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Elastic wave energy scattering and propagation in composite structures.

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To my family.

Abstract

This thesis considers the problem of high-frequency wave energy scattering and propagation in composite structures. In particular, structures made of the two-dimensional plate- or shell-like elements with composite laminate material are of interest. We propose an extension of the Dynamical Energy Analysis (DEA) method for such structures. For this purpose, we develop a semi-analytical and a hybrid Finite Element (FE) and Wave and Finite Element (WFE) methods to compute wave energy reflection/transmission at junctions of arbitrarily layered composite plates.

In the first part of the thesis, a brief review of high-frequency numerical methodologies is presented. Advantages and limitations of each method are discussed. Furthermore, a review of analytical and numerical methods for calculating wave propagation characteristics such as dispersion relations, group velocity and scattering coefficients is presented.

In the second part of the thesis, the main theoretical basics of the DEA method, Classical Laminated Plate theory and the WFE method are demonstrated.

In the third part of the thesis, semi-analytical method for computing the energy scattering coefficients of structural junctions made up of thin composite laminated plates is developed. Expressions quantifying transmission and reflection coefficients as a function of the frequency and the angle of incidence are derived. An effective scattering matrix for a plate with multiple finite stiffeners attached to it is obtained.

In the fourth part of the thesis, a hybrid FE/WFE model that predicts the scattering properties for different junctions of two-dimensional anisotropic composite plates is developed. The influence of the angle of incidence and the frequency on the distribution of the power flow of incident bending, shear and longitudinal type waves is investigated. A detailed comparison with semi-analytical evaluations of scattering coefficients derived in the third part of the thesis is presented. The method gives for the first time a detailed recipe for computing scattering coefficients for the generic case of an arbitrary number of composite plates connected at a junction without restrictions on the angles at which the plate meet or the orientation of the principal axis of individual plates.

In the last part of the thesis, the theoretical base of the DEA method for composite structures is developed and discussed. The findings of the third and fourth parts of the thesis are used to derive the stationary wave energy density arising in the composite structure due to a harmonic point and edge sources. Numerical results for the cases of a polygonally shaped plate, an L-shaped composite plate and an electric vehicle gearbox are presented.

Declaration

The work presented in this thesis is the result of my research project as a PhD student at the University of Nottingham and was conducted between October 2016 and October 2020. I declare that the work is my own and has not been submitted for a degree elsewhere.

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List of Publications

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Contents

Abstract	i
Declaration	ii
Acknowledgements	iii
List of Publications	iv
1 Introduction	1
1.1 Motivation and literature review	1
1.1.1 Review of high-frequency methods	2
1.1.2 Review of methods for evaluation of wave propagation characteristics	5
1.2 Aims and objectives	13
1.3 Thesis outline	14
2 Background	16
2.1 Dynamical Energy Analysis for isotropic structures	16
2.1.1 Wave equation	17
2.1.2 The free-space Green function	18
2.1.3 Eikonal approximation and ray tracing	20
2.1.4 Liouville equation and boundary mapping	23
2.2 Classical Laminated Plate theory	29
2.3 The Wave Finite Element method for composite plates	33
2.3.1 Statement of the problem	33
2.3.2 Periodic structure theory and eigenvalue problem	35
2.4 Conclusion	38
3 Energy scattering properties of line joints connecting compos-	

ite plates	40
3.1 Introduction	40
3.2 Statement of the problem	41
3.2.1 Governing equations of motion	42
3.3 Dispersion relations, the group velocity and Snell's law	43
3.3.1 Relations between incoming and outgoing waves at junctions for composite plates	44
3.3.2 Snell's law for composite plates - an example	47
3.4 Derivation of the dynamic stiffness matrix	52
3.4.1 Energy flow and the response of the plate	52
3.4.2 The dynamic stiffness matrix	57
3.5 Reflection and transmission at plate junctions	59
3.6 Stiffened plate	61
3.6.1 A composite plate with one stiffener	64
3.6.2 A composite plate with a pair of stiffeners	68
3.6.3 A composite plate with multiple stiffeners	70
3.7 Computational results	72
3.7.1 Reflection at the free edge of a composite plate	72
3.7.2 Two composite plates joined at a right angle	77
3.7.3 Three composite plates joined in a T-junction	83
3.7.4 Stiffened composite plates	85
3.8 Conclusion	94
4 Wave and Finite Element Analysis of wave propagation in composite plates	95
4.1 Introduction	95
4.2 Summary of the WFE method for composite plates	96
4.3 Calculation of wave numbers and group velocity	98
4.3.1 Incoming, outgoing waves and wave basis	99
4.3.2 Group velocity estimation	102

4.4	Computation of scattering coefficients at junctions of composite plates	104
4.4.1	Reflection matrix for a single plate system	104
4.4.2	Scattering at the shared edge between shell plates	105
4.4.3	Hybrid FE/WFE method of computation of scattering coefficients in three-dimensional solid plates	107
4.5	Numerical results	112
4.5.1	Cross-ply laminated plates	113
4.5.2	Angle-ply composite plates	119
4.6	Conclusion	122
5	Dynamical Energy Analysis for composite structures	126
5.1	Introduction	126
5.2	Ray tracing for composite structures	127
5.2.1	Hamilton function and boundary mapping	127
5.2.2	Transformation between 2D and 4D phase space densities	130
5.3	Basis set expansion of the problem	131
5.3.1	Boundary integral operator	132
5.3.2	Initial phase space densities	135
5.3.3	Stationary energy and intensity	139
5.4	Numerical results	142
5.4.1	Test case for validation	143
5.4.2	Two composite plates joined at a right angle	150
5.4.3	Electric Vehicle Gearbox	153
5.5	Conclusion	158
6	Conclusion	160
6.1	Main Contributions	160
6.2	Further work	161
	Bibliography	164
	Appendices	180

A	Special laminated plate stiffness coefficients derivation	180
A.1	Cross-ply symmetric laminates	180
A.2	Angle-ply symmetric laminates	181
B	Plane stress-reduced stiffnesses	182
C	List of Abbreviations	183

List of Tables

- 3.1 Engineering constants of Epoxy Carbon UD (230 GPa) material
used for individual laminas of a composite plate. 47
- 5.1 Engineering constants of aluminium and stainless steel materials.144

List of Figures

2.1	The set up of the local coordinate system at the boundary $X = (s, p_{ })$. A ray emitting from the point $\mathbf{r}_0$ reach the point $\mathbf{r}_s$, which has the local position s on the boundary Γ . The momentum vector $\mathbf{p}$ has normal and tangential components $p_{\perp}$ and $p_{ }$ at the point s , and θ denotes the angle between the ray connecting $\mathbf{r}_0$ and s and the normal to Γ	25
2.2	The lay-up configuration of a composite laminate.	30
2.3	A periodic segment of a composite plate with two different alternating plies modelled with SOLID185 (a) and SHELL181 elements (b) in Ansys. The degrees of freedom are grouped into internal $\mathbf{q}_I$, edge $\mathbf{q}_L, \mathbf{q}_R, \mathbf{q}_B, \mathbf{q}_T$ and corner $\mathbf{q}_{LB}, \mathbf{q}_{RB}, \mathbf{q}_{LT}, \mathbf{q}_{RT}$ degrees of freedom.	33
3.1	The set up of N plates joined together at a common interface along the x_g axis.	41
3.2	Local coordinate system, displacements and tractions on the common edge “e” of the j th plate.	42
3.3	Schematic of an L-junction of two orthotropic plates. The angle ψ_2 here is set to 90° . Red and blue lines represent the fiber direction of the plates. The local angles of orientations ϕ_1 and ϕ_2 are both set to 45°	47

- 3.4 Bending wave vector curve in a 45° rotated orthotropic plate at a frequency 3000 Hz (left) and a schematic representation of incoming/outgoing waves at the junction of two identical plates (right). Blue dots represent wave numbers related to outgoing waves while red squares correspond to incoming waves. Wave number components $k_x^{\max}$ and k_x^* equal to 62.96 m^{-1} and 29.34 m^{-1} , respectively. 48
- 3.5 Group velocity ray picture for bending waves at junction of two identical 45° rotated orthotropic plates at a frequency 3000 Hz. Blue lines highlight the region of incoming rays, which represents the negative refractive index phenomenon. 50
- 3.6 Longitudinal and shear wave vector curves in a 45° rotated orthotropic plate at a frequency 3000 Hz. Dots represent wave numbers related to outgoing waves while squares - incoming waves. Wave number components k_L^* and k_S^* equal to 6 m^{-1} and 9.3 m^{-1} , respectively - the critical values for longitudinal and shear waves 50
- 3.7 Bending wave vector curve in a $45^\circ / -45^\circ / 45^\circ / -45^\circ / 45^\circ$ composite plate at a frequency 3000 Hz (left) and a group velocity ray picture for bending waves at the junction of two identical plates (right). Blue dots represent wave numbers related to outgoing waves while red squares correspond to incoming waves. Wave number components $k_x^{\max}$ and k_x^* equal to 53 m^{-1} and 10.43 m^{-1} , respectively. Blue lines highlight the region of incoming rays, which represents the negative refractive index phenomenon. 51

- 3.8 Longitudinal and shear wave vector curves in a $45^\circ/-45^\circ/45^\circ/-45^\circ/45^\circ$ composite plate at a frequency 3000 Hz. Dots represent wave numbers related to outgoing waves while squares - incoming waves. Wave number components k_L^* and k_S^* equal to 3.71 m^{-1} and 7.8 m^{-1} , respectively - the critical values for longitudinal and shear waves 51
- 3.9 Decay direction angles of various evanescent waves as a function of a ply direction angle in an orthotropic plate with material parameters from Table 3.1. 53
- 3.10 Contour plot of the real part of the out-of-plane displacement produced by the outgoing evanescent bending wave in a 45° -rotated orthotropic plate plotted in the (x, y) plane. Brighter colours represent positive z values, whilst darker ones denote negative z values. 54
- 3.11 A ground composite plate with m stiffener plates attached onto it. The ground plate is considered infinite in both directions, whereas stiffeners are infinite only in their local x coordinate directions. 62
- 3.12 The schematic of incoming (shown in red) and outgoing (shown in blue) waves travelling in the stiffened plate. A_1^- and A_N^- are the incoming waves from infinity. 62
- 3.13 A schematic view of a stiffened plate in the yz plane. Here, a_{ij}^- represent the amplitudes of incoming waves, and a_{ij}^+ , the amplitudes of outgoing waves. 64
- 3.14 A schematic view of a stiffener plate in the xy plane. Blue lines represent the direction of the i th ply of the plate. 66
- 3.15 A schematic view of a plate with a pair of finite stiffener plates in the yz plane. Here, a_{ij}^- represent the amplitudes of incoming waves, and a_{ij}^+ , the amplitudes of outgoing waves. 68

- 3.16 A schematic view of a stiffened plate with two stiffeners represented by effective scattering matrices s_{eff}^k and s_{eff}^{k+1} in the yz plane. Here, a_{ij}^- represent the amplitudes of incoming waves, and a_{ij}^+ , the amplitudes of outgoing waves. 71
- 3.17 Wave vector (left) and dispersion (right) curves for a cross-ply composite plate. The wave vector curves are plotted for a fixed frequency $f = 3000$ Hz, whereas the dispersion curves are plotted for a fixed wave number component $k_x = 5\text{m}^{-1}$. Squares and circles denote wave numbers of incoming and outgoing waves, respectively. The bending wave and dispersion curves are scaled with a factor of $1/3$ for the sake of clarity. 73
- 3.18 Energy reflection coefficients of a regular cross-ply symmetric laminate at the free edge for various incident waves as a function of wave number component k_x at a frequency 3000 Hz. The lamination scheme is $0^\circ/90^\circ/0^\circ/90^\circ/0^\circ$ 73
- 3.19 Energy reflection coefficients of a regular cross-ply symmetric laminate at the free edge for various incident waves as a function of frequency at a wave number component $k_x = 5\text{m}^{-1}$. The lamination scheme is $0^\circ/90^\circ/0^\circ/90^\circ/0^\circ$ 74
- 3.20 Energy reflection coefficients of a regular angle-ply symmetric laminate at the free edge for various incident waves as a function of wave number component k_x at a frequency 3000 Hz. The lamination scheme is $45^\circ/-45^\circ/45^\circ/-45^\circ/45^\circ$ 75
- 3.21 Energy reflection coefficients of a regular angle-ply symmetric laminate at the free edge for various incident waves as a function of frequency at a wave number component $k_x = 5\text{ m}^{-1}$. The lamination scheme is $45^\circ/-45^\circ/45^\circ/-45^\circ/45^\circ$ 76

- 3.22 Energy scattering coefficients of an L-joint of regular cross-ply symmetric laminates with the lamination scheme $0^\circ/90^\circ/0^\circ/90^\circ/0^\circ$ for various incident waves as a function of wave number component k_x at a frequency 3000 Hz. 77
- 3.23 Energy scattering coefficients of an L-joint of regular cross-ply symmetric laminates with the lamination scheme $0^\circ/90^\circ/0^\circ/90^\circ/0^\circ$ for bending and longitudinal incident waves as a function of frequency for a wave number component $k_x = 0 \text{ m}^{-1}$ 78
- 3.24 Energy scattering coefficients of an L-joint of regular cross-ply symmetric laminates with the lamination scheme $0^\circ/90^\circ/0^\circ/90^\circ/0^\circ$ for various incident waves as a function of frequency for a wave number component $k_x = 5 \text{ m}^{-1}$ 78
- 3.25 Energy scattering coefficients of an L-joint of regular angle-ply symmetric laminates with the lamination scheme $45^\circ/-45^\circ/45^\circ/-45^\circ/45^\circ$ for various incident waves as a function of wave number component k_x at a frequency 3000 Hz. 79
- 3.26 Energy scattering coefficients of an L-joint of regular angle-ply symmetric laminates with the lamination scheme $45^\circ/-45^\circ/45^\circ/-45^\circ/45^\circ$ for various incident waves as a function of frequency for a wave number component $k_x = 0 \text{ m}^{-1}$ 80
- 3.27 Energy scattering coefficients of an L-joint of regular angle-ply symmetric laminates with the lamination scheme $45^\circ/-45^\circ/45^\circ/-45^\circ/45^\circ$ for various incident waves as a function of frequency for a wave number component $k_x = 5 \text{ m}^{-1}$ 81
- 3.28 Schematic representation of a T-junction connecting three orthotropic plates. The angles ψ_2 and ψ_3 here are set to 180° and 90° , respectively. Red, blue and green lines represent the ply direction of the plates. The local angles of orientations ϕ_1 and ϕ_3 are both set to 45° , whereas ϕ_2 is equal to -45° 82

- 3.29 Energy scattering coefficients of an T-joint of regular angle-ply symmetric laminates with the lamination scheme $45^\circ / -45^\circ / 45^\circ / -45^\circ / 45^\circ$ for various incident waves as a function of wave number component k_x at a frequency 3000 Hz. 83
- 3.30 Energy scattering coefficients of an T-joint of regular angle-ply symmetric laminates with the lamination scheme $45^\circ / -45^\circ / 45^\circ / -45^\circ / 45^\circ$ for various incident waves as a function of frequency at a wave number component $k_x = 5 \text{ m}^{-1}$ 84
- 3.31 Energy scattering coefficients and resonance conditions of a stiffened plate for an incident bending wave as a function of wave number component k_x at a frequency 3000 Hz. The length of the stiffener plate is 30 cm. The lamination scheme of plates is $45^\circ / -45^\circ / 45^\circ / -45^\circ / 45^\circ$ 87
- 3.32 Energy scattering coefficients and resonance conditions of a stiffened plate for an incident bending wave as a function of frequency at a wave number component $k_x = 0 \text{ m}^{-1}$. The length of the stiffener plate is 30 cm. The lamination scheme of plates is $45^\circ / -45^\circ / 45^\circ / -45^\circ / 45^\circ$ 88
- 3.33 Energy scattering coefficients and resonance conditions of a stiffened plate for an incident bending wave as a function of frequency at a wave number component $k_x = 5 \text{ m}^{-1}$. The length of the stiffener plate is 30 cm. The lamination scheme of plates is $45^\circ / -45^\circ / 45^\circ / -45^\circ / 45^\circ$ 89
- 3.34 Energy scattering coefficients and resonance conditions of a symmetrically stiffened plate for an incident bending wave as a function of wave number component k_x at a frequency 3000 Hz. The lengths of the stiffener plates are 30 cm. The lamination scheme of plates is $0^\circ / 90^\circ / 0^\circ / 90^\circ / 0^\circ$ 90

- 3.35 Energy scattering coefficients of a symmetrically stiffened plate for an incident bending wave as a function of frequency at a wave number components $k_x = 0 \text{ m}^{-1}$ and $k_x = 5 \text{ m}^{-1}$. The lengths of the stiffener plates are 30 cm. The lamination scheme of plates is $0^\circ/90^\circ/0^\circ/90^\circ/0^\circ$ 91
- 3.36 Contour plots of energy scattering coefficients of a symmetrically stiffened plate with four pairs of stiffeners for an incident bending wave as a function of frequency and wave number component k_x . The spacings between stiffeners and lengths of the stiffeners are 30 cm. The lamination scheme of plates is $0^\circ/90^\circ/0^\circ/90^\circ/0^\circ$ 92
- 3.37 Contour plots of energy scattering coefficients of a symmetrically stiffened plate with four pairs of stiffeners for an incident bending wave as a function of frequency and phase angle $\theta = \arctan(k_y/k_x)$. The spacings between stiffeners and lengths of the stiffeners are 30 cm. The lamination scheme of plates is $0^\circ/90^\circ/0^\circ/90^\circ/0^\circ$ 93
- 4.1 Bending wave vector curve in a $45^\circ/-45^\circ/45^\circ/-45^\circ/45^\circ$ composite plate at a frequency 3000 Hz. Blue dots represent wave numbers related to outgoing waves while red squares correspond to incoming waves. Angles $\alpha_{1(2)}$ and $\theta_{1(2)}$ represent group and wave vector or phase angles, respectively. The wave number component $k_x^{\max}$ is equal to 53 m^{-1} 100
- 4.2 A schematic representation of N plates connected via the shared edge. The plates are modelled using two-dimensional SHELL181 elements. The x axis of all plates is the same. 106
- 4.3 A schematic representation of the joint connecting N plates. The x axis of all plates are aligned with the x_J axis of the joint. 108

- 4.4 A finite element model of the joint connecting N plates. The degrees of freedom are grouped into internal $\mathbf{q}_I$, edge $\mathbf{q}_B, \mathbf{q}_T, \mathbf{q}_{L,1}, \dots, \mathbf{q}_{L,N}$ and corner $\mathbf{q}_{LB,1}, \dots, \mathbf{q}_{LB,N}, \mathbf{q}_{LT,1}, \dots, \mathbf{q}_{LT,N}$ degrees of freedom. 109
- 4.5 Wave vector (left) and dispersion (right) curves for a cross-ply composite plate. The wave vector curves are plotted for a fixed frequency $f = 3000$ Hz, whereas the dispersion curves are plotted for a fixed wave number component $k_x = 5\text{m}^{-1}$. Squares and circles denote wave numbers of incoming and outgoing waves, respectively. 113
- 4.6 Energy scattering coefficients of an L-junction of two SHELL181-based cross-ply laminated plates for various incident waves plotted with respect to the wave number component k_x at a frequency $f = 3000$ Hz. $\hat{t}_{ij}^{nm}$ denote semi-analytical scattering coefficients obtained using the method presented in Chapter 3. . 114
- 4.7 Energy scattering coefficients of an L-junction of two SOLID185-based cross-ply laminated plates for various incident waves plotted with respect to the wave number component k_x at a frequency $f = 3000$ Hz. $\hat{t}_{ij}^{nm}$ denote semi-analytical scattering coefficients obtained using the method presented in Chapter 3. . 115
- 4.8 Energy scattering coefficients of an L-junction of two SHELL181-based cross-ply laminated plates for various incident waves plotted with respect to the frequency at the fixed wave number component $k_x = 0 \text{ m}^{-1}$. $\hat{t}_{ij}^{nm}$ denote semi-analytical scattering coefficients obtained using the method presented in Chapter 3. . 116
- 4.9 Energy scattering coefficients of an L-junction of two SOLID185-based cross-ply laminated plates for various incident waves plotted with respect to the frequency at the fixed wave number component $k_x = 0 \text{ m}^{-1}$. $\hat{t}_{ij}^{nm}$ denote semi-analytical scattering coefficients obtained using the method presented in Chapter 3. . 116

- 4.10 Energy scattering coefficients of an L-junction of two SHELL181-based cross-ply laminated plates for various incident waves plotted with respect to the frequency at the fixed wave number component $k_x = 5 \text{ m}^{-1}$. $\hat{t}_{ij}^{nm}$ denote semi-analytical scattering coefficients obtained using the method presented in Chapter 3. . 117
- 4.11 Energy scattering coefficients of an L-junction of two SOLID185-based cross-ply laminated plates for various incident waves plotted with respect to the frequency at the fixed wave number component $k_x = 5 \text{ m}^{-1}$. $\hat{t}_{ij}^{nm}$ denote semi-analytical scattering coefficients obtained using the method presented in Chapter 3. . 118
- 4.12 Wave vector (left) and dispersion (right) curves for an angle-ply composite plate. The wave vector curves are plotted for a fixed frequency $f = 3000 \text{ Hz}$, whereas the dispersion curves are plotted for a fixed wave number component $k_x = 5 \text{ m}^{-1}$. Squares and circles denote wave numbers of incoming and outgoing waves, respectively. 120
- 4.13 Energy scattering coefficients of an L-junction of two SHELL181-based angle-ply laminated plates for various incident waves plotted with respect to the wave number component k_y at a frequency $f = 3000 \text{ Hz}$. $\hat{t}_{ij}^{nm}$ denote semi-analytical scattering coefficients obtained using the method presented in Chapter 3. . 121
- 4.14 Energy scattering coefficients of an L-junction of two SOLID185-based angle-ply laminated plates for various incident waves plotted with respect to the wave number component k_y at a frequency $f = 3000 \text{ Hz}$. $\hat{t}_{ij}^{nm}$ denote semi-analytical scattering coefficients obtained using the method presented in Chapter 3. . 122

4.15	Energy scattering coefficients of an L-junction of two SHELL181- and SOLID185-based angle-ply laminated plates for various incident waves plotted with respect to the frequency at the fixed wave number component $k_x = 0 \text{ m}^{-1}$. $\hat{t}_{ij}^{nm}$ denote semi-analytical scattering coefficients obtained using the method presented in Chapter 3.	124
4.16	Energy scattering coefficients of an L-junction of two SHELL181- and SOLID185-based angle-ply laminated plates for various incident waves plotted with respect to the frequency at the fixed wave number component $k_x = 5 \text{ m}^{-1}$. $\hat{t}_{ij}^{nm}$ denote semi-analytical scattering coefficients obtained using the method presented in Chapter 3.	125
5.1	An example of an electric vehicle gearbox being meshed using two-dimensional shell elements. Colours represent elements with different thicknesses.	127
5.2	A schematic representation of the boundary mapping procedure in the triangular mesh element. Green ovals depict wave vector curves. The orientation of the principal material axes is defined in the element's coordinate system (x_g, y_g, z_g) . A ray starting at the position s of edge a points in the direction of the outgoing group velocity vector $\mathbf{c}_g^+$. The group velocity vector is defined by the momentum component $p_{ }$. The ray hits edge a' at the position s' with the new momentum component $p'_{ }$ defined in the coordinate system (e'_x, e'_y) . However, momentum vectors $\mathbf{p}$ and $\mathbf{p}'$ are equal.	129

- 5.3 A representation of the geometrical relations used for computation of the position integral in Equation (5.19). A ray emitting from edge a can only hit edge a' if its direction angle $\alpha^+ > \xi^-$. $D(s, s')$ is the ray length. $s_{\min(\max)}$ is the minimum (maximum) value of the position s in the local coordinate system (e_x, e_y) such that a ray with the direction angle α^+ reach edge a' at some position s' 134
- 5.4 Geometrical relations between differentials $d\phi$ and $d\alpha^-$. Rays emitting from the point source at $\mathbf{r}_0$ intersect edge a at the incoming angles α^- . Similarly as in the case considered in Figure 5.3, rays can only reach edge a if their direction angles $\alpha^- \in [\xi^-, \xi^+]$ 137
- 5.5 Partitioning of the typical wave vector curve (depicted by green ovals) between edges of the mesh element. Red dashed arcs represent $\omega_a^{-1}(\omega_0)$ in Equation (5.36). They are defined by imposing the outgoing group velocity vector angle α^+ to be in range $[\xi^-, \xi^+]$, so that outgoing rays from edge a can reach the point $\mathbf{r}$ 139
- 5.6 Partitioning of the shear wave vector curve (depicted in green) between edges of the mesh element. Red dashed arcs represent $\omega_a^{-1}(\omega_0)$ in Equation (5.36). They are defined by imposing the outgoing group velocity vector angle α^+ to be in range $[\xi^-, \xi^+]$, so that outgoing rays from edge a can reach the point $\mathbf{r}$ 140
- 5.7 A polygonally shaped semi-infinite plate meshed with two-dimensional triangular shell elements. An edge and a point sources are assumed to be applied on the plate on the vertical and horizontal edges, respectively. The diagonal edge is free, and rays hitting this edge reflect back into the plate. Reflection of rays at other edges is prohibited thus imitating “infinity” of the plate through these edges. 143

- 5.8 A geometrical illustration of the ray paths (shown in red) starting from the edge source at the vertical edge. Wave vector curves (shown in green) are a circle for the isotropic material case (a) and an oval for the composite material case (b). 144
- 5.9 The mean acceleration response levels of the polygonally shaped semi-infinite isotropic plate due to an edge source as a function of the number of Legendre polynomials N_m . The acceleration is given in m/s^2 145
- 5.10 The mean acceleration response levels of the polygonally shaped semi-infinite composite plate due to an edge source as a function of the number of Legendre polynomials N_m 147
- 5.11 The intensity vector field of the isotropic (a,c) and composite (b,d) plate cases arising due to an edge source at the vertical edge as a function of the number of Legendre polynomials used in the basis set. 148
- 5.12 The mean acceleration response levels of the polygonally shaped semi-infinite isotropic plate due to an edge source as a function of the number of Legendre polynomials N_m . The acceleration is given in m/s^2 149
- 5.13 The mean acceleration response levels of the polygonally shaped semi-infinite isotropic plate due to an edge source as a function of the number of Legendre polynomials N_m . The acceleration is given in m/s^2 150
- 5.14 An L-shaped composite plate meshed with two-dimensional triangular shell elements. A point source is assumed to be applied on the middle of the ground plate. 151

- 5.15 The mean DEA and FEM acceleration response levels of an L-shaped composite plate due to a harmonic point excitation force of 1 N at the frequency $f = 3150$ Hz. FEM results are frequency-averaged over one-third octave band with the centre frequency $f = 3150$ Hz. The acceleration is given in m/s^2 152
- 5.16 The intensity vector field of an L-shaped composite plate arising due to a point source at the middle of the ground plate. Vector colouring represents intensity amplitudes. 153
- 5.17 The FE meshes of the full gearbox housing (a) and the gearbox housing without stiffeners along the longitudinal axis of the cylindrical part (b). 153
- 5.18 The mean DEA and FEM acceleration response levels of the gearbox housing of an electric vehicle powertrain due to a harmonic point excitation force of 1 N at the frequency $f = 8000$ Hz. The acceleration is given in mm/s^2 154
- 5.19 The intensity vector field of the gearbox housing with stiffeners (a) and without stiffeners (b) arising due to a point source at its front side. The structure colouring represents the mean acceleration levels of the bending mode. 155
- 5.20 Two modification methods of the FE model of the non-stiffened structure to obtain the same structural mass as the full structure. The first one (a) is to compute and assign an effective value of material density to the cylindrical part 2, which is different from 1, where the structure is made of aluminium. The second one (b) is to compute and assign effective thickness of values of shell elements in parts 1 and 2 by calculating the total volume of removed stiffeners and keeping the same material density. 156

5.21 The nodes of the FE mesh, where the acceleration response amplitudes were computed. Input node and nodes 1, 2 and 3 are located on the non-stiffened part, whereas nodes 4 and 5 are on the stiffened part of the structure. 157

5.22 Comparison of mean acceleration amplitudes (given in dB), calculated with different methods on the node positions, described in Figure 5.21, at the frequency $f = 8000$ Hz. 158

Chapter 1

Introduction

In this introduction, we describe the problem of interest and provide motivation for the work presented in this thesis. Further on, we present a review of existing literature and discuss relevant mathematical methods. Finally, we outline the structure of the thesis, summarising key ideas and results from each chapter.

1.1 Motivation and literature review

Composites are nowadays widely used within many transport sectors such as aerospace, automotive and naval architecture industries [1, 2]. In comparison to isotropic materials such as aluminium and stainless steel, composites provide comparable stiffness and strength whilst being significantly lighter [3]. Furthermore, the mechanical characteristics of fibre-reinforced composites can be tailored to suit particular purposes [1, 4]. Over the past several decades, such advantages of composites have led to growing usage of them to construct primary structural components in the aerospace and automotive industries.

However, despite their superior structural characteristics, composites exhibit reduced vibro-acoustic performance levels due to the large variety of waves propagating in them. Modelling noise and vibration in composite structures play an essential role in the various stages of their life cycle. For example, at the design phase of a vehicle, vibro-acoustic analysis can help figure out zones potentially prone to excessive noise. At the post-built stage, non-destructive testing techniques help monitor and assess structures' performance to avoid

sudden failure. Therefore, there is a need for quick and accurate methods of evaluating the vibrational response of composite structures.

1.1.1 Review of high-frequency methods

Structural vibrations of a complex structure are in general modelled using deterministic schemes such the Finite Element (FE) [5–7], Finite Difference (FD) [8, 9] or Boundary Element (BE) methods [10–12]. These methods are particularly useful in providing the full phase and amplitude information of the wave field in the low-frequency regime. However, at higher frequencies, these methods become inefficient and computationally expensive. This is because wavelengths are much smaller than the characteristic dimensions of the structure in the high-frequency range. Many degrees of freedom are required to accurately represent the wave field on the wavelength scale.

At higher frequencies, statistical approaches, energy and ray-tracing methods are favoured. The main statistical approach used extensively in the industry today is Statistical Energy Analysis (SEA) [13–16]. This method is based on several assumptions that are often hard or not possible to validate *a priori* [17, 18]. The structure is divided into subsystems, in which wave fields are assumed to be diffuse. SEA computes the steady-state averaged energy levels in subsystems. Since wave fields in anisotropic media are non-diffuse [19–22], the application of SEA to composite structures might be significantly error-prone [23].

To account for non-diffuse vibration fields and to represent the energy distribution locally in subsystems, several energy methods such as Wave Intensity Analysis (WIA) [24, 25], Energy Finite Element Analysis (EFEA) [26–30] and radiosity method [31–34] have been developed over the past several decades. The WIA and EFEA methods both rely on the reverberant wave field assumption; that is, the vibrational response of the structure is assumed to be a superposition of plane waves. Therefore, in the presence of strong sources and heavy damping condition, both methods underestimate wave energy lev-

els around the excitation points and overestimate the energy levels in the far field [35–37], since the direct field is dominant over the reverberant part in such cases. In EFEA, energy flow is required to be equally probable in all directions, and the reflection of energy at boundaries follows Lambert’s law [35, 37, 38]. In other words, coupling between subsystems is represented by angle-averaged energy scattering coefficients - these are called coupling loss factors in SEA. Another limitation of the EFEA method concerns the vibrational response of two-dimensional systems such as plates and membranes. The far field solution is proportional to $1/\sqrt{r}$, where r is the distance between source and receiver, whereas it must be proportional to $1/r$ in such systems. Other limitations and the validity region of the EFEA method have been presented in several works [35–40]. Nevertheless, the EFEA method has been applied to calculate the vibrational response of composite structures [41]. The radiosity method is technically a ray-tracing technique, and it is a special case of the radiative transfer method in vibro-acoustics [33]. However, this method also employs Lambert’s law to represent the directivity of energy reflection at boundaries. The ray-tracing methods include ray- and beam-tracing techniques [42–47], the radiative transfer method [33, 47–49] and Dynamical Energy Analysis (DEA) [50–53]. Unlike diffusive methods, ray-tracing methods do not employ the statistical law of energy reflection distribution at boundaries. Instead, waves are treated as uncorrelated rays that carry energy along the structure and reflect according to Snell’s law. Full ray paths can be computed once an initial source point and a ray direction are provided. Furthermore, one can retain full amplitude and phase information in the ray- and beam-tracing techniques, whereas no phase information is included in SEA and diffusive methods. This is important when considering transient time-dependent sources [43, 54]. However, proper ray- and beam-tracing techniques often are much more computationally expensive than SEA and diffusive methods [43, 45, 55]. In fact, the wave energy density at a certain point r is computed by summing over contributions from all rays that reach the point r starting from the source points. Therefore,

to guarantee the sufficient density of rays at the point r , one would need to increase the number of rays to a value that is not determined a priori or by any analytical methods, but rather empirically [50, 56].

The radiative transfer and DEA methods are based on the integral equation for the density of propagated rays. The phase information is no longer retained as in SEA and diffusive methods, and only spatial- and (or) frequency-averaged energy amplitudes are computed. Nevertheless, the deterministic law of reflection is retained as in the pure ray-tracing method. In other words, detailed information about waves' reflection and transmission behaviour at structural discontinuities is required and routinely used.

The radiative transfer method is based on the *Huygens principle* [32]: the vibrational field is assumed to be a superposition of the direct field arising from actual sources and the reverberant field emerging from fictive sources placed at the boundary. Consequently, by expressing the energy density and intensity as a sum of the direct and fictive source contributions, a Fredholm integral equation of the second kind is postulated for the fictive sources [33]. Diffuse and specular types of energy reflection for the determination of fictive sources have been considered in [32, 39, 57] and [33, 49], respectively. The former case reduces the radiative transfer method to the radiosity method and subsequently to the SEA method. The resulting equation for the fictive sources is solved numerically by the BE method for a limited number of cases. The correspondent software is called “CeReS” [39]. It is mentioned by Le Bot in [39] that only the case of diffuse energy reflection is implemented in the software, thus limiting its practical application to homogeneous isotropic structures. Moreover, the specular reflection case yields the functional equation, which depends on the contributions from a possibly infinite number of ray paths, thus increasing the practical complexity of the method [49].

Finally, the DEA method is a recently developed technique to evaluate the mean vibrational response of complex structures in high-frequency regimes. This method is implemented directly on the FE meshes [58, 59], thus avoiding

time-consuming remodelling. The main idea is similar to the Huygens principle considered in the radiative transfer method. Namely, the wave energy density is represented as a sum of direct and reverberant field contributions. In contrast to the radiative transfer method, the reverberant part is a solution of the Liouville equation [51] and is expressed in terms of an integral transfer operator T relating densities of rays between mesh cell boundaries as

$$\rho_{\infty}(\mathbf{r}, \mathbf{p}) = \sum_{n=1}^{\infty} T^n \rho_0(\mathbf{r}, \mathbf{p}) = (1 - T)^{-1} \rho_0(\mathbf{r}, \mathbf{p}). \quad (1.1)$$

Infinite ray reflection events are naturally obtained by the above sum, and in dissipative systems, this yields a linear system of equations relating the initial stationary ray densities on the union of all mesh cell boundaries $\rho_0(\mathbf{r}, \mathbf{p})$ and $\rho_{\infty}(\mathbf{r}, \mathbf{p})$, respectively. Note that the ray densities are expressed in phase space; that is, the position $\mathbf{r}$ and the momentum vectors $\mathbf{p}$ are considered as independent variables.

The DEA method has been successfully applied to model the mean vibrational response of a stiffened double-hull structure and a shock tower of a vehicle [59], the floor structure of caravan cars [60, 61], an agricultural tractor [53, 62] and a vehicle cavity [63]. Still, as the radiative transfer method, the DEA method has been developed and applied only for isotropic structures. This motivates the development of an extension of the DEA method for composite structures, and it is one of the primary concerns of this thesis.

1.1.2 Review of methods for evaluation of wave propagation characteristics

To accomplish this goal, one would need to compute wave propagation characteristics such as dispersion relations, the group velocity vector and scattering coefficients at discontinuities of composite structures.

Dispersion relations

Dispersion relations play an important role in many parts of the mathematical modelling of the wave propagation problem. They can be computed by solv-

ing governing equations of harmonic motion for composite laminates obtained using analytical approaches such as the Equivalent Single Layer (ESL) [64–68] and the layerwise theories [69–73] or numerical methods such as the spectral element (SE) [74–78] and the wave finite element (WFE) [79–85] methods. Extensive literature reviews of different analytical and numerical methods for composite laminated plates can also be found in [86–88].

The ESL theories include the Classical Laminated Plate (CLP) theory, First-order Shear Deformation Laminated (FSDL) and Higher-order Shear Deformation Laminated (HSDL) theories. In these theories, three-dimensional elasticity equations of a laminated plate are reduced to a two-dimensional problem using the Kirchhoff-Love or shear deformation hypotheses.

The CLP theory neglects the transverse shear strain, which can lead to substantial estimation errors for thick laminates and sandwich plates [89–92]. Nevertheless, for thin plates, this theory is quick and simple in providing reasonable approximate solutions of elastic problems and dispersion curves of thin laminates [64, 65, 93, 94].

The FSDL theory is an improved version of the CLP theory for thick laminated and sandwich plates. The main difference is that the transverse shear strain takes a non-null constant value in the thickness direction. However, to account for correct representation of the transverse shear stress, the shear correction factors that vary for different laminates are included in this method, [95–97]. Ghinet and Atalla [68] computed dispersion relations of waves propagating in sandwich composite plates using the FSDL theory. The sandwich plate considered in this work were made of transversely isotropic skins and isotropic rigid foam core, thus making the plate to be effectively isotropic. Dispersion relations for heading directions of 0° , 45° and 90° were presented in [98] for a symmetrically laminated composite made up of seven layers. Mejdí and Atalla [99] extended this approach to laminate panels stiffened by composite stiffeners and computed their vibration and acoustic responses.

The HSDL theories are based on the non-linear variation of transverse strain

and stress in the thickness direction. They include several different formulations of the mid-plane displacement field as a function of the thickness direction variable. The Third-order Shear Deformation Laminated (TSDL) theory has been developed by Reddy [67, 100]. It was applied for computing the deflection, the buckling load and natural frequencies of composite laminated plates in [67, 101, 102].

In contrast to ESL methods, all layers of a laminate are treated individually in the layerwise theories. Consequently, the number of unknown variables depends on the number of layers, which can yield large complex models, thus increasing the computational cost. In [71, 98], the Discrete Laminate (DL) theory was applied to obtain dispersion curves of a curved sandwich composite panel, and the results were compared with the FSDL theory estimations. Ghinet and Atalla [103] utilised the DL method to calculate the vibration response of a multi-layer construction consisting of composite and viscoelastic layers.

The SE method relies on constructing the exact dynamic stiffness matrix from the general solution of governing differential equations in the frequency domain. The response is assumed to be the superposition of wave modes at different frequencies based on Discrete Fourier Transformation theory. The Fast Fourier Transform can be applied to get the solution in the time domain. An excellent review of the method and its applications can be found in [74, 104]. Most of the literature on the SE method concerned one-dimensional waveguides such as beams and rods. These waveguides are assumed to be homogeneous and extended to infinity in one direction with arbitrary cross-sections. The dispersion curves of composite beams were obtained and analysed using spectral finite elements in [105–107]. Two-dimensional waveguides such as plates were considered as well; however, several assumptions either on boundary conditions (simply supported or cantilever) or on the material of plates were put so that one could reduce the two-dimensional waveguide problem to a one-dimensional beam-type problem. Datta et al. [75] considered composite plates

with transversely isotropic laminas having the symmetry axis aligned with either one of the plane dimensions, thus allowing to separate the problem into two one-dimensional beam waveguide problems. Dispersion relations for a composite plate consisting of four layers were computed. A similar approach was employed by Mukdadi et al. [77, 108], in which the dispersion relations of a layered plate with rectangular cross-section were obtained. Moreover, the SE method was also applied to an infinite composite plate for the first time, and dispersion curves with directional dependence over the plane dimensions were obtained. Recent works considered the wavelet spectral finite element method to account for short waveguides, where the effect of boundaries was included in the model [109–112].

The WFE method is a technique to study wave motion in periodic structures. For such structures, the dynamic vibro-acoustic behaviour of the whole structure can be described through the analysis of a single period [113]. A periodic cell is modelled using conventional FE methods. Mass, damping and stiffness matrices thus obtained are used to construct the dynamic stiffness matrix. Periodic structure theory is then applied, and the eigenvalue problem is postulated which eigenvalues and eigenvectors can be used to calculate dispersion relations and wave modes, respectively. Moreover, wave modes can be used to compute the reflection/transmission coefficients of joints in coupled structures. One of the main advantages of this method is that since only one periodic segment is used, the size of the WFE model does not depend on the dimensions of the waveguide, and the computational cost of the method is low. In addition, since a single periodic cell is discretised using conventional FE matrices, the full potential of existing conventional FE tools can be exploited. The WFE method was originally proposed by Mead in [114], where he studied the harmonic wave propagation in one-dimensional periodic systems. A huge contribution to the analysis of wave propagation in various periodic structures using a FE model of a single periodic section has been made by Abdel-Rahman [115]. The free wave propagation in one-dimensional isotropic and composite waveguides was anal-

ysed by Mace et al. in [80] and Duhamel et al. in [116]. Dispersion relations of waves were computed using one-dimensional WFE method for structures such as stiffened cylinders [117], car tyre [118], thin-walled structures [119, 120], inhomogeneous cylindrical [121] and fluid-filled pipes [122, 123], sandwich beams and panels [124, 125] and laminated cylinders [126]. Numerous applications of the one-dimensional WFE method can be found in the PhD work of Waki [84]. Also, Waki [127] outlined several numerical issues of the WFE method and how to alleviate errors associated with these issues. It is worth mentioning a work of Mencik and Ichchou [128], where they suggested a substructuring technique to address the problem of numerical issues for the case of multi-layered structures.

The basics of the WFE method for two-dimensional periodic systems was also presented in the work of Mead [114]. Later, Mead and Parthan [79] showed how the problem of defining the dispersion relations in the general direction over the plate's plane dimensions could be reduced to an array of one-dimensional WFE problems with varying periodic distances. This approach has been recently used by Chronopoulos [129] to compute the wave slowness and group velocity curves of an orthotropic graphite-epoxy monolithic plate. Rigour mathematical models of the WFE method for two-dimensional periodic isotropic and composite systems have been developed by Manconi and Mace [82, 83, 130]. Several forms of the eigenvalue problem that lead to the computation of dispersion relations were postulated. Alimonti et al. [131] extended these works by presenting a contour integral method to compute the non-linear eigenvalue problem arising from governing equations of motion upon fixing frequency and propagation direction. Dispersion relations were computed for two-dimensional arbitrarily thick layered panels in [132, 133] and periodic textile composites in [134].

Group velocity

Calculation of the group velocity vector is essential for determining the directions of rays carrying the wave energy. Many authors calculated the group velocity for one- and two-dimensional systems using various FE-based approaches. For example, Mace et al. [80], and later Ichchou et al. [135] considered energy velocity, the ratio between time-averaged energy flow and total energy density, utilising the fact that energy velocity is equal to group velocity in undamped systems. They expressed energy velocity in terms of wave modes, stiffness and mass matrices and the length of a periodic unit cell of one-dimensional systems. Ichchou et al. [135] also computed group velocity by applying a finite central difference scheme to the definition $\frac{\partial \omega}{\partial k}$. It is required that dispersion curves are categorised into distinct propagating branches. This drawback was outlined as well by Finnveden [136]. In his work, the group velocity is expressed in terms of the derivatives of the spectral form of the equations of motion. A similar approach has been recently proposed by Cicirello et al. [137]. They have analysed the first and second-order sensitivity of the general eigenvalue problems, including ones as in Equations (4.6) and (4.8), thus expressing group velocity in terms of wave number sensitivities. However, all referenced works considered one-dimensional systems, and the group velocity scalars are only obtained along the x or y directions, which is not sufficient to define the group velocity field in two-dimensional systems. Langley [138], and later Wang et al. [139] derived the same expressions for group velocity vector components for two-dimensional curved shells and laminated plates, respectively. The energy skew angle, that is, the angular divergence between group and phase velocities, was calculated using geometric considerations of wave vector curves and sensitivity analysis of governing equations of motion in [129]. Chronopoulos et al. [140] studied the sensitivity of propagating waves in two-dimensional composite plates and computed $\frac{\partial \omega}{\partial k}$ for a given set of wave number component (k_x, k_y) . However, the component $\frac{\partial \omega}{\partial k}$ is only the projec-

tion of the group velocity vector on the wave vector [19, 21, 138]. Zhao et al. [141] computed group velocity vectors of Lamb waves in unidirectional and angle-ply composite plates using the 3D elasticity theory.

Scattering coefficients

Most of the research on the calculation of scattering coefficients focused on isotropic plates so far, such as early work done by Cremer et al. [142] considering the structure-borne sound transmission of a flexural wave for right-angled joints of thin plates. Later, Craven and Gibbs [143, 144] and Wöhle et al. [145, 146] included the in-plane wave modes in their analyses together with considering up to four plates joined together. Langley and Heron [147] computed scattering coefficients for structural junctions connecting an arbitrary number of thin, isotropic plates along a rigid beam. A simplified treatment using a line-junction approximation is also described in [147]; that is, boundary conditions and force-balance equations are considered along a 1D line at the centre of the junction only. Mace [148] demonstrated important properties of the scattering coefficients such as reciprocity and conservation of energy on the example of several rods attached to a junction member. Mees and Vermeir [149] analysed the bending wave transmission loss in the system of plates connected by a hinge or by an elastic interlayer. McCollum and Cuschieri [150] studied the flexural behaviour of right-angled thick finite plates using a mobility power flow approach. They also considered both in-plane and out-of-plane wave scattering in right-angled thick semi-infinite plates [151]. Langley investigated wave reflection and transmission coefficients for structural junctions between curved panels and beams [138]. The occurrence of the negative group velocity phenomenon in cylindrical structures has been outlined in this work. Skeen and Kessissoglou [152] computed transmission coefficients for finite and semi-infinite coupled plate structures. Mencik and Ichchou introduced the hybrid FE/WFE method for calculating reflection and transmission coefficients for one-dimensional waveguides coupled longitudinally [81]. In recent years,

this method has been developed and extended to other types of junctions [153–158] and to two-dimensional waveguides [159, 160]. However, structures considered in those works mainly were isotropic or were consisting of layers of isotropic materials.

Beyond the case of wave propagation in isotropic materials, Bosmans et al. [93] studied the scattering properties of orthotropic plate junctions with principal material axes aligned with the plate coordinates, that is, so-called *specialty orthotropic* plates. However, no details on the derivation are given, and results are presented only for the particular case of bending wave transmission loss in right-angled plates, so-called L-junctions. It is not clear whether the approach derived in [93] is limited to specialty orthotropic plates or can be extended to an arbitrary number of plates meeting at the junction with different orientation of the principal material axes. Moreover, detailed information on the reflection/transmission behaviour of all propagating modes at complex junctions is needed for a DEA treatment. This includes information about the angle-of-incidence dependence of scattering coefficients and mode conversion. Karunasena and Shah [161] studied reflection of guided waves at the region of bonding material between two composite plates using the hybrid FE and semi-analytical FE method. Lee et al. [162] have presented the scattering coefficients of coupled composite plates with joint compliance and damping using the First-Order laminated plate theory [67]. Again, the principal material axes of laminates considered in these works are aligned with the plate coordinates, effectively reducing the complexity of underlying governing equations. Furthermore, in [162], the shear correction factor is introduced to correct transverse shear stiffness in the laminate, which must be defined for each laminate separately. Chronopoulos [163] computed scattering coefficients at the damaged junction between two composite beams. Later, Apolowo and Chronopoulos [164] computed the scattering coefficients of two multi-layer composite plates coupled longitudinally to localise the structural damage in the context of Structural Health Monitoring. An attempt of extending the work of Mitrou et al.

in [153] to composite plates has been made by Mitrou and Renno in [165]. The results were not reliable as the energy scattering coefficients did not sum to unity as expected in lossless systems [153, 160].

Therefore, there is a need for a detailed derivation of reflection and transmission matrices for waves travelling in structural junctions connecting composite plates at arbitrary angles and without any restrictions on the orientation of principal material axes both with respect to the orientation of the junction and with respect to the orientation in different plates. This is done following two approaches. The first one is extending the work of Langley and Heron [147] to composite laminated plates utilising the CLP theory. The second one is extending the work of Mitrou et al. [160] to arbitrarily layered composite plates. It is important to note that in both approaches, the scattering coefficients are obtained by working in the infinite junction approximation, i.e., solving the wave problem for an incident plane wave assuming that the plates extend to infinity along the junction. This approximation is justified in the high-frequency range by the *locality principle*, that is, local vibrational behaviour depends on local properties such as geometry and material properties [39].

1.2 Aims and objectives

This thesis aims to develop an extension of the DEA method for composite structures that can be represented by two-dimensional finite shell elements. The following specific goals are put in place to achieve this aim, namely:

- to develop a semi-analytical method for calculating scattering coefficients at line junctions between composite laminated plates.
- to form a hybrid FE and WFE method for computing reflection and transmission matrices of general junctions of composite laminated plates.
- to analyse and compare the effectiveness and applicability of the two methods above.

- to extend the DEA method for composite structures and compare results with FE method simulations.

1.3 Thesis outline

Chapter 2 aims to present the main theoretical basics of the DEA method for two-dimensional isotropic structures. The solution of Helmholtz and biharmonic wave equations are presented. Furthermore, we identify and discuss the main modifications needed to allow for DEA application on composite structures. Also, we describe the CLP theory for thin laminated plates, which is used in Chapter 3 to set up the governing equations of motion for composite plates. Finally, we review the main equations of the WFE method for composite plates. These are used further in Chapter 4 to derive expressions for scattering coefficients using the hybrid FE/WFE method.

Chapter 3 describes in all generality how to compute energy scattering coefficients of structural junctions made up of thin composite laminated plates in the line junction approximation. Expressions quantifying transmission and reflection coefficients as a function of the frequency and the wave number component k_x are derived. Interesting phenomena such as negative refraction and negative group velocity are observed and analysed. Furthermore, an effective scattering matrix for a plate with multiple finite stiffeners attached to it is computed. The scattering coefficients are computed explicitly for examples of two and three composite plates joined together in an L and T geometry.

A hybrid FE/WFE model that predicts the scattering properties for different junctions of two-dimensional anisotropic composite plates is developed in Chapter 4. The influence of the angle of incidence and the frequency on the distribution of the power flow of incident bending, shear and longitudinal type waves is investigated. A detailed comparison with semi-analytical evaluations of scattering coefficients derived in Chapter 3 is presented. The method gives for the first time a detailed recipe for computing scattering coefficients for the generic case of an arbitrary number of composite plates connected at a junction

without restrictions on the angles at which the plate meet or the orientation of the principal axis of individual plates.

In Chapter 5, we present the modified theoretical base of DEA for composite structures. The findings of Chapters 3 and 4 are used to derive the stationary wave energy density arising in the structure due to a point or an edge sources. Numerical results for the cases of a polygonally shaped plate, an electric vehicle gearbox and an L-shaped composite plate are presented.

Finally, Chapter 6 presents concluding remarks with revision of contributions of the work and suggests further potential research.

Chapter 2

Background

In this chapter, we present the background theory needed to proceed with an understanding of the main findings of this thesis presented in Chapters 3, 4 and 5. Section 2.1 describes the basics of the DEA method for two-dimensional isotropic structures. Also, the wave problems that are considered throughout this thesis are reviewed. Furthermore, the main parts of the DEA method that need to be modified to allow for its application to composite structures are identified and discussed. In Section 2.2, we describe the CLP theory for thin laminated plates. It is used in Chapter 3 to set up the governing equations of motion for composite plates, which eventually lead to the computation of the scattering coefficients of various junctions. Finally, Section 2.3 considers the main principles and equations in the WFE method for composite plates modelled using two-dimensional and three-dimensional finite elements. The equations presented in this section will be used further in Chapter 4 to derive expressions for scattering coefficients using the hybrid FE/WFE method.

2.1 Dynamical Energy Analysis for isotropic structures

In this section, we derive the main equations in relation to the DEA method. First, we consider general linear wave equations and show how they can be solved using a sum of the free-space Green functions and the solution of the homogeneous equations in subsection 2.1.1. In subsection 2.1.2, the free-space Green's functions for the Helmholtz and biharmonic wave equations are com-

puted. In subsection 2.1.3, the solution of the homogeneous wave equation is obtained in terms of ray trajectories via high-frequency or so-called *Eikonal* approximation. Subsection 2.1.4 shows that the resulting mean wave energy density can be approximated by the Liouville equation, which is rewritten in terms of the boundary integral operator. The general algorithm of the DEA method is discussed at the end of the section.

2.1.1 Wave equation

We consider stationary problems with continuous monochromatic energy sources. In other words, we assume that linear wave equations in question are not explicitly time-dependent, and driving terms are time-harmonic with a fixed angular frequency ω . This allows us to operate in the frequency domain rather than in the time domain. The general problem of determining the response of a two-dimensional system to a time-harmonic force with the amplitude F_0 at a source point $\mathbf{r}_0 \in \Omega$ can be written as

$$\left(\hat{H} + \omega^2\right) G(\mathbf{r}, \mathbf{r}_0, \omega) = -F_0 \delta(\mathbf{r} - \mathbf{r}_0), \quad \mathbf{r} \in \Omega \subset \mathbb{R}^2, \quad (2.1)$$

where $\hat{H}$ is the linear combination of partial-differential operators with constant coefficients, G is the Green function, and δ is the Dirac delta function [50, 51]. We seek solutions $G(\mathbf{r}, \mathbf{r}_0, \omega)$ in the domain Ω with the boundary Γ . If the linear wave operator $\hat{H}$ is equal to $c^2 \Delta$ with Laplacian operator Δ and positive c denoting the wave velocity, also called the phase velocity, then Equation (2.1) is the inhomogeneous Helmholtz equation. With the appropriate form of the wave velocity c , this equation models vibrations of thin membranes with clamped edges and in-plane deformations of plates due to point source [142, 166, 167]. In such cases, Equation (2.1) can be rewritten as

$$(\Delta + k^2) G(\mathbf{r}, \mathbf{r}_0, \omega) = f_0 \delta(\mathbf{r} - \mathbf{r}_0), \quad \mathbf{r} \in \Omega, \quad (2.2)$$

where $k = \frac{\omega}{c}$ denotes the wave number, and $f_0 = -\frac{F_0}{c^2}$ is the corresponding forcing term.

Another important application of Equation (2.1) is the bending motion of a thin isotropic plate in the Kirchhoff-Love plate theory context. The corresponding linear wave operator $\hat{H}$ is equal to $-\frac{D}{\rho h}\Delta^2$, and the equation takes the following form

$$(\Delta^2 - k^4) G(\mathbf{r}, \mathbf{r}_0, \omega) = f_0 \delta(\mathbf{r} - \mathbf{r}_0), \quad \mathbf{r} \in \Omega, \quad (2.3)$$

where the Green function G now denotes the out-of-plane displacement amplitude, $k^4 = \frac{\rho h}{D}\omega^2$ and $f_0 = \frac{F_0 \rho h}{D}$. Here, ρ is the density of the plate, h is the plate thickness, and $D = \frac{Eh^3}{12(1-\nu^2)}$ is the bending stiffness with Young's modulus E and Poisson ratio ν [1, 142].

Equations (2.2) and (2.3) are supplied with boundary conditions at Γ . These can be Dirichlet, Neumann or mixed Robin conditions for the former case and clamped, free and simply supported boundary conditions for the latter case. The solution of these problems can be found by splitting G into a homogeneous and an inhomogeneous part as

$$G(\mathbf{r}, \mathbf{r}_0, \omega) = G_0(\mathbf{r}, \mathbf{r}_0, \omega) + G_h(\mathbf{r}, \mathbf{r}_0, \omega), \quad (2.4)$$

where G_0 is the free-space Green function, and G_h is a function which satisfies the homogeneous equation

$$(\hat{H} - \omega^2) G_h(\mathbf{r}, \mathbf{r}_0, \omega) = 0, \quad \mathbf{r} \in \Omega, \quad (2.5)$$

and its amplitude is determined by the boundary conditions [51, 168]. The function G_0 describes contributions arising directly from the source, whereas G_h represents the scattering at the boundary Γ . In the next section, we will compute the free-space Green functions for the Helmholtz and biharmonic wave equations.

2.1.2 The free-space Green function

The free-space Green function is defined within an unbounded domain, and it can be obtained by the application of Fourier transform techniques [168,

[169]. Since we consider isotropic structures made of two-dimensional elements, Helmholtz and biharmonic wave equations (2.2) and (2.3) hold for r in the whole two-dimensional space $\mathbb{R}^2$. Without loss of generality, we assume that the input forcing term f_0 is equal to unity. By imposing the Sommerfeld radiation condition to ensure that only outgoing waves are present at infinity [170], one can express the free-space Green function of the Helmholtz equation $G_0(\mathbf{r}, \mathbf{r}_0, \omega)$ as

$$G_H(\mathbf{r}, \mathbf{r}_0, \omega) = -\frac{i}{4} H_0^{(1)}(k|\mathbf{r} - \mathbf{r}_0|) \quad , \quad H_0^{(1)}(z) = \frac{1}{\pi i} \int_{-\infty}^{\infty} e^{iz \cosh u} du \quad , \quad (2.6)$$

where i is the imaginary unit, and $H_0^{(1)}$ is the Hankel function of the first kind and zero order. The asymptotic behaviour for $\omega \rightarrow \infty$ is then given by

$$G_H(\mathbf{r}, \mathbf{r}_0, \omega) \sim -\sqrt{\frac{2}{\pi k|\mathbf{r} - \mathbf{r}_0|}} e^{i(k|\mathbf{r} - \mathbf{r}_0| - \frac{\pi}{4})} \quad . \quad (2.7)$$

For the case of the biharmonic wave equation, the free-space Green function satisfies

$$(\Delta^2 - k^4) G_B(\mathbf{r}, \mathbf{r}_0, \omega) = (\Delta - k^2)(\Delta + k^2) G_B(\mathbf{r}, \mathbf{r}_0, \omega) = \delta(\mathbf{r} - \mathbf{r}_0) \quad , \quad \mathbf{r} \in \mathbb{R}^2 \quad . \quad (2.8)$$

Now, the Green function G_B can be written as $G_B = -\frac{1}{2k^2} (G_H - G_M)$ [167, 171], where G_M is the Green function of the modified Helmholtz equation

$$(\Delta - k^2) G_M(r, r_0, \omega) = \delta(\mathbf{r} - \mathbf{r}_0) \quad , \quad \mathbf{r} \in \mathbb{R}^2 \quad . \quad (2.9)$$

Its solution can be readily obtained from Equation (2.6) by replacing k with ik yielding

$$G_M(\mathbf{r}, \mathbf{r}_0, \omega) = -\frac{i}{4} H_0^{(1)}(ik|\mathbf{r} - \mathbf{r}_0|) = \frac{1}{2\pi} K_0(k|\mathbf{r} - \mathbf{r}_0|) \quad , \quad (2.10)$$

with K_0 being the modified Bessel function of the second kind.

Finally, the free-space Green function of the biharmonic wave equation can be

written as

$$G_B(\mathbf{r}, \mathbf{r}_0, \omega) = \frac{1}{2k^2} \left(\frac{i}{4} H_0^{(1)}(k|\mathbf{r} - \mathbf{r}_0|) - \frac{1}{2\pi} K_0(k|\mathbf{r} - \mathbf{r}_0|) \right), \quad (2.11)$$

and the corresponding high-frequency asymptotic form reads

$$G_B(\mathbf{r}, \mathbf{r}_0, \omega) \sim \frac{1}{8k^2} \sqrt{\frac{2}{\pi k|\mathbf{r} - \mathbf{r}_0|}} \left(e^{i(k|\mathbf{r} - \mathbf{r}_0| - \frac{\pi}{4})} - e^{-k|\mathbf{r} - \mathbf{r}_0|} \right). \quad (2.12)$$

2.1.3 Eikonal approximation and ray tracing

Having computed the free-space Green functions, we can concentrate on the homogeneous part G_h . For the sake of simplicity, we operate with the homogeneous Helmholtz equation

$$\begin{aligned} (\Delta + k^2) G_h(\mathbf{r}, \omega) &= 0, \mathbf{r} \in \Omega \\ G_h(\mathbf{r}_s, \omega) &= -G_0(\mathbf{r}_s, \omega), \mathbf{r}_s \in \Gamma \end{aligned}, \quad (2.13)$$

where we assume Dirichlet boundary conditions $G(\mathbf{r}_s, \omega) = 0$. Generalising the approach to the biharmonic and other wave equations is straightforward [50]. In the high frequency approximation, one can represent G_h in terms of slow varying amplitude A and general phase function ϕ as

$$G_h(\mathbf{r}, \omega) = A(\mathbf{r}) e^{i\omega\phi(\mathbf{r})}, \mathbf{r} \in \Omega. \quad (2.14)$$

Substituting this expression into Equation (2.5) and separating the real from the imaginary part yields the following two equations

$$\begin{aligned} |\nabla\phi|^2 &= \frac{1}{c^2} + \frac{1}{\omega^2} \frac{\Delta A}{A} \\ A \Delta\phi + 2\nabla A \nabla\phi &= \nabla \cdot (A^2 \nabla\phi) = 0 \end{aligned}. \quad (2.15)$$

Now, assuming that $\frac{\Delta A}{A} \ll k^2$, which is consistent with the slow variation of the amplitude, we can drop the ω^{-2} term in the first equation in (2.15) as $\omega \rightarrow \infty$ and get the *eikonal* equation for the phase function alone

$$|\nabla\phi| = \frac{1}{c} = \frac{k}{\omega} \quad (2.16)$$

Now, defining $\mathbf{p} \equiv \nabla\phi$ as the *momentum vector* and setting the Hamilton function $H(\mathbf{r}, \mathbf{p}) = c|\mathbf{p}|$ yields the Hamilton-Jacobi equation

$$H(\mathbf{r}, \mathbf{p}) = 1. \quad (2.17)$$

This equation can be solved by the *method of characteristics* [45, 172, 173].

First, we define ray trajectories $\mathbf{r} = \mathbf{r}(\tau)$ by the following ODEs

$$\frac{d\mathbf{r}}{d\tau} = \nabla_{\mathbf{p}} H. \quad (2.18)$$

The system of equations in (2.18) is under-determined, therefore we need equations for $\mathbf{p} = \mathbf{p}(\tau)$. To accomplish that, we differentiate (2.17) with respect to $\mathbf{r}$ as

$$\nabla_{\mathbf{r}} H(\mathbf{r}, \mathbf{p}) + \nabla_{\mathbf{p}} H(\mathbf{r}, \mathbf{p}) \nabla \mathbf{p} = 0, \quad (2.19)$$

and $\mathbf{p}$ — with respect to τ as

$$\frac{d\mathbf{p}}{d\tau} = \nabla_{\mathbf{p}} \frac{d\mathbf{r}}{d\tau}. \quad (2.20)$$

Now, combining Equations (2.18), (2.19) and (2.20) gives Hamilton's equations for ray-trajectories $(\mathbf{r}(\tau), \mathbf{p}(\tau))$

$$\begin{aligned} \frac{d\mathbf{r}}{d\tau} &= \nabla_{\mathbf{p}} H = c \frac{\mathbf{p}}{|\mathbf{p}|}, \\ \frac{d\mathbf{p}}{d\tau} &= -\nabla_{\mathbf{r}} H. \end{aligned} \quad (2.21)$$

Note that if the phase velocity c is constant in the domain Ω , then $\frac{d\mathbf{p}}{d\tau}$ is zero, and consequently, the ray trajectories are straight lines. Now, if we write the boundary conditions $G_h(\mathbf{r}_s, \omega)$ in the form

$$G_h(\mathbf{r}_s, \omega) = A_0(\mathbf{r}_s) e^{i\omega\phi_0(\mathbf{r}_s)} \quad \text{for } \mathbf{r}_s \in \Gamma, \quad (2.22)$$

then initial conditions for the system of ODEs in (2.21) can be stated as

$$\begin{aligned} \mathbf{r}(0) &= \mathbf{r}_s \\ \mathbf{p}(0) &= \nabla\phi(\mathbf{r}(0)) = \nabla\phi_0(\mathbf{r}_s) \end{aligned} \quad \text{for } \mathbf{r}_s \in \Gamma \quad (2.23)$$

The solution of Equation (2.21) with the mentioned initial conditions are bicharacteristics $(\mathbf{r}, \mathbf{p})$ which are curves in the 4-dimensional (4D) *phase space* $\mathbb{R}^{2 \times 2}$. Now, in order to find a solution $G_h(\mathbf{r}', \omega)$ at a point $\mathbf{r}'$, we need to find all bicharacteristic curves $(\mathbf{r}, \mathbf{p})$ that pass through the point $\mathbf{r}'$ starting on the boundary points $\mathbf{r}_s \in \Gamma$. In general, there can be infinitely many such bicharacteristic curves, and we can compute the phase function $\phi_j(\mathbf{r}')$ obtained from the ray trajectory starting at the boundary point $\mathbf{r}_{s,j}$ in the following way

$$\begin{aligned} \frac{d\phi}{d\tau} = \nabla \phi \frac{d\mathbf{r}}{d\tau} = \mathbf{p} \frac{d\mathbf{r}}{d\tau} \rightarrow \phi(\mathbf{r}', \tau) &= \phi(\mathbf{r}, 0) + \int_0^\tau \mathbf{p} \frac{d\mathbf{r}}{d\tau} d\tau \rightarrow \\ \phi_j(\mathbf{r}') &= \phi_0(\mathbf{r}_{s,j}) + \int_{\mathbf{r}_{s,j}}^{\mathbf{r}'} \mathbf{p} d\mathbf{r}. \end{aligned} \quad (2.24)$$

Once the phase function ϕ_j is known, the amplitude A_j is the solution of the *transport* equation

$$\begin{aligned} A_j \Delta \phi + 2 \nabla A_j \nabla \phi_j &= \nabla \cdot (A_j^2(\mathbf{r}) \nabla \phi_j(\mathbf{r})) = 0, \quad \mathbf{r} \in \Omega, \\ J_0(\mathbf{r}_{s,j}) &= A_0^2(\mathbf{r}_{s,j}) \frac{\partial \phi_0(\mathbf{r}_{s,j})}{\partial n}, \quad \mathbf{r}_{s,j} \in \Gamma, \end{aligned} \quad (2.25)$$

where J_0 is the incoming flux normal to the boundary Γ [51, 174]. Finally, the solution $G_h(\mathbf{r}, \omega)$ to the problem (2.13) can be approximated as

$$G_h(\mathbf{r}, \omega) = \sum_j |A_j(\mathbf{r})| e^{i\omega \phi_j(\mathbf{r}) - i\frac{\pi}{2} m_j}, \quad \mathbf{r} \in \Omega, \quad (2.26)$$

where m_j are the Maslow indices [172, 173]. Since the phase information is not of interest in the DEA method, the details about the Maslow indices are omitted. In what follows, we will be interested in the wave energy density $\epsilon(\mathbf{r}, \omega)$ arising from the reflections off the boundary. It can be approximated as

$$\begin{aligned} \epsilon(\mathbf{r}, \omega) &\propto |G_h(\mathbf{r}, \omega)|^2 = \sum_{j, j'} |A_j A_{j'}| e^{i\omega(\phi_j - \phi_{j'}) - i\frac{\pi}{2}(m_j - m_{j'})} = \\ &= \sum_j |A_j|^2 + \sum_{j \neq j'} |A_j A_{j'}| e^{i\omega(\phi_j - \phi_{j'}) - i\frac{\pi}{2}(m_j - m_{j'})}. \end{aligned} \quad (2.27)$$

Here, the first term represents a smooth part of the wave energy density, while the second term is associated with oscillations on the scale of the wavelength.

The latter can be made negligibly small after averaging over a frequency band centred on ω . Therefore, the total mean wave energy density can be approximated as

$$\epsilon_{tot}(\mathbf{r}, \omega) \propto |G_0(\mathbf{r}, \mathbf{r}_0, \omega)|^2 + \rho(\mathbf{r}, \omega), \quad \rho(\mathbf{r}, \omega) = \sum_j |\bar{A}_j|^2. \quad (2.28)$$

In the next section, we present the connection between the smooth part of the wave energy density $\rho(\mathbf{r}, \omega)$ and the boundary integral representation of the phase space density.

2.1.4 Liouville equation and boundary mapping

The mean wave energy density $\rho(\mathbf{r}, \omega)$ can be well approximated by the phase space density $\rho(\mathbf{r}, \mathbf{p}, \omega)$ passing through a point $\mathbf{r}$ as

$$\rho(\mathbf{r}, \omega) = \int_{\mathbb{R}^2} \rho(\mathbf{r}, \mathbf{p}, \omega) d\mathbf{p}. \quad (2.29)$$

The phase space density $\rho(\mathbf{r}, \mathbf{p}, \omega)$ represents the density of non-interacting particles following trajectories governed by Hamilton equations (2.21) [50]. It satisfies the stationary *Liouville equation* [45, 51] that has the following form

$$\{H, \rho\} = 0, \quad (2.30)$$

where $\{\cdot, \cdot\}$ are the Poisson brackets which have the following definition

$$\{f(\mathbf{r}, \mathbf{p}), g(\mathbf{r}, \mathbf{p})\} = \nabla_{\mathbf{r}} f \nabla_{\mathbf{p}} g - \nabla_{\mathbf{p}} f \nabla_{\mathbf{r}} g. \quad (2.31)$$

The solution of Equation (2.30) can be written as

$$\rho(\mathbf{r}, \mathbf{p}, \omega) = \sum_j |A_j(\mathbf{r}, \omega)|^2 \delta(\mathbf{p} - \nabla \phi_j(\mathbf{r}, \omega)), \quad (2.32)$$

where $A_j(\mathbf{r}, \omega)$ and $\phi_j(\mathbf{r}, \omega)$ are solutions of the eikonal and transport equations (2.16) and (2.25), respectively [45, 174]. It is clear that the phase space density $\rho(\mathbf{r}, \mathbf{p}, \omega)$ given by Equation (2.32) satisfies Equation (2.29). This is also called the *Wigner measure* [45, 175], which represents the weak limit of the Wigner distribution as $\omega \rightarrow \infty$.

Since the summation in (2.28) and (2.32) often consists of infinitely many terms especially when considering long-term dynamics, the formulation can be cumbersome and prone to poor convergence [167]. Instead, we will construct the stationary or long-time limit phase space density $\rho(\mathbf{r}, \mathbf{p}, \omega)$ using a boundary mapping technique. First, we map the initial energy density $\rho_0(\mathbf{r}, \mathbf{r}_0, \omega)$ from the point source at $\mathbf{r}_0$ to the boundary Γ using the ray-tracing formula (2.17) and (2.21). The initial energy density at a point source can be written as

$$\rho_0(\mathbf{r}, \omega) = \frac{R}{2\pi c_g} \frac{e^{-\mu|\mathbf{r}-\mathbf{r}_0|}}{|\mathbf{r}-\mathbf{r}_0|}, \quad \mathbf{r} \in \Omega \subset \mathbb{R}^2, \quad (2.33)$$

where R is the source power, c_g is the group velocity and μ is the attenuation factor [53, 176]. The attenuation factor μ is related to the damping loss factor η and the group velocity c_g by $\mu = \eta \frac{\omega}{c_g}$. For instance, the attenuation factor μ is equal to ηk for in-plane waves and $0.5 \eta k$ for bending waves in thin isotropic plates. This is because the group velocity c_g is equal to the phase velocity c in the former case and twice the phase velocity $2c$ in the latter case [13, 142].

Now, the initial phase space density $\rho_0(\mathbf{r}, \mathbf{p}, \omega)$ can be represented as

$$\rho_0(\mathbf{r}, \mathbf{p}, \omega) = \frac{R}{2\pi c_g} \frac{e^{-\mu|\mathbf{r}-\mathbf{r}_0|}}{|\mathbf{r}-\mathbf{r}_0|} \delta(\mathbf{p} - \mathbf{p}_0), \quad \mathbf{p}_0 = |\mathbf{p}| \frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|}, \quad (2.34)$$

where $\mathbf{p}_0$ is the momentum vector of the ray connecting points $\mathbf{r}_0$ and $\mathbf{r}$. Note that the ray travels along the gradient of the Hamilton function $\nabla_{\mathbf{p}} H$ according to Equation (2.21), and in isotropic structures, the ray trajectory is aligned with the momentum vector direction, thus yielding the form of the $\mathbf{p}_0$ as in (2.34). However, as we will see in Section 3.3, Section 4.3 and Chapter 5, in composite structures, the ray trajectories are not parallel to the momentum vector $\mathbf{p}$, but to the group velocity vector $\mathbf{c}_g$, hence, the delta function in Equation (2.34) will have a different form.

It is worth noting that the phase space densities can describe much more general driving sources than just monochromatic point sources. For instance, in the electromagnetic context, phase space densities in the form of Wigner

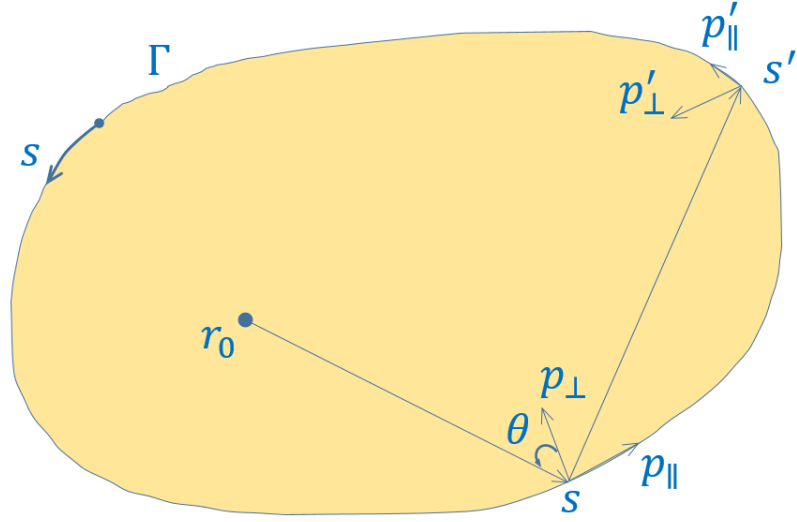


Figure 2.1: The set up of the local coordinate system at the boundary $X = (s, p_{\parallel})$. A ray emitting from the point $\mathbf{r}_0$ reach the point $\mathbf{r}_s$, which has the local position s on the boundary Γ . The momentum vector $\mathbf{p}$ has normal and tangential components $p_{\perp}$ and $p_{\parallel}$ at the point s , and θ denotes the angle between the ray connecting $\mathbf{r}_0$ and s and the normal to Γ .

distributions can represent partially coherent near-homogeneous finite sources [177–181].

Rays emitted isotropically from the point source reach the boundary Γ and produce the phase space density on the boundary $\rho(\mathbf{r}_s, \mathbf{p})$, $\mathbf{r}_s \in \Gamma$. To proceed with the boundary mapping, we need to introduce a coordinate system $X = (s, p_{\parallel})$ on the boundary, where s parametrizes Γ , and $p_{\parallel}$ is the component of the momentum vector $\mathbf{p}$ tangential to Γ at the position s , see Figure 2.1. The component $p_{\perp}$ of the momentum vector $\mathbf{p}$ is normal to the boundary Γ at the position s . Also, we introduce the two-dimensional (2D) boundary phase space density $\rho_{\Gamma}(s, p_{\parallel})$, which will be used to construct the stationary phase space density $\rho^{\infty}(\mathbf{r}, \mathbf{p}, \omega)$. It is important to distinguish between the 2D phase space density $\rho_{\Gamma}(s, p_{\parallel})$ and the density on the full 4D phase space $\rho(\mathbf{r}_s, \mathbf{p})$. The former yields the energy flux through the boundary Γ at position s when integrated over the momentum variables, while the latter produces the energy density at the point $\mathbf{r}_s$ [182]. They can be related as follows

$$\rho(\mathbf{r}_s, \mathbf{p}) = \rho_{\Gamma}(s, p_{\parallel}) \delta \left(\tilde{H}(\mathbf{r}_s, \mathbf{p}) \right), \quad (2.35)$$

where $\tilde{H}(\mathbf{r}, \mathbf{p})$ is the general Hamilton function [50, 51]. In the case of the Helmholtz equation (2.13), $\tilde{H}(\mathbf{r}, \mathbf{p})$ is equal to $c|\mathbf{p}| - 1$ according to Equation (2.17). Now, we apply the delta function substitution rule to the following product of delta functions

$$\frac{cp_{\perp}}{|\mathbf{p}|} \delta(c|\mathbf{p}| - 1) \delta(p_{\parallel} - p_{\parallel,0}) = \delta(\mathbf{p} - \mathbf{p}_0), \quad (2.36)$$

where we combine $\delta(p_{\parallel} - p_{\parallel,0}) \delta(p_{\perp} - p_{\perp,0})$ as $\delta(\mathbf{p} - \mathbf{p}_0)$. Inserting the expression (2.34) for the 4D phase space density into Equation (2.35) and using the relation (2.36) yield the initial 2D phase space density on the boundary as

$$\rho_{\Gamma,0}(s, p_{\parallel}) = \frac{cR}{2\pi c_g} \frac{e^{-\mu|\mathbf{r}_s - \mathbf{r}_0|}}{|\mathbf{r}_s - \mathbf{r}_0|} \cos(\theta(\mathbf{r}_s, \mathbf{r}_0)) \delta(p_{\parallel} - p_{\parallel,0}), \quad (2.37)$$

where $\theta(\mathbf{r}_s, \mathbf{r}_0)$ is the angle between the ray connecting points $\mathbf{r}_0$ and $\mathbf{r}_s$ and the normal to Γ , see Figure 2.1. Rays starting with phase space coordinates $X = (s, p_{\parallel})$ reach the boundary Γ again with phase space coordinates $X' = (s', p'_{\parallel})$, see Figure 2.1.

Now, the idea is to relate the 2D phase space density $\rho_{\Gamma}(X)$ with the new phase space density $\rho'_{\Gamma}(X')$, which is produced after one scattering event of the rays leaving the boundary Γ . This can be achieved by introducing the boundary integral operator T as

$$\rho'_{\Gamma}(X') = \{T\rho_{\Gamma}\}(X') = \int \lambda(X') e^{-\mu D(X, X')} \delta(X' - \Phi(X)) \rho_{\Gamma}(X) dX. \quad (2.38)$$

Here, the function $\Phi(X)$ maps the phase space coordinates $X = (s, p_{\parallel})$ to $X' = (s', p'_{\parallel})$ according to the ray-tracing formula given in (2.21). The term $\lambda(X')$ represents the energy scattering coefficients, that is, the ratios of incoming and outgoing energy fluxes, whereas $D(X, X')$ is the length of the ray trajectory connecting the phase space coordinates X and X' . Equation (2.38) means that the new phase space density ρ'_{Γ} at the phase space point X' is composed of the ray-particles starting at the phase space points X that reach X' whilst being damped along the way with attenuation factor μ and scattered upon hitting the boundary Γ . Note that in undamped systems with perfect

reflection/transmission, that is, when $\mu = 0$ and $\lambda(X') = 1$, the phase space density $\rho'_\Gamma(X')$ is equal to $\rho_\Gamma(\Phi(X))$, which is consistent with phase space volume conservation [45, 172, 173]. The correspondent integral operator is known as the *Perron-Frobenius* operator [50, 53, 172].

Finally, we compute the stationary or long-time limit 2D phase space density $\rho_{\Gamma,\infty}(X)$ as

$$\rho_{\Gamma,\infty}(X) = \sum_{n=1}^{\infty} T^n \rho_{\Gamma,0}(X) = (I - T)^{-1} \rho_{\Gamma,0}(X), \quad (2.39)$$

where $\rho_{\Gamma,0}$ is the initial 2D phase space density on the boundary given by Equation (2.37). Note that for non-vanishing attenuation, the series converges as $n \rightarrow \infty$ [50]. The resulting stationary 4D phase space density $\rho_\infty(\mathbf{r}_s, \mathbf{p})$, $\mathbf{r}_s \in \Gamma$ can be recovered using Equation (2.35) as

$$\rho_\infty(\mathbf{r}_s, \mathbf{p}) = \rho_{\Gamma,\infty}(s, p_\parallel) \delta(\tilde{H}(\mathbf{r}_s, \mathbf{p})) = \rho_{\Gamma,\infty}(s, p_\parallel) \delta(c|\mathbf{p}| - 1). \quad (2.40)$$

Now, one can express the solution of the Liouville equation (2.30) as

$$\rho(\mathbf{r}, \mathbf{p}, \omega) = e^{-\mu|\mathbf{r}-\mathbf{r}_s|} \rho_\infty(\mathbf{r}_s(\mathbf{r}, \mathbf{p}), \mathbf{p}), \quad (2.41)$$

where $\mathbf{r}_s$ is the point of boundary intersection of the ray emanating from the point $\mathbf{r}$ in the direction $-\mathbf{p}$ [53]. Consequently, the mean wave energy density $\rho(\mathbf{r}, \omega)$ at the point $\mathbf{r}$ can be obtained using Equation (2.29).

Having defined the main steps of the DEA method, we still need to address the problem of defining the solution of Equation (2.39). One needs to express the boundary integral operator T in a finite set of basis functions, thus constructing its discrete version. Several basis function sets have been used in the literature. In particular, Tanner [50, 52] used Fourier basis functions both in position and momentum space to represent the operator T . Due to several difficulties such as non-periodic boundary conditions and slow convergence of the associated quadrature rules, Chebyshev polynomial functions with the Gauss-Chebyshev quadrature rule were suggested as an alternative in [183]. It was found in [51, 184] that utilising a basis of functions that is orthogonal in the L^2 space inner

product is useful when considering multi-component fine-meshed systems. The Legendre polynomial functions were used instead of Chebyshev polynomials due to better computational efficiency whilst maintaining fairly good solution approximations.

In this thesis, the Legendre polynomial functions are used to discretise the boundary operator T . However, the implementation of this process and generally of the DEA method will be presented directly for composite structures in Chapter 5. To summarise, the algorithm on how to proceed with the DEA method can be put as

1. Define the Hamilton function (2.17) and the ray-tracing equations (2.21).
2. Compute the initial 2D phase space density using Equation (2.37).
3. Construct the discrete version of the operator T defined in (2.38).
4. Solve Equation (2.39) to obtain the stationary phase space density $\rho_{T,\infty}$.
5. Compute the stationary energy density using Equation (2.29).

(2.42)

To extend the applicability of the DEA method for composite structures, several major steps in this algorithm must be redefined accordingly. For instance, the Hamilton function and ray-tracing formula that define the boundary mapping $\Phi : X \rightarrow X'$ in the definition of the operator T will take more sophisticated forms in the case of composite structures as we shall see throughout this thesis. The scattering coefficients $\lambda(X')$ in (2.38) play a significant role in defining the long-term dynamics of the phase space densities. As outlined in the introduction, Chapters 3 and 4 are devoted to calculating these coefficients for the various junctions of composite laminated plates. Finally, the numerical implementation algorithm, whilst being based on the works of Chappell et al. [51] and Hartmann et al. [53], needs to be changed significantly to suit the case of composite structures. In the following section, the background theory needed for Chapter 3 is presented.

2.2 Classical Laminated Plate theory

In this section, we describe the Classical Laminated Plate theory for composite laminated plates. It is used in Chapter 3 to set up governing equations of motion of composite plates, which eventually leads to the computation of the scattering coefficients of various composite plate junctions.

The CLP theory is one of the Equivalent Single Layer (ESL) theory types, i.e. we model a composite laminated plate with an equivalent single layer [1, 67]. It is an extension of the Kirchhoff-Love plate theory to laminated plates. The following assumptions are made

1. The plate is thin, e.g. thickness is much smaller than length and width dimensions.
2. The plate is made of n linear elastic orthotropic laminae that are perfectly bonded together (see Figure 2.2).
3. Transverse shear strains ε_{xz} , ε_{yz} and transverse normal strain ε_{zz} are negligible.
4. Rotary inertia terms are negligible.
5. The effect of in-plane forces on bending is negligible.

Note that in infinite plates, a plate can be considered thin as long as

$$\lambda_B^{\min} \geq 6h \quad \text{or} \quad k_B^{\max} h < 1 \quad (2.43)$$

where h , $k_B^{\max}$ and $\lambda_B^{\min} = \frac{2\pi}{k_B^{\max}}$ are the laminate thickness, the maximum wave number and the minimum wave length of the bending mode, respectively [142]. According to the assumptions made, the displacement field can be described

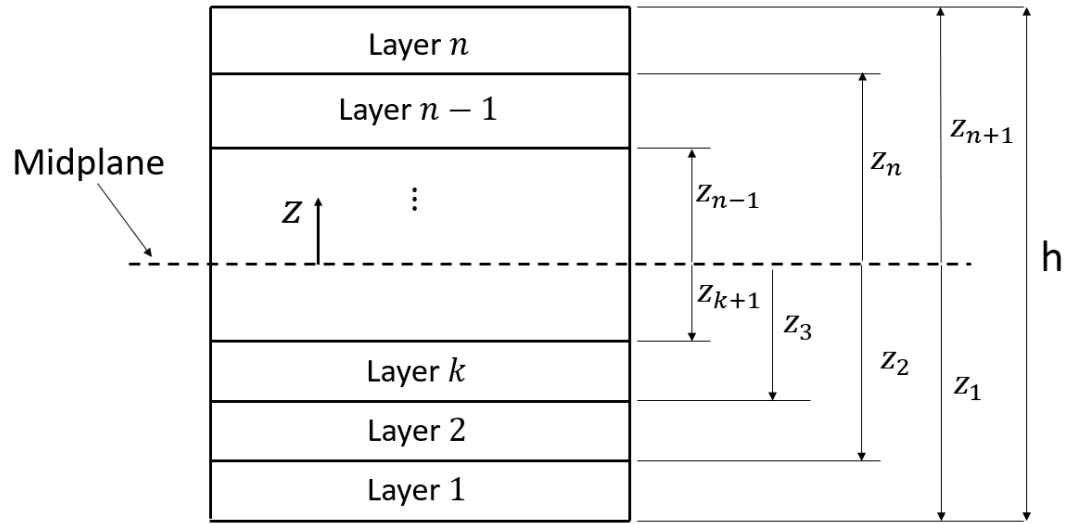


Figure 2.2: The lay-up configuration of a composite laminate.

as

$$\begin{aligned}
 u(x, y, z, t) &= u_0(x, y, t) - z \frac{\partial w_0}{\partial x} \\
 v(x, y, z, t) &= v_0(x, y, t) - z \frac{\partial w_0}{\partial y} \\
 w(x, y, z, t) &= w_0(x, y, t)
 \end{aligned} \tag{2.44}$$

where u_0 , v_0 and w_0 are the displacements along the correspondent axes of a point $(x, y, 0)$ on the mid-plane. Therefore, the infinitesimal strain tensor can be written as

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u_0}{\partial x} \\ \frac{\partial v_0}{\partial y} \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \end{Bmatrix} - z \begin{Bmatrix} \frac{\partial^2 w_0}{\partial x^2} \\ \frac{\partial^2 w_0}{\partial y^2} \\ 2 \frac{\partial^2 w_0}{\partial x \partial y} \end{Bmatrix}. \tag{2.45}$$

Stress components of the k th lamina can be found from the following constitutive relations:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix}^k = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{26} & Q_{26} & Q_{66} \end{bmatrix}^k \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{Bmatrix} \tag{2.46}$$

where the Q_{ij} 's are plane-stress reduced stiffnesses of an orthotropic lamina. If the principle material axes of the k th lamina are not aligned with the natural coordinates of the plate, then all $Q_{ij}^k \neq 0$, and the lamina is *generally*

orthotropic [1]. If they are aligned, then $Q_{16}^k = Q_{26}^k = 0$, and the lamina is *specialy orthotropic*. Details about the relations between the Q_{ij} 's and the more familiar coefficients for specially orthotropic plates as, for example, found in [185], together with the relations to material constants can be found in Appendix B. In-plane stress and moment resultants can be computed via an integration of stresses along the thickness direction as follows

$$\begin{aligned} \begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} dz = \sum_{k=1}^n \int_{z_k}^{z_{k+1}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix}^k dz \\ \begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} &= - \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} z dz = - \sum_{k=1}^n \int_{z_k}^{z_{k+1}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix}^k z dz \end{aligned} \quad (2.47)$$

Now, we can use relations (2.45) and (2.46) to represent the in-plane stress and moment resultants in terms of the displacement field as

$$\begin{aligned} \begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} &= \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{26} & A_{22} & A_{66} \end{bmatrix} \begin{Bmatrix} \frac{\partial u_0}{\partial x} \\ \frac{\partial v_0}{\partial y} \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \end{Bmatrix} - \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{26} & B_{22} & B_{66} \end{bmatrix} \begin{Bmatrix} \frac{\partial^2 w_0}{\partial x^2} \\ \frac{\partial^2 w_0}{\partial y^2} \\ 2 \frac{\partial^2 w_0}{\partial x \partial y} \end{Bmatrix} \\ \begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} &= \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{26} & D_{22} & D_{66} \end{bmatrix} \begin{Bmatrix} \frac{\partial^2 w_0}{\partial x^2} \\ \frac{\partial^2 w_0}{\partial y^2} \\ 2 \frac{\partial^2 w_0}{\partial x \partial y} \end{Bmatrix} - \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{26} & B_{22} & B_{66} \end{bmatrix} \begin{Bmatrix} \frac{\partial u_0}{\partial x} \\ \frac{\partial v_0}{\partial y} \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \end{Bmatrix}, \end{aligned} \quad (2.48)$$

where A_{ij} , B_{ij} and D_{ij} are called extensional, bending-extensional coupling and bending stiffnesses, defined as follows

$$(A_{ij}, B_{ij}, D_{ij}) = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{ij}(1, z, z^2) dz = \sum_{k=1}^n \int_{z_k}^{z_{k+1}} Q_{ij}^k(1, z, z^2) dz. \quad (2.49)$$

If individual layers are homogeneous, the integrations in Equation 2.49 can be

replaced by summations in the following way:

$$\begin{aligned} A_{ij} &= \sum_{k=1}^n Q_{ij}^k (z_{k+1} - z_k) \\ B_{ij} &= \frac{1}{2} \sum_{k=1}^n Q_{ij}^k (z_{k+1}^2 - z_k^2) . \\ D_{ij} &= \frac{1}{3} \sum_{k=1}^n Q_{ij}^k (z_{k+1}^3 - z_k^3) \end{aligned} \quad (2.50)$$

Finally, the governing equations of free motion can be written in terms of in-plane stress and moment resultants as follows

$$\begin{aligned} \frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} &= I_0 \frac{\partial^2 u_0}{\partial t^2} - I_1 \frac{\partial^2}{\partial t^2} \left(\frac{\partial w_0}{\partial x} \right) \\ \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} &= I_0 \frac{\partial^2 v_0}{\partial t^2} - I_1 \frac{\partial^2}{\partial t^2} \left(\frac{\partial w_0}{\partial y} \right) , \\ \frac{\partial^2 M_{xx}}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_{yy}}{\partial y^2} &= -I_0 \frac{\partial^2 w_0}{\partial t^2} + I_1 \frac{\partial^2}{\partial t^2} \left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \right) \end{aligned} \quad (2.51)$$

where $[I_0, I_1] = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho_0 [1, z] dz$ - the mass moments of inertia with ρ_0 being the material density. Transversal tractions Q_x and Q_y which do not appear in the equations of motion are important when formulating the boundary conditions. They can be expressed in terms of moments as follows

$$\begin{aligned} Q_x &= -\frac{\partial M_{xx}}{\partial x} - \frac{\partial M_{xy}}{\partial y} \\ Q_y &= -\frac{\partial M_{yy}}{\partial y} - \frac{\partial N_{xy}}{\partial x} . \end{aligned} \quad (2.52)$$

These tractions together with the twisting moment contributions $-\frac{\partial M_{xy}}{\partial x}$ and $-\frac{\partial M_{xy}}{\partial y}$ define the effective shear forces V_x, V_y as follows:

$$\begin{aligned} V_x &= Q_x - \frac{\partial M_{xy}}{\partial y} \\ V_y &= Q_y - \frac{\partial M_{xy}}{\partial x} . \end{aligned} \quad (2.53)$$

Note that for laminates that consist of an odd number of laminae and are symmetric, that is, when the ply stacking sequence, material, and ply thicknesses are symmetric around the mid-plane, all bending-extension stiffnesses $B_{ij} = 0$, and the mass moment of inertia $I_1 = 0$. Therefore, for such laminates, the equations of motion (2.51) become uncoupled, i.e. the first two equations gov-

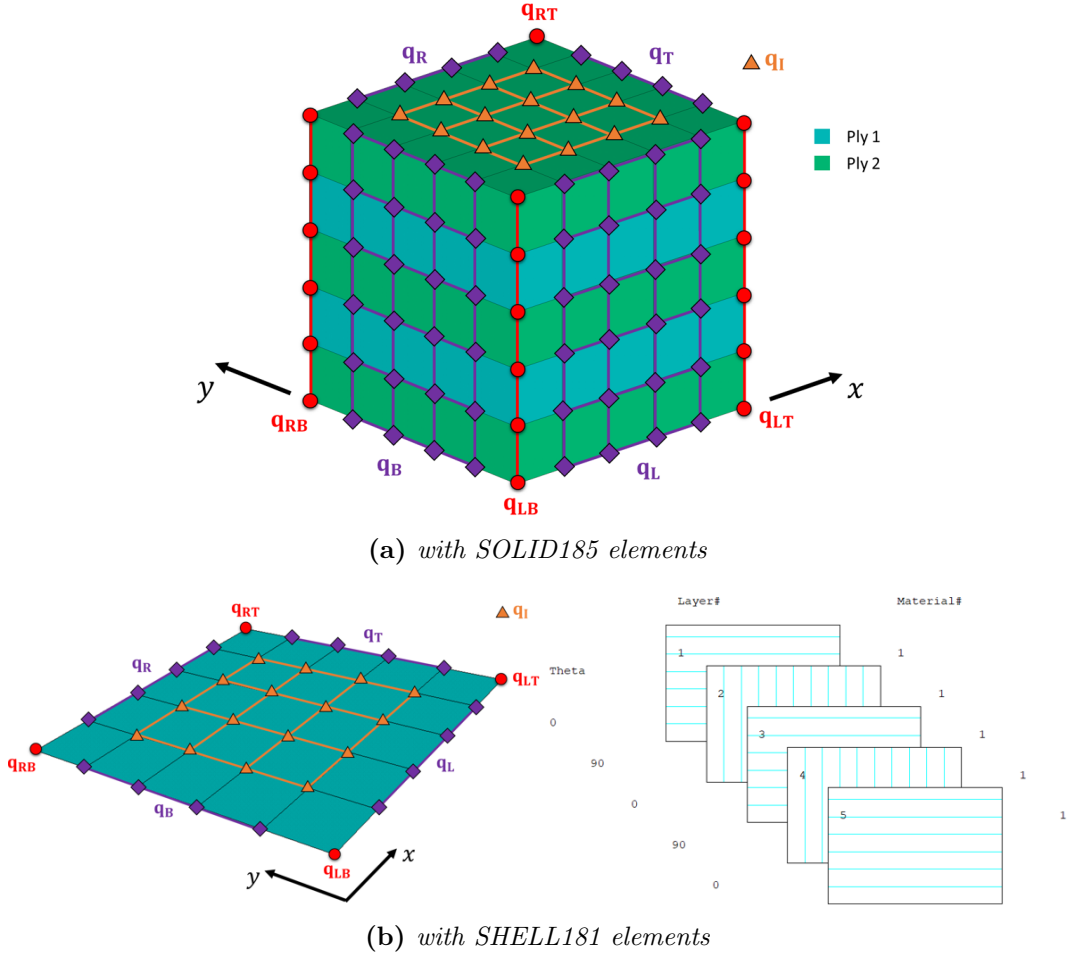


Figure 2.3: A periodic segment of a composite plate with two different alternating plies modelled with *SOLID185* (a) and *SHELL181* elements (b) in Ansys. The degrees of freedom are grouped into internal $\mathbf{q}_I$, edge $\mathbf{q}_L$, $\mathbf{q}_R$, $\mathbf{q}_B$, $\mathbf{q}_T$ and corner $\mathbf{q}_{LB}$, $\mathbf{q}_{RB}$, $\mathbf{q}_{LT}$, $\mathbf{q}_{RT}$ degrees of freedom.

ern the in-plane motion of the laminate, whereas the last equation describes the out-of-plane motion of the laminate. This concludes the review of the CLP theory for thin composite laminates. In the next section, the WFE method for composite two-dimensional waveguides is reviewed.

2.3 The Wave Finite Element method for composite plates

2.3.1 Statement of the problem

We consider a unit cell of a composite plate with arbitrary lay-up through the thickness direction and plane dimensions d_x and d_y . It can be modelled using two-dimensional shell elements with a composite lay-up or three-

dimensional solid elements stacked up one on top of the other, representing different composite layers. For instance, in ANSYS, the former can be represented by the linear element type SHELL181, and the latter by the linear element type SOLID185. Figure 2.3 presents a unit cell of a five-layer plate meshed with SOLID185 and SHELL181 elements and a nodal displacements vector labelling convention. A SOLID185 element consists of eight nodes with three translational degrees of freedom per node, whereas a SHELL181 element consists of four nodes with three translational and three rotational degrees of freedom per node. The nodal degrees of freedom are grouped into internal $\mathbf{q}_I$, edge $\mathbf{q}_L$, $\mathbf{q}_R$, $\mathbf{q}_B$, $\mathbf{q}_T$ and corner $\mathbf{q}_{LB}$, $\mathbf{q}_{RB}$, $\mathbf{q}_{LT}$, $\mathbf{q}_{RT}$ degrees of freedom. Accordingly, the nodal displacements vector $\mathbf{q}$ can be organised as $\mathbf{q} = \left\{ \mathbf{q}_{LB} \ \mathbf{q}_{RB} \ \mathbf{q}_B \ \mathbf{q}_L \ \mathbf{q}_R \ \mathbf{q}_I \ \mathbf{q}_{LT} \ \mathbf{q}_{RT} \ \mathbf{q}_T \right\}^T$. The nodal forces vector $\mathbf{f}$ is arranged in the same manner. The number of degrees of freedom must be the same for each pair of edges on opposite faces. The number of mesh cells in the x , y and z direction are labelled by n_x , n_y and n_z . The number of degrees of freedom per edge is labelled as m ; for plates modelled with SOLID185 elements $m = 3(n_z + 1)$, whereas SHELL181 elements based plates have $m = 6$ since there is only one node in the z direction. Consequently, the sizes of nodal displacement sub-vectors can be represented as

$$\begin{aligned} |\mathbf{q}_{L(R)B(T)}| &= m, \quad |\mathbf{q}_{L(R)}| = m(n_x - 1), \\ |\mathbf{q}_{B(T)}| &= m(n_y - 1), \quad |\mathbf{q}_I| = m(n_x - 1)(n_y - 1). \end{aligned} \quad (2.54)$$

The governing equation of motion of the unit cell can be written as

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{C}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{f}, \quad (2.55)$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are mass, damping and stiffness matrices, respectively. Note that these matrices are symmetric. The dimension of Equation (2.55) is $m(n_x + 1)(n_y + 1)$. We assume that the structure undergoes a harmonic motion at angular frequency ω , therefore, $\ddot{\mathbf{q}} = -\omega^2\mathbf{q}$ and $\dot{\mathbf{q}} = i\omega\mathbf{q}$. Introducing this

expression into Equation (2.55) yields

$$[\mathbf{K} + i\omega\mathbf{C} - \omega^2\mathbf{M}] \mathbf{q} = \mathbf{f}. \quad (2.56)$$

The matrix $\mathbf{C}$ can be composed of different components including layer- and material-wise viscous and structural damping matrices, but in this work, we assume that $\mathbf{C} = \frac{\eta}{\omega}\mathbf{K}$, where η is a uniform structural damping coefficient. Therefore, Equation (2.56) can be written as

$$[\mathbf{K}(1 + i\eta) - \omega^2\mathbf{M}] \mathbf{q} = \mathbf{f}. \quad (2.57)$$

2.3.2 Periodic structure theory and eigenvalue problem

We consider a plane wave travelling across the plate to be of the form $e^{-ik_x x - ik_y y + i\omega t}$, where k_x and k_y are x and y components of the wave vector $\mathbf{k}$. Periodic structure theory [186, 187] requires that the displacement vector between adjacent nodes and opposite sides differs only by a propagation factor λ . For the nodes placed along the x direction (see Figure 2.3), we obtain

$$\begin{Bmatrix} \mathbf{q}_{LT} \\ \mathbf{q}_{RT} \\ \mathbf{q}_B \end{Bmatrix} = \lambda_x \begin{Bmatrix} \mathbf{q}_{LB} \\ \mathbf{q}_{RB} \\ \mathbf{q}_T \end{Bmatrix}, \quad (2.58)$$

where $\lambda_x = e^{-ik_x d_x}$ is the propagation factor in the x direction. Now, we denote as $\mathbf{q}_X^j$, $X = L, R, I$ displacement sub-vectors of nodes placed at $x_j = d_x j / n_x$, $j = 1, 2, \dots, n_x - 1$. Consequently, as in Equation (2.58), one can relate internal and edge degrees of freedom to bottom ones as

$$\begin{Bmatrix} \mathbf{q}_L^j \\ \mathbf{q}_R^j \\ \mathbf{q}_I^j \end{Bmatrix} = e^{-ik_x d_x j / n_x} \begin{Bmatrix} \mathbf{q}_{LB} \\ \mathbf{q}_{RB} \\ \mathbf{q}_B \end{Bmatrix} = \lambda_x^{j/n_x} \begin{Bmatrix} \mathbf{q}_{LB} \\ \mathbf{q}_{RB} \\ \mathbf{q}_B \end{Bmatrix}. \quad (2.59)$$

Using Equations (2.58) and (2.59), we can express the nodal displacements vector $\mathbf{q}$ in terms of displacement sub-vectors of nodes $\mathbf{q}_{red} = \left\{ \mathbf{q}_{LB} \quad \mathbf{q}_{RB} \quad \mathbf{q}_B \right\}^T$

as

$$\mathbf{q} = \mathbf{T} \mathbf{q}_{red}, \quad \mathbf{T} = \begin{bmatrix} \mathbf{I}_1 & 0 & 0 \\ 0 & \mathbf{I}_1 & 0 \\ 0 & 0 & \mathbf{I}_2 \\ \vdots & & \\ \lambda_x^{j/n_x} \mathbf{I}_1 & 0 & 0 \\ 0 & \lambda_x^{j/n_x} \mathbf{I}_1 & 0 \\ 0 & 0 & \lambda_x^{j/n_x} \mathbf{I}_2 \\ \vdots & & \\ \lambda_x \mathbf{I}_1 & 0 & 0 \\ 0 & \lambda_x \mathbf{I}_1 & 0 \\ 0 & 0 & \lambda_x \mathbf{I}_2 \end{bmatrix}, \quad j = 1, 2, \dots, n_x - 1, \quad (2.60)$$

where $\mathbf{I}_1$ and $\mathbf{I}_2$ are identity matrices with dimensions m and $m(n_y - 1)$. The dimension of the matrix $\mathbf{T}$ is $(m(n_x + 1)(n_y + 1), m(n_y + 1))$, and hence, Equation (2.60) can be used to reduce the dimension of Equation (2.57). In fact, premultiplying both sides of Equation (2.57) by conjugate transpose $\mathbf{T}^H$ yields

$$\mathbf{T}^H [\mathbf{K} (1 + i\eta) - \omega^2 \mathbf{M}] \mathbf{T} \mathbf{q}_{red} = \mathbf{f}_{red}, \quad (2.61)$$

and

$$\mathbf{f}_{red} = \begin{Bmatrix} \tilde{\mathbf{f}}_L \\ \tilde{\mathbf{f}}_R \\ \tilde{\mathbf{f}}_O \end{Bmatrix} = \mathbf{T}^H \mathbf{f} = \begin{Bmatrix} \mathbf{f}_{LB} + \sum_{j=1}^{n_x-1} \lambda_x^{-j/n_x} \mathbf{f}_L^j + \lambda_x^{-1} \mathbf{f}_{LT} \\ \mathbf{f}_{RB} + \sum_{j=1}^{n_x-1} \lambda_x^{-j/n_x} \mathbf{f}_R^j + \lambda_x^{-1} \mathbf{f}_{RT} \\ \mathbf{f}_B + \sum_{j=1}^{n_x-1} \lambda_x^{-j/n_x} \mathbf{f}_I^j + \lambda_x^{-1} \mathbf{f}_T \end{Bmatrix}, \quad (2.62)$$

where $\mathbf{f}_X^j$, $X = L, R, I$ denote force sub-vectors of nodes placed at $x_j = d_x j / n_x$, $j = 1, 2, \dots, n_x - 1$. Internal nodal forces $\mathbf{f}_I = 0$ in the absence of external forces, and force equilibrium between opposite sides yields $\lambda_x \mathbf{f}_B + \mathbf{f}_T = 0$. Therefore, the reduced nodal forces vector $\tilde{\mathbf{f}}_O = 0$. Relabelling $\mathbf{q}_{red}$ in Equa-

tion (2.61) as $\left\{ \tilde{\mathbf{q}}_L \quad \tilde{\mathbf{q}}_R \quad \tilde{\mathbf{q}}_O \right\}^T$ yields the following equations of motion:

$$\begin{bmatrix} \mathbf{D}_{LL} & \mathbf{D}_{LR} & \mathbf{D}_{LO} \\ \mathbf{D}_{RL} & \mathbf{D}_{RR} & \mathbf{D}_{RO} \\ \mathbf{D}_{OL} & \mathbf{D}_{OR} & \mathbf{D}_{OO} \end{bmatrix} \begin{Bmatrix} \tilde{\mathbf{q}}_L \\ \tilde{\mathbf{q}}_R \\ \tilde{\mathbf{q}}_O \end{Bmatrix} = \begin{Bmatrix} \tilde{\mathbf{f}}_L \\ \tilde{\mathbf{f}}_R \\ 0 \end{Bmatrix}, \quad (2.63)$$

where $\mathbf{D} = \mathbf{T}^H [\mathbf{K} (1 + i\eta) - \omega^2 \mathbf{M}] \mathbf{T}$ is the reduced dynamic stiffness matrix. The dimension of the square matrix $\mathbf{D}$ is $m(n_y + 1)$. Now, we can reduce further the dimension of Equation (2.63) via dynamic condensation of internal degrees of freedom $\tilde{\mathbf{q}}_O$ and obtain the following equation

$$\begin{bmatrix} \tilde{\mathbf{D}}_{LL} & \tilde{\mathbf{D}}_{LR} \\ \tilde{\mathbf{D}}_{RL} & \tilde{\mathbf{D}}_{RR} \end{bmatrix} \begin{Bmatrix} \tilde{\mathbf{q}}_L \\ \tilde{\mathbf{q}}_R \end{Bmatrix} = \begin{Bmatrix} \tilde{\mathbf{f}}_L \\ \tilde{\mathbf{f}}_R \end{Bmatrix}, \quad (2.64)$$

where

$$\begin{aligned} \tilde{\mathbf{D}}_{LL} &= \mathbf{D}_{LL} - \mathbf{D}_{LO} \mathbf{D}_{OO}^{-1} \mathbf{D}_{OL} \\ \tilde{\mathbf{D}}_{LR} &= \mathbf{D}_{LR} - \mathbf{D}_{RO} \mathbf{D}_{OO}^{-1} \mathbf{D}_{OR} \\ \tilde{\mathbf{D}}_{RL} &= \mathbf{D}_{RL} - \mathbf{D}_{LO} \mathbf{D}_{OO}^{-1} \mathbf{D}_{OL} \\ \tilde{\mathbf{D}}_{RR} &= \mathbf{D}_{RR} - \mathbf{D}_{RO} \mathbf{D}_{OO}^{-1} \mathbf{D}_{OR} \end{aligned} \quad (2.65)$$

The term $\mathbf{D}_{OO}^{-1}$ is computed for each frequency of interest, thus increasing the computational cost. To reduce it, one can approximate $\mathbf{D}_{OO}^{-1}$ by expanding in a power series in ω^2 [127] as

$$\begin{aligned} \mathbf{D}_{OO}^{-1} &= \left(\mathbf{I} + \frac{\omega^2}{1 + i\eta} \tilde{\mathbf{K}}_{OO}^{-1} \tilde{\mathbf{M}}_{OO} + O \left(\left(\frac{\omega^2}{1 + i\eta} \tilde{\mathbf{K}}_{OO}^{-1} \tilde{\mathbf{M}}_{OO} \right)^2 \right) \right) \tilde{\mathbf{K}}_{OO}^{-1} \frac{1}{1 + i\eta} \approx \\ &\approx \left(\mathbf{I} + \frac{\omega^2}{1 + i\eta} \tilde{\mathbf{K}}_{OO}^{-1} \tilde{\mathbf{M}}_{OO} \right) \tilde{\mathbf{K}}_{OO}^{-1} \frac{1}{1 + i\eta} \end{aligned} \quad (2.66)$$

Now, substituting this expression into Equations (2.65) gives

$$\begin{aligned} \tilde{\mathbf{D}} = & \left(\begin{bmatrix} \tilde{\mathbf{K}}_{LL} & \tilde{\mathbf{K}}_{LR} \\ \tilde{\mathbf{K}}_{RL} & \tilde{\mathbf{K}}_{RR} \end{bmatrix} - \begin{bmatrix} \tilde{\mathbf{K}}_{LO} \\ \tilde{\mathbf{K}}_{RO} \end{bmatrix} \tilde{\mathbf{K}}_{OO}^{-1} \begin{bmatrix} \tilde{\mathbf{K}}_{OL} \\ \tilde{\mathbf{K}}_{OR} \end{bmatrix} \right) (1 + i\eta) - \omega^2 \left(\begin{bmatrix} \tilde{\mathbf{M}}_{LL} & \tilde{\mathbf{M}}_{LR} \\ \tilde{\mathbf{M}}_{RL} & \tilde{\mathbf{M}}_{RR} \end{bmatrix} - \right. \\ & \left. - \begin{bmatrix} \tilde{\mathbf{K}}_{LO} \\ \tilde{\mathbf{K}}_{RO} \end{bmatrix} \tilde{\mathbf{K}}_{OO}^{-1} \begin{bmatrix} \tilde{\mathbf{M}}_{OL} \\ \tilde{\mathbf{M}}_{OR} \end{bmatrix} - \begin{bmatrix} \tilde{\mathbf{M}}_{LO} \\ \tilde{\mathbf{M}}_{RO} \end{bmatrix} \tilde{\mathbf{K}}_{OO}^{-1} \begin{bmatrix} \tilde{\mathbf{K}}_{OL} \\ \tilde{\mathbf{K}}_{OR} \end{bmatrix} + \begin{bmatrix} \tilde{\mathbf{K}}_{LO} \\ \tilde{\mathbf{K}}_{RO} \end{bmatrix} \tilde{\mathbf{K}}_{OO}^{-1} \tilde{\mathbf{M}}_{OO} \tilde{\mathbf{K}}_{OO}^{-1} \begin{bmatrix} \tilde{\mathbf{K}}_{OL} \\ \tilde{\mathbf{K}}_{OR} \end{bmatrix} \right), \end{aligned} \quad (2.67)$$

where $\tilde{\mathbf{K}}_{XY} = \mathbf{T}^* \mathbf{K}_{XY} \mathbf{T}$ and $\tilde{\mathbf{M}}_{XY} = \mathbf{T}^* \mathbf{M}_{XY} \mathbf{T}$ for $X, Y = L, R, O$. The component $\tilde{\mathbf{K}}_{OO}^{-1}$ can be precomputed for the range of frequencies considered, thus decreasing the computational cost. Finally, by applying the periodic structure theory and force equilibrium in the y direction, which can be written as

$$\tilde{\mathbf{q}}_R = \lambda_y \tilde{\mathbf{q}}_L, \quad \tilde{\mathbf{f}}_R = -\lambda_y \tilde{\mathbf{f}}_L, \quad \lambda_y = e^{-ik_y d_y}, \quad (2.68)$$

one can get from Equation (2.64) the following eigenvalue problem for the propagation factor λ_y

$$\mathbf{S} \begin{Bmatrix} \tilde{\mathbf{q}}_L \\ \tilde{\mathbf{f}}_L \end{Bmatrix} = \lambda_y \begin{Bmatrix} \tilde{\mathbf{q}}_L \\ \tilde{\mathbf{f}}_L \end{Bmatrix} \quad \text{with} \quad \mathbf{S} = \begin{bmatrix} -\tilde{\mathbf{D}}_{LR}^{-1} \tilde{\mathbf{D}}_{LL} & \tilde{\mathbf{D}}_{LR}^{-1} \\ -\tilde{\mathbf{D}}_{RL} + \tilde{\mathbf{D}}_{RR} \tilde{\mathbf{D}}_{LR}^{-1} \tilde{\mathbf{D}}_{LL} & -\tilde{\mathbf{D}}_{RR} \tilde{\mathbf{D}}_{LR}^{-1} \end{bmatrix}. \quad (2.69)$$

Solving Equation (2.69) yields the propagation factors λ_y and, eventually, the wave number components k_y .

This concludes the review of the WFE method for composite plates.

2.4 Conclusion

This chapter aimed to review the background theory needed to understand the main findings of this thesis presented in the following chapters. In Section 2.1, we have described the basics of the DEA method for two-dimensional isotropic structures. Furthermore, we have reviewed the wave problems that are considered throughout this thesis. Finally, we have identified and discussed the main

parts of the DEA method that need to be modified to allow for its application to composite structures.

One of the main parts to be changed is the calculation of scattering coefficients for composite plates. These coefficients are obtained via two approaches - the semi-analytical approach based on the CLP theory for thin laminated plates and the hybrid FE/WFE approach for general composite plates. In Section 2.2, we have described the CLP theory for thin laminated plates. It will be used in Chapter 3 to set up the governing equations of motion for composite plates, which eventually lead to the computation of the scattering coefficients of various junctions. Finally, in Section 2.3, we have considered the main principles and equations in the WFE method for composite plates modelled using two-dimensional and three-dimensional finite elements. The equations presented in this section will be used further in Chapter 4 to derive expressions for scattering coefficients using the hybrid FE/WFE method.

Chapter 3

Energy scattering properties of line joints connecting composite plates

3.1 Introduction

In this chapter, the semi-analytical method for computing scattering coefficients of line joints connecting composite laminated plates is derived. Composite laminates are modelled in the context of the CLP theory introduced in Section 2.2. We restrict our study to symmetric laminates; that is, the laminae are symmetrically placed around the mid-plane of the laminate. As shown in Section 2.2, in such a case, the governing equations of motion (2.51) become uncoupled, thus reducing the computational complexity of the model. There are two specific cases which we will have a look at that are described in Appendix A, namely, regular cross- and angle-ply laminates with alternating ply direction angles ($0^\circ/90^\circ$) and $(-\alpha/\alpha)$, $0^\circ < \alpha < 90^\circ$, respectively. The method presented here will be used to validate a wave finite element approach of extracting reflection and transmission coefficients in Chapter 4.

This chapter is organised as follows. In Section 3.2, the governing equations of motion for composite symmetric laminated plates are presented. Then, dispersion relations and group velocities for in-plane and out-of-plane waves are given in Section 3.3. The importance of modifications to Snell's law for composite plates is discussed using an example configuration. In Section 3.4, the wave dynamic stiffness matrix is then introduced, which relates displacements and

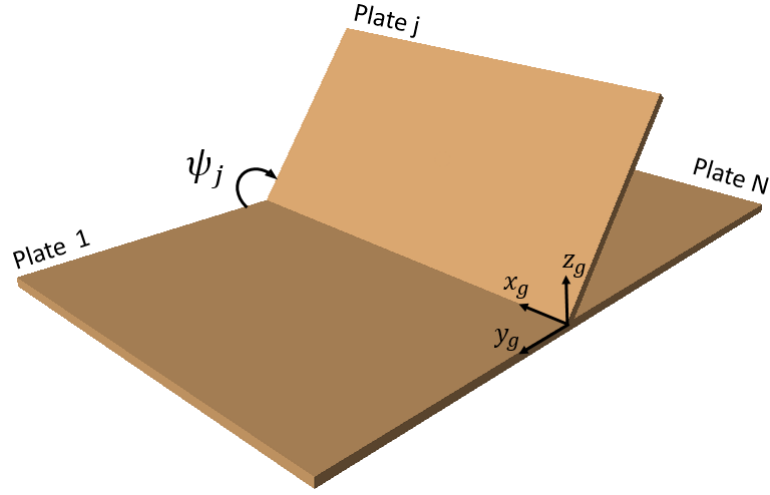


Figure 3.1: The set up of N plates joined together at a common interface along the x_g axis.

forces at the junction. In Section 3.5, the individual dynamic stiffness matrices for plates are then assembled into a global equation via the application of continuity and equilibrium conditions at the junction. Consequently, we obtain the scattering matrix formulation for a junction of several semi-infinite plates. As a final step, in Section 3.6, we derive an effective scattering matrix for a plate with multiple finite stiffeners attached to it. In Section 3.7, numerical case studies for two and three coupled composite plates are presented.

3.2 Statement of the problem

In this section, we outline the mathematical basis of the wave scattering problem. We consider N semi-infinite thin laminated plates connected along a lossless junction as shown in Figure 3.1. The plates consist of n generally orthotropic laminae. The shared edge of the plates is aligned with the x_g axis of the global coordinate system (x_g, y_g, z_g) . All plates are semi-infinite in the positive direction of their local y_j axis. The rotation angle ψ_j describes the position of the j th plate relative to the y_g axis. The position, displacements and tractions on the j th plate are defined with respect to the local coordinate system (x_j, y_j, z_j) , where the local and global x axis are identical, see Figure 3.2. Note that the positive direction of the y_j axis always points away

from the junction.

3.2.1 Governing equations of motion

As outlined in Section 3.1, we consider symmetric laminated plates, and therefore, all bending-extension coefficients B_{ij} are equal to zero. Substituting (2.48) into (2.51) yields the governing equations of motion for the j th plate in terms of displacements u_j, v_j, w_j as

$$\begin{aligned} A_{11} \frac{\partial^2 u}{\partial x^2} + 2A_{16} \frac{\partial^2 u}{\partial x \partial y} + A_{66} \frac{\partial^2 u}{\partial y^2} + A_{16} \frac{\partial^2 v}{\partial x^2} + (A_{12} + A_{66}) \frac{\partial^2 v}{\partial x \partial y} + A_{26} \frac{\partial^2 v}{\partial y^2} &= \rho h \frac{\partial^2 u}{\partial t^2}, \\ A_{22} \frac{\partial^2 v}{\partial y^2} + 2A_{26} \frac{\partial^2 v}{\partial x \partial y} + A_{66} \frac{\partial^2 v}{\partial x^2} + A_{16} \frac{\partial^2 u}{\partial x^2} + (A_{12} + A_{66}) \frac{\partial^2 u}{\partial x \partial y} + A_{26} \frac{\partial^2 u}{\partial y^2} &= \rho h \frac{\partial^2 v}{\partial t^2}, \\ D_{11} \frac{\partial^4 w}{\partial x^4} + 4D_{16} \frac{\partial^3 w}{\partial x^3 \partial y} + 2(D_{12} + 2D_{66}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + 4D_{26} \frac{\partial^3 w}{\partial x \partial y^3} + D_{22} \frac{\partial^4 w}{\partial y^4} &= -\rho h \frac{\partial^2 w}{\partial t^2}. \end{aligned} \quad (3.1)$$

Recall that coefficients A_{ij} and D_{ij} are computed using Equation (2.49), and for special cases of cross- and angle-ply laminates these coefficients are obtained by formula given in Appendices (A.1) and (A.2). We omit the plate index j in Equations (3.1) and whenever we talk about a specific plate for ease of notation; we emphasise here that all quantities, including the material parameters, are plate dependent. The equations of motion for in-plane and out-of-plane motion in the plate are uncoupled and so are the relations between in-plane displacements u_e, v_e to in-plane tractions N_{xy}, N_{yy} and out-of-plane displace-

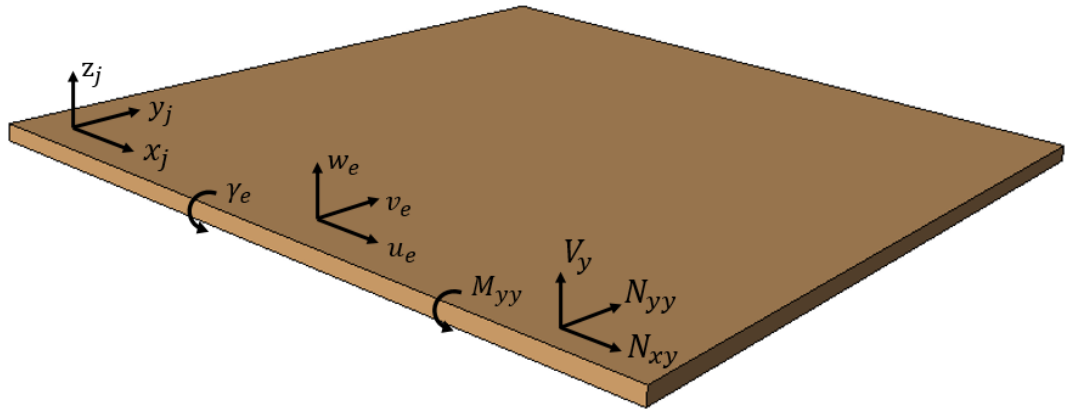


Figure 3.2: Local coordinate system, displacements and tractions on the common edge “e” of the j th plate.

ment w_e and rotation θ_e to effective shear force V_y and bending moment M_{yy} at a junction or edge “e” of that plate j , see Figure 3.2. In what follows, we will use the sub-index “e”, whenever we want to stress that a quantity is taken at the edge of a plate j , that is, at $y_j = 0$, following the notation in [147]. By expanding Equation (2.48), the elastic tractions and moments can be written in the following way

$$\begin{aligned} N_{yy} &= A_{22} \frac{\partial v}{\partial y} + A_{12} \frac{\partial u}{\partial x} + A_{26} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \\ N_{xy} &= A_{16} \frac{\partial u}{\partial x} + A_{26} \frac{\partial v}{\partial y} + A_{66} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \\ V_y &= -D_{22} \frac{\partial^3 w}{\partial y^3} - 4D_{26} \frac{\partial^3 w}{\partial x \partial y^2} - (D_{12} + 4D_{66}) \frac{\partial^3 w}{\partial x^2 \partial y} - 2D_{16} \frac{\partial^3 w}{\partial x^3} \\ M_{yy} &= D_{22} \frac{\partial^2 w}{\partial y^2} + D_{12} \frac{\partial^2 w}{\partial x^2} + 2D_{26} \frac{\partial^2 w}{\partial x \partial y}. \end{aligned} \quad (3.2)$$

In what follows, the twisting moment M_{xy} will be important in calculating the energy flow of a bending mode. It is expressed in terms of displacements as

$$M_{xy} = D_{16} \frac{\partial^2 w}{\partial x^2} + D_{26} \frac{\partial^2 w}{\partial y^2} + 2D_{66} \frac{\partial^2 w}{\partial x \partial y}. \quad (3.3)$$

The angle of rotation γ is approximated as $\frac{\partial w}{\partial y}$ (see Equation (2.44)). Finally, it is noted that Equations (3.1) and (3.2) with appropriate extensional and bending coefficients A_{ij} and D_{ij} describe the structural behaviour of a generally orthotropic plate, that is, an orthotropic plate with principal material axes not necessarily aligned with its local coordinate system. Therefore, we can also analyse the vibrational motion of individual orthotropic lamina of a composite plate and, consequently, the scattering properties of the junctions of orthotropic plates.

3.3 Dispersion relations, the group velocity and Snell's law

Before discussing reflection and transmission coefficients for incoming waves at a specific angle of incidence, it is worth considering the relations between the angle of incoming and outgoing waves at an interface, that is, the relation

equivalent of Snell's law for composite plates.

3.3.1 Relations between incoming and outgoing waves at junctions for composite plates

Since the directions of the group and phase velocities do not necessarily coincide in non-isotropic media [166], formulating the relations between the angles of incoming and outgoing waves at interfaces between plates with different material properties and thus the generalisation of Snell's law is less straightforward. The connection between group and phase velocities is provided by the dispersion relation, which needs to be studied in detail. Given that energy is transported along the group velocity vectors, the relation between incoming and outgoing group velocity directions is essential for calculating ray trajectories in the context of DEA.

We consider a plane wave of the form $e^{-ik_x x - ik_y y + i\omega t}$ travelling across the plate, where k_x and k_y are the x and y components of the wave vector $\mathbf{k}$, and ω is the angular frequency. Substituting this plane wave solution into the 4th order bending equation (3.1) yields the characteristic equation for bending waves

$$D_{11} k_x^4 + 4D_{16} k_x^3 k_y + 2(D_{12} + 2D_{66}) k_x^2 k_y^2 + 4D_{26} k_x k_y^3 + D_{22} k_y^4 - \rho h \omega^2 = 0. \quad (3.4)$$

Following the same procedure using the first two equations in (3.1), the characteristic equation for in-plane waves can be expressed as

$$\begin{aligned} (A_{11} k_x^2 + 2A_{16} k_x k_y + A_{66} k_y^2 - \rho h \omega^2) (A_{66} k_x^2 + 2A_{26} k_x k_y + A_{22} k_y^2 - \rho h \omega^2) - \\ - (A_{16} k_x^2 + (A_{12} + A_{66}) k_x k_y + A_{26} k_y^2)^2 = 0. \end{aligned} \quad (3.5)$$

For fixed ω , the solutions to Equations (3.4) and (3.5) give rise to a closed curve in (k_x, k_y) space describing the *wave vector* curves for bending, shear and longitudinal waves. We note that only real solutions (k_x, k_y) describe propagating waves.

We are interested in the dynamic response of a plate j to an incident plane wave

of the form $e^{-ik_x x + ik_y y + i\omega t}$ travelling towards the junction on plate j' . Compatibility conditions at the junction yield the response of the plate j in the form $e^{-ik_x x - i\mu y + i\omega t}$, that is, k_x and ω are *common to all plates on the edge "e"*. The y component of the wave vector, here denoted μ , is then computed from the dispersion relations (3.4) and (3.5). For the out-of-plane motion, one obtains four solutions of Equation (3.4) which come in pairs of roots $\mu^\pm$. One pair is either real or complex, corresponding to propagating or evanescent bending waves, respectively. We denote this pair as $\mu_{B_1}^\pm$ where the superscripts "+" and "-" represent outgoing and incoming waves. The other pair of roots denoted $\mu_{B_2}^\pm$ is always complex; thus, the corresponding bending waves are always evanescent. Evanescent waves are usually characterised with purely imaginary values of $\mu^\pm$, so that the correspondent displacements increase or decay exponentially along its direction without oscillation [142, 147, 166]. In such waves, energy can only be carried through the interaction of a pair of waves [188, 189]. One might argue that for complex $\mu^\pm$ with $\text{Re}(\mu^\pm) \neq 0$, the solution would oscillate while decaying or increasing exponentially. Therefore, the correspondent waves must be attenuating and carrying energy flux. However, in the absence of damping, such waves are still evanescent with the inclined angle of the decay with respect to the y axis - more on this in subsection 3.4.1.

Similarly, the dispersion relations (3.5) can be solved for the unknown μ yielding a characteristic equation with four roots $\mu_{L,S}^\pm$ that represent real or complex incoming and outgoing quasi-longitudinal and quasi-shear waves, here denoted as L and S , respectively. Valid plate responses, that is, outgoing waves with μ_X^+ , $X = B_1, B_2, L$ and S either oscillate with a positive energy flux along the y axis or decay exponentially with increasing y ; again, we will discuss this in more detail in subsection 3.4.1. In the example to follow below, we will only consider the propagating branches and explain the relation between the directions of incoming and outgoing waves, which follows from the continuity condition $k_x = \text{const.}$

For this, we also need the group velocity vector $\mathbf{c}_g$, which gives the direction

of the wave energy flow [190]. The standard definition $\mathbf{c}_g = \partial\omega/\partial\mathbf{k}$ is not convenient for our purposes here, since the dependence of the angular frequency ω on the wave vector $\mathbf{k}$ is only implicitly given through the dispersion relations (3.4) and (3.5). Instead, one can use

$$(c_{gx}, c_{gy}) = - \left(\frac{\partial \tilde{H}}{\partial k_x}, \frac{\partial \tilde{H}}{\partial k_y} \right) \div \frac{\partial \tilde{H}}{\partial \omega} \quad (3.6)$$

to find the components of the group velocities c_{gx} and c_{gy} [139, 190] for different modes. Here, $\tilde{H} = \tilde{H}(k_x, k_y, \omega) = 0$ represents the dispersion relations (3.4) or (3.5), respectively, and it is the general Hamilton function that defines the ray trajectories in composite plates; we have seen a form of it for isotropic plates in Equation (2.35).

It is important to note that after introducing the polar coordinate system as $k_x = k \sin \theta$, $k_y = k \cos \theta$, the dispersion relations can be recast as

$$H_l(k, \theta) = \beta_l(\theta)k^l = \omega, \quad l = 1, 2, \quad (3.7)$$

where $l = 1, 2$ represents the cases of in-plane and out-of-plane waves, accordingly. Furthermore, the coefficients $\beta_l(\theta)$ are given as follows:

$$\beta_1^{L,S}(\theta) = \sqrt{\frac{2(C_1 C_2 - C_3^2)}{\rho h (C_1 + C_2 \mp \sqrt{(C_1 - C_2)^2 + 4C_3^2})}} \quad , \quad \beta_2(\theta) = \sqrt{\frac{D}{\rho h}} \quad (3.8)$$

with

$$\begin{aligned} C_1 &= A_{11} \sin^2 \theta + 2A_{16} \sin \theta \cos \theta + A_{66} \cos^2 \theta \\ C_2 &= A_{66} \sin^2 \theta + 2A_{26} \sin \theta \cos \theta + A_{22} \cos^2 \theta \\ C_3 &= A_{16} \sin^2 \theta + (A_{12} + A_{66}) \sin \theta \cos \theta + A_{26} \cos^2 \theta \\ D &= D_{11} \sin^4 \theta + 4D_{16} \sin^3 \theta \cos \theta + 2(D_{12} + 2D_{66}) \sin^2 \theta + \\ &\quad + \cos^2 \theta + 4D_{26} \sin \theta \cos^3 \theta + D_{22} \cos^4 \theta \end{aligned} \quad (3.9)$$

While this form of the dispersion relations is simpler than ones in (3.4) and (3.5), one can only extract the propagating modes out of them. The evanescent modes are essential in the calculation of the scattering coefficients as we shall

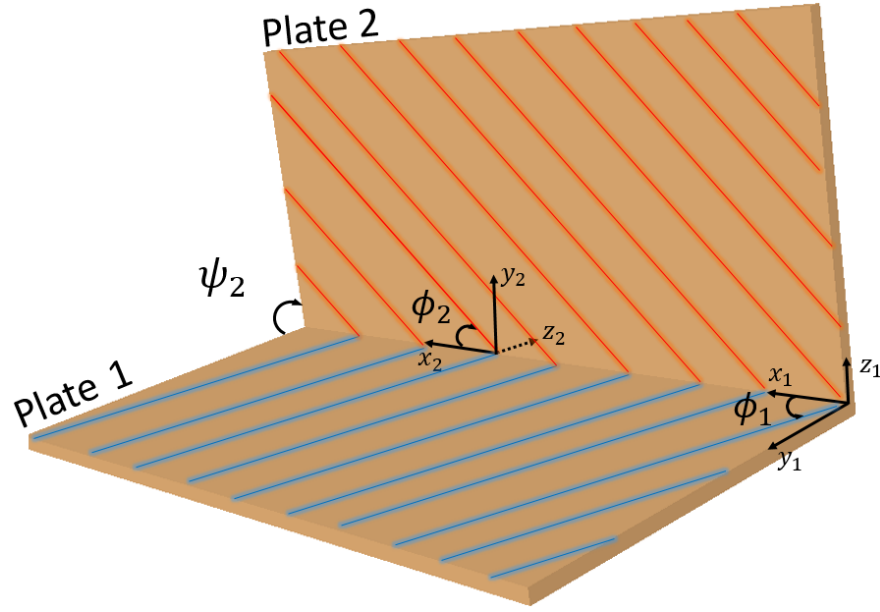


Figure 3.3: Schematic of an L-junction of two orthotropic plates. The angle ψ_2 here is set to 90° . Red and blue lines represent the fiber direction of the plates. The local angles of orientations ϕ_1 and ϕ_2 are both set to 45° .

E_1 (N/m ²)	E_2 (N/m ²)	E_3 (N/m ²)	G_{12} (N/m ²)	G_{23} (N/m ²)	G_{13} (N/m ²)
121×10^9	8.6×10^9	8.6×10^9	4.7×10^9	3.1×10^9	4.7×10^9
ν_{12}	ν_{23}	ν_{13}	ρ (kg/m ³)		
0.27	0.4	0.27	1490		

Table 3.1: Engineering constants of Epoxy Carbon UD (230 GPa) material used for individual laminas of a composite plate.

see in the following sections; hence, Equations (3.4) and (3.5) will be used throughout this chapter. Still, as we will see in Chapter 5, Equation (3.7) is useful for relating the boundary and spatial phase space densities.

3.3.2 Snell's law for composite plates - an example

We will discuss some of the peculiarities of the interplay between incoming and outgoing wave directions at interfaces between composite plates by looking at a specific example. For the sake of clarity, we will consider first generally orthotropic plates meeting at an angle ψ_2 as shown in Figure 3.3. These plates can be viewed as composite laminates that have only one lamina with local ply direction angles ϕ_1 and ϕ_2 . The actual value of ψ_2 is not essential for the

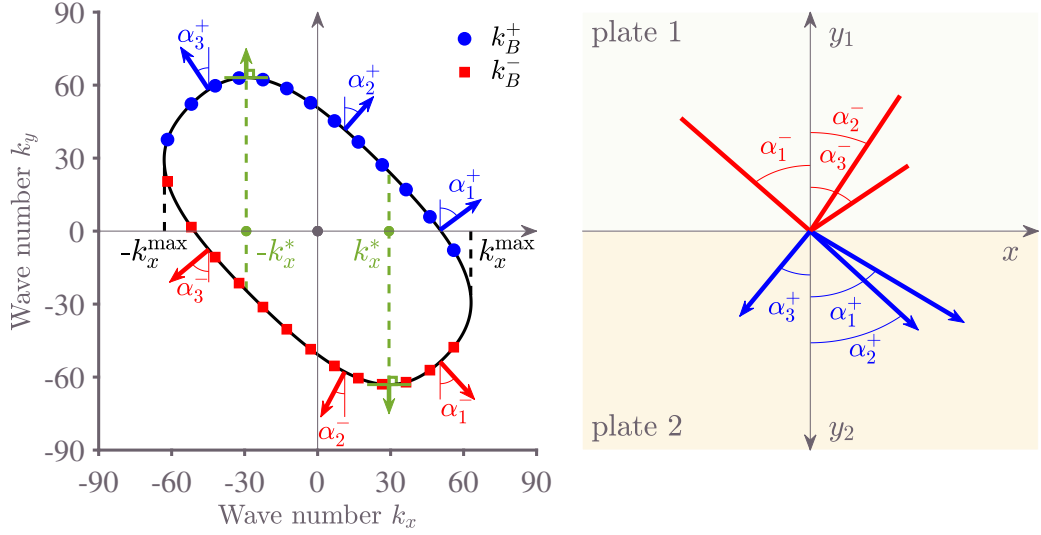


Figure 3.4: Bending wave vector curve in a 45° rotated orthotropic plate at a frequency 3000 Hz (left) and a schematic representation of incoming/outgoing waves at the junction of two identical plates (right). Blue dots represent wave numbers related to outgoing waves while red squares correspond to incoming waves. Wave number components $k_x^{\max}$ and k_x^* equal to 62.96 m^{-1} and 29.34 m^{-1} , respectively.

discussion in this section; it is set to $\psi_2 = 90^\circ$ in Figure 3.3.

The material characteristics of the plates are given in Table 3.1. The thicknesses of plates are both equal to 5 mm. Appendix B describes the relations connecting these material parameters to the stiffness coefficients in Equations (3.1). The orientation of the principal material axes of the two plates with respect to the interface and with respect to each other is important; here, we define the local angles of rotation of the material axes as $\phi_1 = \phi_2 = 45^\circ$ as shown in Figure 3.3. We work at a frequency of 3000 Hz here, a frequency value consistent with the thin plate assumptions described by condition (2.43) - at the chosen frequency $k_B^{\max} h = 0.31 < 1$. The left hand side of Figure 3.4 shows the bending wave vector curve at a frequency 3000 Hz obtained from Equation (3.4) in the local coordinate system displaying the 45° rotation. Note that the wave vector curve is the same for both plates in their local coordinate system. The range of k_x values which allow for propagating bending waves at 3000 Hz is between $(-k_x^{\max}, k_x^{\max})$. It is emphasised again that the angle ψ_2 between the two plates can take arbitrary values here; for the sake of clarity, the local coordinates of the plates, (x, y_i) with $i = 1$ or 2 , are drawn in the

same plane on the right-hand side of Figure 3.4 and in Figure 3.5 below.

As we are interested in the energy flow across interfaces, the relevant angles of incidence, reflection and transmission are those obtained from the group velocity vectors, Equation (3.6); they point along the gradient vectors to the wave vector curves (denoted by the angles $\alpha_j^\pm$ in Figure 3.4). Here, blue symbols correspond to velocity vectors with a positive y component describing waves transporting energy away from the junction, and red symbols describe incoming energy fluxes. The continuity condition $k_x = \text{const}$ now connects incoming with outgoing group velocity directions on the wave vector curve: red squares (incoming) with blue dots (outgoing). This leads to a peculiar effect here: in the range $k_x^* < |k_x| < k_x^{\max}$, propagating waves, for example with angles of incidence $\alpha_{1,3}^-$, are transmitted or reflected keeping their direction of travel in the x direction, see the right side of Figure 3.4. This is what one usually finds at refracting interfaces. However, for $-k_x^* < k_x < k_x^*$, a wave, for example with incident angle α_2^- , is scattered reversing its direction in the x -direction, see α_2^+ in Figure 3.4. This gives rise to negative refraction for these values of k_x . The phenomenon is displayed in more detail in Figure 3.5 representing group velocity rays transmitting from one plate to another; the region in blue shows the negative refraction phenomenon.

Furthermore, note that at specific values of $k_x < -k_x^*$, the group velocity vector points in the opposite y direction compared to the corresponding wave vector, see Figure 3.4. This implies that the individual wavefronts travel away from the junction, whereas the wave energy travels towards it. Similar behaviour can be encountered for wave vectors in the right part of the wave vector curve; for specific values of $k_x > k_x^*$ the outgoing wavefronts travel towards the junction whereas the wave energy propagates away from it.

Interesting features can also be seen for in-plane wave vector curves, as shown in Figure 3.6. The shape of the longitudinal wave vector curve is similar to that of the bending wave. Therefore, it also exhibits negative refraction and opposition of phase and group velocity vector directions for specific fixed values

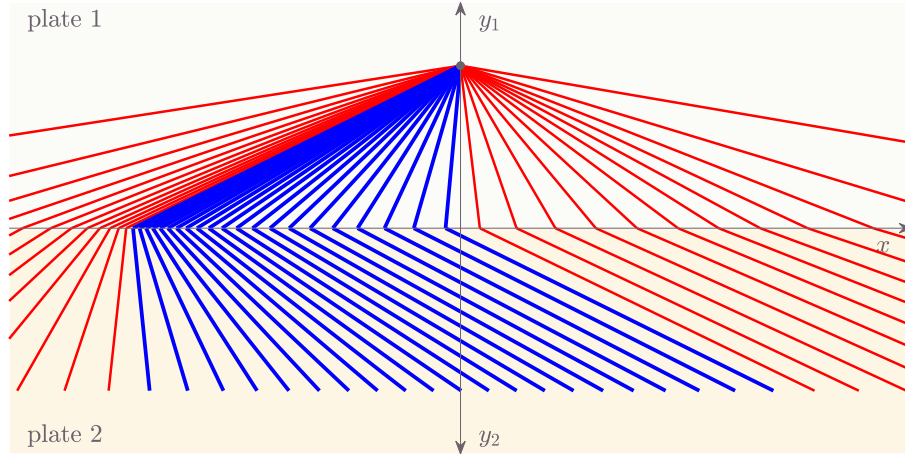


Figure 3.5: Group velocity ray picture for bending waves at junction of two identical 45° rotated orthotropic plates at a frequency 3000 Hz. Blue lines highlight the region of incoming rays, which represents the negative refractive index phenomenon.

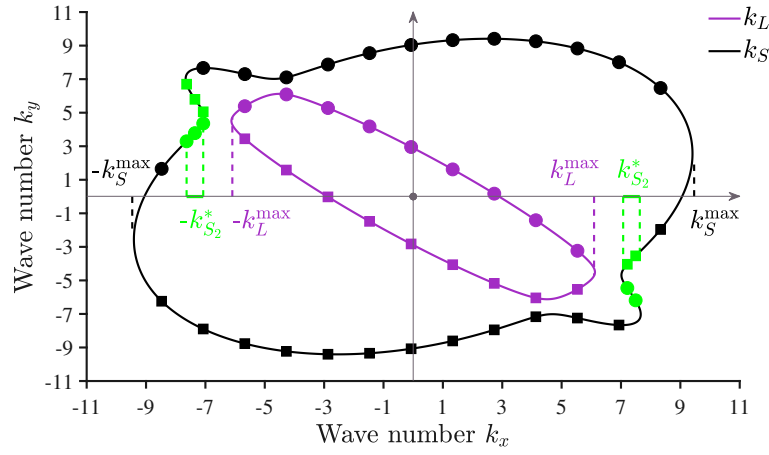


Figure 3.6: Longitudinal and shear wave vector curves in a 45° rotated orthotropic plate at a frequency 3000 Hz. Dots represent wave numbers related to outgoing waves while squares - incoming waves. Wave number components k_L^* and k_S^* equal to 6 m^{-1} and 9.3 m^{-1} , respectively - the critical values for longitudinal and shear waves

of k_x . The shear wave vector curve shows additional features: there exists a set of values of k_x (labelled $k_{S_2}^*$), for which there are two pairs of incoming and outgoing shear waves for each k_x . The second pair of shear waves exhibit behaviour similar to what has been described for bending waves. In fact, individual wavefronts at $k_x \in k_{S_2}^*$ travel away from the junction, whereas the wave energy is travelling towards it. Conversely, the wave energy is travelling away from the junction with individual wavefronts travelling towards it at $k_x \in -k_{S_2}^*$. Furthermore, the two pairs of shear waves can couple with each other so that an incoming wave can convert into two distinct shear waves. Finally,

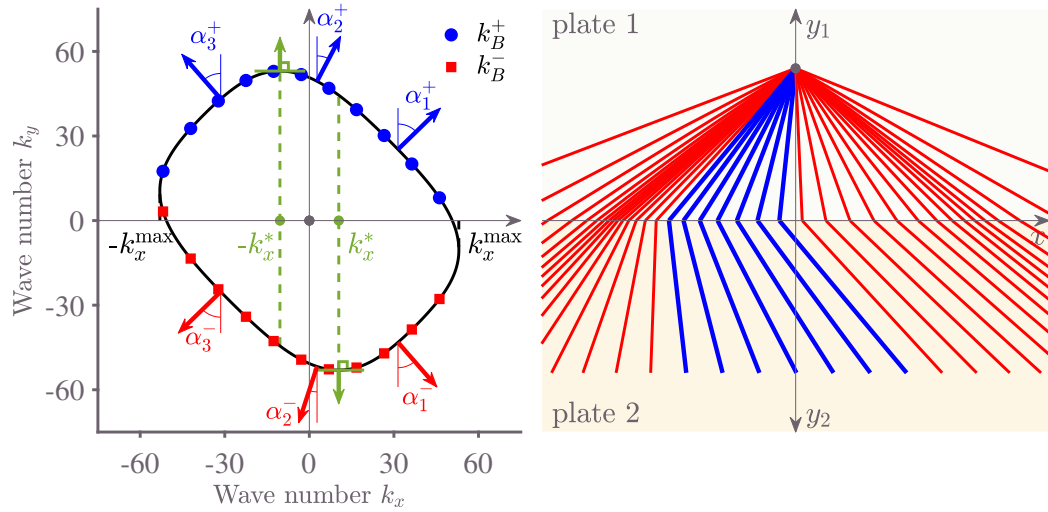


Figure 3.7: Bending wave vector curve in a $45^\circ / -45^\circ / 45^\circ / -45^\circ / 45^\circ$ composite plate at a frequency 3000 Hz (left) and a group velocity ray picture for bending waves at the junction of two identical plates (right). Blue dots represent wave numbers related to outgoing waves while red squares correspond to incoming waves. Wave number components $k_x^{\max}$ and k_x^* equal to 53 m^{-1} and 10.43 m^{-1} , respectively. Blue lines highlight the region of incoming rays, which represents the negative refractive index phenomenon.

propagating longitudinal and shear waves can only exist for k_x values in the range $(-k_L^{\max}, k_L^{\max})$ and $(-k_S^{\max}, k_S^{\max})$, respectively. Beyond these intervals, the corresponding propagating waves become attenuating.

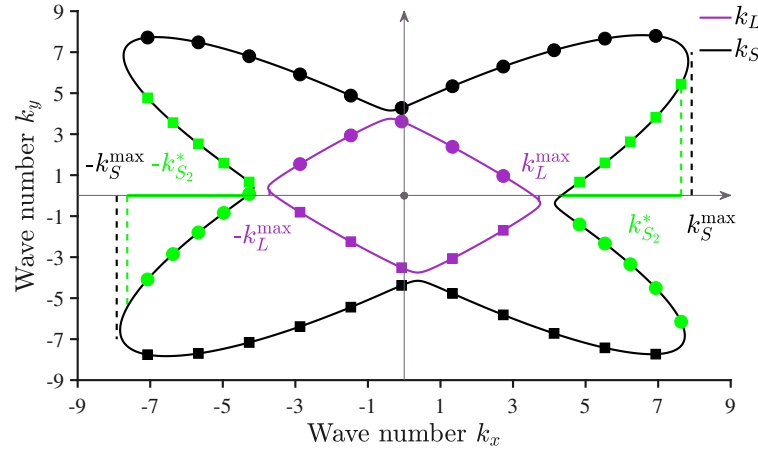


Figure 3.8: Longitudinal and shear wave vector curves in a $45^\circ / -45^\circ / 45^\circ / -45^\circ / 45^\circ$ composite plate at a frequency 3000 Hz. Dots represent wave numbers related to outgoing waves while squares - incoming waves. Wave number components k_L^* and k_S^* equal to 3.71 m^{-1} and 7.8 m^{-1} , respectively - the critical values for longitudinal and shear waves

Now, we consider two identical angle-ply laminates with 5 orthotropic laminas of total thicknesses of 5 mm and with the correspondent lamination schemes

$45^\circ/-45^\circ/45^\circ/-45^\circ/45^\circ$. The material of individual laminas is given in Table 3.1, and Section A.2 describes how to compute extensional and bending stiffness coefficients A_{ij} and D_{ij} for governing equations of motion (3.1). The bending wave vector curve is shown on the left-hand side of Figure 3.7. We note that negative refraction occurs in a smaller range $[-k_x^*, k_x^*]$ than in the previous case. This can also be seen on the right-hand side of Figure 3.7. Regarding the in-plane wave vector curves, the shape of the shear wave vector curve changes drastically, see Figure 3.8. Two pairs of incoming and outgoing shear waves are present for larger ranges $k_x \in -k_{S_2}^*$ and $k_x \in k_{S_2}^*$ than in the generally orthotropic plate case. We will come back to this example in subsection 3.7.2.

3.4 Derivation of the dynamic stiffness matrix

Next, we are interested in the local dynamic stiffness matrix for the plate j . This matrix relates boundary forces to boundary displacements produced by the waves emerging from a junction in the form $e^{-ik_x x - i\mu y + i\omega t}$. As discussed in subsection 3.3.1, k_x and ω are common to all plates meeting at an edge “e” and the y components of the wave vectors, μ , are obtained from the dispersion relations (3.4) and (3.5) with solutions forming real or complex pairs $\mu_X^\pm$ with $X = B_1, B_2, L$ or S .

3.4.1 Energy flow and the response of the plate

The bending wave time-averaged energy flow in the y direction can be written as [191]

$$J_B = \frac{1}{2} \operatorname{Re} \left(i\omega \begin{bmatrix} w & \gamma & \frac{\partial w}{\partial x} \end{bmatrix}^* \begin{bmatrix} Q_y \\ M_{yy} \\ M_{xy} \end{bmatrix} \right), \quad (3.10)$$

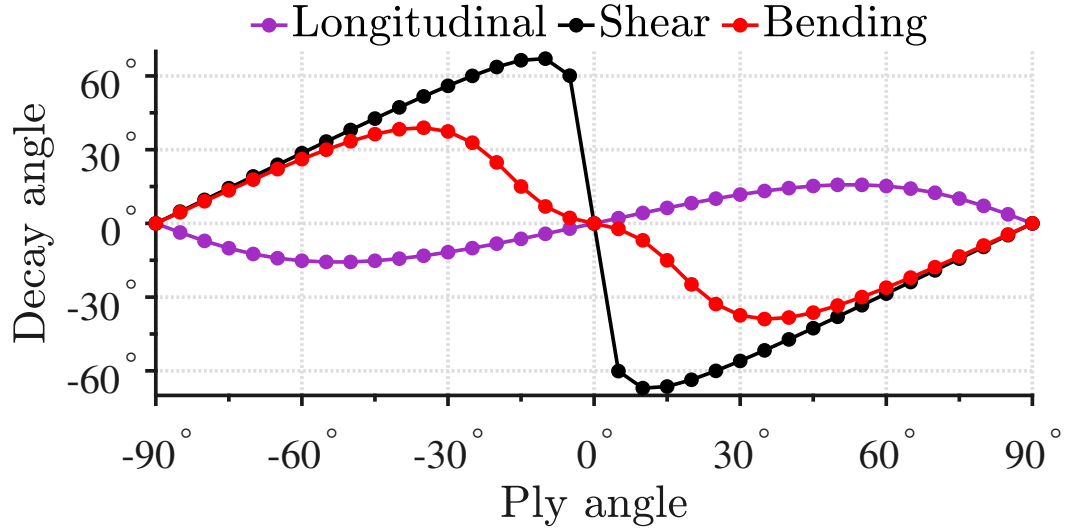


Figure 3.9: Decay direction angles of various evanescent waves as a function of a ply direction angle in an orthotropic plate with material parameters from Table 3.1.

where $*$ denotes complex conjugation and $\gamma = \frac{\partial w}{\partial y}$ as mentioned in subsection 3.2.1. Now, introducing $w = e^{-ik_x x - i\mu_{B1} y + i\omega t}$ into Equation (3.10) yields

$$J_{B1,2}^{\pm} = \frac{1}{2} \operatorname{Re} \left(i\omega \begin{bmatrix} 1 & -i\mu_{B1,2}^{\pm} & -ik_x \end{bmatrix}^* \begin{bmatrix} Q_{y1,2}^{\pm} \\ M_{yy1,2}^{\pm} \\ M_{xy1,2}^{\pm} \end{bmatrix} \right). \quad (3.11)$$

Furthermore, $Q_{y1,2}^{\pm}$, $M_{yy1,2}^{\pm}$ and $M_{xy1,2}^{\pm}$ are given by Equations (3.2) and (3.3) with $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ replaced by $-ik_x$ and $-i\mu_{B1,2}^{\pm}$, respectively.

As stated in subsection 3.3.1, propagating incoming or outgoing waves are those with real wave number components $\mu_{B1}^{\pm}$ giving rise to an energy flux to or away from the junction, that is, $J_{B1}^{\pm} < 0$ or $J_{B1}^{\pm} > 0$, respectively. If the $\mu_{B1,2}^{\pm}$ are complex, then the out-of-plane displacements of outgoing or incoming waves are of the form

$$w^{\pm} = e^{-ik_x x \mp i \operatorname{Re}(\mu^{\pm}) y \mp \operatorname{Im}(\mu^{\pm}) y + i\omega t}, \quad (3.12)$$

and hence, they appear to be oscillating while decaying away or increasing exponentially towards the junction. This seems to cause energy transport as well, since the corresponding energy flux $J_{B1,2}^{\pm} \neq 0$. However, as we mentioned in subsection 3.3.1, these waves are still evanescent and can only exponentially

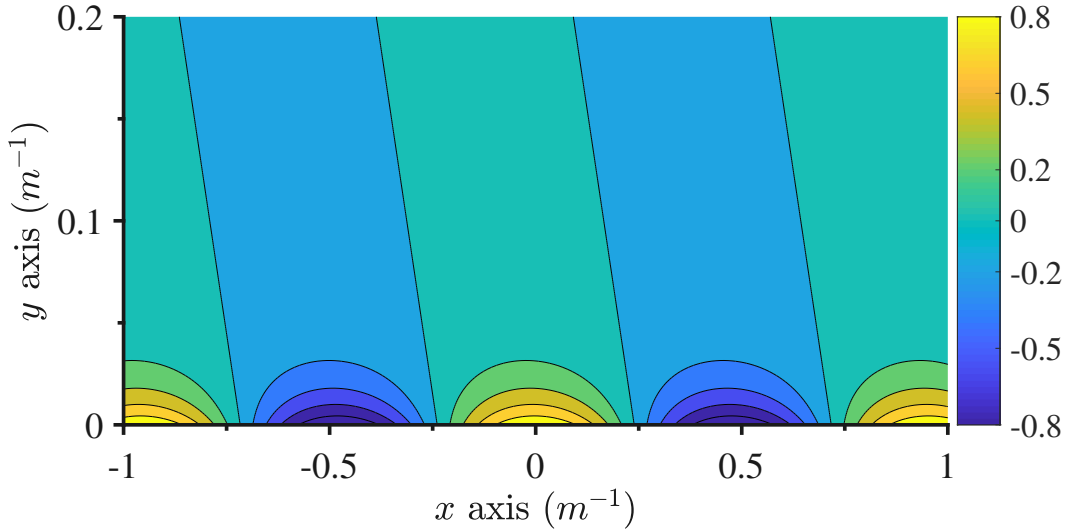


Figure 3.10: Contour plot of the real part of the out-of-plane displacement produced by the outgoing evanescent bending wave in a 45° -rotated orthotropic plate plotted in the (x, y) plane. Brighter colours represent positive z values, whilst darker ones denote negative z values.

decay away or increase towards the junction with the decay/increase direction axis being not aligned with the y axis, in general [192, 193]. The angle of decay/increase of evanescent waves depends on the angle of orientation of principal material axes of plies of the plate, i.e. on the ply direction angle. It can be computed as $\arctan(\text{Re}(\mu^\pm)/k_x)$. Figure 3.9 presents the decay direction angles of longitudinal, shear and bending evanescent waves with respect to the y axis as a function of the ply direction angle of an orthotropic plate with material parameters from Table 3.1. It can be noted that the decay of evanescent waves can be inclined with respect to the y axis with angles up to 60° ; for example, the line corresponding to the evanescent shear wave in Figure 3.9. We can see that the decay angles are zero for ply direction of angles 0° and $\pm 90^\circ$, that is, the evanescent waves decay/increase along the y axis, and the correspondent $J_{B1,2}^\pm = 0$, as expected. In such cases, the plate is *specialty orthotropic* if it consists of only one layer (see Section A.1 for details) or balanced if it consists of multiple layers [1].

Figure 3.10 presents the contour plot of the real part of the out-of-plane displacement field created by the outgoing bending evanescent wave in a 45° -rotated orthotropic plate. The displacement at $y = 0$ along the x axis is

oscillating, as expected. In contrast, at $y > 0$ it decays away along the inclined null-lines, i.e., lines at which displacement is zero (black straight lines in Figure 3.10). This leads to the oscillating displacement shape projection along the y axis, which is why the energy flux along the y axis $J_B^\pm$ appears to be non-zero.

Once the appropriate bending wave number roots $\mu_{B1,2}^+$ are defined, the out-of-plane response of the plate can be expressed as

$$w^+ = \alpha_{B1}^+ e^{-ik_x x - i\mu_{B1}^+ y + i\omega t} + \alpha_{B2}^+ e^{-ik_x x - i\mu_{B2}^+ y + i\omega t}, \quad (3.13)$$

where the constants α_{B1}^+ and α_{B2}^+ are amplitudes of the outgoing bending waves.

For the in-plane motion, the response of the plate takes the form

$$\begin{aligned} v &= \Phi_v e^{-ik_x x - i\mu y + i\omega t} \\ u &= \Phi_u e^{-ik_x x - i\mu y + i\omega t}. \end{aligned} \quad (3.14)$$

Equation (3.5) can be solved for the unknown μ yielding a characteristic equation with four roots $\mu_{L,S}^\pm$ that represent incoming and outgoing quasi-longitudinal and quasi-shear waves, here denoted again as L and S , respectively. Similar to the out-of-plane case, a valid outgoing solution produces a positive energy flow in the y direction if the correspondent wave is propagating, or it is associated with complex-valued $\mu_{L,S}^+$ with $\text{Im}(\mu_{L,S}^+) < 0$; that is, the correspondent wave is exponentially decaying as $y \rightarrow \infty$. Note that purely imaginary solutions of Equation (3.5) $\mu_{L,S}^+$, which are present in specially orthotropic plates, produce no energy flow in the y direction. On the other hand, complex-valued solutions can appear to produce energy flow along the y axis, as discussed previously in the out-of-plane case.

The time-averaged energy flow in the y direction of in-plane waves can be generally written as

$$J = \frac{1}{2} \text{Re} \left(i\omega \begin{bmatrix} v & u \end{bmatrix}^* \begin{bmatrix} N_{yy} \\ N_{xy} \end{bmatrix} \right). \quad (3.15)$$

Introducing Equations (3.14) into Equation (3.15) yields the energy flow expressions for the longitudinal and shear wave modes, that is,

$$J_L^\pm = \frac{1}{2} \operatorname{Re} \left(i\omega \begin{bmatrix} 1 & \Phi_L^\pm \end{bmatrix}^* \begin{bmatrix} N_{yyL}^\pm \\ N_{xyL}^\pm \end{bmatrix} \right) \quad , \quad J_S^\pm = \frac{1}{2} \operatorname{Re} \left(i\omega \begin{bmatrix} \Phi_S^\pm & 1 \end{bmatrix}^* \begin{bmatrix} N_{yyS}^\pm \\ N_{xyS}^\pm \end{bmatrix} \right) \quad (3.16)$$

with

$$\begin{aligned} \Phi_L^\pm &= - \frac{A_{26} \mu_L^{\pm 2} + k_x \mu_L^\pm (A_{12} + A_{66}) + A_{16} k_x^2}{A_{66} \mu_L^{\pm 2} + 2 A_{16} k_x \mu_L^\pm + A_{11} k_x^2 - \rho h \omega^2} \\ \Phi_S^\pm &= - \frac{A_{26} \mu_S^{\pm 2} + k_x \mu_S^\pm (A_{12} + A_{66}) + A_{16} k_x^2}{A_{22} \mu_S^{\pm 2} + 2 A_{26} k_x \mu_S^\pm + A_{66} k_x^2 - \rho h \omega^2} \end{aligned} \quad (3.17)$$

where $\begin{bmatrix} 1 & \Phi_L^\pm \end{bmatrix}^T$ and $\begin{bmatrix} \Phi_S^\pm & 1 \end{bmatrix}^T$ are the eigenvectors in the Φ_v, Φ_u basis corresponding to the wave numbers $\mu_L^\pm$ and $\mu_S^\pm$ for incoming and outgoing modes. $N_{yyL,S}^\pm$ and $N_{xyL,S}^\pm$ are given by substituting $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ by $-ik_x$ and $-i\mu_{L,S}^\pm$ in Equations (3.2), respectively.

The in-plane response of the plate can be written as

$$\begin{bmatrix} v^+ \\ u^+ \end{bmatrix} = \alpha_L^+ \begin{bmatrix} 1 \\ \Phi_L^+ \end{bmatrix} e^{-ik_x x - i\mu_L^+ y + i\omega t} + \alpha_S^+ \begin{bmatrix} \Phi_S^+ \\ 1 \end{bmatrix} e^{-ik_x x - i\mu_S^+ y + i\omega t} \quad , \quad (3.18)$$

where α_L^+ and α_S^+ are the amplitudes of outgoing quasi-longitudinal and quasi-shear waves. This particular choice of eigenvectors ensures the correct representation of the displacement field in the case of $k_x = 0$ and $A_{16} = A_{26} = 0$, that is, when the incident plane wave vector is directed normal to the junction and the plate is *quasi*-specially orthotropic. Then $\Phi_L^+ = \Phi_S^+ = 0$, and the response of the plate can be expressed as

$$\begin{aligned} v^+ &= \alpha_L^+ e^{-i\tilde{\mu}_L^+ y + i\omega t} \\ u^+ &= \alpha_S^+ e^{-i\tilde{\mu}_S^+ y + i\omega t} \end{aligned} \quad (3.19)$$

The values $\tilde{\mu}_L^+ = \sqrt{\rho \omega^2 / Q_{22}}$ and $\tilde{\mu}_S^+ = \sqrt{\rho \omega^2 / Q_{66}}$ agree then with the wave numbers of purely longitudinal and shear waves, respectively, and the in-plane

response of the plate consists of purely longitudinal and shear displacements in the y and x directions.

3.4.2 The dynamic stiffness matrix

Using Equation (3.13), one can define the out-of-plane displacement w^+ and rotation γ^+ at the common intersection “e” at $y = 0$ in terms of the amplitudes $\alpha_{B_1}^+$ and $\alpha_{B_2}^+$, that is,

$$\begin{bmatrix} w_e^+ \\ \gamma_e^+ \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -i\mu_{B_1}^+ & -i\mu_{B_2}^+ \end{bmatrix} \begin{bmatrix} \alpha_{B_1}^+ \\ \alpha_{B_2}^+ \end{bmatrix} e^{-ik_x x + i\omega t}. \quad (3.20)$$

The elastic tractions involving the out-of-plane displacement can be written in terms of the same amplitudes $\alpha_{B_1}^+$ and $\alpha_{B_2}^+$ by inserting (3.13) in (3.2), that is,

$$\begin{aligned} V_{y_e}^+ &= \sum_{m=1}^2 \left(D_{22} i\mu_{B_m}^{+3} - 4D_{26} \mu_{B_m}^{+2} i k_x - (D_{12} + 4D_{66}) i\mu_{B_m}^+ k_x^2 - 2D_{16} i k_x^3 \right) \alpha_{B_m}^+ e^{-ik_x x + i\omega t} \\ M_{yy_e}^+ &= - \sum_{m=1}^2 \left(D_{22} \mu_{B_m}^{+2} + D_{12} k_x^2 + 2D_{26} \mu_{B_m}^+ k_x \right) \alpha_{B_m}^+ e^{-ik_x x + i\omega t}. \end{aligned} \quad (3.21)$$

Finally, combining Equations (3.20) with Equations (3.21) yields a relation between the elastic tractions $V_{y_e}^+$, $M_{yy_e}^+$ and the edge displacement w_e^+ and rotation γ_e^+ at $y = 0$, that is,

$$\begin{bmatrix} V_{y_e}^+ \\ M_{yy_e}^+ \end{bmatrix} = K_{w\gamma}^+ \begin{bmatrix} w_e^+ \\ \gamma_e^+ \end{bmatrix} \quad (3.22)$$

with

$$\begin{aligned} K_{w\gamma}^+(1, 1) &= -D_{22} i\mu_{B_1}^+ \mu_{B_2}^+ (\mu_{B_1}^+ + \mu_{B_2}^+) + 4D_{26} \mu_{B_1}^+ \mu_{B_2}^+ i k_x - 2D_{26} i k_x^3 \\ K_{w\gamma}^+(1, 2) &= -D_{22} (\mu_{B_1}^{+2} + \mu_{B_2}^{+2} + \mu_{B_1}^+ \mu_{B_2}^+) + 4D_{26} (\mu_{B_1}^+ + \mu_{B_2}^+) k_x + (D_{12} + 4D_{66}) k_x^2 \\ K_{w\gamma}^+(2, 1) &= D_{22} \mu_{B_1}^+ \mu_{B_2}^+ - D_{12} k_x^2 \\ K_{w\gamma}^+(2, 2) &= -D_{22} i (\mu_{B_1}^+ + \mu_{B_2}^+) - 2D_{26} i k_x \end{aligned} \quad (3.23)$$

$K_{w\gamma}^+$ is a block part of the dynamic stiffness matrix for out-of-plane displacement w_e^+ and rotation γ_e^+ .

Next, we consider the corresponding part for in-plane motion. The in-plane displacements v_e^+ and u_e^+ at the shared edge are given as

$$\begin{bmatrix} v_e^+ \\ u_e^+ \end{bmatrix} = \begin{bmatrix} 1 & \Phi_S^+ \\ \Phi_L^+ & 1 \end{bmatrix} \begin{bmatrix} \alpha_L^+ \\ \alpha_S^+ \end{bmatrix} e^{-ik_x x + i\omega t}. \quad (3.24)$$

Now, inserting (3.18) into (3.2) as

$$\begin{aligned} N_{yy_e}^+ &= -i \left(A_{22}\mu_L^+ + A_{12}\Phi_L^+ k_x + A_{26}(\Phi_L^+ \mu_L^+ + k_x) \right) \alpha_L^+ e^{-ik_x x + i\omega t} \\ &\quad - i \left(A_{22}\Phi_S^+ \mu_S^+ + A_{12}k_x + A_{26}(\mu_S^+ + \Phi_S^+ k_x) \right) \alpha_S^+ e^{-ik_x x + i\omega t} \\ N_{xy_e}^+ &= -i \left(A_{26}\mu_L^+ + A_{16}\Phi_L^+ k_x + A_{66}(\Phi_L^+ \mu_L^+ + k_x) \right) \alpha_L^+ e^{-ik_x x + i\omega t} \\ &\quad - i \left(A_{26}\Phi_S^+ \mu_S^+ + A_{16}k_x + A_{66}(\mu_S^+ + \Phi_S^+ k_x) \right) \alpha_S^+ e^{-ik_x x + i\omega t} \end{aligned} \quad (3.25)$$

and using (3.24) yield a relation between the in-plane tractions and the displacements of the outgoing in-plane waves, that is,

$$\begin{bmatrix} N_{yy_e}^+ \\ N_{xy_e}^+ \end{bmatrix} = K_{vu}^+ \begin{bmatrix} v_e^+ \\ u_e^+ \end{bmatrix} \quad (3.26)$$

with

$$\begin{aligned} K_{vu}^+(1, 1) &= cA_{22}(\mu_L^+ - \Phi_L^+ \Phi_S^+ \mu_S^+) + A_{26}(c\Phi_L^+(\mu_L^+ - \mu_S^+) + k_x) \\ K_{vu}^+(1, 2) &= cA_{26}(\mu_S^+ - \Phi_L^+ \Phi_S^+ \mu_L^+) + cA_{22}\Phi_S^+(\mu_S^+ - \mu_L^+) + A_{12}k_x \\ K_{vu}^+(2, 1) &= cA_{26}(\mu_L^+ - \Phi_L^+ \Phi_S^+ \mu_S^+) + A_{66}(c\Phi_L^+(\mu_L^+ - \mu_S^+) + k_x) \\ K_{vu}^+(2, 2) &= cA_{66}(\mu_S^+ - \Phi_L^+ \Phi_S^+ \mu_L^+) + cA_{26}\Phi_S^+(\mu_S^+ - \mu_L^+) + A_{16}k_x \end{aligned}, \quad (3.27)$$

where $c = \frac{1}{i(1 - \Phi_L^+ \Phi_S^+)}$. The matrix K_{vu}^+ is the part of the dynamic stiffness matrix related to in-plane motion.

We now define the elastic tractions and displacements at the junction edge as

$F = (N_{yy_e}, N_{xy_e}, V_{y_e}, M_{yy_e})^T$ and $U = (v_e, u_e, w_e, \gamma_e)^T$ and write

$$F_j^+ = K_j^+ U_j^+ \quad (3.28)$$

for the relation between the displacements of an outgoing wave at the edge of plate j and the associate forces, where the 4×4 dynamic stiffness matrix K_j^+ is a block matrix defined through (3.22) and (3.26). An associated stiffness matrix K_j^- relating displacements of the incoming waves U_j^- with forces F_j^- can be obtained from K_j^+ by changing the corresponding wave number components μ^+ by μ^- in the sub-matrices related to in-plane and out-of-plane motion. In what follows, it will also be important to consider the relation between displacements and mode amplitudes. We denote $A^\pm = (\alpha_L^\pm, \alpha_S^\pm, \alpha_{B_1}^\pm, \alpha_{B_2}^\pm)^T$ as the vector of amplitudes of incoming or outgoing modes, and write

$$U_j^\pm = H_j^\pm A_j^\pm \quad (3.29)$$

on the edge of plate j , where $H_j^\pm$ is the block-diagonal matrix obtained from (3.20) and (3.24). Here, H_j^- is obtained from H_j^+ by changing the wave number components μ^+ by μ^- . In the next step, we will derive the global dynamic stiffness matrices of each plate junction and associated scattering matrices using force equilibrium conditions and continuity conditions at the junction.

3.5 Reflection and transmission at plate junctions

Assuming no external forces are applied at the junction, one can write the force equilibrium and continuity conditions at an edge shared between N plates as

$$\sum_{j=1}^N R_j F_j = 0, \quad (3.30)$$

$$U_j = R_j^T U \quad \text{for all } j = 1, \dots, N, \quad (3.31)$$

where $F_j = F_j^+ + F_j^-$ and $U_j = U_j^+ + U_j^-$ is the total force and displacement at the edge of plate j . U denotes the displacement common to all plates (continuity) and R_j is the rotation matrix from the local coordinate system

(x_j, y_j, z_j) to the global coordinate system (x_g, y_g, z_g) , that is,

$$R_j = \begin{bmatrix} \cos \psi_j & 0 & -\sin \psi_j & 0 \\ 0 & 1 & 0 & 0 \\ \sin \psi_j & 0 & \cos \psi_j & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (3.32)$$

Rewriting (3.30) using (3.28), we obtain

$$\sum_{j=1}^N R_j K_j^+ U_j^+ = - \sum_{j=1}^N R_j K_j^- U_j^- \quad (3.33)$$

$$\Rightarrow \sum_{j=1}^N R_j K_j^+ U_j = \sum_{j=1}^N R_j (K_j^+ - K_j^-) U_j^-. \quad (3.34)$$

Using (3.31), we can now deduce the common displacement vector U as a function of the incoming waves, that is,

$$U = \left(\sum_{j=1}^N R_j K_j^+ R_j^T \right)^{-1} \sum_{n=1}^N R_n (K_n^+ - K_n^-) U_n^-. \quad (3.35)$$

Inserting (3.35) into the N matrix equations (3.31), one obtains for $m = 1, \dots, N$

$$U_m^+ = R_m^T \left(\sum_{j=1}^N R_j K_j^+ R_j^T \right)^{-1} \left(\sum_{n=1}^N R_n (K_n^+ - K_n^-) U_n^- \right) - U_m^-, \quad (3.36)$$

and writing this in terms of the mode amplitudes using (3.29), we obtain

$$A_m^+ = (H_m^+)^{-1} R_m^T \left(\sum_{j=1}^N R_j K_j^+ R_j^T \right)^{-1} \left(\sum_{n=1}^N R_n (K_n^+ - K_n^-) H_n^- A_n^- \right) - (H_m^+)^{-1} H_m^- A_m^-. \quad (3.37)$$

Equation (3.37) gives relations between incoming and outgoing wave mode amplitudes and can be interpreted as defining the matrix elements of a $4N \times 4N$ scattering matrix S . We note that by setting $N = 1$ in Equation (3.37), we can compute wave reflection coefficients of different modes at the free edge of a composite plate. In fact, the elastic tractions $F_j = F_j^+ + F_j^-$ are equal to zero at the free edge of the j th plate. Using Equations (3.28) and (3.29), we

can write

$$\begin{aligned} K_j^+ H_j^+ A_j^+ + K_j^- H_j^- A_j^- &= 0 \\ A_j^+ &= - (K_j^+ H_j^+)^{-1} K_j^- H_j^- A_j^- . \end{aligned} \quad (3.38)$$

From the other side, setting $N = 1$ in Equation (3.37) yields

$$\begin{aligned} A_j^+ &= (H_j^+)^{-1} (K_j^+)^{-1} (K_j^+ - K_j^-) H_j^- A_j^- - (H_j^+)^{-1} H_j^- A_j^- , \\ A_j^+ &= - (H_j^+)^{-1} (K_j^+)^{-1} K_j^- H_j^- A_j^- \end{aligned} \quad (3.39)$$

which is exactly the same expression as in Equation (3.38).

Finally, we compute the energy scattering coefficients, that is, the ratio between outgoing and incident energy fluxes. Writing the matrix elements of S in the form $s_{ij}^{nm}(\omega, k_x)$ for an incoming wave of type i in plate n and a reflected or transmitted wave of type j in plate m (at angular frequency ω and wave number component k_x), we obtain for the associated energy fluxes

$$t_{ij}^{nm}(\omega, k_x) = \begin{cases} \frac{J_{j,m}^+}{J_{i,n}^-} |s_{ij}^{nm}|^2 & \text{if wave } j \text{ is propagating.} \\ 0 & \text{otherwise.} \end{cases} \quad (3.40)$$

Here, $J_{i,n}^-$ ($J_{j,m}^+$) is the incoming (outgoing) energy flux of type i (j) on plate n (m) given by either (3.16) for in-plane modes or (3.11) for out-of-plane motion. It is noted here that the sum of energy scattering coefficients over the outgoing modes equals one, that is,

$$\sum_{m=1}^N \sum_j t_{ij}^{nm} = 1 \quad (3.41)$$

due to energy conservation. We will use this relation as a check when considering numerical applications of the method developed.

3.6 Stiffened plate

This section considers several composite plates finite in their respective y coordinate directions mounted onto a ground composite plate from one or both sides. Figure 3.11 describes the configuration of a plate with m stiffeners at-

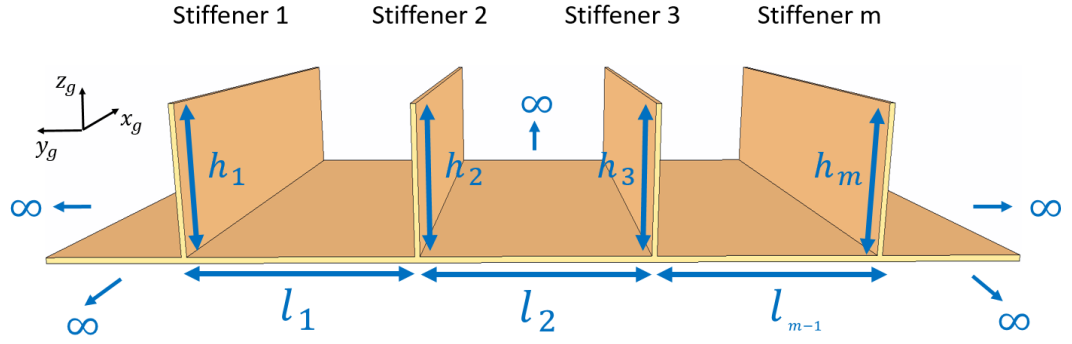


Figure 3.11: A ground composite plate with m stiffener plates attached onto it. The ground plate is considered infinite in both directions, whereas stiffeners are infinite only in their local x coordinate directions.

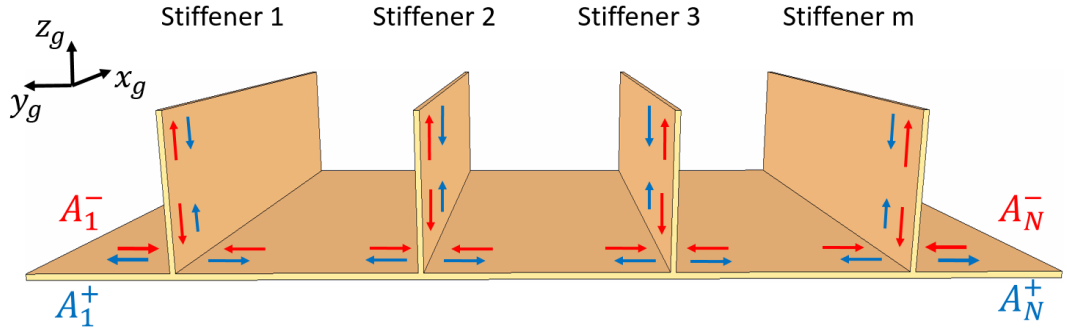


Figure 3.12: The schematic of incoming (shown in red) and outgoing (shown in blue) waves travelling in the stiffened plate. A_1^- and A_N^- are the incoming waves from infinity.

tached to the ground plate from its top side, thus forming T-shaped junctions locally. The lengths of stiffener plates are h_k , $k = 1, \dots, m$, and distances between stiffeners are l_k , $k = 1, \dots, m-1$. For the case of stiffeners attached from both sides of a ground plate, we assume that they are symmetrically placed around the ground plate and made of the same material - minor changes in the method's development would be needed to remove this assumption. We are interested in the scattering coefficients of incoming waves travelling from infinity towards the plate with stiffeners taking into account that the excitation can also enter into stiffeners and get reflected at their respective free ends as shown with red and blue arrows in Figure 3.12. This leads to resonance phenomena, that is, at certain wave number component and angular frequency values k_x and ω , a perfect transmission or reflection of incoming wave energy can occur. These effects were previously presented and analysed for beams with the symmetric constraint [194], for beams with periodic stiffeners [195]

and for plates with periodic stiffeners [196]. We discuss the conditions at which resonances are attained in the last parts of subsections 3.6.1, 3.6.2 and 3.6.3. As previously stated, we want to compute an effective scattering matrix S_{eff} of the following form

$$\begin{bmatrix} A_1^+ \\ A_N^+ \end{bmatrix} = S_{\text{eff}} \begin{bmatrix} A_1^- \\ A_N^- \end{bmatrix}, \quad (3.42)$$

where $A_{1,N}^\pm$ are the amplitudes of incoming/outgoing waves travelling from infinity towards the first junctions of the stiffened plate that the waves meet. Note that this particular setup is based on the short wavelength assumption. Specifically, the longest wavelengths of various modes that can propagate in the plate need to be smaller than the stiffener spacings and lengths. This assumption allows to neglect an always attenuating bending wave B_2 in the derivation of the effective scattering matrix since it rapidly decays along stiffener lengths and distances between them, thus carrying no wave energy - it is still used for the calculation of junction scattering coefficients as per Equation (3.37). We denote the propagating bending waves as B (instead of B_1) for simplicity.

In order to compute S_{eff} , we first derive an effective scattering matrix for a plate with one or a pair of stiffeners mounted on it in subsections 3.6.1 and 3.6.2. Using calculated effective scattering matrices, we derive an effective scattering matrix for a plate with two stiffeners or two pairs of stiffeners in subsection 3.6.3. Recursively, we compute an effective scattering matrix for a composite plate with several stiffeners. Finally, the effective energy scattering coefficients $t_{ij,\text{eff}}^{nm}$ can be computed as

$$t_{ij,\text{eff}}^{nm}(\omega, k_x) = \begin{cases} \frac{J_{j,m}^+}{J_{i,n}^-} |s_{ij,\text{eff}}^{nm}|^2 & \text{if wave } j \text{ is propagating.} \\ 0 & \text{otherwise.} \end{cases} \quad n, m = 1, N \quad (3.43)$$

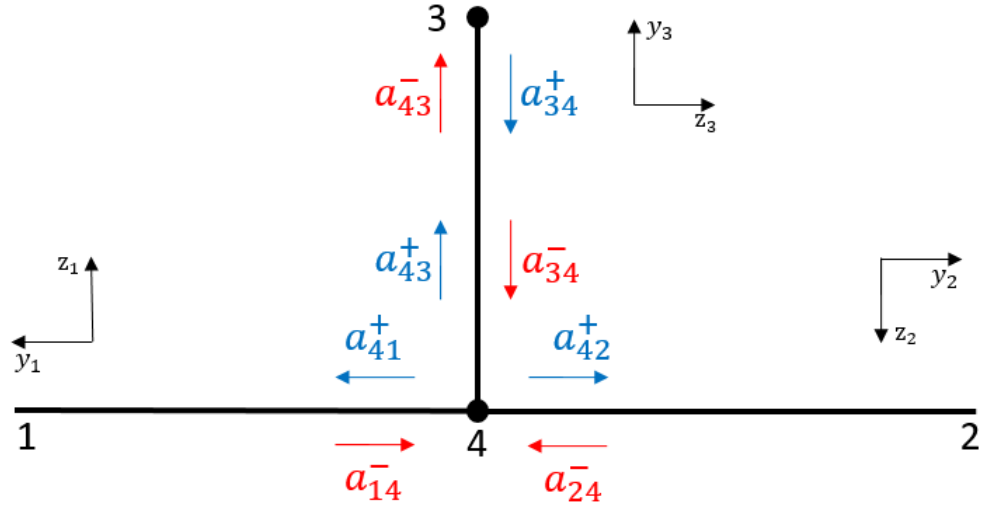


Figure 3.13: A schematic view of a stiffened plate in the yz plane. Here, a_{ij}^- represent the amplitudes of incoming waves, and a_{ij}^+ , the amplitudes of outgoing waves.

3.6.1 A composite plate with one stiffener

We give here the details for computing the scattering coefficients for a plate with a stiffener k attached. The stiffener plate has a finite length of h_k . We are interested in the reflection and transmission coefficients between points 1 and 2 in Figure 3.13 considering that the excitation can also enter the stiffener at point 4 and being reflected at the free end at point 3. In order to compute the energy scattering coefficients (3.40) relating plate one and two, one needs to compute the associate scattering matrix s_{eff} from the scattering matrix at junction 4, but including waves travelling into the stiffener and being reflected at the end of the stiffener at 3.

Following the treatment in [197] extended to the elastodynamic case in [198], we now introduce the amplitudes of incoming and outgoing waves at each of the plate segments shown in Figure 3.13, that is,

$$a_{ij}^{\pm} = \begin{bmatrix} a_L^{\pm} \\ a_S^{\pm} \\ a_B^{\pm} \end{bmatrix}_{ij} \quad i, j \in (1, 2, 3, 4), \quad (3.44)$$

where the subscript ij is related to a wave travelling from i to j . Following

this notation, we write the desired effective scattering matrix s_{eff} as

$$\begin{bmatrix} a_{41}^+ \\ a_{42}^+ \end{bmatrix} = s_{\text{eff}} \begin{bmatrix} a_{14}^- \\ a_{24}^- \end{bmatrix}. \quad (3.45)$$

From Equations (3.37) and (3.38), we can write the two scattering matrices at the junction 4 and free edge 3 as

$$\begin{bmatrix} a_{41}^+ \\ a_{42}^+ \\ a_{43}^+ \end{bmatrix} = S^{(4)} \begin{bmatrix} a_{14}^- \\ a_{24}^- \\ a_{34}^- \end{bmatrix} = \begin{bmatrix} \rho_{11} & \tau_{21} & \tau_{31} \\ \tau_{12} & \rho_{22} & \tau_{32} \\ \tau_{13} & \tau_{23} & \rho_{33} \end{bmatrix} \begin{bmatrix} a_{14}^- \\ a_{24}^- \\ a_{34}^- \end{bmatrix} \quad (3.46)$$

and

$$a_{34}^+ = S^{(3)} a_{43}^- = \tilde{\rho}_{33} a_{43}^-. \quad (3.47)$$

The sub-matrices in (3.46) contain the scattering coefficients s_{ij}^{nm} from plate n to m of mode type i to j in the form

$$\rho_{nn} = \begin{bmatrix} s_{LL}^{nn} & s_{LS}^{nn} & s_{BL}^{nn} \\ s_{LS}^{nn} & s_{SS}^{nn} & s_{BS}^{nn} \\ s_{LB}^{nn} & s_{SB}^{nn} & s_{BB}^{nn} \end{bmatrix}, \quad \tau_{nm} = \begin{bmatrix} s_{LL}^{nm} & s_{LS}^{nm} & s_{BL}^{nm} \\ s_{LS}^{nm} & s_{SS}^{nm} & s_{BS}^{nm} \\ s_{LB}^{nm} & s_{SB}^{nm} & s_{BB}^{nm} \end{bmatrix}. \quad (3.48)$$

Note that $\tilde{\rho}_{33}$ in Equation (3.47) denotes the reflection matrix at node 3 and is different from ρ_{33} corresponding to reflection of waves on node 4 incoming from plate 3.

The first two rows of matrix equation in (3.46) can render the effective scattering matrix s_{eff} if amplitudes a_{34}^- are expressed in terms of amplitudes a_{14}^- and a_{24}^- . To accomplish that, we first note that the amplitudes a_{43}^+ of outgoing waves are related to the amplitudes a_{43}^- of the incoming waves as

$$a_{43}^- = P a_{43}^+, \quad P = \text{diag} \left(\exp(-i\mu^+ h_k) \right), \quad (3.49)$$

where $\mu^+ = \begin{bmatrix} \mu_L^+ & \mu_S^+ & \mu_B^+ \end{bmatrix}^T$ and $\text{diag}(\exp(-i\mu^+ h_k))$ represents the diagonal matrix with $\exp(-i\mu^+ h_k)$ on its diagonal (see [197], [198]). The same applies

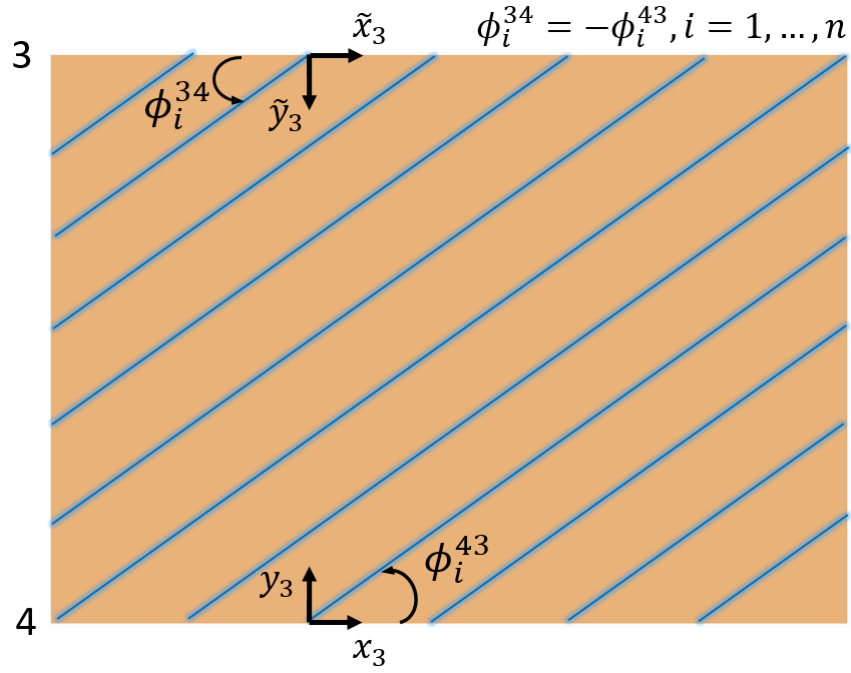


Figure 3.14: A schematic view of a stiffener plate in the xy plane. Blue lines represent the direction of the i th ply of the plate.

for the pair of amplitudes of outgoing and incoming waves a_{34}^+ and a_{34}^- , that is,

$$a_{34}^- = \tilde{P} a_{34}^+ \quad , \quad \tilde{P} = \text{diag} \left(\exp(-i\tilde{\mu}^+ h_k) \right). \quad (3.50)$$

We note that $\tilde{\mu}^+$ are computed in the local coordinate system of the semi-infinite plate with the free edge. If we label the local ply direction angles of the semi-infinite plate at point 4 as $\phi_1^{43}/\phi_2^{43}/\dots/\phi_n^{43}$, then the local ply direction angles of the semi-infinite plate at point 3 must satisfy

$$\phi_i^{34} = -\phi_i^{43}, i = 1, \dots, n \quad (3.51)$$

to ensure that the stiffener plate is uniform and that waves with amplitudes $a_{43(34)}^-$ and $a_{43(34)}^+$ have the same angles of propagation, see Figure 3.14. This entails

$$\tilde{\mu}^+ \neq \mu^+ \Rightarrow \tilde{P} \neq P \quad \text{if} \quad \phi_i^{43(34)} \neq 0^\circ, 90^\circ \quad , \quad i = 1, \dots, n. \quad (3.52)$$

Now, to eliminate a_{34}^- in the matrix equation (3.46), we consider its third row,

that is,

$$a_{43}^+ = \tau_{13}a_{14}^- + \tau_{23}a_{24}^- + \rho_{33}a_{34}^-. \quad (3.53)$$

By using (3.49), (3.50) and (3.47), one obtains

$$a_{34}^- = \tilde{P} \tilde{\rho}_{33} P a_{43}^+, \quad (3.54)$$

which combined with (3.53) yields the following matrix equation for a_{34}^- :

$$\left(I - \tilde{P} \tilde{\rho}_{33} P \rho_{33} \right) a_{34}^- = \tilde{P} \tilde{\rho}_{33} P \left(\tau_{13}a_{14}^- + \tau_{23}a_{24}^- \right), \quad (3.55)$$

where I is a 3-by-3 identity matrix. Finally, by introducing the solution of Equation (3.55) into Equation (3.46), we can define the effective scattering matrix s_{eff} as

$$\begin{bmatrix} a_{41}^+ \\ a_{42}^+ \end{bmatrix} = s_{\text{eff}} \begin{bmatrix} a_{14}^- \\ a_{24}^- \end{bmatrix} = \begin{bmatrix} \rho_{11}^* & \tau_{21}^* \\ \tau_{12}^* & \rho_{22}^* \end{bmatrix} \begin{bmatrix} a_{14}^- \\ a_{24}^- \end{bmatrix} \quad (3.56)$$

with

$$\begin{aligned} \rho_{11}^* &= \rho_{11} + \tau_{31} \left(I - \tilde{P} \tilde{\rho}_{33} P \rho_{33} \right)^{-1} \tilde{P} \tilde{\rho}_{33} P \tau_{13} \\ \tau_{21}^* &= \tau_{21} + \tau_{31} \left(I - \tilde{P} \tilde{\rho}_{33} P \rho_{33} \right)^{-1} \tilde{P} \tilde{\rho}_{33} P \tau_{23} \\ \tau_{12}^* &= \tau_{12} + \tau_{32} \left(I - \tilde{P} \tilde{\rho}_{33} P \rho_{33} \right)^{-1} \tilde{P} \tilde{\rho}_{33} P \tau_{13} \\ \rho_{22}^* &= \rho_{22} + \tau_{32} \left(I - \tilde{P} \tilde{\rho}_{33} P \rho_{33} \right)^{-1} \tilde{P} \tilde{\rho}_{33} P \tau_{23} \end{aligned} \quad (3.57)$$

The effective energy scattering coefficients $t_{ij,\text{eff}}^{nm}$, $n, m \in \{1, 2\}$ can be computed from s_{eff} using (3.43). Note that $\tilde{P} \tilde{\rho}_{33} P \rho_{33}$ is sub-unitary due to the sub-unitarity of ρ_{33} , and s_{eff} is thus not singular.

A resonance condition can be formulated, that is, resonances are attained at wave numbers k_x and frequencies ω values giving rise to local minima of

$$\left| \det \left(I - \tilde{P} \tilde{\rho}_{33} P \rho_{33} \right) \right|. \quad (3.58)$$

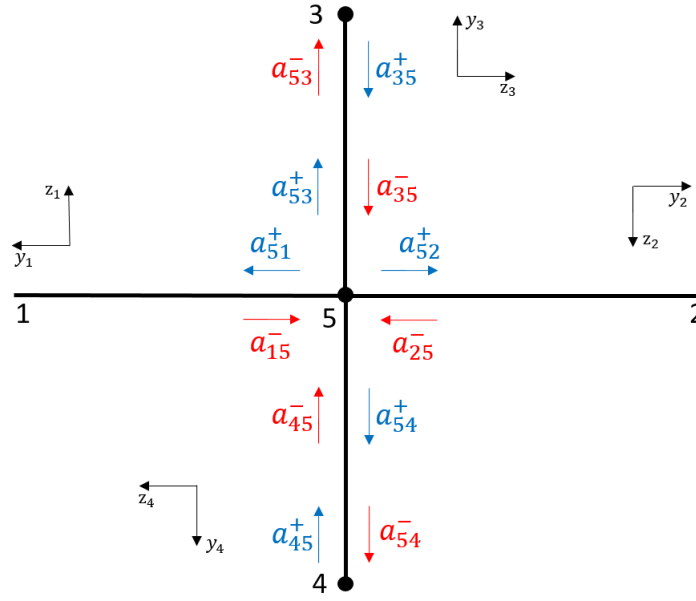


Figure 3.15: A schematic view of a plate with a pair of finite stiffener plates in the yz plane. Here, a_{ij}^- represent the amplitudes of incoming waves, and a_{ij}^+ , the amplitudes of outgoing waves.

3.6.2 A composite plate with a pair of stiffeners

In this subsection, we consider a composite ground plate stiffened with two stiffener plates symmetrically attached from both sides, forming a cross-shaped junction. This particular system prevents the generation of outgoing in-plane waves from an incoming bending wave [196], and therefore, the corresponding effective scattering matrix has null entries $s_{BL,\text{eff}}^{11}$ and $s_{BS,\text{eff}}^{12}$. Figure 3.15 describes the configuration of plates in the yz plane. As in the previous case, waves with amplitudes a_{15}^- impinging on the junction point 5 can be transmitted into plate 2 with amplitudes a_{52}^+ , can be reflected with amplitudes a_{15}^- and can be transmitted to the stiffener plates with amplitudes a_{53}^+ and a_{54}^+ . At the free ends of the stiffeners, waves reflect and become incoming on the junction point 5 again, thus creating further scattering. Accordingly, we are interested in the reflection and transmission coefficients between points 1 and 2, that is,

$$\begin{bmatrix} a_{51}^+ \\ a_{52}^+ \end{bmatrix} = s_{\text{eff}} \begin{bmatrix} a_{15}^- \\ a_{25}^- \end{bmatrix}. \quad (3.59)$$

Following the approach described for the case of one stiffener plate in the previous subsection, we express the scattering matrices at the junction point 5 and free ends 3 and 4 as

$$\begin{bmatrix} a_{51}^+ \\ a_{52}^+ \\ a_{53}^+ \\ a_{54}^+ \end{bmatrix} = S^{(5)} \begin{bmatrix} a_{15}^- \\ a_{25}^- \\ a_{35}^- \\ a_{45}^- \end{bmatrix} = \begin{bmatrix} \rho_{11} & \tau_{21} & \tau_{31} & \tau_{41} \\ \tau_{12} & \rho_{22} & \tau_{32} & \tau_{42} \\ \tau_{13} & \tau_{23} & \rho_{33} & \tau_{43} \\ \tau_{14} & \tau_{24} & \tau_{34} & \rho_{44} \end{bmatrix} \begin{bmatrix} a_{15}^- \\ a_{25}^- \\ a_{35}^- \\ a_{45}^- \end{bmatrix}, \quad \begin{aligned} a_{35}^+ &= S^{(3)} a_{53}^- = \tilde{\rho}_{33} a_{53}^- \\ a_{45}^+ &= S^{(4)} a_{54}^- = \tilde{\rho}_{44} a_{54}^- \end{aligned}. \quad (3.60)$$

Expressing amplitudes a_{35}^- and a_{45}^- in terms of amplitudes a_{15}^- and a_{25}^- would yield the scattering matrix s_{eff} from the first matrix equation in (3.60). To accomplish that, we first note that similarly as in Equations (3.49) and (3.50), we can relate the amplitudes $a_{53(35)}^-$ and $a_{54(45)}^-$ with $a_{53(35)}^+$ and $a_{54(45)}^+$ as follows

$$\begin{aligned} a_{53}^- &= P a_{53}^+ \\ a_{35}^- &= \tilde{P} a_{35}^+ \\ a_{54}^- &= \tilde{P} a_{54}^+ \\ a_{45}^- &= P a_{45}^+ \end{aligned}, \quad \text{with} \quad \begin{aligned} P &= \text{diag} \left(\exp(-i\mu^+ h_k) \right) \\ \tilde{P} &= \text{diag} \left(\exp(-i\tilde{\mu}^+ h_k) \right) \end{aligned}. \quad (3.61)$$

The matrix P is used for the relations between pairs of amplitudes a_{53}^- , a_{53}^+ and pairs a_{45}^- , a_{45}^+ . This is because the stiffener plates are similar in material and geometrical parameters. The coordinate systems of the semi-infinite stiffener plates used to compute scattering matrices at the junction point 5 and the free end 4 are equivalent. Accordingly, the matrix $\tilde{P}$ relate amplitudes of outgoing waves a_{35}^+ and a_{45}^+ with amplitudes of incoming waves a_{35}^- and a_{45}^- , respectively. By combining (3.61) with the second and third matrix equations in (3.60), one obtains

$$\begin{aligned} a_{35}^- &= \tilde{P} \tilde{\rho}_{33} P a_{53}^+ \\ a_{45}^- &= P \tilde{\rho}_{44} \tilde{P} a_{54}^+ \end{aligned}. \quad (3.62)$$

Substituting these expressions to the third and fourth rows of the first equation in (3.60) yields the matrix equation for $\begin{bmatrix} a_{35}^- & a_{45}^- \end{bmatrix}^T$ as

$$\begin{bmatrix} I - \tilde{P}\tilde{\rho}_{33}P\rho_{33} & -\tilde{P}\tilde{\rho}_{33}P\tau_{43} \\ -P\tilde{\rho}_{44}\tilde{P}\tau_{34} & I - P\tilde{\rho}_{44}\tilde{P}\rho_{44} \end{bmatrix} \begin{bmatrix} a_{35}^- \\ a_{45}^- \end{bmatrix} = \begin{bmatrix} \tilde{P}\tilde{\rho}_{33}P\tau_{13} & \tilde{P}\tilde{\rho}_{33}P\tau_{23} \\ P\tilde{\rho}_{44}\tilde{P}\tau_{14} & P\tilde{\rho}_{44}\tilde{P}\tau_{24} \end{bmatrix} \begin{bmatrix} a_{15}^- \\ a_{25}^- \end{bmatrix}. \quad (3.63)$$

Finally, we can insert the solution of Equation (3.63) into (3.60) thus computing the effective scattering matrix s_{eff} as follows

$$s_{\text{eff}} = \begin{bmatrix} \rho_{11} & \tau_{21} \\ \tau_{12} & \rho_{22} \end{bmatrix} + \begin{bmatrix} \tau_{31} & \tau_{41} \\ \tau_{32} & \tau_{42} \end{bmatrix} \begin{bmatrix} I - \tilde{P}\tilde{\rho}_{33}P\rho_{33} & -\tilde{P}\tilde{\rho}_{33}P\tau_{43} \\ -P\tilde{\rho}_{44}\tilde{P}\tau_{34} & I - P\tilde{\rho}_{44}\tilde{P}\rho_{44} \end{bmatrix}^{-1} \begin{bmatrix} \tilde{P}\tilde{\rho}_{33}P\tau_{13} & \tilde{P}\tilde{\rho}_{33}P\tau_{23} \\ P\tilde{\rho}_{44}\tilde{P}\tau_{14} & P\tilde{\rho}_{44}\tilde{P}\tau_{24} \end{bmatrix} \quad (3.64)$$

The resonance conditions for which a perfect transmission or reflection of incoming waves can occur are defined by the wave number and angular frequency values k_x and ω at which local minima of

$$\left| \det \left(\begin{bmatrix} I - \tilde{P}\tilde{\rho}_{33}P\rho_{33} & -\tilde{P}\tilde{\rho}_{33}P\tau_{43} \\ -P\tilde{\rho}_{44}\tilde{P}\tau_{34} & I - P\tilde{\rho}_{44}\tilde{P}\rho_{44} \end{bmatrix} \right) \right| \quad (3.65)$$

are obtained.

3.6.3 A composite plate with multiple stiffeners

Now, having computed the scattering coefficients for a plate with one or a pair of stiffener plates attached to it, we can formulate the scattering matrix for a plate with two stiffener plates and, eventually, with m stiffeners using recursion. First, we consider a plate with two stiffeners k and $k+1$ replaced by the correspondent effective matrices s_{eff}^k and s_{eff}^{k+1} as shown in Figure 3.16. The distance between points 2 and 3 is l_k . Following the same approach as in the previous subsection, we write the two effective scattering matrices as

$$\begin{bmatrix} a_{21}^+ \\ a_{23}^+ \end{bmatrix} = \begin{bmatrix} \rho_{11} & \tau_{31} \\ \tau_{13} & \rho_{33} \end{bmatrix} \begin{bmatrix} a_{12}^- \\ a_{32}^- \end{bmatrix}, \quad \begin{bmatrix} a_{32}^+ \\ a_{34}^+ \end{bmatrix} = \begin{bmatrix} \rho_{22} & \tau_{42} \\ \tau_{24} & \rho_{44} \end{bmatrix} \begin{bmatrix} a_{23}^- \\ a_{43}^- \end{bmatrix}. \quad (3.66)$$

The amplitudes $a_{23(32)}^-$ are related to $a_{23(32)}^+$ in the similar way as in (3.49) and (3.50),

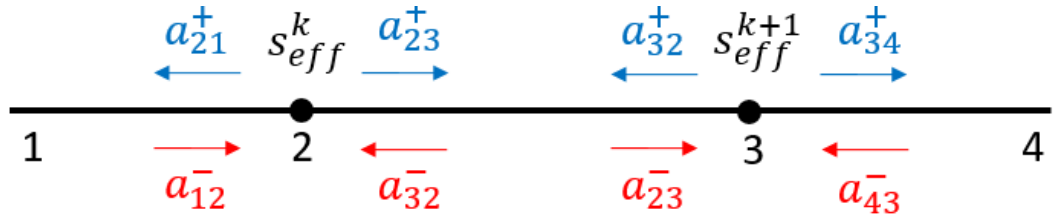


Figure 3.16: A schematic view of a stiffened plate with two stiffeners represented by effective scattering matrices s_{eff}^k and s_{eff}^{k+1} in the yz plane. Here, a_{ij}^- represent the amplitudes of incoming waves, and a_{ij}^+ , the amplitudes of outgoing waves.

that is,

$$\begin{cases} a_{23}^- = P a_{23}^+ & , \quad P = \text{diag}(e^{-\mu^+ l_k}) \\ a_{32}^- = \tilde{P} a_{32}^+ & , \quad \tilde{P} = \text{diag}(e^{-\tilde{\mu}^+ l_k}) \end{cases} \quad (3.67)$$

As highlighted in subsection 3.6.1, P is not equal to $\tilde{P}$ in general. Combining the second row of matrix equations (3.66) with (3.67), we find

$$\begin{aligned} a_{23}^- &= P \tau_{13} a_{12}^- + P \rho_{33} a_{32}^- \\ a_{32}^- &= \tilde{P} \rho_{22} a_{23}^- + \tilde{P} \tau_{42} a_{43}^- \end{aligned} \quad (3.68)$$

Now, solving (3.68) in terms of a_{12}^- and a_{43}^- yields the following:

$$\begin{aligned} a_{23}^- &= \left(I - P \rho_{33} \tilde{P} \rho_{22} \right)^{-1} \left(P \tau_{13} a_{12}^- + P \rho_{33} \tilde{P} \tau_{42} a_{43}^- \right) \\ a_{32}^- &= \left(I - \tilde{P} \rho_{22} P \rho_{33} \right)^{-1} \left(\tilde{P} \rho_{22} P \tau_{13} a_{12}^- + \tilde{P} \tau_{42} a_{43}^- \right) \end{aligned} \quad (3.69)$$

Finally, we substitute these equations into equations (3.66) and obtain the effective scattering matrix for a plate with two stiffeners attached as

$$\begin{bmatrix} a_{21}^+ \\ a_{34}^+ \end{bmatrix} = s_{eff}^* \begin{bmatrix} a_{12}^- \\ a_{43}^- \end{bmatrix} = \begin{bmatrix} \rho_{11}^* & \tau_{41}^* \\ \tau_{14}^* & \rho_{44}^* \end{bmatrix} \begin{bmatrix} a_{12}^- \\ a_{43}^- \end{bmatrix} \quad (3.70)$$

with

$$\begin{aligned} \rho_{11}^* &= \rho_{11} + \tau_{31} \left(I - \tilde{P} \rho_{22} P \rho_{33} \right)^{-1} \tilde{P} \rho_{22} P \tau_{13} \\ \tau_{41}^* &= \tau_{31} \left(I - \tilde{P} \rho_{22} P \rho_{33} \right)^{-1} \tilde{P} \tau_{42} \\ \tau_{14}^* &= \tau_{24} \left(I - P \rho_{33} \tilde{P} \rho_{22} \right)^{-1} P \tau_{13} \\ \rho_{44}^* &= \rho_{44} + \tau_{24} \left(I - P \rho_{33} \tilde{P} \rho_{22} \right)^{-1} P \rho_{33} \tilde{P} \tau_{42} \end{aligned} \quad (3.71)$$

We note that the values of wave numbers k_x and frequencies ω at which the resonances occur can be found by calculating local minima of

$$\left| \det \left(\mathbf{I} - \tilde{\mathbf{P}} \rho_{22} \mathbf{P} \rho_{33} \right) \right| \quad \text{and} \quad \left| \det \left(\mathbf{I} - \mathbf{P} \rho_{33} \tilde{\mathbf{P}} \rho_{22} \right) \right|. \quad (3.72)$$

Finally, we can compute the total effective scattering matrix S_{eff} in Equation (3.42) via the following algorithm:

1. Compute local effective scattering matrices $s_{\text{eff}}^k, k = 1, \dots, m$ of T-type junctions using Equations (3.57) and (3.64).
2. Compute s_{eff}^* that replaces the local effective scattering matrices s_{eff}^1 and s_{eff}^2 using Equation (3.71).
3. Repeat step 2 for effective scattering matrices s_{eff}^* and $s_{\text{eff}}^k, k = 3, \dots, m$.
4. Compute effective energy scattering coefficients using Equation (3.43).

(3.73)

3.7 Computational results

In this section, several numerical applications of the method discussed earlier are presented. We first consider the reflection of various incoming waves' power at the free edge of a composite plate, then we turn to the reflection and transmission coefficients of composite plates joined together in the form of L and T junctions. Finally, the effective energy scattering coefficients of a stiffened composite plate with one and multiple stiffeners are shown in the last example. Only the scattering coefficients for the propagating waves are shown. The attenuating contributions are, of course, considered in full in the actual computations.

3.7.1 Reflection at the free edge of a composite plate

We consider a semi-infinite regular symmetric laminated plate with the free edge along the x axis. The plate has five layers, and two lamination schemes are considered. Namely, the first one is $0^\circ/90^\circ/0^\circ/90^\circ/0^\circ$, that is, a cross-ply lami-

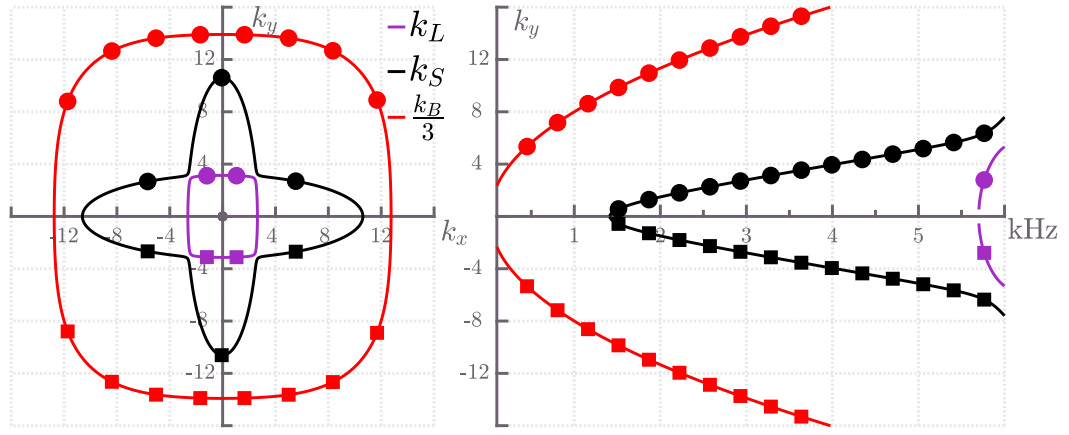


Figure 3.17: Wave vector (left) and dispersion (right) curves for a cross-ply composite plate. The wave vector curves are plotted for a fixed frequency $f = 3000$ Hz, whereas the dispersion curves are plotted for a fixed wave number component $k_x = 5 \text{ m}^{-1}$. Squares and circles denote wave numbers of incoming and outgoing waves, respectively. The bending wave and dispersion curves are scaled with a factor of $1/3$ for the sake of clarity.

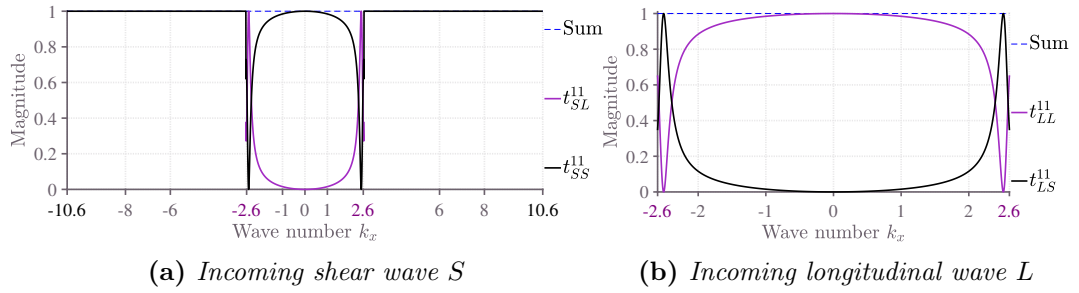


Figure 3.18: Energy reflection coefficients of a regular cross-ply symmetric laminate at the free edge for various incident waves as a function of wave number component k_x at a frequency 3000 Hz. The lamination scheme is $0^\circ/90^\circ/0^\circ/90^\circ/0^\circ$.

nated plate, and the second one is $45^\circ/-45^\circ/45^\circ/-45^\circ/45^\circ$, that is, an angle-ply laminated plate. The material of the orthotropic lamina used in the laminate is given in Table 3.1. The total thickness of the laminate is 5 mm. The energy reflection coefficients can be computed using Equations (3.38) and (3.40). Figure 3.17 presents wave vector and dispersion curves of the cross-ply laminated plate for a fixed frequency $f = 3000$ Hz on the left side and a fixed wave number component $k_x = 5 \text{ m}^{-1}$ on the right side, respectively. Since the plate consists of laminas with ply direction angles of 0° and 90° , the extensional coefficients A_{16} , A_{26} and bending coefficients D_{16} , D_{26} are equal to zero (see Section A.1 for details). Therefore, according to Equations (3.4) and (3.5), the incoming and outgoing wave number components are equal in magnitude, i.e.

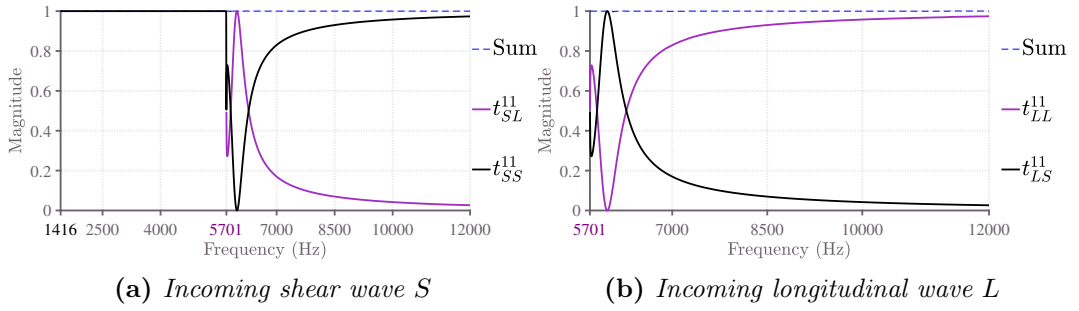


Figure 3.19: Energy reflection coefficients of a regular cross-ply symmetric laminate at the free edge for various incident waves as a function of frequency at a wave number component $k_x = 5 \text{ m}^{-1}$. The lamination scheme is $0^\circ/90^\circ/0^\circ/90^\circ/0^\circ$.

$$k_y^+ = -k_y^-.$$

We note that the longitudinal waves can propagate for $|k_x| \leq 2.6 \text{ m}^{-1}$ at a fixed frequency $f = 3000 \text{ Hz}$ and for $f \geq 5701 \text{ Hz}$ at a fixed wave number component $k_x = 5 \text{ m}^{-1}$. Regarding the shear waves, they can propagate for $|k_x| \leq 10.6 \text{ m}^{-1}$ at a fixed frequency $f = 3000 \text{ Hz}$ and for $f \geq 1416 \text{ Hz}$ at a fixed wave number component $k_x = 5 \text{ m}^{-1}$.

Figure 3.18 presents the energy reflection coefficients of the cross-ply laminated plate for incoming shear and longitudinal waves as a function of the wave number component k_x at a fixed frequency $f = 3000 \text{ Hz}$. An incoming longitudinal wave is fully reflected without mode conversion for $|k_x| \leq 1 \text{ m}^{-1}$, see Figure 3.18b. The corresponding energy reflection coefficient has the form t_{LL}^{11} . For $1 \text{ m}^{-1} < |k_x| \leq 2.6 \text{ m}^{-1}$, mode conversion to the reflected shear wave takes place with the energy reflection coefficient t_{LS}^{11} . Vice-versa, for the same range of k_x , an incoming shear wave is reflected with mode conversion to the longitudinal wave in Figure 3.18a - the corresponding energy reflection coefficients is labelled as t_{SL}^{11} . Pure reflection of shear waves occurs for $|k_x| \leq 1 \text{ m}^{-1}$ and $2.6 \text{ m}^{-1} < |k_x| \leq 10.6 \text{ m}^{-1}$ with the energy reflection coefficient t_{SS}^{11} .

Figure 3.19 presents the energy reflection coefficients of the cross-ply laminated plate for incoming shear and longitudinal waves as a function of frequency at a fixed wave number component $k_x = 5 \text{ m}^{-1}$. An incoming shear wave is reflected without mode conversion at frequencies where no energy can be reflected by longitudinal waves, that is, at frequencies $1416 \text{ Hz} \leq f < 5701 \text{ Hz}$,

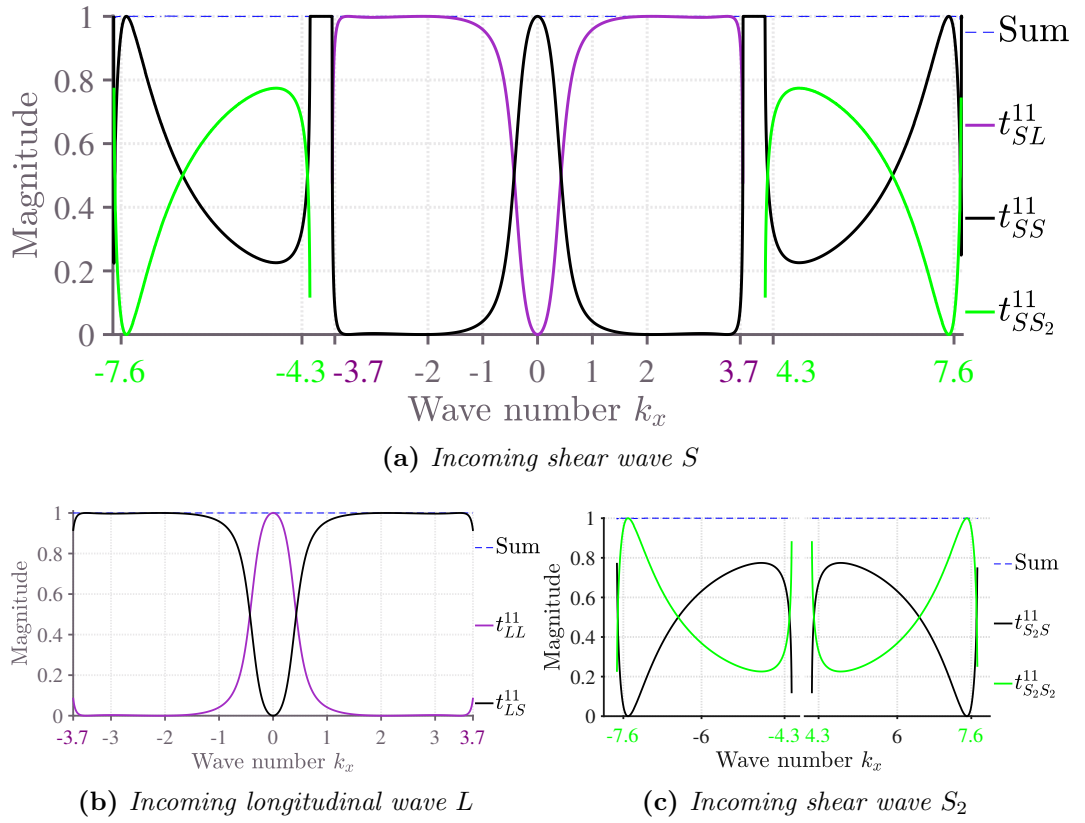


Figure 3.20: Energy reflection coefficients of a regular angle-ply symmetric laminate at the free edge for various incident waves as a function of wave number component k_x at a frequency 3000 Hz. The lamination scheme is $45^\circ/-45^\circ/45^\circ/-45^\circ/45^\circ$.

see Figure 3.19a. At frequencies $f \geq 5701$ Hz, the energy reflection coefficients of incoming shear and longitudinal waves share similar shapes, that is, $t_{SL}^{11} = t_{LS}^{11}$ and $t_{SS}^{11} = t_{LL}^{11}$.

Figure 3.20 presents the energy reflection coefficients of the angle-ply laminated plate for incoming shear and longitudinal waves at a fixed frequency $f = 3000$ Hz. Pure reflection of longitudinal and shear waves without mode conversion occurs only in the vicinity of $k_x = 0$, see Figure 3.20a and Figure 3.20b. The phenomenon of mode conversion is observed again only between specific critical values: for an incoming shear mode, the reflected longitudinal wave is present for the k_x values between $\pm 3.7 \text{ m}^{-1}$. Furthermore, for $4.3 \text{ m}^{-1} \leq |k_x| \leq 7.6 \text{ m}^{-1}$, mode conversion to the second shear wave (labelled S_2) can occur. We encountered two pairs of incoming/outgoing shear wave numbers whilst considering shear wave vector curves in Figure 3.8. Vice-versa, the incoming second shear wave can couple to both types of shear waves, see

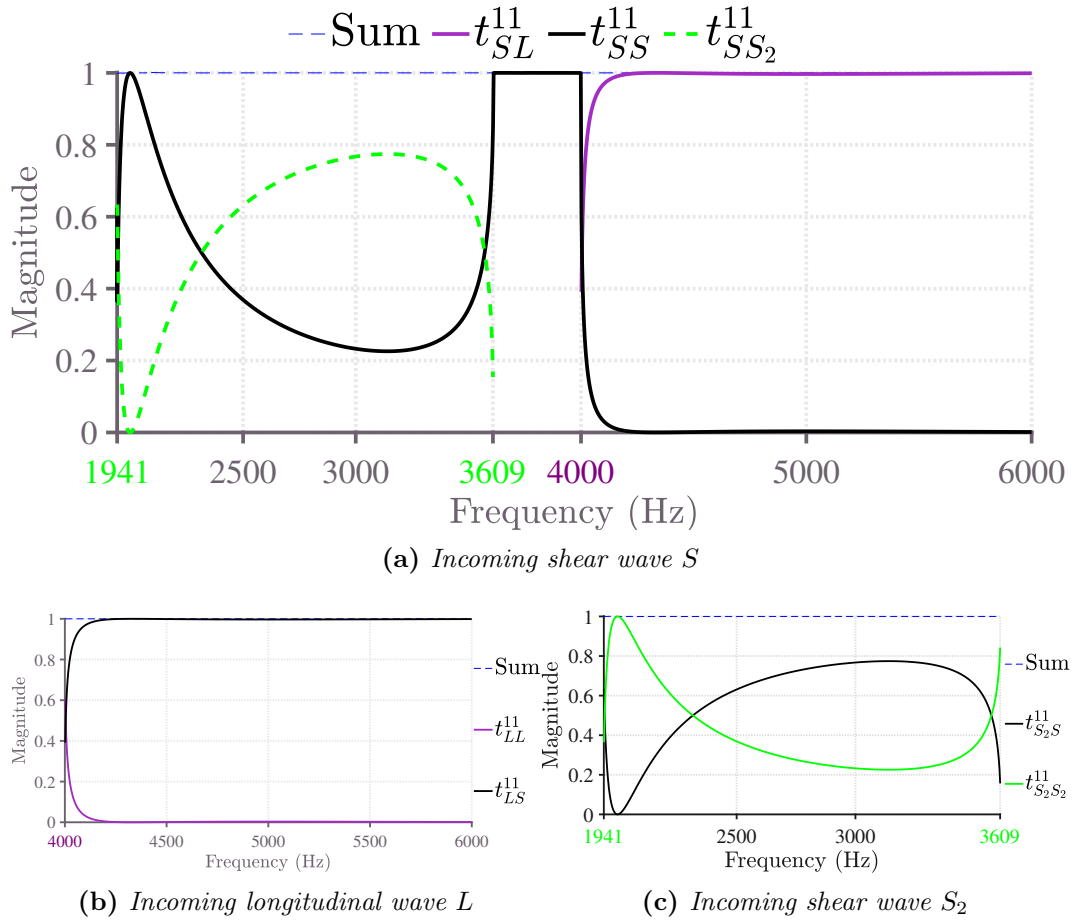


Figure 3.21: Energy reflection coefficients of a regular angle-ply symmetric laminate at the free edge for various incident waves as a function of frequency at a wave number component $k_x = 5 \text{ m}^{-1}$. The lamination scheme is $45^\circ / -45^\circ / 45^\circ / -45^\circ / 45^\circ$

Figure 3.20c.

If the wave number component k_x is fixed and the frequency is varied, one can find critical frequencies at which mode conversion phenomena start to occur. For example, in the case of an incident shear wave with $k_x = 5 \text{ m}^{-1}$, almost all energy is carried back by a longitudinal wave for $f \geq 4000 \text{ Hz}$, see Figure 3.21. Accordingly, in the same range of frequencies, an incoming longitudinal wave is almost fully reflected as a shear wave in Figure 3.21b. Mode conversion from incident shear wave to the second shear wave S_2 occurs in the frequency range $f \in [1941, 3609] \text{ Hz}$, and the energy reflection coefficients $t_{SS_2}^{11}$, t_{SS}^{11} and $t_{S_2S}^{11}$, $t_{S_2S_2}^{11}$ are equal, respectively.

The bending wave power is fully reflected at the free edge without mode conversion at all frequencies and wave number components k_x , that is, the energy

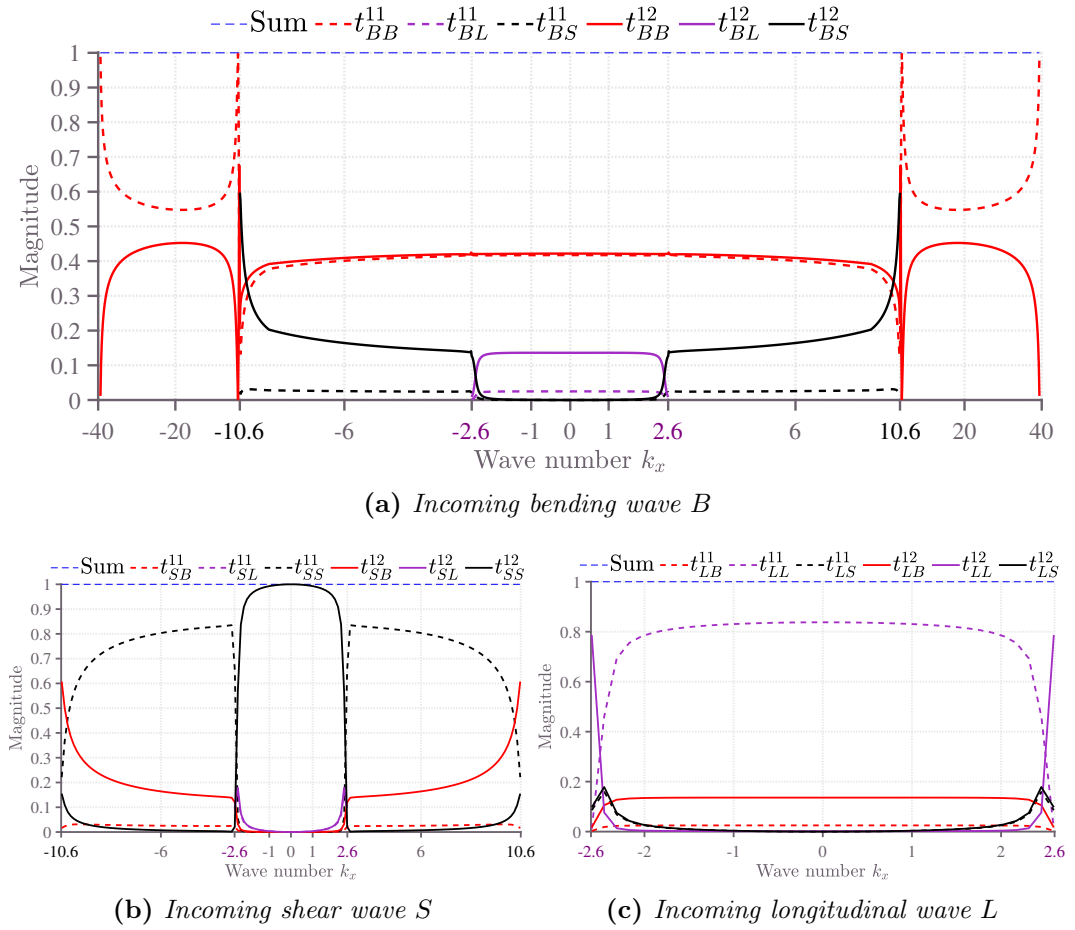


Figure 3.22: Energy scattering coefficients of an L -joint of regular cross-ply symmetric laminates with the lamination scheme $0^\circ/90^\circ/0^\circ/90^\circ/0^\circ$ for various incident waves as a function of wave number component k_x at a frequency 3000 Hz.

reflection coefficient $t_{BB}^{11} = 1$. Therefore, it is not presented in this subsection. Note that the energy reflection coefficients sum up to one according to Equation (3.41) which is used here as a check of consistency of the results (dashed blue lines in Figures 3.18, 3.19, 3.20, 3.21).

3.7.2 Two composite plates joined at a right angle

In this example, we consider two joined identical regular angle- and cross-ply laminated plates with the correspondent lamination schemes $45^\circ/-45^\circ/45^\circ/-45^\circ/45^\circ$ and $0^\circ/90^\circ/0^\circ/90^\circ/0^\circ$, respectively.

We fix the inter-plate angle to $\psi = 90^\circ$, that is, the plate configuration has the form of an L-junction, similar to one in Figure 3.3. First, we consider a junction of two cross-ply laminated plates.

Figure 3.22 presents the energy scattering coefficients of the L-junction of

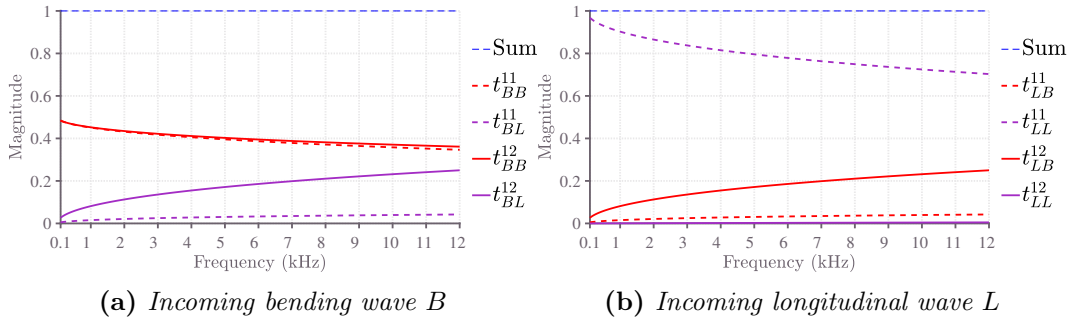


Figure 3.23: Energy scattering coefficients of an L -joint of regular cross-ply symmetric laminates with the lamination scheme $0^\circ/90^\circ/0^\circ/90^\circ/0^\circ$ for bending and longitudinal incident waves as a function of frequency for a wave number component $k_x = 0 \text{ m}^{-1}$.

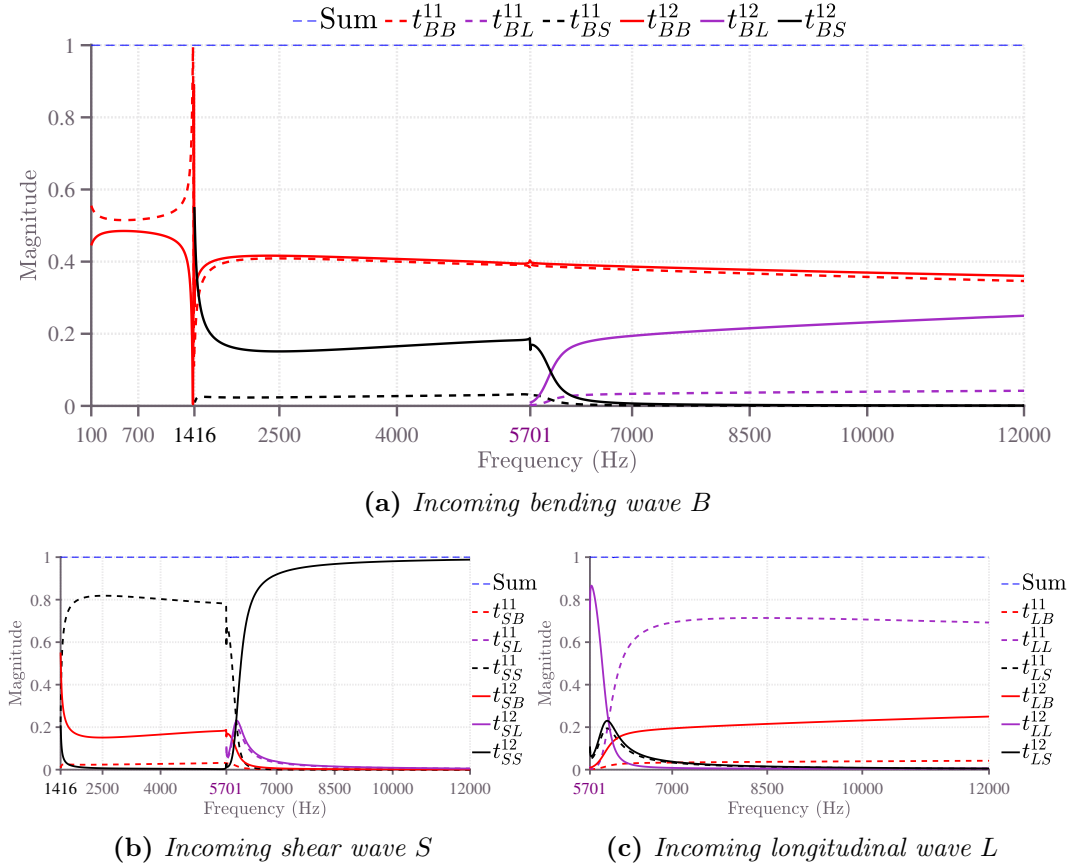


Figure 3.24: Energy scattering coefficients of an L -joint of regular cross-ply symmetric laminates with the lamination scheme $0^\circ/90^\circ/0^\circ/90^\circ/0^\circ$ for various incident waves as a function of frequency for a wave number component $k_x = 5 \text{ m}^{-1}$.

cross-ply laminated plates for incident bending, shear and longitudinal modes as a function of the wave number component k_x at a frequency $f = 3000 \text{ Hz}$. Since the wave number solutions of Equations (3.4) and (3.5) $\mu_{L,S,B}^- = \mu_{L,S,B}^+$, all energy scattering coefficients are symmetrical around $k_x = 0 \text{ m}^{-1}$. Longitudinal and shear modes are propagating for the same range of wave number

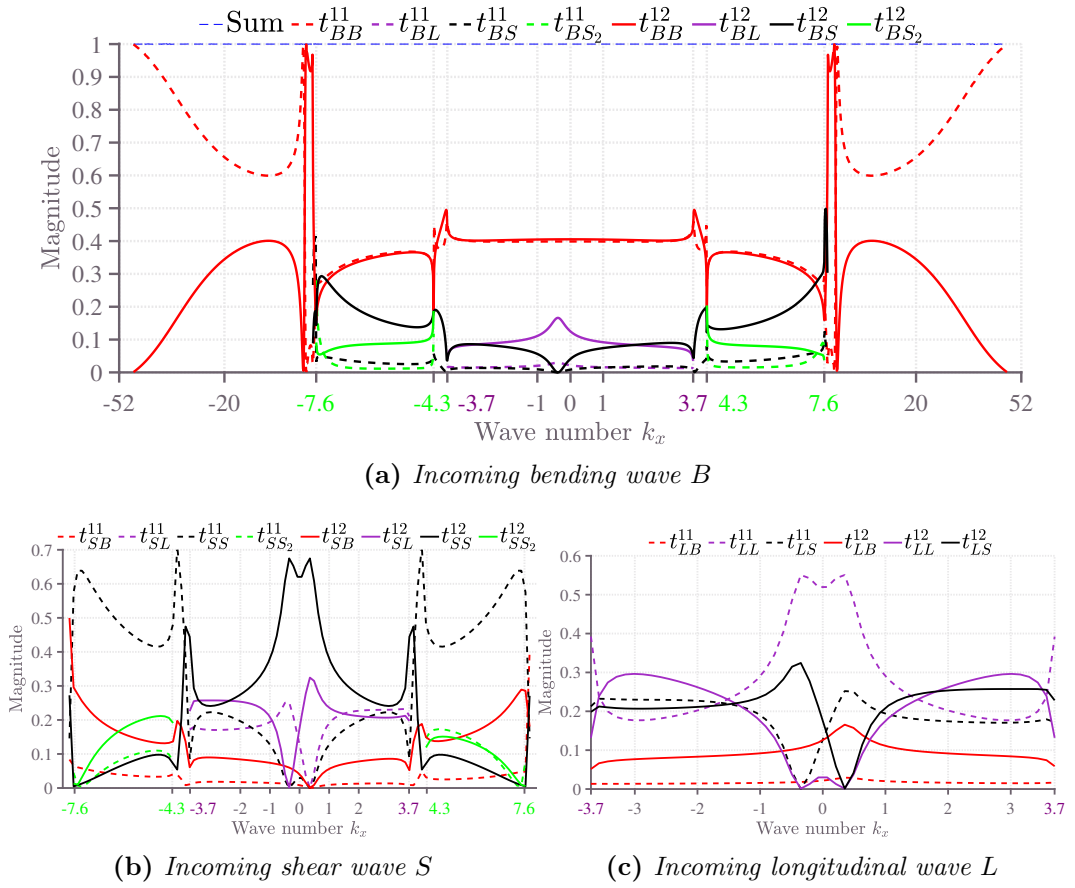


Figure 3.25: Energy scattering coefficients of an L-joint of regular angle-ply symmetric laminates with the lamination scheme $45^\circ / -45^\circ / 45^\circ / -45^\circ / 45^\circ$ for various incident waves as a function of wave number component k_x at a frequency 3000 Hz.

components k_x as in Figure 3.18, as expected. Consequently, outgoing longitudinal and shear waves can only occur in the narrow vicinity of normal incidence for an incoming bending wave. The phase angles of outgoing propagating longitudinal waves are limited by $\sim 3^\circ$ and $\sim 11.5^\circ$ with respect to the y axis. Furthermore, the energy scattering coefficients $t_{BL}^{11(12)}$ and $t_{BS}^{11(12)}$ are bounded by a value of 0.2 almost everywhere in the correspondent propagating ranges of k_x . On the other hand, the bending energy reflected and transmitted coefficients are equal to each other in these ranges of k_x . As can be seen in Figure 3.22a, curves corresponding to the longitudinal mode are smoothly transformed to ones corresponding to the shear mode, thus indicating the similar nature of coupling to both in-plane modes. In Figure 3.22b, an incoming shear wave energy is almost fully transmitted in the range $|k_x| \leq 2.6 \text{ m}^{-1}$, and the curve is similar to one corresponding to the shear wave reflection at

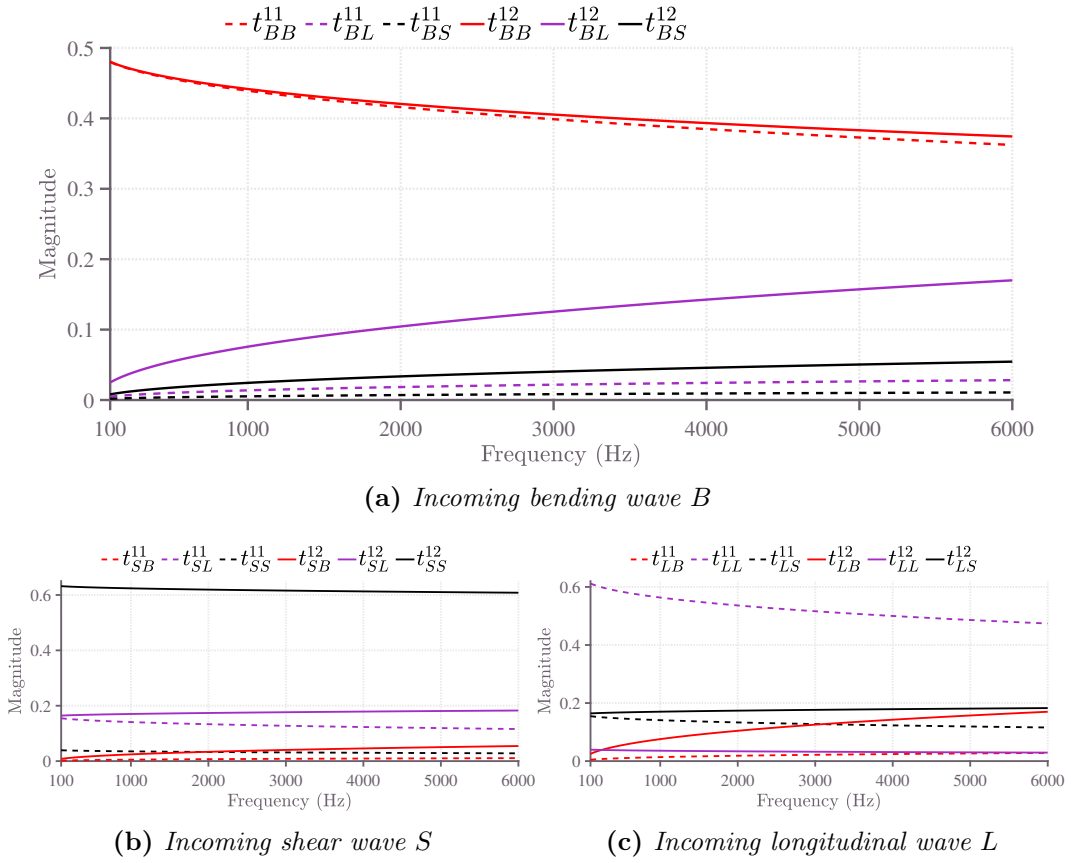


Figure 3.26: Energy scattering coefficients of an L -joint of regular angle-ply symmetric laminates with the lamination scheme $45^\circ / -45^\circ / 45^\circ / -45^\circ / 45^\circ$ for various incident waves as a function of frequency for a wave number component $k_x = 0 \text{ m}^{-1}$.

the free edge in Figure 3.18a. Incoming longitudinal wave energy is mainly reflected at the shared edge between plates, see Figure 3.22c.

If we consider only normal incidence of incoming waves and vary frequency, we can note that coupling can only occur between longitudinal and bending waves, see Figures 3.23a and 3.23b. Furthermore, the shear wave energy is fully transmitted at all frequencies, that is, $t_{SS}^{12} = 1$, and hence, it is not shown in Figure 3.23. At $k_x = 5 \text{ m}^{-1}$, coupling to all modes is present and occurred at the correspondent critical frequencies. For instance, at frequencies $f \geq 1416 \text{ Hz}$, an incoming bending wave power P is equally distributed between reflected and transmitted bending waves with $0.4P$ and is converted to shear waves and longitudinal waves (at $f \geq 5701 \text{ Hz}$) both with $\sim 0.2P$. Figure 3.25 shows the energy scattering coefficients of the L -junction of angle-ply laminated plates for incident bending, shear or longitudinal modes as a

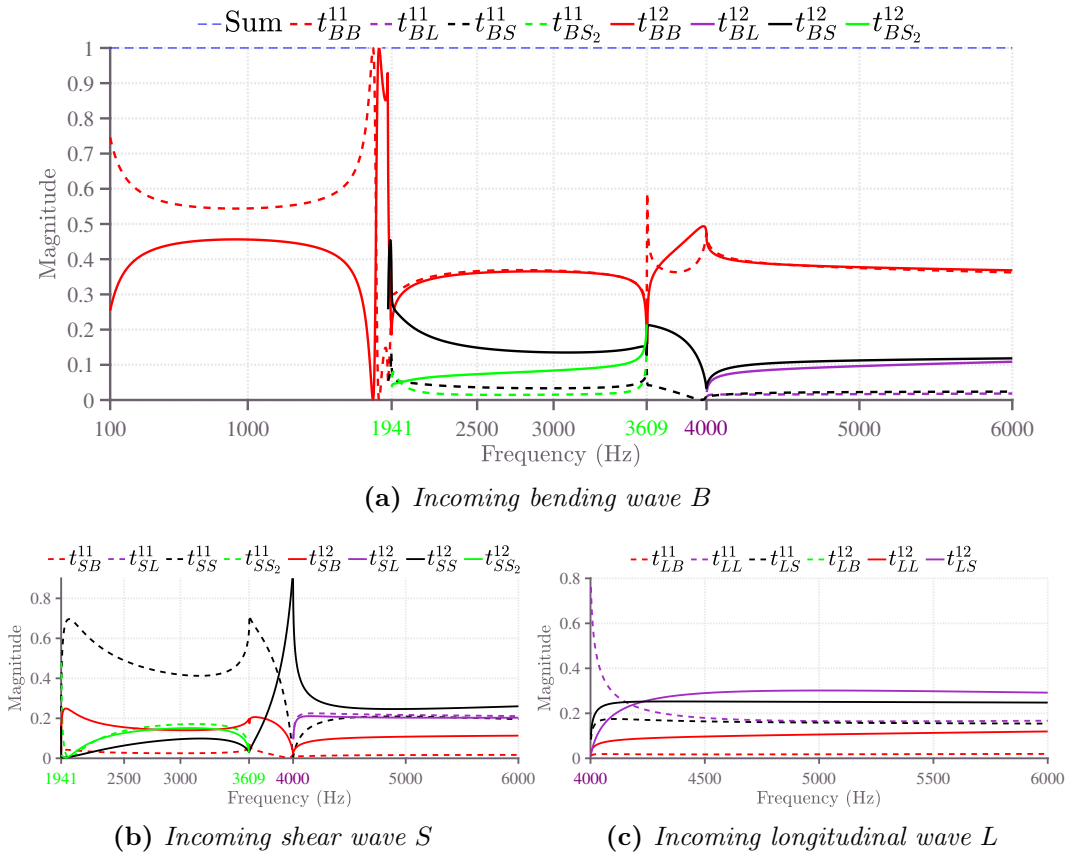


Figure 3.27: Energy scattering coefficients of an L-joint of regular angle-ply symmetric laminates with the lamination scheme $45^\circ / -45^\circ / 45^\circ / -45^\circ / 45^\circ$ for various incident waves as a function of frequency for a wave number component $k_x = 5 \text{ m}^{-1}$.

function of the wave number component k_x at 3000 Hz. Note that reflection and transmission coefficients of bending, shear and longitudinal waves without mode conversion (having the form t_{XX}^{ij} with $X = B, L, S$ and $i, j = 1, 2$) are symmetric around $k_x = 0$ whereas coefficients describing scattering between different modes, that is, $t_{XX'}^{ij}$ with $X \neq X'$ are non-symmetric. Again, the mode conversion can only occur between certain critical values: for an incoming bending mode, these are the k_x values between $\pm 7.8 \text{ m}^{-1}$ for reflected or transmitted shear waves, $\pm 3.7 \text{ m}^{-1}$ for longitudinal waves and $4.3 \text{ m}^{-1} \leq |k_x| \leq 7.6 \text{ m}^{-1}$ for second shear waves. For example, an incoming bending wave power P with $k_x \sim -7.2 \text{ m}^{-1}$ is reflected and transmitted, both with $0.35P$, converted to the shear transmitted wave with $0.2P$ and the second shear transmitted wave with $0.1P$, see Figure 3.25a.

If the frequency is varied with a fixed wave number component k_x , one can

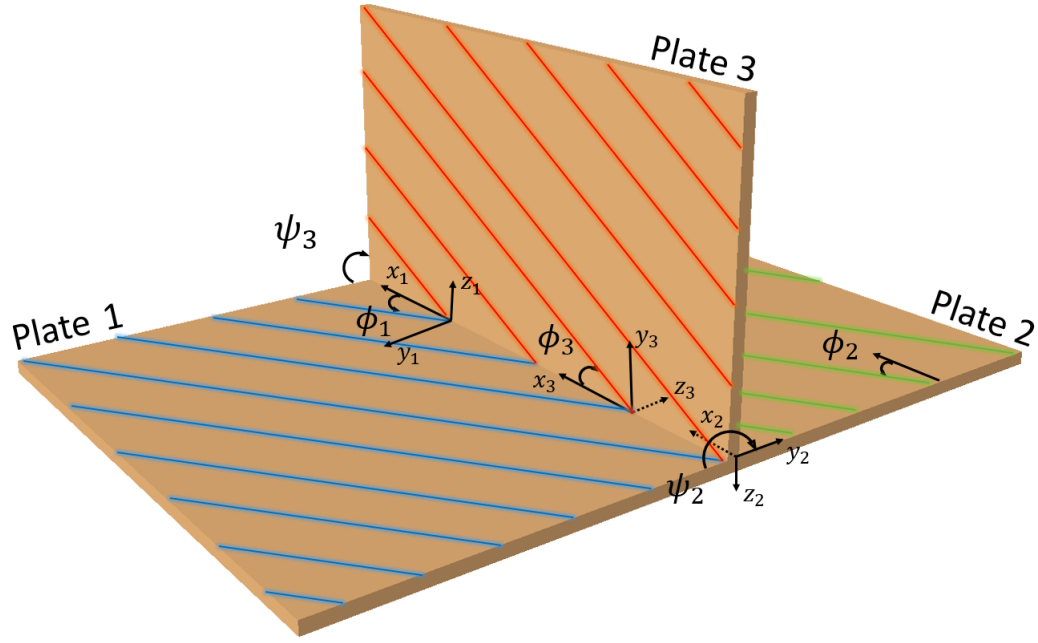
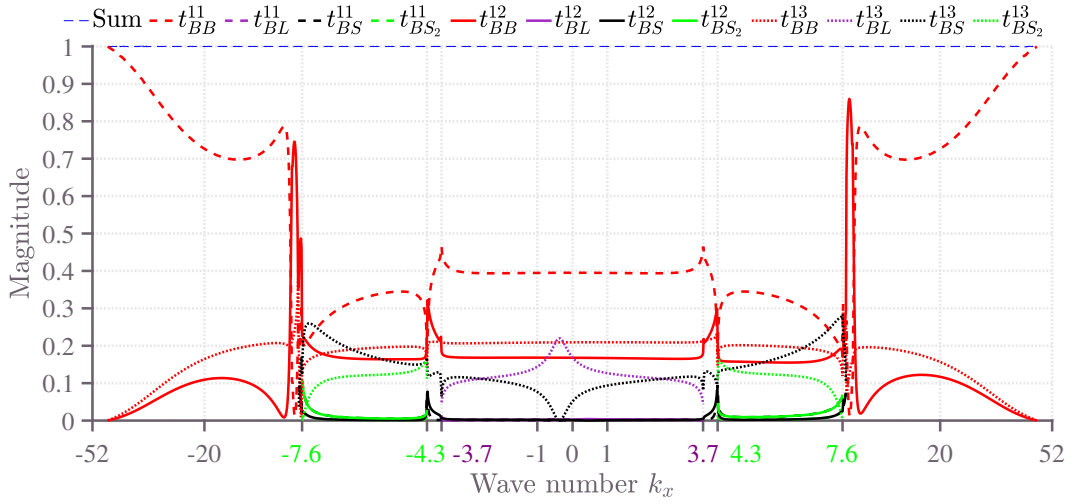
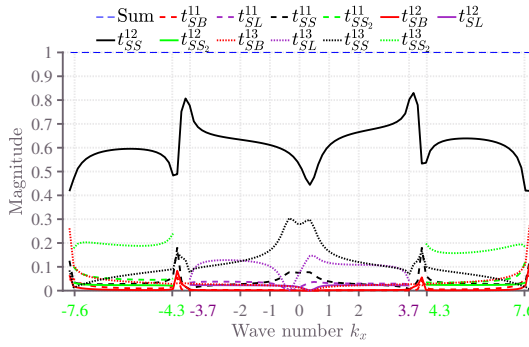


Figure 3.28: Schematic representation of a T-junction connecting three orthotropic plates. The angles ψ_2 and ψ_3 here are set to 180° and 90° , respectively. Red, blue and green lines represent the ply direction of the plates. The local angles of orientations ϕ_1 and ϕ_3 are both set to 45° , whereas ϕ_2 is equal to -45° .

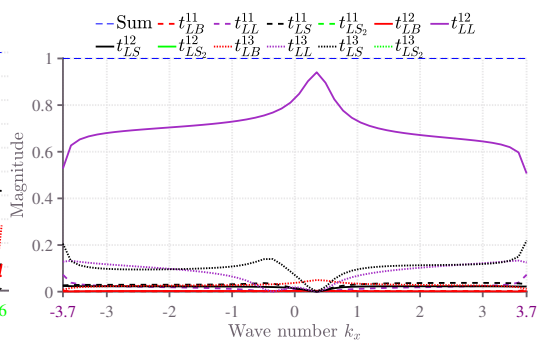
note several things. For a normally incident waves, that is, at a fixed wave number component $k_x = 0 \text{ m}^{-1}$, all modes are propagating for all frequencies, see Figure 3.26. An incoming bending wave energy flux is equally divided between reflected and transmitted bending wave energy fluxes for frequencies $100 \text{ Hz} \leq f \leq 6000 \text{ Hz}$. Mode conversion to the longitudinal wave becomes more significant as frequency increases, although all coupled energy scattering coefficients having the form t_{XY}^{ij} , with $X \neq Y$ and $i, j = 1, 2$ do not exceed 0.2. In Figure 3.26b, the incoming shear wave energy is mostly reflected with a constant value of 0.6 over the range of frequencies considered, whereas the energy scattering coefficient t_{LL}^{11} decreases with increasing frequency, see Figure 3.26c. If the wave number component $k_x = 5 \text{ m}^{-1}$, one can find critical frequencies at which mode conversion phenomena start to occur, see Figure 3.27. As in subsection 3.7.1, outgoing shear waves become propagating at $f \geq 1941 \text{ Hz}$, whereas outgoing longitudinal waves become propagating for $f \geq 4000 \text{ Hz}$, see Figure 3.27. Mode conversion from incident bending and shear waves to the second shear wave S_2 occurs in the frequency range $f \in [1941, 3609] \text{ Hz}$.



(a) Incoming bending wave B



(b) Incoming shear wave S



(c) Incoming longitudinal wave L

Figure 3.29: Energy scattering coefficients of an T-joint of regular angle-ply symmetric laminates with the lamination scheme $45^\circ / -45^\circ / 45^\circ / -45^\circ / 45^\circ$ for various incident waves as a function of wave number component k_x at a frequency 3000 Hz.

3.7.3 Three composite plates joined in a T-junction

In this example, we consider three composite plates joined in the form of a T-junction with angles $\psi_1 = 0^\circ$, $\psi_2 = 180^\circ$ and $\psi_3 = 90^\circ$. The composites considered are the same as in the case of the L-junction. Figure 3.28 shows an illustrative example of such a configuration of composite plates with only one ply per thickness. In Figure 3.28, the local ply direction angles are $\phi_1 = \phi_3 = 45^\circ$ and $\phi_2 = -45^\circ$, and ϕ_2 is given a value of -45° to ensure that plates 1 and 2 form a uniform ground plate. In the similar manner, the local ply direction angles of laminated plates considered are $\phi_1^1 / \dots / \phi_1^5 = \phi_3^1 / \dots / \phi_3^5 = 45^\circ / -45^\circ / 45^\circ / -45^\circ / 45^\circ$, and $\phi_2^1 / \dots / \phi_2^5 = -45^\circ / 45^\circ / -45^\circ / 45^\circ / -45^\circ$. The case of cross-ply and laminated plates is omitted here for the sake of

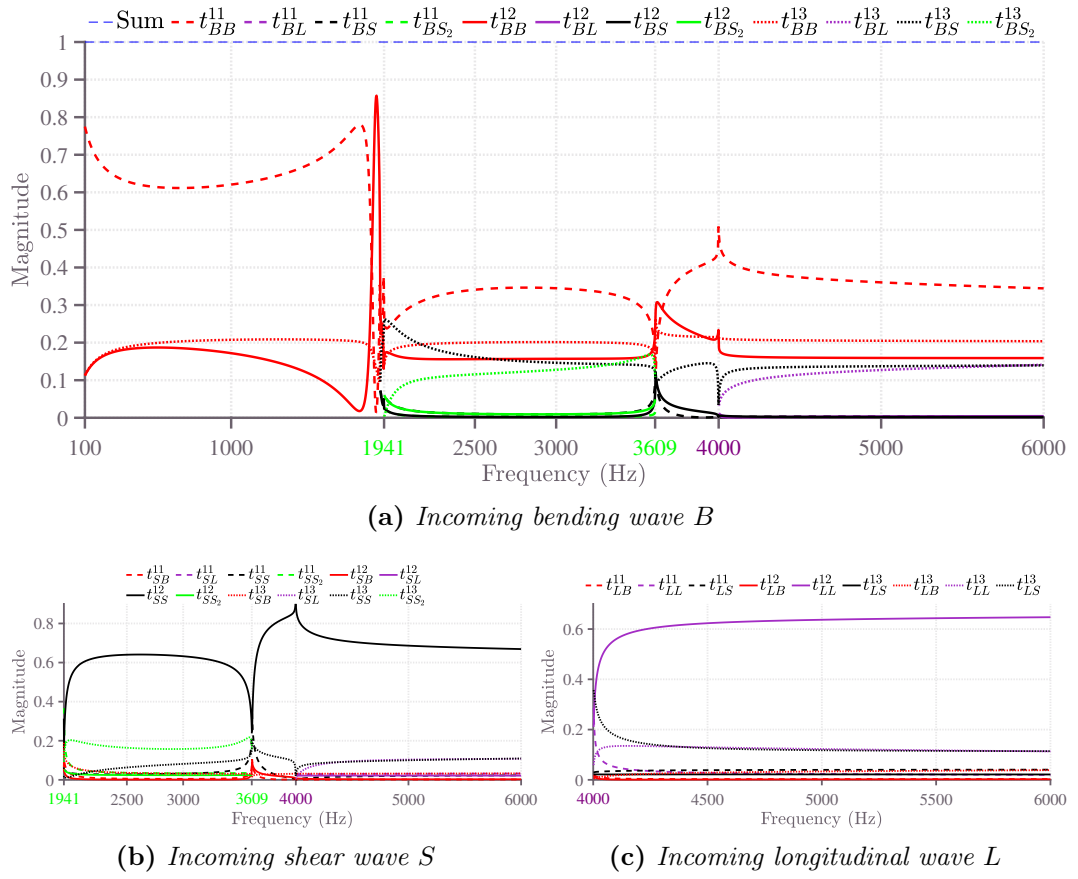


Figure 3.30: Energy scattering coefficients of an T-joint of regular angle-ply symmetric laminates with the lamination scheme $45^\circ / -45^\circ / 45^\circ / -45^\circ / 45^\circ$ for various incident waves as a function of frequency at a wave number component $k_x = 5 \text{ m}^{-1}$.

brevity. However, the energy scattering coefficients for junctions of cross-ply and general-ply plates are computed for comparison with the hybrid FE/WFE method based results in Chapter 4.

Figure 3.29 presents the energy scattering coefficients of the T-junction for incoming bending, shear and longitudinal waves as a function of wave number component k_x at a frequency 3000 Hz. We note that for varying wave number component k_x , most of the incoming shear and longitudinal incoming power is transmitted to the second plate with $\psi_2 = 180^\circ$, which can be explained by the fact that the first and second plates constitute a uniform ground plate, see Figure 3.29b and Figure 3.29c. However, the incoming bending wave power is mostly reflected back at the junction with minor deviations at critical wave number components $|k_x| = 3.7 \text{ m}^{-1}$, 4.3 m^{-1} , 7.6 m^{-1} , 7.8 m^{-1} , corresponding to longitudinal, second shear and shear modes, see Figure 3.29a. We note that

as opposed to the case of the L-junction, symmetry of wave energy reflection and transmission coefficients without mode conversion is no longer present.

In Figure 3.30, the dependence of the energy scattering coefficients on the frequency at a fixed wave number component $k_x = 5 \text{ m}^{-1}$ is regarded. As in the fixed frequency case, most of the incoming bending wave energy is reflected at the shared edge except for frequencies 1941 Hz and 3609 Hz at which coupling to shear waves occurs. Incoming shear and longitudinal wave energies are mainly transmitted to the ground plate, see Figures 3.30b and 3.30c. Moreover, mode coupling occurs mainly between in-plane modes with energy scattering coefficients having the form t_{XY}^{ij} with $X = L, S, S_2$ and $i, j = 1, 2, 3$. Finally, the blue dash lines in Figures 3.29 and 3.30 represent sums of energy scattering coefficients for different incoming modes. As expected in the undamped systems, the energy scattering coefficients sum up to unity.

3.7.4 Stiffened composite plates

In the final example, we consider a composite plate with several composite plates of finite length mounted onto and/or under it at 90° , see Figures 3.11 and 3.15. All material properties of the plates are the same as in the case of the L-junction.

In the case of stiffeners put only on one side of the ground plate, the intersections (or common edge) between each stiffener and a ground plate have thus the form of a T-junction with angles $\psi_1 = 0^\circ$, $\psi_2 = 180^\circ$ and $\psi_3 = 90^\circ$ similar to the configuration shown in Figure 3.28. In the case of angle-ply laminated plates, the local ply direction angles are $\phi_1^1 / \dots / \phi_1^5 = \phi_3^1 / \dots / \phi_3^5 = 45^\circ / -45^\circ / 45^\circ / -45^\circ / 45^\circ$, and $\phi_2^1 / \dots / \phi_2^5 = -45^\circ / 45^\circ / -45^\circ / 45^\circ / -45^\circ$ - this is to ensure that plates 1 and 2 form a uniform ground plate.

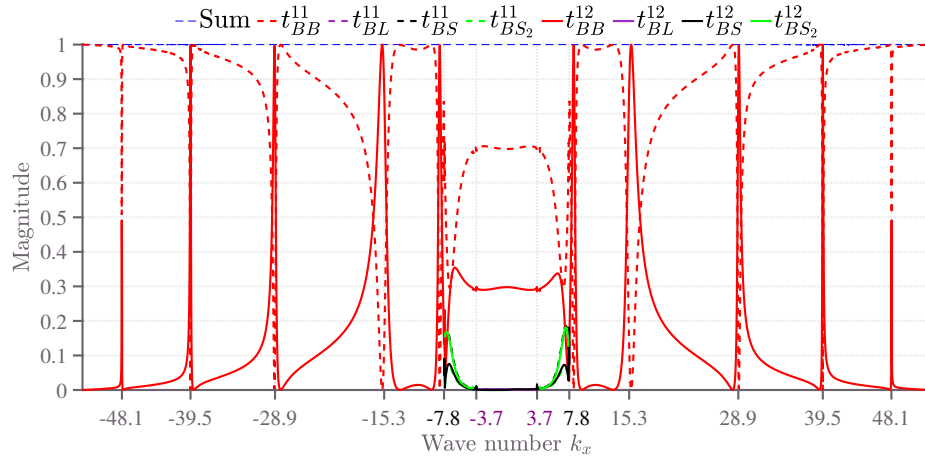
In the case of stiffeners put on both sides of the ground plate, the intersections form a “+” shape with angles $\psi_1 = 0^\circ$, $\psi_2 = 180^\circ$, $\psi_3 = 90^\circ$ and $\psi_4 = 270^\circ$ as in Figure 3.15.

In this case, the local ply direction angles are $\phi_1^1 / \dots / \phi_1^5 = \phi_3^1 / \dots / \phi_3^5 = 45^\circ / -$

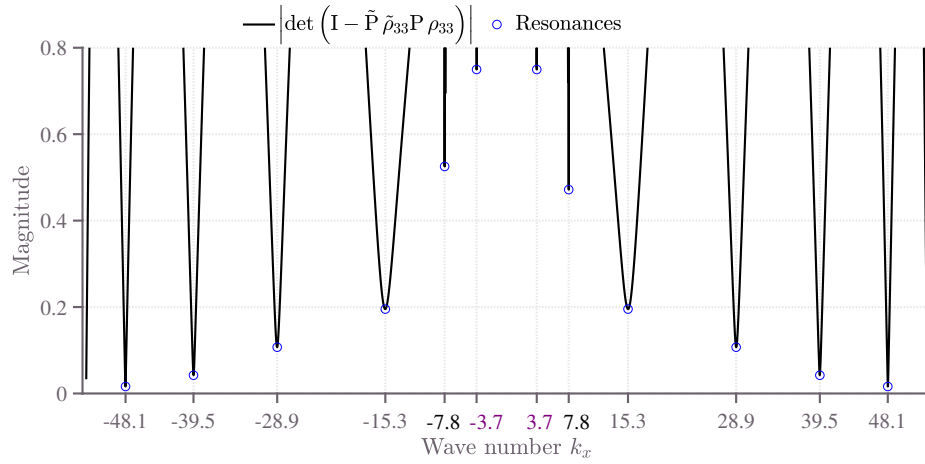
$45^\circ/45^\circ/-45^\circ/45^\circ$, and $\phi_2^1/\dots/\phi_2^5 = \phi_4^1/\dots/\phi_4^5 = -45^\circ/45^\circ/-45^\circ/45^\circ/-45^\circ$ thus ensuring that plates 1 and 2 form a uniform ground plate and that plates 3 and 4 - a uniform stiffener plate. We will also consider the cross-ply laminated plates; in this case, all local ply direction angles are $0^\circ/90^\circ/0^\circ/90^\circ/0^\circ$, since changing signs of ply direction angles from 90° to -90° do not change the wave number solutions and correspondent wave shape modes.

We are interested, how the energy flux generated by an incident wave in the ground plate is partitioned between reflected and transmitted outgoing energy fluxes again in the ground plate, see Equation (3.43). To achieve this, one can follow the algorithm proposed in (3.73). The scattering coefficients computed in subsections 3.7.1 and 3.7.3 are used to produce effective scattering coefficients in the stiffened plate. In fact, $S^{(3)}$ and $S^{(4)}$ in Equation (3.46) denote scattering coefficients of a free edge and of a T-junction of composite plates.

First, we consider a composite plate with one stiffener plate mounted onto it. Figure 3.31 shows the energy scattering coefficients and the absolute value of the determinant in Equation (3.58) as a function of the wave number component k_x for a stiffener of length $l = 30$ cm with an incident bending wave at frequency 3000 Hz. The longest bending wavelength at this frequency is equal to 15.7 cm; thus, the short-wavelength approximation discussed in Section 3.6 is validated. Critical wave number components k_x for longitudinal and shear waves remain the same as discussed before, and only bending waves can propagate for $|k_x| > 7.8 \text{ m}^{-1}$. Note that the effective bending energy scattering coefficients are symmetric around $k_x = 0$, whilst in-plane mode related coefficients are slightly asymmetric in $|k_x| \leq 7.8 \text{ m}^{-1}$. Furthermore, one can observe a resonant behaviour of the reflected and transmitted energy flux coefficients t_{BB}^{11} and t_{BB}^{12} at specific k_x values, which related to the resonance condition (3.58). At most of the values of k_x , which satisfy this condition, see Figure 3.31b, an incoming wave is either totally transmitted or reflected, see Figure 3.31a. Critical wave number component values $|k_x| = 3.7$ yield local minima of $|\det(\mathbf{I} - \tilde{\mathbf{P}}\tilde{\rho}_{33}\mathbf{P}\rho_{33})|$, however, the effective bending reflection



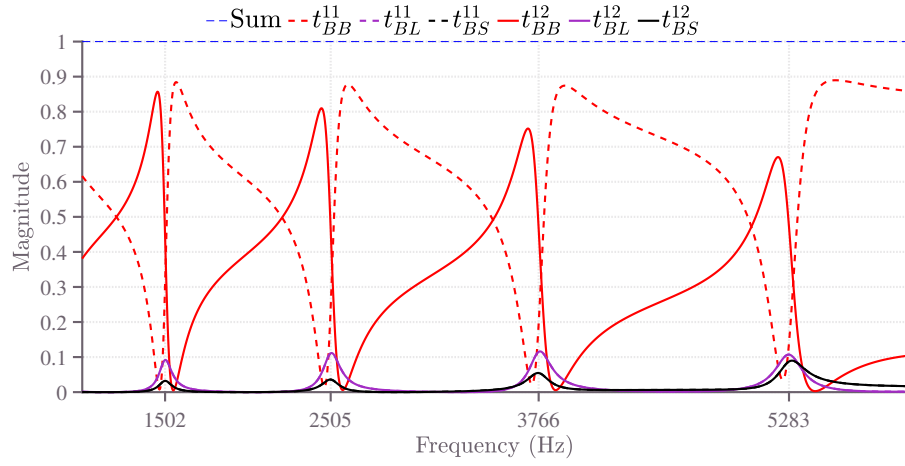
(a) Energy scattering coefficients.



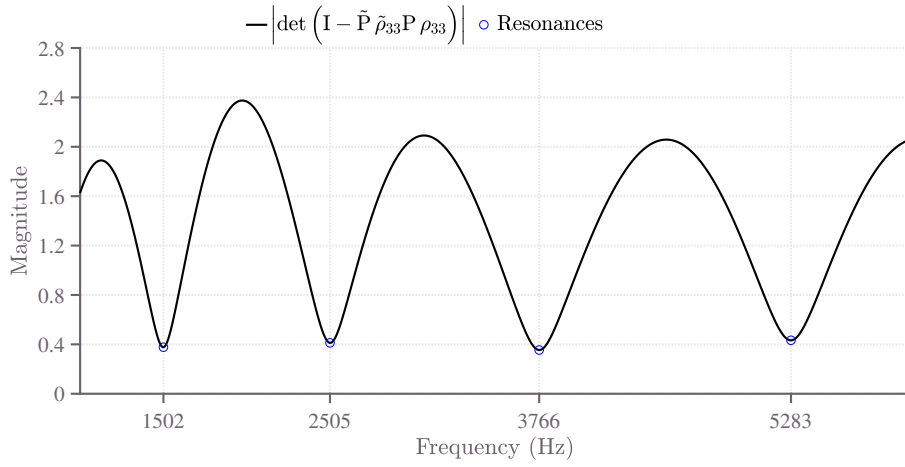
(b) The absolute value of the determinant as in Equation (3.58) and resonances.

Figure 3.31: Energy scattering coefficients and resonance conditions of a stiffened plate for an incident bending wave as a function of wave number component k_x at a frequency 3000 Hz. The length of the stiffener plate is 30 cm. The lamination scheme of plates is $45^\circ / -45^\circ / 45^\circ / -45^\circ / 45^\circ$.

and transmission coefficients do not exhibit resonant behaviour. A similar phenomenon can be observed for the case of fixed wave number component k_x and varying frequency. Figure 3.32 presents effective bending energy scattering coefficients and resonances of Equation (3.58) for the frequency range $1000 \text{ Hz} \leq f \leq 6000 \text{ Hz}$ at $k_x = 0$. A short-wavelength approximation still holds for this frequency range, since the maximum bending wavelength at the lowest frequency $f = 1000 \text{ Hz}$ is $\sim 27 \text{ cm} < 30 \text{ cm}$ - the length of the stiffener plate. Mode conversion to shear and longitudinal waves is present only in the vicinity of the resonant frequencies, becoming more significant as frequency increases. Bending reflection and transmission coefficients behave similarly as



(a) Energy scattering coefficients.

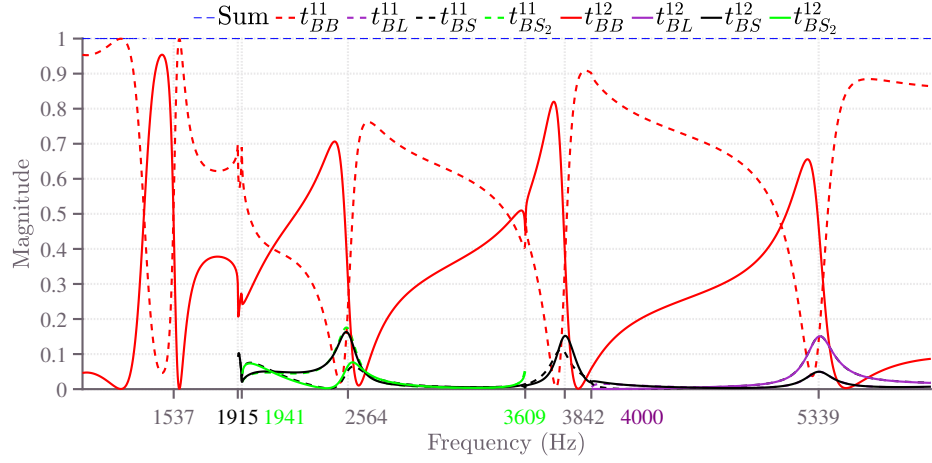


(b) The absolute value of the determinant as in Equation (3.58) and resonances.

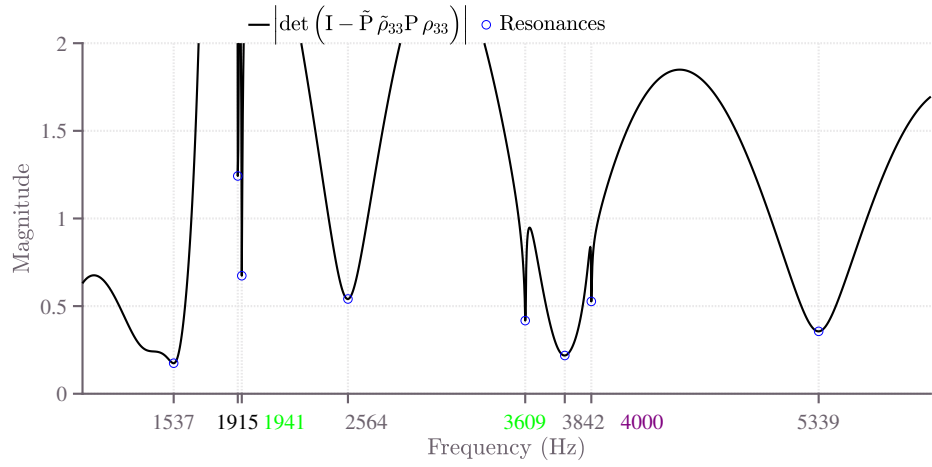
Figure 3.32: Energy scattering coefficients and resonance conditions of a stiffened plate for an incident bending wave as a function of frequency at a wave number component $k_x = 0 \text{ m}^{-1}$. The length of the stiffener plate is 30 cm. The lamination scheme of plates is $45^\circ / -45^\circ / 45^\circ / -45^\circ / 45^\circ$.

in the case of varying k_x , that is, waves are mainly reflected or transmitted except for small coupling to in-plane modes. When $k_x = 5 \text{ m}^{-1}$, total transmission or reflection of the bending energy occurs at the resonant frequency $f = 1537 \text{ Hz}$. Starting at $f = 1915 \text{ Hz}$, an incoming bending wave energy can be reflected and transmitted as both shear modes and longitudinal mode at $f \geq 4000$, see Figure 3.33.

Now, we consider the effective bending energy scattering coefficients of a symmetrically stiffened cross-ply plate as in Figure 3.16. The lamination scheme is the same as before - $0^\circ / 90^\circ / 0^\circ / 90^\circ / 0^\circ$. As mentioned in subsection 3.6.2, this configuration prohibits the generation of the outgoing in-plane modes when



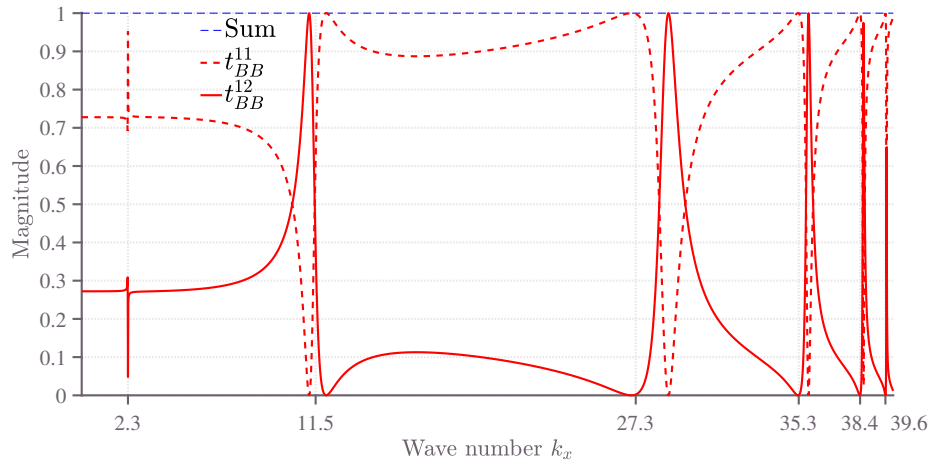
(a) Energy scattering coefficients.



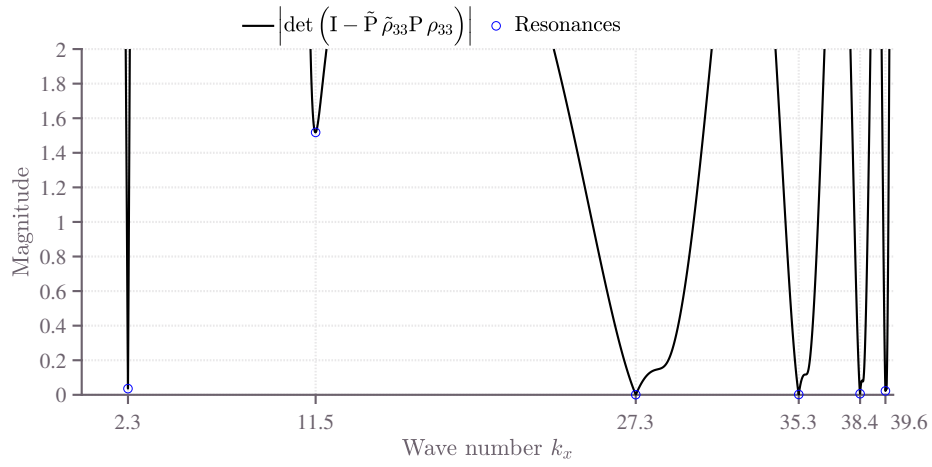
(b) The absolute value of the determinant as in Equation (3.58) and resonances.

Figure 3.33: Energy scattering coefficients and resonance conditions of a stiffened plate for an incident bending wave as a function of frequency at a wave number component $k_x = 5 \text{ m}^{-1}$. The length of the stiffener plate is 30 cm. The lamination scheme of plates is $45^\circ / -45^\circ / 45^\circ / -45^\circ / 45^\circ$.

a bending wave is incident upon this structure. In Figure 3.34, the effective scattering coefficients are plotted with respect to the wave number component k_x at a frequency $f = 3000 \text{ Hz}$. It is clear that there only occurs reflection or transmission of the bending wave without mode conversion. Since the plates considered are specially orthotropic, and scattering coefficients are symmetric around $k_x = 0 \text{ m}^{-1}$, only the right side of the k_x axis is considered. We can note that most of the incoming bending energy is reflected, whilst at the resonant values of k_x , perfect transmission occurs. In Figure 3.35, the effective bending energy scattering coefficients are plotted for the range of frequencies $1000 \text{ Hz} \leq f \leq 6000 \text{ Hz}$ at fixed wave number components



(a) Energy scattering coefficients.



(b) The absolute value of the determinant as in Equation (3.58) and resonances.

Figure 3.34: Energy scattering coefficients and resonance conditions of a symmetrically stiffened plate for an incident bending wave as a function of wave number component k_x at a frequency 3000 Hz. The lengths of the stiffener plates are 30 cm. The lamination scheme of plates is $0^\circ/90^\circ/0^\circ/90^\circ/0^\circ$.

$k_x = 0 \text{ m}^{-1}$ and $k_x = 5 \text{ m}^{-1}$. As opposed to the previous cases, figures for $k_x = 0 \text{ m}^{-1}$ and $k_x = 5 \text{ m}^{-1}$ are similar. Namely, the number of resonant frequencies is the same for both cases, and they are shifted along the frequency axis with increasing k_x . Furthermore, it appears that the scattering coefficients behave in the similar manner at frequencies $f > 1416 \text{ Hz}$, a frequency at which shear waves start to propagate in non-stiffened plate cases, see Figures 3.19 and 3.24. The energy scattering coefficients sum up to one in the absence of damping, as expected.

In the final example, we consider a symmetrically stiffened plate with four pairs of stiffener plates attached to it. The lamination scheme of all plates is

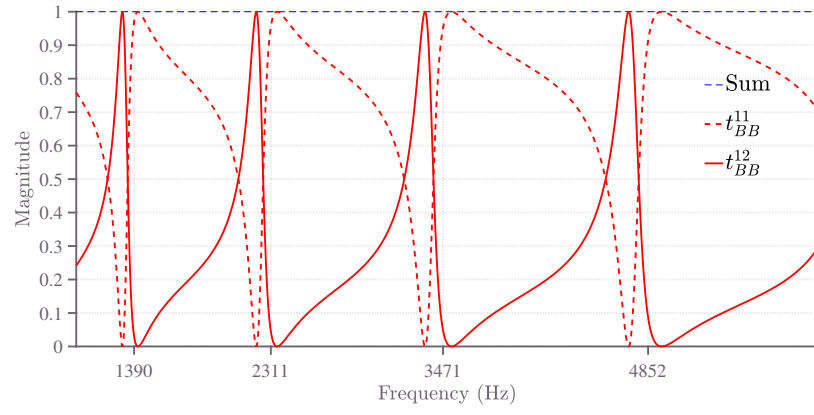
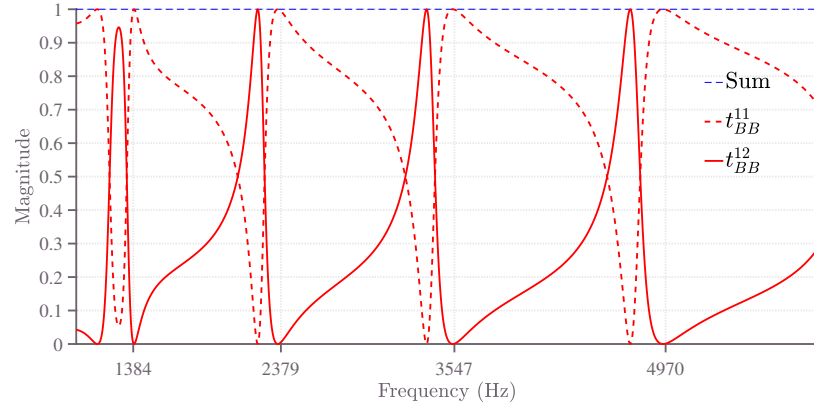
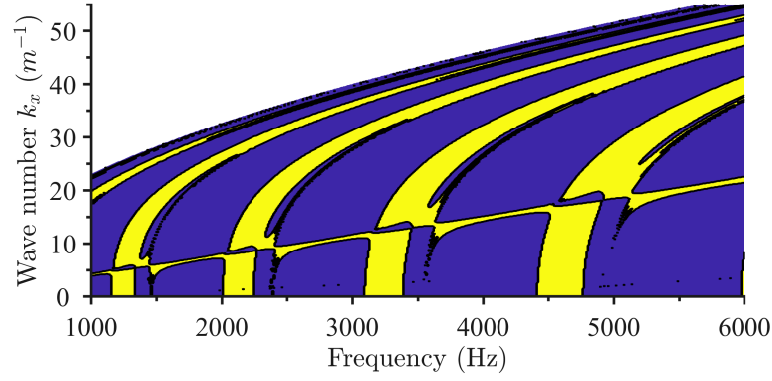
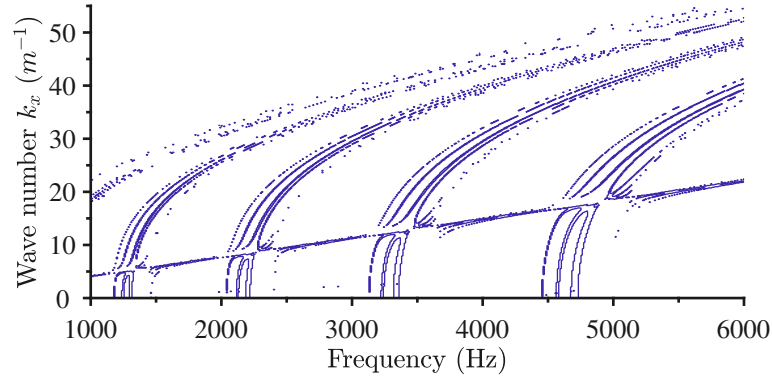
(a) $k_x = 0 \text{ m}^{-1}$.(b) $k_x = 5 \text{ m}^{-1}$.

Figure 3.35: Energy scattering coefficients of a symmetrically stiffened plate for an incident bending wave as a function of frequency at a wave number components $k_x = 0 \text{ m}^{-1}$ and $k_x = 5 \text{ m}^{-1}$. The lengths of the stiffener plates are 30 cm. The lamination scheme of plates is $0^\circ/90^\circ/0^\circ/90^\circ/0^\circ$.

the same as in the previous example. The spacing between pairs of stiffeners is equal to the stiffener lengths 30 cm. In Figure 3.36, the energy scattering coefficients are shown in the form of a contour plot as a function of both the wave number component k_x and frequency f . The wave number component k_x varies between 0 m^{-1} and 57 m^{-1} , whereas frequency is varied from 1000 Hz to 6000 Hz. Resonance conditions are satisfied along the frequency and k_x axes, as before in a stiffened plate with only one or a pair of stiffeners. However, since there are several pairs of stiffeners attached to the ground plate, instead of distinct resonant frequencies and values of k_x , we get so-called *pass* and *stop bands* of frequencies and wave number components k_x , at which total transmission or reflection of an incoming bending wave energy is observed. These bands are represented by blue colours in Figures 3.36a and 3.36b. Several things can



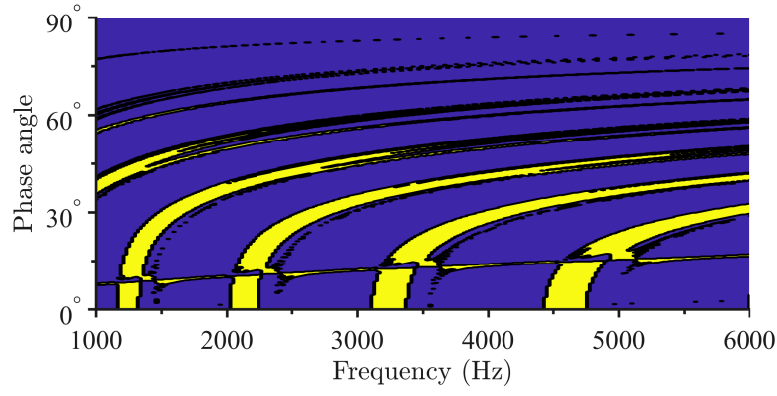
(a) Frequencies and wave numbers k_x at which total energy reflection occurs (coloured in blue). Partial energy transmission are presented in yellow colour.



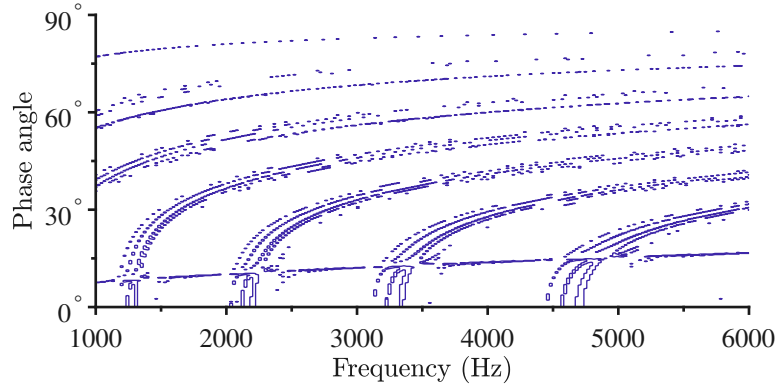
(b) Frequencies and wave numbers k_x at which total energy transmission occurs (coloured in blue).

Figure 3.36: Contour plots of energy scattering coefficients of a symmetrically stiffened plate with four pairs of stiffeners for an incident bending wave as a function of frequency and wave number component k_x . The spacings between stiffeners and lengths of the stiffeners are 30 cm. The lamination scheme of plates is $0^\circ/90^\circ/0^\circ/90^\circ/0^\circ$.

be noted from these figures. Firstly, bending waves can propagate within a bounded surface of possible values of frequency and wave number component k_x defined by Equations (3.4) and (3.5). Secondly, there is no single pass or stop bands at particular frequencies, which is the case in one-dimensional systems like stiffened beams or periodically supported beams. This feature was highlighted for isotropic stiffened plates by Tso and Hansen [196]. However, in that work, stiffeners are assumed to be periodically attached to the ground plate, and using periodic structure theory [113, 186], propagation factors are computed to derive pass and stop bands of the stiffened plate. The method presented in this chapter is suitable for any configuration of the stiffened plate



(a) Frequencies and phase angles $\theta = \arctan(k_x/k_y)$ at which total energy reflection occurs (coloured in blue). Partial energy transmission are presented in yellow colour.



(b) Frequencies and phase angles $\theta = \arctan(k_x/k_y)$ at which total energy transmission occurs (coloured in blue).

Figure 3.37: Contour plots of energy scattering coefficients of a symmetrically stiffened plate with four pairs of stiffeners for an incident bending wave as a function of frequency and phase angle $\theta = \arctan(k_y/k_x)$. The spacings between stiffeners and lengths of the stiffeners are 30 cm. The lamination scheme of plates is $0^\circ/90^\circ/0^\circ/90^\circ/0^\circ$.

with arbitrary material and geometrical parameters. Thirdly, perfect transmission zones presented in blue colour in Figure 3.36b are narrower and more discrete than the yellow zones in Figure 3.36a. This is because yellow zones represent not only total transmission of bending wave energy but also partial transmission, that is, the effective scattering coefficients $0 < t_{BB,\text{eff}}^{12} < 1$. Finally, we can express the effective energy coefficients with respect to the phase angle $\theta = \arctan(k_x/k_y)$, which is varied from 0° to 90° in the cross-ply laminated stiffened plate, see Figure 3.37. This allows to represent pass and stop bands over a fixed (f, θ) plane.

3.8 Conclusion

In this chapter, we have derived the semi-analytical method for computing scattering coefficients of structural junctions made up of composite laminated plates in the line junction approximation. Composite laminates have been modelled in the context of the CLP theory introduced in Section 2.2.

By analysing dispersion relations, we have observed and discussed interesting phenomena such as negative refraction, negative group velocity and evanescent waves decaying along a direction axis inclined to the coordinate axis. Furthermore, we have derived expressions quantifying transmission and reflection coefficients as a function of the frequency and the wave number component k_x . Also, we have computed an effective scattering matrix for a plate with multiple finite stiffeners attached to it. Conditions at which resonances are achieved have been discussed.

Finally, in the numerical case studies, we have computed the scattering coefficients for examples of two and three composite plates joined together in an L and T geometry and effective scattering coefficients for a plate with one and four stiffeners attached to it. The method discussed in this chapter will be used for comparison with a wave finite element approach of extracting reflection and transmission coefficients in Chapter 4.

Chapter 4

Wave and Finite Element Analysis of wave propagation in composite plates

4.1 Introduction

In this chapter, the wave and finite element method based approach for calculation of energy scattering coefficients for arbitrary junctions of composite plates is derived. The wave and finite element method is based on modelling only a periodic segment of a plate using an FE software and applying periodic structure theory to extract wave propagation characteristics such as wave numbers, wave mode shapes and group velocities. Once these are determined for each plate considered, the displacement continuity and force equilibrium conditions at the junction between plates are used to express amplitudes of outgoing waves in terms of amplitudes of incoming waves, thus producing the scattering matrix. The plates considered are modelled using two different finite element types, two-dimensional shell and three-dimensional solid elements. We consider the implication and influence of each element type on the scattering coefficients.

In contrast to the semi-analytical approach presented in Chapter 3, we only require that individual laminas of the plates are homogeneous. Numerical results on selected examples of junctions of composite plates are compared with semi-analytical results from Chapter 3.

This chapter is organised as follows. In Section 4.2, the WFE method for mod-

elling composite plates is briefly reviewed summarising key moments discussed in Section 2.3. An eigenvalue problem whose solutions yield wave numbers and mode shapes is set up. Then, the classification of wave numbers and the wave basis setting are given in Section 4.3. Having established a wave basis representation of displacement and force vectors in individual plates, we combine these solutions with the equations of motion of the joint via application of continuity of displacements and force equilibrium at the joint boundaries, thus producing scattering coefficients in Section 4.4. Finally, Section 4.5 presents numerical case studies for two and three coupled composite plates. The energy scattering coefficients of L junctions of regular cross-ply and angle-ply composite plates are computed and compared with semi-analytical results presented in Section 3.7.

4.2 Summary of the WFE method for composite plates

As presented in Section 2.3, a unit cell of a composite plate with plane dimensions d_x and d_y is modelled using two-dimensional shell elements with a composite lay-up or three-dimensional solid elements stacked up one on top of the other, representing different composite layers. Assuming that the structure undergoes a harmonic vibration with angular frequency ω and no external forces is applied, we can write the governing equation of motion of the unit cell as

$$\mathbf{D}\mathbf{q} = \mathbf{f}, \quad \mathbf{D} = \mathbf{K}(1 + i\eta) - \omega^2\mathbf{M}, \quad (4.1)$$

where $\mathbf{D}$ is the dynamic stiffness matrix with $\mathbf{M}$ and $\mathbf{K}$ being mass and stiffness matrices, respectively. η denotes a uniform structural damping coefficient. This equation relates the displacement and force vectors $\mathbf{q}$ and $\mathbf{f}$. Now, we can use the periodic structure theory [186, 187] to relate displacement and

force sub-vectors along the x axis of the plate as

$$\mathbf{q}_T = \lambda_x \mathbf{q}_B, \quad \mathbf{f}_T = -\lambda_x \mathbf{f}_B, \quad \lambda_x = e^{-ik_x d_x} \quad (4.2)$$

and reduce Equation (4.1) to the form equivalent to Equation (2.64) as

$$\begin{bmatrix} \tilde{\mathbf{D}}_{LL} & \tilde{\mathbf{D}}_{LR} \\ \tilde{\mathbf{D}}_{RL} & \tilde{\mathbf{D}}_{RR} \end{bmatrix} \begin{Bmatrix} \tilde{\mathbf{q}}_L \\ \tilde{\mathbf{q}}_R \end{Bmatrix} = \begin{Bmatrix} \tilde{\mathbf{f}}_L \\ \tilde{\mathbf{f}}_R \end{Bmatrix}, \quad (4.3)$$

where the matrix $\tilde{\mathbf{D}}$ is a dynamically-condensed version of the modified dynamic stiffness matrix $\mathbf{T}^* \mathbf{D} \mathbf{T}$, see subsection 2.3.2. The transformation matrix $\mathbf{T}$ is given by Equation (2.60). The vectors $\tilde{\mathbf{q}}_L$ and $\tilde{\mathbf{q}}_R$ represent the nodal displacement sub-vectors $\mathbf{q}_{LB}$ and $\mathbf{q}_{RB}$ in Figure 2.3, whereas the vectors $\tilde{\mathbf{f}}_L$ and $\tilde{\mathbf{f}}_R$ can be computed as

$$\begin{Bmatrix} \tilde{\mathbf{f}}_L \\ \tilde{\mathbf{f}}_R \end{Bmatrix} = \mathbf{T}^H \mathbf{f} = \begin{Bmatrix} \mathbf{f}_{LB} + \sum_{j=1}^{n_x-1} \lambda_x^{-j/n_x} \mathbf{f}_L^j + \lambda_x^{-1} \mathbf{f}_{LT} \\ \mathbf{f}_{RB} + \sum_{j=1}^{n_x-1} \lambda_x^{-j/n_x} \mathbf{f}_R^j + \lambda_x^{-1} \mathbf{f}_{RT} \end{Bmatrix} \quad (4.4)$$

with n_x denoting the number of mesh cells in the x direction of the periodic cell.

Finally, by applying the periodic structure theory and force equilibrium in the y direction, which can be written as

$$\tilde{\mathbf{q}}_R = \lambda_y \tilde{\mathbf{q}}_L, \quad \tilde{\mathbf{f}}_R = -\lambda_y \tilde{\mathbf{f}}_L, \quad \lambda_y = e^{-ik_y d_y}, \quad (4.5)$$

one can get from Equation (4.3) the following eigenvalue problem for the propagation factor λ_y

$$\mathbf{S} \begin{Bmatrix} \tilde{\mathbf{q}}_L \\ \tilde{\mathbf{f}}_L \end{Bmatrix} = \lambda_y \begin{Bmatrix} \tilde{\mathbf{q}}_L \\ \tilde{\mathbf{f}}_L \end{Bmatrix} \quad \text{with} \quad \mathbf{S} = \begin{bmatrix} -\tilde{\mathbf{D}}_{LR}^{-1} \tilde{\mathbf{D}}_{LL} & \tilde{\mathbf{D}}_{LR}^{-1} \\ -\tilde{\mathbf{D}}_{RL} + \tilde{\mathbf{D}}_{RR} \tilde{\mathbf{D}}_{LR}^{-1} \tilde{\mathbf{D}}_{LL} & -\tilde{\mathbf{D}}_{RR} \tilde{\mathbf{D}}_{LR}^{-1} \end{bmatrix}. \quad (4.6)$$

The dimension of the matrices $\tilde{\mathbf{D}}$ and $\mathbf{S}$ is $2m$, where m is the dimension of the displacement and force vectors $\tilde{\mathbf{q}}_L$ and $\tilde{\mathbf{f}}_L$. Therefore, the solution of

the eigenvalue problem (4.6) consists of $2m$ propagation factors $\lambda_{y,i}$ and the correspondent eigenvectors $\left\{ \phi_{q,i} \ \phi_{f,i} \right\}^T$ provided that the circular frequency ω and the wave number component k_x are fixed. The wave number components $k_{y,i}$ can be computed as

$$k_{y,i} = \ln \left(\frac{\lambda_{y,i}}{-i d_y} \right), \quad i = 1, \dots, 2m. \quad (4.7)$$

Consequently, by varying the wave number component k_x , one can extract wave vector curves (k_x, k_y) for a fixed value of ω . It is worth noting that obtained wave number components $k_{y,i}$ can be real, imaginary or complex, making the correspondent plane waves propagating, evanescent or attenuating in the y direction - more on that in subsection 4.3.1.

4.3 Calculation of wave numbers and group velocity

The solution of the eigenvalue problem (4.6) might be prone to ill-conditioning of the matrix S , since the formulation of this matrix involves calculation of $\tilde{\mathbf{D}}_{LR}^{-1}$ [127]. The equivalent form of the eigenvalue problem can be formulated by eliminating $\tilde{\mathbf{f}}_L$ and $\tilde{\mathbf{f}}_R$ in Equation (4.3) as suggested in [199]:

$$\mathbf{N} \begin{Bmatrix} \tilde{\mathbf{q}}_L \\ \tilde{\mathbf{q}}_R \end{Bmatrix} = \lambda_y \mathbf{L} \begin{Bmatrix} \tilde{\mathbf{q}}_L \\ \tilde{\mathbf{q}}_R \end{Bmatrix}, \quad \mathbf{N} = \begin{bmatrix} 0 & \mathbf{I} \\ -\tilde{\mathbf{D}}_{RL} & -\tilde{\mathbf{D}}_{RR} \end{bmatrix}, \quad \mathbf{L} = \begin{bmatrix} \mathbf{I} & 0 \\ \tilde{\mathbf{D}}_{LL} & \tilde{\mathbf{D}}_{LR} \end{bmatrix}, \quad (4.8)$$

with $\mathbf{I}$ being the m -by- m identity matrix. The matrices $\mathbf{N}$ and $\mathbf{L}$ consist of block parts of the reduced dynamic stiffness matrix $\tilde{\mathbf{D}}$ with no matrix inversion as in Equation (4.6), thus reducing numerical ill-conditioning of the method to some extent. However, there might be a difference of several orders of magnitude between $\mathbf{I}$ and $-\tilde{\mathbf{D}}_{RL(RR)}$ and $\tilde{\mathbf{D}}_{LL(LR)}$; therefore, the condition numbers of the matrices $\mathbf{N}$ and $\mathbf{L}$ can still be large, causing numerical errors in the evaluation of eigenvalues and eigenvectors.

We use the following form of the eigenvalue problem (4.8) proposed by Fan et

al. [157, 158]

$$\tilde{\mathbf{N}} \begin{Bmatrix} \tilde{\mathbf{q}}_L \\ \tilde{\mathbf{q}}_R \end{Bmatrix} = \lambda_y \tilde{\mathbf{L}} \begin{Bmatrix} \tilde{\mathbf{q}}_L \\ \tilde{\mathbf{q}}_R \end{Bmatrix}, \quad \tilde{\mathbf{N}} = \begin{bmatrix} 0 & \sigma \mathbf{I} \\ -\tilde{\mathbf{D}}_{RL} & -\tilde{\mathbf{D}}_{RR} \end{bmatrix}, \quad \tilde{\mathbf{L}} = \begin{bmatrix} \sigma \mathbf{I} & 0 \\ \tilde{\mathbf{D}}_{LL} & \tilde{\mathbf{D}}_{LR} \end{bmatrix}, \quad (4.9)$$

where $\sigma = \frac{\|\tilde{\mathbf{D}}_{RR}\|_2}{m^2}$, $\|\cdot\|_2$ representing the largest singular value of a matrix. This formulation is an improved version of the eigenvalue problem (4.8), and the factor σ is introduced here to reduce the condition number of the matrices $\mathbf{N}$ and $\mathbf{L}$. However, the formulations are still equivalent, and the eigenvalue solutions are the same. In fact, writing the characteristic equation of the eigenvalue problem (4.9) yields

$$0 = \det(\tilde{\mathbf{N}} - \lambda_y \tilde{\mathbf{L}}) = \sigma^m \det(\mathbf{N} - \lambda_y \mathbf{L}) = \det(\mathbf{N} - \lambda_y \mathbf{L}) = 0, \quad (4.10)$$

where we have applied a property of the determinant which tells that if a row of a matrix is multiplied by a non-zero constant, the determinant of that matrix is multiplied by the same constant. Note that eigenvectors in Equation (4.9) consist of left and right nodal displacements sub-vectors. To compute the nodal force sub-vectors $\tilde{\mathbf{f}}_L$, one can apply the following transformation

$$\begin{Bmatrix} \tilde{\mathbf{q}}_L \\ \tilde{\mathbf{f}}_L \end{Bmatrix} = \begin{bmatrix} \mathbf{I} & 0 \\ \tilde{\mathbf{D}}_{LL} & \tilde{\mathbf{D}}_{LR} \end{bmatrix} \begin{Bmatrix} \tilde{\mathbf{q}}_L \\ \tilde{\mathbf{q}}_R \end{Bmatrix}. \quad (4.11)$$

4.3.1 Incoming, outgoing waves and wave basis

Eigensolutions of Equation (4.8) can be separated into m pairs of roots, $\lambda_{y,j}^\pm$; these correspond to negative or incoming waves (with superscripts “−”) and positive or outgoing waves (with superscripts “+”). Furthermore, the waves can be categorised as propagating, evanescent or attenuating. The standard wave classification [80, 160, 200] consists of checking whether $|\lambda_{y,j}| \leq 1$ to identify outgoing or incoming waves, respectively. The transfer matrix $\mathbf{S}$ in Equation (4.6) is assumed to be symplectic, hence the eigenvalues of Equa-

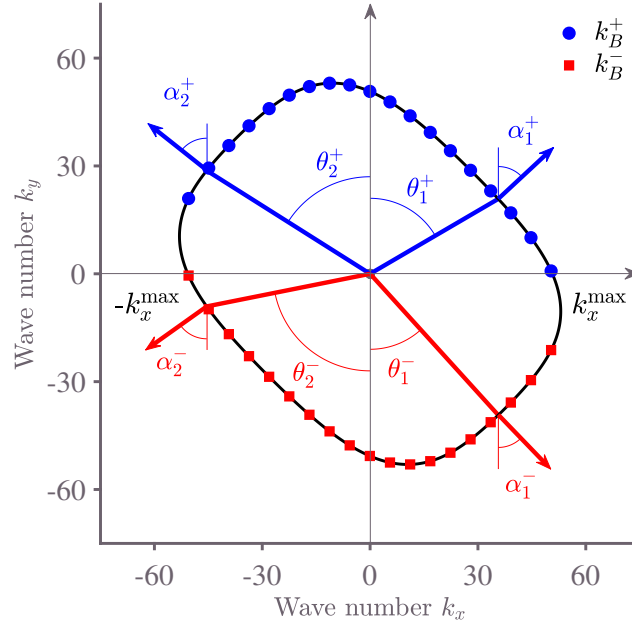


Figure 4.1: Bending wave vector curve in a $45^\circ / -45^\circ / 45^\circ / -45^\circ / 45^\circ$ composite plate at a frequency 3000 Hz. Blue dots represent wave numbers related to outgoing waves while red squares correspond to incoming waves. Angles $\alpha_{1(2)}$ and $\theta_{1(2)}$ represent group and wave vector or phase angles, respectively. The wave number component $k_x^{\max}$ is equal to 53 m^{-1} .

tions (4.6) and (4.8) must appear in pairs $(\lambda_{y,j}, 1/\lambda_{y,j})$. It follows that $k_{y,j}^+$ must equal to $-k_{y,j}^-$, and this is true for isotropic and some special types of composite plates, e.g. cross-ply laminates, which consist of layers with ply direction angles 0° or 90° . However, in general, the transfer matrix $\mathbf{S}$ is not symplectic because it depends on the wave number component k_x via the transformation matrix $\mathbf{T}$, and $k_{y,j}^+ \neq -k_{y,j}^-$. Figure 4.1 shows an example of a bending wave vector curve, where one can see the inequality between incoming and outgoing wave number components k_y represented by red squares and blue dots, respectively. This inequality was demonstrated for a similar wave vector curve by Taupin et al. [201] using the Semi-Analytical Finite Element method. Furthermore, one can note the difference between phase and group angles in composite plates shown with $\theta_{1,2}^\pm$ and $\alpha_{1,2}^\pm$, respectively. This property was highlighted and discussed previously using the semi-analytical approach in Section 3.3.

To correctly categorise incoming/outgoing propagating/attenuating waves, we

establish the following algorithm for the j th wave

$$\begin{aligned}
& \text{if } \text{Re}(k_{y,j}) > c \text{ Im}(k_{y,j}) \Rightarrow \text{the wave } j \text{ is propagating} \\
& \quad \text{if } \text{Re}(i\omega\phi_{q,j}^* \phi_{f,j}) > 0 \Rightarrow \text{the wave } j \text{ is outgoing} \\
& \quad \text{else} \Rightarrow \text{the wave } j \text{ is incoming} \\
& \text{else} \Rightarrow \text{the wave } j \text{ is attenuating} \\
& \quad \text{if } \text{Im}(k_{y,j}) < 0 \Rightarrow \text{the wave } j \text{ is outgoing} \\
& \quad \text{else} \Rightarrow \text{the wave } j \text{ is incoming,}
\end{aligned} \tag{4.12}$$

where $\text{Re}(i\omega\phi_{q,j}^* \phi_{f,j})$ is the energy flux of the j th wave in the positive y direction, and c is a real parameter defined empirically and used to separate propagating from attenuating waves; in this work, it is equal to 10.

Categorising correctly whether a wave is incoming/outgoing and propagating/evanescent/attenuating is not enough to further proceed with the calculation of scattering coefficients. In fact, for a range of frequencies and wave number components k_x , we obtain a set of unclassified branches of propagating, evanescent and attenuating waves. To identify the eigensolutions of Equation (4.9) corresponding to the same wave type, we apply the so-called MAC criterion [132, 202, 203]. For an eigenvector solution $\Phi_i = \left\{ \phi_{q,i}^\pm \quad \phi_{f,i}^\pm \right\}^T$ defined at frequency ω for a fixed wave number component k_x , we find an eigenvector solution $\Phi_j = \left\{ \phi_{q,i}^\pm \quad \phi_{f,i}^\pm \right\}^T$ at frequency $\omega + d\omega$ with sufficiently small $d\omega$ such that

$$M^\pm(\omega) = \frac{(\Phi_i^T(\omega)\Phi_j^*(\omega + d\omega))(\Phi_j^T(\omega)\Phi_i^*(\omega + d\omega))}{(\Phi_i^T(\omega)\Phi_i^*(\omega + d\omega))(\Phi_j^T(\omega)\Phi_j^*(\omega + d\omega))} \tag{4.13}$$

is maximised. The same criterion can be utilised to classify wave types for a range of wave number components k_x and a fixed frequency with corresponding step size dk_x being sufficiently small.

Once the appropriate pairs of wave number components $k_{y,j}^\pm$ and wave mode shapes $\left\{ \phi_{q,j}^\pm \quad \phi_{f,j}^\pm \right\}^T$ are determined, we can express nodal displacements and

forces in the basis of wave mode shapes as follows

$$\begin{Bmatrix} \tilde{\mathbf{q}}_L \\ \tilde{\mathbf{f}}_L \end{Bmatrix} = \sum_{j=1}^m \left(a_j^+ \begin{Bmatrix} \phi_{q,j}^+ \\ \phi_{f,j}^+ \end{Bmatrix} + a_j^- \begin{Bmatrix} \phi_{q,j}^- \\ \phi_{f,j}^- \end{Bmatrix} \right) = \begin{Bmatrix} \Phi_q^+ \mathbf{a}^+ + \Phi_q^- \mathbf{a}^- \\ \Phi_f^+ \mathbf{a}^+ + \Phi_f^- \mathbf{a}^- \end{Bmatrix}, \quad (4.14)$$

where a_j^+ and a_j^- are the amplitudes of the j th outgoing and incoming waves. Using the expression of nodal displacements and forces in terms of wave mode shapes, we are now able to compute the energy scattering matrix at the junction of several plates which will be done in the next section.

4.3.2 Group velocity estimation

Calculation of the group velocity vector is essential for determining directions of rays carrying the wave energy. It is also used to relate the energy density to the velocity field in Chapter 2. Furthermore, it is present in the calculation of spatial density integrals for composite structures in Chapter 5.

As noted in subsection 3.3.1, the group velocity vector $\mathbf{c}_g$ is defined as

$$\mathbf{c}_g = \begin{Bmatrix} c_{g,x} & c_{g,y} \end{Bmatrix}^T = \begin{Bmatrix} \frac{\partial \omega}{\partial k_x} & \frac{\partial \omega}{\partial k_y} \end{Bmatrix}^T. \quad (4.15)$$

Since $k_x = k \cos \theta$ and $k_y = k \sin \theta$, where θ is the wave vector polar angle, the partial derivatives $\frac{\partial \omega}{\partial k}$ and $\frac{\partial \omega}{\partial \theta}$ can be expressed in the form

$$\begin{aligned} \frac{\partial \omega}{\partial k} &= \frac{\partial \omega}{\partial k_x} \frac{\partial k_x}{\partial k} + \frac{\partial \omega}{\partial k_y} \frac{\partial k_y}{\partial k} = \frac{\partial \omega}{\partial k_x} \cos \theta + \frac{\partial \omega}{\partial k_y} \sin \theta \\ \frac{\partial \omega}{\partial \theta} &= \frac{\partial \omega}{\partial k_x} \frac{\partial k_x}{\partial \theta} + \frac{\partial \omega}{\partial k_y} \frac{\partial k_y}{\partial \theta} = -\frac{\partial \omega}{\partial k_x} k \sin \theta + \frac{\partial \omega}{\partial k_y} k \cos \theta \end{aligned} \quad (4.16)$$

It then follows that x and y components of the group velocity vector $\mathbf{c}_g$ may be written as

$$\begin{Bmatrix} c_{g,x} \\ c_{g,y} \end{Bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{Bmatrix} \frac{\partial \omega}{\partial k} \\ \frac{\partial \omega}{k \partial \theta} \end{Bmatrix}. \quad (4.17)$$

Considering the wave vector curve in the form $\omega(k, \theta) = \omega_0$ with a fixed value ω_0 , we can express $\frac{\partial \omega}{\partial \theta}$ in terms of $\frac{\partial \omega}{\partial k}$ and $\frac{\partial k}{\partial \theta}$ using a chain rule as follows

$$0 = d\omega = \frac{\partial \omega}{\partial k} \frac{\partial k}{\partial \theta} + \frac{\partial \omega}{\partial \theta} \Rightarrow \frac{\partial \omega}{\partial \theta} = -\frac{\partial \omega}{\partial k} \frac{\partial k}{\partial \theta}. \quad (4.18)$$

Now, introducing Equation (4.18) into Equation (4.17) yields the final expressions for the group velocity vector components

$$\begin{pmatrix} c_{g,x} \\ c_{g,y} \end{pmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{pmatrix} \frac{\partial \omega}{\partial k} \\ -\frac{\partial \omega}{k \partial k} \frac{\partial k}{\partial \theta} \end{pmatrix}. \quad (4.19)$$

Now, $\frac{\partial \omega}{\partial k}$ and $\frac{\partial k}{\partial \theta}$ need be computed for a polar angle $\theta = \arctan(k_y/k_x)$, hence for a wave number component k_x , and there are several methods to accomplish that.

Firstly, one can approximate these partial derivatives by finite differences as

$$\begin{aligned} \frac{\partial \omega}{\partial k}(k_x) \big|_{\omega=\omega_1} &\approx \frac{\Delta \omega}{k \big|_{\omega_1+\Delta \omega} - k \big|_{\omega_1}}, \\ \frac{\partial k}{\partial \theta}(k_x) \big|_{\theta=\theta_1} &\approx \frac{k \big|_{\theta_1+\Delta \theta} - k \big|_{\theta_1}}{\Delta \theta}, \end{aligned} \quad (4.20)$$

where $\Delta \omega$ and $\Delta \theta$ are sufficiently small increment values of frequency and polar angle. The second equation in (4.20) can be easily utilised in the context of the presented wave finite element method, since wave vector curves are computed at the fixed frequency - an example of a bending wave vector curve is shown in Figure 4.1. However, the first equation in (4.20) requires wave vector solutions at two very close frequencies, ω_1 and $\omega_1 + \Delta \omega$, which increases the computational cost of the approach.

Lastly, one can approximate these derivatives by performing sensitivity analysis on the eigenvalue problem (4.9) in a similar way as in [140] and [137]. Nonetheless, this is beyond the scope of this thesis; therefore, the finite difference method is used to approximate partial derivatives.

Finally, one can calculate the direction angle of the energy flow as

$$\alpha = \arctan \left(\frac{c_{g,y}}{c_{g,x}} \right). \quad (4.21)$$

4.4 Computation of scattering coefficients at junctions of composite plates

In this section, the energy scattering coefficients of the junction of several composite plates are derived. There are three cases considered. Firstly, a wave energy reflection occurring at the free edge of the composite plate is described. Secondly, we compute the wave energy reflection and transmission at the shared edge between plates modelled with SHELL181 elements. Finally, a case of a solid joint connecting several plates modelled with SOLID185 elements is regarded. In all cases, boundary conditions at the junction, such as continuity of displacement vectors and force equilibrium, are postulated.

4.4.1 Reflection matrix for a single plate system

This subsection presents the case of wave energy scattering at the boundary of a composite plate. The boundary can be an edge placed along the x axis for shell elements and a face lying on the (x, z) plane for solid elements. Since we consider a single plate, an incoming wave is fully reflected at the boundary with no transmission, although coupling to different modes can occur. Any boundary condition of a plate can be represented in terms of nodal displacements and forces as follows [153]

$$\mathbf{C}_q \tilde{\mathbf{q}}_L + \mathbf{C}_f \tilde{\mathbf{f}}_L = 0. \quad (4.22)$$

For instance, if free boundary conditions are considered, i.e. all tractions at the boundary are equal to zero, then $\mathbf{C}_q = 0$ and $\mathbf{C}_f = \mathbf{I}$. Furthermore, the same formulation with appropriate matrices $\mathbf{C}_q$ and $\mathbf{C}_f$ can describe the case of the fixed edge shared between plates modelled with SHELL181 elements, more on this in subsection 4.4.2. Introducing Equation (4.14) into Equation (4.22) yields

$$\mathbf{a}^+ = \mathbf{s} \mathbf{a}^-, \quad \text{with} \quad \mathbf{s} = -(\mathbf{C}_q \Phi_q^+ + \mathbf{C}_f \Phi_f^+)^{-1} (\mathbf{C}_q \Phi_q^- + \mathbf{C}_f \Phi_f^-). \quad (4.23)$$

The expression for $\mathbf{s}$ defines a $m \times m$ scattering matrix relating the amplitudes $\mathbf{a}^-$ and $\mathbf{a}^+$ of incoming and outgoing waves, respectively. Specifically, we write the scattering matrix elements in the form $s_{ij}^{nm}(\omega, k_x)$, which relates an incoming wave i in the plate n and a reflected or transmitted wave j in the plate m at angular frequency ω and wave number component k_x . To compute the energy scattering coefficients, one needs to compute the outgoing wave energy flux ratio over incoming wave energy flux. For the associated energy fluxes, we obtain the energy scattering coefficients as

$$t_{ij}^{nm}(\omega, k_x) = \begin{cases} \frac{J_{j,m}^+}{J_{i,n}^-} |s_{ij}^{nm}|^2 & \text{if wave } j \text{ is propagating.} \\ 0 & \text{otherwise.} \end{cases}, \quad (4.24)$$

where

$$\begin{cases} J_{i,n}^- &= |\operatorname{Re}(\mathrm{i}\omega \phi_{q,i,n}^{-*} \phi_{f,i,n}^-)|. \\ J_{j,m}^+ &= |\operatorname{Re}(\mathrm{i}\omega \phi_{q,j,m}^{+*} \phi_{f,j,m}^+)|. \end{cases} \quad (4.25)$$

In the absence of damping, total energy must be conserved, hence the sum of energy scattering coefficients over the outgoing modes equals one, that is,

$$\sum_{m=1}^N \sum_j t_{ij}^{nm} = 1. \quad (4.26)$$

Note that since in this subsection we consider only one plate, the energy scattering coefficients have the form $t_{ij}^{11}(\omega, k_x)$, that is, only wave energy reflection can occur with or without coupling to other wave types.

4.4.2 Scattering at the shared edge between shell plates

In this subsection, we consider N composite plates modelled using SHELL181 elements connected via a shared edge, see Figure 4.2. The local y axes of all plates are directed away from the shared edge. The local x axes of all plates are all coinciding. The displacement continuity and force equilibrium conditions must be fulfilled at the shared edge. Taking the first plate as a reference plate,

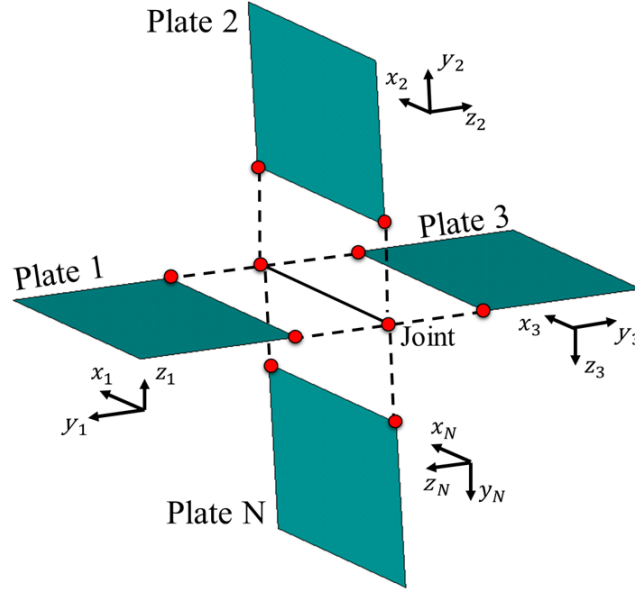


Figure 4.2: A schematic representation of N plates connected via the shared edge. The plates are modelled using two-dimensional SHELL181 elements. The x axis of all plates is the same.

we write

$$\tilde{\mathbf{q}}_{L,1} = \mathbf{R}_2 \tilde{\mathbf{q}}_{L,2} = \dots = \mathbf{R}_N \tilde{\mathbf{q}}_{L,N}, \quad (4.27)$$

where $\tilde{\mathbf{q}}_{L,k}$ denotes the nodal displacement vector of the k th plates, and $\mathbf{R}_k, k = 2, \dots, N$ is a matrix that transforms the local coordinate system of the k th plate to the coordinate system of the first plate. Since the local x_k axis is aligned with the x_1 axis, the transformation matrix $\mathbf{R}_k$ can be written as

$$\mathbf{R}_k = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos(\psi_k) & -\sin(\psi_k) & 0 & 0 & 0 \\ 0 & \sin(\psi_k) & \cos(\psi_k) & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos(\psi_k) & -\sin(\psi_k) \\ 0 & 0 & 0 & 0 & \sin(\psi_k) & \cos(\psi_k) \end{bmatrix} \quad (4.28)$$

where ψ_k denotes the angle of rotation between y_k and y_1 . The matrices R_k are 6-by-6 since there are 6 degrees of freedom in $\mathbf{q}_{L,k}$ for plate modelled with SHELL181 elements. The sum of forces at the shared edge must be zero,

therefore, it follows that

$$\tilde{\mathbf{f}}_{L,1} + \sum_{k=2}^N \mathbf{R}_k \tilde{\mathbf{f}}_{L,k} = 0. \quad (4.29)$$

Now, expressing the nodal displacement and force vectors in the basis of wave mode shapes in each plate using Equation (4.14), we can write

$$\mathbf{C}^+ \mathbf{A}^+ + \mathbf{C}^- \mathbf{A}^- = 0, \quad (4.30)$$

where

$$\mathbf{C}^\pm = \begin{bmatrix} \Phi_{f,1}^\pm & \mathbf{R}_2 \Phi_{f,2}^\pm & \mathbf{R}_3 \Phi_{f,3}^\pm & \cdots & \mathbf{R}_N \Phi_{f,N}^\pm \\ \Phi_{q,1}^\pm & -\mathbf{R}_2 \Phi_{q,2}^\pm & 0 & \cdots & 0 \\ \Phi_{q,1}^\pm & 0 & -\mathbf{R}_3 \Phi_{q,3}^\pm & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Phi_{q,1}^\pm & \cdots & \cdots & \cdots & -\mathbf{R}_N \Phi_{q,N}^\pm \end{bmatrix}, \quad \mathbf{A}^\pm = \begin{Bmatrix} \mathbf{a}_1^\pm \\ \vdots \\ \mathbf{a}_N^\pm \end{Bmatrix}. \quad (4.31)$$

The vectors $\mathbf{A}^\pm$ consist of amplitudes of incoming and outgoing waves $\mathbf{a}_k^\pm$ of the k th plate, $k = 1, \dots, N$. Finally, it follows from Equation (4.30)

$$\mathbf{A}^+ = \mathbf{s} \mathbf{A}^-, \quad \mathbf{s} = -(\mathbf{C}^+)^{-1} \mathbf{C}^-, \quad (4.32)$$

where $\mathbf{s}$ defines a $6N \times 6N$ scattering matrix. Accordingly, we can compute the energy scattering coefficients using Equation (4.24).

4.4.3 Hybrid FE/WFE method of computation of scattering coefficients in three-dimensional solid plates

This subsection considers N composite plates modelled with SOLID185 elements that are connected together via a solid joint element, see Figure 4.3. The local y axes of all plates are directed away from the joint, whereas the x axes are aligned with the x_J axis of the joint. The joint is assumed to be periodic in the x_J axis. According to Figure 4.4, the nodal displacements vector

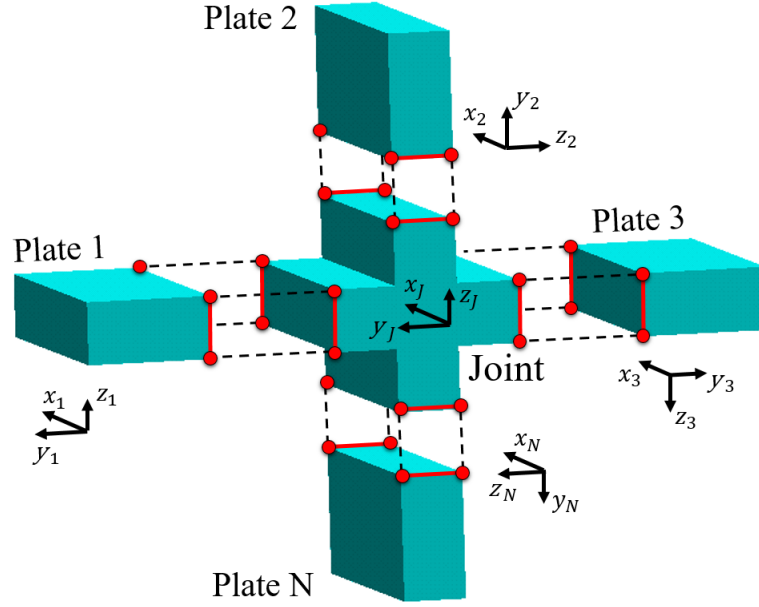


Figure 4.3: A schematic representation of the joint connecting N plates. The x axis of all plates are aligned with the x_J axis of the joint.

of the joint $\mathbf{q}_J$ is organised in the following way

$$\mathbf{q}_J = \left\{ \begin{matrix} \mathbf{q}_E & \mathbf{q}_O \end{matrix} \right\}^T, \quad \begin{matrix} \mathbf{q}_E = \left\{ \mathbf{q}_{LB,1} & \mathbf{q}_{L,1} & \mathbf{q}_{LT,1} & \dots & \mathbf{q}_{LB,N} & \mathbf{q}_{L,N} & \mathbf{q}_{LT,N} \right\} \\ \mathbf{q}_O = \left\{ \mathbf{q}_B & \mathbf{q}_I & \mathbf{q}_T \right\} \end{matrix} \quad (4.33)$$

The nodal displacement vector $\mathbf{q}_O$ consists of degrees of freedom of internal nodes. It is required that the node arrangement on the face containing $\mathbf{q}_{LB,k}$, $\mathbf{q}_{L,k}$ and $\mathbf{q}_{LT,k}$, $k = 1, \dots, N$ is coherent with one on the left face of the k th plate. We recall that m_k and $n_{x,k}$ are the number of degrees of freedom per edge of the k th plate and the number of mesh cells in the x direction of the k th plate, accordingly. Therefore, we enforce that $|\mathbf{q}_{LB,k}| = |\mathbf{q}_{L,k}| = |\mathbf{q}_{LT,k}| = |\tilde{\mathbf{q}}_{L,k}| = m_k$ and $n_{J,k} = n_{x,k}$, where $n_{J,k}$ is the number of mesh cells in the joint element along the x direction of the face containing $\mathbf{q}_{LB,k}$, $\mathbf{q}_{L,k}$ and $\mathbf{q}_{LT,k}$. The nodal forces vector $\mathbf{f}_J$ is arranged in the same way.

The governing equations of motion of the joint are of the same form as in Equation (4.1)

$$\mathbf{D}_J \mathbf{q}_J = \mathbf{f}_J, \quad \mathbf{D}_J = \mathbf{K}_J(1 + i\eta) - \omega^2 \mathbf{M}_J, \quad (4.34)$$

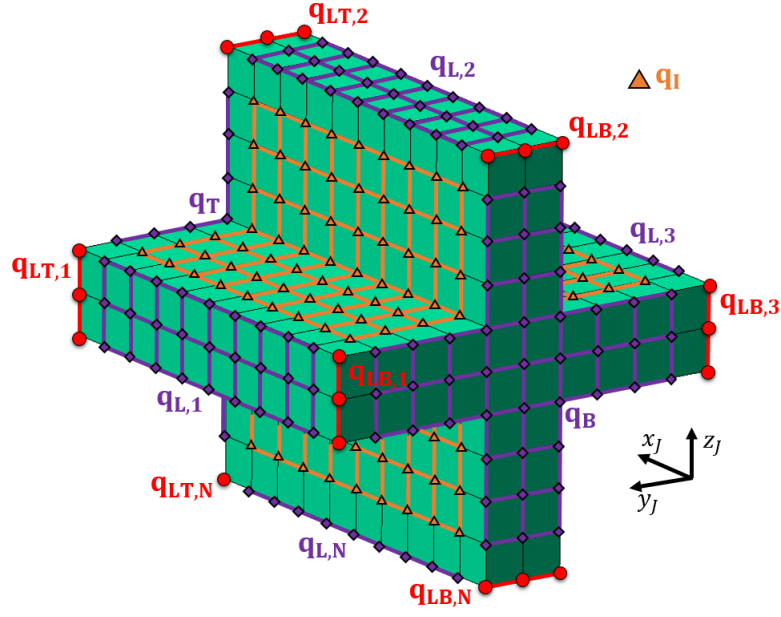


Figure 4.4: A finite element model of the joint connecting N plates. The degrees of freedom are grouped into internal $\mathbf{q}_I$, edge $\mathbf{q}_B, \mathbf{q}_T, \mathbf{q}_{L,1}, \dots, \mathbf{q}_{L,N}$ and corner $\mathbf{q}_{LB,1}, \dots, \mathbf{q}_{LB,N}, \mathbf{q}_{LT,1}, \dots, \mathbf{q}_{LT,N}$ degrees of freedom.

where $\mathbf{K}_J$ and $\mathbf{M}_J$ are the stiffness and mass matrices of the joint. When no external forces are applied on the internal nodes of the joint, Equation (4.34) can be written as

$$\begin{bmatrix} \mathbf{D}_{EE} & \mathbf{D}_{EI} \\ \mathbf{D}_{IE} & \mathbf{D}_{II} \end{bmatrix} \begin{Bmatrix} \mathbf{q}_E \\ \mathbf{q}_O \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_E \\ 0 \end{Bmatrix}, \quad (4.35)$$

where $\mathbf{f}_O = 0$. Consequently, one can remove internal degrees of freedom using dynamic condensation as

$$\mathbf{D}_{J,\text{cond}} \mathbf{q}_E = \mathbf{f}_E, \quad \mathbf{D}_{J,\text{cond}} = \mathbf{D}_{EE} - \mathbf{D}_{EI} \mathbf{D}_{II}^{-1} \mathbf{D}_{IE}. \quad (4.36)$$

Now, following the similar approach as in subsection 2.3.2, we apply periodic structure theory on the nodal displacement sub-vectors along the x_J axis as

follows

$$\mathbf{q}_{J,red} = \mathbf{T}_J \mathbf{q}_E, \quad \mathbf{q}_{J,red} = \left\{ \mathbf{q}_{LB,1} \quad \cdots \quad \mathbf{q}_{LB,N} \right\}^T$$

$$\mathbf{T}_J = \begin{bmatrix} \mathbf{I}_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_x^{j_1/n_{J,1}} \mathbf{I}_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_x \mathbf{I}_1 & 0 & \cdots & 0 \\ 0 & \mathbf{I}_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \lambda_x^{j_2/n_{J,2}} \mathbf{I}_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \lambda_x \mathbf{I}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{I}_N \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_x^{j_N/n_{J,N}} \mathbf{I}_N \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_x \mathbf{I}_N \end{bmatrix}, \quad (4.37)$$

where $\mathbf{I}_k$ is the m_k -by- m_k identity matrix and $j_k = 1, \dots, n_{J,k} - 1$, $k = 1, \dots, N$.

Using Equation (4.37), we can reduce the dimension of Equation (4.36) as

$$\tilde{\mathbf{D}}_J \mathbf{q}_{J,red} = \mathbf{f}_{J,red}, \quad \tilde{\mathbf{D}}_J = \mathbf{T}_J^H \mathbf{D}_{J,cond} \mathbf{T}_J \quad (4.38)$$

with

$$\mathbf{f}_{J,red} = \begin{Bmatrix} \tilde{\mathbf{f}}_{LB,1} \\ \vdots \\ \tilde{\mathbf{f}}_{LB,N} \end{Bmatrix} = \mathbf{T}_J^H \mathbf{f}_J = \begin{Bmatrix} \mathbf{f}_{LB,1} + \sum_{j_1=1}^{n_{J,1}-1} \lambda_x^{-j_1/n_{J,1}} \mathbf{f}_{L,1}^{j_1} + \lambda_x^{-1} \mathbf{f}_{LT,1} \\ \vdots \\ \mathbf{f}_{LB,N} + \sum_{j_N=1}^{n_{J,N}-1} \lambda_x^{-j_N/n_{J,N}} \mathbf{f}_{L,N}^{j_N} + \lambda_x^{-1} \mathbf{f}_{LT,N} \end{Bmatrix}, \quad (4.39)$$

where $\mathbf{f}_{L,i}^{j_i}$ denote force sub-vectors of nodes placed at $x_{j_i} = d_x j_i / n_{L,i}$, $j_i = 1, 2, \dots, n_{L,i} - 1$ and $i = 1, \dots, N$. Equation (4.38) considers only interface nodes, that is, nodes shared between the plates and the joint. The force equilibrium and displacement continuity conditions must be satisfied at these nodes. Specifically, $\mathbf{q}_{LB,k}$ and $\tilde{\mathbf{f}}_{LB,k}$, $k = 1, \dots, N$ denote nodal displacements and forces at nodes shared between the k th plate and the joint. Therefore, one can represent $\mathbf{q}_{LB,k}$ and $\tilde{\mathbf{f}}_{LB,k}$ in the wave mode shapes basis of the k th plate using Equation (4.14) as follows

$$\begin{aligned}\mathbf{q}_{LB,k} &= \mathbf{R}_k \left(\Phi_{\mathbf{q},k}^+ \mathbf{a}_k^+ + \Phi_{\mathbf{q},k}^- \mathbf{a}_k^- \right) \\ \tilde{\mathbf{f}}_{LB,k} &= \mathbf{R}_k \left(\Phi_{\mathbf{f},k}^+ \mathbf{a}_k^+ + \Phi_{\mathbf{f},k}^- \mathbf{a}_k^- \right)\end{aligned}\quad (4.40)$$

where $\mathbf{R}_k$ is the matrix that transforms the local coordinate system of the k th plate to the global coordinate system. Since the local x_k axis is aligned with the global x_J axis, the transformation matrix $\mathbf{R}_k$ can be written as

$$\mathbf{R}_k = \begin{bmatrix} \mathbf{R}_{\text{node}} & 0 & \cdots & 0 \\ 0 & \mathbf{R}_{\text{node}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{R}_{\text{node}} \end{bmatrix}_{m_k \times m_k}, \quad \mathbf{R}_{\text{node}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\psi_k) & -\sin(\psi_k) \\ 0 & \sin(\psi_k) & \cos(\psi_k) \end{bmatrix}\quad (4.41)$$

where ψ_k denotes the angle of rotation between y_k and y_J . Now, we can concatenate individual expressions (4.40) for $\mathbf{q}_{LB,k}$ and $\mathbf{f}_{LB,k}$ to express $\mathbf{q}_{J,\text{red}}$ and $\mathbf{f}_{J,\text{red}}$ in Equation (4.38)

$$\begin{aligned}\mathbf{q}_{J,\text{red}} &= \mathbf{R} \left(\Phi_Q^+ \mathbf{A}^+ + \Phi_Q^- \mathbf{A}^- \right) \\ \mathbf{f}_{J,\text{red}} &= \mathbf{R} \left(\Phi_F^+ \mathbf{A}^+ + \Phi_F^- \mathbf{A}^- \right)\end{aligned}\quad (4.42)$$

with

$$\mathbf{R} = \begin{bmatrix} \mathbf{R}_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbf{R}_N \end{bmatrix}, \quad \Phi_Q^\pm = \begin{bmatrix} \Phi_{\mathbf{q},1}^\pm & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Phi_{\mathbf{q},N}^\pm \end{bmatrix}, \quad \Phi_F^\pm = \begin{bmatrix} \Phi_{\mathbf{f},1}^\pm & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Phi_{\mathbf{f},N}^\pm \end{bmatrix}. \quad (4.43)$$

Substituting Equation (4.42) in Equation (4.38), we obtain

$$\mathbf{A}^+ = \mathbf{s} \mathbf{A}^-, \quad \text{with} \quad \mathbf{s} = - \left(\tilde{\mathbf{D}}_J \mathbf{R} \Phi_Q^+ - \mathbf{R} \Phi_F^+ \right)^{-1} \left(\tilde{\mathbf{D}}_J \mathbf{R} \Phi_Q^- - \mathbf{R} \Phi_F^- \right). \quad (4.44)$$

As in subsection 4.4.2, the expression for $\mathbf{s}$ defines a scattering matrix relating the amplitudes $\mathbf{A}^-$ and $\mathbf{A}^+$ of incoming and outgoing waves, respectively. The dimension of the scattering matrix $\mathbf{s}$ is $\sum_{k=1}^N m_k$, and scattering coefficients have the form $s_{ij}^{nm}(\omega, k_x)$, relating an incoming wave i in the plate n and a reflected or transmitted wave j in the plate m at angular frequency ω and wave number component k_x . Consequently, the energy scattering coefficients are defined as $t_{ij}^{nm}(\omega, k_x)$ according to Equation (4.24). It is reminded that in the absence of damping, total energy must be conserved; hence, the sum of energy scattering coefficients over the outgoing modes equals one, see Equation (4.26).

4.5 Numerical results

In this section, several numerical case examples are presented to demonstrate the applicability of the developed method. We compute energy scattering coefficients of an L-junction of regular symmetric cross- and angle-ply laminated plates for various incoming waves. These results are compared with ones obtained using the semi-analytical approach presented in Chapter 3. In all cases, the energy scattering coefficients are computed with respect to the frequency f and the wave number component k_x . We assume that the system is undamped; however, to facilitate the wave tracking process described in subsection 4.3.1, a small damping coefficient $\eta = 0.00001$ is added.

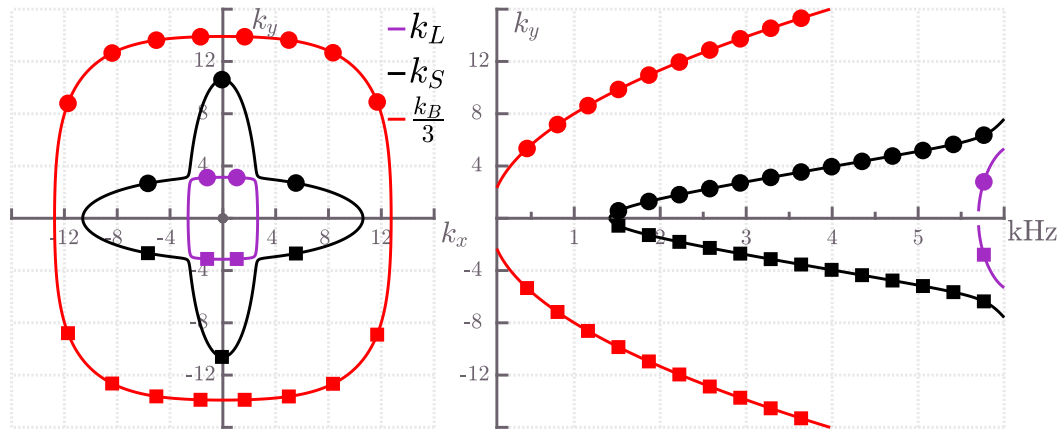


Figure 4.5: Wave vector (left) and dispersion (right) curves for a cross-ply composite plate. The wave vector curves are plotted for a fixed frequency $f = 3000$ Hz, whereas the dispersion curves are plotted for a fixed wave number component $k_x = 5 \text{ m}^{-1}$. Squares and circles denote wave numbers of incoming and outgoing waves, respectively.

4.5.1 Cross-ply laminated plates

A five-layer symmetric cross-ply laminated plate of the total thickness of $h = 0.005$ m is considered. The material characteristics of all layers are the same and given in Table 3.1. The lamination scheme is $0^\circ/90^\circ/0^\circ/90^\circ/0^\circ$.

A periodic cell of length $d_x = 0.001$ m and width $d_y = 0.001$ m is modelled in Ansys with 1 SHELL181 element and 3 SOLID185 elements per ply, i.e. 15 finite solid elements in total. The usage of only one element in the cross-section is justified since the laminates considered are homogeneous in their plane dimensions. However, there must be at least 6-10 FE elements per wavelength to obtain accurate results. In other words, the wave numbers $k \leq \frac{2\pi}{10 \max(d_x, d_y)}$ can be computed accurately. One can use more elements in the cross-section to alleviate round-off errors due to truncation of inertia terms in the dynamic stiffness matrix if needed [127].

As presented in Section 4.3, solving Equations (4.6) or (4.8) yields propagating wave vector pairs (k_x, k_y) for a fixed frequency ω^* or dispersion curves $k_y = k_y(\omega, k_x^*)$ for a fixed wave number component k_x^* . Figure 4.5 presents bending, shear and longitudinal wave vector curves for a fixed frequency $f = 3000$ Hz on the left side and the correspondent dispersion curves for a fixed wave number component $k_x = 5 \text{ m}^{-1}$ on the right side. These numerical dispersion relations

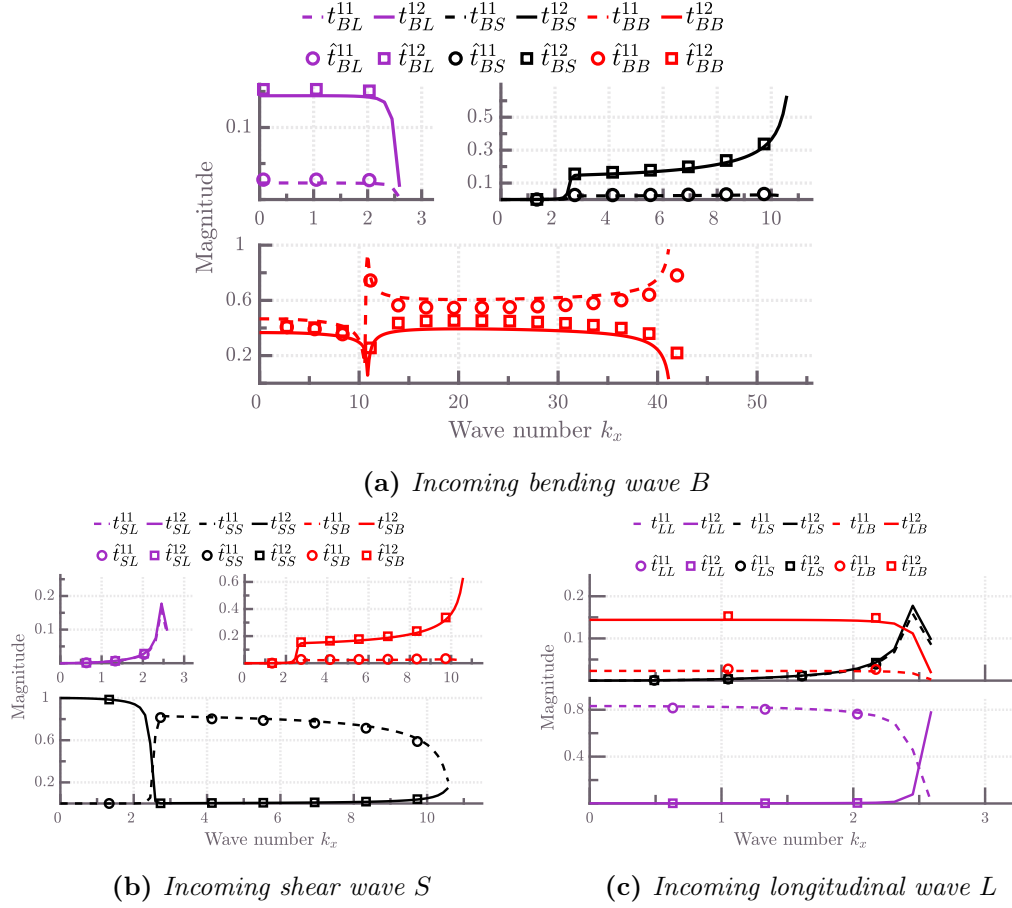


Figure 4.6: Energy scattering coefficients of an L-junction of two SHELL181-based cross-ply laminated plates for various incident waves plotted with respect to the wave number component k_x at a frequency $f = 3000$ Hz. $\hat{t}_{ij}^{nm}$ denote semi-analytical scattering coefficients obtained using the method presented in Chapter 3.

can be used to calculate the group velocity vectors and, therefore, propagation angles of transmitted wave energies via application of (4.19) and (4.21), respectively.

Now, we consider two identical cross-ply composite plates connected at 90° if modelled by SHELL181 elements and through an L-joint modelled as in Figure 4.4 - if modelled by SOLID185 elements. The FE model of the joint consists of 55 SOLID185 elements; thus, the dimensions of the joint stiffness and mass matrices are 164-by-164. The energy scattering coefficients for SHELL181-based plates are computed using Equation (4.32), whereas for SOLID185-based plates - using Equation (4.44). Furthermore, numerical results are compared with semi-analytical energy scattering coefficients obtained in Section 3.7.

Figure 4.6 presents a comparison of energy scattering coefficients obtained

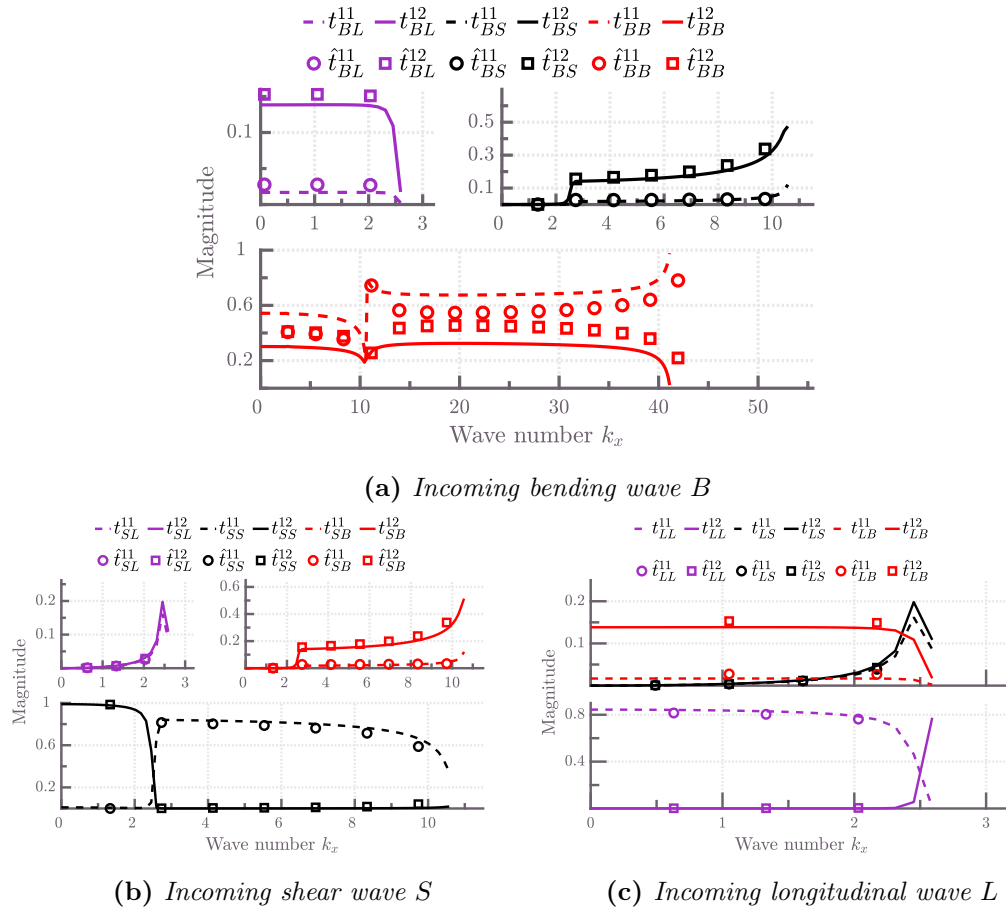


Figure 4.7: Energy scattering coefficients of an L-junction of two SOLID185-based cross-ply laminated plates for various incident waves plotted with respect to the wave number component k_x at a frequency $f = 3000$ Hz. $\hat{t}_{ij}^{nm}$ denote semi-analytical scattering coefficients obtained using the method presented in Chapter 3.

from the SHELL181-based wave finite element approach presented in subsection 4.4.2 and from the theoretical approach developed in Chapter 3 for various incident modes as a function of wave number component k_x at the frequency $f = 3000$ Hz. All solid and dashed lines represent numerical results, whereas circles and squares - semi-analytical results. A slight deviation between numerical and theoretical energy scattering coefficients can be noted for an incoming bending wave results in Figure 4.6a. Overall, however, one can observe an excellent agreement between numerical and analytical results for all cases of incoming modes. This is an expected result since SHELL181 elements are modelled in ANSYS using the FSDL theory [67, 204]. The CLP theory presented in Section 2.2 and used in Chapter 3 and the FSDL theories are expected to give identical results [204].

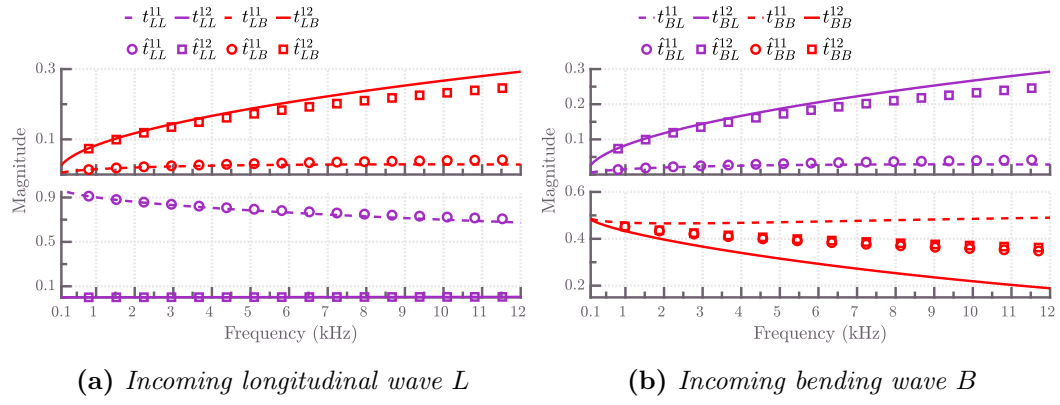


Figure 4.8: Energy scattering coefficients of an L -junction of two SHELL181-based cross-ply laminated plates for various incident waves plotted with respect to the frequency at the fixed wave number component $k_x = 0 \text{ m}^{-1}$. $\hat{t}_{ij}^{nm}$ denote semi-analytical scattering coefficients obtained using the method presented in Chapter 3.

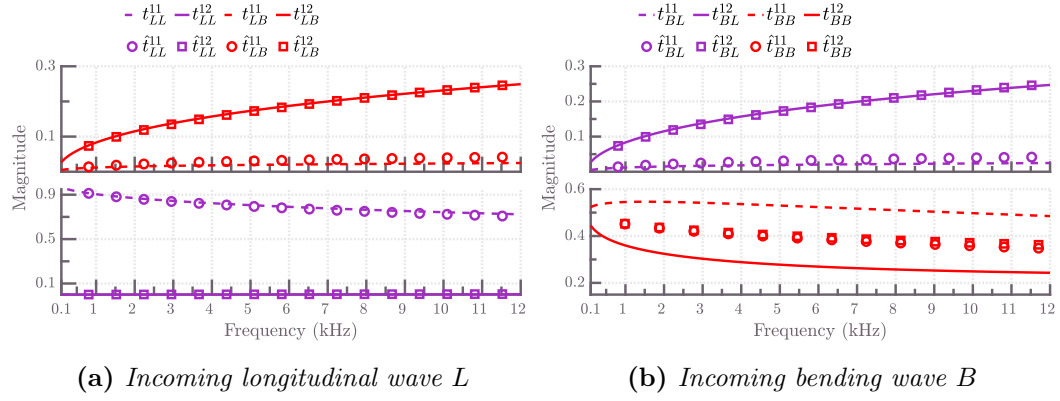


Figure 4.9: Energy scattering coefficients of an L -junction of two SOLID185-based cross-ply laminated plates for various incident waves plotted with respect to the frequency at the fixed wave number component $k_x = 0 \text{ m}^{-1}$. $\hat{t}_{ij}^{nm}$ denote semi-analytical scattering coefficients obtained using the method presented in Chapter 3.

If we consider SOLID185-based plates and a solid L-joint, then discrepancies between numerical and theoretical results appear. For instance, from Figure 4.7a it can be noted that theoretical results seem to underestimate energy reflection and hence, overestimate energy transmission of an L-junction of plates. In fact, the maximum difference observed between bending numerical and theoretical results is $\sim 20 - 22\%$. This can be referred to the fact that we used the analytical model based on the assumption that the joint can be represented as a shared line between plates. This assumption breaks down at higher frequencies since the influence of the joint size becomes more significant. Notably, the shear strain becomes more critical in the dynamic response

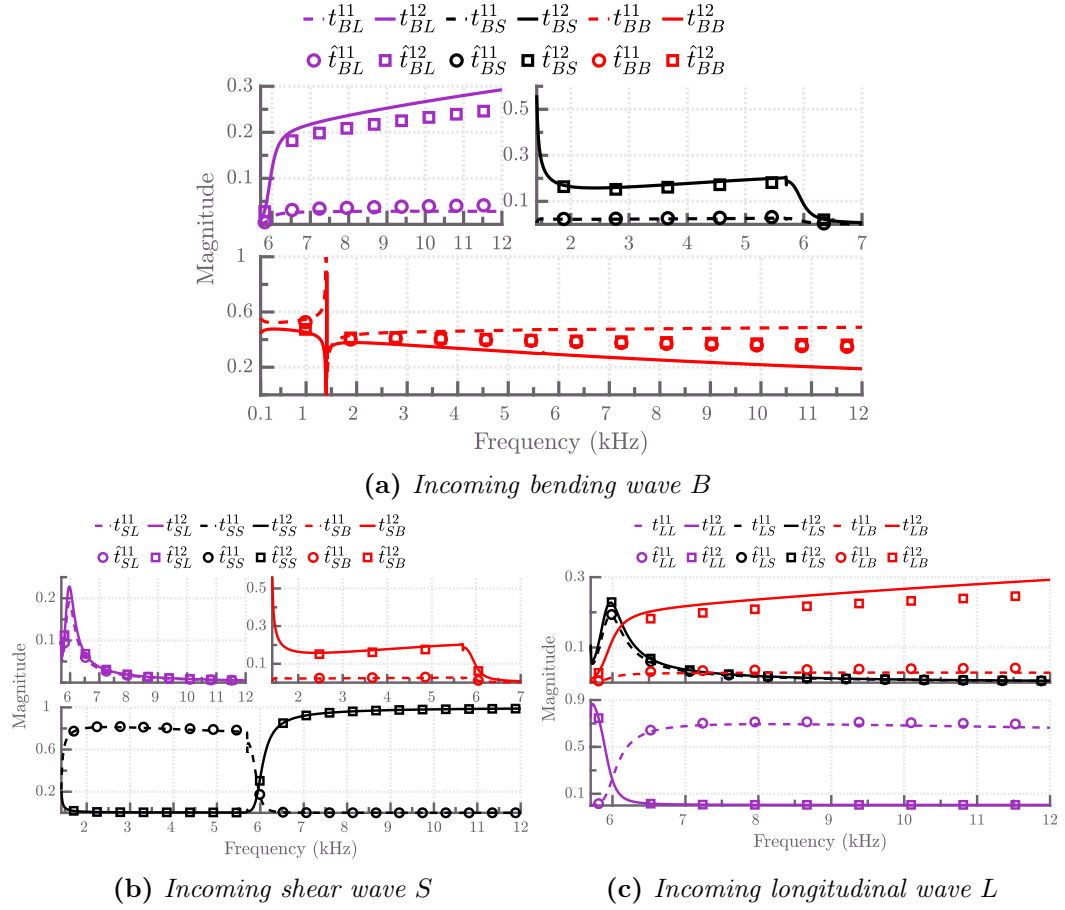


Figure 4.10: Energy scattering coefficients of an L-junction of two SHELL181-based cross-ply laminated plates for various incident waves plotted with respect to the frequency at the fixed wave number component $k_x = 5 \text{ m}^{-1}$. $\hat{t}_{ij}^{nm}$ denote semi-analytical scattering coefficients obtained using the method presented in Chapter 3.

of the joint and this effect is not considered in the analytical model [160, 205]. The energy scattering coefficients for incoming in-plane modes are in excellent agreement. Hence, modelling thin plates and junctions of thin plates using SHELL181 elements or semi-analytically is sufficient for accurate estimation of in-plane wave energy scattering coefficients.

If the wave number component k_x is fixed, then one can compare energy scattering coefficients for a range of frequencies. Figures 4.8 and 4.9 represent energy scattering coefficients obtained from the SHELL181- and SOLID185-based approaches for incident longitudinal and bending modes as a function of frequency at the wave number component $k_x = 0 \text{ m}^{-1}$. An incident shear wave is fully reflected without mode conversion for the wave number component $k_x = 0$, that is, $t_{SS}^{11} = 1$. A great agreement of semi-analytical and numerical

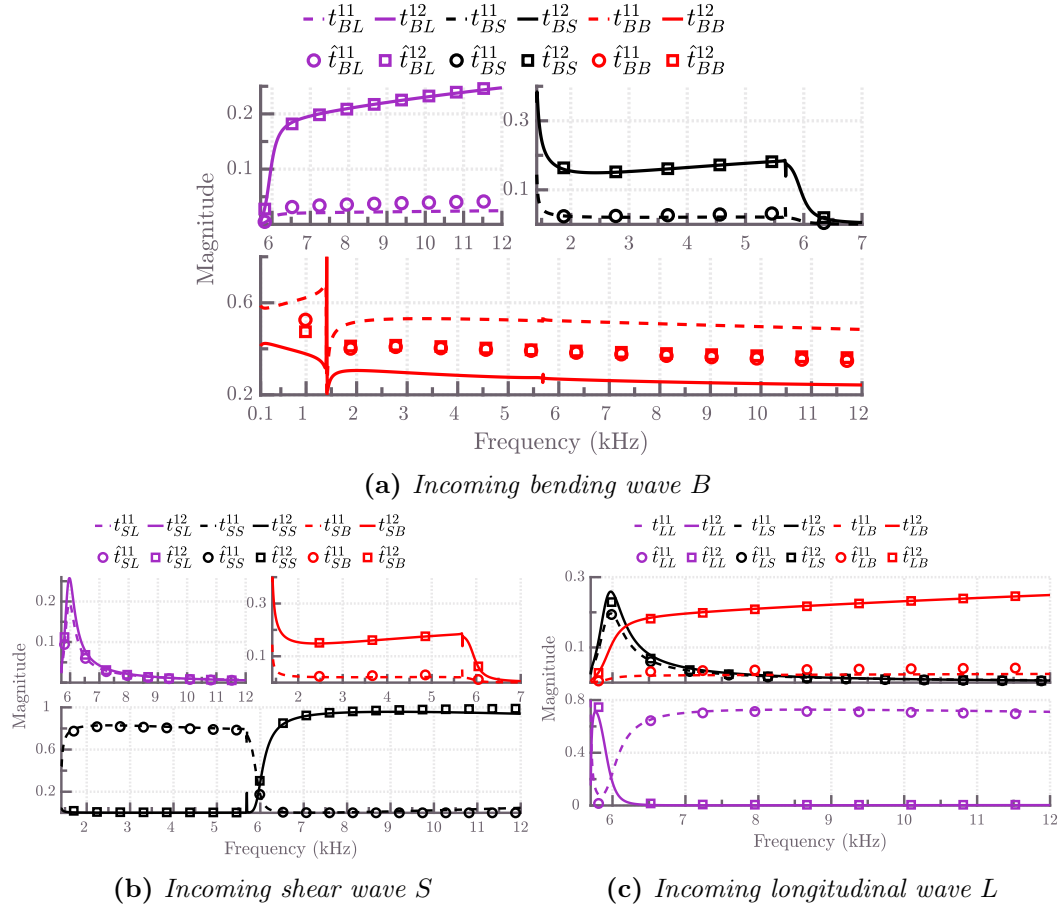


Figure 4.11: Energy scattering coefficients of an L -junction of two SOLID185-based cross-ply laminated plates for various incident waves plotted with respect to the frequency at the fixed wave number component $k_x = 5 \text{ m}^{-1}$. $\hat{t}_{ij}^{nm}$ denote semi-analytical scattering coefficients obtained using the method presented in Chapter 3.

results for longitudinal energy scattering coefficients can be noted. The longitudinal mode is propagating at all frequencies at the wave number component $k_x = 0$. Regarding the incident bending mode, the semi-analytical approach predicts that the bending energy reflection and transmission coefficients are nearly same in magnitude across the range of frequencies $f \in [0.1, 12] \text{ kHz}$, see Figures 4.8b and 4.9b. However, the bending energy reflection coefficient obtained from the hybrid FE/WFE approach is larger than the transmission coefficient, and the magnitude difference increases with the frequency, becoming 0.3 at $f = 12 \text{ kHz}$. Nevertheless, the maximum difference between numerical and semi-analytical scattering coefficients is ~ 0.2 .

A mode coupling phenomenon can be observed when considering the dependence of energy scattering coefficients on the frequency at the wave number

component $k_x \neq 0 \text{ m}^{-1}$. For instance, in Figures 4.10 and 4.11, energy scattering coefficients are computed for a fixed wave number component $k_x = 5 \text{ m}^{-1}$ and obtained from SHELL181- and SOLID185-based hybrid FE/WFE approach, respectively. An incident bending wave generates outgoing shear and longitudinal waves at frequencies $1416 \text{ Hz} \leq f < 5701 \text{ Hz}$ and $f \geq 5701 \text{ Hz}$, respectively, see Figures 4.10a and 4.11a. The energy scattering coefficients of coupled shear-bending waves are equal, that is, $t_{SB}^{11(12)} = t_{BS}^{11(12)}$. The same applies for shear-longitudinal wave coupling, that is, $t_{SL}^{11(12)} = t_{LS}^{11(12)}$. Overall, a great agreement between numerical and semi-analytical results can be noted with small deviations for incident bending wave energy scattering coefficients. The summation to unity of the energy reflection and transmission coefficients validates the numerical results obtained.

4.5.2 Angle-ply composite plates

A five-layer angle-ply laminated plate of the total thickness of $h = 0.005 \text{ m}$ is considered. The material characteristics of the individual layers are the same as in the previous example. The lamination scheme is $45^\circ / -45^\circ / 45^\circ / -45^\circ / 45^\circ$. A periodic cell of length $d_x = 0.001 \text{ m}$ and width $d_y = 0.001 \text{ m}$ is again modelled in Ansys with 1 SHELL181 element and 15 SOLID185 elements.

Figure 4.12 presents bending, shear and longitudinal wave vector curves of an angle-ply laminated plate for a fixed frequency $f = 3000 \text{ Hz}$ on the left side and the correspondent dispersion curves for a fixed wave number component $k_x = 5 \text{ m}^{-1}$ on the right side. Note that there are two shear wave dispersion curves present on the right side of Figure 4.12. The second shear waves (denoted as S_2 and plotted in green) exhibit a negative group velocity phenomenon as in wave vector curves at a fixed frequency.

A comparison of energy scattering coefficients obtained from the SHELL181-based numerical and theoretical methods is given in Figure 4.13 for various incident modes as a function of wave number component k_x at a frequency $f = 3000 \text{ Hz}$. Again, numerical and theoretical energy scattering coefficients

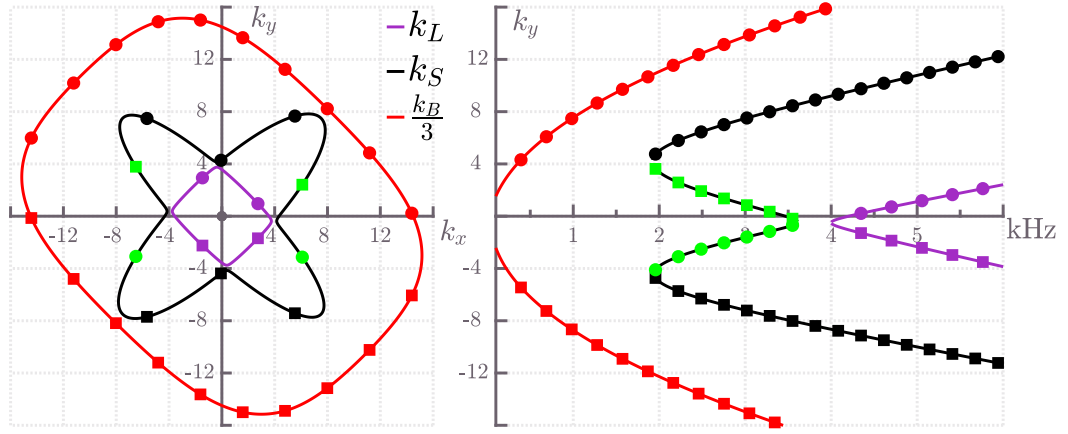


Figure 4.12: Wave vector (left) and dispersion (right) curves for an angle-ply composite plate. The wave vector curves are plotted for a fixed frequency $f = 3000$ Hz, whereas the dispersion curves are plotted for a fixed wave number component $k_x = 5 \text{ m}^{-1}$. Squares and circles denote wave numbers of incoming and outgoing waves, respectively.

agree well for all incoming modes with slight deviation for an incoming bending mode.

A drastic change in the shape of energy reflection and transmission coefficients can be seen for an incoming bending wave in the case of SOLID185-based numerical results, see Figure 4.14. For instance, at the range of wave number component $10 \text{ m}^{-1} < |k_x| < 20 \text{ m}^{-1}$, numerical energy transmission is around 0.6 whereas semi-analytical method estimates energy transmission at only 0.4. Vice-versa, numerical reflection coefficient has value of 0.4, whereas theoretical reflection - 0.6, see Figure 4.14a. The same discrepancy can be noted between numerical results obtained from SOLID185- and SHELL181-based approaches. This phenomenon can be explained by the fact that the ESL-theories behind the semi-analytical and SHELL181-based approaches produce effective one-layer plates joined along the shared edge, thus losing complexity of the connection between individual layers of plates at the junction.

Numerical energy scattering coefficients for in-plane modes are in excellent agreement with semi-analytical results. As in the case of cross-ply laminated plates, curves for shear and bending coupled waves vary identically for a range of wave number components $|k_x| < 7.6 \text{ m}^{-1}$ - the correspondent scattering coefficients are $t_{SB}^{11(12)} = t_{BS}^{11(12)}$. Similarly, the scattering coefficients of longitu-

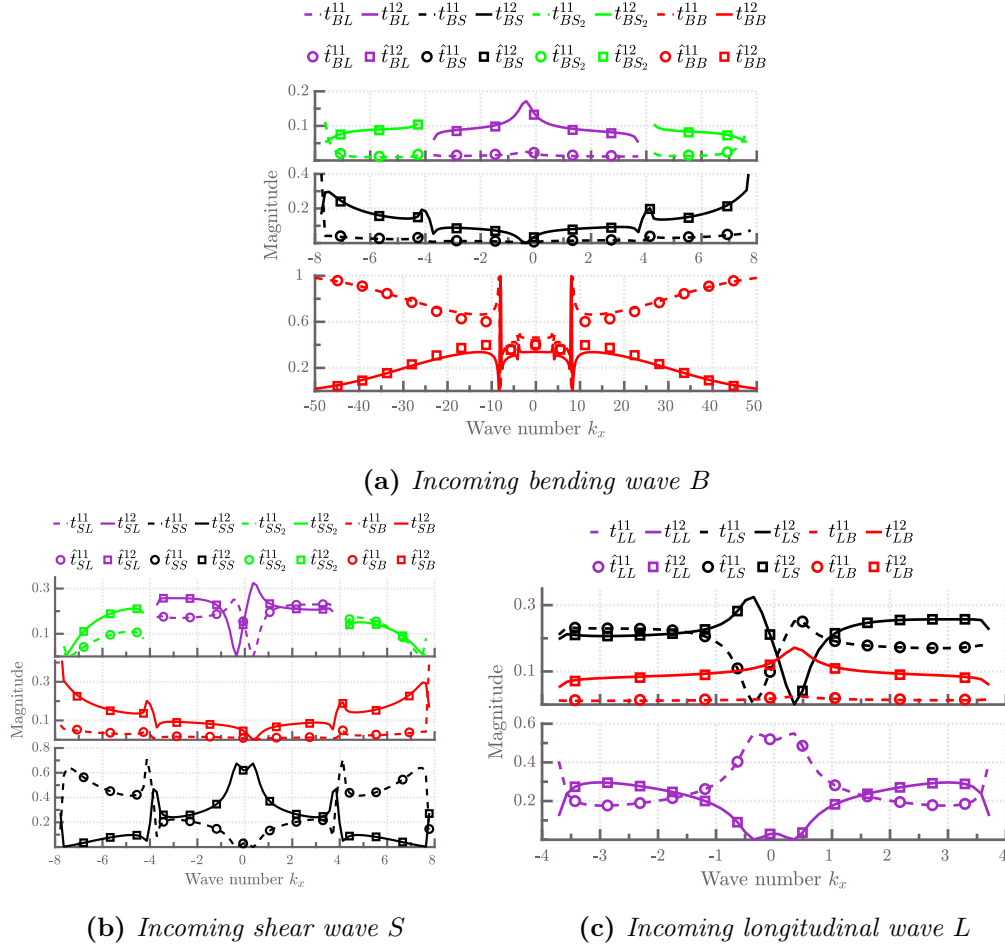


Figure 4.13: Energy scattering coefficients of an L -junction of two SHELL181-based angle-ply laminated plates for various incident waves plotted with respect to the wave number component k_y at a frequency $f = 3000$ Hz. $\hat{t}_{ij}^{nm}$ denote semi-analytical scattering coefficients obtained using the method presented in Chapter 3.

dinal and bending coupled waves $t_{LB}^{11(12)}$ and $t_{BL}^{11(12)}$ are equal for $|k_x| < 3.7 \text{ m}^{-1}$. Furthermore, energy scattering coefficients of longitudinal and shear coupled waves are symmetric around $k_x = 0 \text{ m}^{-1}$, that is, $t_{SL}^{11(12)}(k_x) = t_{LS}^{11(12)}(-k_x)$.

Figures 4.15 and 4.16 represent energy scattering coefficients as a function of frequency at the fixed wave number components $k_x = 0 \text{ m}^{-1}$ and $k_x = 5 \text{ m}^{-1}$, respectively. It can be noted that numerical scattering coefficients based on the SHELL181 finite elements agree well with semi-analytical ones for all incident waves. This is an expected result, since the semi-analytical and SHELL181-based numerical approaches use approximately equal formulations of the thin laminated plate at the frequencies considered, see Figures 4.15a, 4.15c and 4.15e for $k_x = 0 \text{ m}^{-1}$ and Figures 4.16a, 4.16c and 4.16e for $k_x = 5 \text{ m}^{-1}$.

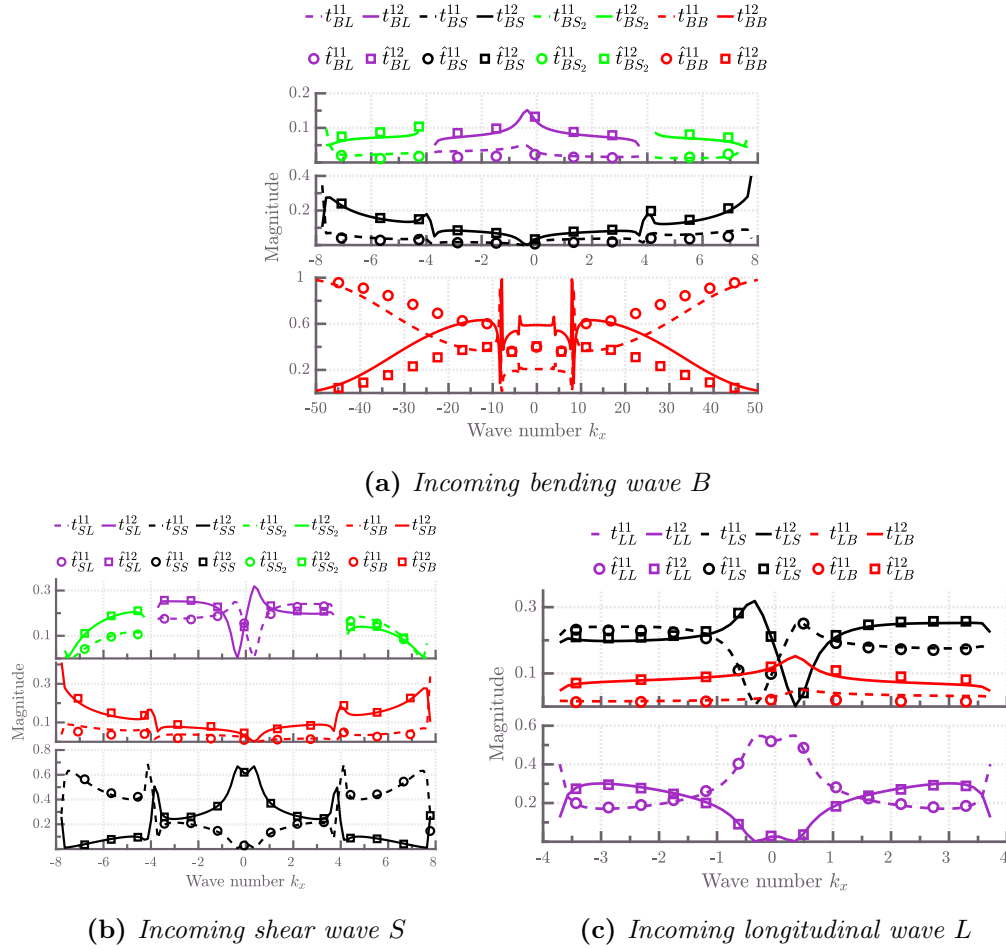


Figure 4.14: Energy scattering coefficients of an L -junction of two *SOLID185*-based angle-ply laminated plates for various incident waves plotted with respect to the wave number component k_y at a frequency $f = 3000$ Hz. $\hat{t}_{ij}^{nm}$ denote semi-analytical scattering coefficients obtained using the method presented in Chapter 3.

On the other hand, in Figures 4.15b and 4.16b, numerical results based on the *SOLID185* finite elements predict higher bending wave energy reflection across all frequencies considered for both values of the wave number component k_x .

4.6 Conclusion

This chapter aimed to derive the hybrid FE/WFE method to calculate of energy scattering coefficients for arbitrary junctions of composite plates. The approach is based on modelling joints with finite elements with boundary conditions given by the solutions of the WFE method for the composite plates in the infinite half-spaces connected to the joint. The displacement and force nodal vectors are expressed in the basis of positive and negative waves, and

satisfying boundary conditions at the connection between the plates and the joint yields the scattering matrix. The method gives for the first time a detailed recipe for computing scattering coefficients for the generic case of an arbitrary number of composite plates connected at a junction without restrictions on the angles at which the plate meet or the orientation of the principal axis of individual plates.

The plates considered have been modelled using two different finite element types, two-dimensional SHELL181 and three-dimensional SOLID185 elements, and we have considered the implication and influence of each element type on the scattering coefficients. Also, numerical results on selected examples of junctions of composite plates have been compared with semi-analytical results obtained from Chapter 3. It has been found that modelling thin plates and their junctions using SHELL181 elements or semi-analytically using the method from Chapter 3 is sufficient for accurate estimation of in-plane wave energy scattering coefficients. However, the SOLID185-based approach provides more accurate energy scattering coefficients of the incident bending wave at higher frequencies since the influence of the shear strain becomes significant, which is neglected in the semi-analytical method and approximated by shear correction factors in the SHELL181-based approach. Nevertheless, in the selected examples, the maximum difference observed between bending numerical and theoretical results is $\sim 20 - 22\%$. The results of this chapter can be used for the computation of wave energy distribution in the SEA method in the form of coupling loss factors and as well as for angle-of-incidence dependent scattering coefficients entering the DEA method in Chapter 5.

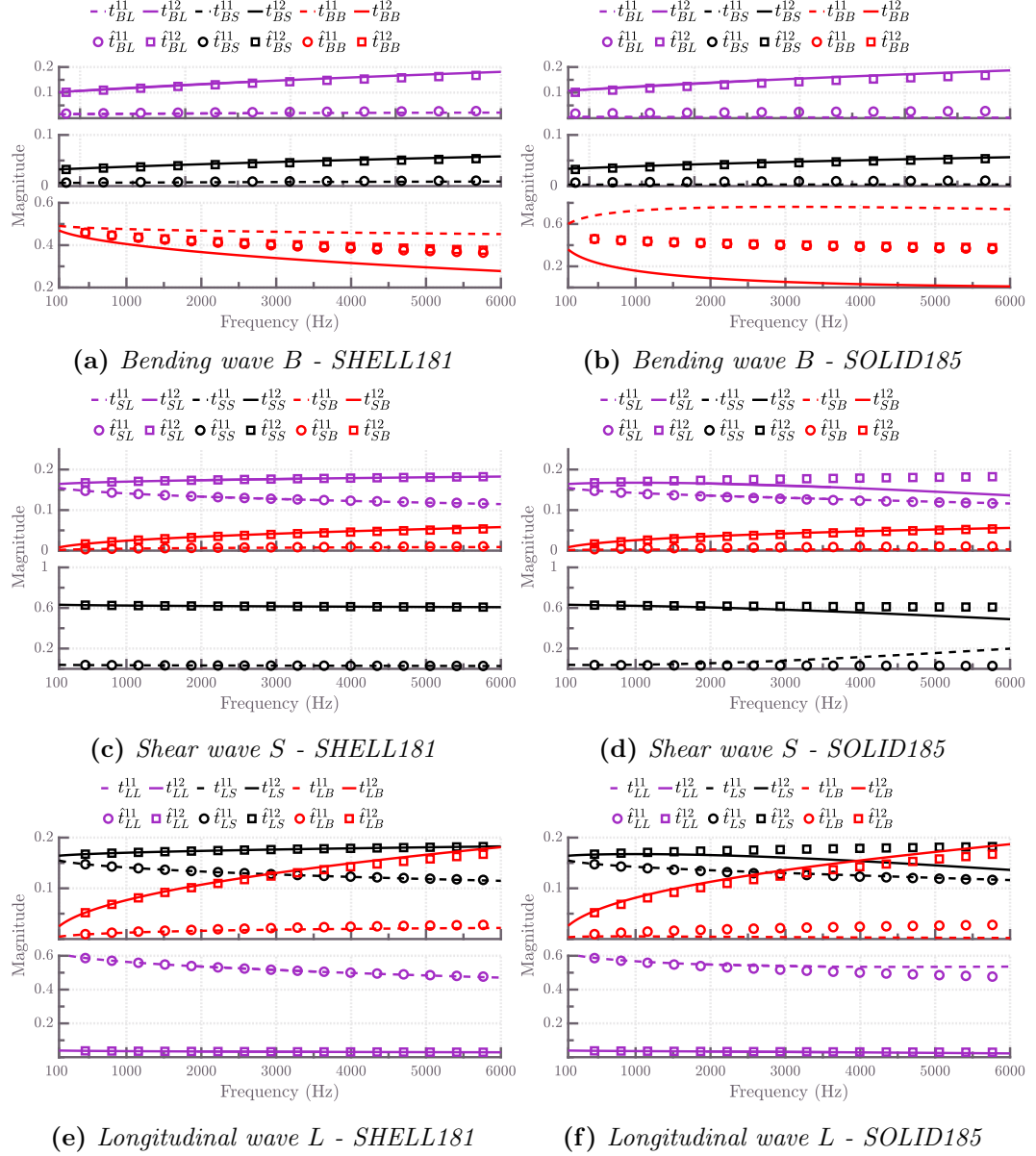


Figure 4.15: Energy scattering coefficients of an L-junction of two SHELL181- and SOLID185-based angle-ply laminated plates for various incident waves plotted with respect to the frequency at the fixed wave number component $k_x = 0 \text{ m}^{-1}$. $\hat{t}_{ij}^{nm}$ denote semi-analytical scattering coefficients obtained using the method presented in Chapter 3.

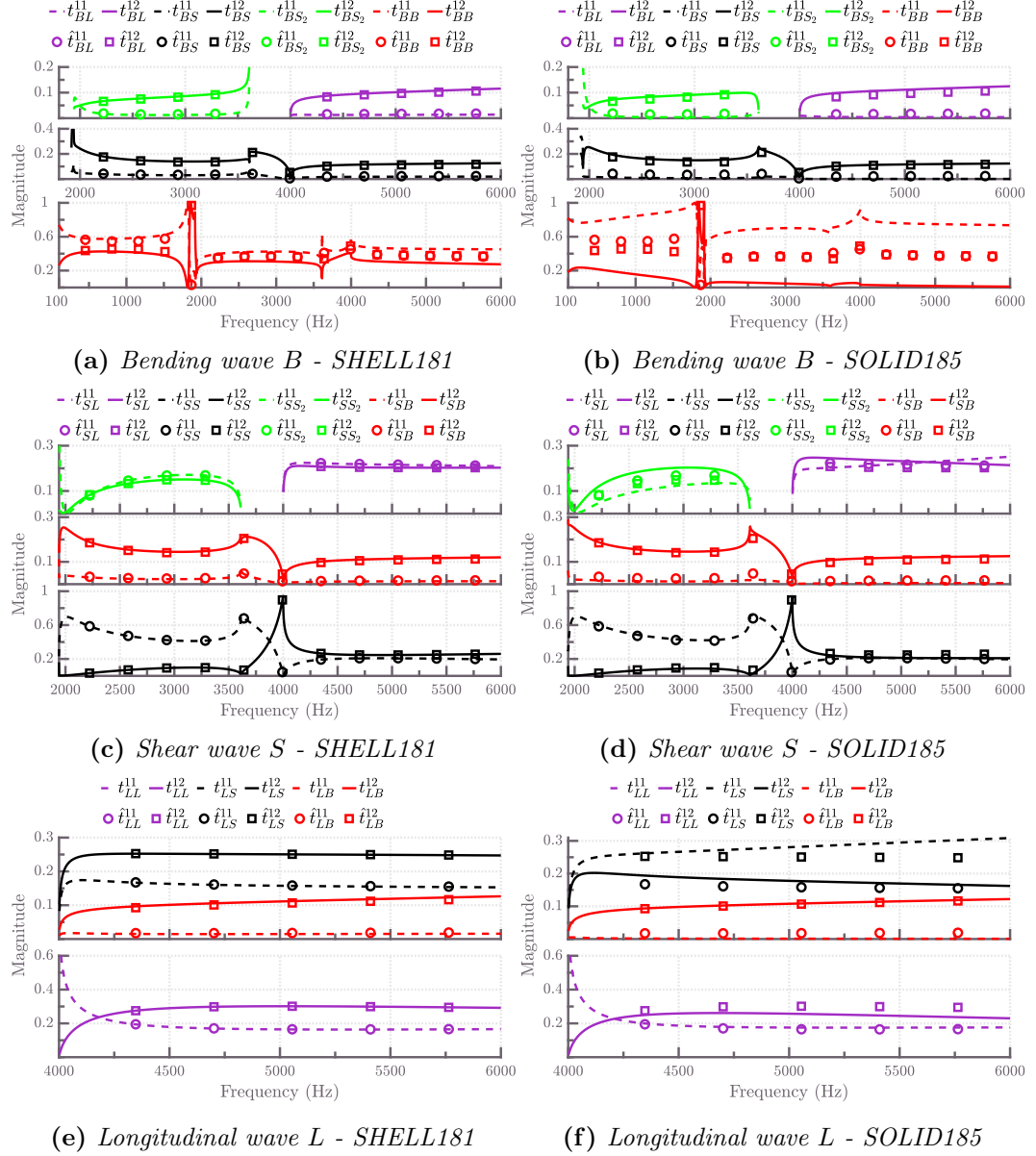


Figure 4.16: Energy scattering coefficients of an L-junction of two SHELL181- and SOLID185-based angle-ply laminated plates for various incident waves plotted with respect to the frequency at the fixed wave number component $k_x = 5 \text{ m}^{-1}$. $\hat{t}_{ij}^{nm}$ denote semi-analytical scattering coefficients obtained using the method presented in Chapter 3.

Chapter 5

Dynamical Energy Analysis for composite structures

5.1 Introduction

Having computed and validated the scattering coefficients for the composite laminated plate junctions in Chapters 3 and 4, respectively, we will next consider the implementation of the DEA method for composite structures. The main ideas of this method were presented in Chapter 2, and the general algorithm was given in (2.42). The actual implementation of the method will be performed on two-dimensional FE triangular meshes.

This chapter is organised as follows. In Section 5.2, we first discuss how the domain Ω is meshed using two-dimensional triangular cells. In subsection 5.2.1, we consider how the ray tracing formula presented in Equation (2.21) are changed in the case of composite laminated plates. Furthermore, they define the linear transformation between the boundary (2D) and full (4D) phase space densities described in subsection 5.2.2. In Section 5.3, we introduce a finite basis of functions onto which we project the phase space densities and the boundary operator T , which is given by Equation (2.38). The discretisation of the boundary integral operator T on the triangular meshes is discussed in subsection 5.3.1. The initial boundary phase space density due to a point or an edge source is also expressed in terms of the finite basis functions in subsection 5.3.2. The stationary energy density and the intensity vector field are computed in subsection 5.3.3. Finally, Section 5.4 considers numerical case

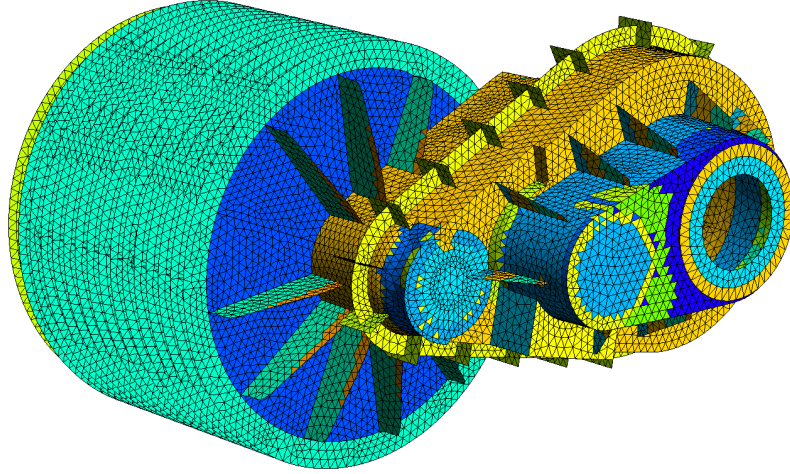


Figure 5.1: *An example of an electric vehicle gearbox being meshed using two-dimensional shell elements. Colours represent elements with different thicknesses.*

studies for single and two coupled composite plates. Also, it presents part of the work performed during the industrial placement at Romax Technology.

5.2 Ray tracing for composite structures

From here on, we assume that the whole domain Ω is subdivided into subdomains $\Omega_i, i = 1, \dots, N$, with N being the total number of subsystems. For instance, this can be naturally achieved by meshing the domain using two-dimensional finite shell elements in triangular form. It is also assumed that all material and geometrical parameters remain constant within the mesh cells. For instance, a shell-element based mesh cell must be homogeneous in material parameters and of uniform thickness. Figure 5.1 presents an example of a two-dimensional finite element mesh for an electric vehicle gearbox, where different colours represent different thicknesses of the shell elements.

5.2.1 Hamilton function and boundary mapping

The wave energy travels between neighbouring mesh cells through edges. Now, the trajectories of the ray-particles carrying the wave energy are governed by the ray-tracing map defined by Equation (2.21). This was outlined in subsection 2.1.3 for isotropic structures with the Hamilton function $H(\mathbf{r}, \mathbf{p}) = c|\mathbf{p}| = 1$. As was seen in subsection 3.3.1, the Hamilton function for composite

plates can be written in the similar manner as

$$H(\mathbf{r}, \mathbf{p}) = \beta_l \left(\arctan \left(\frac{p_{\parallel}}{p_{\perp}} \right) \right) |\mathbf{p}|^l = \omega, \quad l = 1, 2. \quad (5.1)$$

Here, the momentum vector $\mathbf{p}$ is equal to the wave vector $\mathbf{k}$ with its amplitude $|\mathbf{p}|$ being the wave number k , and consequently, from here on, we will denote the wave number components k_x and k_y as $p_{\parallel}$ and $p_{\perp}$, respectively. The function $\beta_l(\theta)$ is given by Equation (3.8). The parameter $l = 1, 2$ represents propagating in-plane and bending modes, respectively.

The Hamilton function in (5.1) represents only in-plane and out-of-plane modes in the context of the CLP theory presented in Chapter 3. The more general Hamilton function for arbitrary composite plates can be obtained numerically using the WFE method described in Chapter 4. It then can describe not only classical longitudinal, shear and bending modes but also higher-order Lamb modes. For the sake of clarity, the Hamilton function (5.1) will be used throughout this chapter; however, all calculations can be applied to the numerical Hamilton function obtained from the WFE method.

As per Equation (2.21), we can write the equations of motion of the ray-particles as

$$\begin{bmatrix} \frac{dx}{d\tau} \\ \frac{dy}{d\tau} \\ \frac{dp_{\parallel}}{d\tau} \\ \frac{dp_{\perp}}{d\tau} \end{bmatrix} = \begin{bmatrix} \frac{\partial H}{\partial p_{\parallel}} \\ \frac{\partial H}{\partial p_{\perp}} \\ -\frac{\partial H}{\partial x} \\ -\frac{\partial H}{\partial y} \end{bmatrix} = \begin{bmatrix} \left(lp_{\parallel} + \frac{\partial \beta_l}{\partial \theta} p_{\perp} \right) \frac{\omega}{k^2} \\ \left(lp_{\perp} - \frac{\partial \beta_l}{\partial \theta} p_{\parallel} \right) \frac{\omega}{k^2} \\ 0 \\ 0 \end{bmatrix}, \quad l = 1, 2. \quad (5.2)$$

Here, $(x, y, p_{\parallel}, p_{\perp})$ are the 4D coordinates in the full phase space domain. Since $H(r, p) = \omega$, then $\frac{\partial H}{\partial p_{\parallel}}$ and $\frac{\partial H}{\partial p_{\perp}}$ are the components of the group velocity vector $\mathbf{c}_g$. Therefore, ray-particles travel along the direction of the group velocity vector. As in the case of isotropic structures considered in subsection 2.1.3, the ray-particle trajectories are straight lines, since $\frac{d\mathbf{p}}{d\tau} = 0$.

The boundary mapping procedure described in subsection 2.1.4 is based on

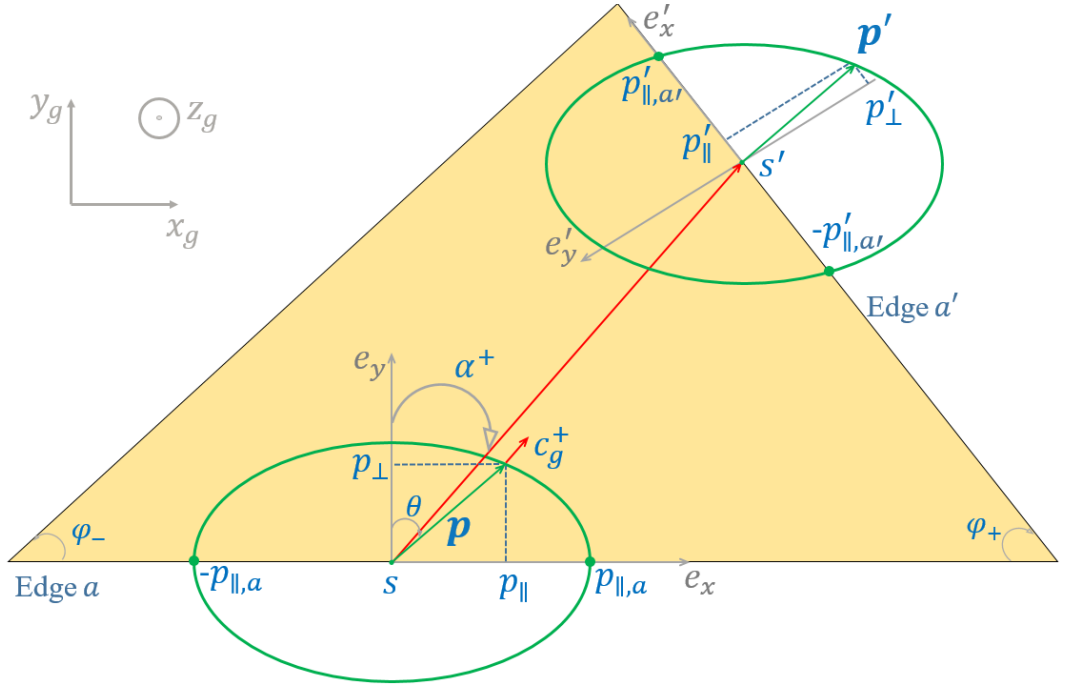


Figure 5.2: A schematic representation of the boundary mapping procedure in the triangular mesh element. Green ovals depict wave vector curves. The orientation of the principal material axes is defined in the element's coordinate system (x_g, y_g, z_g) . A ray starting at the position s of edge a points in the direction of the outgoing group velocity vector $\mathbf{c}_g^+$. The group velocity vector is defined by the momentum component $p_{||}$. The ray hits edge a' at the position s' with the new momentum component $p'_{||}$ defined in the coordinate system (e'_x, e'_y) . However, momentum vectors $\mathbf{p}$ and $\mathbf{p}'$ are equal.

the integral operator T :

$$\rho'_\Gamma(X') = \{T\rho_\Gamma\}(X') = \int \lambda(X') e^{-\mu D(X, X')} \delta(X' - \Phi(X)) \rho_\Gamma(X) dX, \quad (5.3)$$

where $\rho_\Gamma(X)$ is the 2D phase space density on the boundary Γ . The mapping function $\Phi : X \rightarrow X'$ is now described by the ray-tracing equations (5.2). Figure 5.2 shows the schematic representation of this mapping between the boundary phase space coordinates $X = (s, p_{||})$ and $X' = (s', p'_{||})$ at edges a and a' of a single mesh cell, respectively. Here, the green ovals represent the wave vector curves of a particular mode, and they are the same for both edges because the material parameters are constant in the mesh cell. The momentum component $p_{||}$ is used to compute the outgoing group velocity vector $\mathbf{c}_g^+$ (depicted in red) and the direction angle of the trajectory α^+ measured with respect to the local e_y axis. Note that $|p_{||}| \leq p_{||,a}$, and the maximum

value $p_{||,a}$ can be found from the dispersion relations. Furthermore, the value of $p_{||,a}$ changes for different propagating modes. The red line connecting points s and s' represents the trajectory of the ray-particle. Now, since the momentum vector $\mathbf{p}$ is conserved, the vector $\mathbf{p}'$ is equal to $\mathbf{p}$ in the global coordinate system (x_g, y_g) of the mesh cell. However, at the local edge coordinate system (e'_x, e'_y) the components $p'_{||}$ and $p'_{\perp}$ will be different from $p_{||}$ and $p_{\perp}$. Also, the maximum values $p_{||,a}$ and $p'_{||,a'}$ are also different. The component $p'_{||}$ is further used to compute the scattering coefficients $\lambda(p'_{||})$. These coefficients have been previously obtained in the context of the classical thin laminated plate in Equation (3.43) and using the WFE method in Equation (4.24).

5.2.2 Transformation between 2D and 4D phase space densities

Equation (5.2) defines the linear transformation Π between $(s, \tau, p_{||}, \omega)$ and the full phase space coordinates $(x, y, p_{||}, p_{\perp})$ [53]:

$$\Pi : \begin{bmatrix} s \\ \tau \\ p_{||} \\ \omega \end{bmatrix} \mapsto \begin{bmatrix} x \\ y \\ p_{||} \\ p_{\perp} \end{bmatrix} + \frac{\omega\tau}{k^2} \begin{bmatrix} lp_{||} + \frac{\partial\beta_l}{\partial_l\partial\theta}p_{\perp} \\ lp_{\perp} - \frac{\partial\beta_l}{\partial_l\partial\theta}p_{||} \\ 0 \\ 0 \end{bmatrix}. \quad (5.4)$$

The Jacobian matrix $D\Pi$ can be written as

$$D\Pi = \frac{\partial(x, y, p_{||}, p_{\perp})}{\partial(s, \tau, p_{||}, \omega)} = \begin{bmatrix} 1 & \frac{\partial H}{\partial p_{||}} & 0 & 0 \\ 0 & \frac{\partial H}{\partial p_{\perp}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{\partial p_{\perp}}{\partial p_{||}} & \frac{\partial p_{\perp}}{\partial \omega} \end{bmatrix}. \quad (5.5)$$

Now, at a fixed angular frequency $\omega = \omega_0$, one can compute $\frac{\partial p_{\perp}}{\partial p_{||}}$ using Equation (5.1) in the following way

$$0 = d\omega = \frac{\partial H}{\partial p_{||}} dp_{||} + \frac{\partial H}{\partial p_{\perp}} dp_{\perp} \rightarrow \frac{\partial p_{\perp}}{\partial p_{||}} = -\frac{c_{g,x}}{c_{g,y}}, \quad l = 1, 2. \quad (5.6)$$

This relation will also be important in the computation of the stationary energy and intensity densities in subsection 5.3.3. Finally, the determinant of the Jacobian matrix $\det(D\Pi) = \frac{\partial H}{\partial p_\perp} \frac{\partial p_\perp}{\partial \omega} = 1$, therefore, the transformation Π is phase space volume preserving, and the full phase space density can be related to the 2D phase space density as

$$\rho(x, y, p_\parallel, p_\perp) = \rho_\Gamma(x, p_\parallel) \delta(\omega(p_\parallel, p_\perp) - \omega_0). \quad (5.7)$$

Recall that the 2D phase space density represents the directional energy flux through the boundary Γ at the position x and momentum component $p_\parallel$. The relation (5.7) will be used to define the initial 2D phase space density $\rho_{\Gamma,0}(s, p_\parallel)$ and the stationary energy density $\rho^\infty(\mathbf{r})$ in further subsections.

5.3 Basis set expansion of the problem

To proceed further with the DEA method, we need to introduce a finite basis set of functions onto which the 2D phase space densities and the boundary integral operator T will be projected. As outlined in Chapter 2, Legendre polynomials will be used for this purpose because of their orthogonality property in the L^2 space. They can be defined by *Bonnet's recursion formula* as

$$\begin{aligned} P_0(x) &= 1, P_1(x) = x, \quad x \in [-1, 1]. \\ (m+1)P_{m+1}(x) &= (2m+1)xP_m(x) - mP_{m-1}(x), \quad m \geq 2. \end{aligned} \quad (5.8)$$

Legendre polynomials are orthogonal with respect to the L^2 -norm on the interval $-1 \leq x \leq 1$:

$$\int_{-1}^1 P_n(x) P_m(x) dx = \frac{2}{2m+1} \delta_{nm}, \quad (5.9)$$

where δ_{mn} denotes the Kronecker delta. Since the boundary operator T and phase space densities $\rho_\Gamma(s, p_\parallel)$ operate on the 2D phase space $(s, p_\parallel)$, the basis functions must be defined accordingly both for position and momentum spaces.

We define the basis functions in a similar way as in [53]:

$$F_{\mathbf{n}=(a,m,l)}(s, p_{||}) = \frac{1}{\sqrt{L_a, p_{||,a,l}}} \mathbb{1}_a(s) P_m \left(\frac{p_{||}}{p_{||,a,l}} \right), \quad \mathbb{1}_a(s) = \begin{cases} 1 & \text{for } s \text{ on edge } a. \\ 0 & \text{otherwise} \end{cases}, \quad (5.10)$$

where $\mathbb{1}_a$ is the characteristic function of edge a , and L_a is the length of edge a . The multi-index $\mathbf{n} = (a, m, l)$ combines the index of the edge a , the degree m of the Legendre polynomial and the index l of the propagating mode, where $l = L, S, B$ represents longitudinal, shear and bending modes propagating in thin plates, see subsection 3.3.1. As outlined earlier, the higher-order Lamb modes can be taken into account by computing the Hamilton function numerically in the WFE method context, thus enlarging the dimension of the index l .

The Legendre polynomials are written for the transformed argument $|x| \leq 1 \rightarrow |p_{||}| \leq p_{||,a,l}$. By defining the inner product as

$$(u, v) = \int u(X) v(X) dX, \quad (5.11)$$

we get the orthogonality property of the basis functions:

$$(F_{\mathbf{n}}, F_{\mathbf{n}'}) = \frac{2}{2m+1} \delta_{a,a'} \delta_{m,m'} \delta_{l,l'}. \quad (5.12)$$

5.3.1 Boundary integral operator

After introducing the finite basis set of functions, we can now express the phase space densities and the operator T in this basis. The 2D phase space densities $\rho_{\Gamma}(s, p_{||})$ and $\rho'_{\Gamma}(s', p'_{||})$ can be approximated as

$$\rho_{\Gamma}(s, p_{||}) \approx \sum_{\mathbf{n}} \rho_{\mathbf{n}} F_{\mathbf{n}}(s, p_{||}), \quad \rho'_{\Gamma}(s', p'_{||}) \approx \sum_{\mathbf{n}'} \rho'_{\mathbf{n}'} F_{\mathbf{n}'}(s', p'_{||}), \quad (5.13)$$

where $\rho'_{\Gamma}(s', p'_{||})$ is the 2D phase space density on edge a' obtained after a single application of the boundary integral operator T , that is, $\rho'_{\Gamma} = \{T \rho_{\Gamma}\}$.

The coefficients $\rho_{\mathbf{n}}$ and $\rho'_{\mathbf{n}'}$ are given as

$$\rho_{\mathbf{n}} = \frac{(\rho_{\Gamma}, F_{\mathbf{n}})}{(F_{\mathbf{n}}, F_{\mathbf{n}})}, \quad \rho'_{\mathbf{n}'} = \frac{(\rho'_{\Gamma}, F_{\mathbf{n}'})}{(F_{\mathbf{n}'}, F_{\mathbf{n}'})}. \quad (5.14)$$

Consequently, the coefficients $\rho'_{\mathbf{n}}$ are connected with $\rho_{\mathbf{n}}$ through the following relation

$$\rho'_{\mathbf{n}'} = \sum_{\mathbf{n}} T_{\mathbf{n}'\mathbf{n}} \rho_{\mathbf{n}}, \quad T_{\mathbf{n}'\mathbf{n}} = \frac{(F_{\mathbf{n}'}, T F_{\mathbf{n}})}{(F_{\mathbf{n}'}, F_{\mathbf{n}})}, \quad (5.15)$$

where $T_{\mathbf{n}'\mathbf{n}}$ is the matrix representation of the boundary integral operator T . After substituting Equation (5.3) into the matrix $T_{\mathbf{n}'\mathbf{n}}$, we obtain the following integral form:

$$\begin{aligned} T_{\mathbf{n}'\mathbf{n}} &= \frac{2m' + 1}{2} \iint F_{\mathbf{n}'}(X') \lambda(X') e^{-\mu D(X, X')} \delta(X' - \Phi(X)) F_{\mathbf{n}}(X) dX dX' = \\ &= \frac{2m' + 1}{2} \int \lambda(\Phi(X)) e^{-\mu D(X, \Phi(X))} F_{\mathbf{n}'}(\Phi(X)) F_{\mathbf{n}}(X) dX. \end{aligned} \quad (5.16)$$

Finally, in the chosen finite set of basis functions, the matrix elements $T_{\mathbf{n}'\mathbf{n}}$ can be written as

$$\begin{aligned} T_{\mathbf{n}'\mathbf{n}} &= \frac{2m' + 1}{2\sqrt{L_a L_{a'} p_{\parallel, a, l} p'_{\parallel, a', l'}}} \\ &\int_{0-p_{\parallel, a, l}}^{L_a p_{\parallel, a, l}} \int \lambda(p'_{\parallel}) e^{-\mu D(s, s')} \mathbb{1}_{a'}(s') \mathbb{1}_a(s) P_m\left(\frac{p_{\parallel}}{p_{\parallel, a, l}}\right) P_{m'}\left(\frac{p'_{\parallel}}{p'_{\parallel, a', l'}}\right) dp_{\parallel} ds. \end{aligned} \quad (5.17)$$

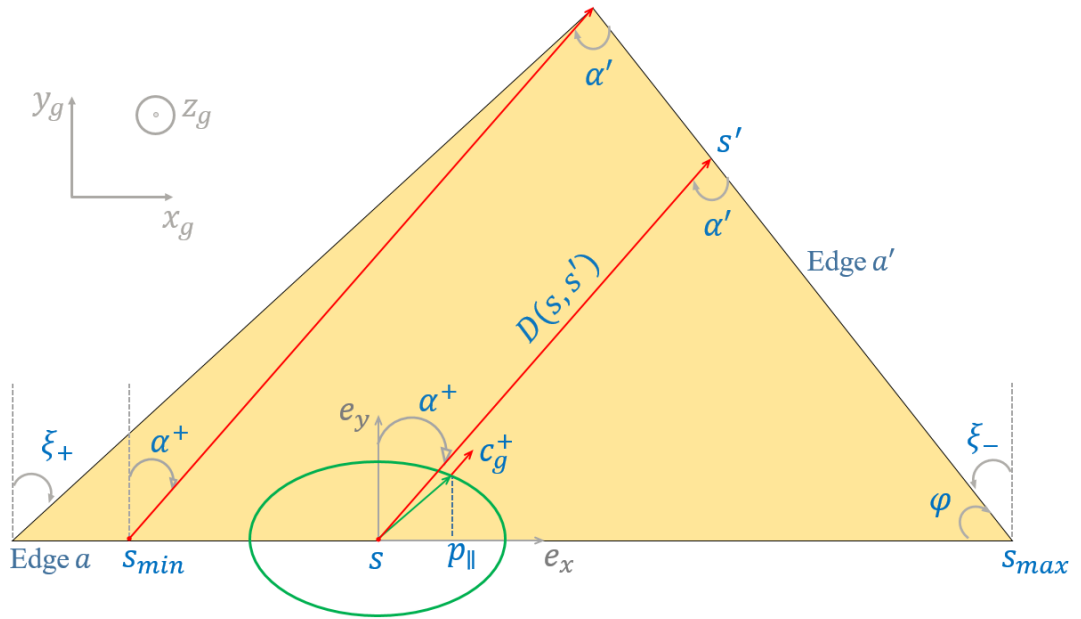
The double integral can be reduced to a single integral with respect to the momentum component $p_{\parallel}$ as follows:

$$T_{\mathbf{n}'\mathbf{n}} = \frac{2m' + 1}{2\sqrt{L_a L_{a'} p_{\parallel, a, l} p'_{\parallel, a', l'}}} \int_{-p_{\parallel, a, l}}^{p_{\parallel, a, l}} \lambda(p'_{\parallel}) h(\mu, p_{\parallel}) P_m\left(\frac{p_{\parallel}}{p_{\parallel, a, l}}\right) P_{m'}\left(\frac{p'_{\parallel}}{p'_{\parallel, a', l'}}\right) dp_{\parallel} \quad (5.18)$$

with

$$h(\mu, p_{\parallel}) = \int_0^{L_a} e^{-\mu D(s, s')} \mathbb{1}_{a'}(s') \mathbb{1}_a(s) ds = \int_{s_{\min}}^{s_{\max}} e^{-\mu D(s, s')} ds. \quad (5.19)$$

The function $h(\mu, p_{\parallel})$ can be computed analytically. In fact, $D(s, s')$ is the distance between points s and s' in Figure 5.3, and it can be computed using



the sine rule in the triangle connecting points s , s' and $s_{\max}$ as

$$D(s, s') = \frac{s_{\max} - s}{\sin \alpha'} \sin \phi = \frac{s_{\max} - s}{\cos(\phi - \alpha^+)} \sin \phi, \quad (5.20)$$

$$h(\mu, p_{||}) = \frac{1 - e^{-\tilde{\mu}(s_{\max} - s_{\min})}}{\tilde{\mu}}, \quad \tilde{\mu} = \frac{\mu \sin \phi}{\cos(\phi - \alpha^+)}, \quad \mu \neq 0. \quad (5.21)$$

Now, we need to compute $s_{\max} - s_{\min}$, which is equal to $h(0, p_{||})$, according to Equation (5.19). For $\alpha^+ \in [-\frac{\pi}{2}, \xi_-]$, $h(0, p_{||})$ is equal to zero, since no rays with such direction angles α^+ can reach the edge a' from the edge a , see Figure 5.3. On the other hand, for $\alpha^+ \in [\xi_+, \frac{\pi}{2}]$, $h(0, p_{||})$ is equal to L_a , that is, the length of the edge a , because all points on the edge a can be mapped to the edge a' by the rays with such direction angles α^+ . Finally, for $\alpha^+ \in (\xi_-, \xi_+)$, using the sine rule in the triangle connecting points $s_{\min}$, $s_{\max}$ and the top vertex of the mesh cell, one gets $h(0, p_{||}) = L_{a'} \frac{\cos(\phi - \alpha^+)}{\cos \alpha^+}$. We can summarise the definition of

$h(0, p_{||})$ as follows

$$h(0, p_{||}) = \begin{cases} 0 & \text{for } p_{||} : \alpha^+(p_{||}) \in [-\frac{\pi}{2}, \xi_-] \\ L_{a'} \frac{\cos(\phi - \alpha^+)}{\cos \alpha^+} & \text{for } p_{||} : \alpha^+(p_{||}) \in (\xi_-, \xi_+) \\ L_a & \text{for } p_{||} : \alpha^+(p_{||}) \in [\xi_+, \frac{\pi}{2}] \end{cases} \quad (5.22)$$

Recall that if the material of the structure is isotropic, then all ray direction angles α^+ are equal to the wave vector angles θ^+ , thus simplifying calculation of the function $h(\mu, p_{||})$. Furthermore, since wave number k is constant for any direction angle θ , the variable of integration in Equation (5.18) can be changed from the momentum component $p_{||}$ to the wave vector angle θ as follows

$$dp_{||} = k \cos \theta d\theta, \quad (5.23)$$

thus removing the need of wave vector curves and rendering the integrand to be defined by the direction angle θ alone [53]. It is worth noting that if the number of Legendre polynomials used in the basis set is equal to one, then the formulation in Equation (5.18) is equivalent to the SEA approximation of the system [50, 51].

5.3.2 Initial phase space densities

In this subsection, we define and discretise the initial boundary phase space density $\rho_{r,0}(s, p_{||})$ arising due to a point or an edge source. First, we consider the case of a point source at a position $\mathbf{r}_0$, which is in the interior of the mesh cell. The full phase space density at a point $\mathbf{r}$ is written in the similar way as in Equation (2.34):

$$\rho_0(\mathbf{r}, \mathbf{p}) = \frac{R}{|\mathbf{c}_{g,0}|} \frac{e^{-\mu|\mathbf{r}-\mathbf{r}_0|}}{2\pi|\mathbf{r}-\mathbf{r}_0|} \delta(\mathbf{p}-\mathbf{p}_0), \quad \mathbf{p}_0 : \mathbf{c}_{g,0} = |\mathbf{c}_{g,0}| \frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|}, \quad (5.24)$$

where the momentum vector $\mathbf{p}_0$ is such that the group velocity vector $\mathbf{c}_{g,0}(\mathbf{p}_0)$ points in the direction of $\mathbf{r}-\mathbf{r}_0$. Note that the attenuation factor μ depends on the direction of the energy flow, since $\mu = \eta \frac{\omega}{|\mathbf{c}_{g,0}|}$ by definition.

Now, one needs to determine the 2D phase space density $\rho_{a,0}(s, p_{||})$ on edge a

of the mesh cell that surrounds the interior source point $\mathbf{r}_0$. To accomplish this, we use the following equation relating delta functions similarly as in Equation (2.36):

$$\frac{\partial \omega}{\partial p_{\perp}} \delta(\omega(p_{\parallel}, p_{\perp}) - \omega_0) \delta(p_{\parallel} - p_{\parallel,0}) = \delta(\mathbf{p} - \mathbf{p}_0). \quad (5.25)$$

Applying this substitution rule to Equation (5.7), we obtain

$$\rho_{a,0}(s, p_{\parallel}) = \frac{R}{2\pi} \frac{e^{-\mu D(s, \mathbf{r}_0)}}{D(s, \mathbf{r}_0)} \cos \alpha^{-}(s, \mathbf{r}_0) \delta(p_{\parallel} - p_{\parallel,0}(s, \mathbf{r}_0)), \quad (5.26)$$

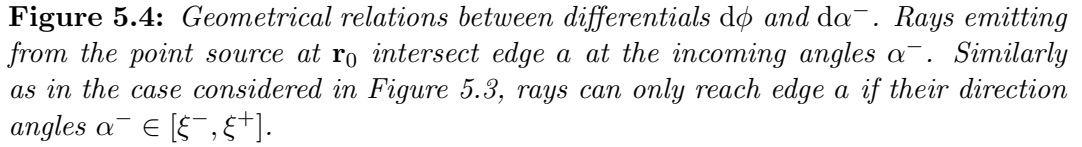
where $\cos \alpha^{-} = \frac{\partial \omega}{\partial p_{\perp}} \frac{1}{|\mathbf{c}_{g,0}^{-}|}$ is the direction cosine of the incoming group velocity vector $\mathbf{c}_{g,0}^{-}$. Projecting the phase space density $\rho_{a,0}(s, p_{\parallel})$ onto the finite basis as in (5.13) yields the coefficients $\rho_{\mathbf{n},0}$ as follows:

$$\rho_{\mathbf{n},0} = \frac{R}{4\pi} \frac{2m+1}{\sqrt{L_a p_{\parallel,a,l}}} \int_0^{L_a} \lambda(p_{\parallel,0}(s, \mathbf{r}_0)) e^{-\mu D(s, \mathbf{r}_0)} P_m \left(\frac{p_{\parallel,0}(s, \mathbf{r}_0)}{p_{\parallel,a,l}} \right) \frac{\cos \alpha^{-}}{D(s, \mathbf{r}_0)} ds. \quad (5.27)$$

Here, the additional term $\lambda(p_{\parallel,0})$ is included in the formulation to amount for possible scattering at the edge a .

Computing numerically the integral in Equation (5.27) is not straightforward, since there is an additional step in defining the momentum component $p_{\parallel,0} = p_{\parallel,0}(s, \mathbf{r}_0)$, which is used later to compute the scattering coefficients $\lambda(p_{\parallel,0})$. As mentioned earlier, since the wave number is constant in isotropic structures, the momentum component $p_{\parallel,0}$ is not needed in such cases, and determining the polar angle $\alpha^{-} = \theta^{-}$ for a given position s is sufficient to define all terms of the integrand in Equation (5.27). While this task can be accomplished using simple geometrical relations in isotropic structures [53], it is not the case for composite structures. One needs to change the variable of integration from s to $p_{\parallel}$ in Equation (5.27), which will be shown next. As can be seen in Figure 5.4, the following relation between differentials ds and $d\phi$ can be established:

$$d\phi = \frac{\cos \alpha^{-}}{D(s, \mathbf{r}_0)} ds, \quad (5.28)$$


$$\begin{aligned} d\phi &= d \arctan \left(\frac{c_{g,y}}{c_{g,x}} \right) = \frac{1}{1 + \left(\frac{c_{g,y}}{c_{g,x}} \right)^2} \left(\frac{\partial \left(\frac{c_{g,y}}{c_{g,x}} \right)}{\partial p_{||}} + \frac{\partial \left(\frac{c_{g,y}}{c_{g,x}} \right)}{\partial p_{\perp}} \frac{\partial p_{\perp}}{\partial p_{||}} \right) dp_{||} = \\ &= \frac{1}{|\mathbf{c}_g|^2} \left(c_{g,x} \frac{\partial c_{g,y}}{\partial p_{||}} - c_{g,y} \frac{\partial c_{g,x}}{\partial p_{||}} - \frac{c_{g,x}^2}{c_{g,y}} \frac{\partial c_{g,y}}{\partial p_{\perp}} + c_{g,x} \frac{\partial c_{g,x}}{\partial p_{\perp}} \right) dp_{||}, \end{aligned} \quad (5.29)$$
$$\begin{aligned} \rho_{\mathbf{n},0} &= \frac{R}{4\pi} \frac{2m+1}{\sqrt{L_a p_{||,a,l}}} \int_{p_{||,\min}}^{p_{||,\max}} \lambda(p_{||}) e^{-\mu D(s, \mathbf{r}_0)} P_m \left(\frac{p_{||}}{p_{||,a,l}} \right) f(p_{||}) dp_{||}, \\ f(p_{||}) &= \frac{1}{|\mathbf{c}_g|^2} \left(c_{g,x} \frac{\partial c_{g,y}}{\partial p_{||}} - c_{g,y} \frac{\partial c_{g,x}}{\partial p_{||}} - \frac{c_{g,x}^2}{c_{g,y}} \frac{\partial c_{g,y}}{\partial p_{\perp}} + c_{g,x} \frac{\partial c_{g,x}}{\partial p_{\perp}} \right). \end{aligned} \quad (5.30)$$

The limits of the integral $p_{\parallel,\min}$ and $p_{\parallel,\max}$ are such that the corresponding incoming ray angles $\alpha_{\min}^-$ and $\alpha_{\max}^-$ are equal to ξ^- and ξ^+ , respectively, see Figure 5.4. This is to ensure that rays incoming from the source point $\mathbf{r}_0$ intersect edge a . Furthermore, the distance $D(s_0, \mathbf{r}_0)$ is equal to $h_{\perp}/\cos \alpha^-$, where $h_{\perp}$ is the height of the triangle connecting the point $\mathbf{r}_0$ and edge a . The described procedure is applied accordingly for other edges of the mesh cell. The case of a source along edge a can be modelled by the following phase space density

$$\rho_{a,0}(s, p_{\parallel}) = \tilde{R} \cos \alpha^+ \mathbb{1}_a(s) \delta(p_{\parallel} - p_{\parallel,0}), \quad (5.31)$$

where $\tilde{R}$ is the intensity of the energy flow in the direction angle α^+ [53]. The angle α^+ is defined by the momentum component $p_{\parallel,0}$, as was discussed in subsection 5.2.1. The coefficients $\rho_{\mathbf{n},0}$ after discretisation are then given as

$$\begin{aligned} \rho_{\mathbf{n},0} &= \frac{2m+1}{2\sqrt{L_a p_{\parallel,a,l}}} \tilde{R} \cos \alpha^+ P_m \left(\frac{p_{\parallel,0}}{p_{\parallel,a,l}} \right) \int_0^{L_a} \mathbb{1}_a(s) ds = \\ &= \frac{2m+1}{2} \sqrt{\frac{L_a}{p_{\parallel,a,l}}} \tilde{R} \cos \alpha^+ P_m \left(\frac{p_{\parallel,0}}{p_{\parallel,a,l}} \right). \end{aligned} \quad (5.32)$$

Having discretised the initial 2D phase space density $\rho_{\Gamma,0}$ and the boundary integral operator T , we can compute the stationary or long-time limit 2D phase space density $\rho_{\Gamma,\infty}$. Writing the coefficients $\rho_{\mathbf{n},\infty}$ and $\rho_{\mathbf{n},0}$ as arrays ϱ_{∞} and ϱ_0 allows to establish the discrete version of Equation (2.39) as

$$\varrho_{\infty} = \sum_{k=0}^{\infty} \mathbf{T}^k \varrho_0 \rightarrow \varrho_{\infty} = (\mathbf{I} - \mathbf{T})^{-1} \varrho_0 \quad (5.33)$$

where $\mathbf{T}$ is the matrix form of the boundary operator T and $\mathbf{I}$ is the identity matrix. The size of matrices $\mathbf{T}$ and $\mathbf{I}$ is $n_a n_m n_l$, where n_a is the number of edges in the meshed structure, n_m is the number of Legendre polynomials used for discretisation of the momentum component $p_{\parallel}$, and n_l is the number of propagating modes in the structure. In thin composite plates, n_l is equal to 3 - longitudinal, shear and bending modes. For thick composite plates modelled by the WFE method, n_l increases with frequency.

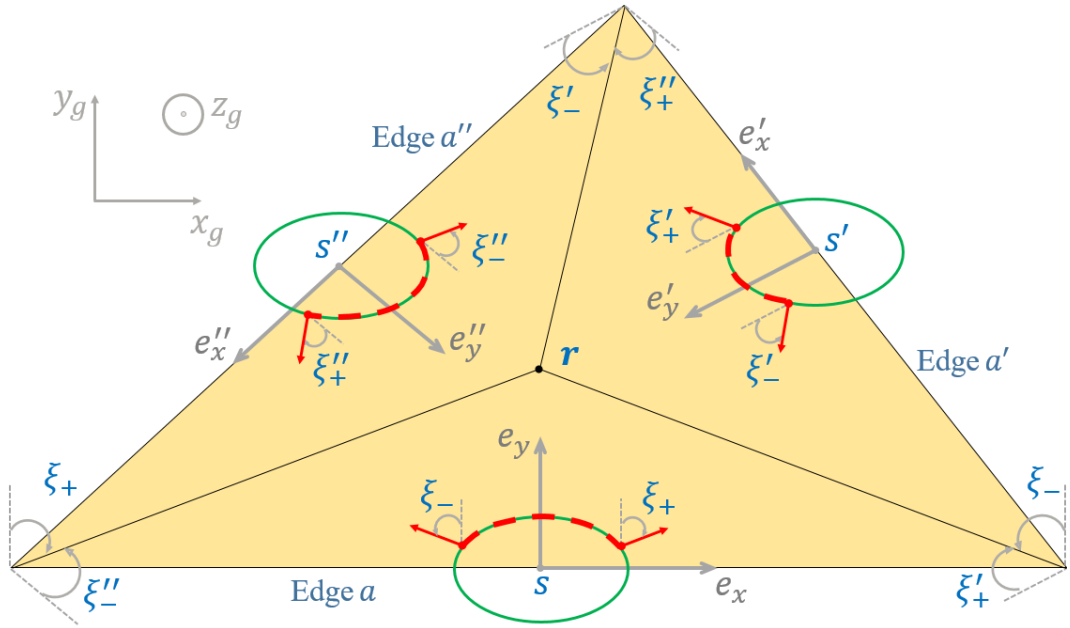


Figure 5.5: Partitioning of the typical wave vector curve (depicted by green ovals) between edges of the mesh element. Red dashed arcs represent $\omega_a^{-1}(\omega_0)$ in Equation (5.36). They are defined by imposing the outgoing group velocity vector angle α^+ to be in range $[\xi^-, \xi^+]$, so that outgoing rays from edge a can reach the point $\mathbf{r}$.

5.3.3 Stationary energy and intensity

As a final step, we need to compute the stationary energy density $\rho_\infty(\mathbf{r})$ defined by the following relation:

$$\rho_\infty(\mathbf{r}) = \int \rho_\infty(\mathbf{r}, \mathbf{p}) d\mathbf{p}. \quad (5.34)$$

The integrand can be developed further using Equations (5.7) and (2.41) as

$$\rho_\infty(\mathbf{r}, \mathbf{p}) = e^{-\mu D(s(\mathbf{r}, \mathbf{p}), \mathbf{r})} \rho_{\Gamma, \infty}(s(\mathbf{r}, \mathbf{p}), p_\parallel) \delta(\omega(p_\parallel, p_\perp) - \omega_0). \quad (5.35)$$

Substituting this relation into Equation (5.34) and using the *coarea formula* for delta functions [169] yields

$$\rho_\infty(\mathbf{r}) = \int_{\omega^{-1}(\omega_0)} \rho_{\Gamma, \infty}(s, p_\parallel) e^{-\mu D(s, \mathbf{r})} \frac{d\sigma}{|\mathbf{c}_g|}. \quad (5.36)$$

Here, $\omega^{-1}(\omega_0)$ is all solution pairs $(p_\parallel, p_\perp)$ such that $\omega(p_\parallel, p_\perp) = \omega_0$, and $d\sigma$ is the measure on the curve $\omega^{-1}(\omega_0)$. After expanding the 2D phase space

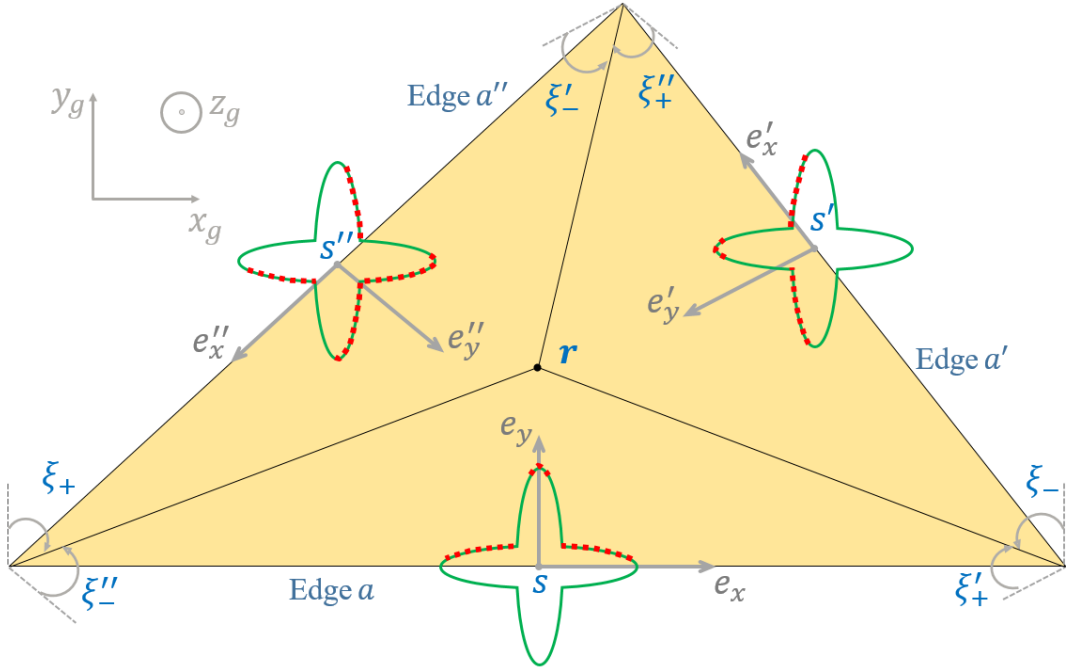


Figure 5.6: Partitioning of the shear wave vector curve (depicted in green) between edges of the mesh element. Red dashed arcs represent $\omega_a^{-1}(\omega_0)$ in Equation (5.36). They are defined by imposing the outgoing group velocity vector angle α^+ to be in range $[\xi^-, \xi^+]$, so that outgoing rays from edge a can reach the point $\mathbf{r}$.

density $\rho_{\Gamma, \infty}$ as in Equation (5.13), we obtain

$$\begin{aligned} \rho_{\infty}(\mathbf{r}) &= \sum_{\mathbf{n}} \frac{\rho_{\mathbf{n}, \infty}}{\sqrt{L_a p_{\parallel, a, l}}} \int_{\omega_a^{-1}(\omega_0)} \frac{e^{-\mu D(s, \mathbf{r})}}{|\mathbf{c}_g|} \mathbb{1}_a(s) P_m \left(\frac{p_{\parallel}}{p_{\parallel, a, l}} \right) d\sigma = \\ &= \sum_{\mathbf{n}} \frac{\rho_{\mathbf{n}, \infty}}{\sqrt{L_a p_{\parallel, a, l}}} \int_{\omega_a^{-1}(\omega_0)} \frac{e^{-\mu D(s, \mathbf{r})}}{|\mathbf{c}_g|} P_m \left(\frac{p_{\parallel}}{p_{\parallel, a, l}} \right) d\sigma \end{aligned}, \quad (5.37)$$

where $\mathbf{n} = (a, m, l)$ includes indices of edges forming the mesh cell containing the point $\mathbf{r}$. The domain of integration $\omega_a^{-1}(\omega_0)$ is the part of the wave vector curve at a fixed frequency ω_0 that produces rays connecting the point $\mathbf{r}$ and some point s on the edge a . Figure 5.5 presents an illustration of how the wave vector curve (depicted by green ovals) is partitioned as $\omega_a^{-1}(\omega_0)$ (depicted by red dashed arcs) between edges a , a' and a'' . It is clear that only rays with angles of the outgoing group velocity vectors $\alpha^+ \in [\xi_-, \xi_+]$ can reach the point $\mathbf{r}$ starting from the edge a . The same applies for edges a' and a'' . The wave vector curve considered in Figure 5.5 is divided smoothly into three parts, each corresponding for one edge. However, this is not always the case. In

fact, for the propagating shear mode of a composite laminate considered in subsection 3.3.2, the partition is not smooth, see Figure 5.6. Note that in the local coordinate systems of edges a' and a'' , there exist values of $p_{||}$ for which there are two outgoing shear waves. This feature was previously highlighted in Figures 3.6 and 3.8.

Once the partitioned parts of the wave vector curve are determined, we can change the variable of integration from $d\sigma$ to $dp_{||}$ via the following relation:

$$d\sigma = \sqrt{dp_{||}^2 + dp_{\perp}^2} = \sqrt{1 + \left(\frac{\partial p_{\perp}}{\partial p_{||}}\right)^2} dp_{||} = \sqrt{1 + \left(\frac{c_{g,x}}{c_{g,y}}\right)^2} dp_{||} = \frac{dp_{||}}{\cos \alpha}, \quad (5.38)$$

where $\frac{\partial p_{\perp}}{\partial p_{||}}$ is defined by Equation (5.6). The clockwise orientation of the curve $\omega^{-1}(\omega_0)$ in the coordinate system of the mesh cell x_g, y_g is consistent with the variation of the momentum component $p_{||}$ in the respective local coordinate systems of edges. After changing the variable of integration in Equation (5.36), we obtain:

$$\rho_{\infty}(\mathbf{r}) = \sum_{\mathbf{n}} \frac{\rho_{\mathbf{n},\infty}}{\sqrt{L_a p_{||,a,l}}} \int_{\Omega_{a,p_{||}}} \frac{e^{-\mu D(s,\mathbf{r})}}{|\mathbf{c}_g| \cos \alpha^+} P_m \left(\frac{p_{||}}{p_{||,a,l}} \right) dp_{||}, \quad (5.39)$$

where $\Omega_{a,p_{||}}$ covers all momentum components $p_{||}$ such that the momentum vectors $\mathbf{p}$ produce outgoing waves in the local coordinate system e_x, e_y of the edge a and $\omega_a(p_{||}, p_{\perp}) = \omega_0$. In the case of an oval-shaped wave vector curve considered in Figure 5.5, $\Omega_{a,p_{||}}$ is equal to $[p_{||,\min}, p_{||,\max}]$ with $p_{||,\min(\max)}$ being the momentum component $p_{||}$ such that the outgoing angle $\alpha^+ = \xi_{-(+)}$. Finally, to compute the total energy density at a point $\mathbf{r}$ in the interior of the same polygon as the source point $\mathbf{r}_0$, we need to include the contribution of the initial energy density at that point. Technically, the total energy density $\epsilon(\mathbf{r})$ can be written as

$$\epsilon(\mathbf{r}) = \rho_{\infty}(\mathbf{r}) + \mathbb{1}_{\mathbf{r}_0}(\mathbf{r}) \frac{R}{|\mathbf{c}_{g,0}|} \frac{e^{-\mu|\mathbf{r}-\mathbf{r}_0|}}{2\pi|\mathbf{r}-\mathbf{r}_0|}, \quad (5.40)$$

where $\mathbb{1}_{\mathbf{r}_0}(\mathbf{r}) = 1$ if the points $\mathbf{r}$ and $\mathbf{r}_0$ are in the interior of the same mesh cell, and 0 otherwise.

One can also compute the intensity vector field $\mathbf{I}(\mathbf{r})$ which is defined as

$$\mathbf{I}(\mathbf{r}) = \int \rho_{\infty}(\mathbf{r}, \mathbf{p}) \mathbf{c}_{\mathbf{g}} d\mathbf{p}. \quad (5.41)$$

After proceeding similarly to the case of the stationary energy density, one obtains:

$$\mathbf{I}(\mathbf{r}) = \sum_{\mathbf{n}} \frac{\rho_{\mathbf{n},\infty}}{\sqrt{L_a p_{\parallel,a,l}}} \int_{\Omega_{a,p_{\parallel}}} \begin{bmatrix} \tan \alpha^+ \\ 1 \end{bmatrix} e^{-\mu D(s,\mathbf{r})} P_m \left(\frac{p_{\parallel}}{p_{\parallel,a,l}} \right) dp_{\parallel}. \quad (5.42)$$

The intensity vector field $\mathbf{I}(\mathbf{r})$ allows estimating the direction of mean energy flow at the point $\mathbf{r}$, which can be helpful to determine energy transfer paths.

5.4 Numerical results

In this section, we consider several numerical case examples to demonstrate the application of the DEA method, both the classical method for isotropic structures and the modified version presented in this chapter for composite structures. In all examples we compute the mean acceleration, which can be estimated as

$$\bar{a}(\mathbf{r}) = \sqrt{\epsilon(\mathbf{r}) \frac{\omega^2}{\rho_{\text{vol}}(\mathbf{r}) h(\mathbf{r})}}, \quad (5.43)$$

where $\epsilon(\mathbf{r})$ is the total mean energy density given by Equation (5.40), $\rho_{\text{vol}}(\mathbf{r})$ and $h(\mathbf{r})$ are the volumetric mass density and the thickness of the polygon containing the point $\mathbf{r}$, respectively.

This section is organised as follows. In subsection 5.4.1, we compute the mean response of the polygonally shaped plate due to a point and an edge sources. Isotropic and composite materials are considered to demonstrate how the wave energy is transported and scattered in each of these cases. Also, the influence of the number of Legendre polynomials on the results is investigated. Then, in subsection 5.4.2, we compute and compare the DEA and FEM frequency-averaged responses of an L-shaped angle-ply laminated plate subjected to a harmonic point excitation force.

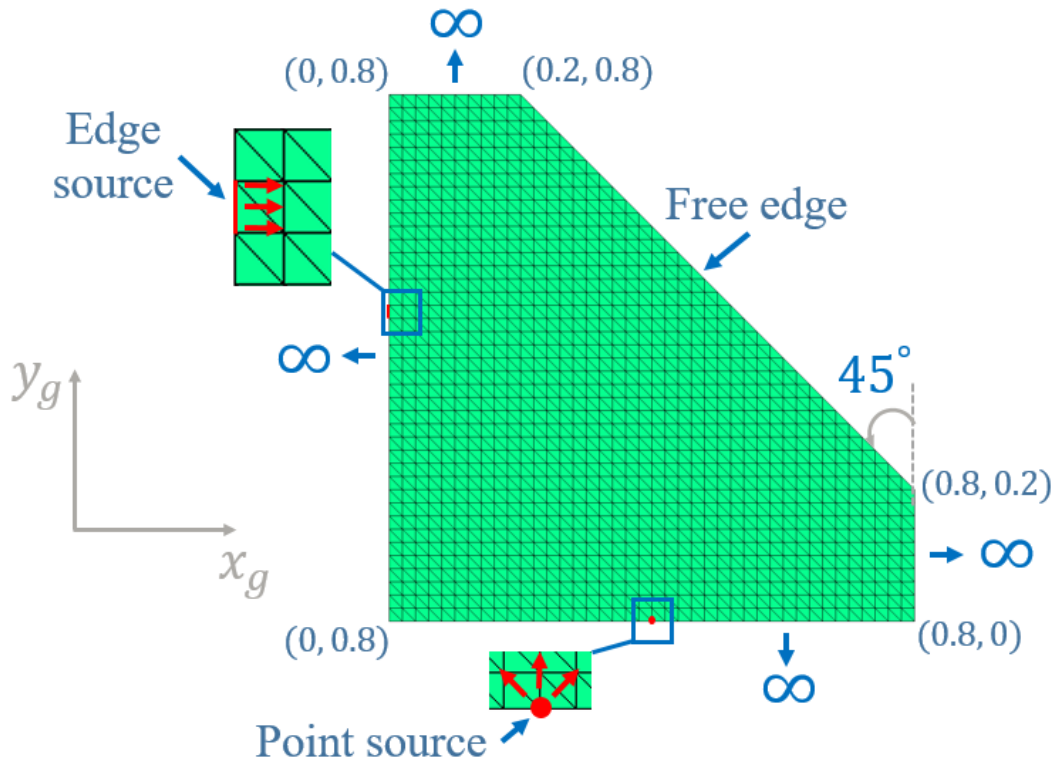


Figure 5.7: A polygonally shaped semi-infinite plate meshed with two-dimensional triangular shell elements. An edge and a point sources are assumed to be applied on the plate on the vertical and horizontal edges, respectively. The diagonal edge is free, and rays hitting this edge reflect back into the plate. Reflection of rays at other edges is prohibited thus imitating “infinity” of the plate through these edges.

Finally, in subsection 5.4.3, we turn to the case of an electric vehicle gearbox, where we compare the results from the classical DEA and the full FEM simulation. We present a limitation of the classical DEA approach on this example and demonstrate how the composite DEA can provide more accurate results. The work presented in this subsection was performed during the industrial placement at “Romax Technology” company.

5.4.1 Test case for validation

We consider a semi-infinite plate with one free edge as shown in Figure 5.7. The mesh consists of 1216 points and 2300 triangular shell elements. The plate thickness is equal to 5 mm. The horizontal edges of the plate are 20 and 80 cm in length, the same applies for the vertical edges. The diagonal edge of the plate is inclined at 45° to the vertical axis and equal to 84.85 cm in length. The “infinity” condition at the straight edges is achieved by prohibiting reflection

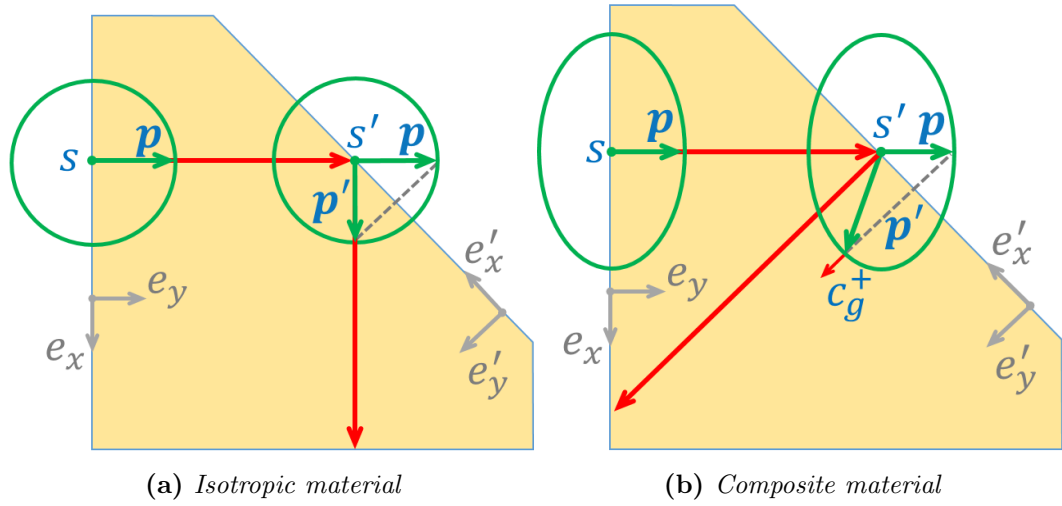


Figure 5.8: A geometrical illustration of the ray paths (shown in red) starting from the edge source at the vertical edge. Wave vector curves (shown in green) are a circle for the isotropic material case (a) and an oval for the composite material case (b).

	Aluminium	Stainless Steel
Young's Modulus (N/m ²)	71×10^9	210×10^9
Poisson's ratio	0.33	0.3
Density (kg/m ³)	2740	7800

Table 5.1: Engineering constants of aluminium and stainless steel materials.

of rays hitting these edges. The correspondent elements of the matrix $\mathbf{T}$ in (5.33) are equal to zero. This setup allows us to work with a small geometrical mesh model and consider local energy scattering events without diffusive type mixing of infinite number of reflections that would occur in a finite model.

First, we assume an edge source of strength $R = 1$ at the vertical edge connecting points with coordinates $(0, 48)$, $(0, 50)$. Other parameters of the initial energy density given by Equation (5.31) are $l = B$, $\eta = 0$, $\omega = 3000$ Hz and $p_{n,0} = 0$, that is, the bending mode is excited at the edge of the undamped plate, and the momentum vector $\mathbf{p}$ points in the direction normal to the edge.

It is important to emphasise that the stationary energy density solution given by the infinite sum $\sum_{k=0}^{\infty} T^k \rho_{\Gamma,0}$ is still converging in the absence of damping because of artificial “infinity” condition at the vertical and horizontal edges.

We consider two plate materials: the first one is isotropic stainless steel, and

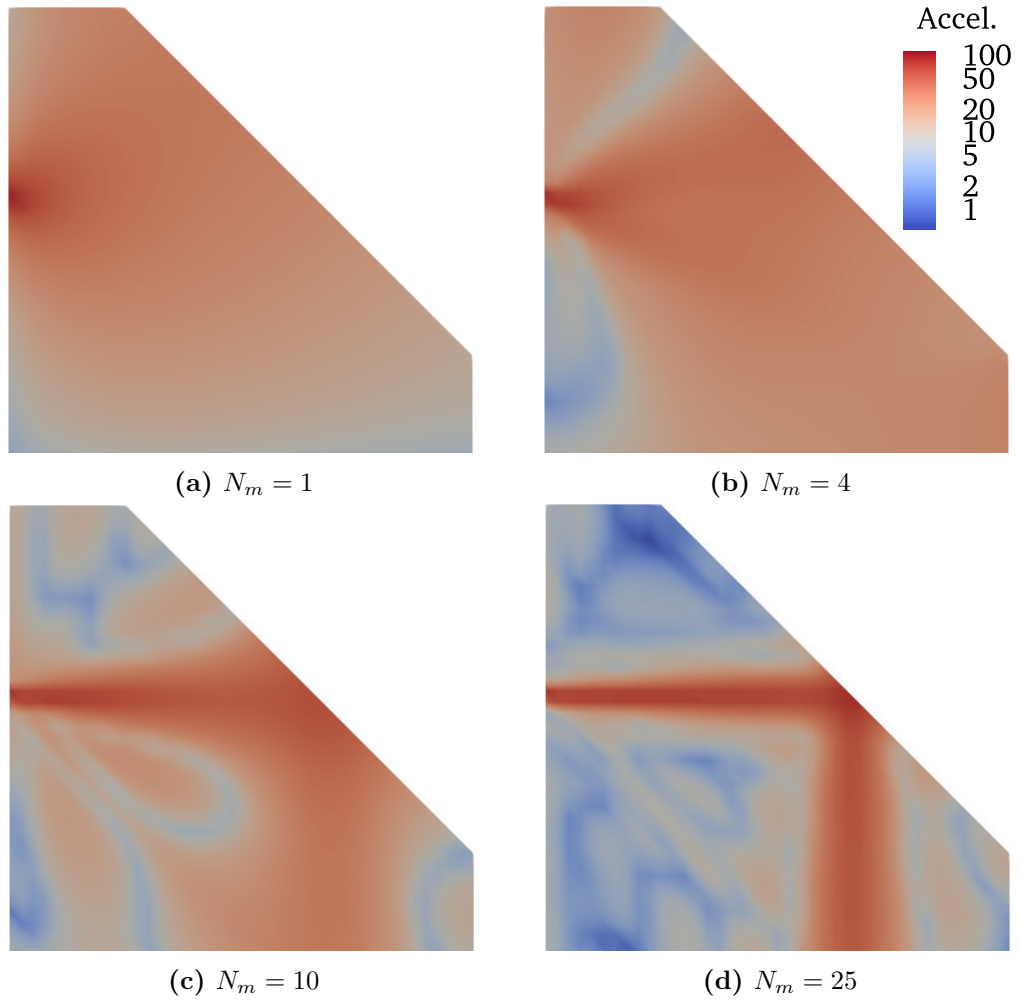


Figure 5.9: The mean acceleration response levels of the polygonally shaped semi-infinite isotropic plate due to an edge source as a function of the number of Legendre polynomials N_m . The acceleration is given in m/s^2 .

the second one is a composite regular cross-ply laminate with the lamination scheme $0^\circ/90^\circ/0^\circ/90^\circ/0^\circ$, considered in-depth in Sections 3.7 and 4.5. The material parameters of stainless steel and the composite laminae are given in Tables 5.1 and 3.1, respectively.

The schematic illustrations presented in Figure 5.8 can give an idea of how initial energy might distribute over the structure in isotropic and composite material cases.

In the case of isotropic material, the Hamilton function fixes the amplitude of the momentum vector $|\mathbf{p}|$; therefore, the curve of possible momentum values has the form of a circle, see Figure 5.8a. Furthermore, the wave energy travels along the momentum vector since the group velocity and momentum vectors

are collinear and co-directed. Rays emitting from the vertical edge reflect from the free diagonal edge at the right angle and travel until hitting and leaving the horizontal edge due to the “infinity” condition imposed. Reflection at the right angle happens because the momentum components $p_{||}$ and $p'_{||}$ must be equal due to continuity condition, see subsection 3.3.1. In isotropic structures, this condition yields *Snell’s law* stating that the angles of incidence and reflection must be equal.

In the case of composite material, the Hamilton function does not fix the amplitude of the momentum vector, and the wave energy travels along the direction of the group velocity vector. The bending wave vector curve has the form of an oval, see Figure 5.8b. Upon reaching the diagonal edge, rays reflect in the direction of the outgoing group velocity vector $\mathbf{c}_g^+$ and which is derived from the continuity condition $p_{||} = p'_{||}$. The angles of incidence and reflection are not equal in this case.

Figures 5.9 and 5.10 present the mean acceleration response amplitudes of the isotropic and composite plate due to an edge source as a function of the number of Legendre polynomials used in discretisation of the momentum component $p_{||}$. As mentioned at the end of subsection 5.3.1, when only the first Legendre polynomial $P_0(x) = 1$ is used, the DEA method works just like the SEA method. The wave energy field is diffusive and consists of a superposition of rays travelling uniformly in all directions, see Figures 5.9a and 5.10a for the isotropic and composite material cases, respectively. As the number of Legendre polynomials used in the basis set increases, the vibrational response of the structure changes so that the initial energy density at the source edge propagates in a less diffusive way through the structure. At $N_m = 25$, one can notice a specular type of reflection of the energy density at the free edge, following the geometrical ray paths described in Figure 5.8. Therefore, as the number of Legendre polynomials increases, the DEA method works like a full ray-tracing method [50, 51].

One can compute the intensity vector field $\mathbf{I}(\mathbf{r})$ using Equation (5.42). Fig-

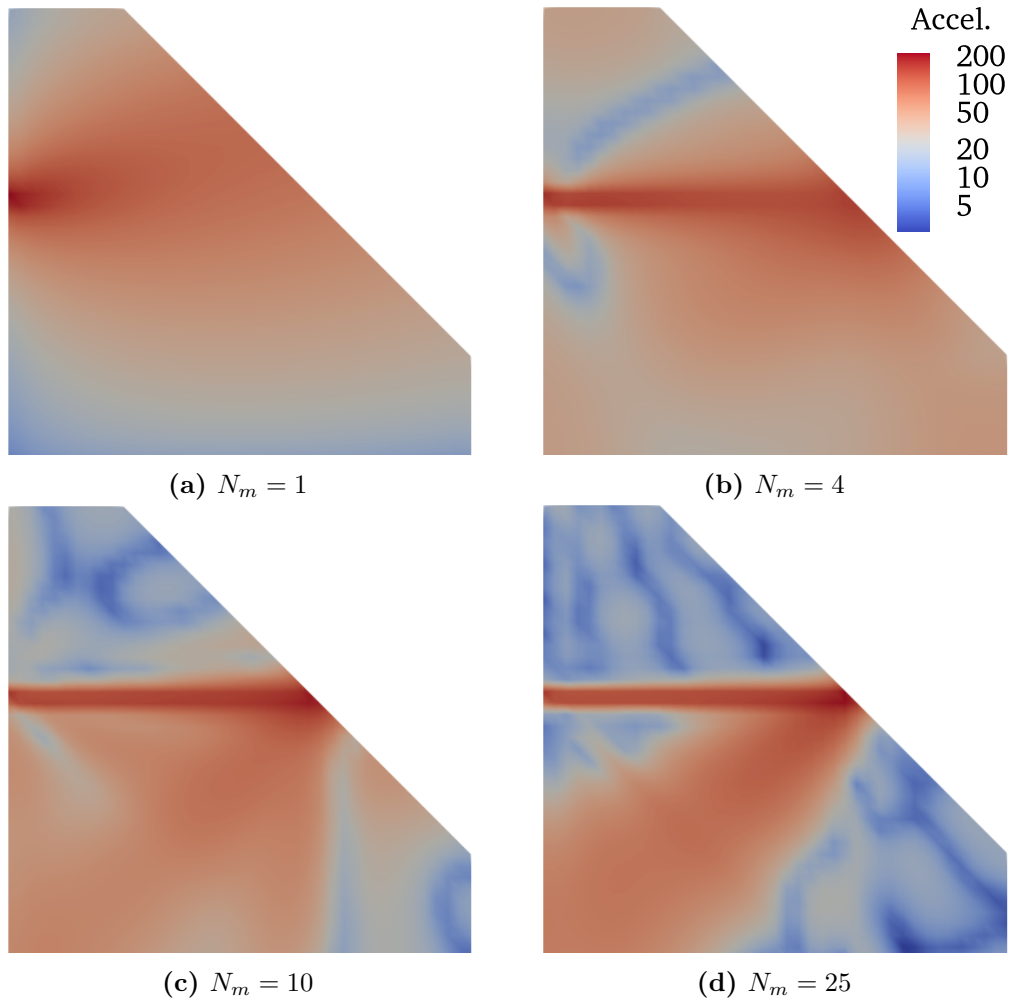


Figure 5.10: *The mean acceleration response levels of the polygonally shaped semi-infinite composite plate due to an edge source as a function of the number of Legendre polynomials N_m .*

Figure 5.11 presents the intensity vector field in the isotropic and composite material cases and how this field varies as a function of the number of Legendre polynomials. In the diffuse field case, that is, when $N_m = 1$, one can clearly see that the wave energy is transported in all directions, see Figures 5.11a and 5.11b for the isotropic and composite material cases. Nevertheless, in the lower part of the plate, the intensity vectors are nearly normal to the horizontal edge in the isotropic material case and directed towards the left corner in the composite material case, visually representing that the group velocity vector field governs the wave energy paths. In the full ray-tracing approximation, that is, when $N_m = 25$, the intensity vector field is consistent with the mean acceleration distribution over the structure, see Figures 5.11c and 5.11d.

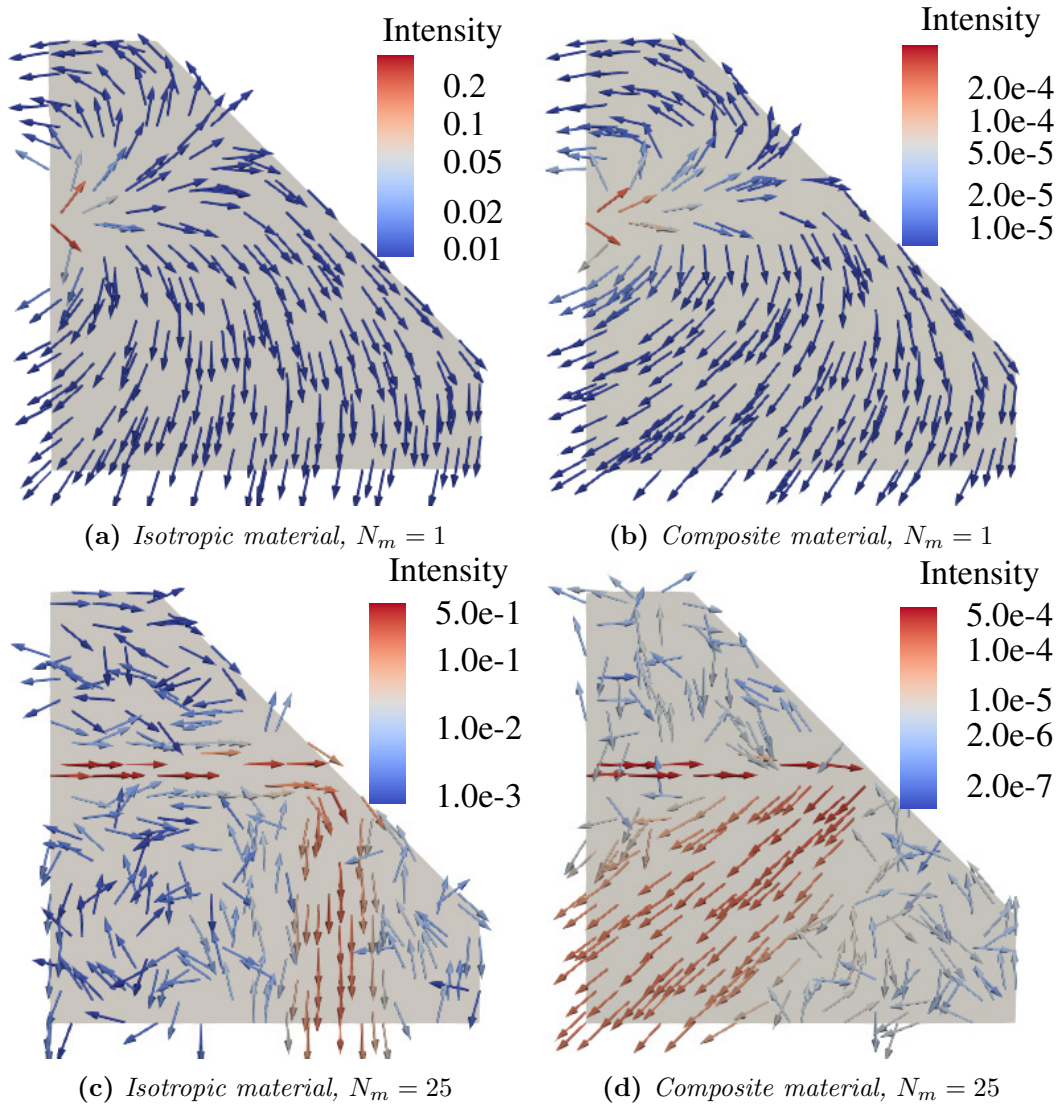


Figure 5.11: The intensity vector field of the isotropic (a,c) and composite (b,d) plate cases arising due to an edge source at the vertical edge as a function of the number of Legendre polynomials used in the basis set.

Now, we consider a point source of strength $R = 1$ at the point $(0.4, 0)$ on the horizontal edge, and the bending mode is excited at this point. In experimental setups, this is usually achieved by using the shaker excitation method. Even though one might argue that the shaker cannot be physically applied on the edge of the structure, it is reminded that the structure is assumed to be infinite through the vertical and horizontal edges by imposing no reflection of the wave energy on them.

Figures 5.12 and 5.13 represent the mean acceleration response levels arising due to the point source at the horizontal edge in the isotropic and composite

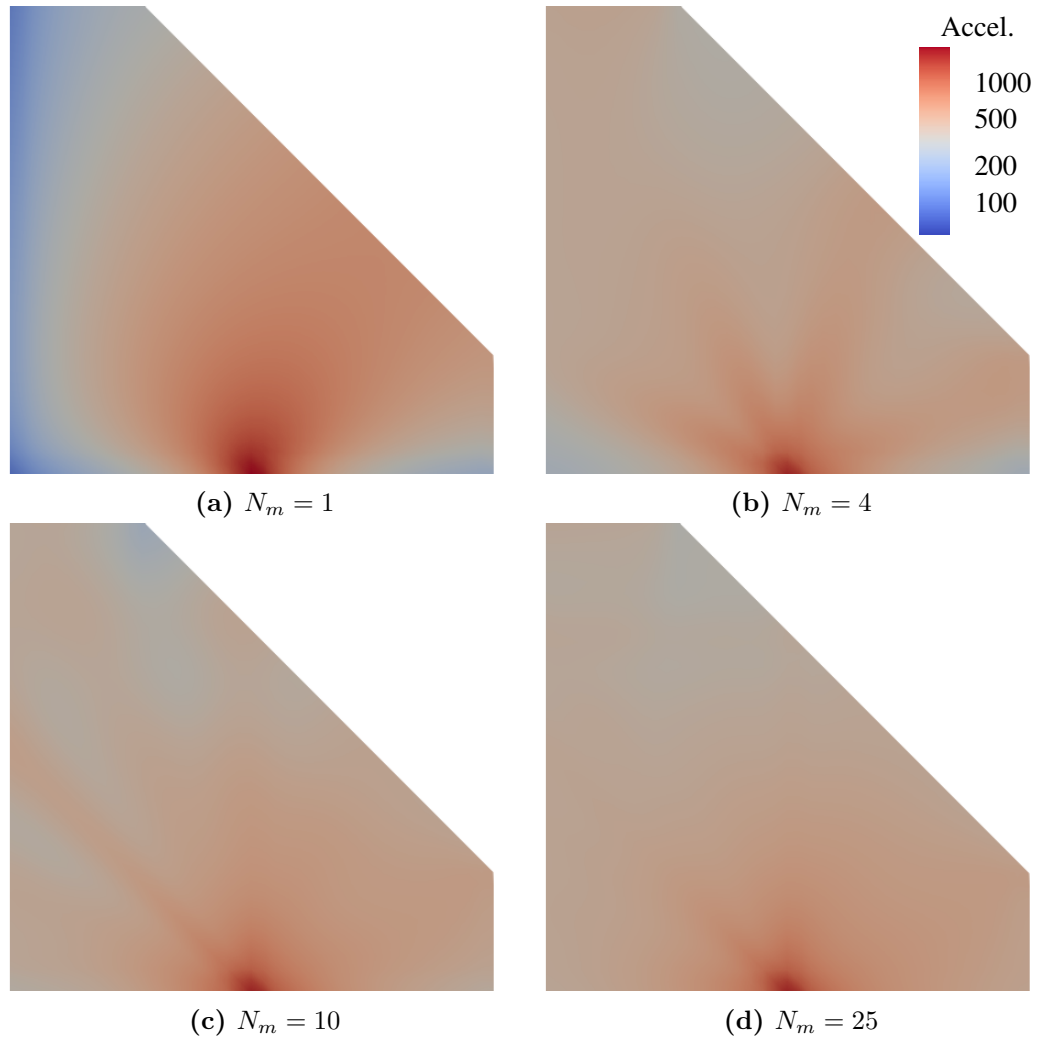


Figure 5.12: The mean acceleration response levels of the polygonally shaped semi-infinite isotropic plate due to an edge source as a function of the number of Legendre polynomials N_m . The acceleration is given in m/s^2 .

material cases, respectively. As the number of Legendre polynomials used to represent the momentum space increases, the vibrational response of the plate becomes more diffusive; however, one can still see individual ray stripes reflecting from the free edge in Figures 5.12c and 5.13c. Nevertheless, it appears that using four Legendre polynomials in the basis set is sufficient to adequately predict the vibrational response of the structure to a point source. Finally, as discussed in subsection 3.7.1, the bending wave energy is fully reflected at the free edge without mode coupling at the frequency considered; therefore, the wave energy is not stored in in-plane modes.

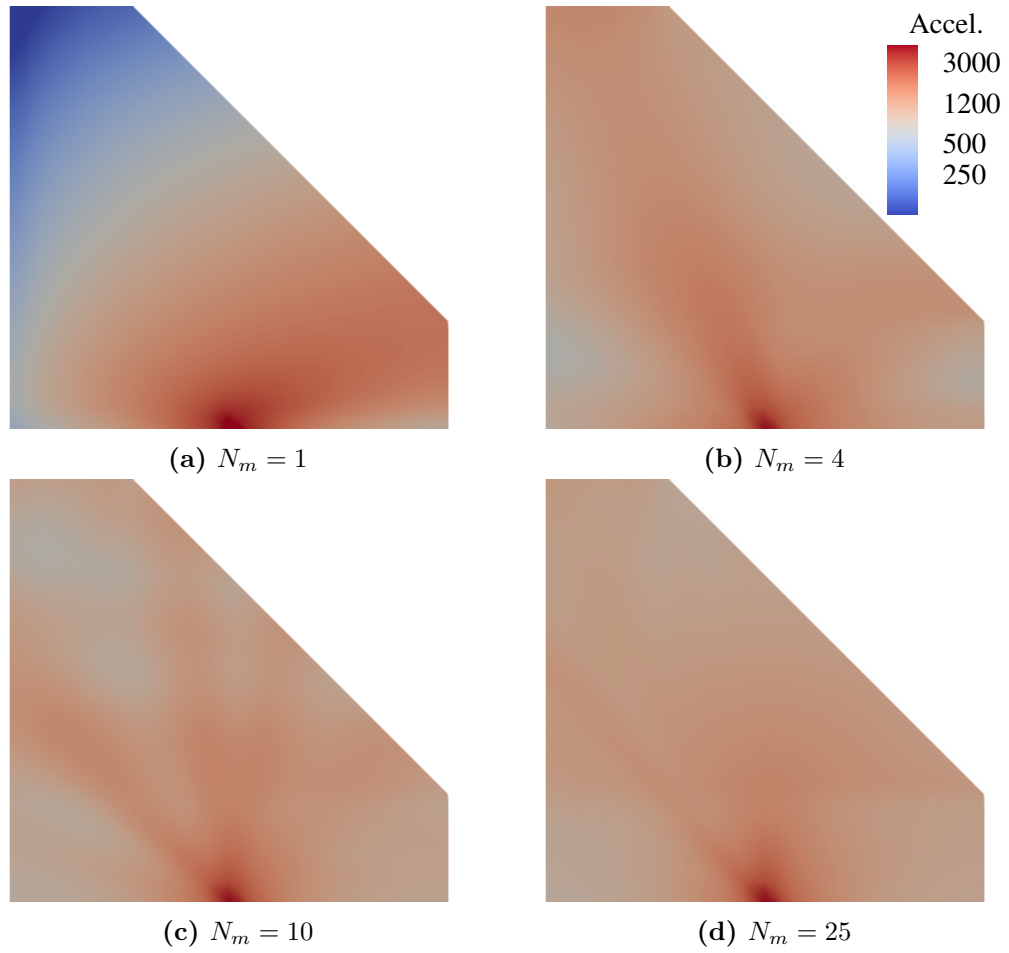


Figure 5.13: The mean acceleration response levels of the polygonally shaped semi-infinite isotropic plate due to an edge source as a function of the number of Legendre polynomials N_m . The acceleration is given in m/s^2 .

5.4.2 Two composite plates joined at a right angle

In this subsection, we compute the mean acceleration response levels of an L-shaped composite plate due to a harmonic point excitation force at the middle of the ground plate. The geometrical configuration of the model is shown in Figure 5.14. The constitutive plates are regular angle-ply laminates with the lamination scheme $45^\circ / -45^\circ / 45^\circ / -45^\circ / 45^\circ$ and the total thickness of 5 mm. Hysteretic damping of 5% is assumed.

Figure 5.15 shows a comparison of mean acceleration levels obtained from DEA and frequency-averaged FEM simulations at the frequency $f = 3150$ Hz. Four Legendre polynomials were used in the basis set to discretise the momentum space. DEA acceleration amplitudes are averaged between three propagating modes in the structure: bending, shear, and longitudinal modes. In Fig-

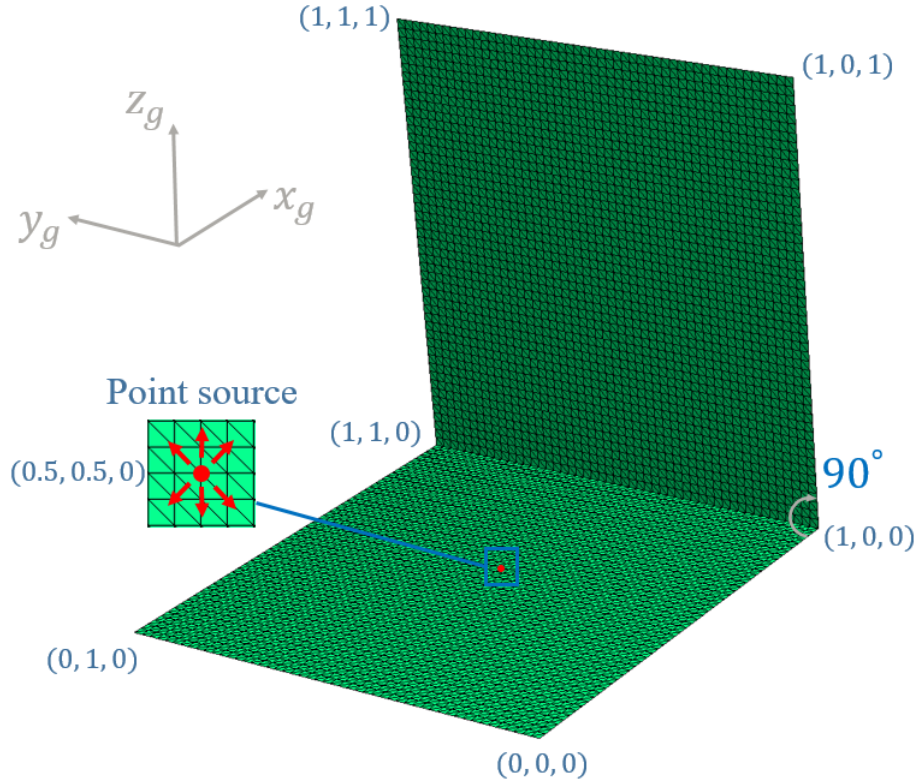


Figure 5.14: An L-shaped composite plate meshed with two-dimensional triangular shell elements. A point source is assumed to be applied on the middle of the ground plate.

ures 5.15c and 5.15d, lines representing high acceleration levels (coloured in red and yellow) are inclined at 45° and -45° with respect to the global x axis. This is because the wave vector curves are 45° rotated with respect to the global x axis, see Figures 3.7 and 3.8. The wave energy scatters from corners of the ground plate and propagates into the L-plate. The overall DEA and FEM acceleration distributions agree well, despite the presence of phase shifts at the wavelength scale in the FEM result, see Figures 5.15a and 5.15b.

The direction of energy flow can be found out by computing the intensity vector field. In contrast to the polygonally shaped plate case, mode coupling occurs at the junction between plates; therefore, we can visualise the energy flow of all modes. Figure 5.16a presents the intensity vector field of the bending mode. Note that energy is transported in all directions as expected. However, the highest intensity values in the ground plate occur along directions of 45° and -45° with respect to the global x axis. The intensity field of the

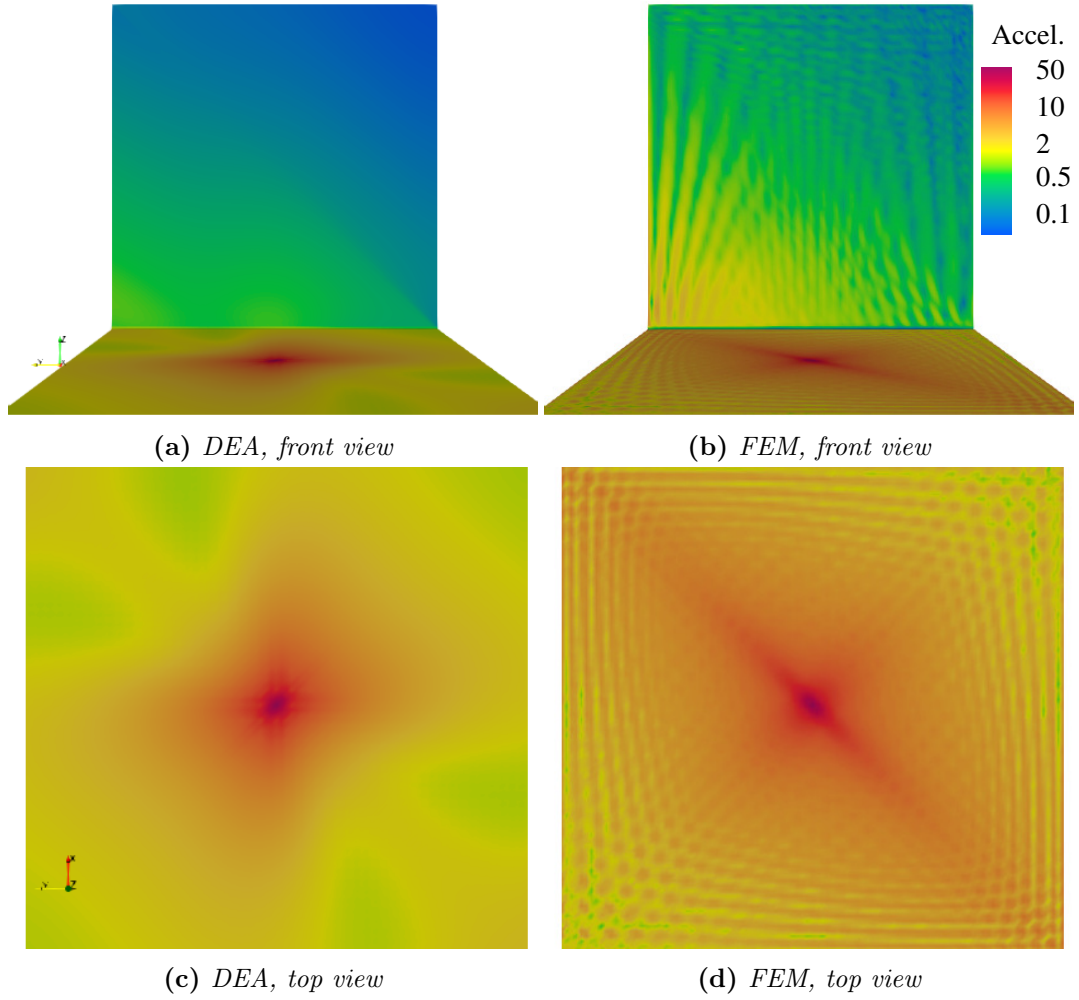


Figure 5.15: The mean DEA and FEM acceleration response levels of an L-shaped composite plate due to a harmonic point excitation force of 1 N at the frequency $f = 3150$ Hz. FEM results are frequency-averaged over one-third octave band with the centre frequency $f = 3150$ Hz. The acceleration is given in m/s^2 .

bending mode is in accordance with the mean acceleration distribution in the L-plate since the bending mode mainly governs the energy transport, whereas the contribution from in-plane modes is small. Nevertheless, one can note how the energy stored in the longitudinal mode propagates through the structure, see Figure 5.16b. The most significant intensity values arise at the corner area of the junction between plates. This is consistent with the intensity distribution of the bending mode, having the highest values at the correspondent corner area at the ground plate.

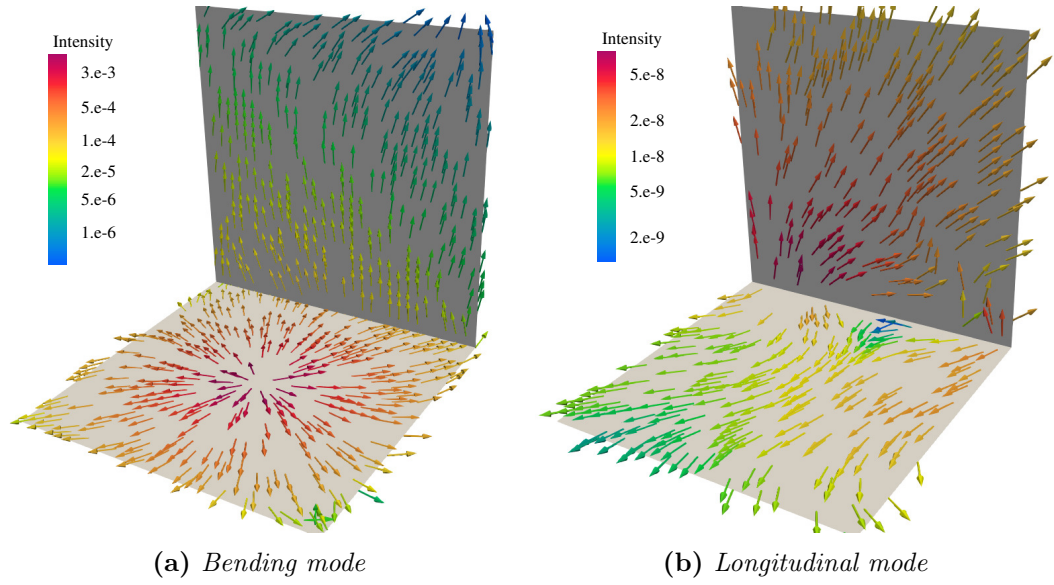


Figure 5.16: The intensity vector field of an L-shaped composite plate arising due to a point source at the middle of the ground plate. Vector colouring represents intensity amplitudes.

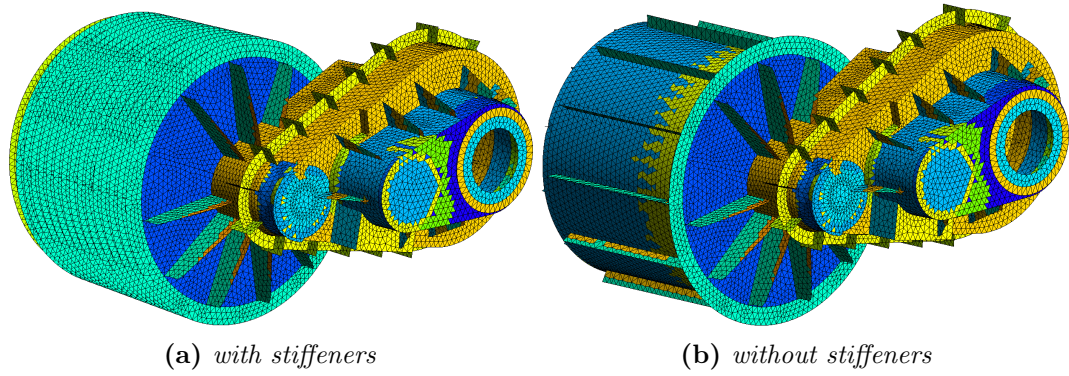


Figure 5.17: The FE meshes of the full gearbox housing (a) and the gearbox housing without stiffeners along the longitudinal axis of the cylindrical part (b).

5.4.3 Electric Vehicle Gearbox

This subsection presents part of the work performed during an industrial placement at Romax Technology. A gearbox housing of the electric vehicle powertrain is considered - its geometrical mesh model is shown in Figure 5.17a. The structure consists of a lower stiffened cylindrical part and an upper non-stiffened part. The FE model contains 96672 nodes and 48713 triangular shell elements of size ~ 3 mm. The structure is made of isotropic aluminium with material parameters given in Table 5.1. The volume of the structure is 6176.3 cm^3 , and its mass is 17 kg. The goal of the work was to perform a modal frequency response analysis of the structure subjected to a harmonic

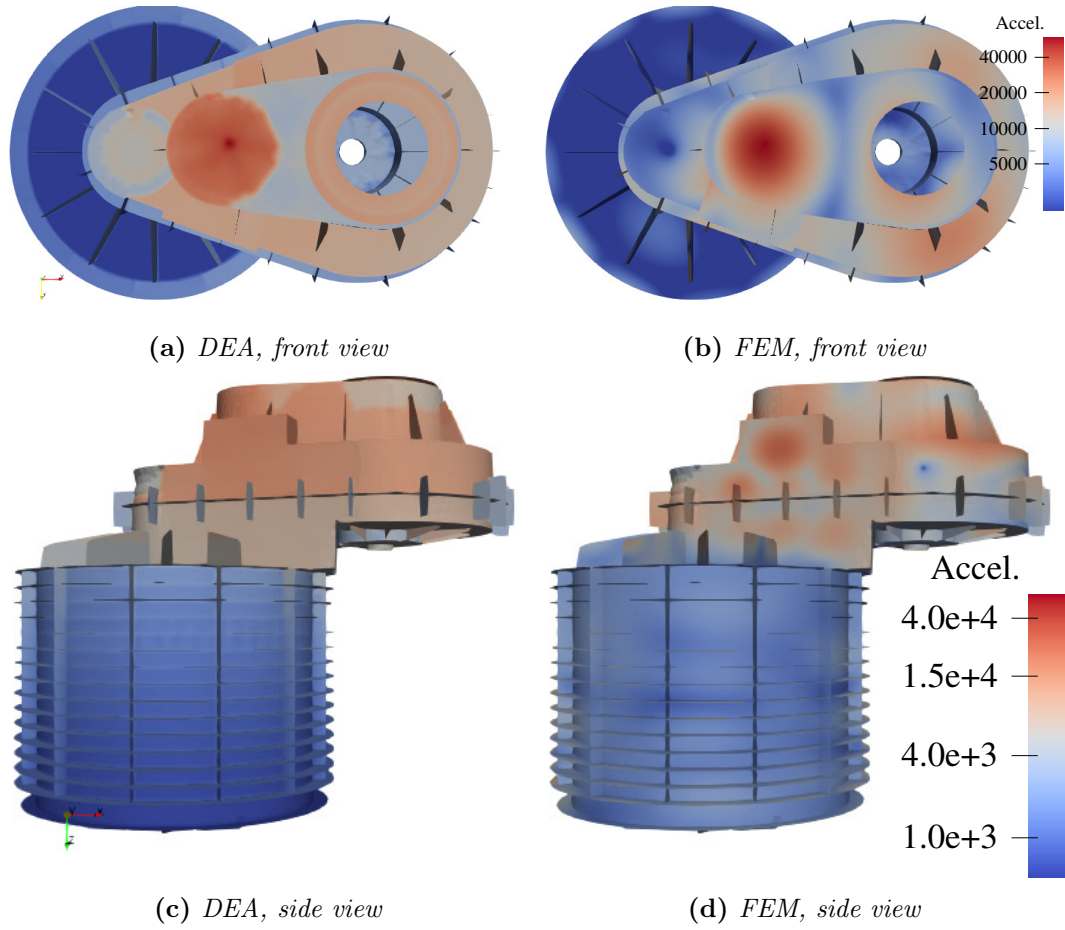


Figure 5.18: The mean DEA and FEM acceleration response levels of the gearbox housing of an electric vehicle powertrain due to a harmonic point excitation force of 1 N at the frequency $f = 8000$ Hz. The acceleration is given in mm/s^2 .

point excitation force and compare results obtained from the FEM and DEA methods for frequencies between 5 kHz and 12.5 kHz. However, for the sake of brevity, results will be shown only for the frequency $f = 8000$ Hz. The FEM results are frequency-averaged over one third octave bands, whereas DEA results are computed at the centre frequencies of these bands. The structure is meshed in the FEM software “Hypermesh”, and the FEM simulations were performed using the software “Optistruct”. A harmonic point excitation force of 1 N is applied at the front side of the structure in the positive Z direction, and hysteretic damping of 0.5% is assumed.

Figure 5.18 presents DEA and FEM acceleration response amplitudes of the gearbox housing at the centre frequency $f = 8000$ Hz. It is noted that the DEA results agree reasonably well with the FEM results across the upper non-

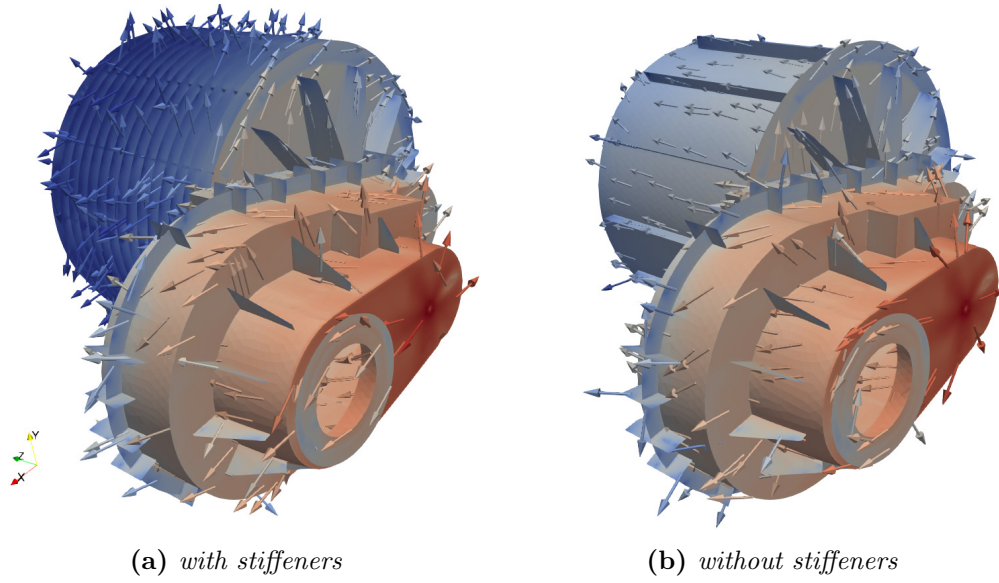


Figure 5.19: The intensity vector field of the gearbox housing with stiffeners (a) and without stiffeners (b) arising due to a point source at its front side. The structure colouring represents the mean acceleration levels of the bending mode.

cylindrical part of the structure. However, it appears that the DEA acceleration levels decrease faster than the correspondent FEM levels in the stiffened part of the structure. During the project, it was found that this discrepancy is connected to the limited applicability of the DEA method at frequencies, where the correspondent wavelengths are of the same order as characteristic lengths of the structure. In fact, the average length between stiffeners along cylinder's longitudinal axis is only 10 mm, and the bending wavelength is ~ 104 mm at the frequency $f = 8000$ Hz. Consequently, the individual stiffeners are not resolved on the wavelength scale. In contrast, in DEA, rays are assumed to interact with each stiffener via scattering events, thus enormously increasing their overall path lengths $D(X, X')$. Therefore, the overall damping term $\exp(-\mu D(X, X'))$ in Equation (5.3) is overestimated, thus decreasing the acceleration levels at higher rate compared to the FEM results. Figure 5.19a presents the intensity vector field obtained from the DEA method at the frequency $f = 8000$ Hz, and one can see that the wave energy scatters at stiffeners of the cylindrical part.

The following steps were performed to resolve this problem:

Firstly, stiffeners along the longitudinal axis of the cylindrical part were re-

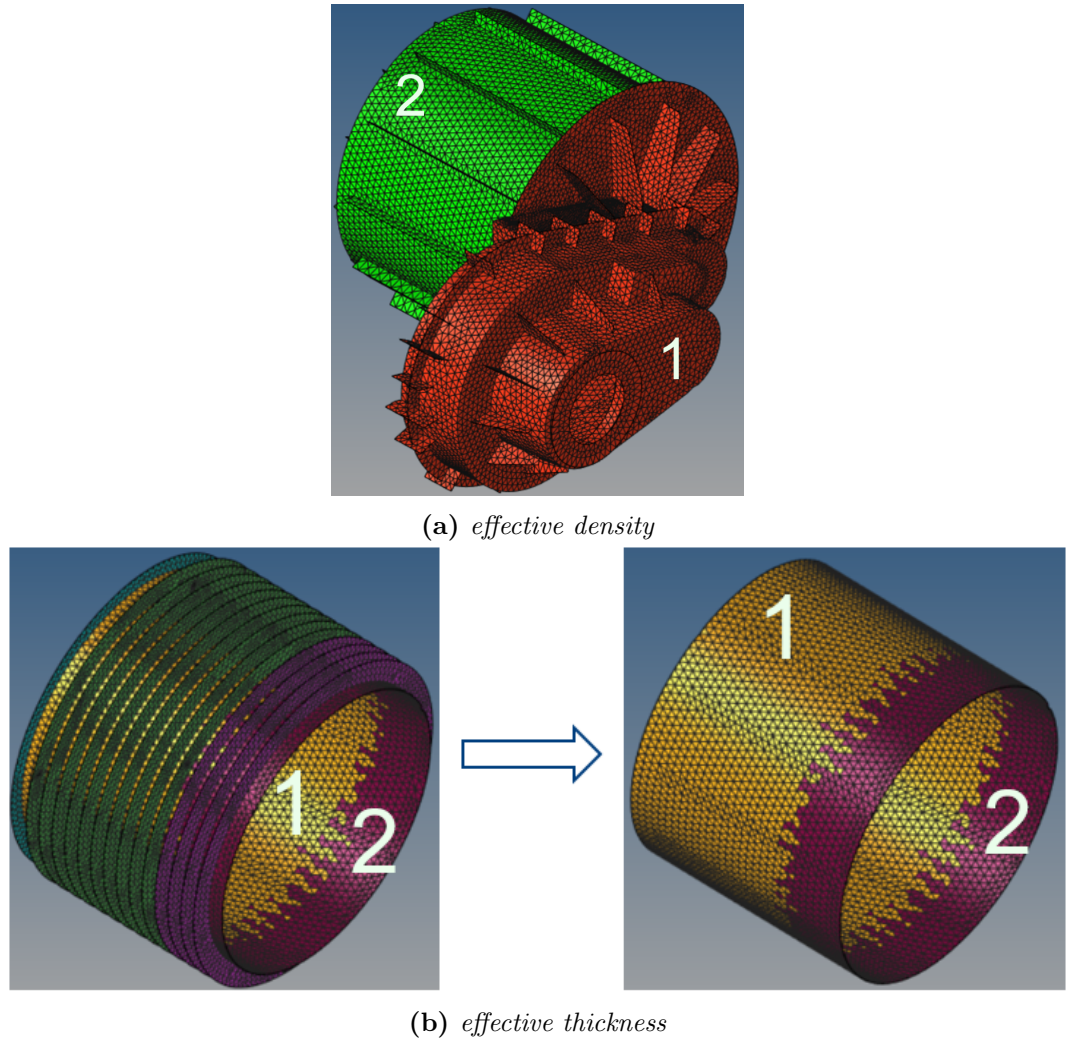


Figure 5.20: Two modification methods of the FE model of the non-stiffened structure to obtain the same structural mass as the full structure. The first one (a) is to compute and assign an effective value of material density to the cylindrical part 2, which is different from 1, where the structure is made of aluminium. The second one (b) is to compute and assign effective thickness of values of shell elements in parts 1 and 2 by calculating the total volume of removed stiffeners and keeping the same material density.

moved from the model, and new FEM and DEA simulations were performed on the non-stiffened gearbox housing. Figure 5.17b presents its FE model, and Figure 5.19b shows the intensity vector field produced by the point source. Secondly, since DEA results of the non-stiffened structure cannot be compared with the FEM results of the initial structure due to the mass difference of 2.9 kg, further modifications of the model were performed. Namely, to obtain the same structural mass, an effective value of material density was computed and assigned to the non-stiffened cylindrical part, see Figure 5.20a. In this

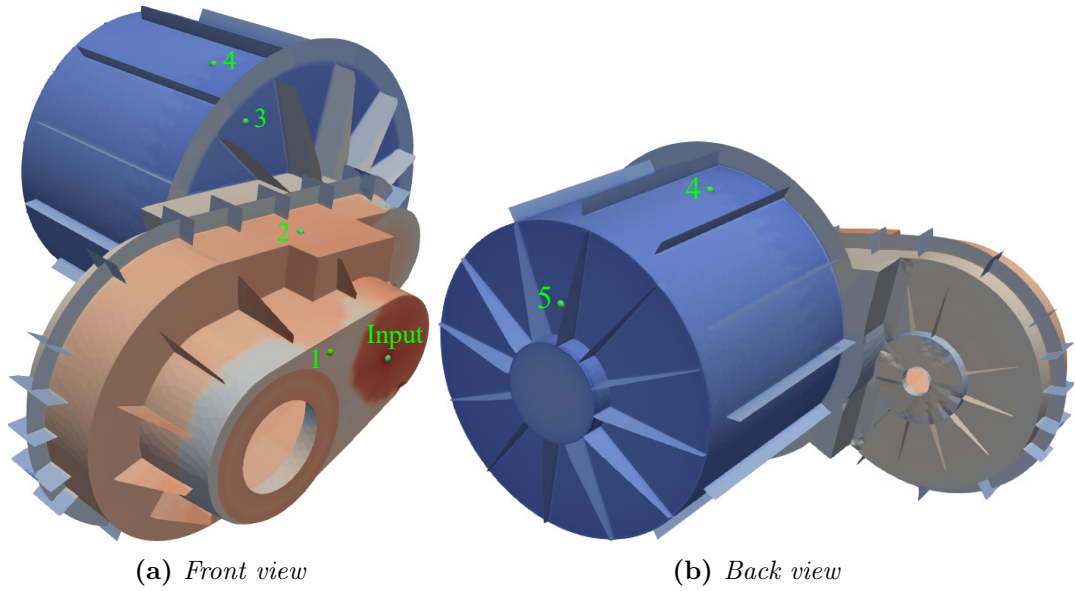


Figure 5.21: The nodes of the FE mesh, where the acceleration response amplitudes were computed. Input node and nodes 1, 2 and 3 are located on the non-stiffened part, whereas nodes 4 and 5 are on the stiffened part of the structure.

figure, finite elements on the part labelled 1 are made of aluminium as before. On the other hand, elements on the part labelled 2 have a new value of material density 4032.55 kg/m^3 .

The same mass was also achieved by a different method. Namely, effective thickness values were assigned to elements on the non-stiffened cylindrical part by calculating the total volume of removed stiffeners and keeping the same material density, see Figure 5.20b. In this figure, shell elements on the part, labelled 1, have thickness value 8.86 mm - before removing stiffeners, and 15.51 mm - after. Elements on the part, labelled 2, have thickness value 10.04 mm - before removing stiffeners, and 15.76 mm - after.

Finally, another method to resolve the issue of artificially damped acceleration levels is to compute and assign the effective orthotropic material parameters to the cylindrical part based on its geometry. These parameters can be computed using the formula presented, for example, in [206] and in Figure 3.28 of [142]. Following this approach, one can obtain results using a composite DEA method presented in this chapter. However, this task was not accomplished due to time restrictions and is regarded as further work to be done outside of the scope of the current thesis.

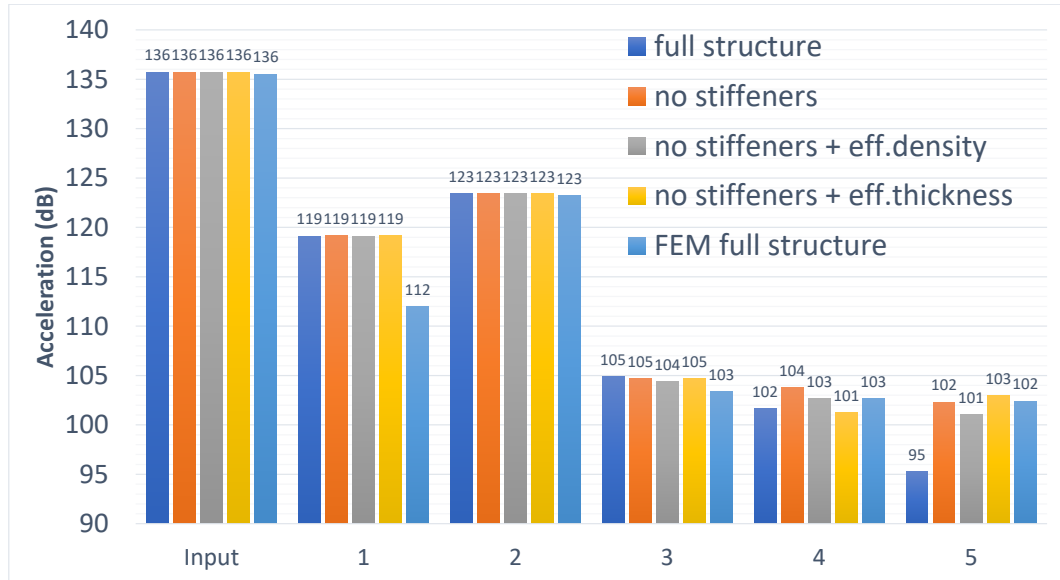


Figure 5.22: Comparison of mean acceleration amplitudes (given in dB), calculated with different methods on the node positions, described in Figure 5.21, at the frequency $f = 8000$ Hz.

Several nodes on the structure are chosen for the point-by-point data comparison, and they are described in Figure 5.21. Note that the input and first three nodes are located on the non-cylindrical part of the structure, whereas nodes 4 and 5 are on the cylindrical part. A point-by-point comparison of different DEA results with the FEM ones at the frequency $f = 8000$ Hz can be found in Figure 5.22. The accelerations are given in the dB scale. We can note that the acceleration levels at the input and first three nodes agree well for all different cases. This is because these nodes are located on the unchanged non-cylindrical part of the structure. The acceleration levels obtained from DEA on the complete structure (denoted in dark blue) are much lower than in other cases at node 5. This discrepancy was discussed earlier, and it was also seen in Figure 5.18c. The *mass smearing* approaches do not appear to change much in acceleration levels at nodes 4 and 5.

5.5 Conclusion

This chapter has considered the extension and implementation of the DEA method for composite structures that can be meshed by two-dimensional shell elements. This has been accomplished for the first time in this work, and we

have used the findings of Chapters 3 and 4 to derive the stationary wave energy density arising in the structure due to a point or an edge source. To achieve this, first, we have discussed how the ray tracing formula that governs the trajectories of energy rays is modified in the case of composite laminated plates. The boundary integral operator T has been discretised on the triangular FE meshes, using Legendre polynomials. We have shown how the expressions for initial 2D phase space densities need to be changed in composite structures. Solving the linear system consisting of the initial 2D phase space density and the boundary integral operator yields the stationary 2D phase space density, which is used to compute the stationary wave energy density at any point of the structure.

Finally, we have considered several numerical case studies such as a polygonally shaped plate, an L-shaped composite plate and an electric vehicle gearbox. In the first case, we have computed the mean response of the polygonally shaped plate due to a point and an edge sources. Isotropic and composite materials have been considered to demonstrate how the wave energy is transported and scattered in each of these cases. Also, the influence of the number of Legendre polynomials on the results has been investigated.

In the second case, we have computed and compared the DEA and FEM frequency-averaged responses of an L-shaped angle-ply laminated plate subjected to a harmonic point excitation force. Also, we have computed the intensity vector field of the bending and longitudinal modes.

In the last case, we have computed and compared the results from the classical DEA and the full FEM simulation for an electric gearbox housing. We have presented a limitation of the classical DEA approach on this example and demonstrated how the composite DEA could provide more accurate results. This work has been performed during the industrial placement at “Romax Technology” company.

Chapter 6

Conclusion

In this chapter, a brief overview of the main findings of the thesis is provided. Furthermore, we discuss further research that could serve as a basis for the extension of the work.

6.1 Main Contributions

In this section, we briefly review the main goals and contributions of the thesis. We have been interested in the problem of high-frequency wave energy scattering and propagation in composite structures. In particular, structures made of the two-dimensional plate- or shell-like elements with composite laminate material were considered. We wanted to provide an extension of the DEA method to such structures.

Chapter 2 aimed to present the main theoretical basics of the DEA method for two-dimensional isotropic structures. Furthermore, the solution of Helmholtz and biharmonic wave equations have been presented. Finally, we have identified and discussed the main modifications needed to allow for DEA application on composite structures.

Chapter 3 has described in all generality how to compute energy scattering coefficients of structural junctions made up of thin composite laminated plates in the line junction approximation. Expressions quantifying transmission and reflection coefficients as a function of the frequency and the wave number component k_x have been derived. Interesting phenomena such as negative re-

fraction and negative group velocity have been observed and analysed. As a final step, an effective scattering matrix for a plate with multiple finite stiffeners attached to it has been derived and computed for the cases of one and four stiffeners. The scattering coefficients have been computed explicitly for examples of two and three composite plates joined together in an L and T geometry.

A hybrid FE/WFE model that predicts the scattering properties for different junctions of two-dimensional anisotropic composite plates has been developed in Chapter 4. The influence of the angle of incidence and the frequency on the distribution of the power flow of incident bending, shear and longitudinal type waves has been investigated. A detailed comparison with semi-analytical evaluations of scattering coefficients derived in Chapter 3 has been presented. The method gives for the first time a detailed recipe for computing scattering coefficients for the generic case of an arbitrary number of composite plates connected at a junction without restrictions on the angles at which the plate meet or the orientation of the principal axis of individual plates.

Finally, in Chapter 5, we have presented the modified theoretical base of DEA for composite structures. The findings of Chapters 3 and 4 have been used to derive the stationary wave energy density arising in the structure due to a point or an edge sources. Numerical results for the cases of a polygonally shaped plate, an electric vehicle gearbox and an L-shaped composite plate has been presented.

6.2 Further work

The current thesis has provided a solid ground for further research regarding wave energy scattering and propagation in composite structures. Several possible continuation works have been identified. These include

- The thin plate assumption used in theory presented in Chapter 3 can be relaxed, and the correspondent governing equations can be derived

from the First- or Third-Order laminated theories [67]. This would allow us to consider relatively thick composite plates in the derivation of energy scattering coefficients and to validate the WFE method results for such structures. Also, one can extend the curved shell theory presented, for example, in [98, 138] and compute energy scattering coefficients of composite shell-plate junctions.

- The work presented in Section 3.6 has considered the case of the stiffened plate. Following the approach presented in [194, 195, 207], one can extend this work to the case of periodic grillages consisting of composite beams and plates. This would allow us to compute effective wave vector curves of such metamaterials for the whole frequency domain, not only for high frequencies. In structures consisting of such metamaterials, one can compute effective scattering coefficients at their junction parts.
- The WFE method presented in Chapter 4 can be extended for curved composite structures and their junctions with composite plates. This can be easily done by following the approach presented in [208, 209].
- The extension of the DEA method for composite structures presented in Chapter 5 is based on the same assumptions as of the classical DEA method. In particular, the wavelengths of propagating modes must be shorter than the characteristic dimensions of the structure. We have seen the implication of breaking this assumption in the case of an electric vehicle gearbox considered in subsection 5.4.3. In the low-frequency regime, stiffened plates can be regarded as orthotropic plates, allowing application of the presented composite DEA method.
- The other limitation of the DEA method is that the curved parts of the structure are considered as the set of plate-like elements connected in the mesh. Energy rays follow not straight, but rather geodesic lines on curved structures [175, 210]. This limitation can be circumvented by allowing

the momentum vector $\mathbf{p}$ to be dependent on the curved geometry, thus complexifying the Hamilton equations (5.2).

- Finally, it is assumed that energy rays scatter in a specular manner at discontinuities of a structure. This restriction can be relaxed by representing the kernel of the phase-space density propagating operator T as a sum of specular and diffusive components [211]. This would allow accounting for irregular or random surfaces.

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Appendix A

Special laminated plate stiffness coefficients derivation

In this appendix, we describe how extensional, bending-extension coupling and bending stiffness coefficients of regular cross- and angle-ply symmetric laminates with alternating material angle of orientation between layers are related to plane-stress reduced stiffnesses of an orthotropic lamina (see [1, 212]).

A.1 Cross-ply symmetric laminates

We consider a cross-ply symmetric laminate consisting of n specially orthotropic layers of the same material with principal material directions alternating between 0° and 90° . The number of layers n must be odd for the laminate to be symmetric. The odd-numbered layers are reinforced in the x axis of the laminate. The principal direction of the even-numbered layers coincides with the y axis of the laminate. Furthermore, the thicknesses of the odd-numbered layers are all equal. The same applies for the even-numbered layers, however, this thickness can be different from that of the odd-numbered layers. We denote as m the ratio of total thickness of the odd-numbered layers to that of the even-numbered layers. Further, F is defined as the ratio of principal stiffnesses of a layer, i.e. $F = Q_{22}/Q_{11} = E_2/E_1$. The extensional and bending stiffnesses

of the laminate can be expressed as

$$\begin{aligned}
A_{11} &= h \frac{m+F}{1+m} Q_{11} \\
A_{22} &= h \frac{1+mF}{1+m} Q_{11} \\
A_{12} &= h Q_{12} \\
A_{66} &= h Q_{66} \\
D_{11} &= \frac{h^3}{12} [(F-1)P+1] Q_{11} \\
D_{22} &= \frac{h^3}{12} [(1-F)P+F] Q_{11} \\
D_{12} &= \frac{h^3}{12} Q_{12} \\
D_{66} &= \frac{h^3}{12} Q_{66}
\end{aligned}$$

with

$$P = \frac{1}{(1+m)^3} + \frac{m(n-3)[m(n-1)+2(n+1)]}{(n^2-1)(1+m)^3} \quad (\text{A.1})$$

A.2 Angle-ply symmetric laminates

We consider an angle-ply symmetric laminate consisting of n generally orthotropic layers of the same material with principal material directions alternating between $-\alpha$ and α , $\alpha \in (0^\circ, 90^\circ)$. The number of layers n must be odd for the laminate to be symmetric. The odd-numbered layers are oriented at $-\alpha$, the even-numbered layers are oriented at α with respect to the x axis of the laminate. All layers are assumed to have the same thickness. The extensional and bending stiffnesses of the laminate can be expressed as

$$\begin{aligned}
A_{11} &= h Q_{11} & D_{11} &= \frac{h^3}{12} Q_{11} \\
A_{22} &= h Q_{22} & D_{22} &= \frac{h^3}{12} Q_{22} \\
A_{12} &= h Q_{12} & D_{12} &= \frac{h^3}{12} Q_{12} \\
A_{66} &= h Q_{66} & D_{66} &= \frac{h^3}{12} Q_{66} \\
A_{16} &= \frac{h}{n} Q_{16} & D_{16} &= \frac{h^3}{12} \left[\frac{3n^2-2}{n^3} \right] Q_{16} \\
A_{26} &= \frac{h}{n} Q_{26} & D_{26} &= \frac{h^3}{12} \left[\frac{3n^2-2}{n^3} \right] Q_{26}
\end{aligned} \quad (\text{A.2})$$

Appendix B

Plane stress-reduced stiffnesses

The plane stress-reduced stiffnesses Q_{ij} of a generally orthotropic lamina in (2.46) can be related to the stiffness coefficients $\bar{Q}_{ij}$ of the same lamina described in a local coordinate system aligned with the principal material coordinate axes as follows:

$$\begin{aligned}
 Q_{11} &= \bar{Q}_{11} \cos^4 \phi + 2 (\bar{Q}_{12} + 2\bar{Q}_{66}) \sin^2 \phi \cos^2 \phi + \bar{Q}_{22} \sin^4 \phi, \\
 Q_{12} &= (\bar{Q}_{11} + \bar{Q}_{22} - 4\bar{Q}_{66}) \sin^2 \phi \cos^2 \phi + \bar{Q}_{12} (\sin^4 \phi + \cos^4 \phi), \\
 Q_{22} &= \bar{Q}_{11} \sin^4 \phi + 2 (\bar{Q}_{12} + 2\bar{Q}_{66}) \sin^2 \phi \cos^2 \phi + \bar{Q}_{22} \sin^4 \phi, \\
 Q_{16} &= (\bar{Q}_{11} - \bar{Q}_{12} - 2\bar{Q}_{66}) \sin \phi \cos^3 \phi + (\bar{Q}_{12} - \bar{Q}_{22} + 2\bar{Q}_{66}) \sin^3 \phi \cos \phi, \\
 Q_{26} &= (\bar{Q}_{11} - \bar{Q}_{12} - 2\bar{Q}_{66}) \sin^3 \phi \cos \phi + (\bar{Q}_{12} - \bar{Q}_{22} + 2\bar{Q}_{66}) \sin \phi \cos^3 \phi, \\
 Q_{66} &= (\bar{Q}_{11} + \bar{Q}_{22} - 2\bar{Q}_{12} - 2\bar{Q}_{66}) \sin^2 \phi \cos^2 \phi + \bar{Q}_{66} (\sin^4 \phi + \cos^4 \phi).
 \end{aligned} \tag{B.1}$$

Here, ϕ is the angle of rotation of the principal material coordinate system with respect to the local coordinate system of the lamina. Furthermore, $\bar{Q}_{ij}$ can be expressed in terms of the material constants as follows:

$$\begin{aligned}
 \bar{Q}_{11} &= \frac{E_x}{1 - \nu_{xy}\nu_{yx}}, \quad \bar{Q}_{22} = \frac{E_y}{1 - \nu_{xy}\nu_{yx}}, \\
 \bar{Q}_{12} &= \nu_{xy}\bar{Q}_{22}, \quad \bar{Q}_{66} = G_{xy}, \quad \nu_{yx} = \nu_{xy} \frac{E_y}{E_x}.
 \end{aligned} \tag{B.2}$$

Note, that for $\phi = \pi n/2, n \in \mathbb{Z}$, the plane stress-reduced stiffnesses Q_{16} and Q_{26} are equal to zero, the corresponding laminas are called *specialy orthotropic*.

Appendix C

List of Abbreviations

FE	Finite Element
FD	Finite Difference
BE	Boundary Element
SEA	Statistical Energy Analysis
WIA	Wave Intensity Analysis
EFEA	Energy Finite Element Analysis
DEA	Dynamical Energy Analysis
ESL	Equivalent Single Layer
SE	Spectral Element
WFE	Wave Finite Element
CLP	Classical Laminated Plate
FSDL	First-order Shear Deformation Laminated
HSDL	Higher-order Shear Deformation Laminated
TSDL	Third-order Shear Deformation Laminated