.64(i) add $\frac{5\pi}{30}$ of $c(\Delta) \leq \frac{10\pi}{30}$ to $c(\hat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$ and we add the remainder to $c(\hat{\Delta}_2)$ or $c(\hat{\Delta}_3)$ as follows. If $d(\hat{v}_2) = 3$ then $l(\hat{v}_2) \in \bar{2}\bar{3}\{\bar{1}, \bar{2}, \bar{4}, k\}$ each forces at least one split on $\hat{\Delta}_3$ and $c(\hat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$ and add $c(\Delta)$ to $c(\hat{\Delta}_3)$; if $d(\hat{v}_2) \geq 4$ and forces at least one split on $c(\hat{\Delta}_2)$ then add $c(\Delta) \leq \frac{5\pi}{30}$ to $c(\hat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$; while if it does not force any split on $c(\hat{\Delta}_2)$ then again add $c(\Delta)$ to $c(\hat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$.

Δ is given by Figures 4.64(ii)-(vi):

Then $c(\hat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$ and we add $c(\Delta) \leq \frac{5\pi}{30}$ to $c(\hat{\Delta}_4)$. Note that in Figures 4.64(v)-(vi) Δ may receives extra $\frac{2\pi}{30}$ from $\hat{\Delta}_2$ if $\hat{\Delta}_2$ plays the role of Δ in Figure 4.67(iii) but this forces $d(v_2) = 5$ and 2 splits on $\hat{\Delta}_3$ in Figures 4.64(v)-(vi) and we add the extra $\frac{2\pi}{30}$ to $\hat{\Delta}_3$.

Δ is given by Figures 4.64(vii):

First let $d(v_3) = 4$. Then $l(v_3) \in \bar{1}33\{\bar{2}, 3, 4, \bar{4}, k\}$. If $l(v_3) \in \bar{1}33\{\bar{2}, 4, \bar{4}, k\}$ then this forces at least one split on $\hat{\Delta}_4$ and so $c(\hat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$. Add $c(\Delta) \leq \frac{5\pi}{30}$ to $c(\hat{\Delta}_4)$, see Figure 4.84(i). If $l(v_3) = \bar{1}333$ then this forces two splits on $\hat{\Delta}_3$ and so $c(\hat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$. Add $c(\Delta) \leq \frac{5\pi}{30}$ to $c(\hat{\Delta}_3)$, see Figure 4.84(ii). Now let $d(v_3) = 5$. Then $c(\Delta) \leq \frac{2\pi}{30}$. If $l(v_3)$ forces at least one split on $\hat{\Delta}_4$ then add $c(\Delta)$ to $c(\hat{\Delta}_4) \leq \frac{-8\pi}{30}$, see Figure 4.84(iii); or if $l(v_3)$ forces two splits on $\hat{\Delta}_3$ then add $c(\Delta)$ to $c(\hat{\Delta}_3) \leq \frac{-8\pi}{30}$, see Figure 4.84(iv). If $l(v_3)$ forces only one split on $\hat{\Delta}_3$ then $l(v_3)$ and $l(\hat{v}_3)$ involve τ -powers and either $d(\hat{v}_2) \geq 4$ so we add $c(\Delta)$ to $c(\hat{\Delta}_3) \leq c(3, 3, 3, 4, 5) = \frac{-3\pi}{30}$, see Figure 4.84(v); or $d(\hat{v}_2) = 3$. If $d(\hat{v}_2) = 3$ then either this forces at least one split on $\hat{\Delta}_3$ and we add $c(\Delta)$ to $c(\hat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$, see Figure 4.84(vi); or $l(\hat{v}_2)$ forces at least two split on $\hat{\Delta}_2$ and we add $c(\Delta)$ to $c(\hat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$, see Figure 4.84(vii). Now assume $l(v_3)$ does not force any splits on $\hat{\Delta}_3$. Then $l(v_3) \in \{\bar{1}333\{4, \bar{4}\}, \bar{1}3312\}$ in which case checking possible labellings shows that either there is a further split on $\hat{\Delta}_4$ or $d(\hat{v}_4) \geq 4$ and add $c(\Delta)$ to $c(\hat{\Delta}_4) \leq$

$c(3, 3, 3, 4, 5) = \frac{-3\pi}{30}$, see Figure 4.84(viii).

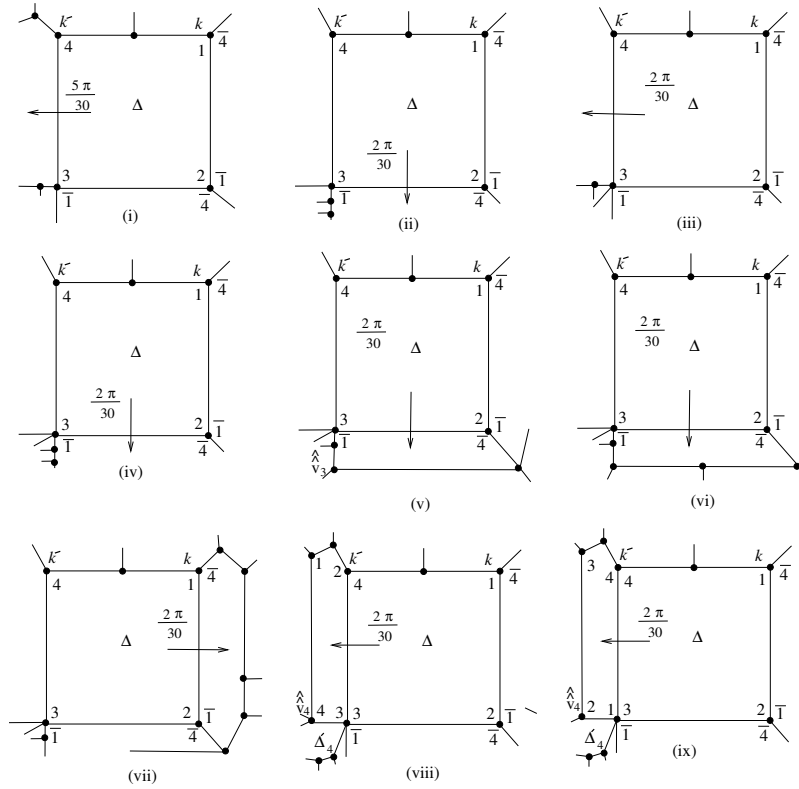


Figure 4.84: $d(u_1) \geq 3$.

Δ is given by Figures 4.64(viii):

A similar argument to above works here. If $d(v_3) = 4$ then this implies $l(v_3) = \bar{1}31\bar{3}$ which forces two splits on both of $c(\hat{\Delta}_3)$ and $c(\hat{\Delta}_4)$. Add $c(\Delta) \leq \frac{\pi}{6}$ to $c(\hat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 3, 4) = \frac{-\pi}{2}$ as in Figure 4.84(i). Also $d(v_3) = 5$ either forces at least one splits on $c(\hat{\Delta}_4)$; or two splits on $c(\hat{\Delta}_3)$; or one split on $c(\hat{\Delta}_3)$, or $l(v_3) \in \{\bar{1}31\bar{2}\bar{4}, \bar{1}3142\}$ each forces at least two splits on Δ'_4 . In the first case and we add $c(\Delta) \leq \frac{2\pi}{30}$ to $c(\hat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$ as in Figure 4.84(iii). In the second case we add $c(\Delta) \leq \frac{2\pi}{30}$ to $c(\hat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$ as in Figure 4.84(iv). Note that in this case Δ may receives extra $\frac{2\pi}{30}$ from $\hat{\Delta}_4$ if $\hat{\Delta}_4$ plays the role of Δ in Figure 4.67(iv) and we add the sum $\leq \frac{4\pi}{30}$ to $c(\hat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$. In the third case either $d(\hat{v}_2) \geq 4$ and we add $c(\Delta) \leq \frac{2\pi}{30}$ to $c(\hat{\Delta}_3) \leq c(3, 3, 3, 4, 5) = \frac{-3\pi}{30}$ as in Figure 4.84(v); or $d(\hat{v}_2) = 3$. If $d(\hat{v}_2) = 3$ then either it forces a split on $\hat{\Delta}_3$ and we add $c(\Delta) \leq \frac{2\pi}{30}$ to

$c(\widehat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$ as in Figure 4.84(vi); or $d(\widehat{v}_2) = 3$ and it forces two splits on $\widehat{\Delta}_2$ and we add $c(\Delta) \leq \frac{2\pi}{30}$ to $c(\widehat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$ as in Figure 4.84(vii). In the forth case, that is $l(v_3) \in \{\overline{1}31\overline{2}4, \overline{1}3142\}$, see Figure 4.84(ix), if $d(\widehat{v}_4) \geq 4$ then we add $c(\Delta)$ to $c(\widehat{\Delta}_4) \leq c(3, 3, 3, 4, 5) = \frac{-3\pi}{30}$; While if $d(\widehat{v}_4) = 3$ then $\widehat{\Delta}_4$ plays the role of Δ in Figures 4.67(vi) or 4.67(viii) and this forces further two splits on Δ'_4 and we add $c(\Delta)$ to $c(\Delta'_4) \leq c(3, 3, 3, 3, 3, 3, 5) = \frac{-4\pi}{30}$ and we distribute $c(\widehat{\Delta}_4)$ as in Figures 4.67(vi) or 4.67(viii).

Δ is given by Figures 4.64(ix):

A similar argument to the argument used in Figure 4.64(i) to distribute the remaining $\frac{5\pi}{30}$ to $\widehat{\Delta}_2$ and $\widehat{\Delta}_3$ can be applied here to distribute $c(\Delta)$ to $c(\widehat{\Delta}_2)$ and $c(\widehat{\Delta}_3)$.

Δ is given by Figures 4.65(i)-(ii):

A similar argument to the one use in Figure 4.64(i)-(ii) where we send $\frac{5\pi}{30}$ to $c(\widehat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$. In Figures 4.65(i) we add the remaining $\frac{5\pi}{30}$ to $c(\widehat{\Delta}_4) \leq \frac{-5\pi}{30}$ when $d(\widehat{v}_4) \geq 4$ or when $l(\widehat{v}_4) \in 4\overline{1}\{\overline{1}, \overline{3}, k\}$ which forces at least one split on $c(\widehat{\Delta}_4)$; while we add the remaining $\frac{5\pi}{30}$ to $c(\widehat{\Delta}_1) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$ when $d(\widehat{v}_4) = 3$ and $l(\widehat{v}_4) = 4\overline{1}4$ which forces at least two splits on $c(\widehat{\Delta}_1)$.

Δ is given by Figure 4.65(iii):

We add $c(\Delta) \leq \frac{5\pi}{30}$ to $c(\widehat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$.

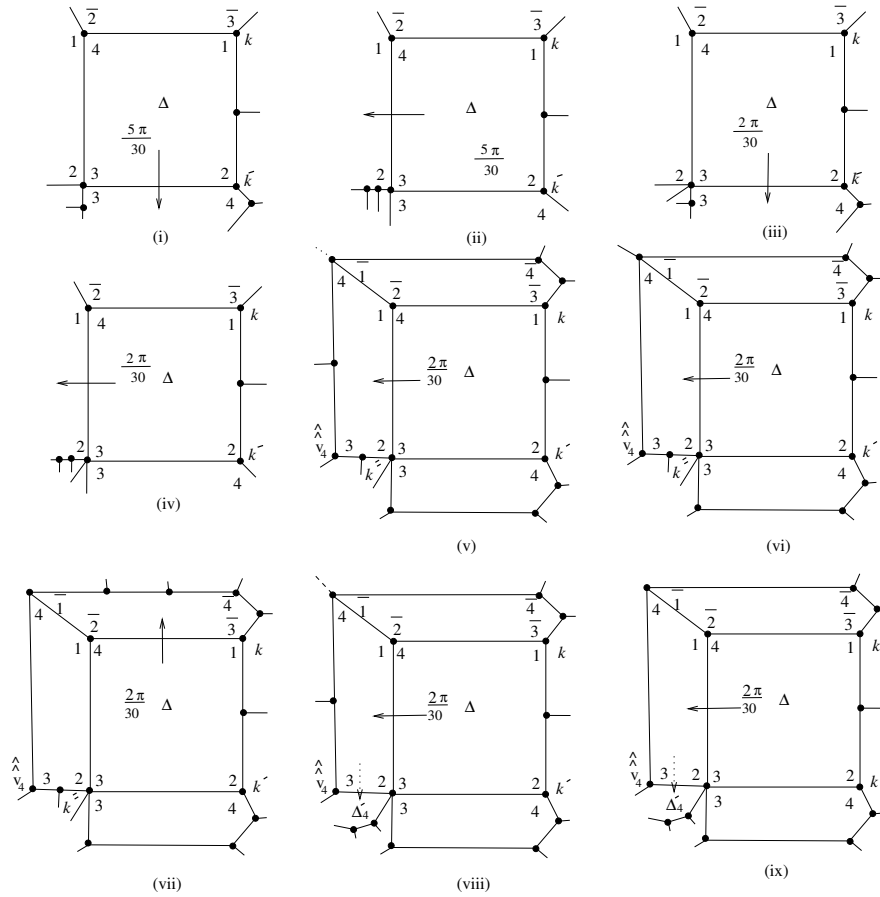
Δ is given by Figure 4.65(iv):

Add $c(\Delta) \leq \frac{5\pi}{30}$ to $c(\widehat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-20\pi}{30}$.

Δ is given by Figure 4.65(v):

First let $d(v_3) = 4$. Then $l(v_3) \in 332\{1, 3, 4, \overline{4}, k\}$. If $l(v_3) \in 332\{1, 4, \overline{4}, k\}$ then this forces at least one split on $\widehat{\Delta}_3$ and we add $c(\Delta) \leq \frac{5\pi}{30}$ to $c(\widehat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$, see Figure 4.85(i); if $l(v_3) = 3323$ then this forces at least two splits on $\widehat{\Delta}_4$ and we add $c(\Delta)$ to $c(\widehat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$, see Figure 4.85(ii). Now let $d(v_3) = 5$. Then $c(\Delta) \leq \frac{2\pi}{30}$. If $l(v_3)$ forces at least one split on $\widehat{\Delta}_3$ then again add $c(\Delta)$ to $c(\widehat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$, see Figure 4.85(iii). Suppose otherwise. If $l(v_3)$ forces at least two splits on $\widehat{\Delta}_4$ then add $c(\Delta)$ to $c(\widehat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$, see Figure 4.85(iv); if not but $l(v_3)$ forces only

one split on $\hat{\Delta}_4$ then add $c(\Delta)$ to $c(\hat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$ when $d(\hat{v}_4) = 3$ with $l(\hat{v}_4) \in 4\bar{1}\{\bar{1}, \bar{3}, k\}$; or to $c(\hat{\Delta}_4) \leq c(3, 3, 3, 3, 4, 5) = \frac{-3\pi}{30}$ when $d(\hat{v}_4) \geq 4$; or add $c(\Delta)$ to $c(\hat{\Delta}_1) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-3\pi}{30}$ when $d(\hat{v}_4) = 3$ with $l(\hat{v}_4) = 4\bar{1}4$ which forces at least two splits on $\hat{\Delta}_1$, see Figures 4.85(v)-(vii). Now assume $l(v_3)$ does not force any splits on $\hat{\Delta}_4$. Then $l(v_3) \in 332\{\bar{1}\bar{1}, \bar{1}\bar{2}, 43, \bar{4}3\}$ each forces at least two splits on Δ'_4 , see Figure 4.85(viii)-(ix). If $l(\hat{v}_4)$ forces two splits on $\hat{\Delta}_4$ or two splits on $\hat{\Delta}_1$ then add $c(\Delta)$ to $c(\hat{\Delta}_4)$ or $c(\hat{\Delta}_1)$ as appropriate. Otherwise add $c(\Delta)$ to $c(\hat{\Delta}_4)$ and distribute the sum as follows. If $d(\hat{v}_4) = 3$ and it forces only one split on $\hat{\Delta}_4$ then $l(\hat{v}_4) = 4\bar{1}k$, see Figure 4.85(viii), and either $d(\hat{v}_4) \geq 4$ and we add $c(\Delta)$ to $c(\hat{\Delta}_4) \leq c(3, 3, 3, 3, 4, 5) = \frac{-3\pi}{30}$; or $l(\hat{v}_4)$ involves a τ -power, in which case $\hat{\Delta}_4$ plays the role of Δ in Figures 4.68(i)-(iii), and forces at least one split on Δ'_4 and $c(\Delta) + c(\hat{\Delta}_4) \leq \frac{4\pi}{30}$ and $c(\Delta'_4) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$. Add the sum to $c(\Delta'_4)$. Now let $d(\hat{v}_4) \geq 4$. If $l(\hat{v}_4)$ forces one split on $\hat{\Delta}_4$ then we add $c(\Delta) \leq \frac{2\pi}{30}$ to

Figure 4.85: $d(u_2) \geq 3$.

$c(\widehat{\Delta}_4) \leq c(3, 3, 3, 4, 5) = \frac{-3\pi}{30}$ as in Figure 4.85(viii); while if it does not force any splits on $\widehat{\Delta}_4$ then $c(\widehat{\Delta}_4) \leq c(3, 3, 4, 5) = \frac{7\pi}{30}$ when $d(\widehat{v}_4) = 3$ and we distribute the sum $c(\Delta) + c(\widehat{\Delta}_4) \leq \frac{9\pi}{30}$ as in Figure 4.62(v)-(viii) in which Δ plays the role of $\widehat{\Delta}_4$; or $c(\widehat{\Delta}_4) \leq c(3, 4, 4, 5) = \frac{2\pi}{30}$ when $d(\widehat{v}_4) \geq 4$ and $c(\Delta'_4) \leq c(3, 3, 3, 3, 4) = \frac{-5\pi}{30}$ and we distribute the sum $c(\Delta) + c(\widehat{\Delta}_4) \leq \frac{4\pi}{30}$ as in Figure 4.83(vi)-(vii) in which Δ plays the role of $\widehat{\Delta}_4$.

Δ is given by Figures 4.65(vi)-(viii):

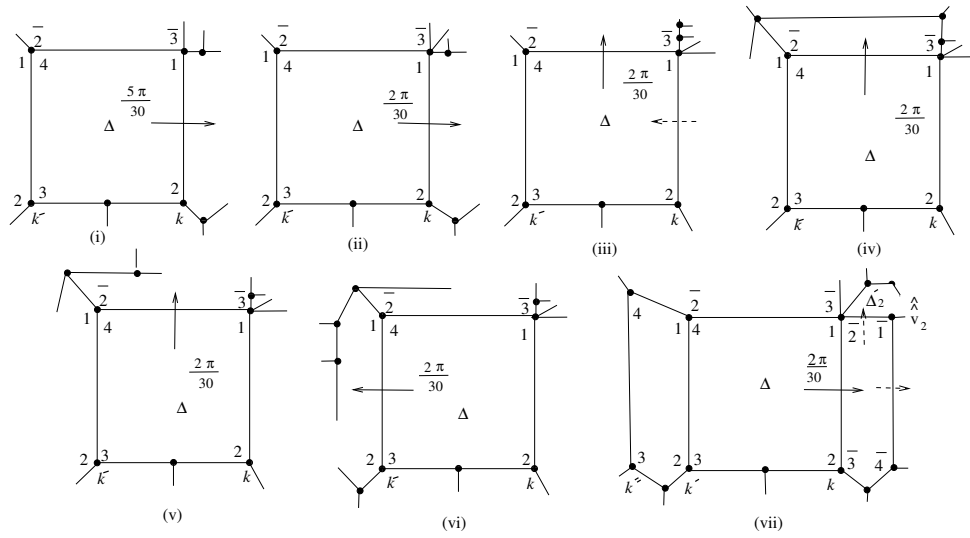
Add $c(\Delta) \leq \frac{5\pi}{30}$ to $c(\widehat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$. Note that in Figure 4.65(vii) Δ may receive extra $\frac{2\pi}{30}$ when $d(v_4) = 5$ and $\widehat{\Delta}_1$ plays the role of Δ in Figure 4.68(i) and we add the extra $\frac{2\pi}{30}$ to $c(\widehat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$ as this forces 2 splits on $\widehat{\Delta}_2$.

Δ is given by Figure 4.66(i):

Add $c(\Delta) \leq \frac{10\pi}{30}$ to $c(\widehat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$.

Δ is given by Figure 4.66(ii):

Let $d(v_1) = 4$. Then this forces $l(v_1) \in \overline{312}\{\overline{1}, k\}$ which forces at least one split on $\widehat{\Delta}_2$ and we add $c(\Delta) \leq \frac{5\pi}{30}$ to $c(\widehat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$, see Figure 4.86(i). Moreover letting $d(v_1) = 5$ implies $c(\Delta) \leq \frac{2\pi}{30}$. If $l(v_1)$ forces at least one split on $\widehat{\Delta}_2$ then add $c(\Delta) \leq \frac{2\pi}{30}$ to $c(\widehat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$, see Figure 4.86(ii). Suppose otherwise. If $l(v_1)$ forces at least two splits on $\widehat{\Delta}_1$ then add $c(\Delta)$ to $c(\widehat{\Delta}_1) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$. Note that in this case Δ may receive extra $\frac{2\pi}{30}$ from $\widehat{\Delta}_2$ if $\widehat{\Delta}_2$ plays the role of Δ in Figure 4.68(ii) and the sum $\frac{4\pi}{30}$ is sent to $c(\widehat{\Delta}_1)$, see Figure 4.86(iii); if $l(v_1)$ forces only one split on $\widehat{\Delta}_1$ then add $c(\Delta)$ to $c(\widehat{\Delta}_1) \leq c(3, 3, 3, 4, 5) = \frac{-3\pi}{30}$ when $d(\widehat{v}_4) \geq 4$ or when $d(\widehat{v}_4) = 3$ with $l(\widehat{v}_4) = 4\overline{1}\{\overline{1}, 4, k\}$, see Figures 4.86(iv)-(v); and add $c(\Delta)$ to $c(\widehat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$ when $d(\widehat{v}_4) = 3$ with $l(\widehat{v}_4) = 4\overline{1}\{\overline{3}, 3\}$ which forces at least 2 splits on $\widehat{\Delta}_4$, see Figure 4.86(vi); Now assume $l(v_1)$ does not force any splits on $\widehat{\Delta}_1$. Then $l(v_1) \in \{\overline{3121\overline{3}}, \overline{312\overline{3}2}\}$ each forces at least two splits on Δ'_2 . Add $c(\Delta)$ to $c(\widehat{\Delta}_2) \leq c(3, 3, 3, 4, 5) = \frac{-3\pi}{30}$ when $d(\widehat{v}_2) \geq 4$ or add $c(\Delta) + c(\widehat{\Delta}_2) \leq \frac{4\pi}{30}$ to $c(\Delta'_2) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$ when $d(\widehat{v}_2) = 3$, see Figure 4.86(vii). Note that when $d(\widehat{v}_2) = 3$ $\widehat{\Delta}_1$ plays the role of Δ in Figure 4.68(ii) or Figure 4.68(vi).

Figure 4.86: $d(u_3) \geq 3$.

Δ is given by Figures 4.66(iii)-(iv), Figures 4.66(vi)-(viii):

Add $c(\Delta)$ to $c(\hat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$. Note that in Figure 4.66(vi) Δ may receive extra $\frac{2\pi}{30}$ from $\hat{\Delta}_2$ which plays the role of Δ in Figure 4.66(xii) but this forces $d(v_1) \geq 5$ in Figure 4.66(vi) and we distribute the extra $\frac{2\pi}{30}$ to $\hat{\Delta}_1$ which would be forced to have 2 splits.

Δ is given by Figure 4.66(v):

A similar argument to the one used in Figure 4.86 with $d(v_1) = 4$ forces $l(v_1) \in \bar{3}14\{1, 2, \bar{3}, 4, k\}$ which all, except $l(v_1) = \bar{3}144$, forces at least one split on $\hat{\Delta}_2$ and we add $c(\Delta) \leq \frac{5\pi}{30}$ to $c(\hat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$ as in Figure 4.86(ii). When $l(v_1) = \bar{3}144$ then this forces at least two splits on $\hat{\Delta}_1$ and we add $c(\Delta) \leq \frac{5\pi}{30}$ to $c(\hat{\Delta}_1) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$, see in Figure 4.87(i). Moreover $d(v_1) = 5$ forces $c(\Delta) \leq \frac{2\pi}{30}$ and either it forces at least one split on $\hat{\Delta}_2$ so we add $c(\Delta)$ to $c(\hat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$ as in Figure 4.86(iii); or it forces two splits on $\hat{\Delta}_1$ so we add $c(\Delta)$ to on $c(\hat{\Delta}_1) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$ as in Figure 4.86(ii). Note that in this case Δ can receive extra $\frac{2\pi}{30}$ from $\hat{\Delta}_2$ if $\hat{\Delta}_2$ plays the role of Δ in 4.66(xii) and we add the sum $\leq \frac{4\pi}{30}$ to $c(\hat{\Delta}_2)$; or it forces exactly one split on $\hat{\Delta}_1$ and as in Figures 4.86(iii)-(vi) we add $c(\Delta)$ to $c(\hat{\Delta}_1) \leq c(3, 3, 3, 4, 5) = \frac{-3\pi}{30}$ or to $c(\hat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$; or $l(v_1) \in \bar{3}14\{\bar{1}\bar{1}, 4\bar{3}\}$ each forces at least two splits on Δ'_2 and we add $c(\Delta)$ to $c(\hat{\Delta}_2) \leq c(3, 3, 3, 4, 5) = \frac{-3\pi}{30}$ when $d(\hat{v}_2) \geq 4$

or add $c(\Delta) + c(\widehat{\Delta}_2) \leq \frac{4\pi}{30}$ to $c(\Delta'_2) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$ when $d(\widehat{v}_2) = 3$, see Figure 4.87(ii). Note that when $d(\widehat{v}_2) = 3$ (that is only possible if $l(v_1) = \overline{3}14\overline{1}\overline{1}$) then $\widehat{\Delta}_1$ plays the role of Δ in Figure 4.66(xii).

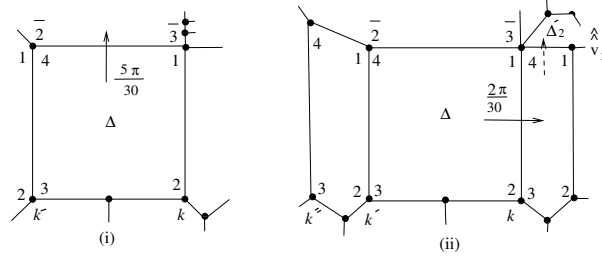


Figure 4.87: $d(u_3) \geq 3$.

Δ is given by Figure 4.66(ix):

A similar argument to the one used in Figure 4.59(vi) can be used here and we add $c(\Delta) \leq \frac{5\pi}{30}$ to $c(\widehat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$ when $d(\widehat{v}_4) = 3$ or when $d(\widehat{v}_4) = 4$ with at least one split on $\widehat{\Delta}_4$; and we add $c(\Delta)$ to $c(\widehat{\Delta}_1) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$ otherwise.

Δ is given by Figure 4.66(x):

Add $c(\Delta) \leq \frac{5\pi}{30}$ to $c(\widehat{\Delta}_1) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-20\pi}{30}$.

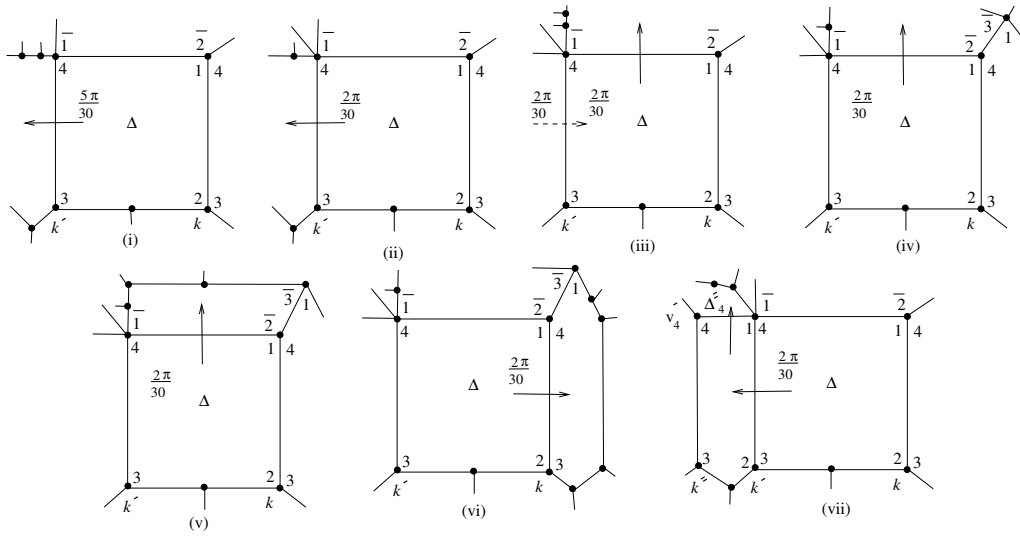
Δ is given by Figure 4.66(xi):

Add $c(\Delta) \leq \frac{5\pi}{30}$ to $c(\widehat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$.

Δ is given by Figure 4.66(xii):

First let $d(v_4) = 4$. Then $l(v_4) = 14\overline{1}\overline{4}$ which forces at least 2 splits on each of $\widehat{\Delta}_1$ and $\widehat{\Delta}_4$ and we add $c(\Delta) \leq \frac{5\pi}{30}$ to $c(\widehat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 3, 4) = \frac{-15\pi}{30}$, see Figure 4.88(i). Now let $d(v_4) = 5$ then $c(\Delta) \leq \frac{2\pi}{30}$. If $l(v_4)$ forces at least one split on $\widehat{\Delta}_4$ and we add $c(\Delta)$ to $c(\widehat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$, see Figure 4.88(ii). Suppose otherwise. If $l(v_4)$ forces at least two splits on $\widehat{\Delta}_1$ then add $c(\Delta)$ to $c(\widehat{\Delta}_1) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$. In this case note that Δ may receive extra $\frac{2\pi}{30}$ from $\widehat{\Delta}_4$ if $\widehat{\Delta}_4$ plays the role of Δ in Figure 4.66(v) and we add the sum $\leq \frac{4\pi}{30}$ to $c(\widehat{\Delta}_1)$, see Figure 4.88(iii). If $l(v_4)$ forces only one split on $\widehat{\Delta}_1$ then add $c(\Delta)$ to $c(\widehat{\Delta}_1) \leq c(3, 3, 3, 4, 5) = \frac{-3\pi}{30}$ when $d(\widehat{v}_1) \geq 4$ or when $d(\widehat{v}_1) = 3$ with

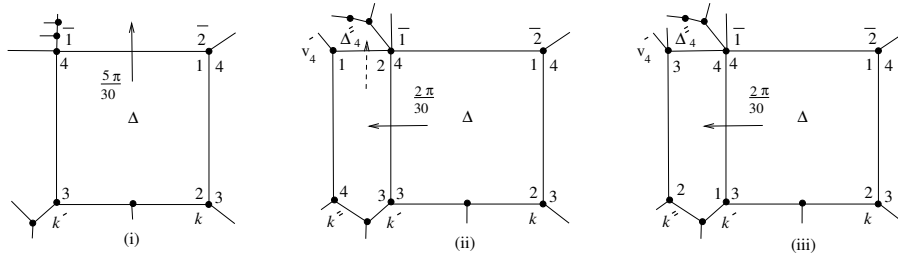
$l(\widehat{v}_1) \in \overline{31}\{1, 2, \overline{4}, k\}$ which forces at least one split on $\widehat{\Delta}_1$, see Figure 4.88(iv)-(v); or add $c(\Delta)$ to $c(\widehat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$ when $d(\widehat{v}_1) = 3$ with $l(\widehat{v}_1) = \overline{31}\overline{3}$ which forces at least 2 splits on $\widehat{\Delta}_4$, see Figure 4.88(vi). If $l(v_4)$ does not force any splits on $\widehat{\Delta}_1$ then $l(v_4) \in 14\overline{1}2\{3, \overline{3}\}$ each forces at least two splits on Δ'_4 , see Figure 4.88(vii). Similarly to above argument if $d(v'_4) \geq 4$ then add $c(\Delta)$ to $c(\widehat{\Delta}_4) \leq c(3, 3, 3, 4, 5) = \frac{-3\pi}{30}$; while if $d(v'_4) = 3$ then there are two cases. We add the sum $c(\Delta) + c(\widehat{\Delta}_4) \leq \frac{4\pi}{30}$ to $c(\Delta'_4) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$ in which case $d(v'_4) = 3$ and $\widehat{\Delta}_4$ plays the role of Δ in Figure 4.66(v) while we add $\frac{2\pi}{30}$ to each of $c(\Delta'_4) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$ and $\widehat{\Delta}_4$ in which case $d(v'_4) = 3$ and $\widehat{\Delta}_4$ plays the role of Δ in Figure 4.66(vi).

Figure 4.88: $d(u_3) \geq 3$.

Δ is given by Figure 4.66(xiii):

We use similar distribution as in Figure 4.88. Letting $d(v_4) = 4$ forces $l(v_4) \in 24\overline{1}\{\overline{1}, 2, k\}$ and we add $c(\Delta) \leq \frac{5\pi}{30}$ to $c(\widehat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$ when $l(v_4) \in 24\overline{1}\{2, k\}$ which forces at least one split on $\widehat{\Delta}_4$ as in Figure 4.88(i); and add $c(\Delta)$ to $c(\widehat{\Delta}_1) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$ otherwise as the labelling would force at least two splits on $\widehat{\Delta}_1$, see Figure 4.89(i). Now let $d(v_4) = 5$. Then if $l(v_4)$ forces at least one split on $\widehat{\Delta}_4$ we add $c(\Delta)$ to $c(\widehat{\Delta}_4) \leq \frac{-8\pi}{30}$ as in Figure 4.88(ii). Suppose otherwise. If $l(v_4)$ forces two splits on $\widehat{\Delta}_1$ then we add $c(\Delta)$ to $c(\widehat{\Delta}_1) \leq \frac{-8\pi}{30}$ as in Figure 4.88(iii); while if $l(v_4)$ forces one split on $\widehat{\Delta}_1$ then we add $c(\Delta)$ to $c(\widehat{\Delta}_1)$ or

$c(\widehat{\Delta}_2)$ as in Figures 4.88(iv)-(vi). Note that $d(\widehat{v}_1) = 3$ implies $l(\widehat{v}_1) \in \overline{31}\{1, 2, \overline{4}, k\}$ each forces at least one split on $\widehat{\Delta}_1$. If $l(v_4)$ does not force any splits on $\widehat{\Delta}_4$ nor on $\widehat{\Delta}_1$ then $l(v_4) \in 24\overline{1}\{23, 2\overline{3}, 3\overline{1}, \overline{3}1\}$ each forces at least two splits on Δ''_4 , see Figure 4.89(ii). If $d(v'_4) \geq 4$ then add $c(\Delta)$ to $c(\widehat{\Delta}_4) \leq c(3, 3, 3, 4, 5) = \frac{-3\pi}{30}$; while if $d(v'_4) = 3$ (which only occur when $l(v_4) = 24\overline{1}23$) we add the sum $c(\Delta) + c(\widehat{\Delta}_4) \leq \frac{4\pi}{30}$ to $c(\Delta''_4) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$ where $\widehat{\Delta}_4$ plays the role of Δ in Figure 4.68(iii).


 Figure 4.89: $d(u_3) \geq 3$.

Δ is given by Figure 4.66(xiv):

Again we use the same distribution as above. Letting $d(v_4) = 4$ forces $l(v_4) \in 44\overline{1}\{\overline{2}, 3, \overline{3}, 4, k\}$ and we add $c(\Delta) \leq \frac{5\pi}{30}$ to $c(\widehat{\Delta}_1) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$ when $l(v_4) = 44\overline{1}4$ as this forces 2 splits on $\widehat{\Delta}_1$ as in Figure 4.89(i); or add it to $c(\widehat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$ otherwise as in Figure 4.88(i) since the labelling forces at least one split on $\widehat{\Delta}_4$. Also $d(v_4) = 5$ forces $c(\Delta) \leq \frac{2\pi}{30}$ and we add $c(\Delta)$ to $c(\widehat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$ if $l(v_4)$ forces at least one split on $\widehat{\Delta}_4$ as in Figure 4.88(ii). Suppose otherwise. If $l(v_4)$ forces two splits on $\widehat{\Delta}_1$ then we add $c(\Delta)$ to $c(\widehat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$ as in Figure 4.88(iii); while if $l(v_4)$ forces one split on $\widehat{\Delta}_1$ then we add $c(\Delta)$ to $c(\widehat{\Delta}_1)$ or $c(\widehat{\Delta}_2)$ as in Figure 4.88(iv)-(vi). Note that $d(\widehat{v}_1) = 3$ implies $l(\widehat{v}_1) = \overline{31}\{1, 2, \overline{4}, \overline{3}, k\}$. If $l(v_4)$ does not force any splits on $\widehat{\Delta}_1$ then $l(v_4) \in 44\overline{1}\{21, 22, 34, \overline{3}4\}$ which forces at least two splits on Δ''_4 each, see Figure 4.89(iii). Checking shows that $d(v'_4) \geq 4$ and we add $c(\Delta)$ to $c(\widehat{\Delta}_4) \leq c(3, 3, 3, 4, 5) = \frac{-3\pi}{30}$.

Δ is given by Figure 4.67(i):

$c(\Delta) \leq \frac{10\pi}{30}$. Add $\frac{5\pi}{30}$ to $c(\widehat{\Delta}_1) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$ and add the remaining to $c(\widehat{\Delta}_2)$ or $c(\widehat{\Delta}_3)$ as follows. If $d(\widehat{v}_2) = 3$ then $l(\widehat{v}_2) = \overline{2}3k''$ which

in turn forces at least one more split on $\widehat{\Delta}_2$ and we add the remaining $\frac{5\pi}{30}$ to $c(\widehat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$. If $d(\widehat{v}_2) = 4$ and forces at least one split on $\widehat{\Delta}_3$ then we add the remaining $\frac{5\pi}{30}$ to $c(\widehat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$. Otherwise add the remaining $\frac{5\pi}{30}$ to $c(\widehat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$.

Δ is given by Figure 4.67(ii):

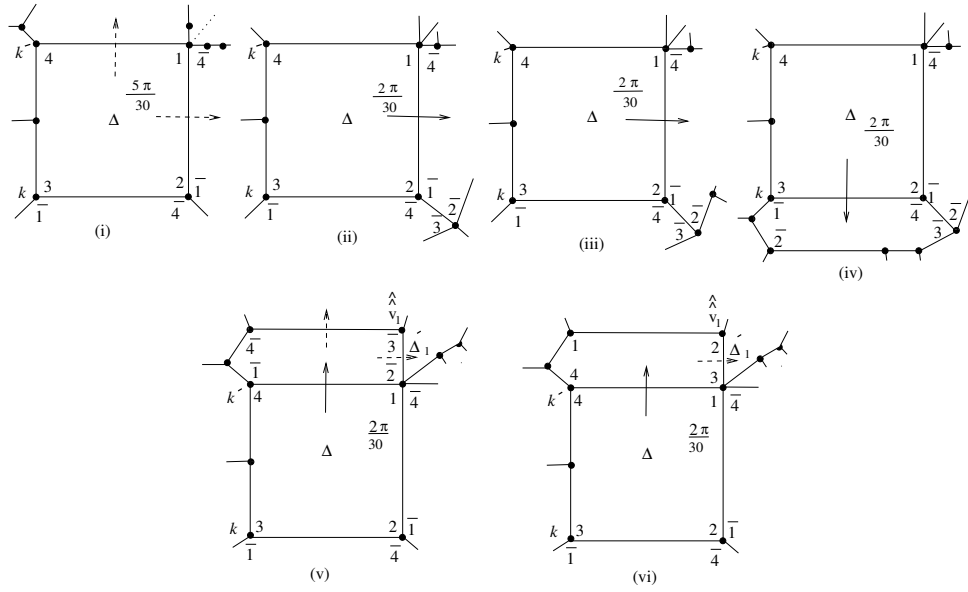
$c(\Delta) \leq \frac{10\pi}{30}$. Add $\frac{5\pi}{30}$ to $c(\widehat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$ and add the remaining $\frac{5\pi}{30}$ to $c(\widehat{\Delta}_2)$ or $c(\widehat{\Delta}_1)$ as follows. If $d(\widehat{v}_1) = 3$ then this implies $l(\widehat{v}_1) \in \overline{31}\{1, k''\}$ which in turn forces at least one more split on $\widehat{\Delta}_2$ and we add the remaining $\frac{5\pi}{30}$ to $c(\widehat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$. If $d(\widehat{v}_1) = 4$ such that $l(\widehat{v}_1)$ forces a split on $\widehat{\Delta}_1$ then we add the remaining $\frac{5\pi}{30}$ to $c(\widehat{\Delta}_1) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$. Otherwise, that is $d(\widehat{v}_1) = 4$ and $l(\widehat{v}_1)$ forces a split on $\widehat{\Delta}_2$ or $d(\widehat{v}_1) \geq 5$, we add the remaining $\frac{5\pi}{30}$ to $c(\widehat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$.

Δ is given by Figure 4.67(iii):

A similar argument as the one used in Figure 4.86 is applied here. letting $d(v_1) = 4$ implies $l(v_1) \in \overline{214}\{\overline{1}, \overline{2}, k\}$ each forces at least one split on $\widehat{\Delta}_1$ and we add $c(\Delta) \leq \frac{5\pi}{30}$ to $c(\widehat{\Delta}_1) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$; or two splits on $\widehat{\Delta}_2$ and we add $c(\Delta)$ to $c(\widehat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$, see Figure 4.90(i). Moreover letting $d(v_1) = 5$ implies $c(\Delta) \leq \frac{2\pi}{30}$ and either there is at least one split on $\widehat{\Delta}_1$ and again we add $c(\Delta)$ to $c(\widehat{\Delta}_1) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$; or two splits on $\widehat{\Delta}_2$ and we add $c(\Delta)$ to $c(\widehat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$ as in Figure 4.90(i); or $l(v_1)$ forces no splits on $\widehat{\Delta}_1$ and exactly one split on $\widehat{\Delta}_2$ and we distribute $c(\Delta)$ as shown in Figures 4.90(ii)-(iv); or $l(v_1) \in \overline{214}\{\overline{1}, \overline{4}, \overline{2}, \overline{4}, \overline{41}\}$ each forces at least two splits on Δ'_1 , see Figure 4.90(v). In the last case if $d(\widehat{v}_1) \geq 4$ then we add $c(\Delta)$ to $c(\widehat{\Delta}_1) \leq c(3, 3, 3, 4, 5) = \frac{-3\pi}{30}$; while if $d(\widehat{v}_1) = 3$ then $\widehat{\Delta}_1$ plays the role of Δ in Figures 4.64(v)-(vi) and we add $\frac{2\pi}{30}$ to $c(\Delta'_1) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$ and $c(\widehat{\Delta}_1) \leq \frac{2\pi}{30}$ as in Figures 4.64(v)-(vi).

Δ is given by Figures 4.67(iv):

As above letting $d(v_1) = 4$ implies $l(v_1) \in 31\overline{4}\{1, 2, 3, \overline{4}, k\}$ each forces either at least one split on $\widehat{\Delta}_1$ and we add $c(\Delta) \leq \frac{5\pi}{30}$ to $c(\widehat{\Delta}_1) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$; or two splits on $\widehat{\Delta}_2$ and we add $c(\Delta)$ to $c(\widehat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$ as Figure 4.90(i). Moreover letting $d(v_1) = 5$ implies $c(\Delta) \leq \frac{2\pi}{30}$ and either there is at least one split on $\widehat{\Delta}_1$ and again we add $c(\Delta)$ to $c(\widehat{\Delta}_1) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$; or

Figure 4.90: $d(u_4) \geq 3$.

two splits on $\hat{\Delta}_2$ and we add $c(\Delta)$ to $c(\hat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$ as in Figure 4.90(i); or $l(v_1)$ forces no splits on $\hat{\Delta}_1$ and exactly one split on $\hat{\Delta}_2$ and we distribute $c(\Delta)$ as shown in Figures 4.90(ii)-(iv); or $l(v_1) \in 314\bar{1}\bar{2}, \bar{2}\bar{1}, \bar{2}\bar{2}, \bar{4}\bar{3}$ each forces at least two splits on Δ'_1 , see Figure 4.90(vi). In the last case if $d(\hat{v}_1) \geq 4$ then we add $c(\Delta)$ to $c(\hat{\Delta}_1) \leq c(3, 3, 3, 4, 5) = \frac{-3\pi}{30}$; while if $d(\hat{v}_1) = 3$ (can only occur when $l(v_1) = 314\bar{2}\bar{1}$) then $\hat{\Delta}_1$ plays the role of Δ in Figure 4.64(viii) and we add the sum $c(\Delta) + c(\hat{\Delta}_1) \leq \frac{4\pi}{30}$ to $c(\Delta'_1) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$.

Δ is given by Figures 4.67(v)-(viii):

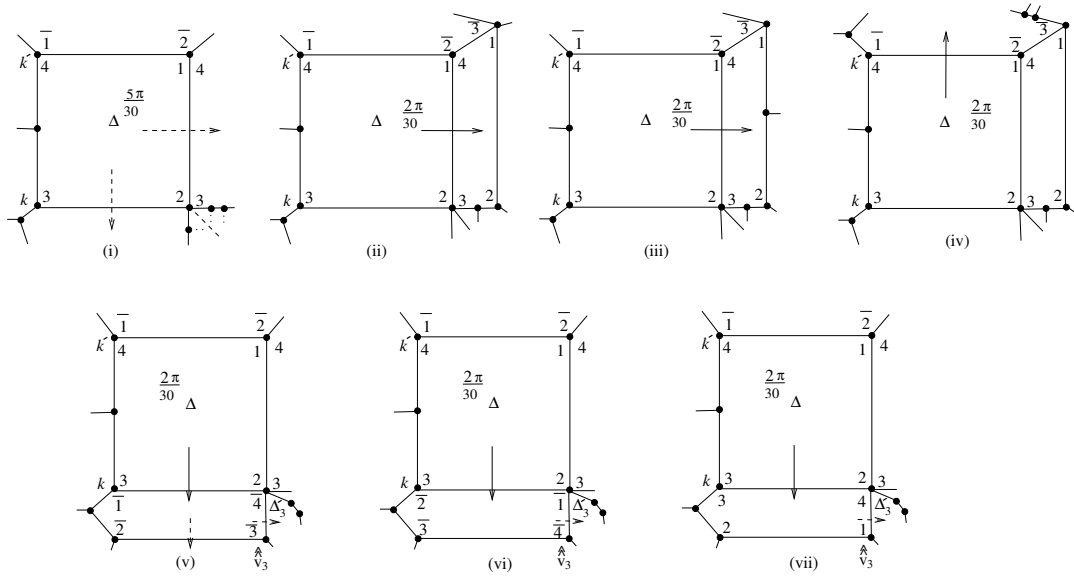
Add $c(\Delta) \leq \frac{5\pi}{30}$ to $c(\hat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$. Note that in Figures 4.67(vi) and 4.67(viii) Δ may receive extra $\frac{2\pi}{30}$ from $\hat{\Delta}_1$ but that forces $d(v_1) \geq 5$ and $l(v_1)$ forces splits on $\hat{\Delta}_2$ and we add the extra $\frac{2\pi}{30}$ to $c(\hat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$.

Δ is given by Figure 4.68(i):

First let $d(v_2) = 4$. Then $l(v_1) \in 324\bar{1}\{1, 2, 3, \bar{4}, k\}$ each forces either at least one split on $\hat{\Delta}_3$ and we add $c(\Delta) \leq \frac{5\pi}{30}$ to $c(\hat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$; or two splits on $\hat{\Delta}_2$ and we add $c(\Delta)$ to $c(\hat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$, see Figure 4.91(i). Now let $d(v_2) = 5$. Then $c(\Delta) \leq \frac{2\pi}{30}$. If $l(v_2)$ forces at least one split on $\hat{\Delta}_3$ or

two splits on $\widehat{\Delta}_2$ then again add $c(\Delta)$ to $c(\widehat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$ or to $c(\widehat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$ as in Figure 4.91(i). Note that when $d(v_2) = 5$ and $l(v_2)$ forces at least one split on $\widehat{\Delta}_3$ then Δ may receive extra $\frac{2\pi}{30}$ from $\widehat{\Delta}_2$ if $\widehat{\Delta}_2$ plays the role of Δ in Figure 4.65(vii) and this forces further split on $\widehat{\Delta}_3$ and we add the sum $\leq \frac{4\pi}{30}$ to $c(\widehat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 3, 5) = \frac{-18\pi}{30}$. This last argument applies to the next 2 Figures 4.68(ii)-(iii).

Suppose otherwise. If $l(v_2)$ forces exactly one split on $\widehat{\Delta}_3$ then we add $c(\Delta)$ to $\widehat{\Delta}_2$ or $\widehat{\Delta}_1$ as shown in Figures 4.91(ii)-(iv). If $l(v_2)$ forces no splits on $\widehat{\Delta}_3$ then $l(v_2) \in 32\overline{4}\{2\overline{2}, 4\overline{3}\}$ each forces at least two splits on Δ'_3 , see Figure 4.91(v). In the last case if $d(\widehat{v}_3) \geq 4$ then we add $c(\Delta)$ to $c(\widehat{\Delta}_3) \leq c(3, 3, 3, 4, 5) = \frac{-3\pi}{30}$; while if $d(\widehat{v}_3) = 3$ (can only occur when $l(v_2) = 32\overline{4}\overline{4}3$) then $\widehat{\Delta}_3$ plays the role of Δ in Figure 4.65(vii) and we add $\frac{2\pi}{30}$ to $c(\Delta'_3) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$ and we distribute $c(\Delta)$ as in Figure 4.65(vii).


 Figure 4.91: $d(u_4) \geq 3$.

Δ is given by Figure 4.68(ii):

As above letting $d(v_2) = 4$ implies $l(v_1) \in 32\overline{1}\{1, 2, k\}$ each forces either at least one split on $\widehat{\Delta}_3$ and we add $c(\Delta) \leq \frac{5\pi}{30}$ to $c(\widehat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$; or two splits on $\widehat{\Delta}_2$ and we add $c(\Delta)$ to $c(\widehat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$ as in Figure 4.91(i). Moreover letting $d(v_2) = 5$ implies $c(\Delta) \leq \frac{2\pi}{30}$ and either $l(v_2)$ forces at

least one split on $\widehat{\Delta}_3$ or two splits on $\widehat{\Delta}_2$ and again as in Figure 4.91(i) we add $c(\Delta)$ to $c(\widehat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$ or to $c(\widehat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$. Note that when $l(v_2)$ forces two splits on $\widehat{\Delta}_2$ then Δ may receive extra $\frac{2\pi}{30}$ from $\widehat{\Delta}_3$ if $\widehat{\Delta}_3$ plays the role of Δ in Figure 4.66(vii) and we add the sum $\leq \frac{4\pi}{30}$ to $c(\widehat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$. If $l(v_2)$ forces exactly one split on $\widehat{\Delta}_3$ then we add $c(\Delta)$ to $\widehat{\Delta}_2$ or $\widehat{\Delta}_1$ as shown in Figures 4.91(ii)-(iv); otherwise $l(v_2) \in 32\bar{1}\{23, 3\bar{1}, 3\bar{2}\}$ each forces at least two splits on Δ'_3 , see Figure 4.91(vi). In the last case if $d(\widehat{v}_3) \geq 4$ then we add $c(\Delta)$ to $c(\widehat{\Delta}_3) \leq c(3, 3, 3, 4, 5) = \frac{-3\pi}{30}$; while if $d(\widehat{v}_3) = 3$ (can only occur when $l(v_2) = 32\bar{1}3\{\bar{1}, \bar{2}\}$) then $\widehat{\Delta}_3$ plays the role of Δ in Figure 4.66(ii) and we add the sum $c(\Delta) + c(\widehat{\Delta}_3) \leq \frac{4\pi}{30}$ to $c(\Delta'_3) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$.

Δ is given by Figure 4.68(iii):

As above letting $d(v_2) = 4$ implies $l(v_1) \in 324\{1, 2, 3, 4, k\}$ each forces either at least one split on $\widehat{\Delta}_3$ and we add $c(\Delta) \leq \frac{5\pi}{30}$ to $c(\widehat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$; or two splits on $\widehat{\Delta}_2$ and we add $c(\Delta)$ to $c(\widehat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$ as in Figure 4.91(i). Moreover letting $d(v_2) = 5$ implies $c(\Delta) \leq \frac{2\pi}{30}$ and either $l(v_2)$ forces at least one split on $\widehat{\Delta}_3$ or two splits on $\widehat{\Delta}_2$ and again as in Figure 4.91(i) we add $c(\Delta)$ to $c(\widehat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$ or to $c(\widehat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$. Note that when $l(v_2)$ forces two splits on $\widehat{\Delta}_2$ then Δ may receive extra $\frac{2\pi}{30}$ from $\widehat{\Delta}_3$ if $\widehat{\Delta}_3$ plays the role of Δ in Figure 4.89(ii) and we add the sum $\leq \frac{4\pi}{30}$ to $c(\widehat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$. If $l(v_2)$ forces exactly one split on $\widehat{\Delta}_3$ then we add $c(\Delta)$ to $\widehat{\Delta}_2$ or $\widehat{\Delta}_1$ as shown in Figures 4.91(ii)-(iv); otherwise $l(v_2) \in 324\{\bar{1}\bar{1}, \bar{1}\bar{2}, 43\}$ each forces at least two splits on Δ'_3 , see Figure 4.91(vii). In the last case if $d(\widehat{v}_3) \geq 4$ then we add $c(\Delta)$ to $c(\widehat{\Delta}_3) \leq c(3, 3, 3, 4, 5) = \frac{-3\pi}{30}$; while if $d(\widehat{v}_3) = 3$ (can only occur when $l(v_2) = 324\bar{1}\{\bar{1}, \bar{2}\}$) then $\widehat{\Delta}_3$ plays the role of Δ in Figure 4.66(xiii) and we add the sum $c(\Delta) + c(\widehat{\Delta}_3) \leq \frac{4\pi}{30}$ to $c(\Delta'_3) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$.

Δ is given by Figures 4.68(iv)-(viii):

In Figures 4.67(iv)-(ix) add $c(\Delta) \leq \frac{5\pi}{30}$ to $c(\widehat{\Delta}_1) \leq c(3, 3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$. Note that Δ may receive extra $\frac{2\pi}{30}$ from $\widehat{\Delta}_3$ if $\widehat{\Delta}_3$ plays the role of Δ in Figure 4.86(vii) and we add the extra $\leq \frac{2\pi}{30}$ to $c(\widehat{\Delta}_2) \leq c(3, 3, 3, 3, 3, 5) = \frac{-8\pi}{30}$ as in this case $d(v_2) = 5$ and lv_2 forces 2 splits on $\widehat{\Delta}_2$.

Δ is given by Figure 4.68(ix):

Add $c(\Delta) \leq \frac{5\pi}{30}$ to $c(\hat{\Delta}_1) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$.

Δ is given by Figures 4.68(x):

Add $c(\Delta) \leq \frac{5\pi}{30}$ to $c(\hat{\Delta}_1) \leq c(3, 3, 3, 3, 3, 4) = \frac{-5\pi}{30}$.

Δ is given by Figures 4.68(xi):

Add $c(\Delta) \leq \frac{5\pi}{30}$ to $c(\hat{\Delta}_3) \leq c(3, 3, 3, 3, 3, 3) = \frac{-10\pi}{30}$.

Let $\hat{\Delta} \in \{\hat{\Delta}_i, \hat{\Delta}'_i, \Delta'_i\}$, $1 \leq i \leq 4$ and assume that $\hat{\Delta}$ is receiving positive curvature from exactly one region. Then as shown in the discussion of Figures 4.59- 4.68 that $c^*(\hat{\Delta}) \leq 0$. Also $\hat{\Delta}$ never receives curvature across a split edge or across more than three edges. Now assume that $\hat{\Delta}$ is receiving positive curvature from two or more neighbouring regions. If $\bar{2}14$ is a vertex labelling of some $l(\hat{v}_i)$ then $\hat{\Delta}$ may receive from 2 adjacent edges and the cases are as follows. (In what follows checking means checking Figures 4.59- 4.90).

First let $l(v_1) = \bar{2}14$. Then checking shows that $d(\hat{\Delta}) \geq 6$. Let $d(\hat{\Delta}) \geq 7$. Then $\hat{\Delta}$ would have 2 edges which split with at least one edge having 2 splits. Then either it receives at most $\frac{5\pi}{30}$ across each of the other two edges and so $c^*(\hat{\Delta}) \leq c(3, 3, 3, 3, 3, 3) + 2(\frac{5\pi}{30}) = 0$; or it receives up to $\frac{15\pi}{30}$ from 2 neighbouring regions but this would force at least one vertex of degree ≥ 4 and so $c^*(\hat{\Delta}) \leq c(3, 3, 3, 3, 3, 4) + \frac{15\pi}{30} = 0$. An example for the former case is when $\hat{\Delta} = \hat{\Delta}_2$ in Figure 4.70(iv) and $l(\hat{v}_2) = \bar{2}\bar{3}\bar{3}$, then $\hat{\Delta}_2$ may receive $\leq \frac{5\pi}{30}$ from $\hat{\Delta}_3$ if it is the region in Figure 4.61(i) and it may receive $\leq \frac{4\pi}{30}$ from Δ . An example for the latter case is when $\hat{\Delta} = \hat{\Delta}_2$ in Figure 4.71(v) which can receive up to $\frac{7\pi}{30}$ from each of Δ and $\hat{\Delta}_3$ but this forces at least one more split on $\hat{\Delta}_2$; another example is $\hat{\Delta} = \hat{\Delta}_4$ in Figure 4.73(iv) where it receives $\leq \frac{10\pi}{30}$ from Δ and it may receive at most $\frac{5\pi}{30}$ from $\hat{\Delta}_1$ if it plays the role of Δ in Figure 4.67(i).

Now let $d(\hat{\Delta}) = 6$. Then either $\hat{\Delta}$ has 2 edges with one split each or one edge with 2 splits. In the former case the only example is $\hat{\Delta} = \hat{\Delta}_4$ in Figure 4.77(iv) where it may receive $\leq \frac{7\pi}{30}$ from $\hat{\Delta}$ and $\leq \frac{2\pi}{30}$ from $\hat{\Delta}_1$ when $d(\hat{v}_4) = 4$ which forces one split on $\hat{\Delta}_4$ and then $c^*(\hat{\Delta}_4) \leq c(3, 3, 3, 3, 4, 4) + \frac{7\pi}{30} + \frac{2\pi}{30} < 0$. In the latter case $\hat{\Delta}$ receives across exactly 2 edges or across 3. If $\hat{\Delta}$ receives across exactly 2 edges then this either forces one vertex of degree ≥ 5 and the most received is no more than $\frac{8\pi}{30}$ in total and $c^*(\hat{\Delta}) \leq c(3, 3, 3, 3, 3, 5) + \frac{5\pi}{30} + \frac{3\pi}{30} = 0$; or it forces one

vertex of degree ≥ 4 and one vertex of degree ≥ 5 and it may receive $\leq \frac{7\pi}{30} + \frac{4\pi}{30}$ and so $c^*(\hat{\Delta}_4) \leq c(3, 3, 3, 3, 4, 5) + \frac{7\pi}{30} + \frac{4\pi}{30} < 0$. An example of the first case is when $\hat{\Delta} = \hat{\Delta}_4$ in Figure 4.80(iii) which receives $\leq \frac{4\pi}{30}$ from Δ and it can as well receive $\leq \frac{4\pi}{30}$ from $\hat{\Delta}_1$ if $\hat{\Delta}_1$ plays the role of Δ in Figure 4.60(i) with $d(v_3) \geq 5$; and another example is $\hat{\Delta} = \hat{\Delta}_2$ in Figure 4.70(v) with $d(\hat{v}_2) \geq 5$ which receives $\leq \frac{5\pi}{30}$ from Δ and can receive $\leq \frac{3\pi}{30}$ from $\hat{\Delta}_3$. An example of the second case is $\hat{\Delta} = \hat{\Delta}_4$ in Figure 4.76(iii) which receives $\leq \frac{7\pi}{30}$ from Δ and may receive $\leq \frac{4\pi}{30}$ from $\hat{\Delta}_1$ if $\hat{\Delta}_1$ plays the role of Δ in Figure 4.80(ii) which forces $d(\hat{v}_4) \geq 4$, and another example is $\hat{\Delta} = \hat{\Delta}_2$ in Figure 4.80(ii) which receives $\leq \frac{4\pi}{30}$ from Δ and may receive $\leq \frac{5\pi}{30}$ from $\hat{\Delta}_3$ if $\hat{\Delta}_3$ plays the role of Δ in Figure 4.61(i). Now let $\hat{\Delta}$ receives across 3 edges. Then placing the third positive region is only possible in the last three examples immediately above but this forces the degree 5 vertex to increase to degree ≥ 6 and the third region would send at most $\frac{3\pi}{30}$ and so in the first example $c^*(\hat{\Delta}_4) \leq c(3, 3, 3, 3, 3, 6) + \frac{3\pi}{30} + \frac{2\pi}{30} + \frac{5\pi}{30} = 0$ and in the other two $c^*(\hat{\Delta}_2) \leq c(3, 3, 3, 3, 4, 6) + \frac{4\pi}{30} + \frac{7\pi}{30} + \frac{3\pi}{30} < 0$.

Now let $l(v_2) = \bar{2}14$. Then checking shows that $d(\hat{\Delta}) \geq 5$. Let $d(\hat{\Delta}) \geq 7$. Then either $\hat{\Delta}$ receives at most $2(\frac{5\pi}{30})$ across 2 edges and $c^*(\hat{\Delta}) \leq c(3, 3, 3, 3, 3, 3, 3) + 2(\frac{5\pi}{30}) = 0$; or $\hat{\Delta}$ receives up to $\frac{15\pi}{30}$ across 2 edges but this would force at least one vertex of degree ≥ 4 and $c^*(\hat{\Delta}) \leq c(3, 3, 3, 3, 3, 3, 4) + \frac{15\pi}{30} = 0$. Examples for the first case are when $\hat{\Delta} = \hat{\Delta}_1$ in Figure 4.63(ii), then $\hat{\Delta}_1$ may receive $\frac{5\pi}{30}$ from each of Δ and $\hat{\Delta}_4$; and when $\hat{\Delta} = \hat{\Delta}_1$ in Figure 4.72(iv) which may receive $\frac{3\pi}{30}$ from $\hat{\Delta}_4$ but this forces at least one more edge to split and Δ may send up to $\frac{7\pi}{30}$ when $d(\hat{v}_4) \geq 5$ or $d(\hat{v}_4) = 4$ and forces a split on $\hat{\Delta}$. Examples for the second case are $\hat{\Delta} = \hat{\Delta}_1$ in Figure 4.61(iii) which may receive $\frac{5\pi}{30}$ from Δ and $\frac{7\pi}{30}$ from $\hat{\Delta}_2$ which forces at least one vertex of degree ≥ 4 ; and $\hat{\Delta} = \hat{\Delta}_1$ in Figure 4.73(v) which may receive $\frac{10\pi}{30}$ from Δ and $\hat{\Delta}_4$ may send up to $\frac{5\pi}{30}$ if it is Δ in Figure 4.66(i).

Now let $d(\hat{\Delta}) = 6$. Then either it has one vertex of degree ≥ 5 and in this case it receives at most from 2 region a sum $\leq \frac{8\pi}{30}$ and so $c^*(\hat{\Delta}) \leq c(3, 3, 3, 3, 3, 5) + \frac{8\pi}{30} = 0$; or it has one vertex of degree ≥ 4 and one vertex of degree ≥ 5 and in this case $\hat{\Delta}$ may receive from two or three regions. An example for the former case is when $\hat{\Delta} = \hat{\Delta}_1$ in Figure 4.70(v) which can receive at most $\leq \frac{3\pi}{30}$ through the edge (4, 1) only if $d(\hat{v}_4) \geq 5$ with $\leq \frac{5\pi}{30}$ from Δ , and another example is when $\hat{\Delta} = \hat{\Delta}_1$ in Figure 4.61(iii) which may receive $\leq \frac{5\pi}{30}$ from Δ and $\leq \frac{3\pi}{30}$ from $\hat{\Delta}_2$

if it forces a vertex of degree ≥ 5 . In the latter case if $\widehat{\Delta}$ receives from 2 regions then it may receive up to $\frac{10\pi}{30}$ and $c^*(\widehat{\Delta}) \leq c(3, 3, 3, 3, 4, 5) + \frac{10\pi}{30} < 0$ as for example $\widehat{\Delta} = \widehat{\Delta}_1$ in Figure 4.77(v) which may receive from 2 regions, namely $\frac{7\pi}{30}$ from Δ and $\frac{3\pi}{30}$ from Δ'_1 ; and another example is when $\widehat{\Delta} = \widehat{\Delta}_1$ in Figure 4.80(ii) which may receive from 2 regions, namely $\leq \frac{4\pi}{30}$ from Δ and $\leq \frac{7\pi}{60}$ from $\widehat{\Delta}_4$. If $\widehat{\Delta}$ receives from 3 regions then checking shows that this is only possible in the last example and a third region (Δ'_1 in Figure 4.80(iv)) can be placed such that it send positive curvature to $\widehat{\Delta} = \widehat{\Delta}_1$. In this case either $\widehat{\Delta} = \widehat{\Delta}_1$ receives extra $\leq \frac{2\pi}{30}$ and at least one vertex is forced to increase from degree 3 to degree 4 and hence $c^*(\widehat{\Delta}_1) \leq c(3, 3, 3, 3, 4, 5) + \frac{4\pi}{30} + \frac{7\pi}{60} + \frac{2\pi}{30} < 0$ (if Δ'_1 as in Figure 4.83(i) for example); or it receives extra $\leq \frac{3\pi}{30}$ and this forces $d(v_3) \geq 6$ which implies that Δ sends $\leq \frac{2\pi}{30}$ (if Δ'_1 as in Figure 4.59(v) for example) and hence $c^*(\widehat{\Delta}) \leq c(3, 3, 3, 3, 3, 6) + \frac{2\pi}{30} + \frac{7\pi}{60} + \frac{3\pi}{30} < 0$.

Now let $d(\widehat{\Delta}) = 5$. Then the only example is $\widehat{\Delta} = \widehat{\Delta}_4$ in Figure 4.85(viii) which receives $\frac{2\pi}{30}$ from Δ and may receive $\frac{2\pi}{30}$ from $\widehat{\Delta}_1$ when $d(\widehat{v}_4) = 5$ and $\widehat{\Delta}_1$ plays the role of Δ in Figure 4.90(ii) and so $c^*(\widehat{\Delta}) \leq c(3, 3, 3, 5, 5) + 2(\frac{2\pi}{30}) < 0$.

Finally let $l(v_4) = \overline{2}14$. Then checking shows that $d(\widehat{\Delta}) \geq 6$. Let $d(\widehat{\Delta}) \geq 7$. Then either it receives a sum $\leq 2(\frac{5\pi}{30})$ and so $c^*(\widehat{\Delta}) \leq c(3, 3, 3, 3, 3, 3, 3) + 2(\frac{5\pi}{30}) = 0$; or it receives a sum $\leq \frac{5\pi}{30} + \frac{7\pi}{30}$ and this forces at least one vertex of degree ≥ 4 and hence $c^*(\widehat{\Delta}) \leq c(3, 3, 3, 3, 3, 3, 4) + \frac{5\pi}{30} + \frac{7\pi}{30} < 0$. An example for the former case is $\widehat{\Delta} = \widehat{\Delta}_2$ in Figure 4.60(iii) which can receive $\leq \frac{5\pi}{30}$ from each of Δ and $\widehat{\Delta}_1$. Also an example for the latter case is when $\widehat{\Delta} = \widehat{\Delta}_2$ in Figure 4.83(i) with $l(v_1) = \overline{1}2\overline{4}$ which can receive $\leq \frac{5\pi}{30}$ from Δ and $\leq \frac{7\pi}{30}$ from $\widehat{\Delta}_1$.

Now let $d(\widehat{\Delta}) = 6$. Then either it has at least one vertex of degree ≥ 5 and in this case it receives from 2 or 3 regions a sum $\leq \frac{8\pi}{30}$ and so $c^*(\widehat{\Delta}) \leq c(3, 3, 3, 3, 3, 5) + \frac{8\pi}{30} = 0$; or receives from 2 regions a sum $\leq \frac{10\pi}{30}$, but this forces at least 2 vertices of degree ≥ 4 and so $c^*(\widehat{\Delta}) \leq c(3, 3, 3, 3, 4, 4) + \frac{10\pi}{30} = 0$. In the former case if $\widehat{\Delta}$ receives from 2 positive regions then an example for this is when $\widehat{\Delta} = \widehat{\Delta}_3$ in Figure 4.64(i) which can receive $\leq \frac{3\pi}{30}$ from $\widehat{\Delta}_2$ only if it forces a vertex of degree ≥ 5 with $\leq \frac{5\pi}{30}$ from Δ , and another example is when $\widehat{\Delta} = \widehat{\Delta}_3$ in Figure 4.80(i) when $l(\widehat{v}_2) = \overline{2}\overline{3}\overline{4}$ can receive $\leq \frac{4\pi}{30}$ from each of Δ and $\widehat{\Delta}_2$. Also if $\widehat{\Delta}$ receives from 3 regions then the only example when $\widehat{\Delta} = \widehat{\Delta}_3$ and $d(\widehat{v}_2) \geq 4$ in Figure 4.80(i) in which case $\widehat{\Delta} = \widehat{\Delta}_3$ receives from $\widehat{\Delta}_2$ and Δ'_3 in Figure 4.60(i) at most $\leq \frac{2\pi}{30}$ each plus $\leq \frac{4\pi}{30}$ from Δ . An example for the latter case is when

$\widehat{\Delta} = \widehat{\Delta}_3$ in Figure 4.78(ii) which can receive $\leq \frac{5\pi}{30}$ from each of Δ and $\widehat{\Delta}_2$ when $d(\widehat{v}_2) \geq 4$ and forces one split on $\widehat{\Delta}_3$.

Suppose now that $\bar{2}14$ is not a vertex labelling in $\widehat{\Delta}$. If $\widehat{\Delta}$ receives $\frac{10\pi}{30}$ from a neighbouring region then this forces at least 2 edges to have two splits each and this leaves one further edge through which $\widehat{\Delta}$ receives at most $\frac{10\pi}{30}$ and so $c^*(\widehat{\Delta}) \leq c(3, 3, 3, 3, 3, 3, 3) + 2(\frac{10\pi}{30}) = 0$. If $\widehat{\Delta}$ receives up to $\frac{\pi}{4}$ from a neighbouring region then this forces at least 1 edge to split and one edge to have two splits and at least one vertex of degree ≥ 4 and it receives at most from 2 regions. So $c^*(\widehat{\Delta}) \leq c(3, 3, 3, 3, 3, 3, 4) + 2(\frac{\pi}{4}) = 0$. Otherwise the most $\widehat{\Delta}$ receives is $\leq \frac{5\pi}{30}$ and this forces at least 2 splits on one edge and a vertex of degree ≥ 4 . If $\widehat{\Delta}$ receives across two edges then this forces at least another vertex to have degree ≥ 4 and hence $c^*(\widehat{\Delta}) \leq c(3, 3, 3, 3, 4, 4) + 2(\frac{5\pi}{30}) = 0$. If receives across exactly 3 edges then this would force the same situation as above with the third region sends at most $\frac{3\pi}{30}$ and at least one of the shared vertices increases the degree from 4 to ≥ 5 and so $c^*(\widehat{\Delta}) \leq c(3, 3, 3, 3, 4, 5) + 2(\frac{5\pi}{30}) + \frac{3\pi}{30} = 0$.

Chapter 5

Proof of Theorem 8

Let D be the tessellation of the 2-sphere presented in Chapter 1 or an amended version of D given in Sections 2.2 and 4.2.1 and assume that we make all the curvature distributions from each positive interior region Δ to its neighbouring region $\hat{\Delta}$ as described in Chapters 2-4. Then as shown $c^*(\hat{\Delta}) \leq 0$ and Lemma 3 is proved. It is now left to prove that if Δ_0 is a boundary region then $\sum c^*(\Delta_0) < 4\pi$. Recall that a boundary region of D is a region that contains the distinguished vertex $\mathbf{v}_0$ and let $d(\mathbf{v}_0) = k_0$. As discussed in Chapter 1 to prove $\sum c^*(\Delta_0) < 4\pi$ it is only needed to prove that $c^*(\Delta_0) < \frac{4\pi}{k_0}$ for each boundary region. Note that the curvature is sent from interior regions and so a boundary region can not receive curvature cross the edges incident to $\mathbf{v}_0$. Note the following:

- in amended D the average curvature sent per edge is $\frac{\pi}{6}$ as estimated in Sections 2.2 and 4.2.1.
- in D without the amendment there are two types of distribution. First regions that receives through a part of an edge used in Chapter 3 for some figures in Cases II(a1), II(a2) and II(a3). However in these cases if v_0 coincide with an existing g-vertex then no positive curvature can be sent cross the edges incident to $\mathbf{v}_0$ and otherwise in these cases curvature sent through an edge can be ignored as the net contribution is ≤ 0 as shown in the discussion for Figure 3.11. So we need only consider regions that receives through a whole edge. Then checking all Cases I, II, III shows that the maximum sum added to any region is $\leq \frac{15\pi}{30}$ across at most three edges except in some incidents in Case III(a1) and Case III(a2) where the sum

of positive curvature sent to one region is $\leq \frac{2\pi}{3}$ across 2 edges. However in the former case the region must be of degree ≥ 6 and in the latter case the region must be of degree ≥ 8 . Moreover when the region is of degree ≤ 5 and it receives curvature from neighbouring regions then it receives $< 2(\frac{\pi}{6})$.

First let Δ_0 be a boundary region in amended D . Then $d(\Delta_0) \geq 3$. Assume that $k_0 \leq 2$. This implies that $d(\Delta_0) = 3$ and so Δ_0 never receives positive curvature from neighbouring regions and:

if $k_0 = 1$ then $c(D) = c^*(\Delta_0) \leq c(k_0, 3, 3, 3) = 2\pi < \frac{4\pi}{k_0}$;

if $k_0 = 2$ then $c^*(\Delta_0) \leq c(k_0, 3, 3, 3) = \pi < \frac{4\pi}{k_0}$.

So assume from now on that $k_0 \geq 3$ and so $d(\Delta_0) \geq 4$. First let $d(\Delta_0) = 4$. Then Δ_0 may receive at most $\frac{\pi}{6}$ from one region and this forces at least one vertex of degree ≥ 4 and hence:

$$c^*(\Delta_0) \leq c(k_0, 3, 3, 4) + \frac{\pi}{6} = 2(\frac{\pi}{k_0}) < \frac{4\pi}{k_0}.$$

Now let $d(\Delta_0) = 5$. Then $c(\Delta_0) \leq c(k_0, 3, 3, 3, 3)$ and Δ_0 may receive at most from two regions and hence:

$$c^*(\Delta_0) \leq c(k_0, 3, 3, 3, 3) + 2(\frac{\pi}{6}) = \frac{4\pi}{k_0} - \frac{2\pi}{3} < \frac{4\pi}{k_0}.$$

Assume now that $d(\Delta_0) \geq 6$. Note that the argument used in Sections 2.2 and 4.2.1 that shows that $c^*(\hat{\Delta}) \leq 0$ for $\hat{\Delta}$ interior region still holds if $n_1 \geq 1$ of the g -vertices of $\hat{\Delta}$ are changed to v_0 or if v_0 is added to $\hat{\Delta}$ as an extra vertex. That is because the curvature has been sent between two t -vertices except for some figures in the Subsection 4.2.1 but even there the occurrence of v_0 has no effect on the estimation done. Indeed the valency of g -vertices of degree ≥ 3 is not needed to offset the curvature that Δ_0 receives. Therefore $c^*(\Delta_0) \leq 0$.

Finally let Δ_0 be a boundary region in D without the amendment. Then Δ_0 contains at least 4 t -vertices and $d(\Delta_0) \geq 4$. Assume $k_0 \leq 2$ and $d(\Delta_0) = n$:

$$\begin{aligned} \text{if } k_0 = 1 \text{ then } c(D) = c^*(\Delta_0) &\leq c(1, 3, 3, \dots, 3) + \frac{2\pi}{3} = \pi[2 - n + 2(1 + \frac{n}{3}) + \frac{2}{3}] \\ &= \pi[2 - n + 2 + \frac{2n}{3} + \frac{2}{3}] = \pi[4 - \frac{n}{3} + \frac{2}{3}] \leq \frac{10\pi}{3} < 4\pi = \frac{4\pi}{k_0}; \end{aligned}$$

$$\begin{aligned} \text{if } k_0 = 2 \text{ then } c^*(\Delta_0) &= c(2, 3, 3, \dots, 3) + (\frac{2\pi}{3}) = \pi[2 - n + 2(\frac{n}{3}) + \frac{2}{3}] \\ &= \pi[2 - n + \frac{2n}{3} + \frac{2}{3}] = \pi[2 - \frac{n}{3} + \frac{2}{3}] \leq 2 - \frac{2\pi}{3} < 2\pi = \frac{4\pi}{k_0}. \end{aligned}$$

Now assume $k_0 \geq 3$. Then $d(\Delta_0) \geq 5$. If $d(\Delta_0) = 5$ then again it receives $\leq 2(\frac{\pi}{6})$ and:

$$c^*(\Delta_0) \leq c(k_0, 3, 3, 3, 3) + 2(\frac{\pi}{6}) < \frac{4\pi}{k_0}.$$

Now let $d(\Delta_0) \geq 6$. Then it may receive $\leq \frac{2\pi}{3}$ from neighbouring regions. Let v_0 coincide with n_1 g -vertices. Then $d(\Delta_0) = n \geq n_1 + 4$ and hence:

$$\begin{aligned}
c^*(\Delta_0) &\leq c(k_0, k_0, \dots, 3, 3, \dots, 3) + \frac{2\pi}{3}. \\
&= (2 - n)\pi + 2\pi\left(\frac{n_1}{k_0} + \frac{n - n_1}{3}\right) + \frac{2\pi}{3} \\
&= \pi\left[(2 - n) + \frac{2(n_1 - 2)}{k_0} + \frac{4}{k_0} + \frac{2(n - n_1)}{3} + \frac{2}{3}\right] \\
&\leq \pi\left[2 - n + \frac{2(n_1 - 2)}{3} + \frac{4}{k_0} + \frac{2(n - n_1)}{3} + \frac{2}{3}\right] \\
&= \pi\left[\frac{4}{3} - \frac{n}{3} + \frac{4}{k_0}\right] < \frac{4\pi}{k_0}.
\end{aligned}$$

Bibliography

- [1] F.Levin. Solution of Equations over Groups. *Bull. Amer. Math. Soc*,68:603-604,(1962).
- [2] R.C. Lyndon and P.E.Schupp.*Combinatorial Group Theory*, Egerbnisse der Mathematik und ihere Grenzgebiete 89(Springer-Verlag, Berline, 1977).
- [3] A.Evangelidou. The Solution for Length Five Equations over Groups.*Communications in Algebra*,35(6):1914-1948,2007.
- [4] M.Gerstenhaber and O. S. Rothaus. The Solution of Sets of Equations in Groups. *Proceeding of the Nathional Academy of Siences of the United States of Amereca*,48(9): 1531-1533, 1962.
- [5] A.A.Klyachko. A Funny Property of a Sphere and Equations over Groups. *Communications in Algebra*,21:2555-2575,1993.
- [6] M.Edjvet. Equations over Groups and a Theorem of Higman, Neumann, and Neumann. *Proceeding of the London Mathematical Society* 62:563-589,1991.
- [7] M.Edjvet and A. Juhász. Non-singular Equations over Groups I.*Algebra Colloquium*18: 221-240,2011.
- [8] M.Edjvet and A. Juhász.Nonsingular Equations over Groups II. *Communications in Algebra*38: 1640-1657, 2010.
- [9] M.Edjvet and A. Juhász.Nonsingular equations over groups III.*University of Nottingham preprint*, 2008.
- [10] J. M.Paul. *Equations over groups and and cyclically presented groups*. PhD. Mathematical school of the University of Nottingham. UK,2010.

- [11] D.L. Johnson. *Presentation of Groups*, Cambridge University Press, 1997.
- [12] S.V. Ivanov and A.A. Klyachko. Solving equations of length at most six over torsion-free groups. *J. Group Theory*, 3:329-337, 2000.
- [13] J. Howie. On pairs of 2-complexes and systems of equations over groups. *J. Reine Angew. Math.*, 324:165-174, 1981.