


# Vortex dynamics in the Ginzburg-Landau and Maxwell-Klein-Gordon models with pinning

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*"Well, Seymour, I made it..."*

- Superintendent Chalmers

*"Owl explained about flux pinning and creep. He had explained this to Pooh and Christopher Robin once before, and had been waiting ever since for a chance to do it again, because it is a thing you can easily explain twice before anybody knows what you are talking about."*

- A. A. Milne, *Winnie-the-Pooh*

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**Abstract**

We study the motion of vortices in a superconductor subject to a perturbed background potential. Such a potential models the presence of pinning sites in the superconductor which the vortices are attracted to. The pinning potential is scaled such that the intervortex interactions also play a role in the vortex dynamics. We use the time-dependent Ginzburg-Landau model to relate the vortex velocities to forces in the system; namely, the gradient of the renormalized energy which accounts for intervortex interactions and the gradient of the pinning potential. We make an effort to include the gauge terms at every step. Additionally, we include both parabolic motion and Schrödinger flow from the outset which results in a mixed-flow dynamical law.

In the second part of the thesis we shift to the Maxwell-Klein-Gordon model with a pinned Ginzburg-Landau potential to model the motion of vortices in a different mathematical framework. The equations are now second-order in time (compared to the first-order in time Ginzburg-Landau equations) so proves that the methods applied in the first part of the thesis are robust and can be adapted to a model with greater complexity. We derive a dynamical law for vortices exposed to the critically scaled pinning potential and experiencing interactions from other vortices in the system.

# Notation and definitions

## Functions of superconductor theory and pinning

| | |
|---|---|
| $\varepsilon$ | material constant which we study in the limit $\varepsilon \to 0$ |
| $u_\varepsilon$ | order parameter |
| $u_*$ | limit of order parameter as $\varepsilon \to 0$ |
| $A_\varepsilon$ | magnetic vector potential |
| $A_*$ | limit of magnetic vector potential as $\varepsilon \to 0$ |
| $\Phi_\varepsilon$ | electric scalar potential |
| $\phi_\varepsilon$ | phase of order parameter according to $u_\varepsilon = \lvert u_\varepsilon \rvert \mathrm{e}^{i\phi_\varepsilon}$ |
| $h_\varepsilon$ | induced magnetic field in superconductor: $h_\varepsilon := \operatorname{curl} A_\varepsilon$ |
| $h_*$ | limit of magnetic field as $\varepsilon \to 0$ |
| $h_{ex}$ | externally applied magnetic field |
| $E_\varepsilon$ | induced electric field in superconductor: $E_\varepsilon := -\partial_t A_\varepsilon - \nabla \Phi_\varepsilon$ |
| $\sigma_\varepsilon$ | conductivity of material inn its normal state |
| $\sigma_0$ | $\sigma_\varepsilon = \sigma_0 / \lvert \log \varepsilon \rvert$ |
| $\gamma_\varepsilon$ | relaxation rate parameter |
| $\alpha_\varepsilon$ | parabolic motion relaxation rate |
| $\alpha_0$ | $\alpha_\varepsilon = \alpha_0 / \lvert \log \varepsilon \rvert$ |
| $\beta_0$ | conservative/Schrödinger motion relaxation rate |
| $p_\varepsilon$ | pinning potential |
| $\eta_\varepsilon$ | pseudo-pinning potential |
| $w_\varepsilon$ | normalized order parameter: $w_\varepsilon = u_\varepsilon / \eta_\varepsilon$ |
| $q_\varepsilon$ | pinning function: $p_\varepsilon^2 = 1 + \frac{q_\varepsilon}{\lvert \log \varepsilon \rvert}$ |
| $Q_\varepsilon$ | pseudo-pinning function: $\eta_\varepsilon^2 = 1 + \frac{Q_\varepsilon}{\lvert \log \varepsilon \rvert}$ |
| $Q_0$ | limit of $Q_\varepsilon$ as $\varepsilon \to 0$ |
| $\nabla_{A_\varepsilon} w_\varepsilon$ | covariant derivative: $\nabla_{A_\varepsilon} w_\varepsilon := \nabla w_\varepsilon - i A_\varepsilon w_\varepsilon$ |
| $\nabla_{A_\varepsilon} w_\varepsilon$ | covariant Laplacian (see page 3) |
| $\partial_{\Phi_\varepsilon} w_\varepsilon$ | covariant derivative: $\partial_{\Phi_\varepsilon} w_\varepsilon := \partial_t w_\varepsilon + i \Phi_\varepsilon w_\varepsilon$ |
| $\partial_{\Phi_\varepsilon} w_\varepsilon$ | second order covariant derivative (see page 12) |

| | |
|---|---|
| $G_\varepsilon$ | Ginzburg-Landau energy functional |
| $\mathcal{G}_\varepsilon^p$ | Ginzburg-Landau energy functional with pinning |
| $G_\varepsilon^{\eta_\varepsilon}$ | Ginzburg-Landau energy functional with pseudo-pinning potential |
| $g_\varepsilon^{\eta_\varepsilon}$ | energy density (see page 30) |
| $e_\varepsilon^{\eta_\varepsilon}(w_\varepsilon)$ | energy density without gauge |
| $e_{A_\varepsilon}^{\eta_\varepsilon}(w_\varepsilon)$ | energy density without magnetic field terms |
| $j_{A_\varepsilon}(w_\varepsilon)$ | supercurrent |
| $j_{A_*}(w_*)$ | limiting supercurrent |
| $J_{A_\varepsilon}(w_\varepsilon)$ | gauge-invariant Jacobian determinant of $w_\varepsilon$ |
| $s_\varepsilon$ | upper bound for distance of Jacobian from its limiting value (Thm 3.9) |
| $V_{A_\varepsilon}^{\Phi_\varepsilon}(w_\varepsilon)$ | gauge-invariant space-time Jacobian determinant of $w_\varepsilon$ |
| $T$ | time for which vortex motion obeys our derived ODEs (see Ch5 & Ch7) |
| $T_\varepsilon^{\eta_\varepsilon}$ | stress-energy tensor in GL model with pinning (see page 70) |
| $\omega(t)$ | difference between solutions to ODE and vortex positions: $\xi(t) - a(t)$ |

**Vortex parameters**

| | |
|---|---|
| $n$ | number of vortices in the system |
| $\xi_i$ | position of $i^{\text{th}}$ vortex |
| $\xi = (\xi_1, ..., \xi_n)$ | vector of vortex positions |
| $d_i$ | degree of $i^{\text{th}}$ vortex |
| $d = (d_1, ..., d_n)$ | vector of vortex degrees |
| $\delta_{\xi_i}$ | delta function located at position $\xi_i$ |
| $a_i$ | $i^{\text{th}}$ solution to equation of motion ODE |
| $a = (a_1, ..., a_n)$ | vector of solutions to ODEs |
| $W(a, d)$ | renormalized energy (see page 5 & 37) |
| $W^Q(a, d)$ | renormalized energy with pinning component (see page 37) |
| $I(r, \varepsilon)$ | energy contained within vortex balls (see page 38) |
| $\gamma_0$ | constant: $\lim_{r\to\infty} \left(I(r, \varepsilon) - \pi \log \frac{r}{\varepsilon}\right)$ (see page 38) |
| $D_\varepsilon$ | excess energy (see page 39) |

## Domains

| | |
|---|---|
| $\Omega_s$ | region of superconducting material |
| $\Omega_{im}$ | small region occupied by impurities |
| $\Omega$ | $\Omega_s \cup \Omega_{im}$ |
| $\partial\Omega$ | boundary of $\Omega$ |
| $r$ | radius of vortex ball |
| $r_a$ | minimum distance between vortices or vortex and $\partial\Omega$ |
| $B_r(a_i)$ | vortex ball of radius $r$ centred at $a_i$ |
| $\Omega_r(a)$ | perforated domain: $\Omega_r(a) := \Omega \setminus \cup_{i=1}^n \overline{B_r(a_i)}$ |
| $\Omega^n$ | space of vector of positions: $a = (a_1, ..., a_n) \in \Omega^n$ |
| $\Omega^{n*}$ | $\{a \in \Omega^n \mid r_a > 0\}$ |
| $\nu$ | outward pointing unit normal vector |
| $\tau$ | unit tangent vector |
| $r_\star$ | optimized radius of vortex balls in Theorem 3.9 |

## Maxwell-Klein-Gordon parameters

| | |
|---|---|
| $c$ | speed of light |
| $m$ | mass of particle in Klein-Gordon model |
| $\hbar$ | reduced Planck's constant |
| $\mu$ | $c/\sqrt{\lvert \log \varepsilon \rvert}$ |
| $H_\varepsilon^{\eta_\varepsilon}$ | Hamiltonian (see page 95) |
| $\mathcal{H}_\varepsilon^{\eta_\varepsilon}$ | Hamiltonian density (see page 95) |
| $\mathcal{T}$ | stress-energy tensor in MKG-pGL model (see page 102) |

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# Chapter 1

# Introduction

There are two models we use in this thesis: the time-dependent Ginzburg-Landau (TDGL) model and the Maxwell-Klein-Gordon model supplemented with a Ginzburg-Landau potential (MKG-GL). The aim is to derive the limiting dynamical law for vortices which occur in both cases.

From the outset we would like to make clear some of the notation which is used throughout the thesis. $B_r(x)$ denotes a ball (always in two dimensions) which is centred at $x$ and has radius $r$, $o_\varepsilon(1)$ represents a quantity which is vanishingly small as $\varepsilon \to 0$, and $C$ represents a constant which may vary from line to line. More notation specific to our problem is introduced when it's necessary.

## 1.1 Ginzburg-Landau model

For certain metals cooled to low enough temperatures, a phase transition occurs in which the material switches from its *normal* state to its *superconducting* state, and with the transition the material develops an array of interesting and counter-intuitive properties. The resistivity of the material vanishes, allowing the flow of electric current without any energy dissipation, and weak magnetic fields are forbidden from entering the material. However, a strong magnetic field can penetrate the material, destroying the superconductivity and forcing

the system back into its normal state. These properties lend themselves to an array of useful real-world devices and machines, so there is much interest in the ability to harness these superconducting powers. Examples of applications include strong electromagnets used in MRI machines, particle accelerators and the anticipated maglev trains of the future.

Suppose we have a superconductor subjected to an externally applied magnetic field, and as the strength of the field increases, the material suddenly transitions into the normal state. This transition is characteristic of a *type-I* superconductor [58]. However, if we increase the strength of the magnetic field and observe an intermediate state in the material in which only some of the magnetic field penetrates, we have a *type-II* superconductor [58]. The intermediate state exists for a certain rage of magnetic field strengths and if the magnetic field strength is further increased, the material again transitions into its normal state. This intermediate state has garnered much interest over the years and it is this regime we study here.

The theory describing the superconducting state from microscopic principles is given by the BCS theory [5]. The Ginzburg-Landau (GL) model of superconductivity can be derived from phenomenological considerations but has recently been shown to arise from the BCS theory on the macroscopic scale, under strict constraints [19].

Suppose the superconductor is an infinitely long cylinder about the $z$-axis. The domain $\Omega \subset \mathbb{R}^2$ is the cross-section of this cylinder perpendicular to the $z$-axis and represents the region of superconductivity. The GL model defines the energy functional of the superconductor in terms of two quantities - an *order parameter* $u : \Omega \to \mathbb{C}$ and the *vector potential* $A : \Omega \to \mathbb{R}^2$ associated the magnetic field induced in the material - and derives further information about the system from this energy. The order parameter is essentially a wave function for superconducting particles ($|u|^2$ is the local density of superconducting part-

icles) and the magnetic vector potential provides the magnetic field $h$ induced in the material according to $h := \text{curl } A = \partial_1 A_2 - \partial_2 A_1$. We find it enlightening to introduce the mathematics of the model alongside the physical description, so without further ado, the *Ginzburg-Landau energy* is

$$G_\varepsilon(u, A) = \frac{1}{2}\int_\Omega |\nabla u - iAu|^2 + \frac{1}{2\varepsilon^2}(1 - |u|^2)^2 + |h - h_{ex}|^2. \tag{1.1}$$

The covariant derivative accounting for the Dirichelet part of the energy is ubiquitous in GL theory and so it is useful to use the abbreviation

$$\nabla_A u := \nabla u - iAu,$$

and later use

$$\Delta_A u := \Delta u - 2iA \cdot \nabla u - iu \text{ div } A - |A|^2 u$$

where $\Delta u$ is the Laplacian of $u$ such that if $\nabla u_\varepsilon = (\partial_1 u_\varepsilon, \partial_2 u_\varepsilon)$, then

$$\Delta u_\varepsilon := \nabla \cdot \nabla u_\varepsilon = \partial_1^2 u_\varepsilon + \partial_2^2 u_\varepsilon$$

and div $A = \partial_1 A_1 + \partial_2 A_2$. We make use of an inner product between complex functions. Suppose $a, b \in \mathbb{C}$, $Re(a)$ is the real part of $a$ and $Im(a)$ is the imaginary part of $a$, then

$$\langle a, b\rangle = Re(a)Re(b) + Im(a)Im(b).$$

If there is some $\lambda \in \mathbb{R}$, then

$$\langle \lambda a, b\rangle = \langle a, \lambda b\rangle = \lambda\langle a, b\rangle.$$

If there are vectors with complex components $X = (X_1, X_2) \in \mathbb{C}^2$ and $Y = (Y_1, Y_2) \in \mathbb{C}^2$, then

$$\langle a, X\rangle = \begin{pmatrix} \langle a, X_1\rangle \\ \langle a, X_2\rangle \end{pmatrix},$$

which is a vector in $\mathbb{R}^2$, and

$$\langle X, Y\rangle = \left\langle \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \right\rangle = \langle X_1, Y_1\rangle + \langle X_2, Y_2\rangle.$$

Note that the inner product $\langle \cdot, \cdot \rangle$ always produces a real-valued function.

If $\nabla u = (\partial_1 u, \partial_2 u)$, $A = (A_1, A_2)$ and $\cdot$ represents the standard scalar product between two real vectors, then we have that

$$\begin{aligned}
|\nabla_A u|^2 &:= \langle \nabla_A u, \nabla_A u \rangle \\
&= \langle \nabla u - iAu, \nabla u - iAu \rangle \\
&= \langle \nabla u, \nabla u \rangle + |A|^2 \langle iu, iu \rangle - 2A \cdot \langle iu, \nabla u \rangle \\
&= \langle \partial_1 u, \partial_1 u \rangle + \langle \partial_2 u, \partial_2 u \rangle + (A_1^2 + A_2^2)\langle u, u \rangle - 2 \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \cdot \begin{pmatrix} \langle iu, \partial_1 u \rangle \\ \langle iu, \partial_2 u \rangle \end{pmatrix} \\
&= \langle \partial_1 u, \partial_1 u \rangle + \langle \partial_2 u, \partial_2 u \rangle + (A_1^2 + A_2^2)\langle u, u \rangle - 2A_1 \langle iu, \partial_1 u \rangle - 2A_2 \langle iu, \partial_2 u \rangle,
\end{aligned}$$

where we have used the property $\langle iu, iu \rangle = \langle u, u \rangle$.

$G_\varepsilon(u, A)$ is actually the energy per length, but we always just refer to it as the energy of the superconductor. $h_{ex}\hat{z}$ is the externally applied magnetic field, which we take to be a constant. The quantity $\varepsilon > 0$ is real-valued and represents a material constant. It is the quantity which distinguishes a type-I superconductor from a type-II superconductor. If $\varepsilon < \sqrt{2}$ (see [32, 49]), the superconductor is type-II, which is exclusively the type of superconductor we consider in this thesis, so henceforth we take $\varepsilon$ to be small. We ultimately consider the behaviour of the superconducting quantities in the limit $\varepsilon \to 0$. With this in mind we work with sequences of functions $(u_\varepsilon, A_\varepsilon)$, where $\varepsilon$ will be used to denote any fixed sequence $\varepsilon_k \to 0$, and we take subsequences of this sequence and still denote these as $\varepsilon \to 0$. We endeavour to include the subscript $\varepsilon$ on the necessary quantities when the $\varepsilon$-dependence plays a role, but sometimes we will simply write $(u, A)$ to avoid notational clutter.

It is natural to consider the form of $(u_\varepsilon, A_\varepsilon)$ which minimizes the energy. To this end, the Euler-Lagrange equations associated to the functional (1.1) give

the *Ginzburg-Landau equations*:

$$\begin{cases} \Delta_{A_\varepsilon} u_\varepsilon + \frac{u_\varepsilon}{\varepsilon^2}(1 - |u_\varepsilon|^2) = 0 & \text{in } \Omega \\ \nabla^\perp h_\varepsilon + \langle iu_\varepsilon, \nabla_{A_\varepsilon} u_\varepsilon \rangle = 0 & \text{in } \Omega \\ \nu \cdot \nabla_{A_\varepsilon} u_\varepsilon = 0 & \text{on } \partial\Omega \\ h_\varepsilon = h_{ex} & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

The symbol $\nabla^\perp := (-\partial_2, \partial_1)$ denotes the gradient rotated by $\frac{\pi}{2}$ anticlockwise in the plane and $\nu$ is the outward pointing normal vector to $\Omega$. Taking $R(a)$ to be the real part of $a \in \mathbb{C}$, and $I(a)$ its imaginary part, $\langle a, b \rangle = R(a)R(b) + I(a)I(b)$ is an inner product producing a real value. If $V = (V_1, V_2) \in \mathbb{C}^2$, then $\langle a, V \rangle = (\langle a, V_1 \rangle, \langle a, V_2 \rangle)$.

The GL equations have the special property that they are unchanged if $(u, A)$ both change in a special way know as a *gauge transformation* [49].

**Definition 1.1.** *Consider any $u : \Omega \to \mathbb{C}$ and any $A : \Omega \to \mathbb{R}^2$. For any (smooth) function $f : \Omega \to \mathbb{R}$ define $v = ue^{if}$ and $B = A + \nabla f$. We say the transformation from $(u, A)$ to $(v, B)$ is a gauge transformation and that a quantity unchanged by a gauge transformation is gauge invariant.*

For solutions to (1.2) to be unique, we must eliminate the gauge invariance by choosing the form of $f$ we want to work with. This is done in Section 2.4.

We note that two trivial solutions to (1.2) exist. If $u \equiv 1$ and $A \equiv 0$ we call it the *superconducting solution*. A zero vector potential forbids the presence of an induced magnetic field in the superconductor. If $u \equiv 0$ and curl $A = h_{ex}$ we call it the *normal solution*. We see in the next section that there exists another, more interesting solution to (1.2).

### 1.1.1 Vortices

Since the work of Abrikosov [1], it is known that for a certain range of magnetic field strengths and small $\varepsilon$, (1.2) admits a periodic solution with

small regions where $|u| = 0$, but the phase of $u$ around these regions has a nonzero winding number. That is to say, normal-state ($|u| = 0$) regions are interspersed with superconducting ($|u| = 1$) regions. The points where $u$ vanishes are called *vortices* and the nonzero winding number of a vortex is called its *degree*. Suppose a superconductor has $n$ vortices located at $a_i \in \Omega$ ($i = 1, ..., n$). Following [49], if we write the order parameter in the form $u_\varepsilon(x) = |u_\varepsilon(x)|\mathrm{e}^{i\phi_\varepsilon(x)}$, where $\phi_\varepsilon$ is a single-valued function, then the degree of a vortex is the integer

$$d_i = \frac{1}{2\pi} \int_{\partial B_r(a_i)} \nabla\phi_\varepsilon \cdot \tau, \tag{1.3}$$

where $\tau$ is is the tangent vector to the ball. In the units of our model $\varepsilon$ approximates the size of the vortex core.

It turns out that at a vortex the induced magnetic field $h$ has a local maximum [49]. A *supercurrent* (flow of superconducting particles) encircles the regions where a magnetic field is induced, acting to shield the superconducting regions from it, but a strong magnetic field permeates the sample and forces it into its normal state.

Abrikosov found that the zeros of $u$ arrange themselves in a hexagonal lattice in order to minimise the energy, which indicates the presence of vortex-vortex repulsion. Studying a simpler GL energy without magnetic field terms, Bethuel, Brezis and Hélein [7] determined the role of vortices in minimising the energy. They introduced a function called the *renormalized energy* which describes the behaviour of vortex-vortex interactions in terms of the positions of the vortices and their degrees:

$$W(a, d) = -\pi \sum_{i \neq j} d_i d_j \log |a_i - a_j| + \text{boundary terms.}$$

Serfaty [51] then confirmed such a function plays a role in minimising the full GL energy with magnetic field. We discuss the renormalized energy further in Chapter 2.

### 1.1.2 Time-dependence

As written in (1.2), the GL equations provide the form of $(u_\varepsilon, A_\varepsilon)$ in the superconductor when the energy is at a critical point and the system is not exposed to external forces. If the superconductor is driven out of equilibrium, it becomes necessary to incorporate time-dependence in the model to accommodate relaxation processes. The equations can be somewhat generalized to incorporate nonstationary processes from the assumption that if the superconductor is driven out of equilibrium, the order parameter should relax back to its equilibrium value, i.e. the solution to (1.2). It is assumed the rate of relaxation of $u_\varepsilon$ is proportional to the deviation of $G_\varepsilon$ from equilibrium, given by the functional derivative $\frac{\partial G_\varepsilon}{\partial u_\varepsilon^*}$, where $u_\varepsilon^*$ is the complex conjugate of $u_\varepsilon$:

$$\gamma_\varepsilon \frac{\partial u_\varepsilon}{\partial t} = -\frac{\partial G_\varepsilon}{\partial u_\varepsilon^*}, \tag{1.4}$$

where $\gamma_\varepsilon$ is positive valued and characterises the relaxation time of the system to equilibrium. More on $\gamma_\varepsilon$ shortly and in the text following Definition 2.3.

These phenomenological assumptions can be justified microscopically in a more detailed discussion of superconductor theory [12, 32], and they find that (1.4) is only valid if the excitations are very small and the system is not shifted much from its equilibrium state. The functional derivative $-\frac{\partial G_\varepsilon}{\partial u_\varepsilon^*}$ actually gives the left-hand side of the first GL equation in (1.2), hence, we have that

$$\gamma_\varepsilon \partial_t u_\varepsilon = \Delta_{A_\varepsilon} u_\varepsilon + \frac{u_\varepsilon}{\varepsilon^2}(1 - |u_\varepsilon|^2). \tag{1.5}$$

Recall Definition 1.1 and note that equation (1.5) isn't gauge invariant. There must be a quantity we are missing which cancels with the time-derivative of the gauge transformation of $u_\varepsilon$. For a time-dependent system, the definition of a gauge transformation must be extended to include the scalar potential $\Phi_\varepsilon : \Omega \times \mathbb{R}^+ \to \mathbb{R}$.

**Definition 1.2.** *Consider any* $u : \Omega \times \mathbb{R}^+ \to \mathbb{C}$, $A : \Omega \times \mathbb{R}^+ \to \mathbb{R}^2$ *and* $\Phi : \Omega \times \mathbb{R}^+ \to \mathbb{R}$. *For any (smooth) function* $f : \Omega \times \mathbb{R}^+ \to \mathbb{R}$, *define*

$$v = ue^{if}, \qquad B = A + \nabla f \qquad \text{and} \qquad \theta = \Phi - \partial_t f.$$

*We say the transformation from $(u, A, \Phi)$ to $(v, B, \theta)$ is a gauge transformation and that a quantity which is unchanged by a gauge transformation is gauge invariant.*

In light of this definition, we modify (1.5) such that

$$\gamma_\varepsilon(\partial_t + i\Phi_\varepsilon)u_\varepsilon = \Delta_{A_\varepsilon} u_\varepsilon + \frac{u_\varepsilon}{\varepsilon^2}(1 - |u_\varepsilon|^2).$$

In time-dependent GL theory, the covariant time-derivative appears frequently so we find it useful to make the abbreviation

$$\partial_{\Phi_\varepsilon} u_\varepsilon := \partial_t u_\varepsilon + i\Phi_\varepsilon u_\varepsilon$$

in what follows.

We can justify the introduction of $\Phi$ by considering that excitations in the system naturally cause electrons to move, and so we must include the description of the electric field in the model. The theory of electromagnetism states that the electric field can be written in the form

$$E_\varepsilon = -\partial_t A_\varepsilon - \nabla\Phi_\varepsilon.$$

Appealing to the theory of electrodynamics (specifically Ampere's law), we see that including the electric field in a gauge-invariant way leads to the second equation of (1.2) becoming

$$\sigma E_\varepsilon = -\nabla^\perp h_\varepsilon - \langle iu_\varepsilon, \nabla_{A_\varepsilon} u_\varepsilon\rangle, \tag{1.6}$$

where $\sigma$ the conductivity of the material in its normal state.

The time-dependent equations in this form with $\gamma_\varepsilon$ a real positive constant were first derived by Schmid [50] and by Gorkov and Eliashberg [21]. However, since the work of Dorsey [17] and Kopnin *et al* [31], it has been known that the relaxation rate $\gamma_\varepsilon$ should be complex. This accounts for the *flux-flow Hall effect* (Ch12, [32]) which is observed in type-II superconductors. The physically relevant relaxation rate is of the form $\gamma_\varepsilon = \alpha_\varepsilon + i\beta_\varepsilon$, with $\alpha_\varepsilon, \beta_\varepsilon \in \mathbb{R}$, and is the

coefficient we adopt in this work. Hence, our *time-dependent Ginzburg-Landau (TDGL) equations* are

$$\begin{cases} (\alpha_\varepsilon + i\beta_\varepsilon)\partial_{\Phi_\varepsilon} u_\varepsilon = \Delta_{A_\varepsilon} u_\varepsilon + \frac{u_\varepsilon}{\varepsilon^2}(1 - |u_\varepsilon|^2) & \text{in } \Omega \\ \sigma E_\varepsilon = -\nabla^\perp h_\varepsilon - \langle i u_\varepsilon, \nabla_{A_\varepsilon} u_\varepsilon \rangle & \text{in } \Omega \\ \nu \cdot \nabla_{A_\varepsilon} u_\varepsilon = 0 & \text{on } \partial\Omega \\ h_\varepsilon = h_{ex} & \text{on } \partial\Omega. \end{cases} \tag{1.7}$$

The equations (1.7) are ill-posed if the gauge is not fixed. Choosing $f$ in Definition 1.2 such that the system obeys the *Coulomb gauge* conditions (see Definition 2.5), then the global existence of solutions $(u_\varepsilon, A_\varepsilon, \Phi_\varepsilon)$ is established and for sufficiently smooth initial data, which follows in Ch. 2, the solutions are highly regular (Remark 1.3, [35]). We treat $(u_\varepsilon, A_\varepsilon, \Phi_\varepsilon)$ as classical solutions to (1.7).

The forces which cause these nonstationary processes may pass to the vortices and in general the positions and degrees of the vortices have a time-dependence: $a_i = a_i(t)$, $d_i = d_i(t)$ for $i = 1, ..., n$ if there are $n$ vortices in the system. It is the dynamics of the vortices which is our main interest in this work. The aim is to be able to derive a dynamical law for the vortex positions, starting with the TDGL equations.

### 1.1.3 Pinning

Suppose we have a type-II superconductor in the mixed state and wish to pass an electric current through it. We will find that the current does not flow freely through the system in the resistance-free manner we expect from a superconductor, but that the current forces the magnetic vortices into motion, causing dissipation in the system. Since the vortex motion causes energy loses in the system, it is of interest to understand ways in which one can restrict the motion of the magnetic vortices. To suppress such movement, it is common to introduce impurities in the material which act to locally lower the

background potential, causing the vortices to be attracted to these sites and become "pinned". The impurity, which may be in its normal or superconducting state, is a different material to the main superconductor. It is effective as a pinning site as long as it provides a small region where the superconducting particle density is lower than what appears in the bulk superconductor [30]. The smaller density of superconducting particles acts to lower the energy penalty associated with the vortex core; thus it is energetically favourable for a vortex to lie through an impurity [10].

Modelling pinning effects requires a modification to the GL model. The new model has a new energy functional $\mathcal{G}_\varepsilon^p(u_\varepsilon, A_\varepsilon)$ which incorporates a function $p(x)$ which describes the superconducting particle density at a pinning site. It appears in the penalty term of the energy [39]:

$$\mathcal{G}_\varepsilon^p(u_\varepsilon, A_\varepsilon) = \frac{1}{2}\int_\Omega |\nabla_{A_\varepsilon} u_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(p^2 - |u_\varepsilon|^2)^2 + |h_\varepsilon - h_{ex}|^2. \tag{1.8}$$

Here the domain is for the superconducting sample $\Omega_s$ and the small regions of the impurities $\Omega_{im}$ are taken together:

$$\Omega := \Omega_s \cup \Omega_{im}.$$

The energy written in this way was first presented by Likharev [39] (after a rescaling of the parameters). Chapman *et al* [9] provided a more thorough discussion and justification for including the pinning in this way, and since then many authors have accepted it as the correct way to incorporate pinning into the model [2, 10, 30, 34, 37, 52]. Chapman *et al* [9] argue that the key difference between the superconducting region and the impurity region is the value of the coefficient of the $|u_\varepsilon|^2$ term in the energy functional, and accounting for the difference in the coefficient of the $|u_\varepsilon|^4$ term amounts to a "minor perturbation".

The early works on the GL model with pinning considered a simplified GL energy; one without gauge terms:

$$F_\varepsilon(\tilde{u}_\varepsilon) := \int_\Omega \frac{1}{2}|\nabla \tilde{u}_\varepsilon|^2 + \frac{1}{4\varepsilon^2}(p^2 - |\tilde{u}_\varepsilon|^2)^2. \tag{1.9}$$

Here $\tilde{u}_\varepsilon$ is not necessarily a solution to TDGL equations. André and Shafrir [3] introduce the normalized order parameter

$$w_\varepsilon = \frac{\tilde{u}_\varepsilon}{p} \tag{1.10}$$

in order to simplify the analysis in determining the minimizer to $F_\varepsilon$. The novelty in making this change of variables is that the energy functional can be written in the form

$$F_\varepsilon(\tilde{u}_\varepsilon) = \tilde{G}_\varepsilon^p(w_\varepsilon) + O(1)$$

where

$$\tilde{G}_\varepsilon^p(w_\varepsilon) := \int_\Omega \frac{p^2}{2}|\nabla w_\varepsilon|^2 + \frac{p^4}{4\varepsilon^2}(1-|w_\varepsilon|^2)^2.$$

André and Shafrir deem this energy functional to be easier to work with and the problem of determining the minimizers is closer to previously existing problems for which there are results in the literature.

Some time later, Lassoued and Mironescu [37] observed that the change of variables given by (1.10) is not valid if $p$ is discontinuous. Instead, they suggest working with

$$w_\varepsilon = \frac{u_\varepsilon}{\eta_\varepsilon} \tag{1.11}$$

where $\eta_\varepsilon$ is the minimizer of the energy $F_\varepsilon(\tilde{\eta}_\varepsilon)$ for $\tilde{\eta}_\varepsilon \in H^1$. It follows that $\eta_\varepsilon$ satisfies the elliptic PDE

$$\begin{cases} \Delta\eta_\varepsilon + \frac{\eta_\varepsilon}{\varepsilon^2}(p^2-\eta_\varepsilon^2) = 0 & \text{in } \Omega \\ \nu\cdot\nabla\eta_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.12}$$

$\eta_\varepsilon$ acts a sort of pseudo-pinning potential and it can be shown that it is close to $p$ in the limit of $\varepsilon \to 0$. The main observation by Lassoued and Mironescu [37] is that the energy functional can be slit in two:

$$F_\varepsilon(\eta_\varepsilon w_\varepsilon) = F_\varepsilon(\eta_\varepsilon) + \tilde{G}_\varepsilon^{\eta_\varepsilon}(w_\varepsilon).$$

In this way, the two problems

i) determining the minimizers of the energy functional $F_\varepsilon$

ii) accounting for a discontinuous $p(x)$

can now be treated separately, for we have that

i) $u_\varepsilon$ is a minimizer of $F_\varepsilon \iff u_\varepsilon = \eta_\varepsilon w_\varepsilon$ and $w_\varepsilon$ is a minimizer of $\tilde{G}_\varepsilon^{\eta_\varepsilon}$

ii) $\eta_\varepsilon$ may be seen as a regularization of $p(x)$.

Since these works, the method of using a normalized order parameter has become standard in the study of the GL functional with pinning. The change of variables allows authors to model quite regular inhomogeneous background potentials [26, 34], discontinuous pinning potentials [30, 37], and periodic pinning sites [16], to name just a few. Some authors take $p(x)$ to be a constant [9, 10], whilst some allow it to have an $\varepsilon$-dependence [2, 34]. In our work, we consider $p_\varepsilon(x)$.

Kachmar [30] has since shown that a similar splitting of the energy occurs for the full (with gauge) GL energy functional:

$$\mathcal{G}_\varepsilon^{p_\varepsilon}(u_\varepsilon, A_\varepsilon) = G_\varepsilon^{\eta_\varepsilon}(w_\varepsilon, A_\varepsilon) + F_\varepsilon(\eta_\varepsilon) \tag{1.13}$$

where

$$G_\varepsilon^{\eta_\varepsilon}(w_\varepsilon, A_\varepsilon) := \int_\Omega \frac{\eta_\varepsilon^2}{2}|\nabla_{A_\varepsilon} w_\varepsilon|^2 + \frac{\eta_\varepsilon^4}{4\varepsilon^2}(1-|w_\varepsilon|^2)^2 + \frac{1}{2}(h_\varepsilon - h_{ex})^2. \tag{1.14}$$

Thus, the problem of minimising $\mathcal{G}_\varepsilon^{p_\varepsilon}(u_\varepsilon, A_\varepsilon)$ amounts to minimising $G_\varepsilon^{\eta_\varepsilon}(w_\varepsilon, A_\varepsilon)$.

Determining the Euler-Lagrange equations for $G_\varepsilon^{\eta_\varepsilon}(w_\varepsilon, A_\varepsilon)$ leads to the TDGL equations with pinning. The the main difference is the addition of an interaction term of the form $\nabla \log \eta_\varepsilon^2 \cdot \nabla_{A_\varepsilon} w_\varepsilon$ in the equation for the evolution of the (normalized) order parameter. See Chapter 2 for the derivation of the TDGL equations with pinning.

We are interested in how a system of vortices evolves under the influence of the pinning potential but also under the repulsive influence of the other vortices. The vortex-vortex interaction is actually quite weak and any strong forces in the system cause its effects to become negligible. In order to retain

both pinning and vortex interaction forces in the dynamical law for the vortices, we must scale the pinning potential so it is on the order of the vortex-vortex interaction. We set $p_\varepsilon(x)$ and $\eta_\varepsilon(x)$ to be

$$p_\varepsilon^2(x) = 1 + \frac{q_\varepsilon(x)}{|\log\varepsilon|} \qquad \text{and} \qquad \eta_\varepsilon^2(x) = 1 + \frac{Q_\varepsilon(x)}{|\log\varepsilon|}$$

which ensures that

$$|1 - p_\varepsilon^2| \leq \frac{C}{|\log\varepsilon|} \qquad \text{and} \qquad |1 - \eta_\varepsilon^2| \leq \frac{C}{|\log\varepsilon|}.$$

We assume $p_\varepsilon(x) \in H^k(\Omega)$, $k \geq 2$ and $\eta_\varepsilon \in H^1(\Omega)$, although the regularity can be improved away from the boundary such that $\eta_\varepsilon \in H^{k+2}(\Omega)$ (see Appendix A, [34] for details). Since our pinning isn't discontinuous we may have followed André and Shafrir [3] by working with $p_\varepsilon$ and dispensed with the normalised order parameter and $\eta_\varepsilon$. However, as previously mentioned, it has become standard practice to work with a normalized order parameter and we hope by doing so it is instructive to the reader who may want to follow our scheme with a discontinuous pinning potential. This *critical scaling*, first discussed by Kurzke *et al* [34], allows the interplay between the vortex-vortex interaction and the vortex-pinning interaction in the limiting dynamical law for the vortices. We will show that both $q_\varepsilon$ and $Q_\varepsilon$ converge to a quantity $Q_0$ as $\varepsilon \to 0$, and to recover the situation without pinning we may set $q_\varepsilon = Q_\varepsilon = 0$.

## 1.2 Maxwell-Klein-Gordon

Following the TDGL equations, some authors [23, 55] then considered dynamics for $u$ which are second-order in time:

$$\partial_\Phi^2 u = \Delta_A u + \frac{u}{\varepsilon^2}(1 - |u|^2), \tag{1.15}$$

where

$$\partial_\Phi^2 u := (\partial_t + i\Phi)(\partial_t + i\Phi)u.$$

Such equations describe different physical phenomena than the previous TDGL equations of superconductivity; (1.15) is one of the equations known as the

*Maxwell-Higgs* equations which are familiar in the theory of particle physics [23]. Building on this and inspired by the recent work of Yu [60], in the later part of the thesis we work with the equation

$$\frac{1}{c^2}\partial_\Phi^2 u + \frac{m^2c^2}{\hbar^2}u = \Delta_A u + \frac{u}{\varepsilon^2}(1-|u|^2), \tag{1.16}$$

where $m, c$ and $\hbar$ are the mass of a particle, speed of light and the reduced Planck's constant (all real numbers), respectively. The mathematical interest in such an equation is that it is a generalization of the Maxwell-Higgs equations through the use of a different operator applied to $u$ on the left-hand side of the equations, but it turns out that the equation has physical significance. (1.16) is one of the *Maxwell-Klein-Gordon* equations which bring together relativistic and quantum effects to describe the motion of particles with spin-0. In what follows we provide some background to the equations to motivate our studies.

The Standard Model of physics describes the fundamental particles and forces of nature, except gravity. It is a theory described by a Lagrangian formulation, and the fundamental quantities are described by fields. Since the Lagrangian is invariant under continuous local gauge transformations (rotations in the phase space where the fundamental fields live), the theory is sometimes called a *gauge field theory*. The Lagrangian in the Standard Model gives rise to evolution equations of the gauge fields, called the *Yang-Mills equations* [11, 15]. The Maxwell-Klein-Gordon equation is a special instance of the Yang-Mills equation, and hence is within the framework of the gauge field theory.

To begin with the mathematical description, we start with the presentation of the Klein-Gordon equation, which itself is of huge physical importance. Suppose that $u : \Omega \times \mathbb{R}^+ \to \mathbb{C}$ is a field whose quanta are spinless particles, the norm squared of $u$ being charge density. Suppose $V(x, |u|)$ is a potential energy density of the fields, then we may consider the following Lagrangian density:

$$\mathcal{L}_1(u) = -\frac{1}{2c^2}|\partial_t u|^2 + \frac{1}{2}|\nabla u|^2 + \frac{m^2c^2}{2\hbar^2}|u|^2 + V(x, |u|). \tag{1.17}$$

Such a Lagrangian admits the *Klein-Gordon equation* supplemented by a potential energy $V' := \frac{\partial V}{\partial u}$:

$$\frac{1}{c^2}\partial_t^2 u + \frac{m^2c^2}{\hbar^2}u = \Delta u + V'(x,u). \tag{1.18}$$

When $V' \equiv 0$, (1.18) is the typical Klein-Gordon equation for a free particle. Using the ansatz $u(x,t) = v(x,t)\exp(imc^2t/\hbar)$ leads to the equation

$$\frac{1}{c^2}\partial_t^2 v + \frac{2im}{\hbar}\partial_t v = \Delta v + V'(x,v). \tag{1.19}$$

The KG equation is a relativistically-invariant version of the Schrödinger equation. The terms $\hbar$ and $c$ makes the connection between quantum mechanics and relativity, and the KG equation emerges as a prominent equation of quantum field theory.

To incorporate the effects of electromagnetism, it is necessary to modify the Lagrangian density:

$$\mathcal{L}_2(u,A,\Phi) = -\frac{1}{2c^2}|\partial_\Phi u|^2 + \frac{1}{2}|\nabla_A u|^2 + \frac{m^2c^2}{2\hbar^2}|u|^2 + V(x,|u|) + \frac{1}{2}|h|^2 - \frac{1}{2}|E|^2. \tag{1.20}$$

Here the derivatives from (1.17) become covariant derivatives to preserve gauge invariance and the *Maxwell Lagrangian density* has been added. The resulting Maxwell-Klein-Gordon equations are the Euler-Lagrange equations of $\mathcal{L}_2$:

$$\begin{cases} \frac{1}{c^2}\partial_\Phi^2 u + \frac{m^2c^2}{\hbar^2}u = \Delta_A u + V'(x,u) \\ \frac{1}{c^2}\partial_t E = -\nabla^\perp h - \langle iu, \nabla_A u\rangle \\ \mathrm{div} E = -\langle iu, \partial_\Phi u\rangle. \end{cases} \tag{1.21}$$

We see that because the Lagrangian includes time-dependent quantities, the evolution of $u$ and $A$ is naturally present. This is in contrast to the GL model were additional arguments were necessary to justify the time-derivatives. It is clear that the key difference between the MKG equations and the TDGL equations is the second order time-derivative compared to just first-order before.

### 1.2.1 MKG with pinned GL potential

Our interest lies in the specific problem of when the potential energy $V$ in (1.20) is of the form of the GL penalty term. Specifically, we set

$$V(x, |u|) = \frac{1}{4\varepsilon^2}(p^2 - |u|^2)^2, \tag{1.22}$$

where $p$ is the pinning potential we introduced in Section 1.1.3.

Based on the discussion in Section 1.1.1, we know that if the potential energy is of the form of the GL energy, the Euler-Lagrange equations admit solutions which describe an array of vortices. Hence, although the MKG equations describe spin-0 particles, in the limit $\varepsilon \to 0$ our equations will have vortex solutions. Indeed, the equation of motion we derive is a vortex dynamical law.

Part of the scope of this thesis is to study how the methods in deriving the vortex dynamical law for TDGL adapt to the MKG setting. The model itself is interesting to look at because it is more general than the TDGL model and more naturally includes the electromagnetic fields (equations for electromagnetic fields are closer to Maxwell's equations), so is more physically relevant, but beyond the physical relevance of the model, we use MKG as a framework for testing the adaptability of the mathematics we use in the TDGL model to a more complicated setting. The benefit of supplementing the MKG equation with a GL potential is that we can call on some results from the first part of the thesis, but we will see that some changes to the calculations will be necessary. We postpone more in depth discussion and the presentation of the MKG equations with pinning to Chapter 6.

## 1.3 Previous work

### 1.3.1 TDGL

Formally, (1.4) shows that the relaxation of $u$ is proportional to the negative derivative of $G$. More rigorously we know this to be true for the negative $L^2$

gradient of $G$ with respect to $u$. Additionally, the right-hand side of the second equation in (1.7) is the negative $L^2$ gradient of the energy with respect to variations in $A$. Hence, the TDGL equations can be written

$$\begin{cases} (\alpha + i\beta)(\partial_t + i\Phi)u = -\nabla_u^{L^2} G_\varepsilon(u, A) & \text{in } \Omega \\ \sigma(\partial_t A + \nabla\Phi) = -\nabla_A^{L^2} G_\varepsilon(u, A) & \text{in } \Omega \\ \nu \cdot (\nabla - iA)u = 0 & \text{on } \partial\Omega \\ \text{curl } A = h_{ex} & \text{on } \partial\Omega. \end{cases}$$

The coefficients $\alpha$ and $\beta$ determine the relaxation time of $u$, and it is clear that they both describe different types of flow. When $\alpha \neq 0$ and $\beta = 0$ the equations are parabolic and we have a purely dissipative gradient flow. When $\alpha = 0$ and $\beta \neq 0$, the motion is conservative. We call it "Schrödinger flow" since the equation resembles a gauged Schrödinger equation.

Much of the work on vortex dynamics considers one type of flow and only very few with the presence of pinning sites. The dynamical law for the vortices derived from the gradient flow equation for $u$ was studied by Spirn [54] and Sandier and Serfaty [48]. Neither consider the role of pinning. Spirn set the external magnetic field to be $h_{ex} = O(1)$ and finds that the position $a_i(t)$ of the $i^{th}$ vortex satisfies the ODE

$$\frac{da_i}{dt} =: \dot{a}_i = -\frac{1}{\pi}\nabla_{a_i} W(a, d).$$

Sandier and Serfaty study the system with a stronger external field, $h_{ex} = O(|\log\varepsilon|)$, and find that the vortices follow the gradient flow of a function of the field rather than the renormalized energy. In a bounded domain the external magnetic field on the boundary decays into the material, creating a background potential felt by the vortices. The stronger field dominates the dynamics and the intervortex interactions don't appear in the vortex motion law.

The Schrödinger flow was studied by Spirn [55]. Spirn found that it is necessary for the initial energy in the system to be precisely bounded with very

little excess energy. The gradient-flow requires much less stringent control on the initial data. A further difference discovered by Spirn is the movement of vortices in the $O(1)$ timescale. Gradient-flow motion only occurs on a $|\log\varepsilon|$ timescale and so in those works time must be accelerated to observe vortex dynamics. The resulting dynamical law from the Schrödinger flow equations is

$$\dot{a}_i^{\perp}(t) = \nabla_{a_i} W(a, d).$$

The first work to study the role of pinning in the dynamics was by Lin [41] in a setting without gauge terms. The vortex dynamics were derived from the equation

$$\partial_t u = \frac{1}{p}\mathrm{div}(p\nabla u) + \frac{u}{\varepsilon^2}(1 - |u|^2).$$

Here the pinning potential $p$ is independent of $\varepsilon$, is strong enough to cause vortex motion in the $O(1)$ timescale and is incorporated in a different way to what we consider. It leads to a vortex motion law of the form

$$\dot{a}_i(t) = -\nabla \log p(a_i(t)).$$

Notice that the strong pinning force dominates the vortex motion law in which the much weaker intervortex-interactions don't play a role.

The first work in the setting of the critically scaled pinning potential is by Kurzke, Marzuola and Spirn [34]. They consider each type of flow separately but set $A = \Phi = 0$. They find that both the pinning potential and intervortex-interactions influence the vortex dynamics.

Only recently has the case of mixed-flow been considered. At fist, the mixed-flow equation with $A = \Phi = 0$ was studied. Kurzke *et al* [33] presented the vortex motion law for a bounded domain and Miot [44] for the whole plane. The result is a dynamical law with a component of vortex velocity parallel and perpendicular to the gradient of the renormalized energy. Kurzke and Spirn [35] then include the gauge terms $(A, \Phi)$ with the mixed-flow to find that in a

bounded domain, the position $a_i(t)$ of a given vortex satisfies the ODE

$$\alpha \dot{a}_i + d_i \beta \dot{a}_i^{\perp} = -\frac{1}{\pi} \nabla_{a_i} W(a, d).$$

The vortex exhibits a combination of gradient descent motion and conservative Schrödinger motion.

Serfaty and Tice [52] consider the situation of mixed-flow and strong pinning but with the added complication of an externally applied electric current on the boundary of the domain. Their electric current and pinning forces are so strong that the intervortex- interactions have no effect on the dynamics. This work provides a novel way to treat gauge terms and identified a special structure in the vortex-current interaction term. They find that the electric current produces a force orthogonal to the velocity of the vortex. This is consistent with the Lorentz force one would expect in this situation. We are slightly wary about the applicability of the TDGL model to a superconductor subject to such large forces, since the model's validity is for only small deviations from equilibrium. Indeed, the authors state that the results are valid for a short time.

Chapman *et al* [9] provide some numerical simulation results which show that a vortex near an impurity will be attracted to the pinning site, and that this will cause the additional effect of reorganising the other vortices in the sample due to the repulsive forces between vortices. For this reason we expect the physically relevant model of pinning in a superconductor should also account for the effect of vortex-vortex interactions.

The novelty of the work in this thesis is that it brings together the concept of mixed-flow in the setting of a critically-scaled pinning potential and includes the gauge terms in every step. This leads to a number of extra terms involving the pinning or electric/magnetic fields which we must deal with. We set $h_{ex} = O(1)$ so that the magnetic field doesn't dominate the dynamics of the vortices, but it is nevertheless considered in the analysis. Additionally, because our interest is

in the interplay of the pinning and the remormalized energy, we don't consider the role of an applied electric current.

### 1.3.2 MKG

Whilst some authors considered the time-derivative of the order parameter equal to the critical point of the energy (see (1.4)), others studied the consequences of setting the second-order time-derivative equal to the critical point of the energy. We briefly mention a few works which focus on the wave equation analogue of the TDGL equation. Lin [42] and Jerrard [25] consider the equation

$$\frac{1}{|\log \varepsilon|}\frac{\partial^2 u}{\partial t^2} = \Delta u + \frac{u}{\varepsilon^2}(1-|u|^2). \tag{1.23}$$

The second-order time derivative carries over to the dynamical law for the vortices and Lin finds that the vortex positions satisfy the following ODE:

$$\ddot{a}_i(t) = -\frac{1}{\pi}\nabla_{a_i} W(a,d). \tag{1.24}$$

Continuous motion is seen on the $O(\sqrt{|\log \varepsilon|})$ timescale and the requirement on the initial data is that the velocities of the vortices are zero and the vortex configuration is of almost minimal energy with given locations of vortices.

Including gauge terms to allow the interaction with the electromagnetic fields, the so-called *Maxwell-Higgs equations* take the following form:

$$\begin{cases} \partial_\Phi^2 u = \Delta_A u + \frac{u}{\varepsilon^2}(1-|u|^2) \\ \partial_t E = -\nabla^\perp h - \langle iu, \nabla_A u\rangle. \end{cases} \tag{1.25}$$

Such equations describe the $U(1)$-gauge Higgs model of particle physics [23]. The equation for the electromagnetic potentials is the more familiar form of Ampere's law from electrodynamics. The equation from (1.6) is a more heuristic equation of motion for the potentials. Spirn [55] records that the limiting dynamical law from the Maxwell-Higgs equations (1.25) is (1.24). Gustafson and Sigal [23] achieve the same dynamical law and record that it desribes the

motion of Nielsen-Olesen vortices or Nambu strings in the Abelian Higgs model of particle physics.

A vortex motion law from the Klein-Gordon equation with a Ginzburg-Landau potential is derived by Yu [59]. The Klein-Gordon equation does not include gauge terms so has the form

$$\frac{1}{c^2}\partial_t^2 u + \frac{c^2}{4}u = \Delta u + \frac{u}{\varepsilon^2}(1-|u|^2) \tag{1.26}$$

for the choice $\hbar = 1$ and $m = \frac{1}{2}$ which Yu makes. If one assumes that $u(x,t) = v(x,t)\exp(ic^2t/2)$, then (1.26) reduces to

$$\frac{1}{c^2}\partial_t^2 v + i\partial_t v = \Delta v + \frac{v}{\varepsilon^2}(1-|v|^2). \tag{1.27}$$

Hence, the KG equation with GL potential defines a mixed flow combining the GL wave equation and Schrödinger flow equation. This allows one to use the information obtained about wave and Schrödinger flows from previous works to determine the vortex motion law. In this spirit, Yu uses the initial data from [25, 42], allows $c^2 = \mu^2|\log\varepsilon|$ and arrives at

$$\frac{1}{\mu^2}\ddot{a}_i(t) + d_i\dot{a}_i^{\perp}(t) = -\frac{1}{\pi}\nabla_{a_i}W(a,d). \tag{1.28}$$

For discussion of the various limits ($c\to\infty$, $\hbar\to 0$, $c=\hbar^{-\alpha}$ for $\alpha>0$) of the KG equation see [43].

Building on his previous work, Yu [60] then considered the gauged version of the Klein-Gordon equation; the Maxwell-Klein-Gordon equations. Again the GL potential was included:

$$\frac{1}{c^2}\partial_\Phi^2 u + \frac{c^2}{4}u = \Delta_A u + \frac{u}{\varepsilon^2}(1-|u|^2) \tag{1.29}$$

$$\frac{1}{c^2}\partial_t E = -\nabla^{\perp}h - \langle iu, \nabla_A u\rangle. \tag{1.30}$$

Here Yu has set $c\hbar = 1$ and $m = 1$ for simplicity. Unfortunately, this equation can't be reduced to a simpler form from a simple ansatz and so it is necessary

to derive the vortex dynamics from the equation (1.29) as it is written - we can't use the known information about GL wave equation or the parabolic/Schrodinger flows. An added complication to this work is the consideration of an applied electric field on the boundary of the domain. This has the effect of inducing an additional form in the vortex dynamical law. If $c^2 = \mu^2 |\log \varepsilon|$, then the vortex dynamical law is

$$\frac{1}{\mu^2}\ddot{a}_i(t) = -\frac{1}{\pi}\nabla_{a_i} W(a,d) - \frac{1}{\mu} d_i f(a_i)\dot{a}_i^{\perp}(t) \tag{1.31}$$

where $f(a_i)$ is determined by the electric field felt from the boundary. The MKG model admits an equation of motion more familiar with Newton's second law.

To our knowledge, no work has been done with the Maxwell-Klein-Gordon equations with a pinned GL potential.

## 1.4 Plan for the thesis

The thesis is essentially split into two parts; results for $(u, A, \Phi)$ satisfying the time-dependent Ginzburg-Landau equations and then for $(u, A, \Phi)$ satisfying the Maxwell-Klein-Gordon equations. In each case the equations are generalized to include the effects of a perturbed background potential.

After setting out some definitions and preliminary results in Chapter 2, we present our first key result in Chapter 3. Here, we show an upper bound on the energy in what we call the "coercivity estimate", which is the first of its kind to include both gauge and pinning terms. Such an estimate on the energy is important in its own right, but proves useful in the later chapters. In Chapter 4 we prove control on the energy and dissipation for some finite time and in Chapter 5 we go about deriving the dynamical law for vortices in the TDGL framework. Chapters 6 and 7 follow the procedure set out previously to derive the vortex motion law in the MKG framework. At any stage, by setting the

pinning potential to 1 so that it is uniform, we immediately recover results for the Ginzburg-Landau model without pinning or Maxwell-Klein-Gordon without pinning, which may be of interest.

# Chapter 2

# Ginzburg-Landau theory with pinning

Now we get further into the mathematics. As well as introducing the key quantities, we establish some notation and definitions which will be used throughout the sequel. The aims of the chapter are to present how we incorporate the pinning potential in to the Ginzburg-Landau model and to establish all the quantities which play a role in deriving the dynamical law of the vortices. We present some preliminary lemmas.

## 2.1 Notation & definitions

Suppose that $\Omega \subset \mathbb{R}^2$ is bounded, simply connected and has $C^1$ boundary. For pairwise distinct points in the domain $a = (a_1, ..., a_n) \in \Omega^n$, we define

$$r_a := \frac{1}{4} \min\{\min_{i \neq j} |a_i - a_j|, \min_i \operatorname{dist}(a_i, \partial\Omega)\}. \tag{2.1}$$

For $r < r_a$, we define the perforated domain

$$\Omega_r(a) := \Omega \backslash \bigcup_{i=1}^{n} \overline{B_r(a_i)},$$

but often just write $\Omega_r$ if the positions of the balls is clear from context. Due to the two-dimensional nature of the system, we first record that the curl of a vector $V = (V_1, V_2) \in \mathbb{R}^2$ produces a scalar: curl $V = \partial_1 V_2 - \partial_2 V_1$. Conversely,

the curl of a scalar $h : \mathbb{R}^2 \to \mathbb{R}$ produces a vector: curl $h = (\partial_2 h, \partial_1 h)$. The divergence of vectors and scalars works in the normal way. Rotation of a vector anticlockwise in the plane by $\pi/2$ is denoted $V^\perp = (-V_2, V_1)$, and for a rotated gradient we write $\nabla^\perp h = (\nabla h)^\perp$. It is worth noting that curl$\nabla^\perp h = \Delta h$. For $a, b \in \mathbb{C}$, the quantity

$$\langle a, b \rangle = R(a)R(b) + I(a)I(b)$$

denotes the scalar product, which is always real. We sometimes refer to it as an *inner product* although it should not be confused with the standard Hermitian form of an inner product on $\mathbb{C}^n$. See Appendix A for some properties of the inner product in the sense we use it. If $X = (X_1, X_2) \in \mathbb{C}^2$, then

$$\langle a, X \rangle = (\langle a, X_1 \rangle, \langle a, X_2 \rangle)$$

is a vector in $\mathbb{R}^2$.

The *order parameter* of our superconductor is $u_\varepsilon : \Omega \times \mathbb{R}^+ \to \mathbb{C}$, the *magnetic vector potential* is $A_\varepsilon : \Omega \times \mathbb{R}^+ \to \mathbb{R}^2$ and the *electric scalar potential* is $\Phi_\varepsilon : \Omega \times \mathbb{R}^+ \to \mathbb{R}$, as they were in (1.7). Throughout the thesis, we use the covariant derivatives

$$\nabla_{A_\varepsilon} u_\varepsilon := \nabla u_\varepsilon - i A_\varepsilon u_\varepsilon$$

$$\Delta_{A_\varepsilon} u_\varepsilon := \Delta u_\varepsilon - 2i A_\varepsilon \cdot \nabla u_\varepsilon - i u_\varepsilon \text{ div } A_\varepsilon - |A_\varepsilon|^2 u_\varepsilon$$

$$\partial_{\Phi_\varepsilon} u_\varepsilon := \partial_t u_\varepsilon + i \Phi_\varepsilon u_\varepsilon,$$

where, if $\nabla u_\varepsilon = (\partial_1 u_\varepsilon, \partial_2 u_\varepsilon)$, then

$$\Delta u_\varepsilon := \nabla \cdot \nabla u_\varepsilon = \partial_1^2 u_\varepsilon + \partial_2^2 u_\varepsilon.$$

We endeavour to include the subscript $\varepsilon$ in the key places but often simply write $(u, A, \Phi)$ for brevity and ease of reading. Limiting values as $\varepsilon \to 0$ are always denoted with a subscript *: $(u_*, A_*, \Phi_*)$. The *magnetic field* is

$$h_\varepsilon := \text{curl } A_\varepsilon, \tag{2.2}$$

whilst the *electric field* is

$$E_\varepsilon := -\partial_t A_\varepsilon - \nabla \Phi_\varepsilon. \tag{2.3}$$

The flow of superconducting particles is referred to as the *supercurrent* and has the following mathematical structure

$$j_{A_\varepsilon}(u_\varepsilon) := \langle i u_\varepsilon, \nabla_{A_\varepsilon} u_\varepsilon \rangle. \tag{2.4}$$

$H_0^1$ denotes the closure of smooth functions with compact support in $\Omega$ in the $H^1$ norm $\|u\|_{H^1}^2 = \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2$. Similarly $W_0^{1,p}$ denotes the closure of smooth functions with compact support in $\Omega$ in the $W^{1,p}$ norm while $W_0^{-1,p}$ is the dual of $W^{1,q}$, where $1/p + 1/q = 1$. The norm defined next is ubiquitous in the results and proofs in the following sections. It is a weak norm on the dual of $W_0^{1,p}(\Omega)$

$$\|\mu\|_{\dot{W}^{-1,q}} := \sup\left\{\mu(\phi) \mid \phi \in W_0^{1,p}(\Omega),\ \ \|\nabla\phi\|_{L^p} \leq 1\right\}, \qquad \frac{1}{p} + \frac{1}{q} = 1.$$

Suppose $X$ is a Banach space with norm $\|\cdot\|_X$. The space $L^p([0,T];X)$ consists of all measurable functions $f : [0,T] \to X$ with

$$\|f\|_{L^p([0,T];X)} := \Big( \int_0^T \|f(t)\|_X^p \, dt \Big)^{\frac{1}{p}} < \infty \qquad (1 \leq p < \infty) \tag{2.5}$$

and

$$\|f\|_{L^\infty([0,T];X)} := \operatorname*{ess\,sup}_{t\in[0,T]} \|f(t)\|_X < \infty. \tag{2.6}$$

## 2.2 Ginzburg-Landau energy with pinning

Following the approach laid out in the previous chapter we derive the TDGL equations with pinning. In Section 1.1.3 we justified including the pinning potential $p_\varepsilon(x)$ inside the penalty term of the GL energy, like so [2, 34]:

$$\mathcal{G}_\varepsilon^{p_\varepsilon}(u_\varepsilon, A_\varepsilon) = \int_\Omega \frac{1}{2}|\nabla_{A_\varepsilon} u_\varepsilon|^2 + \frac{1}{4\varepsilon^2}(p_\varepsilon^2 - |u_\varepsilon|^2)^2 + \frac{1}{2}|h_\varepsilon - h_{ex}|^2. \tag{2.7}$$

Following Lassoued and Mironescu [37] we work with the normalized order parameter $w_\varepsilon : \Omega \times \mathbb{R}^+ \to \mathbb{C}$ which satisfies

$$w_\varepsilon(x,t) = \frac{u_\varepsilon(x,t)}{\eta_\varepsilon(x)}, \tag{2.8}$$

where $\eta_\varepsilon$ is a background potential which is the unique, positive solution of the following minimization problem [34]:

$$\eta_\varepsilon = \arg \min_{\tilde{\eta}_\varepsilon \in H^1(\Omega;\mathbb{R})} \int_\Omega \frac{1}{2}|\nabla \tilde{\eta}_\varepsilon|^2 + \frac{1}{4\varepsilon^2}(p_\varepsilon^2 - \tilde{\eta}_\varepsilon^2)^2 \, dx.$$

It follows that $\eta_\varepsilon$ satisfies the elliptic PDE

$$\begin{cases} 0 = \Delta \eta_\varepsilon + \frac{\eta_\varepsilon}{\varepsilon^2}(p_\varepsilon^2 - \eta_\varepsilon^2) & \text{in } \Omega \\ \nu \cdot \nabla \eta_\varepsilon = 0 & \text{on } \partial\Omega \end{cases} \tag{2.9}$$

where $\nu$ is the unit outward normal vector to the boundary and $\cdot$ denotes the usual dot product between vectors.

We will see in Proposition 2.4 that (2.9) ensures $\eta_\varepsilon$ is close to $p_\varepsilon$. First we show that (2.8) leads to a useful splitting of the energy functional. Kachmar [30] provides this result but we include it here for convenience and provide the extra details in our proof which are omitted from that article.

**Lemma 2.1.** *If $u_\varepsilon = \eta_\varepsilon w_\varepsilon$ and $\eta_\varepsilon$ satisfies (2.9), then the energy functional becomes*

$$\mathcal{G}_\varepsilon^{p_\varepsilon}(u_\varepsilon, A_\varepsilon) = G_\varepsilon^{\eta_\varepsilon}(w_\varepsilon, A_\varepsilon) + F_\varepsilon(\eta_\varepsilon), \tag{2.10}$$

*where*

$$G_\varepsilon^{\eta_\varepsilon}(w_\varepsilon, A_\varepsilon) := \int_\Omega \frac{\eta_\varepsilon^2}{2}|\nabla_{A_\varepsilon} w_\varepsilon|^2 + \frac{\eta_\varepsilon^4}{4\varepsilon^2}(1 - |w_\varepsilon|^2)^2 + \frac{1}{2}|h_\varepsilon - h_{ex}|^2 \tag{2.11}$$

*and*

$$F_\varepsilon(\eta_\varepsilon) := \int_\Omega \frac{1}{2}|\nabla \eta_\varepsilon|^2 + \frac{1}{4\varepsilon^2}(p_\varepsilon^2 - \eta_\varepsilon^2)^2. \tag{2.12}$$

*Proof.* Replace $u_\varepsilon$ by $\eta_\varepsilon w_\varepsilon$ in (2.7):

$$\mathcal{G}_\varepsilon^{p_\varepsilon}(\eta_\varepsilon w_\varepsilon, A_\varepsilon) = \int_\Omega \frac{\eta_\varepsilon^2}{2}|\nabla_{A_\varepsilon} w_\varepsilon|^2 + \frac{1}{2}|w_\varepsilon|^2|\nabla\eta_\varepsilon|^2 + \frac{1}{2}\eta_\varepsilon\nabla\eta_\varepsilon\cdot\nabla|w_\varepsilon|^2 + \int_\Omega \frac{1}{4\varepsilon^2}(p_\varepsilon^2 - \eta_\varepsilon^2|w_\varepsilon|^2)^2 + \frac{1}{2}|h_\varepsilon - h_{ex}|^2. \quad (2.13)$$

Note that

$$\begin{aligned}(p_\varepsilon^2 - \eta_\varepsilon^2|w_\varepsilon|^2)^2 &= \big((p_\varepsilon^2 - \eta_\varepsilon^2) + \eta_\varepsilon^2(1-|w_\varepsilon|^2)\big)^2 \\ &= (p_\varepsilon^2-\eta_\varepsilon^2)^2 + \eta_\varepsilon^4(1-|w_\varepsilon|^2)^2 + 2\eta_\varepsilon^2(p_\varepsilon^2-\eta_\varepsilon^2)(1-|w_\varepsilon|^2).\end{aligned}$$

Thus, (2.13) becomes

$$\begin{aligned}\mathcal{G}_\varepsilon^{p_\varepsilon}(\eta_\varepsilon w_\varepsilon, A_\varepsilon) = \int_\Omega \frac{\eta_\varepsilon^2}{2}|\nabla_{A_\varepsilon} w_\varepsilon|^2 + \frac{\eta_\varepsilon^4}{4\varepsilon^2}(1-|w_\varepsilon|^2)^2 + \frac{1}{2}|h_\varepsilon - h_{ex}|^2 \\ + \int_\Omega \frac{1}{2}|\nabla\eta_\varepsilon|^2 + \frac{1}{4\varepsilon^2}(p_\varepsilon^2-\eta_\varepsilon^2)^2 \\ + \int_\Omega \frac{1}{2}\eta_\varepsilon\nabla\eta_\varepsilon\cdot\nabla|w_\varepsilon|^2 - \frac{1}{2}(1-|w_\varepsilon|^2)|\nabla\eta_\varepsilon|^2 + \frac{\eta_\varepsilon^2}{2\varepsilon^2}(p_\varepsilon^2-\eta_\varepsilon^2)(1-|w_\varepsilon|^2). \quad (2.14)\end{aligned}$$

Since $\eta_\varepsilon$ satisfies (2.9), we multiply the first equation of (2.9) by $\frac{1}{2}\eta_\varepsilon(1-|w_\varepsilon|^2)$ and integrate over $\Omega$:

$$\int_\Omega \frac{\eta_\varepsilon}{2\varepsilon^2}(p_\varepsilon^2-\eta_\varepsilon^2)(1-|w_\varepsilon|^2) = -\int_\Omega \frac{1}{2}(\Delta\eta_\varepsilon)\eta_\varepsilon(1-|w_\varepsilon|^2)$$

$$= \int_\Omega \frac{1}{2}|\nabla\eta_\varepsilon|^2(1-|w_\varepsilon|^2) - \frac{1}{2}\eta_\varepsilon\nabla\eta_\varepsilon\cdot\nabla|w_\varepsilon|^2 - \int_{\partial\Omega}\frac{\eta_\varepsilon}{2}(1-|w_\varepsilon|^2)\ \nu\cdot\nabla\eta_\varepsilon. \quad (2.15)$$

The boundary integral vanishes due to $\nu\cdot\nabla\eta_\varepsilon = 0$ in (2.9), and so (2.15) provides

$$\int_\Omega \frac{1}{2}\eta_\varepsilon\nabla\eta_\varepsilon\cdot\nabla|w_\varepsilon|^2 - \frac{1}{2}(1-|w_\varepsilon|^2)|\nabla\eta_\varepsilon|^2 + \frac{\eta_\varepsilon^2}{2\varepsilon^2}(p_\varepsilon^2-\eta_\varepsilon^2)(1-|w_\varepsilon|^2) = 0. \quad (2.16)$$

Using (2.16) in (2.14) provides the result. □

Since $\eta_\varepsilon$ is defined to minimise $F_\varepsilon$, the problem of minimising $\mathcal{G}_\varepsilon^{p_\varepsilon}(u_\varepsilon, A_\varepsilon)$ amounts to minimising $G_\varepsilon^{\eta_\varepsilon}(w_\varepsilon, A_\varepsilon)$. We now find the Euler-Lagrange equations for the energy functional (2.11).

**Proposition 2.2.** *If $\eta_\varepsilon$ satisfies (2.9), the Euler-Lagrange equations associated to the energy functional with pinning (2.11) are*

$$\begin{cases} 0 = \Delta_{A_\varepsilon} w_\varepsilon + \frac{\eta_\varepsilon^2 w_\varepsilon}{\varepsilon^2}(1-|w_\varepsilon|^2) + \nabla \log \eta_\varepsilon^2 \cdot \nabla_{A_\varepsilon} w_\varepsilon & \text{in } \Omega \\ 0 = \nabla^\perp h_\varepsilon + \eta_\varepsilon^2 j_{A_\varepsilon}(w_\varepsilon) & \text{in } \Omega \\ \nu \cdot \nabla_{A_\varepsilon} w_\varepsilon = 0 & \text{on } \partial\Omega \\ h_\varepsilon = h_{ex} & \text{in } \mathbb{R}^2 \backslash \Omega. \end{cases} \tag{2.17}$$

*Proof.* Suppose $t \in \mathbb{R}$ and $v \in H^1(\Omega, \mathbb{C})$. Applying the minimisation procedure to $G_\varepsilon^{\eta_\varepsilon}(w_\varepsilon, A_\varepsilon)$ in its first variable leads to

$$\frac{d}{dt} G_\varepsilon^{\eta_\varepsilon}(w+tv, A)\Big|_{t=0} = \int_\Omega \eta_\varepsilon^2 \langle \nabla_A w, \nabla_A v\rangle - \frac{\eta_\varepsilon^4}{\varepsilon^2}(1-|w|^2)\langle w, v\rangle. \tag{2.18}$$

By Lemma A.1, we have $\langle \nabla_A w, \nabla_A v\rangle = \operatorname{div}\langle \nabla_A w, v\rangle - \langle \Delta_A w, v\rangle$, where $\langle \nabla_A w, v\rangle = \big(\langle (\partial_1 - iA_1)w, v\rangle, \langle (\partial_2 - iA_2)w, v\rangle\big)$. Thus,

$$\begin{aligned} \int_\Omega \eta_\varepsilon^2 \langle \nabla_A w, \nabla_A v\rangle &= -\int_\Omega \eta_\varepsilon^2 \langle \Delta_A w, v\rangle + \int_\Omega \eta_\varepsilon^2 \operatorname{div}\langle \nabla_A w, v\rangle \\ &= -\int_\Omega \eta_\varepsilon^2 \langle \Delta_A w, v\rangle + \nabla \eta_\varepsilon^2 \cdot \langle \nabla_A w, v\rangle + \int_{\partial\Omega} \eta_\varepsilon^2 \langle \nu \cdot \nabla_A w, v\rangle. \end{aligned}$$

Collecting everything together in (2.18) gives

$$\begin{aligned} \frac{d}{dt} G_\varepsilon^{\eta_\varepsilon}(w+tv, A)\Big|_{t=0} &= \int_{\partial\Omega} \eta_\varepsilon^2 \langle \nu \cdot \nabla_A w, v\rangle \\ &\quad - \int_\Omega \eta_\varepsilon^2 \langle \Delta_A w + \nabla \log \eta_\varepsilon^2 \cdot \nabla_A w + \frac{\eta_\varepsilon^2 w}{\varepsilon^2}(1-|w|^2), v\rangle. \end{aligned}$$

Since this is true for any $v \in H^1(\Omega, \mathbb{C})$, we find that the energy is at a critical point if

$$0 = \Delta_A w + \nabla \log \eta_\varepsilon^2 \cdot \nabla_A w + \frac{\eta_\varepsilon^2 w}{\varepsilon^2}(1-|w|^2) \qquad \text{in } \Omega$$

and $\nu \cdot \nabla_{A_\varepsilon} w_\varepsilon = 0$ on $\partial\Omega$.

Suppose $B \in H^1(\Omega, \mathbb{R}^2)$. Applying the minimising procedure to $G_\varepsilon^{\eta_\varepsilon}(w_\varepsilon, A_\varepsilon)$ for the second variable gives

$$\frac{d}{dt} G_\varepsilon^{\eta_\varepsilon}(w, A+tB)\Big|_{t=0} = \int_\Omega \eta_\varepsilon^2 \langle \nabla_A w, -iBw\rangle + (h - h_{ex})\operatorname{curl} B.$$

We want to integrate by parts on the last term so that we have something multiplied by just $B$. Note that

$$(h - h_{ex})\text{curl } B = \text{curl}((h - h_{ex})B) - \nabla^{\perp}(h - h_{ex}) \cdot B,$$

and then

$$\frac{d}{dt} G_{\varepsilon}^{\eta_\varepsilon}(w, A + tB)\Big|_{t=0} = -\int_{\Omega} \left(\eta_{\varepsilon}^2 \langle iw, \nabla_A w\rangle + \nabla^{\perp}(h - h_{ex})\right) \cdot B + \int_{\partial\Omega} (h - h_{ex}) B \cdot \tau$$

where $\tau$ is the unit tangent vector to the boundary.

Since this is true for any $B \in H^1(\Omega, \mathbb{R}^2)$, we find the energy is at a critical point if

$$0 = \nabla^{\perp}(h - h_{ex}) + \eta_{\varepsilon}^2 \langle iw, \nabla_A w\rangle \qquad \text{in } \Omega$$

and $h = h_{ex}$ on $\partial\Omega$. □

Incorporating the time-dependence of $(u_\varepsilon, A_\varepsilon, \Phi_\varepsilon)$ in the equations in the manner of (1.4) leads to

**Definition 2.3.** *The TDGL equations with pinning are*

$$\begin{cases} (\alpha_\varepsilon + i\beta_0)\partial_{\Phi_\varepsilon} w_\varepsilon = \Delta_{A_\varepsilon} w_\varepsilon + \frac{\eta_\varepsilon^2 w_\varepsilon}{\varepsilon^2}(1 - |w_\varepsilon|^2) + \nabla \log \eta_\varepsilon^2 \cdot \nabla_{A_\varepsilon} w_\varepsilon & \text{in } \Omega \\ \sigma_\varepsilon E_\varepsilon = -\nabla^{\perp} h_\varepsilon - \eta_\varepsilon^2 j_{A_\varepsilon}(w_\varepsilon) & \text{in } \Omega \\ \nu \cdot \nabla_{A_\varepsilon} w_\varepsilon = 0 & \text{on } \partial\Omega \\ h_\varepsilon = h_{ex} & \text{on } \partial\Omega. \end{cases} \tag{2.19}$$

The well-posedness of the problem is discussed in Remark 2.6. We allow the coefficients to have an $\varepsilon$-dependence such that

$$\alpha_\varepsilon = \frac{\alpha_0}{|\log \varepsilon|} \qquad \text{and} \qquad \sigma_\varepsilon = \frac{\sigma_0}{|\log \varepsilon|} \ll 1 \tag{2.20}$$

where $\alpha_0$ and $\sigma_0$ are constants independent of $\varepsilon$. We note that this is a completely natural assumption to have on $\sigma_\varepsilon$. As the size of the vortices

decreases, the normal conductivity in the system decreases. This scaling for $\sigma$ is necessary for sufficient control over the electric field to be able to derive continuous motion of vortices. This is consistent with [48, 54] where this is implicitly used. The precise value of $\sigma_\varepsilon$ is irrelevant for the vortex motion here and in those works. In a more detailed analysis, we expect $\sigma$ to be visible in higher order terms. Dorsey [17] manages to show some $\sigma$-dependence in the coefficients of $\dot{a}$ and $\dot{a}^\perp$ under a certain rescaling of parameters and for an approximate to $\Phi$. It is difficult to reconcile Dorsey's equations with the equations we present here for those reasons. Our upper bound for $\sigma$ ensures that the limiting magnetic field is determined by the vortex positions alone.

As Tice [57] records, vortices do not move until after an amount of time $\tau_\varepsilon = |\log \varepsilon|/f_\varepsilon$, where $f_\varepsilon$ is the dominant force in the system. In the work by Spirn [54], $f_\varepsilon = h_{ex}$ and is independent of $\varepsilon$. This means that in the limit $\varepsilon \to 0$, $\tau_\varepsilon \to \infty$ and the vortices don't move. In the work by Serfaty [48], $f_\varepsilon = h_{ex}(\varepsilon) = C_0|\log \varepsilon|$, so $\tau_\varepsilon = C_0^{-1}$ and the vortices move in the original time scale. However, such strong forces which cause vortex motion in this time scale dominate the vortex dynamics and the vortex-vortex interaction is negligible by comparison. Since we wish to determine the role of the vortex-vortex interactions in the dynamics, it is necessary for the other forces in the system to not scale with $\varepsilon$, hence $\tau_\varepsilon \to \infty$. In order to see vortex motion, we rescale the coefficients of the time-derivatives such that our approach is capable of describing slow processes. See Theorem 3 from Tice's work [57] for a description of the various coefficients one can expect in the dynamical law for different strengths of forcing. We let $\beta_0$ be independent of $\varepsilon$. This choice of scaling for $\alpha$ and $\beta$ is the only one where a mixed motion law can be expected for our choice of $h_{ex} = O(1)$ and $p_\varepsilon^2 = 1 + \frac{q_\varepsilon}{|\log \varepsilon|}$. This logarithmic choice of $\alpha_\varepsilon$ which speeds up the motion is the same logarithmic time scale at which vortex motion becomes apparent in the gradient flow for the ungauged problem [27, 40].

Instead of the full GL energy functional, it is often convenient to work with

the energy density with pinning given by

$$g_\varepsilon^{\eta_\varepsilon}(w_\varepsilon, A_\varepsilon) := \frac{\eta_\varepsilon^2}{2}|\nabla_{A_\varepsilon} w_\varepsilon|^2 + \frac{\eta_\varepsilon^4}{4\varepsilon^2}(1-|w_\varepsilon|^2)^2 + \frac{1}{2}|\text{curl } A_\varepsilon - h_{ex}|^2.$$

Additionally, we make the following definitions:

$$e_\varepsilon^{\eta_\varepsilon}(w_\varepsilon) := \frac{\eta_\varepsilon^2}{2}|\nabla w_\varepsilon|^2 + \frac{\eta_\varepsilon^4}{4\varepsilon^2}(1-|w_\varepsilon|^2)^2$$

and

$$e_{A_\varepsilon}^{\eta_\varepsilon}(w_\varepsilon) := e_\varepsilon^{\eta_\varepsilon}(|w_\varepsilon|) + \frac{\eta_\varepsilon^2}{2}\left|\frac{j_{A_\varepsilon}(w_\varepsilon)}{|w_\varepsilon|}\right|^2.$$

If $w_\varepsilon = |w_\varepsilon|e^{i\phi_\varepsilon}$, then the energy density includes the terms

$$|\nabla_{A_\varepsilon} w_\varepsilon|^2 = |\nabla|w_\varepsilon||^2 + |w_\varepsilon|^2|\nabla\phi_\varepsilon - A_\varepsilon|^2.$$

Since $j_{A_\varepsilon}(w_\varepsilon) = |w_\varepsilon|^2|\nabla\phi_\varepsilon - A_\varepsilon|$ (see Appendix A), it follows that

$$\left|\frac{j_{A_\varepsilon}(w_\varepsilon)}{|w_\varepsilon|}\right|^2 = |w_\varepsilon|^2|\nabla\phi_\varepsilon - A_\varepsilon|^2,$$

and the energy density with pinning can be written compactly as

$$g_\varepsilon^{\eta_\varepsilon}(w_\varepsilon, A_\varepsilon) = e_{A_\varepsilon}^{\eta_\varepsilon}(w_\varepsilon) + \frac{1}{2}|h_\varepsilon - h_{ex}|^2.$$

## 2.3 Pinning regime

Here we outline what it means mathematically for the pinning potential to be critically scaled. The properties for our pinning potential are taken from [34]. Although the perturbation to the background potential comes from $p_\varepsilon(x)$, we essentially work with $\eta_\varepsilon(x)$ which we have greater control of but require to be close to $p_\varepsilon(x)$. Let

$$p_\varepsilon^2(x) = 1 + \frac{q_\varepsilon(x)}{|\log\varepsilon|} \qquad \text{and} \qquad \eta_\varepsilon^2(x) = 1 + \frac{Q_\varepsilon(x)}{|\log\varepsilon|} \tag{2.21}$$

and assume that $q_\varepsilon \to q_0(x)$ in $H^k(\Omega)$ with $k$ made precise in Proposition 2.4. The following proposition shows that $p_\varepsilon$ and $\eta_\varepsilon$ are indeed close.

**Proposition 2.4.** *(Proposition 1.6 of [34]). Let $q_\varepsilon$ satisfy the following*

$$\begin{cases} \|q_\varepsilon(x) - q_0(x)\|_{H^k(\Omega)} \to 0 \qquad \text{as } \varepsilon \to 0, \\ \nabla q_\varepsilon \text{ is compactly supported in } \Omega \text{ for all } \varepsilon, \\ \|\varepsilon^2 q_\varepsilon\|_{H^{k+1}(\Omega)} \to 0 \qquad \text{as } \varepsilon \to 0. \end{cases} \tag{2.22}$$

*If $\eta_\varepsilon$ satisfies (2.9) and (2.21), then $\|Q_\varepsilon - q_0\|_{H^k(\Omega)} \to 0$ as $\varepsilon \to 0$ for integer $k \geq 4$.*

*Proof.* The proof is substantial and can be found in the Appendix of [34]. □

Assuming that the limit of $Q_\varepsilon$, $Q_0$, is bounded above by a constant, it follows from (2.21) that

$$\left\|1 - \eta_\varepsilon^2\right\|_{L^\infty(\Omega)} \leq \frac{C}{|\log \varepsilon|}. \tag{2.23}$$

We sometimes claim that for $\varepsilon$ sufficiently small, $\eta_\varepsilon \geq \frac{1}{2}$. In every instance, the quantities without pinning can be recovered by setting $q_\varepsilon = Q_\varepsilon = 0$.

## 2.4 Choosing a gauge and well-posedness

Now we must make a choice for the gauge we use in the sequel. Our results on the vortex motion are gauge invariant, so valid no matter which gauge we choose, but we must make some sort of choice in order to estimate quantities such as $A_\varepsilon$ and $\Phi_\varepsilon$. If we don't make a choice for the gauge, then these quantities remain so general that we struggle to estimate them.

**Definition 2.5.** *Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain. We say $A_\varepsilon : \Omega \times \mathbb{R}^+ \to \mathbb{R}^2$ satisfies the Coulomb gauge condition if*

$$\begin{cases} \operatorname{div} A_\varepsilon = 0 & \text{in } \Omega \\ \nu \cdot A_\varepsilon = 0 & \text{on } \partial\Omega \\ A_\varepsilon(0) = 0 & \text{initially,} \end{cases} \tag{2.24}$$

*where $\nu$ is the outward pointing unit normal to $\partial\Omega$.*

**Remark 2.6.** *In the Coulomb gauge, the Cauchy problem for equation (2.19) is well posed.* $(w_\varepsilon, A_\varepsilon, \Phi_\varepsilon)$ *are regular enough to be treated as classical solutions, and specifically, they allow the calculation in the proof of Lemma 5.7 to pass through without any problems. This high level of regularity follows from Remark 1.3 in [35] and noting that the equations from that paper are equivalent to those in (2.19) after the substitution of* $u_\varepsilon$ *for* $\eta_\varepsilon w_\varepsilon$*, where* $\eta_\varepsilon$ *is a highly regular function. See [34] for the proof of regularity of* $\eta_\varepsilon$*.*

As a result of $A$ satisfying the Coulomb gauge, we immediately have the following estimates:

**Proposition 2.7.** *(Proposition 3.3 of [49]). Let* $\Omega \subset \mathbb{R}^2$ *be a smooth, bounded, simply connected domain. There exists a constant* $C > 0$ *such that if* $A_\varepsilon : \Omega \to \mathbb{R}^2$ *satisfies the Coulomb gauge condition, then*

$$\|A_\varepsilon\|^2_{H^1(\Omega,\mathbb{R}^2)} \leq C \|\text{curl } A_\varepsilon\|^2_{L^2(\Omega)} \tag{2.25}$$

*and*

$$\|A_\varepsilon\|^2_{H^2(\Omega,\mathbb{R}^2)} \leq C \|\text{curl } A_\varepsilon\|^2_{H^1(\Omega)} \tag{2.26}$$

## 2.5 Jacobian

We recall that there are two quantities which describe a vortex; the position of its centre $a_i(t)$ and its degree $d_i(t)$. We would like to be able to relate the positions and degrees of the vortices to the Ginzburg-Landau quantities $(w_\varepsilon, A_\varepsilon, \Phi_\varepsilon)$. Fortunately such a result exists, but first let us make some definitions necessary to make the connection.

**Definition 2.8.** *The gauge-invariant Jacobian determinant of* $w_\varepsilon$ *is*

$$J_{A_\varepsilon}(w_\varepsilon) := \frac{1}{2}\text{curl}\big(\langle iw_\varepsilon, \nabla_{A_\varepsilon} w_\varepsilon\rangle + A_\varepsilon\big). \tag{2.27}$$

*We simply refer to this as the Jacobian.*

It is well established in the literature that $J_{A_\varepsilon}(w_\varepsilon)$ is close in certain norms to the *limiting vorticity measure* $\pi \sum_{i=1}^n d_i \delta_{a_i}$ [49], and that $a_i$ satisfies an ODE determined by the forces in the system. Additionally,

**Definition 2.9.** *The gauge-invariant space-time Jacobian determinant of $w_\varepsilon$ is*

$$V_{A_\varepsilon}^{\Phi_\varepsilon}(w_\varepsilon) := \langle \partial_{\Phi_\varepsilon} w_\varepsilon, i\nabla_{A_\varepsilon} w_\varepsilon \rangle - \frac{1}{2}(1-|w_\varepsilon|^2)(\partial_t A_\varepsilon + \nabla \Phi_\varepsilon). \tag{2.28}$$

*We simply refer to this as the space-time Jacobian.*

The space-time Jacobian is close in certain norms to $\pi \sum_{i=1}^n d_i \dot{a}_i^\perp \delta_{a_i}$ [52], which encodes information about the transport of vortices.

**Remark 2.10.** *Simply writing $J(w_\varepsilon)$ or $V(w_\varepsilon)$ signifies setting $A = \Phi = 0$ in (2.27) or (2.28).*

**Lemma 2.11.** *For $V_A^\Phi(w)$ as defined in (2.28), we have that*

$$V_A^\Phi(w) = \langle \partial_t w, i\nabla w \rangle - \frac{1}{2}\nabla(\Phi(1-|w|^2)) - \frac{1}{2}\partial_t(A(1-|w|^2)). \tag{2.29}$$

*Proof.* Notice from Appendix A that $\langle iw, w\rangle = 0$ and so a term such as $\langle i\Phi w, Aw\rangle = \Phi A\langle iw, w\rangle$ vanishes. Expanding out the inner product in (2.28) gives

$$\begin{aligned}\langle \partial_t w + i\Phi w, i\nabla w + Aw\rangle &= \langle \partial_t w, i\nabla w\rangle + \Phi\langle w, \nabla w\rangle + A\langle \partial_t w, w\rangle \\ &= \langle \partial_t w, i\nabla w\rangle + \frac{1}{2}\Phi\nabla(|w|^2-1) + \frac{1}{2}A\partial_t(|w|^2-1).\end{aligned}$$

Using this in (2.28) leads to the result. □

The next lemma shows that if we have control of the energy functional, the gauge terms in the Jacobian and space-time Jacobian are vanishingly small as $\varepsilon \to 0$. In other words, the limiting value of $V_{A_\varepsilon}^{\Phi_\varepsilon}(w_\varepsilon)$ coincides with the limit of $V(w_\varepsilon)$ and similar for $J_{A_\varepsilon}(w_\varepsilon)$ and $J(w_\varepsilon)$.

**Lemma 2.12.** *If $G_\varepsilon^{\eta_\varepsilon}(t) \le C|\log\varepsilon|$ and $\int_\Omega |E(t)|^2 \le C|\log\varepsilon|$ for all $t \in [0,T]$, then*

$$\|J_A(w) - J(w)\|_{\dot{W}^{-1,1}(\Omega)} \le o_\varepsilon(1) \qquad \forall t \in [0,T] \tag{2.30}$$

*and*

$$\left\|V_A^\Phi(w) - V(w)\right\|_{\dot{W}^{-1,1}(\Omega\times[0,T])} \le o_\varepsilon(1). \tag{2.31}$$

*Proof.* First consider the Jacobian from (2.27). We have that $\langle iw, \nabla_A w\rangle = \langle iw, \nabla w\rangle - |w|^2 A$, and so

$$J_A(w) = \frac{1}{2}\mathrm{curl}\big(\langle iw, \nabla w\rangle + (1-|w|^2)A\big) = J(w) + \frac{1}{2}\mathrm{curl}((1-|w|^2)A).$$

Thus, by the Cauchy-Schwarz inequality, for any $\zeta \in W_0^{1,\infty}(\Omega)$ with $|\nabla\zeta| \leq 1$,

$$\begin{aligned}
\|J_A(w) - J(w)\|_{\dot{W}^{-1,1}(\Omega)} &= \sup_{\zeta, |\nabla\zeta|\leq 1} \int_\Omega \zeta\big(J_A(w) - J(w)\big) \\
&= \sup_{\zeta, |\nabla\zeta|\leq 1} \frac{1}{2}\int_\Omega \zeta\ \mathrm{curl}\big((1-|w|^2)A\big) \\
&= \sup_{\zeta, |\nabla\zeta|\leq 1} \frac{1}{2}\int_\Omega \nabla^\perp\zeta \cdot (|w|^2-1)A \\
&\leq C\left(\int_\Omega (1-|w|^2)^2\right)^{\frac{1}{2}}\left(\int_\Omega |A|^2\right)^{\frac{1}{2}} \\
&\leq C\varepsilon\sqrt{G_\varepsilon^{\eta_\varepsilon}}\,\|A\|_{L^2(\Omega)}\,. \qquad (2.32)
\end{aligned}$$

By (2.25) and the Sobolev embedding theorem we have $\|A\|_{L^2} \leq \|A\|_{H^1} \leq C(G_\varepsilon^{\eta_\varepsilon} + h_{ex})^{1/2}$. Therefore, (2.32) is bounded by $C\varepsilon(G_\varepsilon^{\eta_\varepsilon} + h_{ex}^2)$. Our assumptions on the energy functional mean we have (2.30).

Now consider the space-time Jacobian. By Lemma 2.11,

$$\begin{aligned}
\int_0^T\int_\Omega \zeta\big(V_A^\Phi(w) - V(w)\big) = \frac{1}{2}\int_0^T\int_\Omega (\nabla\zeta)\Phi(1-|w|^2) \\
+ \frac{1}{2}\Big[\int_\Omega \zeta A(|w|^2-1)\Big]_0^T. \qquad (2.33)
\end{aligned}$$

Consider that

$$\begin{aligned}
\left|\int_0^T\int_\Omega (\nabla\zeta)\Phi(1-|w|^2)\right| &\leq C\left|\int_0^T \Big(\int_\Omega |\Phi|^2\Big)^{\frac{1}{2}}\Big(\int_\Omega (1-|w|^2)^2\Big)^{\frac{1}{2}}\right| \\
&\leq C\varepsilon\left|\int_0^T \|\Phi(t)\|_{L^2(\Omega)}\sqrt{G_\varepsilon^{\eta_\varepsilon}(t)}\right|.
\end{aligned}$$

From the Proof of Lemma 3.1 in [60] we have that $\int_\Omega |\Phi_\varepsilon(t)|^2 \leq C|\log\varepsilon|$ for all $t > 0$, and so in conjunction with the energy bound $G_\varepsilon^{\eta_\varepsilon} \leq C|\log\varepsilon|$, we deduce that

$$\left|\int_0^T\int_\Omega (\nabla\zeta)\Phi(1-|w|^2)\right| = o_\varepsilon(1). \qquad (2.34)$$

Next,

$$\int_\Omega \zeta A(t)(|w(t)|^2-1) \le C\Big(\int_\Omega |A(t)|^2\Big)^{\frac12}\Big(\int_\Omega (1-|w(t)|^2)^2\Big)^{\frac12}$$
$$\le C\varepsilon \,\|A(t)\|_{L^2(\Omega)}\,\sqrt{G_\varepsilon^{\eta_\varepsilon}(t)}. \tag{2.35}$$

Arguing as we did after (2.32), we have that (2.35) is $o_\varepsilon(1)$. Therefore, the right-hand side of (2.33) is $o_\varepsilon(1)$ and in the next step we have (2.31). □

## 2.6 Equation for magnetic field

Without pinning, the magnetic field is related to $u_\varepsilon$ through the equation

$$\sigma_\varepsilon \partial_t h_\varepsilon - \Delta h_\varepsilon + h_\varepsilon = 2J_{A_\varepsilon}(u_\varepsilon). \tag{2.36}$$

To derive a similar equation for the situation with pinning, take the curl of the second equation in (2.19) and add $h_\varepsilon$ to both sides:

$$\sigma_\varepsilon \partial_t h_\varepsilon - \Delta h_\varepsilon + h_\varepsilon = 2J_{A_\varepsilon}(w_\varepsilon) + \operatorname{curl}\big((\eta_\varepsilon^2-1)j_{A_\varepsilon}(w_\varepsilon)\big). \tag{2.37}$$

In the next lemma we show that if the energy is bounded, the extra term resulting from pinning is vanishingly small as $\varepsilon \to 0$, hence (2.36) and (2.37) both coincide in the limit.

**Lemma 2.13.** *Suppose $(w_\varepsilon, A_\varepsilon)$ satisfy*

$$G_\varepsilon^{\eta_\varepsilon}(w_\varepsilon, A_\varepsilon) \le C|\log\varepsilon|. \tag{2.38}$$

*Then*

$$\big\|\operatorname{curl}\big((\eta_\varepsilon^2-1)j_{A_\varepsilon}(w_\varepsilon)\big)\big\|_{\dot W^{-1,1}(\Omega)} \to 0 \qquad \text{as } \varepsilon \to 0. \tag{2.39}$$

*Proof.* For any $\zeta \in W_0^{1,\infty}(\Omega)$ with $\|\nabla\zeta\|_{L^\infty(\Omega)} \le 1$ we have

$$\|\nabla\zeta\|_{L^\infty(\Omega)}\,\big\|\operatorname{curl}\big((\eta_\varepsilon^2-1)j_{A_\varepsilon}(w_\varepsilon)\big)\big\|_{\dot W^{-1,1}(\Omega)} = \sup_\zeta \int_\Omega \zeta\,\operatorname{curl}\big((\eta_\varepsilon^2-1)j_{A_\varepsilon}(w_\varepsilon)\big)$$
$$= \sup_\zeta \int_\Omega \nabla^\perp\zeta\cdot(1-\eta_\varepsilon^2)j_{A_\varepsilon}(w_\varepsilon)$$
$$\le \|\nabla\zeta\|_{L^\infty(\Omega)}\,\big\|1-\eta_\varepsilon^2\big\|_{L^\infty(\Omega)}\,\|j_{A_\varepsilon}(w_\varepsilon)\|_{L^1(\Omega)}.$$

Therefore, by (2.23),

$$\left\|\mathrm{curl}\big((\eta_\varepsilon^2-1)j_{A_\varepsilon}(w_\varepsilon)\big)\right\|_{\dot{W}^{-1,1}(\Omega)} \leq \frac{C}{|\log\varepsilon|}\left\|j_{A_\varepsilon}(w_\varepsilon)\right\|_{L^1(\Omega)}. \tag{2.40}$$

To estimate the supercurrent term we use that $\left|\frac{j_A(w)}{|w|}\right|^2 \leq |\nabla_A w|^2$ and $|1-|w|| \leq |1-|w|^2|$ like so:

$$\begin{aligned}
\left|\int_\Omega j_A(w)\right| &\leq \int_\Omega \left|\frac{j_A(w)}{|w|}\right| |1-|w|| + \int_\Omega \left|\frac{j_A(w)}{|w|}\right| \\
&\leq \left(\int_\Omega \left|\frac{j_A(w)}{|w|}\right|^2\right)^{\frac{1}{2}} \left(\int_\Omega (1-|w|^2)^2\right)^{\frac{1}{2}} + C_\Omega \left(\int_\Omega \left|\frac{j_A(w)}{|w|}\right|^2\right)^{\frac{1}{2}} \\
&\leq \left(\int_\Omega |\nabla_A w|^2\right)^{\frac{1}{2}} \left(\int_\Omega (1-|w|^2)^2\right)^{\frac{1}{2}} + C_\Omega \left(\int_\Omega |\nabla_A w|^2\right)^{\frac{1}{2}} \\
&\leq C\varepsilon G_\varepsilon^{\eta_\varepsilon} + C\sqrt{G_\varepsilon^{\eta_\varepsilon}}
\end{aligned} \tag{2.41}$$

We have used that $\eta_\varepsilon^2 > 1/2$ for $\varepsilon$ sufficiently small, and so non-pinning terms can be bounded by the pinned energy. The energy bound (2.38) and (2.41) mean that the inequality (2.40) provides the result of the lemma. □

## 2.7 Excess energy $D_\varepsilon$

Recall from the start of the Chapter that $a = (a_1, ..., a_n) \in \Omega^n$ and the definition of $r_a$. Set $\Omega^{n*} := \{a \in \Omega^n \mid r_a > 0\}$. For $a \in \Omega^{n*}$ and $d \in \mathbb{Z}^n$, the *canonical harmonic map* $w_* = w_*(a,d)$ is defined as the $\mathbb{S}^1$-valued map (after choosing a gauge) that solves

$$\begin{cases} \mathrm{div}\ j(w_*) = 0 & \text{in } \Omega \\ \mathrm{curl}\ j(w_*) = 2\pi\sum_{i=1}^n d_i\delta_{a_i} & \text{in } \Omega \\ \nu\cdot j(w_*) = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.42}$$

Here $j$ is just $j_A$ with setting $A = 0$. In terms of $u_*$, we may recast (1.3) into the form

$$\int_{\partial B_r(a_i)} \langle iu_*, \partial_\tau u_*\rangle = 2\pi d_i. \tag{2.43}$$

Similarly, the limiting magnetic field $h_*$ is defined as the solution (again after choosing a gauge) to the *London equation* [35, 49]:

$$\begin{cases} -\Delta h_* + h_* = 2\pi \sum_{i=1}^{n} d_i \delta_{a_i} & \text{in } \Omega \\ h_* = h_{ex} & \text{on } \partial\Omega, \end{cases} \tag{2.44}$$

and $A_*$ is defined as the solution to

$$h_* = \text{curl } A_*. \tag{2.45}$$

### 2.7.1 Renormalized energy

We saw in Chapter 1 that there exists a function which describes the interaction between vortices; the renormalized energy. Although the renormalized energy ultimately depends on the positions $a = (a_1, .., a_n)$ and degrees $d = (d_1, ..., d_n)$ of the vortices, it can be written in terms of $(u_*, A_*, h_*)$ [35]:

**Definition 2.14.** *We define the renormalized energy to be*

$$W(a,d) := \lim_{r\to 0} \left( \frac{1}{2} \int_{\Omega_r(a)} |\nabla_{A_*} u_*|^2 + n\pi \log r \right) + \frac{1}{2} \int_{\Omega} |h_* - h_{ex}|^2. \tag{2.46}$$

It is clear that the renormalized energy depends on the domain $\Omega$ but we omit this from our notation of $W(a,d)$. We immediately state a result which estimates the rate of convergence of the limit in (2.46).

**Lemma 2.15.** *(Lemma 5.6 in [35]). For all $n \geq 1$, $a \in \Omega^{n*}$, $d \in \{\pm 1\}^n$ and $r < r_a$, there exists a constant $C$ depending only on $\Omega$ such that*

$$W(a,d) - \frac{1}{2} \int_{\Omega_r(a)} |\nabla_{A_*} u_*|^2 - n\pi \log r - \frac{1}{2} \int_{\Omega} |h_* - h_{ex}|^2 \\ \leq Cn\Big( \Big( \frac{nr}{r_a} \Big)^2 + r \Big( \frac{n}{r_a} + h_{ex} \Big)^2 \Big). \tag{2.47}$$

### 2.7.2 Self-energy

For a system of vortices in a superconductor and external forces absent, we know that the renormalized energy accounts for some of the energy in the system, but now we ask if the vortices themselves contribute an amount of

energy. To answer this question we should consider the energy in the balls which contain the vortex cores. By (1.30) in [35]

$$I(r,\varepsilon) := \inf\Big\{ \int_{B_r} e_\varepsilon(u) \;\Big|\; u \in H^1(B_r;\mathbb{C}),\ u = \mathrm{e}^{i\phi} \text{ on } \partial B_r \Big\}$$

which defines the constant

$$\gamma_0 = \lim_{r\to\infty} \Big( I(r,\varepsilon) - \pi \log \frac{r}{\varepsilon} \Big). \tag{2.48}$$

We have a lemma which estimates the rate of convergence.

**Lemma 2.16.** *(Lemma 16 in [28]). There exists a constant $C$ such that*

$$\Big| \gamma_0 - \Big( I(r,\varepsilon) - \pi \log \frac{r}{\varepsilon} \Big) \Big| \leq C \left( \frac{\varepsilon}{r} \right)^2.$$

We claim that the *self-energy* of a vortex is $\gamma_0 + \pi \log \frac{1}{\varepsilon}$. Thought about in a simple way, the renormalized energy provides the energy in $\Omega_r$ and the self-energy gives the energy in the balls. We will see in the next chapter that when we must estimate the energy in the balls, $I(r,\varepsilon)$ provides a lower bound (Theorem 5 of [28]).

### 2.7.3 Excess energy and well-preparedness

Intuitively, we expect that the minimum energy in the system we study should be close to the sum of self-energies of the vortices, the renormalised energy and the pinning potential energy. Indeed, we want any excess energy to be the energy left in the system for vortex motion, i.e. the energy left in the system when the energy due to vortices, renormalised energy and pinning is taken away, and for it to not be too large. Using the notation

$$Q_0(a) := \sum_{i=1}^{n} Q_0(a_i) \tag{2.49}$$

and furthermore, that

$$W^Q(a,d) := W(a,d) + \pi Q_0(a), \tag{2.50}$$

then we have

**Definition 2.17.** *The excess energy is*

$$D_\varepsilon = D_\varepsilon(w_\varepsilon, A_\varepsilon, a, d; h_{ex}) := G_\varepsilon^{\eta_\varepsilon}(w_\varepsilon, A_\varepsilon) - n(\gamma_0 + \pi|\log\varepsilon|) - W^Q(a,d). \quad (2.51)$$

The final definition of this section sets out the initial conditions we use in the sequel. As should be expected, the choice we make here has a massive influence in how the system evolves in time.

**Definition 2.18.** *A solution is well-prepared if* $(w_\varepsilon(0), A_\varepsilon(0))$ *satisfy*

$$\begin{cases} d_i(0) \in \{-1, 1\} \\ J_{A_\varepsilon}(w_\varepsilon)(0) \to \pi \sum_{i=1}^n d_i(0)\delta_{a_i(0)} \\ \limsup_{\varepsilon\to 0} D_\varepsilon(0) = 0. \end{cases} \quad (2.52)$$

# Chapter 3

# Coercivity estimate

In this chapter we prove a Γ-stability result, which we call the *coercivity estimate*. It shows that if the Jacobian is localised around the vortices, the difference between the configurations $(w_\varepsilon, A_\varepsilon)$ and $(w_*, A_*)$ is bounded by the excess energy in the system. This proves to be useful in deriving the dynamical law for vortices in the later chapters. The coercivity estimate presented here builds on what has appeared in [34] and [35] by bringing together gauge and pinning terms. In some places, elements of the proofs presented in those papers are simplified (see later how supercurrent terms are treated as opposed to in [35]). The proof of Theorem 3.9 is long and requires estimating many different terms, so before attempting it we present a series of lemmas which provide fundamental estimates of some key quantities.

## 3.1 Fundamental estimates

Here we mainly quote results from the literature. We saw in Chapter 2 that the gauge terms $(A_\varepsilon, \Phi_\varepsilon)$ are unaffected by the introduction of pinning into the system, and so any estimates required on the fields or gauge potentials can be taken from works which don't consider pinning.

**Lemma 3.1.** *(Theorem 5 of [28]). There exists a constant $C > 0$ such that if*

$u \in H^1(B_\sigma;\mathbb{C})$ *satisfies*

$$\|J(u) \pm \pi\delta_0\|_{\dot{W}^{-1,1}(B_\sigma)} \leq \frac{\sigma}{4},$$

*then*

$$\gamma_0 + \pi \log \frac{\sigma}{\varepsilon} + o_\varepsilon(1) \leq \int_{B_\sigma} e_\varepsilon(w_\varepsilon) + \frac{C}{\sigma}\Big(\varepsilon\sqrt{\log \frac{\sigma}{\varepsilon}} + \|J(u) \pm \pi\delta_0\|_{\dot{W}^{-1,1}(B_\sigma)}\Big). \quad (3.1)$$

**Lemma 3.2.** *For any $\alpha < \frac{2}{p}$ with $2 \leq p < \infty$ and for $A_\varepsilon$ satisfying the Coulomb gauge, we have the following estimates for vector potential $A_\varepsilon$ and order parameter $w_\varepsilon$ (not necessarily solutions)*

$$\|A_\varepsilon\|_{H^1(\Omega)} \leq C(G_\varepsilon^{\eta_\varepsilon} + h_{ex}^2)^{\frac{1}{2}} \quad (3.2)$$

$$\big\|1 - |w_\varepsilon|^2\big\|_{L^p(\Omega)} \leq C\varepsilon^\alpha (G_\varepsilon^{\eta_\varepsilon})^{\frac{3}{2}}. \quad (3.3)$$

*Here $C$ depends on $\alpha$ and $p$.*

*Proof.* Following the proof of Lemma 4.6 of [35], we note that if $A_\varepsilon$ satisfies the Coulomb gauge condition, it implies the existence of a potential $\Xi_\varepsilon$ which satisfies $A_\varepsilon = \nabla^\perp \Xi_\varepsilon$ in $\Omega$ and $\Xi_\varepsilon = 0$ on $\partial\Omega$. Therefore, $\Delta\Xi_\varepsilon = \text{curl } A_\varepsilon = h_\varepsilon - h_{ex} + h_{ex}$, and since $\|h_\varepsilon - h_{ex}\|_{L^2}$ is bounded above by $G_\varepsilon^{\eta_\varepsilon}$, we have $\|\Delta\Xi_\varepsilon\|_{L^2} \leq C(G_\varepsilon^{\eta_\varepsilon} + h_{ex}^2)^{\frac{1}{2}}$ (see [35] for details). This leads to $\|\Xi_\varepsilon\|_{H_0^2(\Omega)} \leq C(G_\varepsilon^{\eta_\varepsilon} + h_{ex}^2)^{\frac{1}{2}}$, which in turn implies (3.2). In this way we don't directly estimate any quantities purely with the Coulomb gauge condition but instead use it to relate $\|A_\varepsilon\|_{H^1(\Omega)}$ to $\|\text{curl } A_\varepsilon\|_{L^2(\Omega)}$, which is estimated by $G_\varepsilon^{\eta_\varepsilon}$.

To prove the second inequality of the lemma, follow [35] and use that for sufficiently small $\varepsilon$, $\eta_\varepsilon^2 \geq 1/2$, and so

$$\int_\Omega (1 - |w_\varepsilon|^2)^2 \leq \int_\Omega 2\eta_\varepsilon^2 (1 - |w_\varepsilon|^2)^2 \leq C\varepsilon^2 G_\varepsilon^{\eta_\varepsilon}.$$

□

Recall from (2.1) that $r_a := \frac{1}{4}\min\{\min_{i\neq j}|a_i - a_j|, \min_i \text{dist}(a_i, \partial\Omega)\}$. Then we have

**Lemma 3.3.** *(Lemma 5.2 of [35]). Let $h_*$ satisfy (2.44). Then for any $p < 2$,*

$$\|h_*\|_{W^{1,p}(\Omega)} \leq C\Big(\frac{n}{r_a} + h_{ex}\Big) \tag{3.4}$$

$$\|A_*\|_{W^{2,p}(\Omega)} \leq C\Big(\frac{n}{r_a} + h_{ex}\Big) \tag{3.5}$$

*where $C$ depends on $\Omega$ and $p$.*

**Lemma 3.4.** *(Lemma 5.3 of [35]). For any $r \leq r_a$ and any $q < 2$*

$$\|j_{A_*}(w_*)\|_{L^q(\Omega)} \leq C\Big(n^{\frac{1+q}{q}} r_a^{\frac{2-q}{q}} + \frac{n}{r_a} + h_{ex}\Big). \tag{3.6}$$

*Additionally,*

$$\|j_{A_*}(w_*)\|_{L^\infty(\Omega_r)} \leq C\Big(\frac{n}{r} + h_{ex}\Big). \tag{3.7}$$

*Proof.* Estimate (3.6) is given in Lemma 5.3 of [35]. To derive (3.7) we first note that $j_{A_*} = j(w_*) - A_*$. Thus,

$$\|j_{A_*}\|_{L^\infty(\Omega_r)} \leq \|j(w_*)\|_{L^\infty(\Omega_r)} + \|A_*\|_{L^\infty(\Omega)}\,. \tag{3.8}$$

Lemma 5.3 of [35] provides

$$\|j(w_*)\|_{L^\infty(\Omega_r)} \leq \frac{2n}{r} \tag{3.9}$$

for all $r \leq r_a$, and (3.5) implies

$$\|A_*\|_{L^\infty(\Omega)} \leq C\Big(\frac{n}{r_a} + h_{ex}\Big). \tag{3.10}$$

We use the two previous inequalities in (3.8) and note that $\frac{n}{r_a} \leq \frac{n}{r}$ to arrive at (3.7). □

The final lemma of this section is necessary to treat the pinned energy on the vortex ball. It's a gauged version of Proposition 0.1 of [14].

**Lemma 3.5.** *Suppose $(w_\varepsilon, A_\varepsilon)$ satisfy*

$$\int_{\cup_i B_r(a_i)} \frac{1}{2}|\nabla_{A_\varepsilon} w_\varepsilon|^2 + \frac{1}{4\varepsilon^2}(1 - |w_\varepsilon|^2)^2 + \frac{1}{2}|h_\varepsilon|^2 \leq n\pi|\log\varepsilon| + C. \tag{3.11}$$

*Then there exists a constant $M$ such that*

$$\int_{\cup_i B_r(a_i)} \frac{1}{\varepsilon^2}(1 - |w_\varepsilon|^2)^2 \leq M \qquad \forall\varepsilon. \tag{3.12}$$

*Proof.* By Theorem 1.1 of [49], for sufficiently small $\varepsilon$ there exists a $K > 0$ such that, for all $v$,

$$n\pi \log \frac{1}{\varepsilon} - K \leq \int_{\cup_i B_r(a_i)} \frac{1}{2}|\nabla_{A_\varepsilon} v|^2 + \frac{1}{4\varepsilon^2}(1 - |v|^2)^2 + \frac{1}{2}|h_\varepsilon|^2. \tag{3.13}$$

Replace $\varepsilon$ with $2\varepsilon$ in (3.13) and let $v = u_{2\varepsilon}$:

$$n\pi \log \frac{1}{2\varepsilon} - K \leq \int_{\cup_i B_r(a_i)} \frac{1}{2}|\nabla_{A_{2\varepsilon}} u_{2\varepsilon}|^2 + \frac{1}{16\varepsilon^2}(1 - |u_{2\varepsilon}|^2)^2 + \frac{1}{2}|h_{2\varepsilon}|^2. \tag{3.14}$$

Add $\frac{3}{16}\int_{\cup_i B_r(a_i)} \frac{1}{\varepsilon^2}(1 - |w_{2\varepsilon}|^2)^2$ to both sides of (3.14) and then after combining with the upper bound (3.11) it follows that

$$\frac{3}{16}\int_{\cup_i B_r(a_i)} \frac{1}{\varepsilon^2}(1 - |w_{2\varepsilon}|^2)^2 \leq C + K + n\pi \log 2. \tag{3.15}$$

□

## 3.2 Compactness results

Now we state some compactness results which will be used in the following sections. In the next proposition the Jacobian converges to a collection of points independent of time.

**Proposition 3.6.** *(Proposition 2.1 of [35]). Let $(w_\varepsilon, A_\varepsilon)$ and some $K_1 > 0$ satisfy the energy bound $G_\varepsilon(w_\varepsilon, A_\varepsilon) \leq K_1|\log \varepsilon|$, where $h_{ex} = O(1)$. Then, for a subsequence,*

$$\left\| J_{A_\varepsilon}(w_\varepsilon) - \pi \sum_{i=1}^{n} d_i \delta_{a_i} \right\|_{(C_0^{0,\gamma}(\Omega))^*} \to 0$$

*where $\gamma > 0$, $d_i \in \mathbb{Z}$ and $a_i \in \Omega$.*

*Proof.* See Chapters 6 and 7 of [49]. □

**Proposition 3.7.** *(Theorem 2.2 of [35]). Let $(w_\varepsilon, A_\varepsilon)$ and some $K_2 > 0$ satisfy the energy bound*

$$G_\varepsilon(w_\varepsilon, A_\varepsilon) \leq n\pi|\log \varepsilon| + K_2 \tag{3.16}$$

*and suppose the vortices are localized as*

$$\left\| J(w_\varepsilon) - \pi \sum_{i=1}^{n} d_i \delta_{a_i} \right\|_{\dot{W}^{-1,1}(\Omega)} \to 0 \tag{3.17}$$

*where* $a \in \Omega^{n*}$ *and* $d_i \in \{-1, 1\}$. *Then this convergence also holds in* $\dot{W}^{-1,p}$ *for* $1 < p < 2$.

*Proof.* The convergence of the vorticity in the larger space comes from the fact that $\dot{W}^{-1,1}$ convergence implies $(C_0^{0,\gamma})^*$ convergence, and the compact embedding $(C_0^{0,\gamma})^* \subset \dot{W}^{-1,p}$, $1 < p < 2$, implies that the compactness also holds in negative Sobolev spaces. □

**Remark 3.8.** *In the previous two propositions the requirement was that the unpinned energy was bounded. The pinned and unpinned energies are actually quite close to each other in value, and thanks to (B.5), we have that if the pinned energy is bounded by* $C|\log\varepsilon|$, *it implies that the unpinned energy is too.*

## 3.3 Coercivity estimate

We are now in a position to present the main result:

**Theorem 3.9.** *Let* $\Omega \subset \mathbb{R}^2$ *be bounded, open, simply connected and have* $C^1$ *boundary. Suppose* $n \geq 0$, $\xi = (\xi_1, ..., \xi_n) \in \Omega^{n^*}$ *and* $d_i \in \{\pm 1\}$ *for all* $i = 1, .., n$. *Define* $w_* = w_*(\cdot; \xi, d)$ *via (2.42),* $h_* = h_*(\cdot; \xi, d)$ *via (2.44) and* $A_* = A_*(\cdot; \xi, d)$ *via (2.45). Further suppose there is a* $w_\varepsilon \in H^1(\Omega, \mathbb{C})$ *and a divergence-free* $A_\varepsilon \in H^1(\Omega, \mathbb{R}^2)$ *(not necessarily solutions) satisfying*

$$G_\varepsilon(w_\varepsilon, A_\varepsilon) \leq n\pi|\log\varepsilon| + K_2 \tag{3.18}$$

*for some* $K_2 > 0$ *and*

$$\left\| J_{A_\varepsilon}(w_\varepsilon) - \pi \sum_{i=1}^{n} d_i \delta_{\xi_i} \right\|_{\dot{W}^{-1,1}(\Omega)} \leq s_\varepsilon \tag{3.19}$$

*for some* $s_\varepsilon \in [\varepsilon\sqrt{\log(r_\xi/\varepsilon)}, r/4]$ *which satisfies* $s_\varepsilon |\log\varepsilon|^{\frac{1}{3}} \to 0$ *as* $\varepsilon \to 0$. *Suppose also that, as* $\varepsilon \to 0$,

$$|\log\varepsilon|^{\frac{1}{3}} \|A_\varepsilon - A_*\|_{L^p(\Omega)} \to 0 \qquad \text{for } p > 2. \tag{3.20}$$

*Let* $0 < \gamma < 1$ *and then there exists a constant* $C$ *such that*

$$r_\star := C\left(\frac{s_\varepsilon}{G_\varepsilon^{\eta_\varepsilon} + n^2 r_\xi^{-2} + h_{ex}^2}\right)^{\frac{1}{1+\gamma}}, \tag{3.21}$$

*then the excess energy as defined in (2.51) satisfies*

$$\int_{\Omega_{r_\star}(\xi)} e_\varepsilon^{\eta_\varepsilon}(|w_\varepsilon|) + \frac{\eta_\varepsilon^2}{2}\left|\frac{j_{A_\varepsilon}(w_\varepsilon)}{|w_\varepsilon|} - j_{A_*}(w_*)\right|^2 + \frac{1}{2}\int_\Omega |h_\varepsilon - h_*|^2 \le D_\varepsilon + error_1, \tag{3.22}$$

*where*

$$\Omega_{r_\star}(\xi) := \Omega \backslash \bigcup_{i=1}^n \overline{B_{r_\star}(\xi_i)}$$

*and*

$$error_1 = Cn\big(s_\varepsilon^\gamma (G_\varepsilon^{\eta_\varepsilon} + n^2 r_\xi^{-2} + h_{ex}^2)\big)^{\frac{1}{1+\gamma}} + o_\varepsilon(1).$$

*Proof.* Allow us to simply write $j_A$ in place of $j_{A_\varepsilon}(w_\varepsilon)$ and $j_{A_*}$ in place of $j_{A_*}(w_*)$ in the following. By (B.1) we immediately note that (3.18) implies

$$G_\varepsilon^{\eta_\varepsilon}(w_\varepsilon, A_\varepsilon) \le n\pi|\log\varepsilon| + K_3. \tag{3.23}$$

We use this bound repeatedly in estimates involving $G_\varepsilon^{\eta_\varepsilon}$ later on.

Consider that

$$e_\varepsilon^{\eta_\varepsilon}(|w_\varepsilon|) + \frac{\eta_\varepsilon^2}{2}\left|\frac{j_A}{|w_\varepsilon|} - |w_\varepsilon| j_{A_*}\right|^2 = e_\varepsilon^{\eta_\varepsilon}(|w_\varepsilon|) + \frac{\eta_\varepsilon^2}{2}\left|\frac{j_A}{|w_\varepsilon|}\right|^2 + \frac{\eta_\varepsilon^2}{2}|w_\varepsilon|^2|j_{A_*}|^2 - \eta_\varepsilon^2 j_{A_*}\cdot j_A$$

$$= e_{A_\varepsilon}^{\eta_\varepsilon}(w_\varepsilon) + \frac{\eta_\varepsilon^2}{2}(|w_\varepsilon|^2 - 1)|j_{A_*}|^2 + \eta_\varepsilon^2 j_{A_*}\cdot(j_{A_*} - j_A) - \frac{\eta_\varepsilon^2}{2}|j_{A_*}|^2$$

$$= e_{A_\varepsilon}^{\eta_\varepsilon}(w_\varepsilon) - e_{A_*}^{\eta_\varepsilon}(w_*) + \frac{\eta_\varepsilon^2}{2}(|w_\varepsilon|^2 - 1)|j_{A_*}|^2 + \eta_\varepsilon^2 j_{A_*}\cdot(j_{A_*} - j_A). \tag{3.24}$$

Separately, we have that

$$\frac{\eta_\varepsilon^2}{2}\left|\frac{j_A}{|w_\varepsilon|} - |w_\varepsilon| j_{A_*}\right|^2 = \frac{\eta_\varepsilon^2}{2}\left|\frac{j_A}{|w_\varepsilon|}\right|^2 + \frac{\eta_\varepsilon^2}{2}|w_\varepsilon|^2|j_{A_*}|^2 - \eta_\varepsilon^2|w_\varepsilon| j_{A_*}\cdot\frac{j_A}{|w_\varepsilon|}$$

$$= \frac{\eta_\varepsilon^2}{2}\left|\frac{j_A}{|w_\varepsilon|}\right|^2 + \frac{\eta_\varepsilon^2}{2}(|w_\varepsilon|^2-1)|j_{A_*}|^2 + \frac{\eta_\varepsilon^2}{2}|j_{A_*}|^2 - \eta_\varepsilon^2(|w_\varepsilon|-1)j_{A_*}\cdot\frac{j_A}{|w_\varepsilon|} - \eta_\varepsilon^2 j_{A_*}\cdot\frac{j_A}{|w_\varepsilon|}$$

$$= \frac{\eta_\varepsilon^2}{2}\left|\frac{j_A}{|w_\varepsilon|} - j_{A_*}\right|^2 + \frac{\eta_\varepsilon^2}{2}(|w_\varepsilon|^2-1)|j_{A_*}|^2 - \eta_\varepsilon^2(|w_\varepsilon|-1)j_{A_*}\cdot\frac{j_A}{|w_\varepsilon|}. \quad (3.25)$$

Combining (3.24) with (3.25) gives

$$e_\varepsilon^{\eta_\varepsilon}(|w_\varepsilon|) + \frac{\eta_\varepsilon^2}{2}\left|\frac{j_A}{|w_\varepsilon|} - j_{A_*}\right|^2 = e_{A_\varepsilon}^{\eta_\varepsilon}(w_\varepsilon) - e_{A_*}^{\eta_\varepsilon}(w_*) + \eta_\varepsilon^2(|w_\varepsilon|-1)j_{A_*}\cdot\frac{j_A}{|w_\varepsilon|}$$
$$+ \eta_\varepsilon^2 j_{A_*}\cdot(j_{A_*} - j_A). \quad (3.26)$$

For any $0 \le r \le r_\xi$, recalling Definition 2.51 for excess energy and (2.47), we have

$$D_\varepsilon = \int_{\Omega_r(\xi)} e_{A_\varepsilon}^{\eta_\varepsilon}(w_\varepsilon) - e_{A_*}(w_*) + \frac{1}{2}\int_\Omega |h_\varepsilon - h_{ex}|^2 - |h_* - h_{ex}|^2$$
$$+ \sum_{i=1}^n \Big[\int_{B_r(\xi_i)} e_{A_\varepsilon}^{\eta_\varepsilon}(w_\varepsilon) - \big(\gamma_0 + \pi\log\frac{r}{\varepsilon} + \pi Q_0(\xi_i)\big)\Big] + error_3. \quad (3.27)$$

where

$$error_3 = O\Big(Cn\Big(\frac{nr}{r_a}\Big)^2\Big) + O\Big(Cnr\Big(\frac{n}{r_a} + h_{ex}\Big)^2\Big).$$

Using that

$$\frac{1}{2}\int_\Omega |h_\varepsilon - h_{ex}|^2 - |h_* - h_{ex}|^2 = \frac{1}{2}\int_\Omega |h_\varepsilon - h_*|^2 - \int_\Omega (h_* - h_{ex})(h_* - h_\varepsilon), \quad (3.28)$$

we combine (3.26), (3.27) and (3.28) to arrive at

$$\int_{\Omega_r(\xi)} e_\varepsilon^{\eta_\varepsilon}(|w_\varepsilon|) + \frac{\eta_\varepsilon^2}{2}\left|\frac{j_A}{|w_\varepsilon|} - j_{A_*}\right|^2 + \frac{1}{2}\int_\Omega |h_\varepsilon - h_*|^2 = D_\varepsilon + \sum_{j=1}^5 I_j + error_3, \quad (3.29)$$

where

$$I_1 = \sum_{i=1}^n \Big[\gamma_0 + \pi\log\frac{r}{\varepsilon} + \pi Q_0(\xi_i) - \int_{B_r(\xi_i)} e_{A_\varepsilon}^{\eta_\varepsilon}(w_\varepsilon)\Big]$$

$$I_2 = \int_{\Omega_r(\xi)} e_{A_*}(w_*) - e_{A_*}^{\eta_\varepsilon}(w_*)$$

$$I_3 = \int_{\Omega_r(\xi)} \eta_\varepsilon^2(|w_\varepsilon|-1)j_{A_*}\cdot\frac{j_A}{|w_\varepsilon|}$$

$$I_4 = \int_{\Omega_r(\xi)} \eta_\varepsilon^2 j_{A_*}\cdot(j_{A_*} - j_A)$$

$$I_5 = \int_\Omega (h_* - h_{ex})(h_* - h_\varepsilon).$$

We estimate the terms $I_1$-$I_5$. In what follows we ask for the radius of the balls to be

$$r = \max\{|\log\varepsilon|^{-\frac{1}{3}}, \sqrt{s_\varepsilon}\} \quad \text{such that} \quad \frac{1}{r} = \min\left\{|\log\varepsilon|^{\frac{1}{3}}, \frac{1}{\sqrt{s_\varepsilon}}\right\}. \tag{3.30}$$

This allows us to control terms where $1/r$ is multiplied by $1/|\log\varepsilon|$, for instance. Let us consider each term in turn.

We rewrite $I_1$ such that

$$I_1 = \sum_{i=1}^{n}\Big[\gamma_0 + \pi\log\frac{r}{\varepsilon} + \pi Q_0(\xi_i) - \int_{B_r(\xi_i)} e_\varepsilon^{\eta_\varepsilon}(w_\varepsilon) + \int_{B_r(\xi_i)} \eta_\varepsilon^2 j_A\cdot A_\varepsilon + \frac{\eta_\varepsilon^2}{2}A_\varepsilon^2|w_\varepsilon|^2\Big]. \tag{3.31}$$

The reason for tearing apart the energy density in this way is that because we have $s_\varepsilon \le \frac{r}{4}$ we are free to apply Lemma 3.1 to deduce that

$$\gamma_0 + \pi\log\frac{r}{\varepsilon} \le \int_{B_r(\xi_i)} e_\varepsilon(w_\varepsilon) + \frac{C}{r}\Big(\varepsilon\sqrt{\log\frac{r}{\varepsilon}} + s_\varepsilon\Big) + o_\varepsilon(1) \qquad \forall i.$$

Since we assumed that $s_\varepsilon \ge \varepsilon\sqrt{\log(r_\xi/\varepsilon)}$, we have

$$\gamma_0 + \pi\log\frac{r}{\varepsilon} \le \int_{B_r(\xi_i)} e_\varepsilon(w_\varepsilon) + \frac{C}{r}s_\varepsilon + o_\varepsilon(1). \tag{3.32}$$

Whilst this is an estimate for $e_\varepsilon(w_\varepsilon)$, we require an estimate for the energy density with pinning $e_\varepsilon^{\eta_\varepsilon}(w_\varepsilon)$. Specifically, we would like to show that

$$\int_{B_r}\eta_\varepsilon^2 e_\varepsilon(w_\varepsilon) \le \int_{B_r} e_\varepsilon^{\eta_\varepsilon}(w_\varepsilon) + error. \tag{3.33}$$

Consider that

$$\eta_\varepsilon^2 e_\varepsilon(w_\varepsilon) + \frac{\eta_\varepsilon^2(\eta_\varepsilon^2-1)}{4\varepsilon^2}(1-|w_\varepsilon|^2)^2 = e_\varepsilon^{\eta_\varepsilon}(w_\varepsilon). \tag{3.34}$$

If $\eta_\varepsilon^2 \ge 1$ then (3.33) immediately follows from (3.34). However, if $\eta_\varepsilon^2 < 1$ then we must show that $\frac{\eta_\varepsilon^2(1-\eta_\varepsilon^2)}{4\varepsilon^2}(1-|w_\varepsilon|^2)^2$ can be controlled by an small error term. By using $|\eta_\varepsilon^2| \le C$, (2.23) and (3.12),

$$\int_{B_r(\xi_i)}\frac{\eta_\varepsilon^2(1-\eta_\varepsilon^2)}{4\varepsilon^2}(1-|w_\varepsilon|^2)^2 \le \frac{CM}{|\log\varepsilon|} = o_\varepsilon(1). \tag{3.35}$$

Hence, we have (3.33) with the error term being $o_\varepsilon(1)$.

Let $q_\varepsilon(r,x_0)=\inf_{x\in B_r(x_0)}Q_\varepsilon(x)$. By (3.32) and (3.33) we can say

$$\begin{aligned}\Big(1+\frac{q_\varepsilon(r,\xi_i)}{|\log\varepsilon|}\Big)\Big(\gamma_0+\pi\log\frac{r}{\varepsilon}-\frac{C}{r}s_\varepsilon-o_\varepsilon(1)\Big)&\le\inf_{x\in B_r(\xi_i)}(\eta_\varepsilon^2(x))\int_{B_r(\xi_i)}e_\varepsilon(w_\varepsilon)\\&\le\int_{B_r(\xi_i)}\eta_\varepsilon^2 e_\varepsilon(w_\varepsilon)\\&\le\int_{B_r(\xi_i)}e_\varepsilon^{\eta_\varepsilon}(w_\varepsilon)+o_\varepsilon(1).\quad(3.36)\end{aligned}$$

Hence,

$$\begin{aligned}&\gamma_0+\pi\log\frac{r}{\varepsilon}+\pi Q_0(\xi_i)-\int_{B_r(\xi_i)}e_\varepsilon^{\eta_\varepsilon}(w_\varepsilon)\\&\quad\le\gamma_0+\pi\log\frac{r}{\varepsilon}+\pi Q_0(\xi_i)-\Big(1+\frac{q_\varepsilon(r,\xi_i)}{|\log\varepsilon|}\Big)\Big(\gamma_0+\pi\log\frac{r}{\varepsilon}-\frac{C}{r}s_\varepsilon-o_\varepsilon(1)\Big)\\&\quad\le\pi\big(Q_0(\xi_i)-q_\varepsilon(r,\xi_i)\big)-C\frac{\log r}{|\log\varepsilon|}+\frac{C}{r}s_\varepsilon+o_\varepsilon(1)\\&\quad\le C\frac{\log|\log\varepsilon|}{|\log\varepsilon|}+\frac{C}{r}s_\varepsilon+o_\varepsilon(1)=\frac{C}{r}s_\varepsilon+o_\varepsilon(1).\end{aligned}\tag{3.37}$$

To estimate $I_1$ it remains to estimate the gauge contribution in (3.31). For any $p>2$, let $\gamma<1-\frac{2}{p}$, then using Lemma 3.2 and (3.23) we have

$$\begin{aligned}\Big\|\eta^2 j_A\cdot A_\varepsilon+\frac{\eta^2}{2}|w_\varepsilon|^2A_\varepsilon^2\Big\|_{L^1(B_r)}&\le C\Big\|\eta^2\frac{j_A}{|w|}\cdot A\Big\|_{L^1(B_r)}+\frac{C}{2}\left\|\eta^2A^2\right\|_{L^1(B_r)}\\&\quad+C\Big\|\eta^2\frac{j_A}{|w|}\cdot A(|w|-1)\Big\|_{L^1(B_r)}+\frac{C}{2}\left\|\eta^2A^2(|w|^2-1)\right\|_{L^1(B_r)}\end{aligned}$$

$$\begin{aligned}&\le C\left\|\eta^2\nabla_A w\right\|_{L^2}\|A\|_{L^p}\,r^{1-\frac{2}{p}}+Cr\left\|\eta^2\right\|_{L^\infty}\|A\|_{L^4}^2\\&\qquad+C\left\|\eta^2\nabla_A w\right\|_{L^2}\|A\|_{L^p}\||1-|w|\|_{L^{\frac{2p}{p-2}}}+C\|A\|_{L^4}^2\left\|\eta^2(1-|w|^2)\right\|_{L^2}\\&\le C(G_\varepsilon^\eta+h_{ex}^2)r^\gamma+C(G_\varepsilon^\eta+h_{ex}^2)r+C((G_\varepsilon^\eta)^3+h_{ex}^2)\varepsilon^\gamma+C((G_\varepsilon^\eta)^{\frac{3}{2}}+h_{ex}^4)\varepsilon\\&\le C(G_\varepsilon^{\eta_\varepsilon}+h_{ex}^2)r^\gamma+o_\varepsilon(1).\end{aligned}\tag{3.38}$$

Inequalities (3.37) and (3.38) together provide

$$I_1\le\frac{Cn}{r}s_\varepsilon+Cnr^\gamma(G_\varepsilon^{\eta_\varepsilon}+h_{ex}^2)+o_\varepsilon(1).\tag{3.39}$$

We now consider $I_2$. By definition,

$$I_2 = \frac{1}{2}\int_{\Omega_r(\xi)} (1-\eta_\varepsilon^2)|j_{A_*}|^2.$$

Combining the pinning estimate (2.23) with the limiting supercurrent estimate (3.7) gives

$$I_2 \leq C\left\|1-\eta_\varepsilon^2\right\|_{L^\infty(\Omega)} \|j_{A_*}\|_{L^\infty(\Omega_r)} \leq \frac{C}{|\log\varepsilon|}\Big(\frac{n}{r}+h_{ex}\Big). \tag{3.40}$$

We use (3.30) to conclude that $I_2 = o_{\varepsilon,s_\varepsilon}(1)$.

Next, since $(1-|w_\varepsilon|) \leq |1-|w_\varepsilon|^2|$, Cauchy-Schwarz implies

$$I_3 \leq \left\|\eta_\varepsilon^2(1-|w_\varepsilon|^2)\right\|_{L^2(\Omega)} \|j_{A_*}\|_{L^\infty(\Omega_r)} \left\|\frac{j_A}{|w_\varepsilon|}\right\|_{L^2(\Omega)} \leq C\varepsilon\Big(\frac{n}{r}+h_{ex}\Big)G_\varepsilon^{\eta_\varepsilon},$$

having used (3.7). Using the condition on $r$ (3.30) and the condition on $G_\varepsilon^{\eta_\varepsilon}$ (3.23), we have that $I_3 = o_\varepsilon(1)$.

To estimate $I_4$ requires some work. Proposition 3.11 shows that $I_4 = o_\varepsilon(1)$.

Finally, we estimate $I_5$. After integration by parts on $I_5$ we have

$$I_5 = -\int_\Omega \nabla^\perp h_* \cdot (A_* - A_\varepsilon).$$

Using (3.4) and (3.20) we have, for $p > 2$,

$$I_5 \leq C\Big(\frac{n}{r_\xi}+h_{ex}\Big)\|A-A_*\|_{L^p(\Omega)} = o_\varepsilon(1).$$

Bringing the estimates for the five terms together in (3.29) gives

$$\int_{\Omega_r(\xi)} e_\varepsilon^{\eta_\varepsilon}(|w_\varepsilon|) + \frac{\eta_\varepsilon^2}{2}\left|\frac{j_A}{|w_\varepsilon|} - j_{A_*}\right|^2 + \frac{1}{2}\int_\Omega |h_\varepsilon - h_*|^2$$
$$\leq D_\varepsilon + C\frac{n}{r}s_\varepsilon + Cnr^\gamma(G_\varepsilon^{\eta_\varepsilon}+h_{ex}^2) + error_3 + o_\varepsilon(1).$$

Consider the error term

$$error = C\frac{n}{r}s_\varepsilon + Cnr^\gamma(G_\varepsilon^{\eta_\varepsilon}+h_{ex}^2) + Cn\Big(\frac{nr}{r_\xi}\Big)^2 + Cnr\Big(\frac{n}{r_\xi}+h_{ex}\Big)^2 + o_\varepsilon(1).$$

Since $(\frac{r}{r_\xi})^2 \leq \frac{r}{r_\xi}$ and $r < r^\gamma$, we have that

$$error \leq C\frac{n}{r}s_\varepsilon + Cnr^\gamma\Big(G_\varepsilon^{\eta_\varepsilon} + h_{ex}^2 + \frac{n^2}{r_\xi^2}\Big) + o_\varepsilon(1).$$

If we optimise the error over $r$ we find the optimum radius is $r_\star$, given by (3.21), and yields an error term given by $error_1$. Hence we have (3.22). □

Before treating the supercurrent integral $I_4$, it is necessary to state a lemma from [7]. The method we use to estimate $I_4$ circumvents complications involving the domain of integration in [35] and is in fact closer to what appears in [48], although the proof of their Lemma 3.7 is for a gauge-free system. We show the necessary convergences through weak-convergence arguments based on compactness, rather than the explicit argument of [35].

**Lemma 3.10.** *(Lemma I.3 in [7]). Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^2$, and let $\omega_i$, $i = 1, ..., n$, be smooth disjoint subdomains of $\Omega$, such that $\Omega_\omega = \Omega \backslash \bigcup_{i=1}^n \overline{\omega}_i$ is connected. Let $v$ be a function satisfying*

$$\begin{cases} \Delta v = 0 & \text{in } \Omega_\omega \\ \int_{\partial\omega_i} \partial_\nu v = 0 & \forall i \\ \partial_\nu v = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.41}$$

*Then*

$$\sup_{\Omega_\omega} v - \inf_{\Omega_\omega} v \leq \sum_{i=1}^n \left( \sup_{\partial\omega_i} v - \inf_{\partial\omega_i} v \right). \tag{3.42}$$

**Proposition 3.11.** *Suppose $(w_\varepsilon, A_\varepsilon, w_*, A_*)$ are defined as in Theorem 3.9. Then we have that for every small $r > 0$,*

$$I_4 := \int_{\Omega_r(\xi)} \eta_\varepsilon^2 j_{A_*} \cdot (j_{A_*} - j_A) = o_\varepsilon(1). \tag{3.43}$$

*Proof.* Take $\varepsilon_k \to 0$ such that

$$|I_4(\varepsilon_k)| \to \limsup_{\varepsilon\to 0} |I_4(\varepsilon)|.$$

To begin, we use that

$$|I_4| \leq \left\|\eta_\varepsilon^2\right\|_{L^\infty(\Omega)} \|j_{A_*} \cdot (j_{A_*} - j_A)\|_{L^1(\Omega_r)} \leq C \|j_{A_*} \cdot (j_{A_*} - j_A)\|_{L^1(\Omega_r)} \tag{3.44}$$

so that we can focus on just estimating the supercurrent terms. Rather than estimating the supercurrent terms directly, we employ a function which is close to the limiting supercurrent as $r \to 0$ and has useful integration properties. Let $\Upsilon_0$ be defined such that

$$\begin{cases} \Delta \Upsilon_0 = 2\pi \sum_{i=1}^n d_i \delta_{a_i} - h_* & \text{in } \Omega \\ \Upsilon_0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.45}$$

The second condition is a Dirichlet boundary condition (see [48]). The subscript 0 indicates that $\Upsilon$ does not depend on the radius of the balls, $r$. It follows that $\Upsilon_0$ satisfies $j_{A_*} = \nabla^\perp \Upsilon_0$ in $\Omega$. Now consider a function $\Psi_r$ which satisfies the following

$$\begin{cases} -\Delta \Psi_r = h_* & \text{in } \Omega_r(\xi) \\ \Psi_r = \text{constant} = c_i & \text{on } \partial B_r(\xi_i) \\ \int_{\partial B_r(a_i)} \partial_\nu \Psi_r + \tau \cdot A_* = 2\pi d_i & \forall i \\ \Psi_r = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.46}$$

If we can show $\Psi_r$ is a good approximation for $\Upsilon_0$ as $r \to 0$, we will use $\nabla^\perp \Psi_r$ to replace $j_{A_*}$ in $I_4$. We seek to use Lemma 3.10. Notice that if $v = \Psi_r - \Upsilon_0$, the criteria (3.41) are satisfied. Hence, by (3.42)

$$\sup_{\Omega_r}(\Psi_r - \Upsilon_0) - \inf_{\Omega_r}(\Psi_r - \Upsilon_0) \leq \sum_{i=1}^n \Big( \sup_{\partial B_r(\xi_i)} (\Psi_r - \Upsilon_0) - \inf_{\partial B_r(\xi_i)} (\Psi_r - \Upsilon_0) \Big). \tag{3.47}$$

Since $\Psi_r$ is a constant on each $\partial B_r(a_i)$, we have for a $C$ independent of on $r$ that

$$\sum_{i=1}^n \Big( \sup_{\partial B_r(\xi_i)} (\Psi_r - \Upsilon_0) - \inf_{\partial B_r(\xi_i)} (\Psi_r - \Upsilon_0) \Big) = \sum_{i=1}^n \Big( \sup_{\partial B_r(\xi_i)} \Upsilon_0 - \inf_{\partial B_r(\xi_i)} \Upsilon_0 \Big) \leq Cr.$$

Here, we have used that $\sup(-f) = -\inf(f)$ and $-\inf(-f) = \sup(f)$ and the last inequality follows due to Lemma 3.12.

Since $\Psi_r - \Upsilon_0 = 0$ on $\partial\Omega$, we deduce from (3.47) that (again $C$ independent on $r$)

$$\|\Psi_r - \Upsilon_0\|_{L^\infty(\Omega_r)} \leq Cr. \tag{3.48}$$

It follows via standard interior gradient bounds for harmonic functions that for every compact subset $K$ of $\Omega\backslash\{\xi_i\}$, one has

$$\|\Psi_r - \Upsilon_0\|_{C^k(K)} \leq C(K)r \tag{3.49}$$

for all sufficiently small $r$.

We estimate an approximation to $I_4$ using $\Psi_r$ with the intention of passing to the limit $r \to 0$ to recover the original integral for $I_4$ we wish to estimate. Consider

$$I_4' = \int_{\Omega_r} \nabla^\perp \Psi_r \cdot \big(\nabla^\perp \Psi_r - j_A\big). \tag{3.50}$$

Integration by parts gives

$$\begin{aligned} I_4' &= -\int_{\Omega_r} \Psi_r\big(\Delta\Psi_r - \text{curl } j_A\big) - \sum_{i=1}^n \int_{\partial B_r(\xi_i)} \Psi_r\big(\partial_\nu \Psi_r - \tau \cdot j_A\big) \\ &= 2\int_{\Omega_r} \Psi_r J_A(w) - \int_{\Omega_r} \Psi_r\big(h_\varepsilon + \Delta\Psi_r\big) - \sum_{i=1}^n \int_{\partial B_r(\xi_i)} \Psi_r\big(\partial_\nu \Psi_r - \langle iw, \tau \cdot \nabla_A w\rangle\big). \end{aligned}$$

The first integral on the right-hand side can be estimated by $\|J_A\|_{\dot{W}^{-1,1}(\Omega_r)} \leq s_\varepsilon$ and that $\|\nabla\Psi_r\| = \|j_{A_*}\| \leq C(\frac{n}{r} + h_{ex})$. By (3.30) and the condition in the hypothesis that $s_\varepsilon |\log\varepsilon|^{\frac{1}{3}} \to 0$ as $\varepsilon \to 0$, the first integral is $o_\varepsilon(1)$. Use the properties of $\Psi_r$ from (3.46) to write

$$I_4' = o_\varepsilon - \int_{\Omega_r} \Psi_r\big(h_\varepsilon - h_*\big) + \sum_{i=1}^n c_i\Big[-2\pi d_i + \int_{\partial B_r(\xi_i)} \tau \cdot A_* + \langle iw, \tau \cdot \nabla_A w\rangle\Big]. \tag{3.51}$$

Consider that the integral involving magnetic field is

$$-\int_{\Omega_r} \Psi_r\big(h_\varepsilon - h_*\big) = \int_{\Omega_r} \nabla^\perp \Psi_r \cdot \big(A_\varepsilon - A_*\big) + \sum_{i=1}^n \int_{\partial B_r(\xi_i)} \Psi_r\ \tau \cdot \big(A_\varepsilon - A_*\big). \tag{3.52}$$

Thus, using (3.52) in (3.51) gives

$$\begin{aligned} I_4' = o_\varepsilon &+ \int_{\Omega_r} \nabla^\perp \Psi_r \cdot \big(A_\varepsilon - A_*\big) \\ &+ \sum_{i=1}^n c_i\Big[-2\pi d_i + \int_{\partial B_r(\xi_i)} \tau \cdot \big(A_\varepsilon - A_*\big) + \tau \cdot A_* + \langle iw, \tau \cdot \nabla_A w\rangle\Big]. \end{aligned}$$

$$= o_\varepsilon + \int_{\Omega_r} \nabla^\perp \Psi_r \cdot \big(A_\varepsilon - A_*\big) + \sum_{i=1}^{n} c_i \Big[ -2\pi d_i + \int_{\partial B_r(\xi_i)} \tau \cdot A_\varepsilon + \langle iw, \tau \cdot \nabla w \rangle - \tau \cdot |w_\varepsilon|^2 A_\varepsilon \Big]$$

$$= o_\varepsilon + \int_{\Omega_r} \nabla^\perp \Psi_r \cdot \big(A_\varepsilon - A_*\big) + \sum_{i=1}^{n} c_i \Big[ -2\pi d_i + \int_{\partial B_r(\xi_i)} \langle iw, \partial_\tau w \rangle + \tau \cdot (1 - |w_\varepsilon|^2) A_\varepsilon \Big]. \tag{3.53}$$

Using (3.20) with the fact that $\|\nabla \Psi_r\| = \|j_{A_*}\| \leq C(\frac{n}{r} + h_{ex})$ means that the first integral on the right hand side of (3.53) is $o_\varepsilon(1)$.

Now we want to show that the boundary integral is $o_\varepsilon(1)$. First recall that from the estimate for $I_1$ (3.39) we have that

$$n\pi |\log \varepsilon| - \int_{\cup_i B_r(\xi_i)} e^{\eta_\varepsilon}_{A_\varepsilon}(w_\varepsilon) \leq n\pi \log \frac{1}{r} - n\gamma_0 - \pi \sum_{i=1}^{n} Q_0(\xi_i) + error_1 + o_\varepsilon(1). \tag{3.54}$$

Using this with the upper bound on the energy (3.23) we deduce that

$$G^{\eta_\varepsilon}_\varepsilon(w_\varepsilon, A_\varepsilon; \Omega_r) \leq K_3 + n\pi |\log \varepsilon| - \int_{\cup_i B_r(\xi_i)} e^{\eta_\varepsilon}_{A_\varepsilon}(w_\varepsilon) \leq K_4(r). \tag{3.55}$$

In light of (3.55) and $\|\eta^2_\varepsilon\|_{L^\infty(\Omega)} \leq C$, we have that $\|\nabla_{A_\varepsilon} w_\varepsilon\|^2_{L^2(\Omega_r)} \leq CK_4(r) \leq K_5(r)$, where $K_5$ is a constant independent of $\varepsilon$. Since

$$|\nabla_{A_\varepsilon} w_\varepsilon|^2 = |\nabla w_\varepsilon|^2 + |w_\varepsilon|^2 |A_\varepsilon|^2 - 2A_\varepsilon \cdot \langle iw_\varepsilon, \nabla w_\varepsilon \rangle,$$

we deduce that

$$\int_{\Omega_r} |\nabla w_\varepsilon|^2 \leq \int_{\Omega_r} |\nabla_{A_\varepsilon} w_\varepsilon|^2 + 2 \int_{\Omega_r} |A_\varepsilon| |j(w_\varepsilon)|. \tag{3.56}$$

By (3.9) and (3.10), (3.56) implies

$$\limsup_{\varepsilon \to 0} \int_{\Omega_r} |\nabla w_\varepsilon|^2 \leq K_5(r) + \frac{Cn}{r}\Big(\frac{n}{r_a} + h_{ex}\Big) \leq K_6(r).$$

We deduce from this that there exists a subsequence of $\varepsilon \to 0$ such that $\nabla w_\varepsilon$ converges weakly in $L^2(\Omega_r)$.

Additionally, from (3.55) we have that $\big\|\frac{1}{\varepsilon}(1 - |w_\varepsilon|^2)\big\|^2_{L^2(\Omega_r)} \leq C^2 K_4(r) \leq K_7(r)$ where $K_7$ is a constant independent of $\varepsilon$. Subsequently, we have that

there exists an $\mathbb{S}^1$-valued map $u_*$ which $w_\varepsilon$ converges to. The convergence of $w_\varepsilon$ and its gradient in combination implies $w_\varepsilon \rightharpoonup u_*$ in $H^1(\Omega_r)$.

Now, inspired by the proof of Theorem VI.1 in [7] (see their (14) in Chapter VI), we have that Fubini's theorem implies there exists a radius $s$ such that $r < s < r_1$, where $r_1 > 0$, which satisfies

$$\int_{\partial B_s(\xi_i)} (1 - |w_\varepsilon|^2)^2 \leq C\varepsilon^2. \tag{3.57}$$

Thus, because $A_\varepsilon$ is bounded in $L^p(\partial B_r(\xi_i))$ due to (3.2) and trace theory, (3.57) implies that

$$\int_{\partial B_r(\xi_i)} \tau \cdot (1 - |w_\varepsilon|^2) A_\varepsilon \to 0. \tag{3.58}$$

Following [48], we claim that $u_*$ has the same degrees $d_i$ on $\partial B_r(\xi_i)$ as $w_\varepsilon$ has. As a consequence, we have that (see (2.43)), as $\varepsilon \to 0$,

$$\int_{\partial B_r(\xi_i)} \langle i w_\varepsilon, \partial_\tau w_\varepsilon \rangle \to \int_{\partial B_r(\xi_i)} \langle i u_*, \partial_\tau u_* \rangle = 2\pi d_i. \tag{3.59}$$

Using (3.58) and (3.59) in (3.53) shows $I_4' \to 0$ as $\varepsilon \to 0$.

Taking $r_0$ to be fixed and using that $\nabla^\perp \Psi_r \to \nabla^\perp \Upsilon_0$ uniformly in every compact subset of $\Omega \backslash \{\xi_i\}$; passing to the limit $r \to 0$ in (3.50) yields

$$\int_{\Omega_{r_0}} j_{A_*} \cdot \big(j_{A_*} - j_A\big) = o_\varepsilon(1),$$

which means $I_4 = o_\varepsilon(1)$ due to (3.44). □

**Lemma 3.12.** *Suppose $\Upsilon_0$ satisfies (3.45). Then*

$$\sum_{i=1}^{n} \Big( \sup_{\partial B_r(a_i)} \Upsilon_0 - \inf_{\partial B_r(a_i)} \Upsilon_0 \Big) \leq Cr. \tag{3.60}$$

*Proof.* For the position $z \in \Omega$ let $\Lambda = \sum_{i=1}^n d_i \log|z - a_i|$. Then due to (3.45), we have that $\Delta(\Upsilon_0 - \Lambda) = -h_*$. By (3.4),

$$\|h_*\|_{W^{1,p}(\Omega)} \leq C\Big(\frac{n}{r_a} + h_{ex}\Big),$$

so by standard elliptic estimates we deduce that

$$\|\Upsilon_0 - \Lambda\|_{W^{3,p}(\Omega)} \leq C\Big(\frac{n}{r_a} + h_{ex}\Big)$$

which, by Sobolev embedding for $p = 1$ ($W^{2,1} \hookrightarrow C^0$ so $W^{3,1} \hookrightarrow C^1$), implies

$$\|\Upsilon_0 - \Lambda\|_{C^1(\Omega)} \leq C\Big(\frac{n}{r_a} + h_{ex}\Big).$$

Since

$$\Upsilon_0 = \Lambda(r) + \text{smooth function}$$

and

$$\Lambda(r) = d_i \log|z - a_i| + \sum_{j \neq i} d_j \log|z - a_j|, \tag{3.61}$$

we see that $\Upsilon_0$ is smooth on $\partial B_r(a_i)$ because the first term on the right-hand side of (3.61) is constant on $\partial B_r(a_i)$, and the last term, which describes the influence from the other vortices, is smooth on $\partial B_r(a_i)$. The smoothness of $\Upsilon_0$ on the boundary of the balls implies that $|\hat{e}_\theta \cdot \nabla\Upsilon_0| = |\frac{1}{r}\partial_\theta \Upsilon_0| \leq C$ on $\partial B_r(a_i)$, which in turn implies the result. □

We make a few comments about the significance of the coercivity estimate. The functions $(w_\varepsilon, A_\varepsilon)$ determine the position of the Jacobian, whilst the "optimal functions" $(w_*, A_*)$ determine the vortex locations. The coercivity estimate tells us the distance to the set of optimal configurations is approximately the amount of excess energy in the system - Theorem 3.9 shows we have $L^2$ convergence of $\nabla w_\varepsilon - \nabla w_*$ and $h_\varepsilon - h_*$ to zero merely from $W^{-1,p}$ convergence and control of the excess energy. In fact, we see that the rate of convergence of the magnetic field goes like the excess energy. We will see that the statement in Theorem 3.9 is general enough that the conclusions hold for solutions to the TDGL equations and in a slightly weaker form for the MKG-pGL equations, and so the coercivity estimate proves to be a key ingredient for estimating energy and supercurrent terms later in the thesis on our way to deriving the limiting dynamical laws.

As the final action of this chapter, we state a result on the supercurrent which follows from the coercivity estimate and is used in Chapter 5.

**Corollary 3.13.** *If the hypothesis of Theorem 3.9 hold, then*

$$\|j_{A_\varepsilon}(w_\varepsilon) - j_{A_*}(w_*)\|_{L^{4/3}(\Omega)} \leq C\sqrt{D_\varepsilon} + error_2 \tag{3.62}$$

*where*

$$error_2 \leq C\sqrt{error_1} + Cn^{\frac{1}{4}}\sqrt{r_\star}\big(\sqrt{G_\varepsilon} + Cn^{\frac{3}{2}}\big) + o_\varepsilon(1).$$

*Proof.* The proof is essentially the same as in the proof of Theorem 2 of [29]. We reproduce it here for convenience and show how to treat the pinning potential. We start with the observation that

$$\begin{aligned}\|j_A - j_{A_*}\|_{L^{4/3}(\Omega)} \leq \left\| \frac{j_A}{|w|}(|w|-1)\right\|_{L^{4/3}(\Omega)} + \left\| \frac{j_A}{|w|} - j_{A_*}\right\|_{L^{4/3}(\Omega_{r_\star})} \\ + \left\| \frac{j_A}{|w|} - j_{A_*}\right\|_{L^{4/3}(\cup_i B_{r_\star})}\end{aligned} \tag{3.63}$$

We estimate each term on the right hand side in turn. Firstly,

$$\left\| \frac{j_A}{|w|}(|w|-1)\right\|_{L^{4/3}(\Omega)} \leq \|\nabla_A w\|_{L^2(\Omega)} \, \|1-|w|\|_{L^4(\Omega)} \leq \sqrt{\varepsilon} G_\varepsilon^{3/4} = o_\varepsilon(1). \tag{3.64}$$

Next, by applying Hölder's inequality, assuming that $\frac{1}{2} \leq \eta_\varepsilon^2$ for sufficiently small $\varepsilon$ and then using the coercivity estimate we have

$$\begin{aligned}\left\| \frac{j_A}{|w|} - j_{A_*}\right\|_{L^{4/3}(\Omega_{r_\star})} &\leq C \left\| \frac{1}{4}\left| \frac{j_A}{|w|} - j_{A_*}\right|\right\|_{L^2(\Omega_{r_\star})} \\ &\leq C \left\| \frac{\eta_\varepsilon^2}{2}\left| \frac{j_A}{|w|} - j_{A_*}\right|\right\|_{L^2(\Omega_{r_\star})} \\ &\leq C\sqrt{D_\varepsilon + error_1}.\end{aligned} \tag{3.65}$$

Now consider that

$$\left\| \frac{j_A}{|w|} - j_{A_*}\right\|_{L^{4/3}(\cup_i B_{r_\star})} \leq \left\| \frac{j_A}{|w|}\right\|_{L^{4/3}(\cup_i B_{r_\star})} + \|j_{A_*}\|_{L^{4/3}(\cup_i B_{r_\star})}.$$

Applying Hölder's inequality we have

$$\left\| \frac{j_A}{|w|}\right\|_{L^{4/3}(\cup_i B_{r_\star})} \leq (n\pi r_\star^2)^{1/4} \left\| \frac{j_A}{|w|}\right\|_{L^2(\cup_i B_{r_\star})} \leq Cn^{1/4}\sqrt{r_\star G_\varepsilon}. \tag{3.66}$$

Finally, to estimate the limiting supercurrent we use the co-area formula

$$\|j_{A_*}\|_{L^{4/3}(\cup_i B_{r_\star}(a_i))} \leq C\Big(\sum_i \int_0^{r_\star} \Big(\frac{n}{r}\Big)^{\frac{4}{3}} r \, dr\Big)^{\frac{3}{4}} \leq Cn^{\frac{7}{4}}\sqrt{r_\star}. \tag{3.67}$$

Collecting together (3.64)-(3.67) in (3.63) gives

$$\|j_A - j_{A_*}\|_{L^{4/3}(\Omega)} \leq C\sqrt{D_\varepsilon + error_1} + Cn^{1/4}\sqrt{r_\star G_\varepsilon} + Cn^{\frac{7}{4}}\sqrt{r_\star} + o_\varepsilon(1).$$

□

# Chapter 4

# Energy analysis for TDGL solutions

The aim of the next chapter is to show the vortices follow well-defined trajectories and derive the equation of motion these trajectories satisfy. However, before we can say anything about vortex trajectories it is necessary to show we have control on the energy and dissipation in the system, at least for a finite period of time. If the energy is under control then we know that the vortices don't jump around chaotically and can define a smooth vortex trajectory. Furthermore, if the energy in the system is too large we struggle to control other terms when we derive the dynamical law. The aim is to show that many terms are small as $\varepsilon \to 0$ and only a few prominent terms survive in the limiting dynamical law.

To establish the necessary control, we require a prerequisite bound and an equation involving the energy which we refer to as the $\Gamma$-*convergence result* and the *conservation of energy* equation.

## 4.1 $\Gamma$-convergence result

The $\Gamma$-convergence result provides a lower bound on the energy based on the compactness of the vorticity. We use the coercivity estimate of Theorem

3.9 to obtain a lower bound on the energy which depends on the relative vortex positions and pinning potential.

**Proposition 4.1.** *Let $(w_\varepsilon, A_\varepsilon)$ and some $K_2 > 0$ satisfy the energy bound*

$$G_\varepsilon^{\eta_\varepsilon}(w_\varepsilon, A_\varepsilon) \leq n\pi |\log \varepsilon| + K_2 \tag{4.1}$$

*and suppose the vortices are localized as*

$$\left\| J(w_\varepsilon) - \pi \sum_{i=1}^{n} d_i \delta_{a_i} \right\|_{\dot{W}^{-1,1}(\Omega)} \to 0 \tag{4.2}$$

*where $a \in \Omega^{n*}$ and $d_i \in \{-1, 1\}$. Then as $\varepsilon \to 0$,*

$$G_\varepsilon^{\eta_\varepsilon}(w_\varepsilon, A_\varepsilon) \geq n\pi |\log \varepsilon| + n\gamma_0 + W^Q(a, d) - o_\varepsilon(1). \tag{4.3}$$

*Proof.* To establish a lower bound on the energy we note that we can lower the energy by minimising over $A_\varepsilon$. If $\tilde{A}_\varepsilon$ minimises $G_\varepsilon^{\eta_\varepsilon}(w_\varepsilon, \cdot)$, it satisfies the Euler-Lagrange equation

$$\begin{cases} \eta_\varepsilon^2 j_{\tilde{A}_\varepsilon}(w_\varepsilon) + \nabla^\perp \text{curl } \tilde{A}_\varepsilon = 0 & \text{in } \Omega \\ \text{curl } \tilde{A}_\varepsilon = h_{ex} & \text{on } \partial\Omega. \end{cases}$$

Taking the curl of the first equation here leads to the corresponding London equation for $\tilde{h}_\varepsilon = \text{curl } \tilde{A}_\varepsilon$:

$$-\Delta \tilde{h}_\varepsilon + \tilde{h}_\varepsilon = 2J_{\tilde{A}_\varepsilon}(w_\varepsilon) + \text{curl}\big((\eta_\varepsilon^2 - 1) j_{\tilde{A}_\varepsilon}(w_\varepsilon)\big). \tag{4.4}$$

Lemma 2.13 shows that the last term here is $o_\varepsilon(1)$. Lemma 2.12 shows that $J_{\tilde{A}_\varepsilon}(w_\varepsilon)$ converges to the same limit as $J(w_\varepsilon)$ if $\tilde{A}_\varepsilon$ is in the Coulomb gauge, and so by (4.2) the limiting London equation is

$$-\Delta h_* + h_* = 2\pi \sum_{i=1}^{n} d_i \delta_{a_i}. \tag{4.5}$$

Take the difference between equations (4.4) and (4.5) and apply elliptic estimates to achieve $W^{1,p}$ convergence of the magnetic field:

$$\|h_\varepsilon - h_*\|_{W^{1,p}} \leq C \left\| 2J_{\tilde{A}_\varepsilon}(w_\varepsilon) - 2\pi \sum_{i=1}^{n} d_i \delta_{a_i} \right\|_{\dot{W}^{-1,p}} \to 0 \qquad \text{as } \varepsilon \to 0.$$

It follows that $\|A_\varepsilon - A_*\|_{W^{2,p}} \to 0$ $(p < 2)$, and by Sobolev embedding, $\|A_\varepsilon - A_*\|_{L^q} \to 0$ for $1 < q < \infty$. We use this in the coercivity estimate in Theorem 3.9. As the left-hand side of (3.22) is nonnegative, after applying the definition of the excess energy (2.51), the estimate (4.3) follows. □

## 4.2 Conservation of energy

The next step to show that the energy and dissipation in the system are bounded up to a finite time is to derive an equation for the evolution of the energy in time. To derive such an equation, we first compute the time derivative of the energy density and then integrate it over $\Omega$. We see that our choice of boundary conditions for solutions to the TDGL equations (2.19) leads to the vanishing of the more complicated terms in the equation for energy evolution.

Recall that the pinned energy density is

$$g_\varepsilon^{\eta_\varepsilon}(w_\varepsilon, A_\varepsilon) := \frac{\eta_\varepsilon^2}{2}|\nabla_{A_\varepsilon} w_\varepsilon|^2 + \frac{\eta_\varepsilon^4}{4\varepsilon^2}(1 - |w_\varepsilon|^2)^2 + \frac{1}{2}|\text{curl } A_\varepsilon - h_{ex}|^2, \tag{4.6}$$

then we have

**Lemma 4.2.** *For any triple $(w, A, \Phi)$ (not necessarily solutions), the energy density given by (4.6) satisfies*

$$\partial_t g_\varepsilon^{\eta_\varepsilon} = \eta_\varepsilon^2 \text{div}\langle \partial_\Phi w, \nabla_A w\rangle - \langle \partial_\Phi w, \eta_\varepsilon^2 \Delta_A w + \frac{\eta_\varepsilon^4 w}{\varepsilon^2}(1 - |w|^2)\rangle \\ - (h - h_{ex})\text{curl } E + \eta_\varepsilon^2 E \cdot j_A(w). \tag{4.7}$$

*Proof.* Directly computing the derivative of (4.6) gives

$$\begin{aligned} \partial_t g_\varepsilon^{\eta_\varepsilon} &= \eta_\varepsilon^2 \langle \partial_t \nabla_A w, \nabla_A w\rangle - \frac{\eta_\varepsilon^4}{\varepsilon^2}(1 - |w|^2)\langle \partial_t w, w\rangle + (h - h_{ex})\partial_t h \\ &= \eta_\varepsilon^2 \langle \partial_\Phi \nabla_A w, \nabla_A w\rangle - \frac{\eta_\varepsilon^4}{\varepsilon^2}(1 - |w|^2)\langle \partial_\Phi w, w\rangle + (h - h_{ex})\partial_t h. \end{aligned} \tag{4.8}$$

To rewrite the first term on the right-hand side consider that from Lemmas A.1 and A.3,

$$\text{div}\langle \partial_\Phi w, \nabla_A w\rangle = \langle \nabla_A \partial_\Phi w, \nabla_A w\rangle + \langle \partial_\Phi w, \Delta_A w\rangle,$$

and

$$\partial_\Phi \nabla_A w - \nabla_A \partial_\Phi w = iwE,$$

so

$$\operatorname{div}\langle \partial_\Phi w, \nabla_A w\rangle = \langle \partial_\Phi \nabla_A w, \nabla_A w\rangle + \langle \partial_\Phi w, \Delta_A w\rangle - E \cdot j_A(w).$$

Use this, and that from the definition of $E$ we have $\operatorname{curl} E = -\partial_t h$, so that (4.8) becomes (4.7). □

Now we consider the form of $\partial_t g_\varepsilon^{\eta_\varepsilon}$ for when $(w_\varepsilon, A_\varepsilon, \Phi_\varepsilon)$ satisfy the TDGL equations and derive the conservation of energy equation (4.10), which will be used in the next section.

**Lemma 4.3.** *If $(w_\varepsilon, A_\varepsilon, \Phi_\varepsilon)$ are solutions to (2.19) then*

$$\partial_t g_\varepsilon^{\eta_\varepsilon}(w_\varepsilon, A_\varepsilon) = -\alpha_\varepsilon \eta_\varepsilon^2 |\partial_{\Phi_\varepsilon} w_\varepsilon|^2 - \sigma_\varepsilon |E_\varepsilon|^2 + \operatorname{div}\big(\eta_\varepsilon^2 \langle \partial_{\Phi_\varepsilon} w_\varepsilon, \nabla_{A_\varepsilon} w_\varepsilon\rangle\big) - \operatorname{curl}\big((h_\varepsilon - h_{ex})E_\varepsilon\big). \quad (4.9)$$

*Consequently,*

$$\partial_t G_\varepsilon^{\eta_\varepsilon}(w_\varepsilon, A_\varepsilon) + \int_\Omega \alpha_\varepsilon \eta_\varepsilon^2 |\partial_{\Phi_\varepsilon} w_\varepsilon|^2 + \sigma_\varepsilon |E_\varepsilon|^2 = 0. \quad (4.10)$$

*Proof.* Starting with (4.7), we can rewrite the second term on the right-hand side as well as the last two terms by using the TDGL equations (2.19). Recall that the first TDGL equation is

$$(\alpha_\varepsilon + i\beta_0)\partial_{\Phi_\varepsilon} w_\varepsilon = \Delta_{A_\varepsilon} w_\varepsilon + \frac{\eta_\varepsilon^2 w_\varepsilon}{\varepsilon^2}(1 - |w_\varepsilon|^2) + \nabla \log \eta_\varepsilon^2 \cdot \nabla_{A_\varepsilon} w_\varepsilon \quad (4.11)$$

and the second is

$$\eta_\varepsilon^2 j_{A_\varepsilon}(w_\varepsilon) = -\sigma_\varepsilon E_\varepsilon - \nabla^\perp h_\varepsilon. \quad (4.12)$$

Inserting (4.11) into the second term on the right-hand side of (4.7) gives

$$\langle \partial_\Phi w, \eta_\varepsilon^2(\alpha_\varepsilon + i\beta_0)\partial_\Phi w - \nabla \eta_\varepsilon^2 \cdot \nabla_A w\rangle = \alpha_\varepsilon \eta_\varepsilon^2 |\partial_\Phi w|^2 - \nabla \eta_\varepsilon^2 \cdot \langle \partial_\Phi w, \nabla_A w\rangle. \quad (4.13)$$

To treat the terms involving electric and magnetic fields in (4.7), apply (4.12) like so

$$\begin{aligned} -(h-h_{ex})\text{curl } E + \eta_\varepsilon^2 E \cdot j_A(w) &= -(h-h_{ex})\text{curl } E - \sigma_\varepsilon |E|^2 - E \cdot \nabla^\perp h \\ &= -\sigma_\varepsilon |E|^2 - \text{curl}\big((h-h_{ex})E\big). \end{aligned} \tag{4.14}$$

Inserting (4.13) and (4.14) into (4.7) leads to (4.9).

To compute $\partial_t G_\varepsilon^{\eta_\varepsilon}$, integrate (4.9) over $\Omega$ and use that

$$\int_\Omega \text{div}\big(\eta_\varepsilon^2 \langle \partial_{\Phi_\varepsilon} w_\varepsilon, \nabla_{A_\varepsilon} w_\varepsilon \rangle\big) = \int_{\partial\Omega} \eta_\varepsilon^2 \langle \partial_{\Phi_\varepsilon} w_\varepsilon, \nu \cdot \nabla_{A_\varepsilon} w_\varepsilon \rangle$$

and

$$\int_\Omega \text{curl}\big((h_\varepsilon - h_{ex})E_\varepsilon\big) = \int_{\partial\Omega} (h_\varepsilon - h_{ex})\ \tau \cdot E_\varepsilon.$$

After applying the boundary conditions which the solutions satisfy,

$$\begin{cases} \nu \cdot \nabla_{A_\varepsilon} w_\varepsilon = 0 & \text{on } \partial\Omega \\ h_\varepsilon = h_{ex} & \text{on } \partial\Omega, \end{cases}$$

we have (4.10). □

## 4.3 Energy bounds

In this section we prove estimates on the energy and dissipation which are valid for a time $T_0 > 0$ which is independent of $\varepsilon$. Such bounds prove to be useful in the next chapter where we derive the dynamical law since so many quantities can themselves be bounded by the energy.

The final result needed to prove the boundedness of energy is a "product estimate" on the limiting value of the space-time Jacobian. The first result of this kind appeared in [47] but has been generalized to include gauge and pinning terms in more recent works [48, 52]. The first statement of Proposition 4.4, which says that if the energy and dissipation is under good enough control then the Jacobian of $w_\varepsilon$ converges to a measure, isn't necessary to bound $G_\varepsilon^{\eta_\varepsilon}$, but is useful in Chapter 5 so we include it.

**Proposition 4.4.** *For any $(w_\varepsilon, A_\varepsilon, \Phi_\varepsilon)$ defined on the time interval $[0,T]$ which satisfy the energy bounds*

$$G_\varepsilon^{\eta_\varepsilon}(w_\varepsilon, A_\varepsilon)(t) \leq n\pi|\log\varepsilon| + o(|\log\varepsilon|) \qquad \forall t \in [0,T] \tag{4.15}$$

*and*

$$\int_0^T \int_\Omega \alpha_0 \eta_\varepsilon^2 |\partial_{\Phi_\varepsilon} w_\varepsilon|^2 + \sigma_0 |E_\varepsilon|^2 \leq C|\log\varepsilon|; \tag{4.16}$$

*the following hold, up to extraction of a subsequence.*

*1. There exist $J_* \in L^\infty([0,T]; M(\Omega))$ and $V_* \in L^2([0,T]; M(\Omega)^2)$ such that $\partial_t J_* + \operatorname{curl} V_* = 0$ and as $\varepsilon \to 0$, $J_{A_\varepsilon}(w_\varepsilon) \to J_*(t)$ and $V_{A_\varepsilon}^{\Phi_\varepsilon}(w_\varepsilon) \to V_*(t)$ in $(C^{0,1}(\Omega \times [0,T]))^*$. Here we have written $M(\Omega) = (C^0(\Omega))^*$ for the space of bounded Radon measures.*

*2. For any $[t_1,t_2] \subset [0,T]$, $Y \in C^0(\Omega \times [t_1,t_2]; \mathbb{R}^2)$ and $\psi \in C^0(\Omega \times [t_1,t_2])$, we have the bound*

$$\frac{1}{2}\left|\int_{t_1}^{t_2}\int_\Omega V_* \cdot \psi Y\right| \leq \liminf_{\varepsilon\to 0} \frac{1}{|\log\varepsilon|}\left(\int_{t_1}^{t_2}\int_\Omega |\nabla_{A_\varepsilon} w_\varepsilon \cdot Y|^2 \int_{t_1}^{t_2}\int_\Omega |\psi \partial_{\Phi_\varepsilon} w_\varepsilon|^2\right)^{\frac{1}{2}}. \tag{4.17}$$

*3. The mapping $t \to \langle J_*(t), \xi\rangle$ is in $H^1([0,T])$ for any $\xi \in C_c^1(\Omega)$, and*

$$\left|\langle J_*(t_2),\xi\rangle - \langle J_*(t_1),\xi\rangle\right| \leq C\sqrt{t_2-t_1}\liminf_{\varepsilon\to 0}\frac{1}{\sqrt{|\log\varepsilon|}}\left(\int_{t_1}^{t_2}\int_\Omega |\partial_{\Phi_\varepsilon} w_\varepsilon|^2\right)^{\frac{1}{2}}$$

*for any $[t_1,t_2] \subset [0,T]$.*

*Proof.* See Proposition 4.3 of [48] and Proposition 4.4 of [52]. Recall from Lemma 2.12 that the gauge terms in $J_{A_\varepsilon}(w_\varepsilon)$ and $V_{A_\varepsilon}^{\Phi_\varepsilon}(w_\varepsilon)$ vanish in the limit $\varepsilon \to 0$ and hence we can replace $J(w_\varepsilon)$ and $V(w_\varepsilon)$ with their gauged counterparts. □

We are now in a position to show that the energy and dissipation in our system indeed satisfy (4.15) and (4.16). We follow what is done in the proof of Lemma 3.4 of [48] but include the gauge and pinning terms.

**Theorem 4.5.** *Let $(w_\varepsilon, A_\varepsilon, \Phi_\varepsilon)$ solve (2.19) and suppose the initial data are well-prepared, i.e. $(w_\varepsilon(0), A_\varepsilon(0))$ satisfy (2.52). Then there exists a constant $T_0 > 0$ such that, as $\varepsilon \to 0$,*

$$G_\varepsilon^{\eta_\varepsilon}(w_\varepsilon, A_\varepsilon)(t) \leq C|\log\varepsilon| \qquad \forall t \in \big[0, T_0\big] \tag{4.18}$$

*and*

$$\int_0^{T_0}\int_\Omega \alpha_0\eta_\varepsilon^2|\partial_{\Phi_\varepsilon} w_\varepsilon|^2 + \sigma_0|E_\varepsilon|^2 \leq C|\log\varepsilon|. \tag{4.19}$$

*Proof.* Integrating (4.10) in time between $t_1$ and $t_2$, we have

$$G_\varepsilon^{\eta_\varepsilon}(w_\varepsilon, A_\varepsilon)(t_2) - G_\varepsilon^{\eta_\varepsilon}(w_\varepsilon, A_\varepsilon)(t_1) + \int_{t_1}^{t_2}\int_\Omega \alpha_\varepsilon\eta_\varepsilon^2|\partial_{\Phi_\varepsilon} w_\varepsilon|^2 + \sigma_\varepsilon|E_\varepsilon|^2 = 0$$

It follows that $G_\varepsilon^{\eta_\varepsilon}(w_\varepsilon, A_\varepsilon)(t)$ is decreasing. Due to (2.52) we deduce that (4.18) is satisfied.

Now we show that (4.19) is also satisfied. Rescale in time such that $v_\varepsilon(x, |\log\varepsilon|t) = w_\varepsilon(x,t)$, $B_\varepsilon(x, |\log\varepsilon|t) = A_\varepsilon(x,t)$ and $\frac{1}{|\log\varepsilon|}\theta_\varepsilon(x, |\log\varepsilon|t) = \Phi_\varepsilon(x,t)$. In these new variables the TDGL equations (2.19) become (we drop the $\varepsilon$ subscript)

$$(\alpha_0 + i\beta_0|\log\varepsilon|)\partial_\theta v = \Delta_B v + \frac{\eta^2 v}{\varepsilon^2}(1 - |v|^2) + \nabla\log\eta^2 \cdot \nabla_B v \tag{4.20}$$

$$\sigma_0\mathcal{E} = -\nabla^\perp \text{curl } B - \eta^2\langle iv, \nabla_B v\rangle \tag{4.21}$$

where $\mathcal{E} = -\partial_t B - \nabla\theta$. Assume by contradiction that there exists a $\lambda_\varepsilon \ll |\log\varepsilon|$ such that

$$\int_0^{\lambda_\varepsilon}\int_\Omega \alpha_0\eta^2|\partial_\theta v|^2 + \sigma_0|\mathcal{E}|^2 = 1. \tag{4.22}$$

Multiply (4.20) by $\eta^2\partial_\theta v$ such that

$$\alpha_0\eta^2|\partial_\theta v|^2 = \langle\partial_\theta v, \eta^2\Delta_B v + \frac{\eta^4 v}{\varepsilon^2}(1 - |v|^2) + \nabla\eta^2 \cdot \nabla_B v\rangle. \tag{4.23}$$

By (4.7) and (4.21), (4.23) becomes

$$\alpha_0\eta^2|\partial_\theta v|^2 + \sigma_0|\mathcal{E}|^2 = -\partial_t g^\eta(v, B) + \text{div}\big(\eta^2\langle\partial_\theta v, \nabla_B v\rangle\big) - \text{curl}\big((\text{curl } B - h_{ex})\mathcal{E}\big). \tag{4.24}$$

Integrate (4.24) over $\Omega \times [0, \lambda_\varepsilon]$ and use the boundary conditions $\nu \cdot \nabla_B v = 0$ and curl $B = h_{ex}$ on $\partial\Omega$ to arrive at

$$\int_0^{\lambda_\varepsilon} \int_\Omega \alpha_0 \eta^2 |\partial_\theta v|^2 + \sigma_0 |\mathcal{E}|^2 = G^\eta(v(0), B(0)) - G^\eta(v(\lambda_\varepsilon), B(\lambda_\varepsilon)). \tag{4.25}$$

Due to (4.22), (4.25) implies

$$1 = G^\eta(v(0), B(0)) - G^\eta(v(\lambda_\varepsilon), B(\lambda_\varepsilon)). \tag{4.26}$$

However, rescaling in time again such that $y_\varepsilon(x,t) = v_\varepsilon(x, \lambda_\varepsilon t)$, $X_\varepsilon(x,t) = B(x, \lambda_\varepsilon t)$ and $\Psi_\varepsilon(x,t) = \frac{1}{\lambda_\varepsilon}\theta_\varepsilon(x, \lambda_\varepsilon t)$, we have that

$$\int_0^1 \int_\Omega \alpha_0 \eta^2 |\partial_\Psi y|^2 + \sigma_0 |\mathscr{E}|^2 = \lambda_\varepsilon \int_0^{\lambda_\varepsilon} \int_\Omega \alpha_0 \eta^2 |\partial_\theta v|^2 + \sigma_0 |\mathcal{E}|^2 = \lambda_\varepsilon \ll |\log \varepsilon|. \tag{4.27}$$

where $\mathscr{E} = -\partial_t X - \nabla \Psi$. Due to (4.18) we have that

$$\int_\Omega \eta^2 |\nabla_X y|^2 \leq G_\varepsilon^{\eta_\varepsilon}(y, X) \leq C |\log \varepsilon|, \tag{4.28}$$

and since (4.15) and (4.16) are satisfied for $(y, X, \Psi)$ we can apply Proposition 4.3.1. Specifically, using (4.27) and (4.28) in the bound (4.17) leads to the conclusion that $V_*(t) = 0$ for $t \in [0,1]$. Since $\partial_t J_* + \text{curl } V_* = 0$, then $J_*(1) = J_*(0)$; i.e the vortices don't move. Here $J_*(t)$ is the limit of $\text{curl}(\langle i y_\varepsilon, \nabla_{X_\varepsilon} y_\varepsilon \rangle + X_\varepsilon)(t)$. Thus,

$$J_*(t) = J_*(0) = 2\pi \sum_{i=1}^n d_i \delta_{a_i(0)}.$$

Returning to the previous time scaling, we find that $\text{curl}(\langle i v_\varepsilon, \nabla_{B_\varepsilon} v_\varepsilon \rangle + B_\varepsilon)(t) \rightharpoonup J_*(0)$ for every $t \in [0, \lambda_\varepsilon]$. Thus, (4.2) is satisfied and we can apply Proposition 4.1. By (4.3), we deduce that

$$G_\varepsilon^\eta(v, B)(\lambda_\varepsilon) \geq n\pi |\log \varepsilon| + n\gamma_0 + W^Q(a(0), d(0)) - o_\varepsilon(1) = G_\varepsilon^\eta(v, B)(0) - o_\varepsilon(1).$$

Plugging this in to (4.26),

$$\begin{aligned} 1 + G_\varepsilon^\eta(v(\lambda_\varepsilon), B(\lambda_\varepsilon)) &= G_\varepsilon^\eta(v(0), B(0)) \\ &\leq G_\varepsilon^\eta(v(\lambda_\varepsilon), B(\lambda_\varepsilon)) + o_\varepsilon(1) \end{aligned}$$

which leads to $1 + o_\varepsilon(1) \leq 0$; a contradiction. We thus deduce the existence of a $T_0 > 0$ such that

$$\int_0^{T_0|\log\varepsilon|} \int_\Omega \alpha_0 \eta^2 |\partial_\theta v|^2 + \sigma_0 |\mathcal{E}|^2 \leq 1,$$

which rescales to (4.19) for all $\varepsilon$. □

# Chapter 5

# Dynamical law in TDGL setting

In this chapter we determine the dynamical law for vortices. We show that if $(w_\varepsilon, A_\varepsilon, \Phi_\varepsilon)$ are solutions to (2.19), then the Jacobian converges to time-dependent points which we call the vortex positions. We proceed to show that the energy density converges to the same points and these positions satisfy an ODE.

Determining the form of the ODE the vortex positions satisfy amounts to relating the time-derivative of the energy density, curl of the space-time Jacobian (which is related to the evolution of the Jacobian in time) and the forces in the system. The renormalized energy is brought in through a quantity called the *stress-energy tensor* which we define in Section 5.2. Finally, we must pass to the limit $\varepsilon \to 0$ in the equation relating the stress-energy tensor, energy density and space-time Jacobian such that the vortex positions become apparent and only the dominant limiting forces remain. We employ a Gronwall argument to show that the excess energy in the system is vanishingly small in the limit and therefore the vortices obey a well-defined ODE.

## 5.1 Vortex trajectories $\xi_i(t)$

Having established control on the energy in Theorem 4.5, we see that the results of Proposition 4.4 hold and that namely, the Jacobian and space-time

Jacobian converge to measures. As we determine the form of these measures the vortex positions become apparent. We find that the Jacobian of $w_\varepsilon$, as well as the energy density, converge to measures which depend purely on the degrees and positions of the vortices.

**Proposition 5.1.** *Suppose $(w_\varepsilon, A_\varepsilon, \Phi_\varepsilon)$ satisfy (2.19) and that the initial data are well-prepared (2.52). Let $n := n(0)$ and $d_i := d_i(0)$. Then there exist $n$ vortex paths $\xi_i(t) \in H^1([0,T_0];\Omega)$ $(i = 1, ..., n)$ such that, for $t \in [0,T_0]$, as $\varepsilon \to 0$*

$$\left\| J_{A_\varepsilon}(w_\varepsilon) - \pi \sum_{i=1}^{n} d_i \delta_{\xi_i(t)} \right\|_{\dot{W}^{-1,1}(\Omega)} \to 0 \tag{5.1}$$

*and*

$$\left\| V_{A_\varepsilon}^{\Phi_\varepsilon}(w_\varepsilon) - \pi \sum_{i=1}^{n} d_i \delta_{\xi_i(t)} \dot{\xi}_i^{\perp}(t) \right\|_{\dot{W}^{-1,1}(\Omega\times[0,T])} \to 0. \tag{5.2}$$

*Proof.* By (4.18), (4.19) and (B.2), we have

$$G_\varepsilon(t) \le C|\log\varepsilon| \quad \forall t \in [0,T_0] \quad \text{and} \quad \int_0^{T_0}\int_\Omega \alpha_0\eta_\varepsilon^2|\partial_{\Phi_\varepsilon} w_\varepsilon|^2 + \sigma_0|E_\varepsilon|^2 \le C|\log\varepsilon|.$$

Hence, the hypothesis of Propositions 3.6 and 4.4 are satisfied so we deduce that $J_{A_\varepsilon}(w_\varepsilon) \to \pi\sum_{i=1}^{n(t)} d_i(t)\delta_{\xi_i(t)}$ in $(C^{0,1}(\Omega))^*$ for all $t \in [0,T_0]$. Since $(C^{0,1}(\Omega))^*$ is $\dot{W}^{-1,1}(\Omega)$, we also have convergence in this space. By Proposition 3.3 of [48], $n(t) = n(0)$ and $d_i(t) = d_i(0)$, hence we have (5.1). The structure of $V_*$ comes by using the relation $\partial_t J_* + \operatorname{curl} V_* = 0$ and our newfound $J_*$. □

Next, consider the role of the vortex trajectories in the limit of the normalized energy density. The proof follows Proposition 4.4 of [57] but is altered to include the effects of pinning.

**Proposition 5.2.** *If the hypothesis of Lemma 3.1 are satisfied and*

$$G_\varepsilon^{\eta_\varepsilon}(w_\varepsilon(t), A_\varepsilon(t)) \le n\pi|\log\varepsilon| + o(|\log\varepsilon|) \qquad \forall t \in [0,T],$$

*then there exists a subsequence such that*

$$\frac{g_\varepsilon^{\eta_\varepsilon}(w_\varepsilon(t), A_\varepsilon(t))}{|\log\varepsilon|} \overset{*}{\rightharpoonup} \pi\sum_{i=1}^{n}\delta_{\xi_i(t)} \qquad \forall t \in [0,T]. \tag{5.3}$$

*Proof.* The proof of Proposition 4.4 of [57] shows that the normalised energy density converges to a measure. We continue the proof from this point.

Note that the energy functional can be split into the energy in the balls centred at the vortex positions and the energy outside the balls:

$$G_\varepsilon^{\eta_\varepsilon}(w_\varepsilon, A_\varepsilon; \Omega) = G_\varepsilon^{\eta_\varepsilon}(w_\varepsilon, A_\varepsilon; \Omega_r) + G_\varepsilon^{\eta_\varepsilon}(w_\varepsilon, A_\varepsilon; \cup_i B_r(\xi_i)).$$

Recall (3.54). The energy in the balls has a large lower bound:

$$n\pi|\log\varepsilon| \leq G_\varepsilon^{\eta_\varepsilon}(\cup_{i=1}^n B_r(\xi_i)) + o(|\log\varepsilon|). \tag{5.4}$$

Using (5.4) with the upper bound on the total energy implies that

$$G_\varepsilon^{\eta_\varepsilon}(\Omega_r) \leq n\pi|\log\varepsilon| - G_\varepsilon^{\eta_\varepsilon}(\cup_i B_r(\xi_i)) + o(|\log\varepsilon|) \leq o(|\log\varepsilon|).$$

Hence,

$$\frac{1}{|\log\varepsilon|} G_\varepsilon^{\eta_\varepsilon}(\Omega_r) = o_\varepsilon(1).$$

Meanwhile, (5.4) also implies that

$$\pi \leq \liminf_{\varepsilon\to 0} \int_{B_r(\xi_i)} \frac{g_\varepsilon^{\eta_\varepsilon}(w_\varepsilon, A_\varepsilon)(t)}{|\log\varepsilon|}. \tag{5.5}$$

This shows that, although the normalized energy outside of the balls is vanishingly small as $\varepsilon \to 0$, the energy in the balls in nonzero and so the energy must concentrate in the balls. Since $r$ can be arbitrarily small, the above implies that the limiting measure has a structure of the form

$$\frac{g_\varepsilon^{\eta_\varepsilon}(w_\varepsilon, A_\varepsilon)(t)}{|\log\varepsilon|} \to \sum_{i=1}^n \lambda_i(t)\delta_{\xi_i(t)}.$$

The task is to calculate the value of $\lambda_i(t)$. We already know from (5.5) that

$$\lambda_i(t) \geq \pi. \tag{5.6}$$

Now we seek an upper bound for $\lambda_i(t)$. Fix $r \leq r_*/4$ for each $i$ and choose $\zeta_i \in C^1(\Omega)$ satisfying $\mathrm{supp}(\zeta_i) = B_{2r}(\xi_i(t))$, $\zeta_i = 1$ on $B_r(\xi_i(t))$, and $0 \leq \zeta_i \leq 1$. From the assumed upper bound on the energy,

$$\begin{aligned}\sum_{i=1}^n \lambda_i(t) = \lim_{\varepsilon\to 0} \sum_{i=1}^n \int_\Omega \zeta_i \frac{g_\varepsilon^{\eta_\varepsilon}(t)}{|\log\varepsilon|} &\leq \limsup_{\varepsilon\to 0} \int_\Omega \frac{g_\varepsilon^{\eta_\varepsilon}(t)}{|\log\varepsilon|} \\ &\leq \limsup_{\varepsilon\to 0} \max_{t\in[0,T]} \frac{G_\varepsilon^{\eta_\varepsilon}(t;\Omega)}{|\log\varepsilon|} = n\pi. \end{aligned} \tag{5.7}$$

Combining (5.6) and (5.7) means $\lambda_i(t) = \pi$ for all $i = 1, ..., n$ and all $t \in [0, T]$. □

**Remark 5.3.** *Well-prepared solutions to (2.19) satisfy the hypothesis of Proposition 5.2.*

## 5.2 Stress-energy tensor

The previous subsection shows that the limiting Jacobian, limiting space-time Jacobian and limiting normalized energy density are given by the positions of the vortices. Our goal is to relate these positions to the forces in the system; namely the gradient of the renormalized energy and the gradient of the pinning potential. In order to bring in the renormalized energy we introduce the following quantity:

**Definition 5.4.** *Writing $T_\varepsilon^{\eta_\varepsilon}$ in place of $T_\varepsilon^{\eta_\varepsilon}(w_\varepsilon, A_\varepsilon)$ and for $\mathcal{I}$ the $2\times 2$ identity matrix, the stress-energy tensor with pinning is*

$$T_\varepsilon^{\eta_\varepsilon} = \eta_\varepsilon^2 \nabla_{A_\varepsilon} w_\varepsilon \otimes \nabla_{A_\varepsilon} w_\varepsilon - \frac{1}{2}\mathcal{I}\big(\eta_\varepsilon^2 |\nabla_{A_\varepsilon} w_\varepsilon|^2 + \frac{\eta_\varepsilon^4}{2\varepsilon^2}(1 - |w_\varepsilon|^2)^2 - |h_\varepsilon|^2\big). \quad (5.8)$$

When taking the divergence of $T_\varepsilon^{\eta_\varepsilon}$, the terms multiplied by the identity matrix are just a gradient, so then taking the curl of div $T_\varepsilon^{\eta_\varepsilon}$ leaves us with

$$\text{curl div } T_\varepsilon^{\eta_\varepsilon} = \text{curl div}\big(\eta_\varepsilon^2 \nabla_A w \otimes \nabla_A w\big). \quad (5.9)$$

The relationship between the stress-energy tensor and the renormalized energy becomes apparent in the next lemma. First, we define some notation

**Definition 5.5.** *If $X$ and $Y$ are rank-2 tensors, then*

$$X : Y = \sum_{lk} X_{lk} Y_{lk}.$$

**Lemma 5.6.** *(Lemma 5.1 of [35]). Let $\xi \in \Omega^{n*}$ and $d \in \{\pm 1\}^n$. Then $w_* = w_*(\cdot;\xi,d)$, $A_* = A_*(\cdot;\xi,d)$ and the renormalized energy $W(\xi,d)$ satisfy*

$$\int_\Omega \nabla\nabla^\perp\zeta : (j_{A_*}(w_*) \otimes j_{A_*}(w_*)) = -\nabla^\perp\zeta \cdot \nabla_{\xi_i} W(\xi,d) \tag{5.10}$$

*where $\zeta \in C^2(\Omega)$ and $\Delta\zeta$ has support in the neighbourhood of the $\xi_i$'s.*

With Lemma 5.6 in hand and after integrating (5.9) against a test function we have

$$\begin{aligned}\int_\Omega \zeta \text{ curl div } T_\varepsilon^{\eta_\varepsilon} = \int_\Omega \zeta \text{ curl div}\big(\eta_\varepsilon^2(\nabla_A w \otimes \nabla_A w - j_{A_*} \otimes j_{A_*})\big) \\ + \int_\Omega \zeta \text{ curl div}\big((\eta_\varepsilon^2 - 1) j_{A_*} \otimes j_{A_*}\big) - \nabla^\perp\zeta \cdot \nabla_{\xi_i} W(\xi).\end{aligned} \tag{5.11}$$

In order to relate the stress-energy tensor to the energy density and Jacobian, we consider more closely the divergence of $T_\varepsilon^{\eta_\varepsilon}$.

**Lemma 5.7.** *If $(w_\varepsilon, A_\varepsilon)$ satisfy (2.19) then*

$$\begin{aligned}\text{div } T_\varepsilon^{\eta_\varepsilon} = \alpha_\varepsilon \eta_\varepsilon^2 \langle \partial_{\Phi_\varepsilon} w_\varepsilon, \nabla_{A_\varepsilon} w_\varepsilon\rangle - \beta_0 \eta_\varepsilon^2 V_{A_\varepsilon}^{\Phi_\varepsilon}(w_\varepsilon) + \frac{\beta_0}{2}\eta_\varepsilon^2(1-|w_\varepsilon|^2)E_\varepsilon \\ + \sigma_\varepsilon h_\varepsilon E_\varepsilon^\perp - \nabla \log \eta_\varepsilon^2 \big(g_\varepsilon^{\eta_\varepsilon} + \frac{\eta_\varepsilon^4}{4\varepsilon^2}(1-|w_\varepsilon|^2)^2 - \frac{1}{2}|h_\varepsilon - h_{ex}|^2\big)\end{aligned} \tag{5.12}$$

*Proof.* Due to Remark 2.6, $(w_\varepsilon, A_\varepsilon, \Phi_\varepsilon)$ are classical solutions and so the following calculations be performed without any issues. Writing the tensor from (5.8) in component form, we have

$$\big(T_\varepsilon^{\eta_\varepsilon}\big)_{ij} = \eta_\varepsilon^2 \langle \partial_i^A w, \partial_j^A w\rangle - \frac{1}{2}\delta_{ij}\big(\eta_\varepsilon^2\langle \partial_i^A w, \partial_i^A w\rangle + \frac{\eta_\varepsilon^4}{2\varepsilon^2}(1-|w|^2)^2 - |h|^2\big).$$

In the calculations which follow we use the identity from Lemma A.1:

$$\partial_i\langle a,b\rangle = \langle \partial_i^A a, b\rangle + \langle a, \partial_i^A b\rangle.$$

We compute the divergence componentwise:

$$\begin{aligned}(\text{div } T_\varepsilon^{\eta_\varepsilon})_j = \partial_i (T_\varepsilon^{\eta_\varepsilon})_{ij} = (\partial_i \eta_\varepsilon^2)\langle \partial_i^A w, \partial_j^A w\rangle \\ + \eta_\varepsilon^2 \partial_i \langle \partial_i^A w, \partial_j^A w\rangle - \frac{1}{2}(\partial_j \eta_\varepsilon^2)\langle \partial_i^A w, \partial_i^A w\rangle - \frac{\eta_\varepsilon^2}{2}\partial_j\langle \partial_i^A w, \partial_i^A w\rangle \\ - \frac{1}{4\varepsilon^2}(\partial_j \eta_\varepsilon^4)(1-|w|^2)^2 - \frac{\eta_\varepsilon^4}{4\varepsilon^2}\partial_j(1-|w|^2)^2 + \frac{1}{2}\partial_j|h|^2\end{aligned}$$

$$= (\partial_i \eta_\varepsilon^2)\langle \partial_i^A w, \partial_j^A w\rangle + \eta_\varepsilon^2 \langle (\partial_i^A)^2 w, \partial_j^A w\rangle$$
$$+ \eta_\varepsilon^2 \langle \partial_i^A w, \partial_i^A \partial_j^A w\rangle - \frac{1}{2}(\partial_j \eta_\varepsilon^2)\langle \partial_i^A w, \partial_i^A w\rangle - \eta_\varepsilon^2 \langle \partial_i^A w, \partial_j^A \partial_i^A w\rangle$$
$$- \frac{1}{4\varepsilon^2}(\partial_j \eta_\varepsilon^4)(1-|w|^2)^2 + \frac{\eta_\varepsilon^4}{\varepsilon^2}(1-|w|^2)\langle w, \partial_j^A w\rangle + \frac{1}{2}\partial_j |h|^2$$

$$= \eta_\varepsilon^2 \langle (\partial_i^A)^2 w + \frac{\eta_\varepsilon^2}{\varepsilon^2}(1-|w|^2)w + (\partial_i \log \eta_\varepsilon^2)\partial_i^A w, \partial_j^A w\rangle$$
$$+ \eta_\varepsilon^2 \langle \partial_i^A w, \partial_i^A \partial_j^A w - \partial_j^A \partial_i^A w\rangle - \frac{1}{2}(\partial_j \eta_\varepsilon^2)\langle \partial_i^A w, \partial_i^A w\rangle$$
$$- \frac{1}{4\varepsilon^2}(\partial_j \eta_\varepsilon^4)(1-|w|^2)^2 + \frac{1}{2}\partial_j |h|^2.$$

Now we use the commutator relation from Lemma A.4

$$\partial_i^A \partial_j^A w - \partial_j^A \partial_i^A w = -iw(\partial_i A_j - \partial_j A_i) = -iwh\mathbb{J}_{ij}$$

where $\mathbb{J}$ is the symplectic matrix

$$\mathbb{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

which just rotates a vector $\frac{\pi}{2}$ anticlockwise in the plane ($\mathbb{J}v = v^\perp$). Using this we have

$$(\operatorname{div} T_\varepsilon^{\eta_\varepsilon})_j = \eta_\varepsilon^2 \langle (\partial_i^A)^2 w + \frac{\eta_\varepsilon^2}{\varepsilon^2}(1-|w|^2)w + (\partial_i \log \eta_\varepsilon^2)\partial_i^A w, \partial_j^A w\rangle$$
$$- \eta_\varepsilon^2 h \mathbb{J}_{ij}\langle iw, \partial_i^A w\rangle - \frac{1}{2}(\partial_j \eta_\varepsilon^2)\langle \partial_i^A w, \partial_i^A w\rangle$$
$$- \frac{1}{4\varepsilon^2}(\partial_j \eta_\varepsilon^4)(1-|w|^2)^2 + \frac{1}{2}\partial_j |h|^2.$$

Using $\mathbb{J}_{ij} = -\mathbb{J}_{ji}$ we may write

$$\operatorname{div} T_\varepsilon^{\eta_\varepsilon} = \eta_\varepsilon^2 \langle \Delta_A w + \nabla \log \eta_\varepsilon^2 \cdot \nabla_A w + \frac{\eta_\varepsilon^2 w}{\varepsilon^2}(1-|w|^2), \nabla_A w\rangle$$
$$- h(\eta_\varepsilon^2 j_A^\perp - \nabla h) - \frac{\nabla \eta_\varepsilon^2}{2}|\nabla_A w|^2 - \frac{\nabla \eta_\varepsilon^4}{4\varepsilon^2}(1-|w|^2)^2. \quad (5.13)$$

The final step is to note that the energy density satisfies

$$-\frac{\nabla \eta_\varepsilon^2}{2}|\nabla_A w|^2 - \frac{\nabla \eta_\varepsilon^4}{4\varepsilon^2}(1-|w|^2)^2 = -\nabla \log \eta_\varepsilon^2 \Big( g_\varepsilon^{\eta_\varepsilon} + \frac{\eta_\varepsilon^4}{4\varepsilon^2}(1-|w|^2)^2 - \frac{1}{2}|h-h_{ex}|^2 \Big). \quad (5.14)$$

Inserting this in to (5.13) provides

$$\begin{aligned}\operatorname{div} T_\varepsilon^{\eta_\varepsilon} &= \eta_\varepsilon^2\langle \Delta_A w + \nabla \log \eta_\varepsilon^2 \cdot \nabla_A w + \frac{\eta_\varepsilon^2 w}{\varepsilon^2}(1-|w|^2), \nabla_A w\rangle \\ &\quad + h(\nabla h - \eta_\varepsilon^2 j_A^\perp) - \nabla \log \eta_\varepsilon^2 \big(g_\varepsilon^{\eta_\varepsilon} + \frac{\eta_\varepsilon^4}{4\varepsilon^2}(1-|w|^2)^2 - \frac{1}{2}|h-h_{ex}|^2\big). \quad (5.15)\end{aligned}$$

The first equation from (2.19) is

$$(\alpha_\varepsilon + i\beta_0)\partial_{\Phi_\varepsilon} w_\varepsilon = \Delta_{A_\varepsilon} w_\varepsilon + \frac{\eta_\varepsilon^2 w_\varepsilon}{\varepsilon^2}(1-|w_\varepsilon|^2) + \nabla \log \eta_\varepsilon^2 \cdot \nabla_{A_\varepsilon} w_\varepsilon,$$

and the second equation from (2.19) gives $\nabla h_\varepsilon - \eta_\varepsilon^2 j_{A_\varepsilon}^\perp = \sigma_\varepsilon E_\varepsilon^\perp$. Use these relations in (5.15) to arrive at

$$\begin{aligned}\operatorname{div} T_\varepsilon^{\eta_\varepsilon} &= \alpha_\varepsilon \eta_\varepsilon^2 \langle \partial_{\Phi_\varepsilon} w_\varepsilon, \nabla_{A_\varepsilon} w_\varepsilon\rangle + \beta_0 \eta_\varepsilon^2 \langle i\partial_{\Phi_\varepsilon} w_\varepsilon, \nabla_{A_\varepsilon} w_\varepsilon\rangle + \sigma_\varepsilon h_\varepsilon E_\varepsilon^\perp \\ &\quad - \nabla \log \eta_\varepsilon^2 \big(g_\varepsilon^{\eta_\varepsilon} + \frac{\eta_\varepsilon^4}{4\varepsilon^2}(1-|w_\varepsilon|^2)^2 - \frac{1}{2}|h_\varepsilon - h_{ex}|^2\big) \quad (5.16)\end{aligned}$$

By (2.28) we have

$$\langle i\partial_{\Phi_\varepsilon} w_\varepsilon, \nabla_{A_\varepsilon} w_\varepsilon\rangle = -V_{A_\varepsilon}^{\Phi_\varepsilon}(w_\varepsilon) + \frac{1}{2}(1-|w_\varepsilon|^2)E_\varepsilon. \quad (5.17)$$

Use (5.17) in (5.16) to arrive at (5.12). □

Equation (5.12) is the basis for the dynamical law for the vortices. We observe that the term $\eta_\varepsilon^2\langle \partial_{\Phi_\varepsilon} w_\varepsilon, \nabla_{A_\varepsilon} w_\varepsilon\rangle$ appears in the equation for the evolution of the energy density (4.9), and so it is a simple matter to incorporate the time-derivative of $g_\varepsilon^{\eta_\varepsilon}(w_\varepsilon, A_\varepsilon)$ in the equation for div $T_\varepsilon^{\eta_\varepsilon}$.

## 5.3 Limiting dynamical law

This final section of the chapter is split into three parts. The first uses the stress-energy tensor to relate the evolution of the energy density and the evolution of the Jacobian (which we write in terms of the curl of the space-time Jacobian) to the gradient of the pinning potential and the gradient of the renormalized energy. We bring these together in what we call the *evolution inequality*. Then we proceed to estimate the extra quantities which appear in

this inequality.

The final act is to determine the limiting dynamical law for the vortex positions. We expect this dynamical law to be an ODE relating the evolution of $\xi(t)$ to forces from the pinning and renormalized energy. In fact, consider $a(t) = (a_1(t), ..., a_n(t))$ to be the solutions to following the ODEs:

$$\pi\alpha_0\dot{a}_j + \pi\beta_0 d_j \dot{a}_j^{\perp} = -\nabla_{a_j} W(a, d) - \pi\nabla Q_0(a_j). \tag{5.18}$$

These $a(t)$ aren't necessary the vortex positions $\xi(t) = (\xi_1(t), ..., \xi_n(t))$ found in Lemma 5.1, but by letting the difference in positions be

$$\omega(t) = \xi(t) - a(t), \tag{5.19}$$

we seek to apply a Gronwall argument to show that $\omega(t) \equiv 0$ for a finite time, hence the vortices do indeed obey (5.18).

The evolution identity which we are about to present is a key component in setting up the Gronwall inequality we need. However, to relate the stress-energy tensor with the evolution of the energy density $g_\varepsilon$, we require a pair of test functions with a special property. We introduce them now and they will be used throughout the rest of the chapter.

**Definition 5.8.** *Suppose we have the test functions $\phi_k^{(j)}, \psi_k^{(j)} \in W_0^{2,\infty}(\Omega)$ where $j = 1, ..., n$ identifies a given vortex and $k = 1, 2$ identifies the (basis) coordinate in the plane. The form of the functions is as follows:*

$$\psi_k^{(j)} = \hat{e}_k \cdot x\chi_r(x - a_j(0))$$
$$\phi_k^{(j)} = \hat{e}_k \cdot x^{\perp}\chi_r(x - a_j(0))$$

*where $(\hat{e}_1, \hat{e}_2)$ are unit vectors and $\chi_r$ is a characteristic function introduced in Appendix D. Often we simply write $(\psi, \phi)$ in place of $(\psi_k^{(j)}, \phi_k^{(j)})$.*

Using the abbreviations

$$B_r^0 := B_r(a_j(0)) \qquad \text{and} \qquad \Omega_r^0 := \Omega \backslash B_r^0,$$

Lemma D.1 establishes that $\nabla\psi = \nabla^\perp\phi$ in $B_r^0$ and the nature of $\chi_r$ ensures that $\psi = \phi = 0$ in $\Omega_{2r}^0$.

Here we present an inequality which relates all of the key quantities.

**Proposition 5.9.** *(Evolution inequality.) Suppose $(w_\varepsilon, A_\varepsilon, \Phi_\varepsilon)$ are solutions to (2.19) and $(\phi, \psi)$ are the test functions from Definition 5.8. Then the following relation holds:*

$$\int_0^\delta \Big| \int_\Omega \Big( \alpha_\varepsilon \psi \partial_t g_\varepsilon^{\eta_\varepsilon} + \beta_0 \nabla^\perp\phi \cdot V_A^\Phi(w) + \nabla^\perp\phi \cdot \nabla(\log\eta_\varepsilon^2) g_\varepsilon^{\eta_\varepsilon} \Big) + \nabla^\perp\phi \cdot \nabla_{\xi_j} W(\xi, d) \Big| \leq \sum_{i=1}^{10} I_i, \quad (5.20)$$

*where*

$$I_1 = \int_0^\delta \left| \int_\Omega \nabla\nabla^\perp\phi : \big(\eta_\varepsilon^2(\nabla_A w \otimes \nabla_A w - j_{A_*} \otimes j_{A_*})\big) \right|$$

$$I_2 = \int_0^\delta \left| \int_\Omega \nabla\nabla^\perp\phi : \big((\eta_\varepsilon^2 - 1) j_{A_*} \otimes j_{A_*}\big) \right|$$

$$I_3 = \alpha_\varepsilon \int_0^\delta \left| \int_\Omega \psi\big(\alpha_\varepsilon \eta_\varepsilon^2 |\partial_{\Phi_\varepsilon} w_\varepsilon|^2 + \sigma_\varepsilon |E_\varepsilon|^2\big) \right|$$

$$I_4 = \alpha_\varepsilon \int_0^\delta \left| \int_\Omega \nabla\psi \cdot (h_\varepsilon - h_{ex}) E_\varepsilon \right|$$

$$I_5 = \beta_0 \int_0^\delta \left| \int_\Omega \nabla^\perp\phi \cdot (1 - \eta_\varepsilon^2) V_{A_\varepsilon}^{\Phi_\varepsilon}(w_\varepsilon) \right|$$

$$I_6 = \frac{\beta_0}{2} \int_0^\delta \left| \int_\Omega \nabla^\perp\phi \cdot \eta_\varepsilon^2 (1 - |w_\varepsilon|^2) E_\varepsilon \right|$$

$$I_7 = \sigma_\varepsilon \int_0^\delta \left| \int_\Omega \nabla^\perp\phi \cdot h_\varepsilon E_\varepsilon^\perp \right|$$

$$I_8 = \int_0^\delta \left| \int_\Omega \nabla^\perp\phi \cdot \nabla(\log\eta_\varepsilon^2) \frac{\eta_\varepsilon^4}{4\varepsilon^2} (1 - |w_\varepsilon|^2)^2 \right|$$

$$I_9 = \frac{1}{2} \int_0^\delta \left| \int_\Omega \nabla^\perp\phi \cdot \nabla(\log\eta_\varepsilon^2) |h_\varepsilon - h_{ex}|^2 \right|$$

$$I_{10} = \alpha_\varepsilon \int_0^\delta \left| \int_{\Omega_r^0} (\nabla^\perp\phi - \nabla\psi) \cdot \eta_\varepsilon^2 \langle \partial_{\Phi_\varepsilon} w_\varepsilon, \nabla_{A_\varepsilon} w_\varepsilon \rangle \right|.$$

*Proof.* Take (5.12), multiply by the test function $\nabla^\perp\phi$ and integrate over $\Omega$:

$$\begin{aligned}\int_\Omega \nabla^\perp\phi\cdot\operatorname{div} T_\varepsilon^{\eta_\varepsilon} &= \alpha_\varepsilon\int_\Omega \nabla^\perp\phi\cdot\eta_\varepsilon^2\langle\partial_{\Phi_\varepsilon}w_\varepsilon,\nabla_{A_\varepsilon}w_\varepsilon\rangle - \beta_0\int_\Omega\nabla^\perp\phi\cdot\eta_\varepsilon^2V_A^\Phi(w)\\ &+\frac{\beta_0}{2}\int_\Omega\nabla^\perp\phi\cdot\eta_\varepsilon^2(1-|w|^2)E+\sigma_\varepsilon\int_\Omega\nabla^\perp\phi\cdot h_\varepsilon E_\varepsilon^\perp-\int_\Omega\nabla^\perp\phi\cdot\nabla(\log\eta_\varepsilon^2)g_\varepsilon^{\eta_\varepsilon}\\ &-\int_\Omega\nabla^\perp\phi\cdot\nabla(\log\eta_\varepsilon^2)\frac{\eta_\varepsilon^4}{4\varepsilon^2}(1-|w_\varepsilon|^2)^2+\frac12\int_\Omega\nabla^\perp\phi\cdot\nabla(\log\eta_\varepsilon^2)|h_\varepsilon-h_{ex}|^2.\end{aligned}\tag{5.21}$$

Integrate by parts on the term on the left-hand side such that we have

$$-\int_\Omega\phi\ \operatorname{curl}\operatorname{div}T_\varepsilon^{\eta_\varepsilon},$$

then we can write the left-hand side in terms of the renormalised energy using (5.11).

Next, consider the first term on the right-hand side of (5.21). Allow us to write $\mathcal{P}_\varepsilon^{\eta_\varepsilon}$ in place of $\eta_\varepsilon^2\langle\partial_{\Phi_\varepsilon}w_\varepsilon,\nabla_{A_\varepsilon}w_\varepsilon\rangle$ for the next few lines. Since $\nabla^\perp\phi=\nabla\psi$ in $B_r^0$, we have that

$$\begin{aligned}\alpha_\varepsilon\int_\Omega\nabla^\perp\phi\cdot\mathcal{P}_\varepsilon^{\eta_\varepsilon}&=\alpha_\varepsilon\int_{\Omega_r^0}\nabla^\perp\phi\cdot\mathcal{P}_\varepsilon^{\eta_\varepsilon}+\alpha_\varepsilon\int_{B_r^0}\nabla\psi\cdot\mathcal{P}_\varepsilon^{\eta_\varepsilon}\\&=\alpha_\varepsilon\int_{\Omega_r^0}(\nabla^\perp\phi-\nabla\psi)\cdot\mathcal{P}_\varepsilon^{\eta_\varepsilon}+\alpha_\varepsilon\int_\Omega\nabla\psi\cdot\mathcal{P}_\varepsilon^{\eta_\varepsilon}.\end{aligned}\tag{5.22}$$

The justification for working with such test functions comes when we integrate by parts on the last integral in (5.22) such that we have

$$-\alpha_\varepsilon\int_\Omega\psi\ \operatorname{div}\big(\eta_\varepsilon^2\langle\partial_{\Phi_\varepsilon}w_\varepsilon,\nabla_{A_\varepsilon}w_\varepsilon\rangle\big).$$

We use (4.9) to write this in terms of $\partial_t g_\varepsilon^{\eta_\varepsilon}$.

So (5.21) is

$$\begin{aligned}-\int_\Omega\phi\ \operatorname{curl}\operatorname{div}T_\varepsilon^{\eta_\varepsilon}&=-\alpha_\varepsilon\int_\Omega\psi\ \operatorname{div}\big(\eta_\varepsilon^2\langle\partial_{\Phi_\varepsilon}w_\varepsilon,\nabla_{A_\varepsilon}w_\varepsilon\rangle\big)-\beta_0\int_\Omega\nabla^\perp\phi\cdot\eta_\varepsilon^2V_A^\Phi(w)\\&+\alpha_\varepsilon\int_{\Omega_r^0}(\nabla^\perp\phi-\nabla\psi)\cdot\eta_\varepsilon^2\langle\partial_{\Phi_\varepsilon}w_\varepsilon,\nabla_{A_\varepsilon}w_\varepsilon\rangle+\frac{\beta_0}{2}\int_\Omega\nabla^\perp\phi\cdot\eta_\varepsilon^2(1-|w|^2)E\\&+\sigma_\varepsilon\int_\Omega\nabla^\perp\phi\cdot h_\varepsilon E_\varepsilon^\perp-\int_\Omega\nabla^\perp\phi\cdot\nabla(\log\eta_\varepsilon^2)g_\varepsilon^{\eta_\varepsilon}-\int_\Omega\nabla^\perp\phi\cdot\nabla(\log\eta_\varepsilon^2)\frac{\eta_\varepsilon^4}{4\varepsilon^2}(1-|w_\varepsilon|^2)^2\\&+\frac12\int_\Omega\nabla^\perp\phi\cdot\nabla(\log\eta_\varepsilon^2)|h_\varepsilon-h_{ex}|^2,\end{aligned}$$

which becomes

$$\begin{aligned}
&\alpha_\varepsilon \int_\Omega \psi \partial_t g_\varepsilon^{\eta_\varepsilon} + \beta_0 \int_\Omega \nabla^\perp \phi \cdot V_A^\Phi(w) + \int_\Omega \nabla^\perp \phi \cdot \nabla(\log \eta_\varepsilon^2) g_\varepsilon^{\eta_\varepsilon} + \nabla^\perp \phi \cdot \nabla_{\xi_j} W \\
&= \int_\Omega \nabla\nabla^\perp \phi : \big(\eta_\varepsilon^2(\nabla_A w \otimes \nabla_A w - j_{A_*} \otimes j_{A_*})\big) + \int_\Omega \nabla\nabla^\perp \phi : \big((\eta_\varepsilon^2 - 1) j_{A_*} \otimes j_{A_*}\big) \\
&\qquad - \alpha_\varepsilon \int_\Omega \psi\big(\alpha_\varepsilon \eta_\varepsilon^2 |\partial_{\Phi_\varepsilon} w_\varepsilon|^2 + \sigma_\varepsilon |E_\varepsilon|^2\big) + \alpha_\varepsilon \int_\Omega \nabla^\perp \psi \cdot (h_\varepsilon - h_{ex}) E_\varepsilon \\
&\qquad + \beta_0 \int_\Omega \nabla^\perp \phi \cdot (1 - \eta_\varepsilon^2) V_A^\Phi(w) + \alpha_\varepsilon \int_{\Omega_r^0} (\nabla^\perp \phi - \nabla \psi) \cdot \eta_\varepsilon^2 \langle \partial_{\Phi_\varepsilon} w_\varepsilon, \nabla_{A_\varepsilon} w_\varepsilon \rangle \\
&\qquad\qquad + \frac{\beta_0}{2} \int_\Omega \nabla^\perp \phi \cdot \eta_\varepsilon^2 (1 - |w|^2) E + \sigma_\varepsilon \int_\Omega \nabla^\perp \phi \cdot h_\varepsilon E_\varepsilon^\perp \\
&\qquad - \int_\Omega \nabla^\perp \phi \cdot \nabla(\log \eta_\varepsilon^2) \frac{\eta_\varepsilon^4}{4\varepsilon^2} (1 - |w_\varepsilon|^2)^2 + \frac{1}{2} \int_\Omega \nabla^\perp \phi \cdot \nabla(\log \eta_\varepsilon^2) |h_\varepsilon - h_{ex}|^2.
\end{aligned}$$

Take absolute values and integrate in time between 0 and $\delta$ to get (5.20). □

To facilitate passing to the limit in the terms of the evolution inequality we first estimate some of the quantities. In the next series of propositions and lemmas we record the estimates which are used in the final part of this chapter. The reader may choose to skip to the work which proceeds Lemma 5.14 and refer back to these estimates when necessary.

We would like to use the coercivity estimate from Theorem 3.9. However, we must show that $\|A_\varepsilon - A_*\|_{L^q(\Omega)} \to 0$ $(q > 2)$ as $\varepsilon \to 0$ for the hypotheses of that Theorem to be satisfied. This is the object of the next proposition.

**Proposition 5.10.** *Suppose $(w_\varepsilon, A_\varepsilon, \Phi_\varepsilon)$ are solutions to (2.19), $A_*$ is defined by (2.45) and both $A_\varepsilon$ and $A_*$ satisfy the Coulomb gauge. Then, if $1 \le p < 2$, it follows that*

$$\int_0^T \|h_\varepsilon(t) - h_*(t)\|_{W^{1,p}(\Omega)}^p \, dt \to 0 \qquad \text{as } \varepsilon \to 0, \tag{5.23}$$

*and if $q > 2$,*

$$\int_0^T \|A_\varepsilon(t) - A_*(t)\|_{L^q(\Omega)} \, dt \to 0 \qquad \text{as } \varepsilon \to 0. \tag{5.24}$$

*Proof.* The fact that Lemma 2.13 provides

$$\left\|\mathrm{curl}\big((\eta_\varepsilon^2-1)j_{A_\varepsilon}(w_\varepsilon)\big)\right\|_{\dot{W}^{-1,1}(\Omega)}=o_\varepsilon(1)$$

means we can follow Section 2.3 of [35] to achieve the results. □

We require a result on the compactness of the supercurrent in order to establish convergence in later propositions.

**Proposition 5.11.** *Suppose $(w_\varepsilon, A_\varepsilon, \Phi_\varepsilon)$ are solutions to (2.19). Then for any $\chi \in W_0^{1,4}(\Omega\times(0,\delta))$, there holds*

$$\left|\int_0^\delta\int_\Omega \nabla\chi\cdot j_{A_\varepsilon}(w_\varepsilon)\right| \le C\,\|\chi\|_{W_0^{1,4}(\Omega\times(0,\delta))}\left(\sqrt{\alpha_\varepsilon}+\varepsilon\sqrt{G_\varepsilon^{\eta_\varepsilon}}\right). \tag{5.25}$$

*Proof.* Multiply the first equation of (2.19) by $i\eta_\varepsilon^2 w_\varepsilon$ and use identities from Appendix A to get

$$\alpha_\varepsilon\eta_\varepsilon^2\langle iw_\varepsilon,\partial_{\Phi_\varepsilon}w_\varepsilon\rangle+\frac{1}{2}\beta_0\eta_\varepsilon^2\partial_t\big(|w_\varepsilon|^2-1\big)=\mathrm{div}\big(\eta_\varepsilon^2 j_{A_\varepsilon}(w_\varepsilon)\big).$$

Testing against $\chi\in W_0^{1,4}(\Omega\times(0,\delta))$ provides the following estimate

$$\int_0^\delta\int_\Omega \chi\,\mathrm{div}\big(\eta_\varepsilon^2 j_{A_\varepsilon}(w_\varepsilon)\big)=\int_0^\delta\int_\Omega \alpha_\varepsilon\eta_\varepsilon^2\chi\langle iw_\varepsilon,\partial_{\Phi_\varepsilon}w_\varepsilon\rangle-\frac{\beta_0}{2}\eta_\varepsilon^2(|w_\varepsilon|^2-1)\partial_t\chi$$

$$\begin{aligned}\le\sqrt{\alpha_\varepsilon}\,\|\eta_\varepsilon\|_{L^\infty(\Omega)}\,\|\chi\|_{L^4(\Omega\times(0,\delta))}\,\|w_\varepsilon\|_{L^4(\Omega\times(0,\delta))}\,\|\sqrt{\alpha_\varepsilon}\eta_\varepsilon|\partial_{\Phi_\varepsilon}w_\varepsilon|\|_{L^2(\Omega\times(0,\delta))}\\ +C\beta_0\left\|\eta_\varepsilon^2(1-|w_\varepsilon|^2)\right\|_{L^2(\Omega\times(0,\delta))}\|\partial_t\chi\|_{L^4(\Omega\times(0,\delta))}\\ \le C\,\|\chi\|_{W_0^{1,4}(\Omega\times(0,\delta))}\left(\sqrt{\alpha_\varepsilon}+\varepsilon\sqrt{G_\varepsilon^{\eta_\varepsilon}}\right)\end{aligned}$$

where we have used the dissipation bound $\left\|\sqrt{\alpha_\varepsilon}\eta_\varepsilon|\partial_{\Phi_\varepsilon}w_\varepsilon|\right\|_{L^2}\le C$ from (4.19). We have that

$$\left|\int_0^\delta\int_\Omega \nabla\chi\cdot\eta_\varepsilon^2 j_{A_\varepsilon}(w_\varepsilon)\right|\le C\,\|\chi\|_{W_0^{1,4}(\Omega\times(0,\delta))}\left(\sqrt{\alpha_\varepsilon}+\varepsilon\sqrt{G_\varepsilon^{\eta_\varepsilon}}\right)$$

and so by using that $\|\eta_\varepsilon^2\|_{L^\infty(\Omega)}\le C$, we have the result. □

Recall from (2.1) that $r_a:=\frac{1}{4}\min\{\min_{i\neq j}|a_i-a_j|,\min_i \mathrm{dist}(a_i,\partial\Omega)\}$. Then we have

**Proposition 5.12.** *Suppose $(w_\varepsilon, A_\varepsilon, \Phi_\varepsilon)$ are solutions to (2.19), $w_*$ satisfies (2.42), $A_*$ satisfies (2.45) and the initial data is well-prepared (see (2.52)). Additionally, suppose that both $A_\varepsilon$ and $A_*$ satisfy the Coulomb gauge condition. Let $r_0 < \frac{1}{4} r_{a(0)}$ and $\delta > 0$ satisfy*

$$\sup_{t\in[0,\delta]} |\omega(t)| + \delta \sup_{t\in[0,\delta]} |\dot{a}(t)| \leq \frac{r_0}{4}. \tag{5.26}$$

*Then there exists a constant $C > 0$ with the following property: for any $\phi \in W_0^{2,\infty}(\Omega)$ that is affine in $\bigcup_i B_{r_0}(a_i(0))$ and satisfies $\|D^2\phi\|_{L^\infty(\Omega)} \leq C/r_0^2$, the following estimate holds:*

$$\int_0^\delta \left| \int_\Omega \nabla\nabla^\perp \phi : \eta_\varepsilon^2 \big( \nabla_{A_\varepsilon} w_\varepsilon \otimes \nabla_{A_\varepsilon} w_\varepsilon - j_{A_*}(w_*) \otimes j_{A_*}(w_*) \big) \right| \leq C \int_0^\delta D_\varepsilon(t) + o_\varepsilon(1). \tag{5.27}$$

*Proof.* Well-prepared solutions of (2.19) imply the results of Theorem 4.5, Lemma 5.1 and Proposition 5.10, which satisfy the requirements for the coercivity estimate of Theorem 3.9 to hold. Lemma 5.1 implies that $s_\varepsilon$ from Theorem 3.9 converges to 0 as $\varepsilon \to 0$. In turn, this implies that $r_\star \to 0$ as $\varepsilon \to 0$. Let $r_0 > r_\star$, where $r_0$ is independent of $\varepsilon$. The coercivity estimate (3.22) now implies that

$$\int_{\Omega_{r_0}(\xi(t))} e_\varepsilon^{\eta_\varepsilon}(|w_\varepsilon|) + \frac{\eta_\varepsilon^2}{2} \left| \frac{j_{A_\varepsilon}(w_\varepsilon)}{|w_\varepsilon|} - j_{A_*}(w_*) \right|^2 \leq D_\varepsilon(t) + o_\varepsilon(1). \tag{5.28}$$

Condition (5.26) ensures that the vortex balls don't overlap for up to time $\delta$, meaning the domain of integration covers the same area for all $t \in [0, \delta]$ despite the vortex positions changing in time. In fact, the balls don't move much from their initial positions. In other words, (5.28) holds for all $t \in [0, \delta]$.

We can now follow the proof of Proposition 3.5 of [35] with the key difference involving the treatment of the pinning potential in estimating

$$\int_\Omega \nabla\nabla^\perp \phi : \eta_\varepsilon^2 (j_A - j_{A_*}) \cdot j_{A_*}.$$

Consider that if $\|\eta_\varepsilon^2\|_{L^\infty(\Omega)} \leq C$ then

$$\int_\Omega \nabla\nabla^\perp \phi : \eta_\varepsilon^2 (j_A - j_{A_*}) \cdot j_{A_*} \leq C \int_\Omega \nabla\nabla^\perp \phi : (j_A - j_{A_*}) \cdot j_{A_*}.$$

Since our (3.62) and Proposition 5.11 match (4.8) and Proposition 3.6 from [35], we can follow their proof without making any changes. □

**Proposition 5.13.** *Let $\phi \in W_0^{2,\infty}(\Omega)$ satisfy the properties in Proposition 5.12. If $w_*$ is defined by (2.42) and $A_*$ is defined by (2.45), then*

$$\int_0^\delta \left| \int_\Omega \nabla\nabla^\perp \phi : \left( (1-\eta_\varepsilon^2) j_{A_*}(w_*) \otimes j_{A_*}(w_*) \right) \right| \leq o_\varepsilon(1).$$

*Proof.* Recall that $\|1-\eta_\varepsilon^2\|_{L^\infty} \leq C|\log\varepsilon|^{-1}$, and using the property that $\phi$ is affine on the balls allows us to write

$$\int_\Omega \nabla\nabla^\perp \phi : \left( (1-\eta_\varepsilon^2) j_{A_*}(w_*) \otimes j_{A_*}(w_*) \right) \leq \frac{C}{r_0^2 |\log\varepsilon|} \left\| j_{A_*}(w_*) \right\|^2_{L^\infty(\Omega_{r_0}(a_i(0)))}. \tag{5.29}$$

From (3.7) we have

$$\|j_{A_*}(u_*)\|_{L^\infty(\Omega_r)} \leq C\Big(\frac{n}{r} + h_{ex}\Big)$$

which leads to the desired result when in combination with (5.29). □

**Lemma 5.14.** *For $I_3$-$I_{10}$ defined as in Lemma 5.9, $\sum_{i=3}^{10} I_i \leq o_\varepsilon(1)$.*

*Proof.* We choose test functions $(\psi, \phi)$ such that

$$|\psi| + |\phi| \leq C \qquad \text{and} \qquad |\nabla\phi| + |\nabla\psi| \leq \frac{C}{r}. \tag{5.30}$$

For $\varepsilon$ sufficiently small, we have that

$$\left| \int_\Omega \eta_\varepsilon^2 \right| \leq C \qquad \text{and} \qquad \nabla\eta_\varepsilon^2 = \frac{\nabla Q_\varepsilon}{|\log\varepsilon|} \leq o_\varepsilon(1).$$

We recall from (4.19) that the dissipation is bounded like so

$$\int_0^\delta \int_\Omega \alpha_\varepsilon \eta_\varepsilon^2 |\partial_{\Phi_\varepsilon} w_\varepsilon|^2 + \sigma_\varepsilon |E_\varepsilon|^2 \leq C. \tag{5.31}$$

Using (5.31) with the fact that $\alpha_\varepsilon \to 0$, we have that $I_3 \leq o_\varepsilon(1)$.

From (5.24), $h_\varepsilon(t)$ is uniformly bounded in $L^2(\Omega)$ for all $t$; we can therefore use (5.31) and the Cauchy-Schwarz inequality to write

$$I_4 \leq \frac{C\alpha_0}{r|\log\varepsilon|}\left(\int_0^\delta\int_\Omega |h-h_{ex}|^2\right)^{\frac{1}{2}}\left(\int_0^\delta\int_\Omega |E|^2\right)^{\frac{1}{2}} \leq \frac{C}{\sqrt{|\log\varepsilon|}}.$$

Thus, $I_4 \leq o_\varepsilon(1)$.

To estimate $I_5$ we note that $\nabla(\eta_\varepsilon^2-1)\to 0$ and so $\|\nabla\psi(\eta_\varepsilon^2-1)\|_{C^{0,1}(\Omega\times[0,\delta])}\to 0$. Using this with the fact that $V_{A_\varepsilon}^{\Phi_\varepsilon}(w_\varepsilon)$ converges in $(C^{0,1}(\Omega\times[0,\delta]))^*$ (see Lemma 5.1), we have $I_5 \leq o_\varepsilon(1)$.

The bound $\int_\Omega \eta_\varepsilon^4(1-|w_\varepsilon|^2)^2 \leq 4\varepsilon^2 G_\varepsilon^{\eta_\varepsilon} \leq C\varepsilon^2|\log\varepsilon|$ and Hölder's inequality, together with (5.31), show $I_6 \leq o_\varepsilon(1)$. Arguing as we did when estimating $I_4$, we have that $I_7 \leq o_\varepsilon(1)$. We note that (3.12) and (3.55) imply that

$$\int_\Omega \frac{1}{4\varepsilon^2}(1-|w|^2)^2 \leq C.$$

Thus,

$$I_8 \leq C\frac{\|\nabla Q_\varepsilon\|_{L^\infty(\Omega)}}{|\log\varepsilon|}\left\|\frac{1}{4\varepsilon^2}(1-|w|^2)^2\right\|_{L^1(\Omega\times[0,\delta])} \leq \frac{C}{|\log\varepsilon|} \leq o_\varepsilon(1).$$

Finally, knowing that $(h-h_{ex})$ is bounded for all $t$ from (5.24),

$$I_9 \leq C\frac{\|\nabla Q_\varepsilon\|_{L^\infty(\Omega)}}{|\log\varepsilon|}\|h-h_{ex}\|^2_{L^2(\Omega\times[0,\delta])} \leq \frac{C}{|\log\varepsilon|} \leq o_\varepsilon(1).$$

It remains to show

$$I_{10} = \alpha_\varepsilon\int_0^\delta\left|\int_{\Omega_r^0}(\nabla^\perp\phi-\nabla\psi)\cdot\eta_\varepsilon^2\langle\partial_{\Phi_\varepsilon}w_\varepsilon,\nabla_{A_\varepsilon}w_\varepsilon\rangle\right| \leq o_\varepsilon(1).$$

The Cauchy-Schwarz inequality implies that

$$\alpha_\varepsilon\int_0^\delta\int_{\Omega_r^0}\eta_\varepsilon^2\langle\partial_{\Phi_\varepsilon}w_\varepsilon,\nabla_{A_\varepsilon}w_\varepsilon\rangle \leq \sqrt{\alpha_\varepsilon}\Big(\int_0^\delta\int_\Omega \alpha_\varepsilon\eta_\varepsilon^2|\partial_{\Phi_\varepsilon}w_\varepsilon|^2\Big)^{\frac{1}{2}}\Big(\int_0^\delta\int_{\Omega_r^0}\eta_\varepsilon^2|\nabla_{A_\varepsilon}w_\varepsilon|^2\Big)^{\frac{1}{2}}.$$

By the dissipation bound we have

$$\alpha_\varepsilon\int_0^\delta\int_{\Omega_r^0}\eta_\varepsilon^2\langle\partial_{\Phi_\varepsilon}w_\varepsilon,\nabla_{A_\varepsilon}w_\varepsilon\rangle \leq C\sqrt{\alpha_\varepsilon}\Big(\int_0^\delta\int_{\Omega_r^0}\eta_\varepsilon^2|\nabla_{A_\varepsilon}w_\varepsilon|^2\Big)^{\frac{1}{2}}. \tag{5.32}$$

Using the decomposition $|\nabla_A w|^2 = |\nabla|w||^2 + \left|\frac{j_A(w)}{|w|}\right|$, it follows that

$$\int_{\Omega_r^0} \eta_\varepsilon^2 |\nabla_{A_\varepsilon} w_\varepsilon|^2 \leq \int_{\Omega_r^0} e_\varepsilon^{\eta_\varepsilon}(|w_\varepsilon|) + \eta_\varepsilon^2 \left| \frac{j_{A_\varepsilon}(w_\varepsilon)}{|w_\varepsilon|} - j_{A_*}(w_*) \right|^2 + \eta_\varepsilon^2 |j_{A_*}(w_*)|^2,$$

and by (5.28) (for $r > r_0$)

$$\int_{\Omega_r^0} \eta_\varepsilon^2 |\nabla_{A_\varepsilon} w_\varepsilon|^2 \leq \int_{\Omega_r^0} \eta_\varepsilon^2 |j_{A_*}(w_*)|^2 + C D_\varepsilon(t) + o_\varepsilon(1).$$

From (3.7) we deduce that

$$\int_{\Omega_r^0} \eta_\varepsilon^2 |j_{A_*}(w_*)|^2 \leq C |\log r|$$

which means that ultimately (5.32) becomes

$$\alpha_\varepsilon \int_0^\delta \int_{\Omega_r^0} \eta_\varepsilon^2 \langle \partial_{\Phi_\varepsilon} w_\varepsilon, \nabla_{A_\varepsilon} w_\varepsilon \rangle \leq C \sqrt{\alpha_\varepsilon \int_0^\delta D_\varepsilon(t) + |\log r| + C}.$$

Provided $D_\varepsilon(t)$ stays bounded, $I_{10} \leq o_\varepsilon(1)$ and the proof is completed. □

We almost have everything we need to determine the limiting dynamical law. Recall (5.18) and (5.19) and consider the difference in ODEs

$$\mathscr{R}_j = \pi \alpha_0 \dot{\omega}_j + \pi \beta_0 d_j \dot{\omega}_j^\perp \tag{5.33}$$

$$= \pi \alpha_0 \dot{\xi}_j + \pi \beta_0 d_j \dot{\xi}_j^\perp + \nabla_{a_j} W(a(t), d) + \pi \nabla Q_0(a_j). \tag{5.34}$$

The aim of this subsection is to show that $\mathscr{R}_j \equiv 0$ for a sufficiently long time, meaning that the vortices follow trajectories determined by (5.18). As a first step, we have the following relation between $\mathscr{R}_j$, $D_\varepsilon(t)$ and $\omega(t)$:

**Proposition 5.15.** *For $0 \leq t_1 \leq t_2 \leq T_0$, where $T_0$ is the time found in Theorem 4.5, the excess energy satisfies*

$$D_\varepsilon(t_2) + |\omega(t_2)| \leq D_\varepsilon(t_1) + |\omega(t_1)| + C \int_{t_1}^{t_2} \sum_{j=1}^n |\mathscr{R}_j| + o_\varepsilon(1), \tag{5.35}$$

*where $C$ only depends on $\min_{t \in [t_1, t_2]} r_{a(t)}$.*

*Proof.* Using the notation from (2.50), by (5.18) we have that

$$\frac{d}{dt}W^Q(a,d) = \sum_{j=1}^{n} \dot{a}_j \cdot \nabla_{a_j} W^Q(a,d) = -\alpha_0 \pi \sum_{j=1}^{n} |\dot{a}_j|^2 . \tag{5.36}$$

We calculate the difference in energy loses using (4.10) and obtain by means of (5.36)

$$\begin{aligned} D_\varepsilon(t_2) &= G_\varepsilon^{\eta_\varepsilon}(t_2) - n(\gamma_0 + |\log \varepsilon|) - W^Q(a(t_2)) \\ &= G_\varepsilon^{\eta_\varepsilon}(t_2) - G_\varepsilon^{\eta_\varepsilon}(t_1) + W^Q(a(t_1)) - W^Q(a(t_2)) + D_\varepsilon(t_1) \\ &= -\int_{t_1}^{t_2} \int_\Omega \left( \alpha_\varepsilon \eta_\varepsilon^2 |\partial_{\Phi_\varepsilon} w_\varepsilon|^2 + \sigma_\varepsilon |E_\varepsilon|^2 \right) + \alpha_0 \pi \int_{t_1}^{t_2} \sum_{i=1}^{n} |\dot{a}_j|^2 + D_\varepsilon(t_1). \end{aligned}$$

Apply the vortex mobility bound (see (2.8) in [35]),

$$\liminf_{\varepsilon \to 0} \int_{t_1}^{t_2} \int_\Omega \alpha_\varepsilon \eta_\varepsilon^2 |\partial_{\Phi_\varepsilon} w_\varepsilon|^2 \geq \alpha_0 \pi \int_{t_1}^{t_2} \sum_{i=1}^{n} \left| \dot{\xi}_j \right|^2 ,$$

and drop the electric field term, such that we have

$$\begin{aligned} D_\varepsilon(t_2) - D_\varepsilon(t_1) &\leq -\alpha_0 \pi \int_{t_1}^{t_2} \sum_{j=1}^{n} \left( |\dot{\xi}_j|^2 - |\dot{a}_j|^2 \right) + o_\varepsilon(1) \\ &= -\alpha_0 \pi \int_{t_1}^{t_2} \sum_{j=1}^{n} \left( \dot{\xi}_j - \dot{a}_j \right) \cdot \left( \dot{\xi}_j + \dot{a}_j \right) + o_\varepsilon(1) \\ &= -\alpha_0 \pi \int_{t_1}^{t_2} \sum_{j=1}^{n} \dot{\omega}_j \cdot \left( \dot{\omega}_j + 2\dot{a}_j \right) + o_\varepsilon(1) \\ &= -\alpha_0 \pi \int_{t_1}^{t_2} \sum_{j=1}^{n} \left( |\dot{\omega}_j|^2 + 2\dot{\omega}_j \cdot \dot{a}_j \right) + o_\varepsilon(1) \\ &\leq 0 + C \sum_{j=1}^{n} \sup_{[t_1,t_2]} |\dot{a}_j(t)| \int_{t_1}^{t_2} |\dot{\omega}_j| + o_\varepsilon(1). \end{aligned}$$

We see from (5.36) that $|\dot{a}_j(t)|$ can be controlled by $|\nabla_{a_j} W^Q(a,d)|$. Equation (5.5) from [35] provides the bound

$$|\nabla_{a_j} W(a,d)| \leq C\Big( \frac{n}{r_a} + h_{ex} \Big). \tag{5.37}$$

Use (5.37) and the *a priori* assumptions that $\nabla Q_0$ is bounded to write

$$D_\varepsilon(t_2) - D_\varepsilon(t_1) \leq C \int_{t_1}^{t_2} \sum_{j=1}^{n} |\dot{\omega}_j| + o_\varepsilon(1). \tag{5.38}$$

Now we use that $|\dot{\omega}_j| = C|\mathscr{R}_j|$ (this comes from (5.33) - see the proof of Proposition 3.1 in [35]) to arrive at

$$D_\varepsilon(t_2) - D_\varepsilon(t_1) \leq C \int_{t_1}^{t_2} \sum_{j=1}^{n} |\mathscr{R}_j| + o_\varepsilon(1). \tag{5.39}$$

Additionally, we have that

$$|\omega(t_2)| - |\omega(t_1)| = \int_{t_1}^{t_2} \frac{d}{dt}|\omega| \leq \int_{t_1}^{t_2} |\dot{\omega}| \leq C \int_{t_1}^{t_2} \sum_{j=1}^{n} |\mathscr{R}_j|. \tag{5.40}$$

Adding (5.39) to (5.40) gives the desired result. □

Now if we can show that $\sum |\mathscr{R}_j|$ is bounded above by $D_\varepsilon(t) + \omega(t)$, then can apply a Gronwall inequality argument (see Appendix E) to show $\mathscr{R}_j \equiv 0$ for all $j$. To facilitate this we introduce

$$\widetilde{\mathscr{R}}_j := \pi\alpha_0\dot{\xi}_j + \pi\beta_0 d_j \dot{\xi}_j^{\perp} + \nabla_{\xi_j} W(\xi(t), d) + \pi\nabla Q_0(\xi_j), \tag{5.41}$$

which leads to the bound

$$\left|\mathscr{R}_j - \widetilde{\mathscr{R}}_j\right| = \left|\nabla_{a_j} W^Q(a(t), d) - \nabla_{\xi_j} W^Q(\xi(t), d)\right| \leq \frac{C}{r_{a(t)}^2}|\omega_j(t)|. \tag{5.42}$$

Notice that by using the reverse triangle inequality and then (5.42), we have

$$|\mathscr{R}_j| \leq |\widetilde{\mathscr{R}}_j| + |\mathscr{R}_j - \widetilde{\mathscr{R}}_j| \leq |\widetilde{\mathscr{R}}_j| + \frac{C}{r_{a(t)}^2}|\omega_j|. \tag{5.43}$$

The next proposition parleys the estimates of the previous subsection together to estimate $\widetilde{\mathscr{R}_j}$.

**Proposition 5.16.** *For the test functions of Definition 5.8 we have the following bound:*

$$\int_0^\delta \left|\widetilde{\mathscr{R}}_j\right| \leq C \int_0^\delta \liminf_{\varepsilon\to 0} D_\varepsilon(t). \tag{5.44}$$

*Proof.* From the results of the previous subsection we have that (5.20) is bounded like so

$$\int_0^\delta \Big| \int_\Omega \Big( \alpha_\varepsilon \psi \partial_t g_\varepsilon^{\eta_\varepsilon} + \beta_0 \nabla^\perp\phi \cdot V_{A_\varepsilon}^{\Phi_\varepsilon}(w_\varepsilon) + \nabla^\perp\phi \cdot \nabla(\log\eta_\varepsilon^2) g_\varepsilon^{\eta_\varepsilon} \Big) + \nabla^\perp\phi \cdot \nabla_{\xi_j} W(\xi, d) \Big| \leq \int_0^\delta D_\varepsilon(t) + o_\varepsilon(1). \tag{5.45}$$

We now want to understand the behaviour of the left-hand side as $\varepsilon \to 0$. By Lemmas 5.1 and 5.2 we obtain the weak convergence

$$\partial_t \int_\Omega \alpha_\varepsilon \psi_k^{(j)} g_\varepsilon^{\eta_\varepsilon} + \beta_0 \int_\Omega \nabla \psi_k^{(j)} \cdot V_{A_\varepsilon}^{\Phi_\varepsilon}(w_\varepsilon) \rightharpoonup \pi \alpha_0 \partial_t \psi_k^{(j)}(\xi_j) + \pi \beta_0 d_j \nabla \psi_k^{(j)}(\xi_j) \cdot \dot{\xi}_j^\perp$$
$$= \pi \nabla \psi_k^{(j)}(\xi_j) \cdot \big(\alpha_0 \dot{\xi}_j + \beta_0 d_j \dot{\xi}_j^\perp\big). \tag{5.46}$$

We note that $\nabla \psi_k^{(j)} = \nabla^\perp \phi_k^{(j)} = \delta_{ik} e_i$ for $k = 1, 2$ (see Lemma D.1); hence, the right-hand side of (5.46) just consists of the two components of the vector $\pi\big(\alpha_0 \dot{\xi}_j + \beta_0 d_j \dot{\xi}_j^\perp\big)$. Applying Lemma 5.2 once more, we have that

$$\int_\Omega \nabla \psi_k^{(j)} \cdot \nabla(\log \eta_\varepsilon^2) g_\varepsilon^{\eta_\varepsilon} = \int_\Omega \nabla \psi_k^{(j)} \cdot \nabla Q_\varepsilon \frac{g_\varepsilon^{\eta_\varepsilon}}{|\log \varepsilon|} + \int_\Omega \nabla \psi_k^{(j)} \cdot \nabla Q_\varepsilon \frac{g_\varepsilon^{\eta_\varepsilon}}{|\log \varepsilon|} \frac{1 - \eta_\varepsilon^2}{\eta_\varepsilon^2}$$
$$\rightharpoonup \pi \nabla \psi_k^{(j)}(\xi_j) \cdot \nabla Q_0(\xi_j).$$

Noting the previous convergences, from (5.45) we have that

$$\int_0^\delta \big|\widetilde{\mathscr{R}}_j\big| \le \int_0^\delta \liminf_{\varepsilon \to 0} \Big| \int_\Omega \big(\alpha_\varepsilon \psi \partial_t g_\varepsilon^{\eta_\varepsilon} + (\beta_0 V_{A_\varepsilon}^{\Phi_\varepsilon}(w_\varepsilon) + \nabla(\log \eta_\varepsilon^2) g_\varepsilon^{\eta_\varepsilon}) \cdot \nabla \psi\big)$$
$$+ \nabla \psi \cdot \nabla_{\xi_j} W(\xi, d)\Big|$$
$$\le C \int_0^\delta \liminf_{\varepsilon \to 0} D_\varepsilon(t).$$

□

Now we present the main result.

**Theorem 5.17.** *Choose $T_1 < \min\{\delta, T_0\}$ where $\delta$ is as in Proposition 5.12 and $T_0$ as in Theorem 4.5. Then the vortex positions $\xi_i(t) \equiv a_i(t)$ satisfy the limiting ODE's*

$$\alpha_0 \dot{a}_i(t) + \beta_0 d_i \dot{a}_i^\perp(t) = -\frac{1}{\pi} \nabla_{a_i} W(a, d) - \nabla Q_0(a) \qquad \forall t \in [0, T_1]. \tag{5.47}$$

*Proof.* Recall that $\limsup_{\varepsilon \to 0} D_\varepsilon(0) = 0$ for well-prepared data and the assumption that $\omega(0) = 0$. Using this in Proposition 5.15 with $t_1 = 0$ and $t_2 = t$ leads to the inequality

$$\limsup_{\varepsilon \to 0} D_\varepsilon(t) + |\omega(t)| \le C \int_0^t \sum_{j=1}^n |\mathscr{R}_j|. \tag{5.48}$$

By Proposition 5.16 and (5.43) we have

$$\int_0^t \sum_{j=1}^n |\mathscr{R}_j| \leq C \int_0^t \Big( \liminf_{\varepsilon\to 0} D_\varepsilon(s) + \sum_{j=1}^n \frac{C}{r_{a(t)}^2} |\omega_j(s)| \Big) ds. \tag{5.49}$$

Due to the condition (5.26), the vortex balls don't overlap during the time $[0, \delta]$. This means that $r_a(t) > C$ for all $t \in [0, \delta]$. Hence, using (5.49) in (5.48) gives

$$\limsup_{\varepsilon\to 0} D_\varepsilon(t) + |\omega(t)| \leq C \int_0^t \Big( \liminf_{\varepsilon\to 0} D_\varepsilon(s) + C|\omega(s)| \Big) ds. \tag{5.50}$$

It follows from Gronwall's inequality (see Appendix E) that $D_\varepsilon(t) \to 0$ for all $t \in [0, T_1]$ and that $\omega(t) \equiv 0$, meaning $\xi(t) = a(t)$. Using this information in (5.49) provides $\mathscr{R}_j = 0$ for all $t \in [0, T_1]$ and all $j = 1, .., n$. □

The limiting dynamical law given by (5.47) contains both gradient descent motion and conservative motion. Ostensibly, the first order time derivative of the order parameter from the TDGL equation has become the vortex velocity in the limit. Interestingly, the conservative motion term depends on the degree of the vortex and so vortices with opposite degrees travel in opposite directions to each other.

Another interesting feature of the limiting dynamical law in comparison to what exists in the literature is that our critically scaled pinning potential manifests itself as $\nabla Q_0(a_i)$, whereas elsewhere in the literature we find terms of the form $\nabla \log \eta_0^2(a_i)$, if $\lim_{\varepsilon\to 0} \eta_\varepsilon^2 = \eta_0^2$. We would like to comment that our result is actually consistent with this logarithmic form of the pinning force since the critically scaled pinning potential satisfies

$$\begin{aligned} \nabla \log \eta_\varepsilon^2 &= \frac{1}{\eta_\varepsilon^2} \nabla \eta_\varepsilon^2 \\ &= \frac{1}{\eta_\varepsilon^2} \frac{1}{|\log \varepsilon|} \nabla Q_\varepsilon \\ &= \frac{1}{|\log \varepsilon|} \nabla Q_\varepsilon + \frac{1 - \eta_\varepsilon^2}{\eta_\varepsilon^2} \frac{1}{|\log \varepsilon|} \nabla Q_\varepsilon \end{aligned}$$

and since $|1 - \eta_\varepsilon^2| \leq C/|\log \varepsilon|$, we have that

$$\frac{1 - \eta_\varepsilon^2}{\eta_\varepsilon^2} \leq \frac{\frac{C}{|\log \varepsilon|}}{1 + \frac{C}{|\log \varepsilon|}} = \frac{C}{C + |\log \varepsilon|}.$$

Since the last term is $O(|\log\varepsilon|^{-2})$ it is negligible in the limiting quantities (recall that the $O(|\log\varepsilon|^{-1})$ is part of the normalized energy density which converges to vortex positions). All of this is to say that in the limit $\varepsilon \to 0$, $\nabla \log \eta_\varepsilon^2$ is actually presented as $\nabla Q_0$ when $\eta_\varepsilon^2$ is critically scaled, so our results are closer to the pinning forced dynamical laws which currently exist in the literature [10, 41, 52] than it first appears.

# Chapter 6

# Maxwell-Klein-Gordon with pinned Ginzburg-Landau potential

Now we study a different type of dynamics for $(u_\varepsilon, A_\varepsilon)$. The preceding chapters have set out the program for deriving the dynamical law for vortices starting from the TDGL equations and now we seek to follow the same program but from a different starting point; namely, the Maxwell-Klein-Gordon equations supplemented with a pinned Ginzburg-Landau potential (MKG-pGL for short). The key difference is that the MKG-pGL equations are second order in time. However, because the equations contain a lot of the structure of the GL equations we can appeal to the estimates and convergences of quantities in the TDGL model and apply them in this new environment. The aim of the rest of the thesis is to determine if the methods of TDGL readily adapt to the new framework and ultimately arrive at an equation of motion for vortices from these different starting equations.

In what follows, the domains $(\Omega, \Omega_r)$, potentials $(A_\varepsilon, \Phi_\varepsilon)$, fields $(E_\varepsilon, h_\varepsilon)$ and covariant derivatives $(\nabla_A, \partial_\Phi)$ are defined as they were in Section 2.1. Again, $u(x,t) : \Omega \times \mathbb{R}^+ \to \mathbb{C}$ is an order parameter but in the MKG setting, $|u|^2$

describes the charge-density of the particles rather than their number-density, as was the case in the TDGL model. Where the TDGL equations are used to describe phenomenon in superconductors, the MKG equations have relevance in the theory of particle physics, so we shift to a new physical setting for the next chapters. However, our interest is in the structure of the equations rather than their physical significance, so the fact that the *Ginzburg-Landau* (or *"Abrikosov"*) *vortices* become *Klein-Gordon vortices* is just a technical point which we do not become preoccupied with.

## 6.1 Derivation of the equations

The MKG-pGL equations can be derived from the following Lagrangian:

$$\mathcal{L}^{p_\varepsilon} = \frac{1}{2}\int_0^T \int_\Omega -\frac{1}{c^2}|\partial_{\Phi_\varepsilon} u_\varepsilon|^2 + |\nabla_{A_\varepsilon} u_\varepsilon|^2 + \frac{m^2c^2}{\hbar^2}|u_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(p_\varepsilon^2 - |u_\varepsilon|^2)^2 + \frac{1}{2}\int_0^T \int_\Omega |h_\varepsilon - h_{ex}|^2 - \frac{1}{c^2}|E_\varepsilon - E_{ex}|^2.$$

The constants $m$ and $\hbar$ are the mass of a particle and the *reduced Planck constant*, respectively. We see the Lagrangian contains much of the structure of the GL energy functional and in fact we can write

$$\mathcal{L}^{p_\varepsilon} = \int_0^T \mathcal{G}_\varepsilon^{p_\varepsilon}(u_\varepsilon, A_\varepsilon) + \frac{1}{2}\int_0^T \int_\Omega -\frac{1}{c^2}|\partial_{\Phi_\varepsilon} u_\varepsilon|^2 + \frac{m^2c^2}{\hbar^2}|u_\varepsilon|^2 - \frac{1}{c^2}|E_\varepsilon - E_{ex}|^2 \quad (6.1)$$

where $\mathcal{G}_\varepsilon^{p_\varepsilon}$ is defined in (2.7). The extra terms given by the last integral are what take us in to the regime of MKG. To simplify the calculations which follow we set $m = 1$,

$$c\hbar = 1 \quad (6.2)$$

and

$$c = \mu\sqrt{|\log\varepsilon|} \quad (6.3)$$

where $\mu > 0$ is a constant. The condition given by (6.2) is referred to as the *nonrelativistic-semiclassical limit* in the physics literature [43]. Condition (6.3) proves to be the required relaxation rate to see vortex dynamics on the $O(1)$

time-scale.

In order to model the effects of a perturbed background potential we include the parameter $p_\varepsilon$ in the GL potential. We still refer to the trapping effects of the perturbed background potential as "pinning" although technically we are no longer in the setting of a superconductor with impurities. For the reasons set out in Chapter 1 we work with a normalized order parameter

$$w_\varepsilon(x,t) = \frac{u_\varepsilon(x,t)}{\eta_\varepsilon(x)},$$

where the pseudo-pinning potential $\eta_\varepsilon$ is the unique, positive minimizer of the following functional:

$$F_\varepsilon(\tilde{\eta}_\varepsilon) = \int_\Omega \frac{1}{2}|\nabla\tilde{\eta}_\varepsilon|^2 + \frac{1}{4\varepsilon^2}(p_\varepsilon^2 - \tilde{\eta}_\varepsilon^2)^2 \, dx$$

for $\tilde{\eta}_\varepsilon \in H^1(\Omega;\mathbb{R})$. It follows that $\eta_\varepsilon$ satisfies the elliptic PDE

$$\begin{cases} \Delta\eta_\varepsilon + \frac{\eta_\varepsilon}{\varepsilon^2}(p_\varepsilon^2 - \eta_\varepsilon^2) = 0 & \text{in } \Omega \\ \nu\cdot\nabla\eta_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

We again ask for $\eta_\varepsilon$ to be critically scaled so that the renormalized energy plays a role in the limiting vortex dynamics.

We seek to write down the MKG-pGL equations in terms of the quantities $(\eta_\varepsilon, w_\varepsilon, A_\varepsilon, \Phi_\varepsilon)$. The next proposition shows that the Lagrangian can be split in a convenient way to facilitate this.

**Proposition 6.1.** *If $u_\varepsilon = \eta_\varepsilon w_\varepsilon$ and $T > 0$, then*

$$\mathcal{L}^{p_\varepsilon}(u_\varepsilon, A_\varepsilon, \Phi_\varepsilon) = L^{\eta_\varepsilon}(w_\varepsilon, A_\varepsilon, \Phi_\varepsilon) + TF_\varepsilon(\eta_\varepsilon)$$

*where*

$$\begin{aligned} L^{\eta_\varepsilon}(w_\varepsilon, A_\varepsilon, \Phi_\varepsilon) := \int_0^T \int_\Omega & -\frac{\eta_\varepsilon^2}{2c^2}|\partial_{\Phi_\varepsilon} w_\varepsilon|^2 + \frac{\eta_\varepsilon^2}{2}|\nabla_{A_\varepsilon} w_\varepsilon|^2 + \frac{c^4\eta_\varepsilon^2}{2}|w_\varepsilon|^2 \\ & + \frac{\eta_\varepsilon^4}{4\varepsilon^2}(1-|w_\varepsilon|^2)^2 + \frac{1}{2}|h_\varepsilon - h_{ex}|^2 - \frac{1}{2c^2}|E_\varepsilon - E_{ex}|^2 \end{aligned} \quad (6.4)$$

*and*

$$F_\varepsilon(\eta_\varepsilon) = \int_\Omega \frac{1}{2}|\nabla\eta_\varepsilon|^2 + \frac{1}{4\varepsilon^2}(p_\varepsilon^2 - \eta_\varepsilon^2)^2.$$

*Proof.* Take the Lagrangian (6.1) and set $m = 1$, $c\hbar = 1$. Since the Lagrangian contains the Ginzburg-Landau energy $\mathcal{G}_\varepsilon^{p_\varepsilon}$, we use Proposition 2.1 to know how the first integral transforms under the change of variables $u_\varepsilon \mapsto w_\varepsilon$:

$$\int_0^T \mathcal{G}_\varepsilon^{p_\varepsilon}(\eta_\varepsilon w_\varepsilon, A_\varepsilon) = \int_0^T \int_\Omega \frac{\eta_\varepsilon^2}{2}|\nabla_{A_\varepsilon} w_\varepsilon|^2 + \frac{\eta_\varepsilon^4}{4\varepsilon^2}(1-|w_\varepsilon|^2)^2 + \frac{1}{2}|h_\varepsilon - h_{ex}|^2 + \int_0^T F_\varepsilon(\eta_\varepsilon).$$

Then by direct substitution it is a simple matter to deduce that the last integral in (6.1) becomes

$$\frac{1}{2}\int_0^T \int_\Omega -\frac{\eta_\varepsilon^2}{c^2}|\partial_\Phi w|^2 + c^4\eta_\varepsilon^2|w|^2 - \frac{1}{c^2}|E - E_{ex}|^2.$$

□

From the Lagrangian we can derive the MKG-pGL equations. This is the objective of the next proposition.

**Proposition 6.2.** *The Euler-Lagrange equations of $L^{\eta_\varepsilon}$ are*

$$\begin{cases} \frac{1}{c^2}\partial_{\Phi_\varepsilon}^2 w_\varepsilon + c^4 w_\varepsilon = \Delta_{A_\varepsilon} w_\varepsilon + \nabla \log \eta_\varepsilon^2 \cdot \nabla_{A_\varepsilon} w_\varepsilon + \frac{\eta_\varepsilon^2 w_\varepsilon}{\varepsilon^2}(1-|w_\varepsilon|^2) & \text{in } \Omega \times [0,T] \\ \frac{1}{c^2}\partial_t E_\varepsilon = -\nabla^\perp h_\varepsilon - \eta_\varepsilon^2 j_{A_\varepsilon}(w_\varepsilon) & \text{in } \Omega \times [0,T] \\ \text{div } E_\varepsilon = -\eta_\varepsilon^2 \langle i w_\varepsilon, \partial_{\Phi_\varepsilon} w_\varepsilon \rangle & \text{in } \Omega \times [0,T] \\ \nu \cdot \nabla_{A_\varepsilon} w_\varepsilon = 0 & \text{on } \partial\Omega \times [0,T] \\ \nu \cdot E_\varepsilon = \nu \cdot E_{ex} & \text{on } \partial\Omega \times [0,T] \\ h_\varepsilon = h_{ex} & \text{on } \partial\Omega \times [0,T]. \end{cases}$$

*Proof.* Let $k \in \mathbb{R}$ and $v \in C^\infty(\Omega \times [0,T], \mathbb{C})$. Minimising in the first variable of $L^\eta(w, A, \Phi)$ from (6.4) gives

$$\frac{d}{dk} L^\eta(w + kv, A, \Phi)\Big|_{k=0} = \int_0^T \int_\Omega \Big( -\frac{\eta_\varepsilon^2}{c^2}\langle \partial_\Phi w, \partial_\Phi v\rangle + \eta_\varepsilon^2 \langle \nabla_A w, \nabla_A v\rangle + \big(c^4\eta_\varepsilon^2 - \frac{\eta_\varepsilon^4}{\varepsilon^2}(1-|w|^2)\big)\langle w, v\rangle \Big).$$

By Lemmas A.1 and A.2 we have

$$\langle \partial_\Phi w, \partial_\Phi v\rangle = \partial_t\langle \partial_\Phi w, v\rangle - \langle \partial_\Phi^2 w, v\rangle \quad \text{and} \quad \langle \nabla_A w, \nabla_A v\rangle = \operatorname{div}\langle \nabla_A w, v\rangle - \langle \Delta_A w, v\rangle.$$

Thus,

$$\begin{aligned}\frac{d}{dk}L^\eta\Big|_{k=0} &= \int_\Omega \frac{\eta_\varepsilon^2}{c^2}\big(\langle \partial_\Phi w, v\rangle(0) - \langle \partial_\Phi w, v\rangle(T)\big)\\ &+ \int_0^T\int_\Omega \eta_\varepsilon^2 \operatorname{div}\langle \nabla_A w, v\rangle + \eta_\varepsilon^2\langle \frac{1}{c^2}\partial_\Phi^2 w - \Delta_A w + \big(c^4 - \frac{\eta^2}{\varepsilon^2}(1-|w|^2)\big)w, v\rangle.\end{aligned}$$

Note that

$$\int_\Omega \eta_\varepsilon^2 \operatorname{div}\langle \nabla_A w, v\rangle = -\int_\Omega \nabla\eta_\varepsilon^2 \cdot \langle \nabla_A w, v\rangle + \int_{\partial\Omega}\eta_\varepsilon^2\langle \nu\cdot\nabla_A w, v\rangle.$$

Hence,

$$\begin{aligned}\frac{d}{dk}L^\eta\Big|_{k=0} &= \int_\Omega \frac{\eta_\varepsilon^2}{c^2}\left(\langle \partial_\Phi w, v\rangle(0) - \langle \partial_\Phi w, v\rangle(T)\right) + \int_0^T\int_{\partial\Omega}\eta_\varepsilon^2\langle \nu\cdot\nabla_A w, v\rangle\\ &+ \int_0^T\int_\Omega \eta_\varepsilon^2\langle \frac{1}{c^2}\partial_\Phi^2 w - \Delta_A w - \nabla\log\eta_\varepsilon^2\cdot\nabla_A w + \big(c^4 - \frac{\eta^2}{\varepsilon^2}(1-|w|^2)\big)w, v\rangle.\end{aligned}$$

Let $v$ satisfy $v(0) = v(T) = 0$ but otherwise be arbitrary. If $(w, A, \Phi)$ represent a critical point then $\frac{d}{dk}L^\eta|_{k=0} = 0$; we must have that

$$\begin{cases}\frac{1}{c^2}\partial_\Phi^2 w + c^4 w = \Delta_A w + \nabla\log\eta_\varepsilon^2\cdot\nabla_A w + \frac{\eta^2 w}{\varepsilon^2}(1-|w|^2) & \text{in } \Omega\times[0,T]\\ \nu\cdot\nabla_A w = 0 & \text{on } \partial\Omega\times[0,T].\end{cases}$$

Let $B \in C^\infty(\Omega\times[0,T], \mathbb{R}^2)$. Minimising in the second variable gives

$$\begin{aligned}\frac{d}{dk}L^\eta(w, A+kB, \Phi)\Big|_{k=0} = \int_0^T\int_\Omega \Big( &-\eta_\varepsilon^2\langle iw, \nabla_A w\rangle\cdot B + (h-h_{ex})\text{curl } B\\ &+ \frac{1}{c^2}(E-E_{ex})\cdot\partial_t B\Big).\end{aligned}$$

After integration by parts on the terms containing derivatives of $B$ we have

$$\begin{aligned}\frac{d}{dk}L^\eta\Big|_{k=0} = -\int_0^T\int_\Omega \big(\eta_\varepsilon^2\langle iw, \nabla_A w\rangle + \nabla^\perp(h-h_{ex}) + \frac{1}{c^2}\partial_t(E-E_{ex})\big)\cdot B\\ + \int_0^T\int_{\partial\Omega}(h-h_{ex})B\cdot\tau + \int_\Omega (E\cdot B)(T) - (E\cdot B)(0).\end{aligned}$$

Let $B$ satisfy $B(0) = B(T) = 0$ but otherwise be arbitrary. The Lagrangian attains a critical point if the triple $(w, A, \Phi)$ satisfy

$$\begin{cases} 0 = \nabla^\perp h + \frac{1}{c^2}\partial_t E + \eta_\varepsilon^2 j_A(w) & \text{in } \Omega \times [0,T] \\ h = h_{ex} & \text{on } \partial\Omega \times [0,T] \end{cases}$$

which is just Ampère's law of electrodynamics.

Let $\theta \in C^\infty(\Omega \times [0,T], \mathbb{R})$. Minimising in the third variable gives

$$\frac{d}{dk} L^\eta(w, A, \Phi + k\theta)\Big|_{k=0} = \int_0^T \int_\Omega -\frac{\eta_\varepsilon^2}{c^2}\langle iw, \partial_\Phi w\rangle\theta + \frac{1}{c^2}(E - E_{ex}) \cdot \nabla\theta$$

$$= -\int_0^T \int_\Omega \big(\frac{\eta_\varepsilon^2}{c^2}\langle iw, \partial_\Phi w\rangle + \frac{1}{c^2}\mathrm{div}(E - E_{ex})\big)\theta + \int_0^T \int_{\partial\Omega} \frac{\theta}{c^2}(E - E_{ex}) \cdot \nu.$$

The system is at a critical point if

$$\begin{cases} \text{div } E + \eta_\varepsilon^2\langle iw, \partial_\Phi w\rangle = 0 & \text{in } \Omega \times [0,T] \\ E \cdot \nu = E_{ex} \cdot \nu & \text{on } \partial\Omega \times [0,T]. \end{cases}$$

□

Since we are interested in the interaction of the vortices with each other and the pinning potential we set $E_{ex} \equiv 0$. A nonzero external electric field signifies the presence of an electric current entering the system. We refer the reader to [52, 57, 60] for results concerning the dynamics of vortices interacting with an externally applied electric current. Henceforth, the MKG-pGL equations we use are

$$\begin{cases} \frac{1}{c^2}\partial_{\Phi_\varepsilon}^2 w_\varepsilon + c^4 w_\varepsilon = \Delta_{A_\varepsilon} w_\varepsilon + \nabla \log \eta_\varepsilon^2 \cdot \nabla_{A_\varepsilon} w_\varepsilon + \frac{\eta_\varepsilon^2 w_\varepsilon}{\varepsilon^2}(1 - |w_\varepsilon|^2) & \text{in } \Omega \times [0,T] \\ \frac{1}{c^2}\partial_t E_\varepsilon = -\nabla^\perp h_\varepsilon - \eta_\varepsilon^2 j_{A_\varepsilon}(w_\varepsilon) & \text{in } \Omega \times [0,T] \\ \text{div } E_\varepsilon = -\eta_\varepsilon^2\langle iw_\varepsilon, \partial_{\Phi_\varepsilon} w_\varepsilon\rangle & \text{in } \Omega \times [0,T] \\ \nu \cdot \nabla_{A_\varepsilon} w_\varepsilon = 0 & \text{on } \partial\Omega \times [0,T] \\ \nu \cdot E_\varepsilon = 0 & \text{on } \partial\Omega \times [0,T] \\ h_\varepsilon = h_{ex} & \text{on } \partial\Omega \times [0,T]. \end{cases} \tag{6.5}$$

## 6.2 Gauge choice and well-posedness

The MKG-GL equations are

$$\begin{cases} \frac{1}{c^2}\partial^2_{\Phi_\varepsilon} u_\varepsilon + c^A u_\varepsilon = \Delta_{A_\varepsilon} u_\varepsilon + \frac{u_\varepsilon}{\varepsilon^2}(1 - |u_\varepsilon|^2) & \text{in } \Omega \times [0,T] \\ \frac{1}{c^2}\partial_t(\nabla\Phi_\varepsilon + \partial_t A_\varepsilon) = \nabla^\perp h_\varepsilon + j_{A_\varepsilon}(u_\varepsilon) & \text{in } \Omega \times [0,T] \\ \operatorname{div}(\nabla\Phi_\varepsilon + \partial_t A_\varepsilon) = \langle i u_\varepsilon, \partial_{\Phi_\varepsilon} u_\varepsilon \rangle & \text{in } \Omega \times [0,T] \\ \nu \cdot \nabla_{A_\varepsilon} u_\varepsilon = 0 & \text{on } \partial\Omega \times [0,T] \\ \nu \cdot E_\varepsilon = 0 & \text{on } \partial\Omega \times [0,T] \\ h_\varepsilon = h_{ex} & \text{on } \partial\Omega \times [0,T]. \end{cases} \tag{6.6}$$

There are not many results in the literature on the regularity of solutions to the MKG-GL equations, however, regularity of solutions to MKG and KG-GL (6.6 with $A_\varepsilon = \Phi_\varepsilon = 0$; a specific occurrence of the more general *nonlinear Klein-Gordon* equation) considered separately is well established. Moncrief [45] showed global well-posedness for MKG in the Lorentz gauge (see Definition 6.3) for data in $H^2$. Czubak and Pikula [13] showed local well-posedness for MKG in the Coulomb gauge for initial data $A_\varepsilon \in H^{s'}$ and $u_\varepsilon \in H^s$, with $\frac{1}{4} < s' < 1$ and $\frac{1}{2} < s < 1$. The same results should hold for $s', s > 1$ and actually be easier to prove.

For nonlinear KG equations with initial conditions, the local well-posedness of the Cauchy problem in the space $H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ ($N < 4$) is established by Ginibre and Velo [20]. Building on this, Cazenave and Naumkin [8] prove that there exists a local, highly-regular, non-vanishing solution of nonlinear KG in every dimension $N \geq 1$.

For our case, short time existence and regularity at fixed $\varepsilon$ follows by adapting the work of [8, 45], and then $H^1$ solutions should exist globally due to energy conservation. Furthermore, Yu [60] works with the MKG-GL equations and this level of regularity is implicitly used in their lemmas and

should pass over to our situation because there is no reason to expect that our introduction of the pinning terms creates any serious issues; the pinning is sufficiently regular. Specifically, our solutions $(w_\varepsilon, A_\varepsilon, \Phi_\varepsilon)$ to (6.5) are regular enough to allow the calculation in the proof of Proposition 7.5 to pass through without any problems; in fact a similar calculation is done by Yu in their paper [60], but without the pinning terms. The pinning terms present for us don't cause any serious complications because our pinning term has a high level of regularity (see [34]).

The following choice of gauge is more suited to the MKG problem:

**Definition 6.3.** *Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain. We say $A : \Omega \times \mathbb{R}^+ \to \mathbb{R}^2$ and $\Phi : \Omega \times \mathbb{R}^+ \to \mathbb{R}$ satisfy the Lorenz gauge condition if*

$$\begin{cases} \Phi = -\text{div } A & \text{in } \Omega \times \mathbb{R}^+ \\ \nu \cdot A = 0 & \text{on } \partial\Omega \times \mathbb{R}^+ \\ \Phi(0) = 0 & \text{initially,} \end{cases} \tag{6.7}$$

*where $\nu$ is the outward pointing unit normal to $\partial\Omega$.*

## 6.3 Energy analysis for MKG-pGL solutions

Following the program set out for the TDGL model, we seek to show that the energy in the system is bounded for at least a finite time. Starting out in this direction, we have

**Definition 6.4.** *The Hamiltonian is*

$$H_\varepsilon^{\eta_\varepsilon}(w_\varepsilon, A_\varepsilon, \Phi_\varepsilon) = \int_\Omega \mathcal{H}_\varepsilon^{\eta_\varepsilon}(w_\varepsilon, A_\varepsilon, \Phi_\varepsilon), \tag{6.8}$$

*where the Hamiltonian density is*

$$\mathcal{H}_\varepsilon^{\eta_\varepsilon} = \frac{1}{2c^2}|E_\varepsilon|^2 + \frac{1}{2}|h_\varepsilon - h_{ex}|^2 + \frac{\eta_\varepsilon^2}{2c^2}|\partial_{\Phi_\varepsilon} w_\varepsilon|^2 + \frac{\eta_\varepsilon^2}{2}|\nabla_{A_\varepsilon} w_\varepsilon|^2 + \frac{\eta_\varepsilon^4}{4\varepsilon^2}(1 - |w_\varepsilon|^2)^2. \tag{6.9}$$

The Hamiltonian contains the GL energy $G_\varepsilon^{\eta_\varepsilon}(u, A)$ + the kinetic energy (previously known as "dissipation"). We will see that by including both terms

on the same footing (recall that it was the time-derivative of $G_\varepsilon^{\eta_\varepsilon}$ which equalled the dissipation in (4.10)), the task of controlling the total energy in the system is simplified. See Propositions 6.9 and 6.10.

We define the Jacobian and space-time Jacobian as in (2.27) and (2.28). The optimum quantities $(w_*, h_*, A_*)$ satisfy (2.42), (2.44) and (2.45). The renormalized energy is again defined by (2.46).

**Definition 6.5.** *The initial data is said to be well-prepared if for some* $a(0) = (a_1(0), ..., a_n(0)) \in \Omega^n$ *and* $d = (d_1, ..., d_n) \in \{\pm 1\}^n$, $(w_\varepsilon(0), A_\varepsilon(0), \Phi_\varepsilon(0))$ *satisfies*

$$\begin{cases} \big\| J_{A_\varepsilon}(w_\varepsilon)(0) - 2\pi \sum d_i \delta_{a_i(0)} \big\|_{\dot{W}^{-1,1}(\Omega)} \to 0 \qquad \text{as } \varepsilon \to 0 \\ H_\varepsilon^{\eta_\varepsilon}(w_\varepsilon, A_\varepsilon, \Phi_\varepsilon)(0) = n\pi |\log \varepsilon| + n\gamma_0 + W^Q(a(0), d) + o_\varepsilon(1). \end{cases} \tag{6.10}$$

In the next couple of propositions we determine the evolution of the Hamiltonian density for solutions to the MKG-pGL equations. This constitutes a conservation law for the system, from which, the evolution of the Hamiltonian follows.

**Proposition 6.6.** *For any* $(w, A, \Phi)$ *we have that*

$$\partial_t \mathcal{H}_\varepsilon^{\eta_\varepsilon} = \eta_\varepsilon^2 \mathrm{div} \langle \partial_\Phi w, \nabla_A w \rangle + \langle \partial_\Phi w, \eta_\varepsilon^2 \big( \frac{1}{c^2} \partial_\Phi^2 w - \Delta_A w - \frac{\eta_\varepsilon^2 w}{\varepsilon^2}(1 - |w|^2) \big) \rangle$$
$$+ E \cdot \big( \frac{1}{c^2} \partial_t E + \eta_\varepsilon^2 j_A(w) \big) - (h - h_{ex}) \mathrm{curl}\ E. \tag{6.11}$$

*Proof.* Recognising that $\mathcal{H}_\varepsilon^{\eta_\varepsilon} = \frac{1}{2c^2}|E_\varepsilon|^2 + \frac{\eta_\varepsilon^2}{2c^2}|\partial_{\Phi_\varepsilon} w_\varepsilon|^2 + g_\varepsilon^{\eta_\varepsilon}(w_\varepsilon, A_\varepsilon)$, we can use Lemma 4.2 which states that

$$\partial_t g_\varepsilon^{\eta_\varepsilon} = \eta_\varepsilon^2 \mathrm{div} \langle \partial_\Phi w, \nabla_A w \rangle - \langle \partial_\Phi w, \eta_\varepsilon^2 \Delta_A w + \frac{\eta_\varepsilon^4 w}{\varepsilon^2}(1 - |w|^2) \rangle$$
$$- (h - h_{ex}) \mathrm{curl}\ E + \eta_\varepsilon^2 E \cdot j_A(w).$$

Using this with

$$\partial_t \big( \frac{1}{2c^2}|E|^2 + \frac{\eta_\varepsilon^2}{2c^2}|\partial_\Phi w|^2 \big) = \frac{1}{c^2} E \cdot \partial_t E + \frac{\eta_\varepsilon^2}{c^2} \langle \partial_\Phi w, \partial_\Phi^2 w \rangle$$

provides (6.11). □

**Proposition 6.7.** *For $(w_\varepsilon, A_\varepsilon, \Phi_\varepsilon)$ satisfying (6.5) we have*

$$\partial_t\big(\mathcal{H}_\varepsilon^{\eta_\varepsilon} + \frac{\eta_\varepsilon^2 c^4}{2}|w_\varepsilon|^2\big) = \mathrm{div}\big(\eta_\varepsilon^2\langle\partial_{\Phi_\varepsilon} w_\varepsilon, \nabla_{A_\varepsilon} w_\varepsilon\rangle + (h_\varepsilon - h_{ex})E_\varepsilon^\perp\big). \tag{6.12}$$

*Consequently,*

$$\partial_t H_\varepsilon^{\eta_\varepsilon}(t) = \int_\Omega \frac{\eta_\varepsilon^2 c^4}{2}\partial_t(1 - |w_\varepsilon(t)|^2). \tag{6.13}$$

*Proof.* Using the relations from (6.5) in (6.11) gives

$$\begin{aligned}\partial_t\mathcal{H}_\varepsilon^{\eta_\varepsilon} = \eta_\varepsilon^2\mathrm{div}\langle\partial_\Phi w, \nabla_A w\rangle + \langle\partial_\Phi w, \nabla\eta_\varepsilon^2\cdot\nabla_A w - \eta_\varepsilon^2 c^4 w\rangle \\ - E\cdot\nabla^\perp(h - h_{ex}) - (h - h_{ex})\mathrm{curl}\ E.\end{aligned} \tag{6.14}$$

Recall that $\langle\partial_\Phi w, w\rangle = \frac{1}{2}\partial_t|w|^2$. Use this in (6.14) and note that

$$-\mathrm{curl}\big((h - h_{ex})E\big) = \mathrm{div}\big((h - h_{ex})E^\perp\big)$$

then we have (6.12).

To derive (6.13), first integrate (6.12) over the domain $\Omega$ such that

$$\begin{aligned}\partial_t\int_\Omega \mathcal{H}_\varepsilon^{\eta_\varepsilon} + \frac{\eta_\varepsilon^2 c^4}{2}|w|^2 &= \int_\Omega \mathrm{div}\big(\eta_\varepsilon^2\langle\partial_\Phi w, \nabla_A w\rangle + (h - h_{ex})E^\perp\big) \\ &= \int_{\partial\Omega}\eta_\varepsilon^2\langle\partial_\Phi w, \nu\cdot\nabla_A w\rangle + (h - h_{ex})\nu\cdot E^\perp.\end{aligned} \tag{6.15}$$

Then due to the boundary conditions in (6.5), the right-hand side of (6.15) is zero. □

**Remark 6.8.** *Since* $\mathrm{div}\ E^\perp = -\mathrm{curl}\ E = \partial_t h$*, adding* $h_{ex}\mathrm{div}\ E^\perp$ *to both sides of (6.12) gives*

$$\partial_t\big(\mathcal{H}_\varepsilon^{\eta_\varepsilon} + \frac{\eta_\varepsilon^2 c^4}{2}|w|^2 + h_{ex}(h - h_{ex})\big) = \mathrm{div}\big(\eta_\varepsilon^2\langle\partial_\Phi w, \nabla_A w\rangle + hE^\perp\big). \tag{6.16}$$

With (6.13) in hand it is possible to make a statement on the value of the Hamiltonian for a period of time.

**Proposition 6.9.** *Suppose* $\varepsilon\sqrt{H_\varepsilon^{\eta_\varepsilon}(t)} \to 0$ *as* $\varepsilon \to 0$ *for all* $t \in [0,T]$. *If* $(w_\varepsilon, A_\varepsilon, \Phi_\varepsilon)$ *are solutions to (6.5) and satisfy the well-preparedness condition (6.10), then*

$$H_\varepsilon^{\eta_\varepsilon}(t) = n\pi|\log\varepsilon| + n\gamma_0 + W^Q(a(0),d) + o_\varepsilon(1). \tag{6.17}$$

*Proof.* As a first order of business we record that by Cauchy-Schwarz,

$$\left|\int_\Omega \frac{\eta_\varepsilon^2}{2}(1-|w|^2)(t)\right| \le \left(\int_\Omega \frac{\eta_\varepsilon^4}{4}(1-|w|^2)^2(t)\right)^{\frac{1}{2}}\left(\int_\Omega 1\right)^{\frac{1}{2}} \le C\varepsilon\sqrt{H(t)}.$$

Due to the assumption on the Hamiltonian this implies that

$$\left|\int_\Omega \frac{\eta_\varepsilon^2 c^4}{2}(1-|w|^2)(t)\right| = o_\varepsilon(1) \qquad \forall t\in[0,T]. \tag{6.18}$$

Now consider integrating (6.13) in time from 0 to $t$, such that

$$H(t) - H(0) = \int_\Omega \frac{\eta_\varepsilon^2 c^4}{2}(1-|w|^2(t)) - \int_\Omega \frac{\eta_\varepsilon^2 c^4}{2}(1-|w|^2(0)). \tag{6.19}$$

By (6.18), (6.19) implies that $H(t) - H(0) = o_\varepsilon(1)$, and using the well-preparedness condition (6.10) allows us to replace the $H(0)$ term and arrive at (6.17). □

Now we show that the kinetic energy is bounded by a constant for all time.

**Proposition 6.10.** *If the hypotheses of Proposition 6.9 are satisfied, then for all* $t \in [0,T]$

$$\int_\Omega \frac{1}{2c^2}|E_\varepsilon(t)|^2 + \frac{\eta_\varepsilon^2}{2c^2}|\partial_{\Phi_\varepsilon} w_\varepsilon(t)|^2 \le C. \tag{6.20}$$

*Proof.* By (6.17) we have

$$\int_\Omega \frac{1}{2c^2}|E_\varepsilon(t)|^2 + \frac{\eta_\varepsilon^2}{2c^2}|\partial_{\Phi_\varepsilon} w_\varepsilon(t)|^2 = n\pi|\log\varepsilon| + n\gamma_0 + W^Q(a(0),d) + o_\varepsilon(1) - G_\varepsilon^{\eta_\varepsilon}(w_\varepsilon, A_\varepsilon). \tag{6.21}$$

If $d_i = \pm 1$ for all $i = 1,...,n$, then Theorem 1.1 of [49] provides the following estimate for the GL energy on the vortex balls:

$$n\pi|\log\varepsilon| \le G_\varepsilon(\cup_i B_r(a_i)) + C_1. \tag{6.22}$$

As a result of Appendix B, (6.22) becomes

$$n\pi|\log\varepsilon| \leq \Big(1+\frac{K}{|\log\varepsilon|}\Big)^2 G_\varepsilon^{\eta_\varepsilon}(\Omega) + C_1. \tag{6.23}$$

This leads to

$$n\pi|\log\varepsilon| - G_\varepsilon^{\eta_\varepsilon}(\Omega) \leq \frac{2K}{|\log\varepsilon|} G_\varepsilon^{\eta_\varepsilon}(\Omega) + \frac{K^2}{|\log\varepsilon|^2} G_\varepsilon^{\eta_\varepsilon}(\Omega) + C_1. \tag{6.24}$$

Since the hypotheses of Proposition 6.9 are satisfied, we know that

$$G_\varepsilon^{\eta_\varepsilon}(w_\varepsilon, A_\varepsilon; \Omega) \leq H_\varepsilon^{\eta_\varepsilon}(w_\varepsilon, A_\varepsilon; \Omega) \leq C|\log\varepsilon|, \tag{6.25}$$

and so (6.24) becomes

$$n\pi|\log\varepsilon| - G_\varepsilon^{\eta_\varepsilon}(w_\varepsilon, A_\varepsilon; \Omega) \leq C_2 + o_\varepsilon(1). \tag{6.26}$$

Using (6.26) in (6.21) provides the result. □

# Chapter 7

# Dynamical law in MKG-pGL setting

Following the procedure laid out in Chapter 5, we will show that if $(w_\varepsilon, A_\varepsilon, \Phi_\varepsilon)$ are solutions to the MKG-pGL equations (6.5), then the Jacobian and Hamiltonian converge to a collection of time-dependent points in $\Omega$, which we call the vortex positions. We then determine an equation which relates the evolution of the Hamiltonian to the stress-energy tensor. After passing to the limit in this equation and employing a Gronwall argument, we determine the ODE which the vortex positions satisfy.

## 7.1 Vortex trajectories $\xi_i(t)$

In this section we show that the vortex positions become apparent in the limit $\varepsilon \to 0$. The next propositions show that the Jacobian, space-time Jacobian and normalized Hamiltonian all concentrate on the vortex locations. This is one stage in our calculations where we benefit from our choice of pinned GL potential in the MKG equations, since we can piggyback on the results of previous chapters to determine the convergences in the new setting.

**Proposition 7.1.** *Suppose $(w_\varepsilon, A_\varepsilon, \Phi_\varepsilon)$ satisfy (6.5) and the initial data satisfy (6.10). Let $n := n(0)$ and $d_i := d_i(0)$ for all $i = 1, ..., n$. Then there exist $n$*

*trajectories* $\xi_i(t) \in H^1([0,T_0],\Omega)$ *such that, for all* $t \in [0,T_0]$*, as* $\varepsilon \to 0$

$$\left\| J_{A_\varepsilon}(w_\varepsilon) - \pi \sum_{i=1}^{n} d_i \delta_{\xi_i(t)} \right\|_{\dot{W}^{-1,1}(\Omega)} \to 0. \tag{7.1}$$

$$\left\| V_{A_\varepsilon}^{\Phi_\varepsilon}(w_\varepsilon) - \pi \sum_{i=1}^{n} d_i \delta_{\xi_i(t)} \dot{\xi}_i^{\perp}(t) \right\|_{\dot{W}^{-1,1}(\Omega\times[0,T_0])} \to 0. \tag{7.2}$$

*Proof.* Take (6.17) and (6.20) and invoke (B.5) such that we have

$$G_\varepsilon(t) \leq C|\log\varepsilon| \quad \forall t \in [0,T] \qquad \text{and} \qquad \int_0^T \int_\Omega \eta_\varepsilon^2 |\partial_{\Phi_\varepsilon} w_\varepsilon|^2 + |E_\varepsilon|^2 \leq C|\log\varepsilon|.$$

The remainder of the proof is the same as the proof of Proposition 5.1. □

**Proposition 7.2.** *Suppose* $(w_\varepsilon, A_\varepsilon, \Phi_\varepsilon)$ *are solutions to (6.5), (6.10) and* $(A_\varepsilon, A_*)$ *are in the Coulomb gauge. Then*

$$\frac{\mathcal{H}_\varepsilon^{\eta_\varepsilon}(t)}{|\log\varepsilon|} \overset{*}{\rightharpoonup} \pi \sum \delta_{\xi_i(t)} \tag{7.3}$$

*Proof.* Recall that $\mathcal{H}_\varepsilon^{\eta_\varepsilon}(w_\varepsilon, A_\varepsilon, \Phi_\varepsilon) = g_\varepsilon^{\eta_\varepsilon}(w_\varepsilon, A_\varepsilon) + \frac{1}{2c^2}|E_\varepsilon|^2 + \frac{\eta_\varepsilon^2}{2c^2}|\partial_{\Phi_\varepsilon} w_\varepsilon|^2$. If $(w_\varepsilon, A_\varepsilon, \Phi_\varepsilon)$ are well-prepared solutions to (6.5), we have (6.17) and (7.1) and so the convergence from Proposition 5.2 can be used, since we also have the necessary Coulomb gauge condition. Using (6.20) shows the extra terms are small and so we have (7.2). □

## 7.2 Stress-energy tensor

Continuing to follow the program set out in Chapter 5, we now define the stress-energy tensor, this time in the MKG-pGL setting. We will see that the stress-energy tensor we state here contains the one from Chapter 5, so we are again in a position to call on previous results. The reason for introducing the stress-energy tensor becomes clear in the next section when we use it to relate the evolution of the energy density to the renormalised energy and pinning potential.

**Definition 7.3.** *If $\mathcal{I}$ is the $2\times 2$ identity matrix, then the stress-energy tensor is*

$$\mathcal{T} = \eta_\varepsilon^2 \nabla_{A_\varepsilon} w_\varepsilon \otimes \nabla_{A_\varepsilon} w_\varepsilon + \frac{1}{c^2} E_\varepsilon^\perp \otimes E_\varepsilon^\perp - \mathcal{I}\big(\frac{\eta_\varepsilon^2}{2}|\nabla_{A_\varepsilon} w_\varepsilon|^2 + \frac{\eta_\varepsilon^4}{4\varepsilon^2}(1-|w_\varepsilon|^2)^2 - \frac{\eta_\varepsilon^2}{2c^2}|\partial_{\Phi_\varepsilon} w_\varepsilon|^2 + \frac{1}{2c^2}|E_\varepsilon|^2 - \frac{1}{2}|h_\varepsilon - h_{ex}|^2 + \frac{c^4\eta_\varepsilon^2}{2}|w_\varepsilon|^2\big). \quad (7.4)$$

Furthermore, recall the following definition for our tensor product notation

**Definition 7.4.** *If $X$ and $Y$ are rank-2 tensors, then*

$$X : Y = \sum_{lk} X_{lk} Y_{lk}.$$

We seek to use Lemma 5.6, which has the relation

$$\int_\Omega \nabla\nabla^\perp \phi : (j_{A_*}(w_*) \otimes j_{A_*}(w_*)) = -\nabla^\perp \phi \cdot \nabla_{\xi_i} W(\xi, d), \quad (7.5)$$

to relate the stress-energy tensor $\mathcal{T}$ to the renormalized energy. Consider that

$$\begin{aligned}\int_\Omega \phi \text{ curl div } \mathcal{T} &= \int_\Omega \nabla\nabla^\perp\phi : \Big(\eta_\varepsilon^2 \nabla_A w \otimes \nabla_A w + \frac{1}{c^2} E^\perp \otimes E^\perp\Big) \\ &= \frac{1}{c^2}\int_\Omega \nabla\nabla^\perp\phi : \big(E^\perp \otimes E^\perp\big) + \int_\Omega \nabla\nabla^\perp\phi : \eta_\varepsilon^2 \left(\nabla_A w \otimes \nabla_A w - j_{A_*} \otimes j_{A_*}\right) \\ &\quad + \int_\Omega \nabla\nabla^\perp\phi : (\eta_\varepsilon^2 - 1)\left(j_{A_*} \otimes j_{A_*}\right) - \nabla^\perp\phi \cdot \nabla_{\xi_i} W(\xi, d) \quad (7.6)\end{aligned}$$

We study more closely the form of the divergence of $\mathcal{T}$.

**Proposition 7.5.** *For $(w_\varepsilon, A_\varepsilon, \Phi_\varepsilon)$ satisfying (6.5),*

$$\begin{aligned}div\ \mathcal{T} &= \frac{1}{c^2}\partial_t\big(\eta_\varepsilon^2 \langle \partial_\Phi w, \nabla_{A_\varepsilon} w_\varepsilon\rangle + hE^\perp\big) \\ &\quad - \nabla \log \eta_\varepsilon^2 \big(g_\varepsilon^{\eta_\varepsilon} + \frac{\eta_\varepsilon^4}{4\varepsilon^2}(1-|w_\varepsilon|^2)^2 - \frac{1}{2}|h_\varepsilon - h_{ex}|^2 - \frac{\eta_\varepsilon^2}{2c^2}|\partial_\Phi w|^2 + \frac{c^4\eta_\varepsilon^2}{2}|w|^2\big)\end{aligned} \quad (7.7)$$

*Proof.* Recall from Section 6.2 that the solutions $(w_\varepsilon, A_\varepsilon, \Phi_\varepsilon)$ to (6.5) are sufficiently regular for the following calculations to be performed without any issues. If $T_\varepsilon^{\eta_\varepsilon}$ is the stress-energy tensor from (5.8), we have that

$$\begin{aligned}\text{div } \mathcal{T} = \text{div } T_\varepsilon^{\eta_\varepsilon} + \frac{1}{c^2}\text{div}(E^\perp \otimes E^\perp - \frac{1}{2}|E|^2\mathcal{I}) + \frac{\nabla\eta_\varepsilon^2}{2c^2}|\partial_\Phi w|^2 \\ + \frac{\eta_\varepsilon^2}{2c^2}\nabla|\partial_\Phi w|^2 - \frac{c^4}{2}|w|^2\nabla\eta_\varepsilon^2 - \frac{c^4\eta_\varepsilon^2}{2}\nabla|w|^2. \quad (7.8)\end{aligned}$$

We recall from (5.15) that

$$\begin{aligned}\operatorname{div} T_\varepsilon^{\eta_\varepsilon} &= \eta_\varepsilon^2\langle \Delta_{A_\varepsilon} w_\varepsilon + \frac{\eta_\varepsilon^2 w_\varepsilon}{\varepsilon^2}(1-|w_\varepsilon|^2) + \nabla \log \eta_\varepsilon^2 \cdot \nabla_{A_\varepsilon} w_\varepsilon, \nabla_{A_\varepsilon} w_\varepsilon\rangle \\ &\quad + h(\nabla h - \eta_\varepsilon^2 j_A^\perp) - \nabla \log \eta_\varepsilon^2 \big(g_\varepsilon^{\eta_\varepsilon} + \frac{\eta_\varepsilon^4}{4\varepsilon^2}(1-|w_\varepsilon|^2)^2 - \frac{1}{2}|h_\varepsilon - h_{ex}|^2\big).\end{aligned}$$

Use (6.5) to write the divergence of $T_\varepsilon^{\eta_\varepsilon}$ in terms of solutions:

$$\begin{aligned}\operatorname{div} T_\varepsilon^{\eta_\varepsilon} &= \frac{\eta_\varepsilon^2}{c^2}\langle \partial_\Phi^2 w, \nabla_{A_\varepsilon} w_\varepsilon\rangle + c^4\eta_\varepsilon^2\langle w, \nabla_A w\rangle + \frac{1}{c^2} h\partial_t E^\perp \\ &\quad - \nabla \log \eta_\varepsilon^2 \big(g_\varepsilon^{\eta_\varepsilon} + \frac{\eta_\varepsilon^4}{4\varepsilon^2}(1-|w_\varepsilon|^2)^2 - \frac{1}{2}|h_\varepsilon - h_{ex}|^2\big). \quad (7.9)\end{aligned}$$

We can rewrite the first term on the right-hand side. By Appendix A, we have

$$\langle \partial_\Phi^2 w, \nabla_A w\rangle = \partial_t\langle \partial_\Phi w, \nabla_A w\rangle - \langle \partial_\Phi w, \partial_\Phi \nabla_A w\rangle$$

and the commutator identity $\partial_\Phi \nabla_A w = \nabla_A \partial_\Phi w + iwE$. Together they provide

$$\langle \partial_\Phi^2 w, \nabla_A w\rangle = \partial_t\langle \partial_\Phi w, \nabla_A w\rangle - \langle \partial_\Phi w, \nabla_A \partial_\Phi w\rangle - E\langle \partial_\Phi w, iw\rangle.$$

From (6.5) we have $\operatorname{div} E = -\eta_\varepsilon^2\langle iw, \partial_\Phi w\rangle$, and so

$$\eta_\varepsilon^2\langle \partial_\Phi^2 w, \nabla_A w\rangle = \eta_\varepsilon^2\partial_t\langle \partial_\Phi w, \nabla_A w\rangle - \eta_\varepsilon^2\langle \partial_\Phi w, \nabla_A \partial_\Phi w\rangle + \frac{1}{2}\nabla|E|^2.$$

We now use that $\langle \partial_\Phi w, \nabla_A \partial_\Phi w\rangle = \frac{1}{2}\nabla|\partial_\Phi w|^2$ to rewrite the second to last term and insert the result in to (7.9) to arrive at

$$\begin{aligned}\operatorname{div} T_\varepsilon^{\eta_\varepsilon} &= \partial_t\big(\frac{\eta_\varepsilon^2}{c^2}\langle \partial_\Phi w, \nabla_{A_\varepsilon} w_\varepsilon\rangle\big) - \frac{\eta_\varepsilon^2}{2c^2}\nabla|\partial_\Phi w|^2 + \frac{1}{2c^2}\nabla|E|^2 + \frac{c^4\eta_\varepsilon^2}{2}\nabla|w|^2 \\ &\quad + \frac{1}{c^2} h\partial_t E^\perp - \nabla \log \eta_\varepsilon^2 \big(g_\varepsilon^{\eta_\varepsilon} + \frac{\eta_\varepsilon^4}{4\varepsilon^2}(1-|w_\varepsilon|^2)^2 - \frac{1}{2}|h_\varepsilon - h_{ex}|^2\big). \quad (7.10)\end{aligned}$$

To deal with the electric field terms in (7.8), by Appendix C we have that

$$\operatorname{div}(E^\perp \otimes E^\perp - \frac{1}{2}|E|^2 I) = -E^\perp \operatorname{curl} E - \frac{1}{2}\nabla|E|^2 = E^\perp \partial_t h - \frac{1}{2}\nabla|E|^2.$$

Insert this and (7.10) into (7.8), and after some cancellations we have

$$\begin{aligned}\operatorname{div} \mathcal{T} &= \partial_t\big(\frac{\eta_\varepsilon^2}{c^2}\langle \partial_\Phi w, \nabla_{A_\varepsilon} w_\varepsilon\rangle\big) + \frac{1}{c^2} h\partial_t E^\perp + \frac{1}{c^2} E^\perp \partial_t h \\ &\quad - \nabla \log \eta_\varepsilon^2 \big(g_\varepsilon^{\eta_\varepsilon} + \frac{\eta_\varepsilon^4}{4\varepsilon^2}(1-|w_\varepsilon|^2)^2 - \frac{1}{2}|h_\varepsilon - h_{ex}|^2 - \frac{\eta_\varepsilon^2}{2c^2}|\partial_\Phi w|^2 + \frac{c^4\eta_\varepsilon^2}{2}|w|^2\big),\end{aligned}$$

which is (7.7). □

## 7.3 Limiting dynamical law

In this final section we relate the stress-energy tensor to the evolution of the Hamiltonian to determine what we call the *evolution equation* for the key quantities. After estimating the extra quantities which appear in this relation, we pass to the limit $\varepsilon \to 0$ in all terms.

Recall the notation from (2.50) and suppose there are points $a_i(t) \in \Omega$ $(i = 1, ..., n)$ which satisfy the ODEs

$$\begin{cases} \frac{\pi}{\mu^2}\ddot{a}_i(t) = -\nabla_{a_i} W^Q(a, d) \\ a_i(0) = \xi_i(0) \\ \dot{a}_i(0) = 0. \end{cases} \tag{7.11}$$

This is the form of the ODE we expect the vortex positions $\xi_i(t)$ to satisfy. Weakly, $a_i$ satisfies

$$\frac{\pi}{\mu^2}\int_0^T \ddot{\zeta} a_i(t) = -\int_0^T \zeta \nabla_{a_i} W^Q(a(t), d) \tag{7.12}$$

for $\zeta$ any function in $C_c^\infty(0, T)$. If $a(t) = (a_1(t), ..., a_n(t))$ and $\xi(t) = (\xi_1(t), ..., \xi_n(t))$, we set

$$\omega(t) := \xi(t) - a(t), \tag{7.13}$$

and the aim of the final subsection is to use the limiting evolution equation, the ODE for $a_i(t)$ given above, and apply Gronwall's inequality to show that $\omega(t) = 0$ for all $t \in [0, T]$, and thus the vortex motion law has been determined.

In the following we again use the notation $B_r^0 := B_r(a_j(0))$ and $\Omega_r^0 := \Omega \backslash B_r^0$. Here we present the equation which relates all the key quantities.

**Proposition 7.6.** *(Evolution equation.) Suppose $(w_\varepsilon, A_\varepsilon, \Phi_\varepsilon)$ are solutions to (6.5), $w_* = w_*(\xi)$ is defined by (2.42), $A_* = A_*(\xi)$ is defined by (2.45), $\zeta \in C^2([0, T])$ and $(\psi, \phi) \in W_0^{2,\infty}(\Omega)$ are the test functions from Definition 5.8. Then we have the following relation:*

$$\int_0^T \Big( \int_\Omega \big( \ddot{\zeta}\psi \frac{\mathcal{H}_\varepsilon^{\eta_\varepsilon}}{c^2} + \zeta \nabla\psi \cdot \nabla(\log \eta_\varepsilon^2) g_\varepsilon^{\eta_\varepsilon} \big) + \zeta \nabla\psi \cdot \nabla_{\xi_i} W(\xi, d) \Big) = \sum_{i=1}^{10} I_i, \tag{7.14}$$

*where*

$$I_1 = \int_0^T \zeta \int_\Omega \nabla\nabla^\perp\phi : \eta_\varepsilon^2\big(\nabla_{A_\varepsilon} w_\varepsilon \otimes \nabla_{A_\varepsilon} w_\varepsilon - j_{A_*}(w_*) \otimes j_{A_*}(w_*)\big)$$

$$I_2 = \frac{1}{c^2}\int_0^T \zeta \int_\Omega \nabla\nabla^\perp\phi : \big(E_\varepsilon^\perp \otimes E_\varepsilon^\perp\big)$$

$$I_3 = \int_0^T \zeta \int_\Omega \nabla\nabla^\perp\phi : (\eta_\varepsilon^2 - 1)\big(j_{A_*}(w_*) \otimes j_{A_*}(w_*)\big)$$

$$I_4 = \frac{c^2}{2}\int_0^T \dot\zeta \int_\Omega \psi\eta_\varepsilon^2 \partial_t |w_\varepsilon|^2$$

$$I_5 = -\frac{1}{c^2}\int_0^T \ddot\zeta \int_\Omega \psi h_{ex}(h_\varepsilon - h_{ex})$$

$$I_6 = -\int_0^T \zeta \int_\Omega \nabla^\perp\phi \cdot \nabla\eta_\varepsilon^2 \frac{\eta_\varepsilon^2}{4\varepsilon^2}(1 - |w_\varepsilon|^2)^2$$

$$I_7 = \frac{1}{2}\int_0^T \zeta \int_\Omega \nabla^\perp\phi \cdot \nabla(\log\eta_\varepsilon^2)|h_\varepsilon - h_{ex}|^2$$

$$I_8 = \frac{1}{2c^2}\int_0^T \zeta \int_\Omega \nabla^\perp\phi \cdot \nabla\eta_\varepsilon^2 |\partial_{\Phi_\varepsilon} w_\varepsilon|^2$$

$$I_9 = -\frac{c^4}{2}\int_0^T \zeta \int_\Omega \nabla^\perp\phi \cdot \nabla\eta_\varepsilon^2 |w_\varepsilon|^2$$

$$I_{10} = \frac{1}{c^2}\int_0^T \dot\zeta \int_{\Omega_r^0} (\nabla^\perp\phi - \nabla\psi) \cdot (\eta_\varepsilon^2 \langle \partial_{\Phi_\varepsilon} w_\varepsilon, \nabla_{A_\varepsilon} w_\varepsilon\rangle + h_\varepsilon E_\varepsilon^\perp).$$

*Proof.* Take the curl of (7.7), multiply by the test functions $\phi$ and $\zeta$ and integrate by parts:

$$\begin{aligned}\int_0^T \zeta \int_\Omega \nabla\nabla^\perp\phi : \mathcal{T} &= \frac{1}{c^2}\int_0^T \dot\zeta \int_\Omega \nabla^\perp\phi \cdot \big(\eta_\varepsilon^2\langle\partial_\Phi w, \nabla_{A_\varepsilon} w_\varepsilon\rangle + hE^\perp\big) \\ &+ \int_0^T \zeta \int_\Omega \nabla^\perp\phi \cdot \nabla(\log\eta_\varepsilon^2) g_\varepsilon^{\eta_\varepsilon} + \int_0^T \zeta \int_\Omega \nabla^\perp\phi \cdot \nabla(\log\eta_\varepsilon^2)\frac{\eta_\varepsilon^4}{4\varepsilon^2}(1 - |w_\varepsilon|^2)^2 \\ &- \frac{1}{2}\int_0^T \zeta \int_\Omega \nabla^\perp\phi \cdot \nabla(\log\eta_\varepsilon^2)\big(|h_\varepsilon - h_{ex}|^2 + \frac{\eta_\varepsilon^2}{c^2}|\partial_\Phi w|^2 - c^4\eta_\varepsilon^2|w|^2\big). \quad (7.15)\end{aligned}$$

Allow us to write $\mathcal{P}_\varepsilon^{\eta_\varepsilon} := \eta_\varepsilon^2\langle\partial_\Phi w, \nabla_A w\rangle + hE^\perp$ in the following. Notice that because $\nabla\psi = \nabla^\perp\phi$ in $B_r^0$ we have

$$\begin{aligned}\frac{1}{c^2}\int_0^T \dot\zeta \int_\Omega \nabla^\perp\phi \cdot \mathcal{P}_\varepsilon^{\eta_\varepsilon} &= \frac{1}{c^2}\int_0^T \dot\zeta \int_{\Omega_r^0} \nabla^\perp\phi \cdot \mathcal{P}_\varepsilon^{\eta_\varepsilon} + \frac{1}{c^2}\int_0^T \dot\zeta \int_{B_r^0} \nabla\psi \cdot \mathcal{P}_\varepsilon^{\eta_\varepsilon} \\ &= \frac{1}{c^2}\int_0^T \dot\zeta \int_{\Omega_r^0} (\nabla^\perp\phi - \nabla\psi) \cdot \mathcal{P}_\varepsilon^{\eta_\varepsilon} + \frac{1}{c^2}\int_0^T \dot\zeta \int_\Omega \nabla\psi \cdot \mathcal{P}_\varepsilon^{\eta_\varepsilon}. \quad (7.16)\end{aligned}$$

Hence, using (7.6) to rewrite the left-hand side of (7.15) in terms of the renormalized energy and (6.16) to rewrite the last integral of (7.16) in terms of $\partial_t \mathcal{H}_\varepsilon^{\eta_\varepsilon}$, we have (7.14). □

Knowing that the Hamiltonian converges to the vortex positions, the aim is to pass to the limit in (7.14). In the next series of propositions we estimate the terms which appear on the right-hand side of (7.14), some of which contain the magnetic field. From (6.5), the equation for the magnetic field is

$$\frac{1}{c^2}\partial_t^2 h_\varepsilon - \Delta h_\varepsilon + h_\varepsilon = 2J_{A_\varepsilon}(w_\varepsilon) + \operatorname{curl}\big((\eta_\varepsilon^2 - 1) j_{A_\varepsilon}(w_\varepsilon)\big). \tag{7.17}$$

In the TDGL case, $h_\varepsilon$ satisfied a parabolic equation and its convergence could be deduced directly through the use of parabolic estimates and information we have about $J_{A_\varepsilon}(w_\varepsilon)$. In the case of the hyperbolic equation (7.17), the regularity of the right-hand side doesn't necessarily propagate to the solution and we do not have a good convergence for the magnetic field directly. However, we now show that we can establish a weak* in time convergence of the magnetic field. The last term in (7.17) converges weakly to zero as $\varepsilon \to 0$.

**Proposition 7.7.** *If $(w_\varepsilon, A_\varepsilon, \Phi_\varepsilon)$ are solutions to (6.5), then*

$$\|A_\varepsilon - A_*\|_{L^\infty(\Omega;\dot{W}^{-1,\infty}(0,T))} \to 0 \qquad \text{as } \varepsilon \to 0. \tag{7.18}$$

*Proof.* Take the curl of the 2nd MKG-pGL equation such that we have

$$-\Delta h_\varepsilon + h_\varepsilon = 2J_{A_\varepsilon}(w_\varepsilon) + \frac{1}{c^2}\operatorname{curl}\,\partial_t E_\varepsilon + \operatorname{curl}\big((\eta_\varepsilon^2 - 1) j_{A_\varepsilon}(w_\varepsilon)\big). \tag{7.19}$$

Multiply by the test function $\lambda \in C_c^{0,1}(0,T)$ and integrate:

$$-\Delta \int_0^T \lambda h_\varepsilon + \int_0^T \lambda h_\varepsilon = 2\int_0^T \lambda J_{A_\varepsilon}(w_\varepsilon) - \frac{1}{c^2}\int_0^T \partial_t\lambda \operatorname{curl} E_\varepsilon + \int_0^T \lambda \operatorname{curl}\big((\eta_\varepsilon^2 - 1) j_{A_\varepsilon}(w_\varepsilon)\big). \tag{7.20}$$

Since $h_*$ satisfies the London equation (2.44), then

$$-\Delta \int_0^T \lambda h_* + \int_0^T \lambda h_* = 2\pi \sum_{i=1}^n d_i \int_0^T \lambda \delta_{\xi_i(t)}. \tag{7.21}$$

Subtract (7.21) from (7.20) to get

$$-\Delta\int_0^T \lambda(h_\varepsilon - h_*) + \int_0^T \lambda(h_\varepsilon - h_*) = 2\int_0^T \lambda\big(J_{A_\varepsilon}(w_\varepsilon) - \pi\sum_{i=1}^n d_i\delta_{\xi_i(t)}\big) - \frac{1}{c^2}\int_0^T \partial_t\lambda \operatorname{curl} E_\varepsilon + \int_0^T \lambda \operatorname{curl}\big((\eta_\varepsilon^2-1)j_{A_\varepsilon}(w_\varepsilon)\big). \quad (7.22)$$

We estimate the final integral. For any function $\phi \in C_c^\infty(\Omega)$ with $\|\nabla\phi\|_{L^\infty(\Omega)} \le C$, note that

$$\|\nabla\phi\|_{L^2(\Omega)}\left\|\int_0^T \lambda \operatorname{curl}\big((\eta_\varepsilon^2-1)j_A\big)\right\|_{\dot W^{-1,2}(\Omega)} = \int_\Omega \phi\int_0^T \lambda \operatorname{curl}\big((\eta_\varepsilon^2-1)j_A\big) = -\int_0^T \lambda\int_\Omega \nabla^\perp\phi\cdot(\eta_\varepsilon^2-1)j_A$$

$$\le \|\nabla\phi\|_{L^\infty(\Omega)}\big\|\eta_\varepsilon^2-1\big\|_{L^\infty(\Omega)}\|j_A\|_{L^\infty([0,T];L^1(\Omega))}\int_0^T|\lambda|.$$

By the proof of Lemma 2.13, this implies that

$$\left\|\int_0^T \lambda \operatorname{curl}\big((\eta_\varepsilon^2-1)j_A\big)\right\|_{\dot W^{-1,2}(\Omega)} \le \frac{C}{|\log\varepsilon|}\sup_{t\in[0,T]}\big(C\varepsilon G_\varepsilon^{\eta_\varepsilon}(t)+C\sqrt{G_\varepsilon^{\eta_\varepsilon}(t)}\big)\int_0^T|\lambda|,$$

and the GL energy is bounded by $H_\varepsilon^{\eta_\varepsilon}$, which by Proposition 6.17 is bounded by $CT|\log\varepsilon|$ for all $t\in[0,T]$, and so

$$\left\|\int_0^T \lambda \operatorname{curl}\big((\eta_\varepsilon^2-1)j_A\big)\right\|_{\dot W^{-1,2}(\Omega)} = o_\varepsilon(1). \quad (7.23)$$

The remainder of the proof follows the proof of Proposition 4.3 in [60], which we present here for convenience. Lemma 4.4 of [60] records that, for $p>1$,

$$\left\|\lambda\big(J_{A_\varepsilon}(w_\varepsilon) - \pi\sum_{i=1}^n d_i\delta_{\xi_i(t)}\big)\right\|_{\dot W^{-1,p}(\Omega)} \le \|\lambda\|_{L^2([0,T])}\left\|J_{A_\varepsilon}(w_\varepsilon) - \pi\sum_{i=1}^n d_i\delta_{\xi_i(t)}\right\|_{L^2([0,T];\dot W^{-1,p}(\Omega))}. \quad (7.24)$$

Meanwhile, Lemma 4.5 of [60] has

$$\left\|\frac{1}{c^2}\int_0^T \partial_t\lambda \operatorname{curl} E\right\|_{\dot W^{-1,2}(\Omega)} \le \frac{1}{c^2}\|E\|_{L^\infty([0,T];L^2(\Omega))}\int_0^T|\partial_t\lambda|. \quad (7.25)$$

By (7.22)-(7.25), one can show that

$$\int_0^T \lambda(h_\varepsilon - h_*) \in W^{1,2}(\Omega),$$

and

$$\left\| \int_0^T \lambda(h_\varepsilon - h_*) \right\|_{W^{1,2}(\Omega)} \leq \left( \left\| J_{A_\varepsilon}(w_\varepsilon) - \pi \sum_{i=1}^n d_i \delta_{\xi_i(t)} \right\|_{L^2([0,T];\dot{W}^{-1,2}(\Omega))} + \frac{1}{c^2} \|E\|_{L^\infty([0,T];L^2(\Omega))} \right) \int_0^T |\partial_t \lambda| + o_\varepsilon(1). \quad (7.26)$$

In the Coulomb gauge, there exist $\psi_\varepsilon$ and $\psi_*$ such that

$$\Delta(\psi_\varepsilon - \psi_*) = h_\varepsilon - h_* \qquad \text{and} \qquad \nabla^\perp(\psi_\varepsilon - \psi_*) = A_\varepsilon - A_* \qquad \text{in } \Omega$$

and $\psi_\varepsilon = \psi_* = \psi_{ex}$ on $\partial\Omega$, with $\Delta\psi_{ex} = h_{ex}$. Hence,

$$\int_0^T \lambda(A_\varepsilon - A_*) \in W^{2,2}(\Omega) \hookrightarrow C^0(\overline{\Omega}).$$

Moreover, by (7.26),

$$\left\| \int_0^T \lambda(A_\varepsilon - A_*) \right\|_{L^\infty(\Omega)} \leq \left( \left\| J_{A_\varepsilon}(w_\varepsilon) - \pi \sum_{i=1}^n d_i \delta_{\xi_i(t)} \right\|_{L^2([0,T];\dot{W}^{-1,2}(\Omega))} + \frac{1}{c^2} \|E\|_{L^\infty([0,T];L^2(\Omega))} \right) \int_0^T |\partial_t \lambda| + o_\varepsilon(1).$$

That is,

$$\|A_\varepsilon - A_*\|_{L^\infty(\Omega;\dot{W}^{-1,\infty}(0,T))} \leq \left\| J_{A_\varepsilon}(w_\varepsilon) - \pi \sum_{i=1}^n d_i \delta_{\xi_i(t)} \right\|_{L^2([0,T];\dot{W}^{-1,2}(\Omega))} + \frac{1}{c^2} \|E\|_{L^\infty([0,T];L^2(\Omega))} + o_\varepsilon(1). \quad (7.27)$$

In light of Propositions 6.10 and 7.1, (7.27) implies the result. □

As a result of the previous proposition we have that the magnetic field converges weak* in time to the solution of the limiting London equation.

**Corollary 7.8.** *Suppose $h_*$ satisfies (2.44). For all $\zeta \in C_c^{0,1}(0,T)$, as $\varepsilon \to 0$,*

$$\int_0^T \zeta \int_\Omega (h_* - h_{ex})(h_\varepsilon - h_*) \to 0. \quad (7.28)$$

*Proof.* The proof is the same as the proof to Corollary 4.6 of [60] if one replaces $\psi_*$ by $(\psi_* - \psi_{ex})$. □

We now go about estimating the extra quantities which appear in the evolution equation (7.14). The first relies on applying an updated version of the coercivity estimate in the MKG setting.

**Proposition 7.9.** *Suppose $(w_\varepsilon, A_\varepsilon, \Phi_\varepsilon)$ are solutions to (6.5), $w_*$ satisfies (2.42), $A_*$ satisfies (2.45) and the initial data is well-prepared (see (6.10)). Additionally, suppose that both $A_\varepsilon$ and $A_*$ satisfy the Coulomb gauge condition. Let $r_0 < \frac{1}{4} r_{a(0)}$ and $T > 0$ satisfy*

$$\sup_{t\in[0,T]} |\omega(t)| + T \sup_{t\in[0,T]} |\dot{a}(t)| \leq \frac{r_0}{4}. \tag{7.29}$$

*Then there exists a constant $K > 0$ with the following property: for any $\phi \in W_0^{2,\infty}(\Omega)$ that is affine in $\bigcup_i B_{r_0}(a_i(0))$ and satisfies $\|D^2\phi\|_{L^\infty(\Omega)} \leq K/r_0^2$, the following estimate holds:*

$$\left| \int_0^T \zeta \int_\Omega \nabla\nabla^\perp\phi : \eta_\varepsilon^2 \big( \nabla_{A_\varepsilon} w_\varepsilon \otimes \nabla_{A_\varepsilon} w_\varepsilon - j_{A_*}(w_*) \otimes j_{A_*}(w_*) \big) \right| \leq \frac{2K}{r_0^2} \int_0^T |\zeta| D_\varepsilon(t) + o_\varepsilon(1). \tag{7.30}$$

*Proof.* Our first action is to note that

$$\int_0^T \zeta \int_\Omega \nabla\nabla^\perp\phi : \eta_\varepsilon^2 \big( \nabla_{A_\varepsilon} w_\varepsilon \otimes \nabla_{A_\varepsilon} w_\varepsilon - j_{A_*}(w_*) \otimes j_{A_*}(w_*) \big) = J_1 + J_2 + J_3 + J_4, \tag{7.31}$$

where

$$J_1 = \int_0^T \zeta \int_\Omega \nabla\nabla^\perp\phi : \eta_\varepsilon^2 \nabla|w| \otimes \nabla|w|$$

$$J_2 = \int_0^T \zeta \int_\Omega \nabla\nabla^\perp\phi : \eta_\varepsilon^2 \Big( \frac{j_A}{|w|} - j_{A_*} \Big) \otimes \Big( \frac{j_A}{|w|} - j_{A_*} \Big)$$

$$J_3 = \int_0^T \zeta \int_\Omega \nabla\nabla^\perp\phi : \eta_\varepsilon^2 \Big[ \Big( \frac{j_A}{|w|} - j_A \Big) \otimes j_{A_*} + \big( j_A - j_{A_*} \big) \otimes j_{A_*} \Big]$$

$$J_4 = \int_0^T \zeta \int_\Omega \nabla\nabla^\perp\phi : \eta_\varepsilon^2 \Big[ j_{A_*} \otimes \Big( \frac{j_A}{|w|} - j_A \Big) + j_{A_*} \otimes \big( j_A - j_{A_*} \big) \Big].$$

The coercivity estimate of Theorem 3.9 estimates these integrals, however, the estimate is not immediately applicable in the MKG framework. As stated, Theorem 3.9 requires the convergence of $(A_\varepsilon - A_*)$ in $L^q(\Omega)$ $(q > 2)$. In the MKG framework, the best we have is weak convergence of the vector potential (see Proposition 7.7). Relaxing the requirement for $\|A_\varepsilon - A_*\|_{L^q(\Omega)} \to 0$ in Theorem 3.9, the coercivity estimate reads

$$\int_{\Omega_{r_\star}(\xi)} e_\varepsilon^{\eta_\varepsilon}(|w_\varepsilon|) + \frac{\eta_\varepsilon^2}{2}\left|\frac{j_{A_\varepsilon}(w_\varepsilon)}{|w_\varepsilon|} - j_{A_*}(w_*)\right|^2 + \frac{1}{2}\int_\Omega |h_\varepsilon - h_*|^2 \\ \leq D_\varepsilon + \int_\Omega (h_* - h_{ex})(h_\varepsilon - h_*) + \int_{\Omega_r(\xi)} j_{A_*}\cdot(A_\varepsilon - A_*) + error_1.$$

Proposition 7.1 implies that $s_\varepsilon$ from Theorem 3.9 converges to 0 as $\varepsilon \to 0$. In turn, this implies that $r_\star \to 0$ as $\varepsilon \to 0$. Let $r_0 > r_\star$, where $r_0$ is independent of $\varepsilon$. The coercivity estimate (3.22) implies that

$$\int_{\Omega_{r_0}(\xi(t))} e_\varepsilon^{\eta_\varepsilon}(|w_\varepsilon|) + \frac{\eta_\varepsilon^2}{2}\left|\frac{j_{A_\varepsilon}(w_\varepsilon)}{|w_\varepsilon|} - j_{A_*}(w_*)\right|^2 + \frac{1}{2}\int_\Omega |h_\varepsilon - h_*|^2 \\ \leq D_\varepsilon + \int_\Omega (h_* - h_{ex})(h_\varepsilon - h_*) + \int_{\Omega_r} j_{A_*}\cdot(A_\varepsilon - A_*) + o_\varepsilon(1). \quad (7.32)$$

Condition (7.29) ensures that (7.32) holds for all $t \in [0, T]$.

Multiply (7.32) through by the test function $\zeta \in C^\infty([0,T])$ and integrate in time:

$$\int_0^T \zeta \int_{\Omega_{r_0}(\xi(t))} e_\varepsilon^{\eta_\varepsilon}(|w_\varepsilon|) + \frac{\eta_\varepsilon^2}{2}\left|\frac{j_{A_\varepsilon}(w_\varepsilon)}{|w_\varepsilon|} - j_{A_*}(w_*)\right|^2 + \frac{1}{2}\int_0^T \zeta\int_\Omega |h_\varepsilon - h_*|^2 \\ \leq \int_0^T \zeta D_\varepsilon(t) + \int_0^T \zeta\int_\Omega (h_* - h_{ex})(h_\varepsilon - h_*) \\ + \int_0^T \zeta \int_{\Omega_r} j_{A_*}\cdot(A_\varepsilon - A_*) + o_\varepsilon(1). \quad (7.33)$$

By (3.6), the limiting supercurrent is bounded by a constant for all time, hence, Proposition 7.7 and Corollary 7.8 imply that (7.33) is

$$\int_0^T \zeta \int_{\Omega_{r_0}(\xi(t))} e_\varepsilon^{\eta_\varepsilon}(|w_\varepsilon|) + \frac{\eta_\varepsilon^2}{2}\left|\frac{j_{A_\varepsilon}(w_\varepsilon)}{|w_\varepsilon|} - j_{A_*}(w_*)\right|^2 + \frac{1}{2}\int_0^T \zeta\int_\Omega |h_\varepsilon - h_*|^2 \\ \leq \int_0^T \zeta D_\varepsilon(t) + o_\varepsilon(1). \quad (7.34)$$

This "new coercivity estimate" in the MKG setting implies (compare with (3.62))

$$\int_0^T \zeta \left\| j_A - j_{A_*} \right\|_{L^{4/3}(\Omega)} \leq C \int_0^T \zeta \sqrt{D(t)} + o_\varepsilon(1). \tag{7.35}$$

We make the following estimate using (7.34)

$$\begin{aligned} J_1 + J_2 &\leq \frac{K}{r_0^2} \int_0^T |\zeta| \int_{\Omega_{r_0}(\xi(t))} e_\varepsilon^{\eta_\varepsilon}(|w_\varepsilon|) + \frac{\eta_\varepsilon^2}{2} \left| \frac{j_{A_\varepsilon}(w_\varepsilon)}{|w_\varepsilon|} - j_{A_*}(w_*) \right|^2 \\ &\leq \frac{K}{r_0^2} \int_0^T |\zeta| D_\varepsilon(t) + o_\varepsilon(1). \end{aligned} \tag{7.36}$$

Finally, by symmetry, it suffices to treat one of either $J_3$ or $J_4$. Using $\|\eta_\varepsilon^2\|_{L^\infty(\Omega)} \leq C$ with (3.23) from [35] and (7.35) implies $J_3 = o_\varepsilon(1)$. Using this with (7.36) in (7.31) provides the result. □

Now we show that when combining $I_1$ and $I_2$, the excess energy and electric field produces a nice cancellation and the remaining terms are small. This is the most technical estimate and calls on Proposition 7.9, a kinetic energy estimate from [60] and a relationship between the renormalized energy evolution and the $a_i$'s from 7.11.

**Proposition 7.10.** *For $I_1$ and $I_2$ defined as in Proposition 7.6 and $\phi \in W_0^{2,\infty}(\Omega)$ satisfying the properties in Proposition 7.9, there exists a constant $M > 0$ such that*

$$\limsup_{\varepsilon \to 0} (I_1 + I_2) \leq M \int_0^T |\zeta| m(t) + o_\varepsilon(1), \tag{7.37}$$

*where*

$$m(t) := |\dot{\xi}(t) - \dot{a}(t)| + |\xi(t) - a(t)|. \tag{7.38}$$

*Proof.* Proposition 7.9 states that $I_1 \leq \frac{2K}{r_0^2} \int_0^T |\zeta| D_\varepsilon(t) + o_\varepsilon(1)$. Recall from the definition of the excess energy, (2.51), that

$$\begin{aligned} D_\varepsilon(t) &= G_\varepsilon^{\eta_\varepsilon}(t) - n(\gamma_0 + \pi |\log \varepsilon|) - W^Q(\xi(t), d) \\ &= H_\varepsilon^{\eta_\varepsilon}(t) - \frac{1}{2c^2} \int_\Omega \left( \eta_\varepsilon^2 |\partial_{\Phi_\varepsilon} w_\varepsilon|^2 + |E_\varepsilon|^2 \right) - n(\gamma_0 + \pi |\log \varepsilon|) - W^Q(\xi(t), d). \end{aligned} \tag{7.39}$$

Using (6.17) in (7.39) gives

$$D_\varepsilon(t) = -\frac{1}{2c^2}\int_\Omega \left(\eta_\varepsilon^2|\partial_{\Phi_\varepsilon} w_\varepsilon|^2 + |E_\varepsilon|^2\right) + W^Q(\xi(0),d) - W^Q(\xi(t),d) + o_\varepsilon(1). \tag{7.40}$$

With this in hand, we can make the following estimate:

$$\begin{aligned} I_2 + I_1 \leq \frac{K}{c^2 r_0^2}\int_0^T |\zeta| \int_\Omega |E_\varepsilon|^2 - \frac{K}{c^2 r_0^2}\int_0^T |\zeta| \int_\Omega \left(\eta_\varepsilon^2|\partial_{\Phi_\varepsilon} w_\varepsilon|^2 + |E_\varepsilon|^2\right) \\ + \frac{2K}{r_0^2}\int_0^T |\zeta|\big(W^Q(\xi(0),d) - W^Q(\xi(t),d)\big) + o_\varepsilon(1). \end{aligned} \tag{7.41}$$

By Proposition 4.2 of [60], we have for all $0 \leq T_1 \leq T$ and all $\zeta \in C^\infty[0,T_1]$,

$$\liminf_{\varepsilon\to 0}\int_0^{T_1}\int_{B_r(\xi_i(0))} |\zeta|\frac{\eta_\varepsilon^2}{c^2}|\partial_{\Phi_\varepsilon} w_\varepsilon|^2 \geq \frac{\pi}{\mu^2}\int_0^{T_1} |\zeta||\dot{\xi}_i|^2, \qquad i = 1,..,n. \tag{7.42}$$

Thus, using (7.42) in (7.41) provides (let $C_0 := K/r_0^2$ in the following)

$$\begin{aligned} \limsup_{\varepsilon\to 0}(I_1 + I_2) \leq -\frac{\pi C_0}{\mu^2}\int_0^T |\zeta| \sum_{i=1}^n |\dot{\xi}_i(t)|^2 \\ + 2C_0\int_0^T |\zeta|\big(W^Q(\xi(0),d) - W^Q(\xi(t),d)\big) + o_\varepsilon(1). \end{aligned} \tag{7.43}$$

Recall from (7.11) that $\xi_i(0) = a_i(0)$. Thus, $W^Q(\xi(0),d) = W^Q(a(0),d)$. We now do some work using the ODE (7.11) in order to replace $W^Q(a(0),d)$ with $W^Q(a(t),d)$ and a term containing the positions $a_i(t)$.

Note that

$$\frac{d}{dt}W^Q(a(t),d) = \sum_{i=1}^n \nabla_{a_i} W^Q(a(t),d)\cdot \dot{a}_i(t) = -\frac{\pi}{\mu^2}\sum_{i=1}^n \ddot{a}_i\cdot\dot{a}_i. \tag{7.44}$$

Since $\ddot{a}_i\cdot\dot{a}_i = \frac{1}{2}\frac{d}{dt}|\dot{a}_i|^2$, (7.44) implies

$$\begin{aligned} W^Q(a(0)) - W^Q(a(t)) &= -\int_0^t \frac{d}{ds}W^Q(a(s),d) \\ &= \frac{\pi}{2\mu^2}\int_0^t \sum_{i=1}^n \frac{d}{ds}|\dot{a}_i|^2 \\ &= \frac{\pi}{2\mu^2}\sum_{i=1}^n \big(|\dot{a}_i(t)|^2 - |\dot{a}_i(0)|^2\big). \end{aligned} \tag{7.45}$$

By (7.11), $\dot{a}_i(0) = 0$ for all $i$, so using (7.45) in (7.43) provides

$$\limsup_{\varepsilon\to 0}(I_1 + I_2) \leq \frac{\pi C_0}{\mu^2}\int_0^T |\zeta| \sum_{i=1}^n \big(|\dot{a}_i(t)|^2 - |\dot{\xi}_i(t)|^2\big) + 2C_0\int_0^T |\zeta|\big(W^Q(a(t),d) - W^Q(\xi(t),d)\big) + o_\varepsilon(1). \quad (7.46)$$

Allow us to write $(|\dot{a}(t)|^2 - |\dot{\xi}(t)|^2)$ in place of $\sum_{i=1}^n \big(|\dot{a}_i(t)|^2 - |\dot{\xi}_i(t)|^2\big)$. Note that

$$\begin{aligned}\int_0^T (|\dot{a}(t)|^2 - |\dot{\xi}(t)|^2) &= \int_0^T (\dot{a} - \dot{\xi})(\dot{a} + \dot{\xi}) \\ &= \int_0^T -\dot{\omega}(\dot{\omega} + 2\dot{a}) \\ &\leq 0 + C\sup_{t\in[0,T]} |\dot{a}(t)| \int_0^T |\dot{\omega}|.\end{aligned}$$

From (7.11), it follows that $|\dot{a}(t)|$ can be controlled by $|\nabla_{a_i} W^Q(a,d)|$, which in turn is bounded by a constant due to (5.37) and our *a priori* assumptions on the boundedness of $\nabla Q_0$. Hence, (7.46) becomes

$$\limsup_{\varepsilon\to 0}(I_1+I_2) \leq C\int_0^T |\zeta|\Big(|\dot{\omega}(t)|+W^Q(a(t),d)-W^Q(\xi(t),d)\Big)+o_\varepsilon(1). \quad (7.47)$$

We now use that (again by (5.37))

$$\big|W^Q(a,d) - W^Q(\xi,d)\big| = \big|\nabla_{a_i} W^Q(a,d) - \nabla_{\xi_i} W^Q(\xi,d)\big||a-\xi| \leq C|\omega|,$$

and the result follows. □

In the propositions which follow, we proceed to show that all other quantities on the right-hand side of (7.14) are vanishingly small in the limit $\varepsilon \to 0$.

**Proposition 7.11.** *Let $\phi \in W_0^{2,\infty}(\Omega)$ satisfy the properties in Proposition 7.9. If $w_*$ is defined by (2.42) and $A_*$ is defined by (2.45), then*

$$\left|\int_0^T \zeta \int_\Omega \nabla\nabla^\perp\phi : \big((1-\eta_\varepsilon^2) j_{A_*}(w_*) \otimes j_{A_*}(w_*)\big)\right| \leq o_\varepsilon(1).$$

*Proof.* Recall that $\|1-\eta_\varepsilon^2\|_{L^\infty(\Omega)} \le C/|\log\varepsilon|$, and using the affine property of $\phi$ in the balls allows us to write

$$\int_0^T \zeta \int_\Omega \nabla\nabla^\perp\phi : \big((1-\eta_\varepsilon^2) j_{A_*}(w_*)\otimes j_{A_*}(w_*)\big) \le \frac{C}{r_0^2|\log\varepsilon|}\int_0^T |\zeta|\,\|j_{A_*}(w_*)\|^2_{L^\infty(\Omega_{r_0}(a_i(0)))}. \tag{7.48}$$

From (3.7) we have

$$\|j_{A_*}(u_*)\|_{L^\infty(\Omega_r)} \le C\Big(\frac{n}{r}+h_{ex}\Big)$$

which leads to the desired result when in combination with (7.48). □

**Proposition 7.12.** *For the $I_i$ defined in Proposition 7.6, $\sum_{i=4}^{10} I_i = o_\varepsilon(1)$.*

*Proof.* Consider that

$$I_4 = \frac{c^2}{2}\int_0^T \dot\zeta \int_\Omega \psi\eta_\varepsilon^2 \partial_t\big(|w|^2-1\big) = -\frac{c^2}{2}\int_0^T \ddot\zeta \int_\Omega \psi\eta_\varepsilon^2\big(|w|^2-1\big).$$

Using that $\|\psi\|_{L^\infty(\Omega)} \le C$,

$$|I_4| \le C|\log\varepsilon| \int_0^T |\ddot\zeta|\,\big\|\eta_\varepsilon^2(|w|^2-1)\big\|_{L^2(\Omega)} \le C|\log\varepsilon|\int_0^T |\ddot\zeta|\varepsilon\sqrt{H_\varepsilon^{\eta_\varepsilon}}.$$

Since (6.17) implies $H_\varepsilon^{\eta_\varepsilon} \le C|\log\varepsilon|$, and we assume $\int_0^T|\ddot\zeta|$ isn't too large, we have that $I_4 \to 0$ as $\varepsilon\to 0$.

To estimate $I_5$, apply Cauchy-Schwarz and use that $h_{ex} = O(1)$:

$$|I_5| \le \frac{C}{c^2}\int_0^T |\ddot\zeta|\left(\int_\Omega |h_{ex}|^2\right)^{\frac12}\left(\int_\Omega |h-h_{ex}|^2\right)^{\frac12} \le \frac{C}{c^2}\int_0^T |\ddot\zeta|\sqrt{H_\varepsilon^{\eta_\varepsilon}}.$$

Since $H_\varepsilon^{\eta_\varepsilon} \le C|\log\varepsilon|$ we have that $I_5 = o_\varepsilon(1)$.

Arguing as in the proof of Lemma 5.14 we have $|I_6|+|I_7| \le o_\varepsilon(1)$. Next, by (6.20) and $\nabla\log\eta_\varepsilon^2 \le C\frac{\nabla Q_\varepsilon}{|\log\varepsilon|}$,

$$I_8 \le C\,\big\|\nabla\log\eta_\varepsilon^2\big\|_{L^\infty(\Omega)}\int_0^T |\zeta|\,\Big\|\frac{\eta_\varepsilon}{c}|\partial_\Phi w|\Big\|^2_{L^2(\Omega)} \le \frac{C\nabla Q_\varepsilon}{|\log\varepsilon|}\int_0^T|\zeta| \le o_\varepsilon(1).$$

Finally,

$$\begin{aligned}|I_9| &= \left|\frac{c^4}{2}\int_0^T \zeta \int_\Omega \phi \nabla \eta_\varepsilon^2 \ \mathrm{curl}(|w|^2-1)\right| \\ &= \left|\frac{c^4}{2}\int_0^T \zeta \int_\Omega \nabla^\perp \phi \cdot \nabla(\log \eta_\varepsilon^2)\eta_\varepsilon^2(|w|^2-1)\right| \\ &\leq Cc^4 \int_0^T |\zeta| \Big(\int_\Omega |\nabla \log \eta_\varepsilon^2|^2\Big)^{\frac{1}{2}} \Big(\int_\Omega \eta_\varepsilon^4 (|w|^2-1)^2\Big)^{\frac{1}{2}} \\ &\leq C\varepsilon |\log \varepsilon| |\nabla Q_\varepsilon| \int_0^T |\zeta| \sqrt{H_\varepsilon^{\eta_\varepsilon}}.\end{aligned}$$

Thus, $I_9 = o_\varepsilon(1)$.

It remains to show

$$I_{10} = \frac{1}{c^2}\int_0^T \dot{\zeta} \int_{\Omega_r^0} (\nabla^\perp \phi - \nabla \psi) \cdot (\eta_\varepsilon^2 \langle \partial_\Phi w, \nabla_A w\rangle + hE^\perp) \leq o_\varepsilon(1).$$

By the Cauchy-Schwarz inequality

$$\frac{1}{c^2}\int_0^T \dot{\zeta} \int_{\Omega_r^0} \eta_\varepsilon^2 \langle \partial_\Phi w, \nabla_A w\rangle \leq \frac{1}{c}\Big(\int_0^T \dot{\zeta} \int_\Omega \frac{\eta_\varepsilon^2}{c^2} |\partial_\Phi w|^2\Big)^{\frac{1}{2}} \Big(\int_0^T \dot{\zeta} \int_{\Omega_r^0} \eta_\varepsilon^2 |\nabla_A w|^2\Big)^{\frac{1}{2}}.$$

The bound on $\partial_\Phi w$ given by (6.20) leads to

$$\frac{1}{c^2}\int_0^T \dot{\zeta} \int_{\Omega_r^0} \eta_\varepsilon^2 \langle \partial_\Phi w, \nabla_A w\rangle \leq \frac{C}{c}\Big(\int_0^T \dot{\zeta} \int_{\Omega_r^0} \eta_\varepsilon^2 |\nabla_A w|^2\Big)^{\frac{1}{2}}. \tag{7.49}$$

Using the decomposition $|\nabla_A w|^2 = |\nabla |w||^2 + \left|\frac{j_A(w)}{|w|}\right|$, it follows that

$$\int_0^T \dot{\zeta} \int_{\Omega_r^0} \eta_\varepsilon^2 |\nabla_{A_\varepsilon} w_\varepsilon|^2 \leq \int_0^T \dot{\zeta} \int_{\Omega_r^0} e_\varepsilon^{\eta_\varepsilon}(|w_\varepsilon|) + \eta_\varepsilon^2 \left|\frac{j_{A_\varepsilon}(w_\varepsilon)}{|w_\varepsilon|} - j_{A_*}(w_*)\right|^2 + \eta_\varepsilon^2 |j_{A_*}(w_*)|^2,$$

and by (7.34) (for $r > r_0$)

$$\int_0^T \dot{\zeta} \int_{\Omega_r^0} \eta_\varepsilon^2 |\nabla_{A_\varepsilon} w_\varepsilon|^2 \leq \int_0^T \dot{\zeta} \int_{\Omega_r^0} \eta_\varepsilon^2 |j_{A_*}(w_*)|^2 + C\int_0^T \dot{\zeta} D_\varepsilon(t) + o_\varepsilon(1).$$

From (3.7) we deduce that

$$\int_{\Omega_r^0} \eta_\varepsilon^2 |j_{A_*}(w_*)|^2 \leq C|\log r|$$

which means that ultimately (7.49) becomes

$$\frac{1}{c^2}\int_0^T \dot{\zeta} \int_{\Omega_r^0} \eta_\varepsilon^2 \langle \partial_\Phi w, \nabla_A w\rangle \leq \frac{C}{c}\sqrt{\int_0^T \dot{\zeta}\Big(D_\varepsilon(t) + |\log r| + C\Big)}.$$

Using Cauchy-Schwarz once more,

$$\frac{1}{c^2}\int_0^T \dot{\zeta}\int_{\Omega_r^0} hE^{\perp} \leq \frac{1}{c}\Big(\int_0^T \dot{\zeta}\int_{\Omega}\frac{1}{c^2}|E|^2\Big)^{\frac{1}{2}}\Big(\int_0^T \dot{\zeta}\int_{\Omega}|h|^2\Big)^{\frac{1}{2}}.$$

The bound given in (6.20) provides $\int_\Omega c^{-2}|E|^2 \leq C$ and Proposition 3.5 in [60] has $\int_\Omega |h|^2 \leq C$. Hence,

$$\frac{1}{c^2}\int_0^T \dot{\zeta}\int_{\Omega_r^0} hE^{\perp} \leq o_\varepsilon(1).$$

Provided $D_\varepsilon(t)$ stays bounded, $I_{10} \leq o_\varepsilon(1)$ and the proof is completed. □

We now use the previous results to arrive at the limiting dynamical law. We seek to use a Gronwall argument to show that our vortex positions $\xi_i(t)$ (the ones found in Proposition 7.1) actually coincide with the $a_i(t)$ from (7.11) for all $t \in [0,T]$.

**Proposition 7.13.** *The vortex positions $\xi_i(t) = a_i(t)$, $i = 1, ..., n$, satisfy the ODEs*

$$\frac{1}{\mu^2}\ddot{a}_i(t) = -\frac{1}{\pi}\nabla_{a_i}W(a,d) - \sum_{i=1}^{n}\nabla Q_0(a_i) \qquad \forall t \in [0,T], \tag{7.50}$$

*where $T$ satisfies Propositions 6.17 and 7.9.*

*Proof.* Using Propositions 7.10, 7.11 and 7.12 in (7.14) gives

$$\int_0^T \Big(\int_\Omega \big(\ddot{\zeta}\psi\frac{\mathcal{H}_\varepsilon^{\eta_\varepsilon}}{c^2} + \zeta\nabla\psi\cdot\nabla(\log\eta_\varepsilon^2)g_\varepsilon^{\eta_\varepsilon}\big) + \zeta\nabla\psi\cdot\nabla_{\xi_i}W(\xi(t),d)\Big) \leq M\int_0^T |\zeta|m(t) + o_\varepsilon(1). \tag{7.51}$$

We let $\zeta$ be any function in $C_c^\infty(0,T)$ such that $\int_0^T |\ddot{\zeta}| < C$ and recall that the definition of $(\psi,\phi)$ ensures that $|\phi| + |\psi| + |\nabla\phi| + |\nabla\psi| < C$, which was used when estimating quantities in the previous subsection.

By Proposition 7.2 we have that

$$\frac{1}{\mu^2}\int_0^T\int_\Omega \ddot{\zeta}\psi\frac{\mathcal{H}_\varepsilon^{\eta_\varepsilon}}{|\log\varepsilon|} \to \frac{\pi}{\mu^2}\int_0^T \ddot{\zeta}\xi_i \qquad \text{as } \varepsilon \to 0. \tag{7.52}$$

Next, use Proposition 5.2 to deduce that

$$\int_0^T \int_\Omega \zeta \nabla^\perp \phi \cdot \nabla(\log \eta_\varepsilon^2) g_\varepsilon^{\eta_\varepsilon} = \int_0^T \int_\Omega \zeta \nabla \psi \cdot \frac{\nabla Q_\varepsilon}{1 + \frac{Q_\varepsilon}{|\log \varepsilon|}} \frac{g_\varepsilon^{\eta_\varepsilon}}{|\log \varepsilon|}$$
$$\to \pi \int_0^T \zeta \ \nabla Q_0(\xi). \tag{7.53}$$

From (7.52) and (7.53) we see that passing to the limit $\varepsilon \to 0$ in (7.51) provides

$$\frac{\pi}{\mu^2} \int_0^T \ddot{\zeta} \xi_i \leq - \int_0^T \zeta \ \nabla_{\xi_i} W^Q(\xi(t), d) + M \int_0^T |\zeta| m(t). \tag{7.54}$$

Subtract (7.12) from (7.54) such that

$$\frac{\pi}{\mu^2} \int_0^T \ddot{\zeta}(\xi_i - a_i) \leq - \int_0^T \zeta \big( \nabla_{\xi_i} W^Q(\xi(t), d) - \nabla_{a_i} W^Q(a(t), d) \big) + M \int_0^T |\zeta| m(t),$$

which implies

$$\frac{\pi}{\mu^2} \int_0^T |\zeta| \left| \frac{d^2}{dt^2}(\xi_i - a_i) \right| \leq \int_0^T |\zeta| \left| \nabla_{\xi_i} W^Q(\xi(t), d) - \nabla_{a_i} W^Q(a(t), d) \right|$$
$$+ M \int_0^T |\zeta| m(t). \tag{7.55}$$

By (5.42), the gradient of the renormalized energies and pinning potentials can be bounded by $C|\omega(t)|$ and so absorbed into the integral involving $m(t)$.

Since $\zeta$ is arbitrary, we have that

$$\left| \frac{d^2}{dt^2}(\xi(t) - a(t)) \right| \leq C m(t). \tag{7.56}$$

Note that

$$\frac{dm}{dt} = \frac{d}{dt}|\xi - a| + \frac{d}{dt} \left| \frac{d}{dt}(\xi - a) \right|$$
$$\leq \left| \frac{d}{dt}|\xi - a| \right| + \left| \frac{d}{dt} \left| \frac{d}{dt}(\xi - a) \right| \right|.$$

Using that $\left| \frac{d}{dt}|f(t)| \right| \leq \left| \frac{d}{dt} f(t) \right|$, we have

$$\frac{dm}{dt} \leq \left| \frac{d}{dt}(\xi - a) \right| + \left| \frac{d^2}{dt^2}(\xi - a) \right| \leq m + \left| \frac{d^2}{dt^2}(\xi - a) \right|,$$

and then by (7.56),

$$\frac{d}{dt} m(t) \leq C m(t). \tag{7.57}$$

Notice Proposition 7.14 below. Since $\dot{\xi}(0) = 0$, by (7.57) and Gronwall's inequality (see Appendix E), we have that $\xi(t) \equiv a(t)$ for all $t \in [0, T]$ and that the vortex positions satisfy the ODE (7.50). □

**Proposition 7.14.** *(Proposition 4.7 of [60]). The vortex paths $\xi_i(t)$, $i = 1, ..., n$, satisfy $\dot{\xi}_i(0) = 0$ for any $i = 1, ..., n$.*

*Proof.* The proof is the same as in [60] by replacing $W(a, d)$ with $W^Q(a, d)$. □

Our final results concern the limiting behaviour of a several quantities in the model. These results rely on the kinetic energy bound from [60] and our recently discovered ODE (7.50).

**Proposition 7.15.** *(Proposition 4.8 of [60]). As $\varepsilon \to 0$,*

$$\int_0^T D_\varepsilon(t) \to 0, \qquad \frac{1}{c^2}\int_0^T \int_\Omega |E_\varepsilon|^2 \to 0 \tag{7.58}$$

*and*

$$\int_0^T \int_\Omega \frac{\eta_\varepsilon^2}{c^2}|\partial_{\Phi_\varepsilon} w_\varepsilon|^2 \to \frac{\pi}{\mu^2}\int_0^T \sum_{i=1}^n |\dot{a}_i(t)|^2 \tag{7.59}$$

*Proof.* Say $\zeta \equiv 1$ on $[0, T]$. Then Proposition 4.2 of [60] states

$$\frac{\pi}{\mu^2}\int_0^T \sum_{i=1}^n |\dot{a}_i(t)|^2 \leq \liminf_{\varepsilon\to 0} \int_0^T \int_\Omega \frac{\eta_\varepsilon^2}{c^2}|\partial_{\Phi_\varepsilon} w_\varepsilon|^2.$$

Thus,

$$\frac{\pi}{2\mu^2}\int_0^T \sum_{i=1}^n |\dot{a}_i(t)|^2 + \lim_{\varepsilon\to 0}\Big[\int_0^T D_\varepsilon(t) + \frac{1}{2c^2}\int_0^T \int_\Omega |E_\varepsilon|^2\Big]$$
$$\leq \lim_{\varepsilon\to 0}\Big[\int_0^T D_\varepsilon(t) + \frac{1}{2c^2}\int_0^T \int_\Omega |E_\varepsilon|^2 + \eta_\varepsilon^2|\partial_{\Phi_\varepsilon} w_\varepsilon|^2\Big].$$

By (7.40) we then have

$$\frac{\pi}{2\mu^2}\int_0^T \sum_{i=1}^n |\dot{a}_i(t)|^2 + \lim_{\varepsilon\to 0}\Big[\int_0^T D_\varepsilon(t) + \frac{1}{2c^2}\int_0^T \int_\Omega |E_\varepsilon|^2\Big]$$
$$\leq \int_0^T W^Q(a(0), d) - W^Q(a(t), d).$$

And then by (7.45), we have

$$\frac{\pi}{2\mu^2}\int_0^T \sum_{i=1}^n |\dot{a}_i(t)|^2 + \lim_{\varepsilon\to 0}\Big[\int_0^T D_\varepsilon(t) + \frac{1}{2c^2}\int_0^T \int_\Omega |E_\varepsilon|^2\Big]$$
$$\leq \frac{\pi}{2\mu^2}\int_0^T \sum_{i=1}^n |\dot{a}_i(t)|^2.$$

This allows us to deduce (7.58). With these newfound convergences, (7.40) now implies

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega \frac{\eta_\varepsilon^2}{2c^2}|\partial_{\Phi_\varepsilon} w_\varepsilon|^2 = \int_0^T W^Q(a(0)) - W^Q(a(t)).$$

And then (7.45) leads to

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega \frac{\eta_\varepsilon^2}{2c^2}|\partial_{\Phi_\varepsilon} w_\varepsilon|^2 = \frac{\pi}{2\mu^2}\int_0^T \sum_{i=1}^n |\dot{a}_i(t)|^2.$$

□

We see that the second order time-derivative in the MKG-pGL equation for the order parameter carries forward to the limiting dynamical law for the vortex positions. This second-order ODE is reminiscent of Newton's second law, so perhaps more physically realistic than the first-order ODE derived in the TDGL setting. The equation there is closer to a transport equation. The chief difference in deriving the limiting motion laws in the two settings is in how the excess energy is used. In the TDGL setting, the excess energy forms part of the quantity to be estimated using the Gronwall inequality, whereas in the MKG setting we were required to use the form of the excess energy to induce a cancellation with the electric field term which appears as part of the stress-energy tensor. This leads to a Gronwall inequality applied to terms containing just the difference in positions and velocities of the vortices and the ideal points $a_i(t)$. Both methods are otherwise very similar and we don't deem there to be much difference in the difficulty in applying our methods to either setting.

# Chapter 8

# Conclusion

Chapters 3-5 saw us derive the limiting dynamical law as $\varepsilon \to 0$ for vortex positions in a superconductor with pinning sites. The result is an ODE with vortex motion having components both parallel and perpendicular to the gradient of the renormalized energy and pinning potential. One may interpret the results from this work in a mathematical context or a physical context. Mathematically, the novelty of the work presented here is that it combines the mixed-flow dynamics with the critically-scaled pinning potential whilst keeping all the gauge terms throughout the calculations. Hence, the results presented here are among some of the most general available in the literature on vortex dynamics.

The significance of the work in this thesis is as much about the method as it is about the results. The first major result is the coercivity estimate of Chapter 3. It is the first estimate of its kind to combine the pinning potential and gauge terms. It allows one to obtain information about the closeness of the energy to the energy of the optimal configuration. The proof required us to show that the integral of the difference between supercurrent and limiting supercurrent is small in the limit $\varepsilon \to 0$. The method of showing this relied on the weak-convergence of the order parameter in the space outside of the vortex balls. Our method builds on that which appears in [48] by including the pinning and gauge terms, and circumvents the quite involved explicit estimates of [35], where a similar coercivity estimate is presented. In that work they integrate

over a domain which may vary in size and compensate with some additional estimates on the size of the domain in their calculations.

The derivation of the dynamical law is done in a modern way by using the space-time Jacobian and known information about its convergence properties. This circumvents the need to manipulate the evolution equation of Chapter 5 to bring about a certain cancellation in terms containing $\langle i\partial_{\Phi_\varepsilon} w_\varepsilon, \nabla_{A_\varepsilon} w_\varepsilon \rangle$ which is necessary for the proof of the limiting dynamical law in [35], where a similar dynamical law is derived. In all, you could say that this work builds on that of [35], making modifications to certain elements of their proofs.

In terms of the physics, the work presented here allows for greater understanding of the intermediate state of type-II superconductors. Any applications of superconductors with pinning sites may find our work to be of interest. However, a reward for such generality in our results is that we may easily recover the analogous vortex dynamics for systems without pinning sites, mixed-flow or gauge. For instance, we may recover the Gross-Pitaevskii equation by removing the pinning and gauge and setting $\alpha_0 = 0$. The Gross-Pitaevskii equation describes the behaviour of superfluids, Bose-Einstein condensates and is applicable in the field of nonlinear optics, among other things [4, 53].

In Chapters 6 and 7 we derived results in the framework of the Maxwell-Klein-Gordon model supplemented with a Ginzburg-Landau potential containing a pinning parameter. Chapter 6 contains energy considerations and Chapter 7 presents the vortex motion law. The vortices follow the gradient descent of the renormalized energy and pinning potential in an equation representing Newton's second law. The results of this thesis are the first of their kind to contain the critically-scaled pinning potential with MKG, and generalises the previous work done by authors studying the GL wave equation.

Ostensibly, we have shown that the methods of Chapter 5 to derive the

limiting dynamical law in the TDGL model can be employed to derive the limiting dynamical law in the more complex MKG model. In fact, by supplementing the equations with a GL potential, we benefited from being able to call on results from the first part of the thesis. However, some adaptations need to be made to account for the fact that the kinetic energy (formally known as the "dissipation") is now included in the Hamiltonian, but we claim that these are in fact simplifications to the process of proving that the energy in the system is bounded for finite time. Other differences between the methods included the space in which the magnetic field converges and the form of the stress-energy tensor, which leads to a slight change in the approach to using the Gronwall inequality. As with TDGL, the equations presented at every step are general in the sense that they contain all of the gauge terms and pinning terms and it is a simple matter to recover the dynamics for systems without gauge or pinning.

Physically, the MKG equations describe the interaction of a spinless particle with the electromagnetic field. However, the GL potential ensures that this "particle" is in fact a point-vortex in the limit of $\varepsilon \to 0$. Hence, our results may be more applicable to the setting of a type-II superconductor than a general space.

**Future work**

The study of pinning in superconductors remains an active area of research [24, 46]. A natural extension to the work we have presented is to include the effects of an applied electric current on the boundary of the system. This amounts to modifying the boundary conditions we have used. The expectation is that the electric current causes an additional forcing term which causes vortex motion chiefly in the direction perpendicular to the flow of current. This force is known as the Lorentz force. However, the inclusion of both types of flow in the mixed flow (i.e. $\alpha_0 \neq 0$ and $\beta_0 \neq 0$) allows one to identify the anomalous Hall effect observed in superconductors.

Serfaty and Tice [52] have studied the system with a strong electric current applied in addition to strong pinning forces in the TDGL setting and they lose the intervortex-interaction in their limiting dynamical law. A critically-scaled electric current should allow the renormalized energy to still play a role in the vortex dynamics. Yu [60] considers such a system with applied current in the MKG setting. We claim that it should be a simple matter to combine Yu's results with our own to deduce a mixed-flow dynamical law for vortices exposed to electric current, intervortex interactions and a pinning potential. The perpendicular flow allows for conservative motion.

One could consider the MKG-pGL equations with different pinning regimes and how the limiting law is affected. The expectation is that for a stronger pinning potential the effects of the intervortex interactions are lost in the limiting dynamical law.

Although the dynamics of vortices controls almost all the magnetic properties of type-II superconductors, especially those of high-temperature superconductors (see page vi of [32]), it should be appropriate to accommodate the effects of a non-negligible temperature. The expectation is that it produces a stochastic force on the vortex dynamics. Some homogenization arguments may be necessary. If one is interested in a system with large numbers of vortices (unbounded as $\varepsilon \to 0$), one requires a completely different method for extracting the limiting dynamical law. See [18] for the most complete work on the dynamics of large numbers of vortices.

In the MKG setting, we would like to know if one can trivially recover the dynamics from the Maxwell-Higgs system by including a general mass term $m$ throughout (we set $m = 1$) and then considering $m \to 0$, or whether a more refined analysis is necessary. One may also consider the non-relativistic limit $c \to \infty$ or the semiclassical limit $\hbar \to 0$ as in [43].

It would be pleasing to see numerical and experimental validation of our results. Since this is the first work bringing together a perturbed background potential and MKG, we know of no results in this direction, but there are some results for the numerical and experimental study of GL equations with pinning. See [56] for numerical results for a stronger pinning potential and [34] for the critically-scaled pinning.

An interesting different direction one can take is to reconstruct the features of the pinning potential from the motion of the vortices. Such an inverse problem is studied by Lai *et al* [36] for the Gross-Pitaevskii equation (no gauge and $\alpha \equiv 0$).

We observe that when the pinning and gauge terms are neglected, our equations are similar to the *Landau-Lifshitz-Gilbert* equations which describe the dynamics of vortices in a ferromagnetic thin film with a strong in-plane anisotropy [33]. It would be interesting to see where else our equations have physical relevance.

# Appendix A

# Inner-product identities

We record some inner product identities which appear in this work. Let $u(x,t)$ and $v(x,t)$ be complex valued functions, $A(x,t)$ a vector valued function and $\Phi(x,t)$ a scalar valued function. Allow $u(x,t) = \rho(x,t)\mathrm{e}^{i\phi(x,t)}$. The following inner products are satisfied:

$$\langle u, iu\rangle = 0 \tag{A.1}$$

$$\langle iu, iv\rangle = \langle u, v\rangle \tag{A.2}$$

$$\langle u, iv\rangle = -\langle iu, v\rangle \tag{A.3}$$

$$\langle u, \partial_\Phi u\rangle = \langle u, \partial_t u\rangle = \frac{1}{2}\left(u\partial_t u^* + u^*\partial_t u\right) = \rho\partial_t\rho \tag{A.4}$$

$$\langle u, \nabla_A u\rangle = \langle u, \nabla u\rangle = \frac{1}{2}\left(u\nabla u^* + u^*\nabla u\right) = \rho\nabla\rho \tag{A.5}$$

$$\begin{aligned}\langle u, \Delta_A u\rangle &= \langle u, \Delta u\rangle + 2A\cdot\langle iu, \nabla u\rangle - |A|^2|u|^2 \\ &= \frac{1}{2}\left(u\Delta u^* + u^*\Delta u\right) + iA\cdot\left(u\nabla u^* - u^*\nabla u\right) - |A|^2|u|^2 \\ &= \rho\Delta\rho - \rho^2\left(\nabla\phi - A\right)^2\end{aligned} \tag{A.6}$$

$$\langle iu, \partial_\Phi u\rangle = \frac{i}{2}\left(u\partial_t u^* - u^*\partial_t u\right) + \Phi|u|^2 = \rho^2\big(\partial_t\phi + \Phi\big) \tag{A.7}$$

$$\langle iu, \nabla_A u\rangle = \frac{i}{2}\left(u\nabla u^* - u^*\nabla u\right) - A|u|^2 = \rho^2\big(\nabla\phi - A\big) \tag{A.8}$$

$$\begin{aligned}\langle iu, \Delta_A u\rangle &= \langle iu, \Delta u\rangle - 2A\cdot\langle u, \nabla u\rangle - (\mathrm{div}\ A)|u|^2 \\ &= \frac{i}{2}\left(u\Delta u^* - u^*\Delta u\right) - A\cdot\left(u\nabla u^* + u^*\nabla u\right) - (\mathrm{div}\ A)|u|^2 \\ &= \mathrm{div}\big(\rho^2(\nabla\phi - A)\big)\end{aligned} \tag{A.9}$$

**Lemma A.1.** *Let $\Omega \subset \mathbb{R}^2$ be some domain. If $u, v \in H^1(\Omega; \mathbb{C})$, then*

$$\nabla \langle u, v \rangle = \langle \nabla_A u, v \rangle + \langle u, \nabla_A v \rangle.$$

*Proof.* Let $\partial_k^A = \partial_k - iA_k$. Then

$$\begin{aligned}
\partial_k \langle u, v \rangle &= \langle \partial_k u, v \rangle + \langle u, \partial_k v \rangle \\
&= \langle \partial_k u, v \rangle + \langle u, \partial_k v \rangle + \langle -iA_k u, v \rangle + \langle iA_k u, v \rangle \\
&= \langle \partial_k u, v \rangle + \langle -iA_k u, v \rangle + \langle u, \partial_k v \rangle + \langle u, -iA_k v \rangle \\
&= \langle \partial_k^A u, v \rangle + \langle u, \partial_k^A v \rangle.
\end{aligned}$$

□

**Lemma A.2.** *Let $I \subset \mathbb{R}$ be some interval. If $u, v \in H^1(I; \mathbb{C})$, then*

$$\partial_t \langle u, v \rangle = \langle \partial_\Phi u, v \rangle + \langle u, \partial_\Phi v \rangle.$$

*Proof.* The proof is similar to that of the previous lemma. □

**Lemma A.3.** *For $A$, $\Phi$ and $E$ defined in Chapter 1, the following commutator identity holds*

$$\partial_\Phi \nabla_A - \nabla_A \partial_\Phi = iE.$$

*Proof.* The result follows by direct calculation:

$$\begin{aligned}
&(\partial_t + i\Phi)(\nabla - iA)u - (\nabla - iA)(\partial_t + i\Phi)u \\
&\quad = \partial_t \nabla u + i\Phi \nabla u + A\Phi u - i\partial_t(Au) - \nabla \partial_t u - i\nabla(\Phi u) - A\Phi u + iA\partial_t u \\
&\quad = -i(\nabla \Phi + \partial_t A)u.
\end{aligned}$$

□

**Lemma A.4.** *Let $h = \operatorname{curl} A$. The commutator between two covariant derivatives is*

$$\partial_i^A \partial_j^A - \partial_j^A \partial_i^A = -i(\partial_i A_j - \partial_j A_i).$$

*In particular,*

$$\partial_1^A \partial_2^A - \partial_2^A \partial_1^A = -ih.$$

*Proof.* By direct calculation we have

$$(\partial_i^A \partial_j^A - \partial_j^A \partial_i^A)u = (\partial_i - iA_i)(\partial_j - iA_j)u - (\partial_j - iA_j)(\partial_i - iA_i)u$$

$$= \partial_i\partial_j u - iA_i\partial_j u - i\partial_i(A_j u) + A_i A_j - (\partial_j\partial_i u - iA_j\partial_i u - i\partial_j(A_i u) + A_j A_i)$$

$$= -iA_i\partial_j u - iA_j\partial_i u - iu\partial_i A_j + iA_j\partial_i u + iA_i\partial_j u + iu\partial_j A_i$$

$$= -iu(\partial_i A_j - \partial_j A_i).$$

□

# Appendix B

# Comparing pinned energy to unpinned energy

Suppose $G_\varepsilon$ is the traditional Ginzburg-Landau energy without pinning, i.e. $G_\varepsilon = G_\varepsilon^{\eta_\varepsilon}$ when $\eta_\varepsilon \equiv 1$, where $G_\varepsilon^{\eta_\varepsilon}$ is defined in Chapter 2. Consider the following

$$\begin{aligned} G_\varepsilon^{\eta_\varepsilon} &= \int_\Omega \frac{\eta_\varepsilon^2}{2}|\nabla_A w|^2 + \frac{\eta_\varepsilon^4}{4\varepsilon^2}(1-|w|^2)^2 + \frac{1}{2}|h-h_{ex}|^2 \\ &= G_\varepsilon + \int_\Omega \frac{\eta_\varepsilon^2-1}{2}|\nabla_A w|^2 + \frac{\eta_\varepsilon^4-1}{4\varepsilon^2}(1-|w|^2)^2 \\ &= G_\varepsilon + \int_\Omega (\eta_\varepsilon^2-1)\Big[\frac{1}{2}|\nabla_A w|^2 + \frac{1}{4\varepsilon^2}(1-|w|^2)^2 + \frac{\eta_\varepsilon^2}{4\varepsilon^2}(1-|w|^2)^2\Big] \\ &\leq G_\varepsilon + \sup_{x\in\Omega}\big(\eta_\varepsilon^2(x)-1\big)G_\varepsilon + \sup_{x\in\Omega}\big(\eta_\varepsilon^2(x)\big(\eta_\varepsilon^2(x)-1\big)\big)G_\varepsilon. \end{aligned}$$

Suppose there exists $C_1 > 0$ and $C_2 > 0$ such that

$$\sup_{x\in\Omega}\big(\eta_\varepsilon^2(x)-1\big) = \frac{C_1}{|\log\varepsilon|} \qquad \text{and} \qquad \sup_{x\in\Omega}\big(\eta_\varepsilon^2(x)\big(\eta_\varepsilon^2(x)-1\big)\big) = \frac{C_2}{|\log\varepsilon|},$$

then

$$G_\varepsilon^{\eta_\varepsilon} \leq \Big(1 + \frac{C_1}{|\log\varepsilon|} + \frac{C_2}{|\log\varepsilon|}\Big)G_\varepsilon \leq \Big(1 + \frac{C_3}{|\log\varepsilon|}\Big)G_\varepsilon. \tag{B.1}$$

Now we call on an observation made in [34] (see the proof of their Theorem 4.4). Note that for any sequence of $f_\varepsilon \in L^2(\Omega)$ we have, for $K \geq 0$,

$$\Big(1 + \frac{K}{|\log\varepsilon|}\Big)^{-1}\int_\Omega |f_\varepsilon|^2 dx \leq \int_\Omega \eta_\varepsilon^2|f_\varepsilon|^2 dx \leq \Big(1 + \frac{K}{|\log\varepsilon|}\Big)\int_\Omega |f_\varepsilon|^2 dx. \tag{B.2}$$

In particular,

$$\int_\Omega |f_\varepsilon|^2 dx \leq \Big(1 + \frac{K}{|\log \varepsilon|}\Big) \int_\Omega \eta_\varepsilon^2 |f_\varepsilon|^2 dx. \tag{B.3}$$

Furthermore, we claim that

$$\int_\Omega |f_\varepsilon|^2 dx \leq \Big(1 + \frac{K}{|\log \varepsilon|}\Big)^2 \int_\Omega \eta_\varepsilon^4 |f_\varepsilon|^2 dx. \tag{B.4}$$

As a result of (B.3) and (B.4), we have that

$$G_\varepsilon(w_\varepsilon, A_\varepsilon) \leq \Big(1 + \frac{K}{|\log \varepsilon|}\Big)^2 G_\varepsilon^{\eta_\varepsilon}(w_\varepsilon, A_\varepsilon). \tag{B.5}$$

# Appendix C

# Tensor product

**Lemma C.1.** *Let $A \in \mathbb{R}^2$ be some vector. If we have the tensor*

$$T = A \otimes A - \frac{1}{2}|A|^2 \mathcal{I}, \tag{C.1}$$

*where $\mathcal{I}$ is the $2 \times 2$ identity matrix, then*

$$\text{div } T = A(\text{div } A) + A^{\perp}(\text{curl } A). \tag{C.2}$$

*Proof.* If we have the rank-2 tensor

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix},$$

then the divergence of the tensor gives a vector

$$\text{div } T = \begin{pmatrix} \partial_1 T_{11} + \partial_2 T_{12} \\ \partial_1 T_{21} + \partial_2 T_{22} \end{pmatrix}. \tag{C.3}$$

If we have the vector $A = (A_1, A_2)$, the tensor product produces a matrix:

$$A \otimes A = \begin{pmatrix} A_1^2 & A_1 A_2 \\ A_1 A_2 & A_2^2 \end{pmatrix}.$$

Thus, for the $T$ given by (C.1),

$$T = \begin{pmatrix} \frac{1}{2}A_1^2 - \frac{1}{2}A_2^2 & A_1 A_2 \\ A_1 A_2 & \frac{1}{2}A_2^2 - \frac{1}{2}A_1^2 \end{pmatrix}.$$

And so by (C.3)

$$\begin{aligned}\text{div } T &= \begin{pmatrix} A_1\partial_1 A_1 - A_2\partial_1 A_2 + A_1\partial_2 A_2 + A_2\partial_2 A_1 \\ A_1\partial_1 A_2 + A_2\partial_1 A_1 + A_2\partial_2 A_2 - A_1\partial_2 A_1 \end{pmatrix} \\ &= \begin{pmatrix} A_1(\partial_1 A_1 + \partial_2 A_2) \\ A_2(\partial_1 A_1 + \partial_2 A_2) \end{pmatrix} + \begin{pmatrix} -A_2(\partial_1 A_2 - \partial_2 A_1) \\ A_1(\partial_1 A_2 - \partial_2 A_1) \end{pmatrix} \\ &= A(\text{div } A) + A^{\perp}(\text{curl } A)\end{aligned}$$

□

**Lemma C.2.** *If $E$ is the electric field and we have the tensor*

$$T = E^{\perp} \otimes E^{\perp} - \frac{1}{2}|E|\mathcal{I},$$

*where $\mathcal{I}$ is the $2 \times 2$ identity matrix, then*

$$\text{div } T = -E^{\perp}(\text{curl } E) - \frac{1}{2}\nabla|E|^2. \tag{C.4}$$

*Proof.* Replacing $A$ by $E^{\perp}$ in (C.2) gives

$$\text{div } T = E^{\perp}(\text{div } E^{\perp}) - E(\text{curl } E^{\perp}). \tag{C.5}$$

By using the following relations for a general vector $A \in \mathbb{R}^2$

$$\begin{cases} \text{curl } A^{\perp} = \text{div } A \\ \text{div } A^{\perp} = -\text{curl } A, \end{cases} \tag{C.6}$$

we have that (C.5) becomes

$$\text{div } T = -E^{\perp}(\text{curl } E) - E(\text{div } E) = -E^{\perp}(\text{curl } E) - \frac{1}{2}\nabla|E|^2.$$

□

# Appendix D

# Test functions

Suppose we have a characteristic function $\chi_{B_1(0)}(x) \in C_0^\infty(\mathbb{R}^2)$ which satisfies

$$\chi_{B_1(0)}(x) = \begin{cases} 1 & \text{if } x \in B_1(0) \\ 0 & \text{if } x \in \Omega \backslash B_2(0). \end{cases}$$

If $\chi_r(x) = \chi_{B_1(0)}(\frac{x}{r})$, then we have that

$$\chi_r(x) = \begin{cases} 1 & \text{if } x \in B_r(0) \\ 0 & \text{if } x \in \Omega \backslash B_{2r}(0). \end{cases}$$

It follows that

$$\chi_r(x - a_j(0)) = \begin{cases} 1 & \text{if } x \in B_r(a_j(0)) \\ 0 & \text{if } x \in \Omega \backslash B_{2r}(a_j(0)). \end{cases}$$

Now consider the test functions $\phi_k^{(j)}, \psi_k^{(j)} \in W_0^{2,\infty}(\Omega)$ where $j = 1, ..., n$ identifies a given vortex and $k = 1, 2$ identifies the (basis) coordinate in the plane. We specify the form of $\phi_k^{(j)}, \psi_k^{(j)}$:

$$\psi_k^{(j)} = \hat{e}_k \cdot x \chi_r(x - a_j(0)) \tag{D.1}$$

$$\phi_k^{(j)} = \hat{e}_k \cdot x^\perp \chi_r(x - a_j(0)) \tag{D.2}$$

where $\hat{e}$ is a unit vector. Written explicitly, we have

$$\psi_1^{(j)} = x_1 \chi_r(x - a_j(0)) \quad \text{and} \quad \psi_2^{(j)} = x_2 \chi_r(x - a_j(0))$$
$$\phi_1^{(j)} = -x_2 \chi_r(x - a_j(0)) \quad \text{and} \quad \phi_2^{(j)} = x_1 \chi_r(x - a_j(0)).$$

**Lemma D.1.** *If $\psi_k^{(j)}, \phi_k^{(j)}$ are defined by (D.1) and (D.2), then they satisfy $\nabla\psi_k^{(j)} = \nabla^\perp\phi_k^{(j)}$ in $B_r(a_j(0))$ for $k = 1, 2$.*

*Proof.* Note that $\chi_r(x - a_j(0)) = 1$ in $B_r(a_j(0))$. Hence, by direct calculation for $k = 1$

$$\nabla\psi_1^{(j)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \nabla\phi_1^{(j)} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

Thus, $\nabla\psi_1^{(j)}$ and $\nabla^\perp\phi_1^{(j)}$ coincide. And for $k = 2$

$$\nabla\psi_2^{(j)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \text{and} \qquad \nabla\phi_2^{(j)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Thus, $\nabla\psi_2^{(j)}$ and $\nabla^\perp\phi_2^{(j)}$ coincide. □

# Appendix E

# Gronwall inequality

Here we state the two versions of Gronwall's inequality which we used to determine the vortex motion law in Chapters 5 and 7.

**Lemma E.1.** *(Differential form [22].) Take the interval $I : [t_1, t_2] \in \mathbb{R}$, where $t_1 < t_2$. Let $\beta(t)$ and $f(t)$ be real-valued continuous functions defined on $I$. If $f(t)$ is differentiable in the interior $\bar{I}$ of $I$ and satisfies the differential inequality*

$$\frac{d}{dt} f(t) \leq \beta(t) f(t), \qquad t \in \bar{I},$$

*then $f(t)$ satisfies the bound*

$$f(t) \leq f(t_1) \exp\Big( \int_{t_1}^{t} \beta(s) \, \mathrm{d}s \Big), \qquad \forall t \in I.$$

**Lemma E.2.** *(Integral form [6].) Take the interval $I : [t_1, t_2] \in \mathbb{R}$, where $t_1 < t_2$. Let $\alpha(t)$, $\beta(t)$ and $f(t)$ be real-valued continuous functions defined on $I$. Assume that $\beta$ and $f$ are continuous and that the negative part of $\alpha$ is integrable on every closed and bounded subinterval of $I$.*

*(i) If $\beta$ is non-negative and if $f$ satisfies the integral inequality*

$$f(t) \leq \alpha(t) + \int_{t_1}^{t} \beta(s) f(s) \mathrm{d}s, \qquad \forall t \in I,$$

*then $f(t)$ satisfies the bound*

$$f(t) \leq \alpha(t) + \int_{t_1}^{t} \alpha(s) \beta(s) \exp\Big( \int_{s}^{t} \beta(r) \, \mathrm{d}r \Big) \mathrm{d}s, \qquad \forall t \in I.$$

*(ii) If, in addition, the function $\alpha$ is non-decreasing, then*

$$f(t) \leq \alpha(t) \exp\Big( \int_{t_1}^{t} \beta(s)\ \mathrm{d}s \Big), \qquad \forall t \in I.$$

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