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Explicit Class Field Theory
One Dimensional
and Higher Dimensional

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Abstract

This thesis investigates class field theory for one dimensional fields and higher dimensional fields. For one dimensional fields we cover the cases of local fields and global fields of positive characteristic. For higher dimensional fields we study the case of higher local fields of positive characteristic. The main content of the thesis is divided into two parts.

The first part solves several problems directly related to Neukirch's axiomatic class field theory method. We first prove the famous Hilbert 90 Theorem in the case of tamely ramified extensions of local fields in an explicit way. This approach can be of use in understanding the role of the ring structure as opposed to the role of multiplication only in local class field theory. Next, we prove that for every local field, its 'class field theory' is unique. Lastly, we establish the Neukirch axiom for global fields of positive characteristic, which leads to a new approach to class field theory for such fields, an approach that has not appeared in the previous literature.

There are two main successful directions in higher local class field theory, one by Kato and another by Fesenko. While Kato used a technical cohomological method, Fesenko generalised the Neukirch method and gave the first proof of the existence theorem. In the second part of the thesis we deal with the third method in class field theory that works in positive characteristic only, the Kawada-Satake method. We generalise the classical Kawada-Satake method to higher local fields of positive characteristic. We correct substantial mistakes in a paper of Parshin on such class field theory. We develop the first complete presentation of the theory based on the generalised Kawada-Satake method using advanced properties of topological Milnor

K-groups. These advanced properties include Fesenko's theorem about relations of topological and algebraic properties of Milnor K-groups.

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CONTENTS

1

INTRODUCTION

1.1 OVERVIEW OF CLASS FIELD THEORY

One of the greatest tasks of number theory is a complete understanding of the absolute Galois group of the rationals. This fundamental problem is extraordinarily difficult and we cannot expect to answer it anytime soon. At the same time, we have a comprehensive answer in the case of the maximal abelian quotient of this absolute Galois group. This answer is provided by class field theory which is the top achievement of algebraic number theory in the 20th century. Comparing with other parts of algebraic number theory, class field theory has had by far the largest number of applications which include those in the study of class numbers, cyclotomic fields and Iwasawa theory. Objects used in class field theory[1] play central role in Iwasawa–Tate theory of zeta integral and their study of fundamental properties of the Dedekind zeta function. This theory played a key role in the development of the Langlands program.

Class field theory has its roots in Gauss' quadratic reciprocity law. Class field theory studies abelian extensions of global fields such as number fields (characteristic 0) and the function field of a smooth projective curve over a finite field and number fields (positive characteristic). Contributions from Kronecker, Weber, Hilbert, Takagi, Artin, Hasse, Chevalley lead to proofs of its main theorems by the end of the 1930s. One of the key objects in

[1] adeles and ideles

class field theory is the reciprocity map $\Psi_K\colon G(K^{ab}/K) \to C_K$ where C_K is the idele class group of the global field K.[2] The reciprocity map induces an isomorphism $C_K/N_{L/K}C_L \cong G(L/K)^{ab}$ for any finite Galois extension L/K. It was understood in the 1930s that in order to have a comprehensive class field theory it was important to study abelian extensions of local fields, i.e. the completion of global fields with respect to its places. Even though certain parts of global class field theory were derived before the emergence of local class field theory, it became standard to develop the local theory first and then to use it to construct the global theory. If K is a global field, then for each place P there is a corresponding local field K_P. For each local field K_P one has a local reciprocity map $\Psi_P : K_P^\times \to G(L_\mathfrak{P}/K_P)^{ab}$ for Galois extension $L_\mathfrak{P}/K_P$. The idele group $I_K = \prod'_P K_P^\times$ naturally contains the multiplicative group of the original global field $K^\times$ so the idele class group $C_K := I_K/K^\times$ is well-defined. Then the global reciprocity map $\Psi_K : C_K \to G(K^{ab}/K)$ is defined as product of the local reciprocity maps Ψ_P. Artin's reciprocity law says that the product vanishes on the diagonal image of the global elements $K^\times$.

Another crucial part of class field theory is the existence theorem, which provides a one-to-one correspondence between finite abelian extensions of the ground field and open subgroups of finite index in the idele class group (in the case of global fields) and in the multiplicative group of the field (in the case of local fields).

The cohomological method in class field theory was developed in the 1950s. It was a cohomological rephrasing of the Hasse method to use the Brauer group. This method uses Tate cohomology groups $\hat{H}^i(G,F)$ for each $i \in \mathbb{Z}$, which are slightly different from the usual Galois cohomology groups. The (inexplicit) cup product with 2nd cocycle gives a map $\hat{H}^i(G,F) \to \hat{H}^{i+2}(G,F)$. One of the key steps in the proof is that there is an isomorphism $H^2(K^{sep}/K) \cong Br(K) \cong \mathbb{Q}/\mathbb{Z}$. This method can be found

[2]The map is called the reciprocity map since it is a far reaching generalisation of the Gauss reciprocity law.

in many texts including [4, Ch 6]. However, it hides fundamental arithmetic features of number fields behind cohomological computations. One of its key drawbacks is its lack of explicitness: it remains a mystery what is the Galois automorphism to which an element of the idele class group is mapped to via the global reciprocity map.

Generations of number theorists were interested to deduce main results of class field theory from as few axioms as possible. Among such approaches three are very famous: cohomological class formations (Artin–Tate), the Kawada–Satake class formations in positive characteristic, the Neukirch class field theory axioms. These three axiomatic approaches will be reviewed in this thesis (sect. 2.1, 4.1, 3.1 resp.). The advantage of Kawada–Satake's approach to class field theory in positive characteristic is that it is much more explicit and more elementary. Kawada and Satake proved in [20] that class formations exist for general characteristic p fields. The key aspect of this approach is to work with $B(K)^\wedge$ where $B(K) = \varinjlim_n (W_n(K)/\mathscr{P}W_n(K))$ for any characteristic p field K and to use Witt duality which gives a continuous and non-degenerate pairing $G^{ab,p} \times B(K) \to \mathbb{Q}/\mathbb{Z}$. This method also works for local fields of positive characteristic. It is interesting that no paper in geometric Langlands correspondence in positive characteristic uses generalisations of the Kawada–Satake method.

A general axiomatic approach to class field theory, in all characteristics, was developed by Neukirch. His method can be viewed as a culmination of efforts of many mathematicians (Hasse, Artin, Tate, Kawada, Satake, Hazewinkel, Iwasawa) to clarify and simplify class field theory and make it very explicit. Neukirch proposed a very compact group theoretic axiom which implies the reciprocity map for the fields under investigation. More specifically, given a field k, he proves that if a $G(k^{sep}/k)$-module A satisfies very simple properties then we immediately get a reciprocity map $A_K := A^{G(K/k)} \to G(K^{ab}/K)$ satisfying the desired properties. The key to this theory is the existence of a $\hat{\mathbb{Z}}$-extension of the ground field (such as the maximal unramified extension of a local field) and of a so called henselian valuation (which is not necessarily 'henselian' in the usual sense) with respect to this extension. One advantage of this method is that it gives a very explicit map between the domain of the reciprocity map and the Galois group. In his several famous Springer books Neukirch checked that his axiomatic approach works well for local fields and global fields of characteristic zero. However, he did not publish a proof in the case of global function fields. This will be discussed in Ch 3. In the same chapter we will also discuss certain twisted local class field theories and tame local class field theories.

There are various generalisations of class field theory. Perhaps, the three most important generalisations are the Langlands program, higher class field theory, anabelian geometry. All of them are approximately 50 years old.

The Langlands program came from the point of view of representation theory. One of its main conjectures says that for a number field K there is a certain correspondence between a certain class of representations of $GL_n(\mathbb{A}_K)$ and n-dimensional complex linear representation of the absolute Galois group of K. However, the Langlands correspondence does not include any generalisation of the existence theorem. While fundamental achievements have been attained in positive characteristic, most fundamental problems remain open in the case of number fields. Perhaps, the main achievement for number fields so far has been the very partial case of the modularity of elliptic curves over rationals established by Wiles and Taylor. Later, Breuli, Conrad, Diamond and Taylor proved the full Modularity theorem as well. There is also a local version of the Langlands program, but this was not developed by Langlands. There is also a generalisation of local class field theory which works for (generally non-abelian) arithmetically pro-finite extensions of a local field; this was developed by Fesenko and then extended by Ikeda and Serbest. [3]

[3]See [8],[10] and part II Ch 9 of [13] for Fesenko's work and [17] for Ikeda and Serbest's generalisation.

Anabelian geometry is another generalisation of class field theory. It studies rigidities of number fields in the sense of how the knowledge of the absolute Galois group of a number field restores the number field. Here the main result is the Neukirch–Ikeda–Uchida theorem (see [28, Ch 12]) which says that an isomorphism of a number field K corresponds to an outer automorphism of the absolute Galois group of K. Anabelian geometry also studies a similar problem (posed by Grothedieck) for hyperbolic projective curves over number fields, here the main theorem was proved by Mochizuki using some p-adic Hodge theory. A version of anabelian geometry called mono-anabelian geometry was developed by Mochizuki. One of key concepts is mono-analytic structure, when one starts with a ring and then forgets addition, working with the multiplicative structure only, and eventually restoring the ring from the knowledge of the multiplicative structure. One of the developments in modern mathematics is a recent inter-universal Teichmüller theory(IUT) of Mochizuki, which is based on mono-anabelian geometry. The development of anabelian geometry helped to clarify the meaning of axiomatic approaches to class field theory. In the first approximation, the way how one deduces main results of class field theory from class field theory axioms depends on the multiplicative structure only, not on the ring structure. However, to establish the axioms of class field theory one has to use the ring structure of the global and local fields. Fesenko's talk on this issue was given at the RIMS IUT Summit workshop in 2016.[4]. This issue will be discussed in Ch 3.2.

Another generalisation of class field theory is higher class field theory. It studies abelian extensions of the higher dimensional global and local fields such as the function fields of an integral schemes of finite type over $\mathbb{Z}$ or higher local fields. When the dimension is one, this is just the classical class field theory. This theory was contributed to, in chronological order, by Lang, Ihara, Parshin, Kato, Saito, Fesenko and other researchers. Higher class field theory is a very non-trivial generalisation of class field theory,

[4]See [12] which is the power point presentation from the workshop

since various properties in the classical theory do not hold any more. For n-dimensional fields, instead of the multiplicative group of the field or the idele class group one uses the Milnor K_n-groups or topological Milnor K_n-groups or their adelic versions (for more detail see 2.3). We still do not have a full knowledge of the structure of the Milnor K_n-groups or topological Milnor K_n-groups. In fact, some of the information about them, such as Hilbert Theorem 90 comes from the famous theorem of Voevodsky[5] However, we know much more in positive characteristic. The Galois descent, which plays the fundamental role in dimension one, does not hold for higher dimensional fields : for a finite Galois extension L/F of n-dimensional fields, Galois invariant elements of $K_n(L)$ are not isomorphic to $K_n(F)$. One can define adeles for higher global fields, and in dimension two there are two types of adeles: geometric adeles and analytic adeles. See [11] for the details.

Higher local class field theory started with Kato's papers published in 1979 and 1980 ([18]). Independently, Parshin proposed a simpler way to deduce higher local class field theory in positive characteristic in [30] and [31]. At the same time, Fesenko constructed an alternative higher local class field theory of more explicit nature [5] and [6]. Objects used in higher class field theory (and in adelic approach to it developed by Fesenko) play central role in Fesenko–Iwasawa–Tate theory of zeta integral of two dimensional arithmetic schemes and his study of fundamental properties of the zeta and L-function of the schemes and their generic fibre. This theory is expected to play a key role in the development of a two dimensional generalisation of the Langlands program.

These three approaches to higher class field theory correspond to the three aforementioned methods in one dimensional class field theory. The cohomological approach is generalised by Kato, which remains to be the only method which has been extended to higher global structures fully. For each $i \in \mathbb{Z}_{>0}$, Kato defines higher cohomology groups $H^i(F)$ for a higher

[5]The theorem which was originally known as (motivic) Bloch-Kato conjecture before its proof by Voevodsky and many other mathematicians.

local field F, which have the property that if the dimension of F is d, then $H^{d+1}(F) \cong \mathbb{Q}/\mathbb{Z}$. These groups generalise the Brauer group, but unlike the Brauer group, they do not share the same central role in algebra. However the calculation involved in the proofs of Kato's theory are very complicated and the reciprocity map constructed by this method is not explicit, similar to the one dimensional cohomological approach. Fesenko generalised Neukirch's method. He uses topological Milnor K-group (K_n^{top}-group) instead which is a quotient group of the Milnor K-group. Neukirch's axiom does not work directly since the natural map $K_n^{top}(F) \to K_n^{top}(L)$ is not necessarily injective when L is an extension field of F. Fesenko's axiom takes care of this and as in Neukirch's axiom, an explicit reciprocity map is developed. He also proves the existence theorem using sequentially saturated topology. [5] and [6] contain the first published results on existence theorem for higher local fields. Parshin in [30] proposed an approach to higher local class field theory for characteristic p fields, by working with Witt vectors and generalising Kawada and Satake's method. His contributions included the first definition of the topological Milnor K-group, however, his definition was wrong for topological reasons and it was corrected in Fesenko's papers ([9]). Unfortunately, there were other fundamental flaws in Parshin's approach such as a serious gap in the use of his norm map and incorrect topology. Chapter 4 contains the first detailed presentation of a corrected version of Parhin's approach to higher local class field theory in positive characteristic. Previously, Syder developed local-global class field theory in the case of characteristic p and dimension 2 in [33].

1.2 SUMMARY

The main contents of this thesis is divided into two chapters. The first part (Ch 3) mainly investigates various central open questions related to Neukirch's abstract class field theory and to explicit class field theory in general. We start by giving a detailed exposition of his theory.

We note that Neukirch's abstract class field theory does not use ring structures. Thus, the class field theory mechanism (i.e. deriving the reciprociy map and its properties from class field theory axioms) could be considered to be mono-anabelian in the sense of mono-anabelian geometry. However, in order to verify the class field theory axioms for local fields, one needs to use the ring structure. It is an open question to find exactly how much ring structure is needed. On the example of proving Hilbert Theorem 90 explicitly in the case of tamely ramified extension of local fields we demonstrate some of key aspects. We answer a question Neukirch himself asked around 1995. There are two $\hat{\mathbb{Z}}$-extensions of $\mathbb{Q}_p$ and he asked whether it is possible to develop explicit class field theory using the $\hat{\mathbb{Z}}$-extension which is not unramified, in particular whether it is possible to construct a henselian valuation with respect to this other $\hat{\mathbb{Z}}$-extension. We prove that this is not possible. Then we finish the first chapter by establishing class field theory in global fields of positive characteristic using Neukirch's axiomatic method. All of this has never been published before.

The next chapter (Ch 4) is dedicated to higher local class field theory of characteristic p using a generalisation of Kawada-Satake's method. We note that the first section of this second chapter is basically the contents from [20]. In order to develop this theory we had to modify several proofs and at times to take a different approach to previously published results. We fix all the gaps and mistakes in Parshin's approach. We note that [13, Ch 9] suggested how one could fix them but we use our original method and deviate from some of those suggestions.

One of the problems with Parshin's work is that he assumed that the usual topology for higher local fields can be used in higher class field theory. However, as was observed by Fesenko, we need to work with the sequential saturation topology instead. One of key fundamental results we use is Fesenko's theorem that the quotient of topological K-group can be defined algebraically (Theorem 4.2.5), see [9] . The proof of this theorem uses p-adic Hodge theory. Using this theorem one immediately proves that the norm map in the usual Milnor K-theory induces a norm map on K_n^{top}-groups. In order to prove a crucial property of the reciprocity map (Theorem 4.2.12), we need the fact that Parshin's definition of the norm map coincides with the one induced from Milnor K-group. Parshin also overlooked that this fact was required in his proof in [31]. We fix all these mistakes.

2

PREREQUISITES

This thesis contains results from several papers and surveys on class field theory on various types of fields. In this chapter we collect some background knowledges. We note that we will usually use the letter k or K for one dimensional fields, F for higher local fields and L for any extension fields. Also for whole of this thesis we will denote the Galois group of L/K as $G(L/K)$.

2.1 GENERAL CLASS FORMATION

Let k_0 be a field, which we will call the ground field, Ω a fixed infinite separable algebraic extension of k_0 and $t = t(k/k_0)$ the set of all subfields k of Ω containing k_0 such that $[k : k_0] < \infty$. We emphasise that Ω does not have to be the separable closure of k_0. For $k \in t$, let $E(k)$ be an abelian group satisfying the following three axioms:

1. For each pair $k, K \in t$ with $k \subset K$ there is an injective homomorphism $\phi_{k,K}$ of $E(k)$ into $E(K)$. Moreover if $k \subset K \subset L$, $\phi_{K,L} \circ \phi_{k,K} = \phi_{k,L}$.

2. If K/k is a Galois extension then $G = G(K/k)$ acts on $E(K)$ and $\phi_{k,K}(E(k)) = E(K)^G$. Moreover, if $k \subset K \subset L$ and L/k and K/k are Galois then for all $\sigma \in G(L/k)$ and $f \in E(K)$

$$\sigma \cdot \phi_{K,L}(f) = \phi_{K,L}(\sigma|_K \cdot f).$$

3. Let $k, K \in t$, K/k be a Galois extension of degree n with $G(K/k) = G$. Then

$$H^1(G, E(K)) = 0, \; H^2(G, E(K)) \cong \mathbb{Z}/n\mathbb{Z}.$$

We refer to these three axioms as the Class Formation Axiom. We note that axioms for class formation were initially developed by Artin and Tate but they are quite different from the one presented here.[1] Kawada in [19] develops the above axioms and proves that the two class formation axioms are equivalent.

Definition. *Suppose that $E(k)$ satisfies the axioms above. Then we call $\{E(k)|k \in t\}$ a class formation for t and each group $E(k)$, the class module associated to t. Define $N_G : E(K) \to E(k)$ to be the usual trace map defined by $a \mapsto \sum_{\sigma \in G} \sigma a$ for $a \in E(K)$. We may also write $N_{K/k}$ for $N_{G(K/k)}$ when K/k is a Galois extension.*

We state Theorem of Tate which can be found in [1, Ch. 14]. Note that this proof is based on Tate's class formation axiom. Here, $\hat{H}^i$ is the Tate cohomology group. See [1] and [28] for the definition.

Theorem 2.1.1. *Suppose that $\{E(k) : k \in t\}$ is a class formation for some t. Let $k, K \in t$ and K/k be a Galois extension with $G(K/k) = G$. Let $u_{K/k}$ be a generator of $H^2(G, E(K))$. Then for each $i \in \mathbb{Z}$, the map $\alpha \mapsto \alpha u_{K/k}$ given by the cup-product is an isomorphism of groups*

$$\hat{H}^i(G(K/k), \mathbb{Z}) \cong \hat{H}^{i+2}(G, E(k)).$$

Proof. Omitted. □

Using this Theorem (by letting $i = -2$) we obtain the main Theorem of Class Formation:

Theorem 2.1.2. *Let $\{E(k)|k \in t\}$ be a class formation for t. Let $k, K \in t$ and K/k be a Galois extension with $G(K/k) = G$. Then there is a natural isomorphism $G/[G, G] \cong E(k)/N_G(E(K))$ given by the cup-product.*

[1] See [1] for their axioms.

This isomorphism can be described. For this, we define canonical cohomology class. Rest of the content (including the proof) in this section can be found in [19, Ch. 1]. We consider two cases.

- Case 1. $k \subset L \subset K$, K/k Galois, $G = G(K/k)$ and $H = G(K/L)$
- Case 2. We assume further that L/k is a Galois extension and $F = G(L/k)$.

In Case 1, $\mathrm{Res}_{G/H} : H^2(G, E(K)) \to H^2(H, E(K))$ is surjective and $\mathrm{Cor}_{H/G} : H^2(H, (E(K)) \to H^2(G, E(K))$ is injective. In Case 2 $\mathrm{Inf}_{F/G} : H^2(F, E(L)) \to H^2(G, E(K))$ is injective. Then we can choose a generator $\xi_{K/k}$ of $H^2(G, E(K))$ for each Galois extension K/k such that $\mathrm{Res}_{G/H}\xi_{K/k} = \xi_{K/L}$ in Case 1 and $\mathrm{Inf}_{F/G}\xi_{L/k} = [L : k]\xi_{K/k}$ in Case 2 hold. We call $\xi_{K/k}$ the canonical cohomology class of K/k. We note that despite its name this is not unique. Then the isomorphism of Theorem 2.1.2 (see [19]) is given by

$$\sigma \text{ mod } [G, G] \mapsto \prod_{\tau \in G} f_{K/k}(\tau, \sigma)^{-1} \text{ mod } N_G E(K)$$

where $f_{K/k}$ is an arbitrary 2-cocycle in the canonical cohomology class and the map does not depend on the choice of $f_{K/k}$.

The inverse of the isomorphism of Theorem 2.1.2 yields a surjective homomorphism

$$(\ , K/k) : E(k) \to G^{ab}$$

with kernel $N_G E(K)$. This map satisfies the below functoriality condition:

Proposition 2.1.3. *Let K'/k' and K/k be finite Galois extensions such that $K \subset K'$ and $k \subset k'$. Then we have the commutative diagram:*

$$\begin{tikzcd} E(k') \arrow[r, "(\ {,}\ K'/k')"] \arrow[d, "N_{G(k'/k)}"'] & G(K'/k')^{ab} \arrow[d] \\ E(k) \arrow[r, "(\ {,}\ K/k)"] & G(K/k)^{ab} \end{tikzcd}$$

Let $\Omega^{ab}(k)$ be the maximal abelian extension of k in Ω. Then we can define a symbol

$$(a,k) = \varprojlim_{K/k}(a,K/k) \text{ for } a \in E(k).$$

The map $a \to (a,k)$ gives a homomorphism Φ_k called the Generalised Norm Residue map. The image of Φ_k in $G(\Omega^{ab}(k)/k)$ is dense, since for any finite extension K/k, $G(K/k) \subset im(\Phi_k)$ but the kernel is not necessarily 0. Again for the details see [19]. We will use these extensively in Ch 4 of this thesis.

2.2 WITT VECTORS

Here we define and state basic properties of Witt vectors. By Witt vectors we mean p-Witt vectors which is the original object defined by Witt. There has been generalisation of this, which is often denoted the big Witt vectors and this will not be used in this thesis. Contents in this section is taken from [14, Ch 1] unless stated otherwise. See Harder's text [15] for another presentation which gives more motivation.

2.2.1 *Construction of Witt Vectors*

Let k be a field and let p be a prime number. A Witt vector over k is a sequence $(x_0, x_1, x_2, \cdots)$ of elements in k. We aim to put a ring structure on the set of Witt vectors $W(k)$. First we define a Witt Polynomial $\mathcal{W}_n(X_0, X_1, X_2, \cdots, X_n)$ for each $n \geq 0$ by the formula

$$\mathcal{W}_n(X_0, \cdots, X_n) = \sum_{i=0}^{n} p^i X_i^{p^{n-i}}.$$

Then $(\mathcal{W}_0(x_0), \mathcal{W}_1(x_0, x_1), \mathcal{W}_2(x_0, x_1, x_2), \cdots)$ is called the ghost component of the Witt vector and will be denoted $(x^{(0)}, x^{(1)}, x^{(2)}, \cdots)$. Roughly speaking we will obtain the ring structure on $W(k)$ from the ring structure of the ghost component.

To construct Witt vectors more formally, let R be an arbitrary ring and let p be a prime. We define $(R)_0^\infty$ as the set of all sequences $(a_i)_{i\geq 0}$ where $a_i \in R$. Then similarly to the case where k is a field we can define a map from $(R)_0^\infty$ to $(R)_0^\infty$ given by $(a_i) \mapsto (a^{(i)})$ where $a^{(i)}$ is given as above. This map is a bijection if p is invertible in R. Now we state a technical Proposition which is the key for the construction of $W(R)$ for a general integral domain R. Let $A = \mathbb{Z}[X_0, \cdots, X_n, Y_0, \cdots, Y_n]$.

Proposition 2.2.1. *There exist unique polynomials*

$$\omega_n^{(*)}(X_0, \cdots, X_n, Y_0, \cdots, Y_n) \in A,\ n \geq 0$$

satisfying the equations

$$\mathcal{W}_n(X_0, \cdots, X_n) * \mathcal{W}_n(Y_0, \cdots, Y_n) = \mathcal{W}_n(\omega_0^*, \cdots, \omega_n^*)$$

for $n \geq 0$, where $* = +$ or $* = \times$. Moreover the polynomial

$$\omega_n^{(*)}(X_0, \cdots, X_n, Y_0, \cdots, Y_n)^p - \omega_n^{(*)}(X_0^p, \cdots, X_n^p, Y_0^p, \cdots, Y_n^p)$$

belongs to pA.

Proof. Can be found in [14, Ch 1]. Note that $\omega_0^{(+)} = X_0 + Y_0$ and $\omega_0^{(\times)} = X_0 Y_0$ and we can construct the rest inductively but the details will be omitted. □

Using this we can give a ring structure on $(R)_0^\infty$ by the following rule:

$$(a_i) * (b_i) = (\omega_0^{(*)}(a_0, b_0), \omega_1^{(*)}(a_0, a_1, b_0, b_1), \cdots)$$

where $\omega_i^{(*)}$ is the image of the polynomial $\omega_i^{(*)} \in A$ under the canonical homomorphism $\mathbb{Z} \to R$.

If p is invertible in R, then the set of Witt vectors form a ring and the ghost map which sends Witt vector to the ghost component is a bijection. In general, when p is not invertible in R, the property of the set $(R)_0^\infty$ of being a commutative ring under the operation $+$, $\times$ defined above can be expressed via certain equations for the coefficients of the polynomials

$\omega_i^{(*)} \in R[X_0, X_1, \cdots, Y_0, Y_1 \cdots]$. This implies that if a ring R satisfies these conditions, then same is true for its subrings, quotient rings and any polynomial ring over R. Note that every ring can be obtained this way from a ring $\mathcal{R}$ in which p is invertible. Thus, we can deduce that under the image in R of the above operations for $\mathcal{R}$ the set $(R)_0^\infty$ is a commutative ring with unity $(1, 0, 0, \cdots)$. Then this ring is called the Witt ring of R which we will denote by $W(R)$. Now we define the ghost component of Witt vector $X \in W(R)$ in this case. Let $\tilde{X}$ be an element of $W(\mathcal{R})$ such that its image in $W(R)$ is X, where $\mathcal{R}$ is the appropriate ring of characteristic 0. Then the ghost component of X is defined to be the image of the ghost component of $\tilde{X}$ in $(R)_0^\infty$.

2.2.2 *Properties and Operations on Witt Vectors*

From now on we assume that $R = k$ where k is a field of characteristic $p > 0$.

Lemma 2.2.2. *Define the maps $V : W(k) \to W(k)$, $F : W(k) \to W(k)$ by the formulas*

$$V(x_0, x_1, \cdots) = (0, x_0, x_1, \cdots)$$

$$F(x_0, x_1, \cdots) = (x_0^p, x_1^p, \cdots)$$

We denote $F - id = \mathscr{P}$ and let $X, Y \in W(k)$. Then $F(XY) = F(X)F(Y)$, $V(X + Y) = V(X) + V(Y)$, $p \cdot X = FV(X) = VF(X) = V(X) + \mathscr{P}V(X)$ and

$$p^n \cdot X = V^n(X) + \mathscr{P}X_0$$

for some suitable X_0.

Proof. All these properties follow from properties of $\omega_i^{(*)}$. □

Put $W_n(k) = W(k)/V^n(W(k))$. This is a ring consisting of finite sequences $(a_0, \cdots, a_{n-1})$. The maps $V, \mathscr{P}, F$ can all be defined over $W_n(k)$ in an obvious way. Now we define a surjective ring homomorphism $A : W_n(k) \to W_{n-1}(k)$ by the formula

$$A(x_0, x_1, \cdots, x_{n-1}) = (x_0, x_1, \cdots, x_{n-2})$$

whose kernel is $V^{n-1}(W_1(k))$. It can be deduced that on $W_n(k)$

$$\mathscr{P} \circ V = V \circ \mathscr{P}, \; \mathscr{P} \circ A = A \circ \mathscr{P}, \; V \circ A \circ \mathscr{P} X = p \cdot X.$$

Now we state a fundamental result on Witt Vectors which constructs local field of characteristic 0 from a finite field.

Proposition 2.2.3. *Let $k = \mathbb{F}_{p^n}$. For a Witt vector $X = (x_0, x_1, \cdots) \in W(k)$ put*

$$v(X) = min\{i : x_i \neq 0\} \text{ if } \alpha \neq 0, \; v(0) = \infty.$$

Let F be the field of fractions of $W(k)$ and $v : F^\times \to \mathbb{Z}$ the extension of v from $W(k)$. Then v is a discrete valuation on F and F is a complete discrete valuation field of characteristic 0 with ring of integers $W(k)$ and residue field isomorphic to k. In fact F is isomorphic to the unramified extension of $\mathbb{Q}_p$ of degree n.

Topology on $W(\mathbb{F}_q)$ is defined by taking

$$\{V^n(W(\mathbb{F}_q)) : n = 1, 2, \cdots\}$$

as the fundamental system of neighbourhoods of 0. Then $W(\mathbb{F}_q)$ is a compact group, and its topology coincides with the usual topology if we identify $W(\mathbb{F}_q)$ with the ring of integers of the appropriate local field.

Now we finish by stating a Theorem of Witt which classifies cyclic p-extensions of k.

Theorem 2.2.4. *Let K be a cyclic extension of degree p^n. Then K is given by $K = k(y_0, y_1, \cdots, y_{n-1})$ where*

$$\mathscr{P}(y_0, y_1, \cdots, y_{n-1}) = (x_0, x_1, \cdots, x_{n-1}) \qquad x_i \in k \, (i = 0, 1, \cdots, n-1).$$

Conversely all such extensions are cyclic of degree p^n if $x_0 \notin \mathscr{P}k$.

We often write $K = k(\mathscr{P}^{-1} X_n)$ if $X_n = (x_0, \cdots, x_{n-1})$.

Proof. Can be found in [34]. □

Remark. *We finish by making two remarks. The generator of the Galois group can be explicitly written down. Namely the group $G(K/k)$ is generated by σ which acts as follows:*

$$\sigma(y_0, y_1, \cdots, y_{n-1}) = (y_0, y_1, \cdots, y_{n-1}) + (1, 0, 0, \cdots, 0).$$

Second remark is that we can generalise Theorem 2.2.4. Generally, let C_n be a finite submodule of $W_n(k)/\mathscr{P}W_n(k)$. Then $K = k(\mathscr{P}^{-1}a_n : a_n \in C_n)$ is an abelian p-extension of k and $G(K/k) \cong C_n$. The proof of this more general statement can be found in [23].

2.3 HIGHER LOCAL FIELDS AND MILNOR K-GROUPS

Here we provide basic information about higher local fields and their Milnor K-groups. We note that merely reading this section cannot possibly replace the proper study of higher local fields since we only state the minimal information which is required in this thesis. Good introductory texts include [25], which gives a lot of details and [13, Ch 1], which provides more insight.

2.3.1 *Higher Local Fields*

We begin by giving a definition of (non-archimedean) local field of higher dimension.

Definition. *Any finite field is a local field of dimension 0. A complete discrete valuation field F is a local field of dimension $n > 0$ if its residue field $\overline{F}$ is a local field of dimension $n-1$.*

Since a higher local field is more complicated than a usual local field, it has more structure. We recall that a prime in a complete discrete valuation field is an element with valuation 1. In a higher local field we have a sequence of local parameters. Let F be a higher local field and as usual let $\mathcal{O}_F = \{\alpha \in F : v_F(\alpha) \geq 0\}$ be the ring of integers of F where v_F is the discrete valuation of F.

Definition. *Let F be an n-dimensional local field. A sequence of local parameters $t_1, \cdots, t_n \in F$ is a sequence of elements with the following (recursively defined) properties:*

1. *t_n is a prime of F;*
2. *$t_1, \cdots, t_{n-1} \in \mathcal{O}_F^{\times}$, and their reduction $\bar{t}_1, \cdots, \bar{t}_{n-1}$ form a sequence of local parameters for the field $\overline{F}$.*

Then we can state the classification Theorem of higher local fields. Definition of $K\{\{t\}\}$ for a complete discrete valuation field K can be found in any introductory texts on Higher local fields.

Theorem 2.3.1. *Let F be an n-dimensional local field. Then F is isomorphic to a field of one of the following types:*
1. $\mathbb{F}_q((t_1))..((t_n))$ *where $t_1, \cdots, t_n$ is a sequence of local parameters*
2. $K((t_1))...((t_{n-1}))$ *where K is a finite extension of $\mathbb{Q}_p$*
3. *A finite extension of* $\mathbb{Q}_p\{\{t_1\}\}\cdots\{\{t_{r-1}\}\}((t_{r+1}))\cdots((t_{n-1}))$.

The proof is well known and for this see [25]. We shall only work with the first type since Kawada-Satake's method only works in positive characteristic. In this case we say $\mathbb{F}_q$ is the last residue field of F. We now recall some elementary facts about higher rank ring of integers of higher local fields.

Definition. *Let F be a local field of dimension $\geq n$. Then the rank n ring of integers $O_F^{(n)}$ is defined recursively as follows: if $n = 0$ then $O_F^{(0)} = F$, and if $n > 0$ then*

$$O_F^{(n)} := \{x \in \mathcal{O}_F : \overline{x} \in O_{\overline{F}}^{(n-1)}\},$$

where $O_{\overline{F}}^{(n-1)}$ is the rank $n-1$ ring of integers of $\overline{F}$. If F is of dimension n we denote $O_F^{(n)} =: O_F$.

Each $O_F^{(n)}$ is a valuation ring and hence it has a unique maximum ideal $p_F^{(n)}$. We write p_F for the maximum ideal of O_F. Then we can define the higher group of principal units V_F of higher local field of dimension n to be $V_F := 1 + p_F$. We can decompose $F^\times$ further than the usual decomposition $F^\times = t^{\mathbb{Z}} \times \mathcal{O}_F^\times$ where t is any prime of F. Using the sequence of local parameters we can obtain further decomposition for local fields of dimension greater than 1.

Proposition 2.3.2. *Let F be an n-dimensional local field and let $t_1, \cdots, t_n$ be a sequence of local parameters of F. Then*

$$F^\times \cong t_1^{\mathbb{Z}} \times \cdots \times t_n^{\mathbb{Z}} \times \mathbb{F}_q^\times \times V_F.$$

Proof. For this, one could prove that $O_F^{(n-1-i)}[t_i^{-1}] = O_F^{(n-i)}$ and $(O_F^{(n-i)})^\times \cong (O_F^{(n-1-i)})^\times \times t_i^{\mathbb{Z}}$. These statements can be proved via induction using the minimum non-zero prime ideal $\mathfrak{p}_F = p_F^{(1)}$ of all $O_F^{(i)}$. Again, see [25] for the details. $\square$

2.3.2 *Properties of Milnor K-groups of Higher Local Fields*

We begin by defining the Milnor K-group and introducing some basic and essential maps on $K_m(F)$. We mainly follow the presentations of [14, Ch 9]. The n-th Milnor K group of a field F, $K_n(F)$ is defined as follows:

Definition. *We let $K_0(F) = \mathbb{Z}$ and $K_1(F) = F^\times$. For higher n let $I_n = \langle \{\alpha_1 \otimes \cdots \otimes \alpha_n \in (F^\times)^{\otimes n} : \alpha_i + \alpha_j = 1,$ for some $1 \leq i, j \leq n$ and $i \neq j\}\rangle$. Then*

$$K_n(F) := (F^\times)^{\otimes n} / I_n.$$

We write $\{\alpha_1, \cdots, \alpha_n\}$ for elements of $K_n(F)$ and let $\phi : (F^\times)^n \to K_n(F)$ be the symbol map. This map is not surjective in general. It is conventional to write multiplication in $K_n(F)$ additively when $n > 1$.

All Milnor K-groups have the following property:

Lemma 2.3.3. 1. $\{\alpha_1, \cdots, \alpha_n\} = 0$ *if* $\alpha_i + \alpha_j = 0$ *for some* $i \neq j$.
2. $\{\cdots, \alpha_i, \cdots, \alpha_j, \cdots\} = -\{\cdots, \alpha_j, \cdots, \alpha_i, \cdots\}$; *$K_n(F)$ is anti-commutative.*

This elementary result will be very useful for calculations in K-groups later on. Immediately we obtain that $\{x, x\} = \{-1, x\}$. Also with the help of this Lemma we can prove that $K_n(\mathbb{F}_q) = 0$ for $n > 1$.

Now let F be a complete discrete valuation ring with a prime t and $\overline{F}$ be its residue field. Then there is a unique homomorphism $\delta : K_n(F) \to K_{n-1}(\overline{F})$ such that $\delta\{\epsilon_1, \cdots, \epsilon_{n-1}, t\} = \{\overline{\epsilon_1}, \cdots, \overline{\epsilon}_{n-1}\}$ and $\delta\{\epsilon_1, \cdots, \epsilon_n\} = 0$ for $\epsilon_i \in \mathcal{O}_F^\times$ and $\overline{\epsilon}_i$ its image in $\overline{F}$. We call δ the boundary map. For $n = 1$, δ is defined to be the same as discrete valuation of F. The explicit definition can be found in [14, Ch 9].

We briefly recall the definition of a norm map for Milnor K-groups. Write $E = F(X)$ and let v be a non-trivial discrete valuation on E, trivial on F. Define $F(v)$ to be the residue field of E with respect to v. Recall that (for instance see [32]) these valuations are in one-to-one correspondence with monic irreducible polynomials of positive degree over F and $1/X$. The latter corresponds to the valuation v_∞ which is defined as $v_\infty(f(X)/g(X)) = deg(f(X)) - deg(g(X))$ for $f, g \in E[X]$.

Let $j_{F/E}$ be the map $K_n(F) \to K_n(E)$ induced by natural injection of fields $F \hookrightarrow E$. Then we have an exact sequence

$$0 \to K_n(F) \stackrel{j_{F/E}}{\to} K_n(E) \stackrel{\oplus\delta_v}{\to} \oplus_{v \neq v_\infty} K_{n-1}(F(v)) \to 0.$$

The exactness is proved by Bass and Tate. To define the norm map we have to consider slightly varied sequence $0 \to K_n(F) \stackrel{j_{F/E}}{\to} K_n(E) \stackrel{\oplus\delta_v}{\to} \oplus_v K_{n-1}(F(v))$. This sequence is not exact in the terms $\oplus_v K_{n-1}(F(v))$ but exact everywhere else since $\delta_{v_\infty} j_{F/E}(K_n(F)) = 0$.

Now introduce the homomorphism

$$N = \oplus N_v : \oplus K_{n-1}(F(v)) \to K_{n-1}(F).$$

First, let N_{v_∞} be the identity map of $K_n(F_{v_\infty}) = K_n(F)$. For all other v let N_v be the map such that the composition

$$K_n(E)/j_{F/E}K_n(F) \stackrel{\sim}{\to} \oplus_{v \neq v_\infty} K_{n-1}(F(v)) \stackrel{\oplus N_v}{\to} K_{n-1}(F)$$

coincides with

$$-\delta_{v_\infty} : K_n(E)/j_{F/E}K_n(F) \to K_{n-1}(F).$$

Such homomorphisms N_v exist and are uniquely determined. Then $N_v : K(F(v)) \to K(F)$ is a homomorphism of $K(F)$-modules, namely,

$$N_v(j_{F/F(v)}(x) \cdot y) = x \cdot N_v(y) \text{ for } x \in K_n(F), y \in K_m(F(v)).$$

Then we obtain the exact sequence

$$0 \to K_n(F) \stackrel{j_{F/E}}{\to} K_n(E) \stackrel{\oplus\delta_v}{\to} \oplus_v K_{n-1}(F(v)) \stackrel{N}{\to} K_{n-1}(F) \to 0.$$

So for any finite primitive extension $F(\alpha)/F$, we have a norm map since in this case there exists a monic irreducible polynomial $f \in F[X] \subset E$ such that $f(\alpha) = 0$. To define the norm map for a general finite extension $L = F(\alpha_1, \cdots, \alpha_n)$ we put $F_0 = F$, $F_i = F_{i-1}(\alpha_i)$. Then there is the homomorphism $N_{v_i} : K_n(F_i) \to K_n(F_{i-1})$, where v_i is the discrete valuation of the field $F_{i-1}(X)$ which corresponds to α_i. We now denote this map by N_{α_i}. Then we may define $N_{L/F} = N_{\alpha_1} \circ \cdots N_{\alpha_n} : K_n(L) \to K_n(F)$. This map is well-defined; it does not depend on the choice of $\alpha_1, \cdots, \alpha_n$. To prove this, one needs to go through some elementary but lengthy arguments which can be found in [14, Ch 9]. We state some properties of the norm maps which can be found in the same text.

Theorem 2.3.4 (Bass-Tate-Kato). *Let L/F be a finite extension. Then the norm map $N_{L/F} : K_n(L) \to K_n(F)$ defined above has the following properties:*

1. *$N_{L/F}$ coincides with $N_{\alpha_1} \circ \cdots N_{\alpha_n}$ for any $\alpha_1, \cdots \alpha_n \in L$ such that $L = F(\alpha_1, \cdots \alpha_n)$.*
2. *For every subextension M/F in L/F*

$$N_{L/F} = N_{M/F} \circ N_{L/M}.$$

3. *The map $N_{L/F}$ acts on $K_0(L)$ as the multiplication by $[L : F]$ and on $K_1(L)$ as the norm map of fields $N_{L/F} : L^\times \to F^\times$.*
4. *The composition $N_{L/F} \circ j_{F/L}$ coincides with multiplication by $[L : F]$.*
5. *If L_1/F is a normal finite extension with $L_1 \supset L$, then the composition $j_{F/L_1} \circ N_{L/F}$ coincides with $m \sum_i \sigma_i$, where m is the degree of inseparability of L/F and $\sigma_i : K(L) \to K(L_1)$ are induced by distinct embeddings of L in L_1 over F.*
6. *If σ is F-automorphism of L, then $N_{L/F} \circ \sigma = N_{L/F}$, where $\sigma : K_n(L) \to K_n(L)$ is induced by σ.*

We have the following commutative diagram for all complete discrete valuation field F and its finite normal extension L.

$$\begin{tikzcd} K_n(L) \arrow[r, "\delta_L"] \arrow[d, "N_{L/F}"'] & K_{n-1}(\overline{L}) \arrow[d, "N_{\overline{L}/\overline{F}}"] \\ K_n(F) \arrow[r, "\delta_F"] & K_{n-1}(\overline{F}) \end{tikzcd}$$

This diagram is commutative since the images of $\{\epsilon_1, \cdots, \epsilon_{n-1}, t_L\}$ and $\{\epsilon_1, \cdots, \epsilon_n\}$ under $\delta_F \circ N_{L/F}$ and $N_{\overline{L}/\overline{F}} \circ \delta_L$ coincide.

3

EXPLICIT CLASS FIELD THEORY IN ONE DIMENSION

Neukirch's abstract and axiomatic class field is important for numerous reasons. Class field theory always admits a canonical isomorphism $G(L/K) \cong A_K/N_{L/K}A_L$ for each finite abelian extension L/K, where A_K is some appropriate domain depending on K, giving a description of the Galois group of abelian extensions in terms of the arithmetic of the field itself. This map is given explicitly in Neukirch's method unlike most other methods available. Also with cohomological approaches we have to prove many results which at times seem irrelevant. On the other hand Neukirch's method is conceptually easy to understand and it is clear what each step means to the theory. Lastly we mention that the class field theory mechanism of Neukirch is completely group theoretical. We shall discuss this in more detail in Ch 3.2.

3.1 NEUKIRCH'S ABSTRACT CLASS FIELD THEORY AXIOM

Here we recall the main idea behind Neukirch's method on Class field theory. We shall briefly outline the abstract class field theory Neukirch constructs and how this applies to local fields. But before this, we first define and prove some results on H^0 and H^{-1}. Any omitted proof in this section can be found in [27].

3.1.1 H^0 and H^{-1}

In this section we define groups H^0 and H^{-1} which are essential objects for Neukirch's abstract class field theory method. They are existing objects in cohomological method[1] but we define them explicitly so that we do not use cohomology.

Let G be a group and let A be a G-module. If G is finite then every G-module A contains the norm group

$$N_G A = \{N_G a := \sum_{\sigma \in G} \sigma a | a \in A\}.$$

Since $\tau N_G a = \sum_{\sigma \in G} \tau \sigma a = N_G a$, we have $N_G A \subset A^G$. Then we define the norm residue group

$$H^0(G, A) = A^G / N_G A.$$

Now let G be a cyclic group with a generator σ. If A is a G-module, then, in addition to $H^0(G, A)$ we consider the group

$$H^{-1}(G, A) = {}_{N_G}A / I_G A$$

where ${}_{N_G}A = \{a \in A | N_G a = 0\}$ and $I_G A = \{\sigma a - a | a \in A\}$. Those two objects are essential in class field theory and the objects $H^i(G, A)$ can be defined more abstractly for all $i \in \mathbb{Z}$. See [4] for the full definition.

Another concept which will be very useful is the Herbrand quotient. For G, a finite cyclic group and a G-module A, we define the Herbrand quotient of A to be

$$h(G, A) = \frac{\# H^0(G, A)}{\# H^{-1}(G, A)}$$

provided that both these orders are finite. Here we state two properties of the Herbrand quotient:

Proposition 3.1.1. *Let G be a finite cyclic group and let $0 \to A \to B \to C \to 0$ be an exact sequence of G-modules. Then*

$$h(G, B) = h(G, A) \cdot h(G, C),$$

[1] Tate cohomology group; see for instance [28]

in the sense that when two of these quotients are defined, so is the third and the equality holds. If A is a finite G-module, then $h(G, A) = 1$.

Now we introduce semilocal theory which will be useful for the transition between local and global class field theory. The rest of content in this section is taken from [21, p. 183−185].

We consider a finite group G acting on an abelian group A. Assume that A is direct product of subgroups,

$$A = \prod_{i=1}^{s} A_i,$$

and that G permutes these groups A_i transitively. Let G_1 be the decomposition group of A_1, i.e. subgroup of elements which fix A_1. We call (G_1, A_1) its local component. For each $a \in A$ we can uniquely write $a = \sum_{i=1}^{s} a_i$ where $a_i \in A_i$. For each i we choose σ_i such that $\sigma_i A_1 = A_i$. Then we have left coset decomposition

$$G = \cup_{i=1}^{s} \sigma_i G_1.$$

We note that each $a_i \in A_i$ can be written as $\sigma_i a_i'$ for a uniquely determined element $a_i' \in A_1$.

Lemma 3.1.2. *The projection $\pi : A \to A_1$ induces an isomorphism $H^0(G, A) \cong H^0(G_1, A_1)$.*

Proof. We first observe that A^G consists of all elements of the form

$$\sum_{i=1}^{s} \sigma_i a_1$$

with $a_1 \in A_1^{G_1}$. It is clear that such an element is fixed under G. On the other hand if $a = \sum_{i=1}^{s} \sigma_i a_i'$ where $a_i' \in A_1$ for each i, is fixed under G, then for a fixed index j we apply σ_j^{-1} and see that $a_j' = \sigma_j^{-1}\sigma_j a_j'$ is the A_1-component of $\sigma_j^{-1} a = a$. Hence $a_j' = a_1'$ for all j. In particular, an element of A^G is uniquely determined by its first component, and thus the projection gives an isomorphism

$$A^G \cong A_1^{G_1}.$$

On the other hand, for a fixed j and $a_1 \in A_1$ we have

$$N_G(\sigma_j a_1) = \sum_{\sigma \in G} \sigma a_1 = \sum_{i=1}^{s} \sigma_i N_{G_1}(a_1).$$

This shows that $N_G(A)$ consists precisely of those elements of the form

$$\sum_{i=1}^{s} \sigma_i N_{G_1}(a_1)$$

and thus it is clear that $A^G/N_G(A) \cong A^{G_1}/N_{G_1}(A_1)$. $\square$

Lemma 3.1.3. *There is an isomorphism $H^{-1}(G, A) \cong H^{-1}(G_1, A_1)$.*

Proof. Let $a = \sum_{i=1}^{s} \sigma_i a'_i$ where $a'_i \in A_1$. If we denote $\alpha = \sum_{i=1}^{s} a'_i$ we obtain

$$N_G(a) = \sum_{j=1}^{s} \sigma_j N_{G_1}(\alpha).$$

Thus it is easy to see that $N_G(a) = 0$ if and only if $N_{G_1}(\alpha) = 0$. Therefore the map

$$a \mapsto \alpha$$

is a homomorphism

$$\lambda : KerN_G \to KerN_{G_1},$$

which is obviously surjective. We now show that λ maps $I_G A$ into $I_{G_1} A_1$. If $\sigma \in G$, then there is a permutation g of the indices i such that

$$\sigma\sigma_i = \sigma_{g(i)}\tau_{g(i)}$$

with some $\tau_{g(i)} \in G_1$. Hence

$$\lambda(\sigma a - a) = \sum_{i=1}^{s} (\tau_{g(i)} a'_i - a'_i)$$

thus proving our assertion. To conclude the proof, it will suffice to show that if $\lambda(a) = 0$ then $a \in I_G(A)$. However, if $\alpha = a'_1 + \cdots + a'_s = 0$ we can write

$$a = \sum_{i=1}^{s} (\sigma_i a'_i - a'_i),$$

and so $a \in I_G A$. $\square$

Finally, note that if L/K is a finite Galois extension of fields with Galois group $G = G(L/K)$ then $H^1(G, L^\times) = 1$. This result is commonly known as *Hilbert 90* theorem and is of huge importance in local class field theory. We will discuss about this theorem in detail in Ch 3.2.

3.1.2 *Abstract Class Field Theory Mechanism*

Let k_0 be a field and $G = G(k_0^{sep}/k_0)$ be its absolute Galois group. For each finite separable extension K/k_0, denote $G_K = G(k_0^{sep}/K)$. Assume that there is a $\hat{\mathbb{Z}}$-extension $\tilde{k}_0/k_0$ and fix the isomorphism $deg : G(\tilde{k}_0/k_0) \cong \hat{\mathbb{Z}}$.

For every finite extension K/k_0 we set $\tilde{K} = K \cdot \tilde{k}_0$ and $f_K = [K \cap \tilde{k}_0 : k_0]$. Then the map deg induces another isomorphism

$$deg_K = \frac{1}{f_K} deg : G(\tilde{K}/K) \cong \hat{\mathbb{Z}}.$$

Definition. *The element $\varphi_K \in G(\tilde{K}/K)$ with $deg_K(\varphi_K) = 1$ is called the Frobenius over K.*

For every finite extension L/K we set $L^0 = L \cap \tilde{K}$, $f_{L/K} = [L^0 : K] = \frac{f_L}{f_K}$, $\varphi_{L^0/K} = \varphi_K|_{L^0}$. We have $\varphi_L|_{\tilde{K}} = \varphi_K^{f_{L/K}}$. Let $\tilde{L} = L \cdot \tilde{K}$. Then $deg_K : G(\tilde{K}/K) \cong \hat{\mathbb{Z}}$ induces a surjective homomorphism

$$deg_K : G(\tilde{L}/K) \to \hat{\mathbb{Z}},$$

and we set

$$\phi(\tilde{L}/K) = \{\tilde{\sigma} \in G(\tilde{L}/K) : deg_K(\tilde{\sigma}) \in \mathbb{Z}_{>0}\}.$$

It can easily be proved that the map

$$\phi(\tilde{L}/K) \to G(L/K),\ \tilde{\sigma} \mapsto \tilde{\sigma}|_L,$$

is surjective. If $\sigma = \tilde{\sigma}|_L$, then we call $\tilde{\sigma}$ a Frobenius lift of σ.

Let A be G-module. We use multiplicative notation on A since it will be more natural to apply it later on. For each field $k_0 \subset K \subset k_0^{sep}$ let $A_K = A^{G_K}$.

If L/K is a finite extension, then $A_K \subset A_L$ and we have the norm map $N_{L/K} : A_L \to A_K$ defined by

$$N_{L/K}(a) = \prod_{\sigma} a^{\sigma}$$

where σ runs through a system of right representatives of G_K/G_L. Similarly to the usual field norm, it can be checked that this norm map is transitive on towers of field extensions. If L/K is a Galois extension, then A_L is a $G(L/K)$ module and $A_K = A_L^{G(L/K)}$.

Definition. *A henselian valuation of A_{k_0} with respect to deg is a homomorphism*

$$v : A_{k_0} \to \hat{\mathbb{Z}}$$

with the properties

i) $v(A_{k_0}) = Z \supset \mathbb{Z}$ and $Z/nZ \cong \mathbb{Z}/n\mathbb{Z}$ for all $n \in \mathbb{Z}_{>0}$.

ii) $v(N_{K/k_0}A_K) = f_K Z$ for all fields K.

Note that the definition does not imply that v is a discrete valuation, and that what we call "henselian valuation" in this thesis is very different, in general, from what is called "henselian valuation" in valuation theory.

For every field K, the henselian valuation $v : A_{k_0} \to \hat{\mathbb{Z}}$ yields the homomorphism

$$v_K = \frac{1}{f_K} v \circ N_{K/k_0} : A_K \to \hat{\mathbb{Z}}$$

with image Z. We say $\pi_K \in A_K$ with valuation 1 is a prime of A_K.

Furthermore, we now assume that G-module A satisfies the following axiomatic condition:

Class Field Theory Axiom. *For every finite cyclic extension L/K*

$$\#H^i(G(L/K), A) = \begin{cases} [L:K] & \text{for } i = 0 \\ 1 & \text{for } i = -1 \end{cases}$$

Then, for each finite Galois extension L/K we define the reciprocity map

$$r_{L/K} : G(L/K) \to A_K/N_{L/K}A_L$$

given by

$$r_{L/K}(\sigma) = N_{\Sigma/K}(\pi_\Sigma) \bmod N_{L/K}A_L,$$

where Σ is the fixed field of a Frobenius lift $\tilde{\sigma} \in \phi(\tilde{L}/K)$ of $\sigma \in G(L/K)$ and $\pi_\Sigma \in A_\Sigma$ is a prime element. Neukirch proves that this map is well-defined and is a homomorphism.

Remark. *We observe that the Class field theory axiom is similar to the 3rd condition of the class formation axiom in Ch 2.1. However they are quite different in two regards. One difference is that we do not need to calculate the group $H^i(G(L/K), A)$ but instead it suffices to only consider its cardinality. This means verification process is a lot simpler. Another difference is that we only need to check the axioms in the case of cyclic extensions.*[2] *However, class formation can be applied in more general way. In particular we shall see in the next chapter that we can define class formation even when the 'maximal extension' does not contain any $\hat{\mathbb{Z}}$-extension.*

We define a **class field theory** for such a G-module to be a pair of homomorphisms

$$(deg : G(\tilde{k}_0/k_0) \to \hat{\mathbb{Z}},\ v : k_0^\times \to \hat{\mathbb{Z}})$$

where deg is continuous and bijective and v is a henselian valuation with respect to deg. Now we are ready to state the main theorem of Class Field Theory:

Theorem 3.1.4. *If L/K is a finite Galois extension, then*

$$r_{L/K} : G(L/K)^{ab} \to A_K/N_{L/K}A_L$$

is an isomorphism.

The inverse map of $r_{L/K}$ yields a surjective homomorphism $(\ ,L/K)$: $A_K \to G(L/K)^{ab}$ with kernel $N_{L/K}A_L$. This map is called the **norm residue symbol** of L/K. The norm residue symbol satisfies certain functoriality conditions. In particular we have the commutative diagram below (for $L \subset L'$ and $K \subset K'$).

[2] In fact it suffices to check in the case of cyclic extensions of prime degree.

$$\begin{array}{ccc} A_{K'} & \xrightarrow{(\ ,L'/K')} & G(L'/K') \\ {\scriptstyle N_{K'/K}}\downarrow & & \downarrow \\ A_K & \xrightarrow{(\ ,L/K)} & G(L/K) \end{array}$$

Passing to projective limits, the norm residue symbol extends to any infinite Galois extensions since we have the above commutative diagram. In the particular case of the extension $\tilde{K}/K$ we have

Proposition 3.1.5. *$deg_K \circ (\ ,\tilde{K}/K) = v_K$. In particular $(a,\tilde{K}/K) = \varphi_K^{v_K(a)}$.*

An another important feature of classical class field theory is the Existence Theorem. For every field K we introduce a topology in A_K as follows. For each $a \in A_K$ we take the cosets $aN_{L/K}A_L$ as a basis of open neighbourhoods of a, where L/K runs through all finite Galois extensions of K. We call this topology the norm topology of A_K. Note that v_K and $N_{L/K}$ are continuous if L/K is a finite extension. Then we can classify the finite abelian extensions.

Theorem 3.1.6 (Existence Theorem)**.** *The map $L \mapsto \mathcal{N}_L = N_{L/K}A_K$ yields a $1-1$ correspondence between the finite abelian extensions L/K and the open subgroups $\mathcal{N}$ of A_K. Moreover*

$$L_1 \subset L_2 \iff \mathcal{N}_{L_1} \supset \mathcal{N}_{L_2},\ \mathcal{N}_{L_1\cdot L_2} = \mathcal{N}_{L_1} \cap \mathcal{N}_{L_2},\ \mathcal{N}_{L_1\cap L_2} = \mathcal{N}_{L_1}\cdot\mathcal{N}_{L_2}.$$

If L is associated to the open subgroup $\mathcal{N}$ of A_K, then L is called the *class field* of $\mathcal{N}$.

3.1.3 *Local Class Field Theory*

We shall briefly recall results from [27] without proof.

Let $k_0 = \mathbb{F}_p((T))$ or $\mathbb{Q}_p$ and $A = (k_0^{sep})^\times$. Thus $A_K^\times = K^\times$ for any finite separable extension K/k_0. Let v be the usual normalised discrete valuation of k_0 and $\tilde{k}_0 = k_0^{ur}$. Note that in the characteristic p case $k_0^{ur} = \mathbb{F}_p^{sep}\cdot k_0$ and in the characteristic 0 case k_0^{ur} is obtained by adjoining all roots of unity coprime to p. Then there is a canonical isomorphism $deg : G(k_0^{ur}/k_0) \cong \hat{\mathbb{Z}}$. Let K be a finite extension of k_0. Then one can prove the following:

Theorem 3.1.7. *The pair* $(deg : G(k_0^{ur}/k_0) \to \hat{\mathbb{Z}}, v : k_0^{\times} \to \hat{\mathbb{Z}})$ *is a class field theory and* $A = (k_0^{sep})^{\times}$ *satisfies the Class Field Theory Axiom. Therefore, for each Galois extension* L/K *of local fields we obtain a canonical isomorphism*

$$r_{L/K} : G(L/K)^{ab} \to K^{\times}/N_{L/K}L^{\times}.$$

Also the Existence Theorem holds.

Note that the only part which requires work is the proof of the class field theory axiom. In fact, by Hilbert 90, it suffices to calculate the Herbrand quotient only. By the exact sequence

$$0 \to U_L \to L^{\times} \xrightarrow{v_L} \mathbb{Z} \to 0$$

it suffices to prove the following:

Lemma 3.1.8. *Let* L/K *be a finite extension of local field and* U_L *be unit group of* L. *Then we have* $h(G(L/K), U_L) = 1$.

Proof. See Ch 5 of [27]. □

As before, the inverse of $r_{L/K}$ gives the local norm residue symbol

$$(\ , L/K) : K^{\times} \to G(L/K)^{ab}$$

with kernel $N_{L/K}L^{\times}$.

3.2 TAME LOCAL CLASS FIELD THEORY-MONO-ANABELIAN STYLE

One of the themes of anabelian geometry is trying to work with group structures only, even when the object has more underlying-structure. In particular when the object is a field, we try to work with its multiplicative structure only, forgetting about the addition. The classical proof of Hilbert 90 would be an example where this strictly does not apply in that we need the ring structure of the field even though it is a result about the multiplicative structure only. Thus, the original proof of Neukirch local class field theory is not done in the mono-anabelian way. In fact this simply would not be possible in general since the interpretation of the norm map requires the ring structure of the base field if the extension is wildly ramified. It is known that the when L/F is a totally ramified extension of degree p, the interpretation of the norm map on U_s[3] is very complicated. In particular addition is essential for this interpretation. (See (1.5) of [14, Ch 3] for a detailed exposition.) However, if we restrict to the tamely ramified extensions, it might be possible to prove it without much use of additive structure. This is especially important since the proof of Neukirch-Ikeda-Uchida theorem requires Local class field theory.

In the last section, Neukirch's abstract theory was presented in a way which is conceptually easiest to see in the form of Galois group of fields. Nonetheless the theory in fact, only requires the group structure; ring or field structure is not needed to present the axioms of Neukirch. The original presentation of Neukirch's theory makes it clear that this assertion is true. Instead of the absolute Galois group of a field k_0, he uses an abstract profinite group G and we can think of G_K as a closed subgroup of G where K is just an index. See [26] for the detail.

[3] s is the smallest integer such that $\frac{\sigma(\pi_L)}{\pi_L} \in U_s$ for a prime L and $id \neq \sigma \in G(L/F)$

3.2.1 *Calculation of the Norm group*

Now we show that for a local field K, $A = (K^{sep})^\times$ satisfies the class field axiom in an explicit way. First we show that $\#K^\times/N_{L/K}L^\times = [L : K]$ for a cyclic tamely ramified extension of L/K. Of course this is classical but we include it in here for completeness. The commutative diagrams can be found in Ch 3.1 and the calculation of norm groups can be found in Ch 4.1 of [14].

Here $U_{i,K}$ is defined to be $1 + p_K^i$ and similarly for $U_{i,L}$ where p_K is the unique maximum ideal of $\mathcal{O}_K$. $\lambda_K : U_K \to \overline{K}^\times$ is defined to be the residue homomorphism given by $\epsilon \mapsto \overline{\epsilon}$ and $\lambda_{i,K} : U_{i,K} \to \overline{K}$ is defined by $1 + \sum_{k\geq i} a_k t_F^k \mapsto a_i$.

Proposition 3.2.1. *Let L/K be an unramified extension of local fields of degree n. Then we have commutative diagrams*

$$\begin{array}{ccc} L^\times & \xrightarrow{v_L} & \mathbb{Z} \\ {\scriptstyle N_{L/K}}\downarrow & & \downarrow{\scriptstyle \times n} \\ K^\times & \xrightarrow{v_K} & \mathbb{Z} \end{array}$$

$$\begin{array}{ccc} U_L & \xrightarrow{\lambda_L} & \mathbb{F}_{q^n}^\times \\ {\scriptstyle N_{L/K}}\downarrow & & \downarrow{\scriptstyle N_{\mathbb{F}_{q^n}/\mathbb{F}_q}} \\ U_K & \xrightarrow{\lambda_K} & \mathbb{F}_q^\times \end{array}$$

$$\begin{array}{ccc} U_{i,L} & \xrightarrow{\lambda_{i,L}} & \mathbb{F}_{q^n} \\ {\scriptstyle N_{L/K}}\downarrow & & \downarrow{\scriptstyle Tr_{\mathbb{F}_{q^n}/\mathbb{F}_q}} \\ U_{i,K} & \xrightarrow{\lambda_{i,K}} & \mathbb{F}_q \end{array}$$

The proof is simple (in [14, Ch 3]) but requires the use of additive structure of field. Since the norm map and trace map are surjective in finite fields, $N_{L/K}L^\times = t^n \times U_K$. Thus, we immediately obtain that $\#K^\times/N_{L/K}L^\times = n$.

Proposition 3.2.2. *Let L/K be a totally and tamely ramified cyclic extension of local fields of degree n. Then we have commutative diagrams*

$$\begin{tikzcd} L^\times \arrow[r, "v_L"] \arrow[d, "N_{L/K}"'] & \mathbb{Z} \arrow[d, "id"] \\ K^\times \arrow[r, "v_K"] & \mathbb{Z} \end{tikzcd}$$

$$\begin{tikzcd} U_L \arrow[r, "\lambda_L"] \arrow[d, "N_{L/K}"'] & \mathbb{F}_q^\times \arrow[d, "\uparrow n"] \\ U_K \arrow[r, "\lambda_K"] & \mathbb{F}_q^\times \end{tikzcd}$$

$$\begin{tikzcd} U_{ni,L} \arrow[r, "\lambda_{ni,L}"] \arrow[d, "N_{L/K}"'] & \mathbb{F}_q \arrow[d, "\times\bar{n}"] \\ U_{i,K} \arrow[r, "\lambda_{i,K}"] & \mathbb{F}_q. \end{tikzcd}$$

Moreover $N_{L/F}U_{i,L} = N_{L/F}U_{i+1,L}$ unless $n \not| i$.

Here $\uparrow n$ is n-power map and $\times\bar{n}$ is the multiplication by $\overline{n} \in \overline{K}$. The diagrams easily give the description of $N_{L/K}L^\times$ thus we obtain that $\#K^\times/N_{L/K}L^\times = n$ if L/K is a cyclic tamely ramified extension.

Similar diagram exists when $n = p$. However this diagram is a lot more complicated and involves explicit use of addition in one crucial case. Thus, it is fundamentally impossible to analyse the norm map without the use of the addition in this case.[4]

[4]Other diagrams also use addition, for instance in order to define λ_* maps we need to use the fact that all elements of U_i can be written in the from $1 + \sum_i a_i t^i$. Nonetheless the maps on the RHS of the diagrams in Proposition 3.2.1 and 3.2.2 are all multiplicative except for the Trace map of finite field which seems to be necessary.

3.2.2 *Explicit Proof of Hilbert 90 for Local Fields*

Theorem 3.2.3 (Hilbert 90)**.** *Let L/K be a cyclic extension of fields with Galois group $G = G(L/K)$ with generator sigma. If $N_{L/K}x = 1$ then there exists $z \in L^\times$ such that $y/\sigma y = x$.*

We recall the usual proof of Hilbert 90.

Proof. Let L/K be cyclic extension of degree n with σ being the generator of the Galois group. If $x \in ker(N_{L/K})$ then we consider the map $z \mapsto z + x\sigma(z) + \cdots + x\sigma(x)\cdots\sigma^{n-2}(x)\sigma^{n-1}(z) =: y$. Then one confirms that $x\sigma(y) = y$. Since σ^i are linearly independent there exists z such that y is non zero. □

This proof is somewhat unsatisfactory from the point of view of mono-anabelian geometry due to the reason we explained earlier.[5] This is an issue for mono-anabelian geometry since one wants to forget about the additive structure. Here we prove Hilbert 90 for tamely ramified extensions of local field in an explicit way. By no means is this proof mono-abelian since we still make use of addition. We shall denote the map $y \mapsto y/\sigma(y)$ as $(1/\sigma)$ instead of the usual notation $(1-\sigma)$ in this proof only. This is because in this proof we make use of the map $y \mapsto y - \sigma y$ as well which would be more appropriate to have that notation.

Explicit proof of Hilbert 90 for tamely ramified extensions of local fields. Let L/K be a cyclic extension which is tamely ramified. We recall that the multiplicative group of a local field K has a decomposition $K^\times \cong t^{\mathbb{Z}} \times \mathbb{F}_q^\times \times U_{1,K}$ and elements of $U_{1,K}$ can be written uniquely by the following convergent product $\prod_{i\geq 0}(1 + a_i t^i)$ where $a_i \in \mu_K \cong \mathbb{F}_q^\times$.

Let L/K be an unramified extension of degree n and let t be a prime element of L which is also a prime of K. Suppose that $x \in ker(N_{L/K})$ where

[5]We also note that the additive version of this theorem relies on the multiplicative structure, similarly as in the multiplicative Hilbert 90.

$x = t^k \times a \times u$ ($a \in \mu_L$ and $u \in U_{1,L}$). Then this necessarily means that $k = 0$, $N_{L/K}a = 1 = N_{L/K}u$. We note that Hilbert 90 for finite fields follows easily from looking at the cardinality. Thus we may assume that $x \in U_{1,L}$. We observe that

$$(1/\sigma)(1+a_it^i) = (1+a_it^i)(1-\sigma(a_i)t^i+\sigma(a_i^2)t^{2i}+\cdots+(-1)^m\sigma(a_i^m)t^{mi})+\cdots$$

$$= 1+\sum_{m>0}(-1)^m(\sigma(a_i)-a_i)(\sigma(a_i))^{m-1}t^{mi}.$$

Now we suppose that $N_{L/K}(x) = 1$ where $x = 1 + a_jt^j + \cdots$ and $a_j \neq 0$. Then by explicit calculation[6] we see that $a_j \in ker(Tr_{\mathbb{F}_{q^n}/\mathbb{F}_q}) = (1-\sigma)\mathbb{F}_{q^n}$. Thus, there exists $b \in \mathbb{F}_q^\times$ such that $(1/\sigma)(1+bt^j) = 1 + a_jt^j + \cdots$. So $x = ((1/\sigma)(1+bt^j)(1+\sum_{i>j} b_it^i)$ for some b_i, and since this step can be done iteratively we obtain Hilbert 90 in this case.

Now suppose that L/K is a totally ramified extension of degree n. Recall that in this case $n|q-1$ and thus group of n-th root of unity is contained in $K^\times$. Also there exists a prime element t of L such that $L = K(t)$ and $t^n =: T$ is a prime of K. Then $\sigma(t) = \xi t$ where ξ is a primitive n-th root of unity. Again consider the decomposition $L^\times = t^{\mathbb{Z}} \times \mathbb{F}_q^\times \times U_{1,L}$. An element $a \in \mathbb{F}_q^\times$ is in $ker(N_{L/K})$ if and only if a is a power of ξ since the norm map restricted to $\mathbb{F}_q^\times$ is simply $\times\bar{n}$ map. Since $(1/\sigma)t = \xi^{-1}$, it remains to prove that any element $x \in U_{1,L}$ such that $N_{L/K}x = 1$ is in the image of $(1/\sigma)$. Similarly as the unramified case, we make use of the calculation

$$(1/\sigma)(1+at^m) = 1+\sum_{i=1}^{\infty}(-1)^i(\xi^m-1)\xi^{m(i-1)}a^it^{mi}.$$

Suppose that $x \in ker(N_{L/K})$ and that $x = 1 + a_jt^j + \cdots \in U_{j,L}\backslash U_{j+1,L}$ for some $j > 0$. If j is not a multiple of n then we can find b_j such that $(1/\sigma)(1+b_jt^j) = 1 + a_jt^j \mod t^{j+1}$.[7] Using the same argument as in unramified case, we obtain that $x \in im(1/\sigma)$. It now suffices to prove that

[6] Or we could use the commutative diagram in Proposition 3.2.1. Either way this argument is not free of addition.

[7] $b_j = a_j(1-\xi^j)^{-1}$

$x \in U_{kn,L} \backslash U_{kn+1,L}$ cannot be in the kernel of $N_{L/K}$.[8] We write $x = (1 + a_{kn}t^{kn}) \times \prod_{i>kn}(1 + a_i t^i) = (1 + a_{kn}T^k) \times \prod_{i>kn}(1 + a_i t^i)$. Then $N_{L/K}x = (1 + a_{kn}T^k)^n \times N_{L/K}\prod_{i>kn}(1 + a_i t^i)$. We observe that $N_{L/K}(\prod_{i>kn}(1 + a_i t^i)) \in U_{kn+1,F}$ and $(1 + a_{kn}T^k)^n \in U_{kn,K} \backslash U_{kn+1,K}$. Thus we conclude that $N_{L/K}x \neq 1$. $\square$

Remark. *Here we presented an explicit proof of Hilbert 90 of local fields when the extension is not wildly ramified. Initially we had hoped to find a completely group theoretical proof or at least a proof with two parts where one part is addition-free. However, clearly the proof presented here is very far from it. In fact, after this investigation, I believe that it is not possible to separate the role of addition, at least with this 'naive'[9] approach. This is because to do any calculation we need to use the explicit form of the elements of local field and role of addition is essential in order to obtain this form.*

[8] In fact this follows immediately by considering properties of $\lambda_{L,n}$ and the commutative diagrams in Proposition 3.2.2. Here we spell out the argument more explicitly.

[9] by naive I mean this kind of approach which is simply brute force calculation

3.3 UNIQUENESS OF CLASS FIELD THEORY IN LOCAL FIELDS

The class field theory Neukirch develops has $\hat{\mathbb{Z}}$-extension of base field as the main object in the theory. We recall that the maximal unramified extension K^{un} of a local field K is a $\hat{\mathbb{Z}}$ extension. However any p-adic fields have more than one of them so a natural question one could ask is whether a similar theory can be developed with these other $\hat{\mathbb{Z}}$-extensions. In fact, I was informed that Neukirch asked this question to several mathematicians around 1995. The answer is negative and we present the simple calculation showing the impossibility.

We recall that a class field theory used in the abstract class field theory of Neukirch is defined to be a pair of homomorphisms $deg : G(\hat{K}/K) \to \hat{\mathbb{Z}}$ which is continuous and bijective (for some $\hat{K} \supset K$) and $v : K^\times \to \hat{\mathbb{Z}}$ is a henselian valuation with respect to deg. We recall that v is a henselian valuation if $im(v) = Z \subset \hat{\mathbb{Z}}$ contains $\mathbb{Z}$ and for any finite extension L/K,

$$v(N_{L/K}L^\times) = f_{L/K}Z$$

where $f_{L/K} = [L \cap \hat{K} : K]$.

Theorem 3.3.1. *Let K be a local field. Suppose that $(deg : G(\hat{K}/K) \cong \hat{\mathbb{Z}}, v : K^\times \to \hat{\mathbb{Z}})$ is a class field theory. Then $\hat{K} = K^{un}$ and v is the usual discrete valuation on K.*

Proof. Let π be a prime of K and suppose that the residue field of K is $\mathbb{F}_q$. We write K_n for the unique unramified extension of degree n of K for each n, namely the extension field of K obtained by adjoining a primitive (q^n-1)th root of unity. Then we have $N_{K_n/K}K_n^\times = \pi^{n\mathbb{Z}} \times U_K$. We recall that $U_K = \mu_K \times U_1$ and this implies that $v(\mu_K) = 0$ for any homomorphism $v : K^\times \to \hat{\mathbb{Z}}$ since $\hat{\mathbb{Z}}$ does not have any non-trivial zero divisor. We now consider two cases:

Case 1. $\hat{K} \supset K_p$ (i.e. $f_{K_p/K} = p$)

We suppose that there exists a prime $l \neq p$ such that $K_l \not\subset \hat{K}$. Then $f_{K_l} = 1$

so $v(N_{K_l/K}K_l^\times) = Z$. By the description of the norm group of K_l defined above we observe that there exist $m \in \mathbb{Z}$ and $x \in U_1$ such that $v(\pi^{ml}x) = 1$. We see that p cannot divide m since if it did, $\pi^{ml}x \in N_{k_p/k}$, contradicting the assumption $f_{K_p/K} = p$. Thus, U_1 is ml-divisible and there exists x' such that $x'^{ml} = x$. Then $ml \times v(\pi x') = 1$ which is a contradiction. Therefore if $\hat{K} \supset K_p$ then $\hat{K}$ is the maximal unramified extension of K.

Case 2. $\hat{K} \not\supset K_p$ (i.e. $f_{K_p/K} = 1$)

Since K_p is not in $\hat{K}$, there exist $n \in \mathbb{Z}_{>0}$ and $x' \in U_1$ such that $v(\pi^{np}x') = 1$. We suppose that p does not divide n. Then $v(\pi^p x'^{1/n}) = 1/n$, which is not possible. We can repeat this argument to obtain that n must be a power of p, say p^{m-1}. Clearly $K_{p^{m+1}} \cap \hat{K} = K$ as well so we can use the same method as above to conclude that there exist $m_1 > m$ and $x'' \in U_1$ such that $v(\pi^{p^{m_1}}x'') = 1$. This means $v(\pi^{p^{m_1}-p^m}x) = 0$ for $x = x'^{-1}x''$ which in turn implies that $v(U_1) \supset v(N_{K_i/K}K_i^\times) \supset iZ$ where $i = (p^{m_1} - p^m)$.

Now let l be a prime larger than i. Then $v(N_{K_l/K}K_l^\times) = v(\pi^{l\mathbb{Z}} \times U_1) = Z$ since it contains both iZ and lZ. Since p does not divide l, U_1 is l-divisible so the argument used in case 1 can be used again to show an existence of element in $N_{K_l/K}K^\times$, say y such that $v(y^l) = lv(y) = 1$.

Thus we conclude that $\hat{K} = K^{un}$. □

Remark. *As we have mentioned earlier, a local field K necessarily has at least two $\hat{\mathbb{Z}}$-extensions. Let $\hat{K}$ be a $\hat{\mathbb{Z}}$ extension not equal to K^{ur}. Then one may wonder why the obvious candidate (deg : $G(\hat{K}/K) \cong \hat{\mathbb{Z}}$, $v := deg \circ (\ ,\hat{K}/K)$) is not a class field theory where $(\ ,\hat{K}/K)$ is the local norm residue symbol which was defined at the end of chapter 3.1.*

We show what happens when $K = \mathbb{Q}_p$. It is well-known that the abelian part of the absolute Galois group of $\mathbb{Q}_p$ is isomorphic to $\hat{\mathbb{Z}} \times \mathbb{Z}_p^\times$ and that $\mathbb{Z}_p^\times \cong \mathbb{F}_p^\times \times \mathbb{Z}_p$. So we can construct the second $\hat{\mathbb{Z}}$ extension $\hat{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. However v does not satisfy the axioms of henselian valuation. This map has image $\mathbb{Z} \times \mathbb{Z}_p$ where we consider $\mathbb{Z} \subset \prod_{q \neq p} \mathbb{Z}_q$. If we write v explicitly (using the explicit description of local residue symbol which can be found in [27]) we obtain that for $x = p^i \xi u$ where ξ is a $(p-1)$th root of unity and u a principal unit, $v(x) = (i, \log(u))$ where we *view* $\hat{\mathbb{Z}} = \prod_{q \neq p} \mathbb{Z}_q \times \mathbb{Z}_p$. *Now consider* K_p, *the unique unramified extension of degree* p *of* $\mathbb{Q}_p$. *By construction* $K_P \not\subset \mathbb{Q}_p$ *so* $f_{K_p/\mathbb{Q}_p} = 1$[10]. *As shown earlier,* $N_{K_p/\mathbb{Q}_p} K_p^\times = p^{p\mathbb{Z}} \times U_K$. *However the image of this norm group under* v *is* $p\mathbb{Z} \times \mathbb{Z}_p$ *which is not the same as* $f_{K_p/\mathbb{Q}_p}\mathbb{Z} \times \mathbb{Z}_p = \mathbb{Z} \times \mathbb{Z}_p$.

[10] with respect to the second $\hat{\mathbb{Z}}$ extension $\hat{\mathbb{Q}}_p$

3.4 characteristic p global class field theory

3.4.1 *Idele Class Group*

Let $k_0 = \mathbb{F}_p(T)$ and let K be a finite extension of k_0. Then we define I_K, the group of ideles over K to be

$$\prod_P{}' K_P^\times$$

the restricted product of localisation of K at each place P with respect to their units U_P. $K^\times$ is naturally diagonally embedded into I_K and therefore we can define $C_K = I_K/K^\times$, the Idele class group. It is well known that the Idele class group for global fields satisfies the Galois descent condition, that is to say if L/K is a Galois extension, we have $C_L^{G(L/K)} = C_K$.

Define an Idelic norm $|\cdot|_K : I_K \to \mathbb{R}_+$ by

$$|a|_K = \prod_P q_P^{-v_P(a)}$$

where $q_P = q^{deg(P)}$, in another words, the size of the residue field of K_P. This is a homomorphism where the image is $q^{\mathbb{Z}} \subset \mathbb{R}_+$. We denote the kernel of this map by I_K^0. By the product formula, $K^\times \in I_K^0$ and therefore we can define subgroup C_K^0 of C_K by $I_K^0/K^\times$. In exactly the same way as the number field case it can be shown that C_K^0 is compact.

Proposition 3.4.1. $C_K \cong C_K^0 \times \mathbb{Z}$.

For the proof of this proposition we introduce new notation/map. For every place P of K we have the canonical injection

$$[\ ]_P : K_P^\times \to C_K$$

which associates to each $a_P \in K_P^\times$ the class of the Idele

$$[a_P]_P = (\cdots, 1, 1, 1, a_P, 1, 1, 1, \cdots).$$

Proof. Let P be a rational place[11] of K, for instance the infinite place, and let T_P be the prime element of K_P. We now define Γ_K to be the image of

[11] a place of degree 1

the subgroup generated by $[T_P]_P$ in C_K. Then we observe that $\Gamma_K \cong \mathbb{Z}$. By construction Γ_K is mapped isomorphically onto $q^{\mathbb{Z}} \subset \mathbb{R}_+$ by the Idelic norm and hence we obtain $C_K = C_K^0 \times \Gamma_K$. □

We recall that the norm map $N_{L/K} : I_L \to I_K$ for Galois extension L/K is defined as follows[12].

$$N_{L/K}(\alpha) = \prod_{\sigma} \alpha$$

for $\alpha \in I_L$. Clearly $\tau N_{L/K}(\alpha) = N_{L/K}(\alpha)$ for $\tau \in G(L/K)$, $N_{L/K}\alpha \in I_K$.

Proposition 3.4.2. *Let L/K be a finite extension and $\alpha \in I_L$, then the local components of $N_{L/K}(\alpha)$ is given by*

$$N_{L/K}(\alpha)_P = \prod_{\mathfrak{P}|P} N_{L_\beta/K_P}(\alpha_{\mathfrak{P}}).$$

Proof. Same as number field case. See [26, p. 82] □

3.4.2 *Class Field Theory Axiom*

Proving the class field theory axiom for the Idele class group is a central step of Class field theory. There have been many different proofs of it in the case of global fields and they were classically called the first and the second inequality and we write them explicitly. Let K be a function field over $\mathbb{F}_q$. Then we have the following two results.

Theorem 3.4.3 (First inequality)**.** *Let L/K be a cyclic extension of degree n with Galois group $G = G(L/K)$. Then $h(G(L/K), C_L) = n$. In particular*

$$\#H^0(G, C_L) = (C_K : N_{L/K}C_L) \geq n.$$

Note the proof is a lot easier than the number field case as there is no archimedian place. For this proof only we write $h(B) := h(G(L/K), B)$ for each $G(L/K)$ module B.

[12] See [26, p. 82] for the definition for general extension

Proof. Let $U = \prod_{\mathfrak{P}} U_{\mathfrak{P}}$ be the unit ideles of L. Clearly $U \subset I_L^0$ so $UL \subset I_L^0$. The multiplicity of herbrand quotient gives

$$h(I_L/L^\times) = h(I_L/I_L^0)h(I_L^0/UL)h(UL/L).$$

Using Proposition 3.4.1 and third isomorphism theorem we obtain $h(I_L/I_L^0) = h(\mathbb{Z}) = n$. Number of divisor classes of degree zero is finite so I_L^0/UL is a finite group and hence $h(I_L^0/UL) = 1$.

UL/L is isomorphic to $U/(U \cap L)$ as a G-module and hence

$$h(UL/L) = h(U/(U \cap L)) = h(U)h(U \cap L)^{-1}.$$

$U \cap L$ is the multiplicative group of the constant field of L so is finite. Hence $h(U \cap L) = 1$. We can express U as a direct product

$$U = \prod_P (\prod_{\mathfrak{P}|p} U_{\mathfrak{P}}).$$

Let U_P denote any of $U_{\mathfrak{P}}$ for $\mathfrak{P}|P$ then we have for $i = 0, -1$,

$$H^i(G, \prod_{\mathfrak{P}|P} U_{\mathfrak{P}}) \cong H^i(G_P, U_P)$$

for $i = 0, -1$ by semilocal theory and hence

$$H^i(U) \cong \prod_P H^i(G_P, U_P).$$

Recall that by Lemma 3.1.8, for each finite cyclic extension of local fields $L_{\mathfrak{P}}/K_P$ we have $h(G(L_{\mathfrak{P}}/K_P), U_{L_{\mathfrak{P}}}) = 1$ and hence $h(G_P, U_P) = 1$ for all P. Therefore $h(U) = \prod_P h(G_P, U_P) = 1$ and $h(C_L) = n$. □

Theorem 3.4.4 (Second Inequality). *$\#H^0(G, C_L)|n$. In particular $\#H^0(G, C_L) \leq n$ and hence equal to n.*

We omit the full proof in this thesis as this is classically known and very long unlike the proof of the First inequality. The main reference for this is [1, Ch. 6.4]. We note that in the case when n is coprime to p, same argument as the number field can be used. For the case $n = p$ it requires a different argument. Also see [35] for the rearranged version.

3.4.3 *Valuation and Deg*

Let $\tilde{k}_0 = \mathbb{F}_p^{sep} \cdot \mathbb{F}_p(T)$. Then $G(\tilde{k}_0/k_0)$ is topologically generated by the map $x \mapsto x^p$, namely the Frobenius, which we shall denote by φ, and is isomorphic to $\hat{\mathbb{Z}}$ via the map $\varphi \mapsto 1$. Denote this isomorphism *deg*.

Then as in the abstract case for each finite extension K of k_0 we can define an isomorphism

$$deg_K = \frac{1}{f_K} deg : G(\tilde{K}/K) \to \hat{\mathbb{Z}}$$

where $f_K = [K \cap \tilde{k}_0 : k_0]$.

Now we define the map $[\ ,\ L/K]$ for every abelian extension L/K by

$$[\alpha, L/K] = \prod_P (\alpha_P,\ L_P/K_P)$$

where $(\ ,\ L_P/K_P)$ is the local norm residue symbol. Note that for each $\alpha = (\alpha_P) \in I_K$, $(\alpha_P,\ L_P/K_P)$ are equal to 1 for almost all P as almost all α_P are units and almost all extensions L_P/K_P are unramified.

Proposition 3.4.5. *If L/K and L'/K' are two abelian extensions such that $K \subset K'$ and $L \subset L'$ then we have the commutative diagram*

$$\begin{array}{ccc} I_{K'} & \xrightarrow{[\ ,\ L'/K']} & G(L'/K') \\ {\scriptstyle N_{K'/K}}\downarrow & & \downarrow \\ I_K & \xrightarrow{[\ ,\ L/K]} & G(L/K) \end{array}$$

Proof. Exactly same as the number field case. □

In the same way as characteristic 0 case, we can define $[\ ,\ L/K]$ for an infinite extension L of K by taking projective limit. It is clear that Proposition 3.4.5 holds for infinite extension as well.

We define homomorphism

$$v_K : I_K \to \hat{\mathbb{Z}}$$

to be the composite of $[\ ,\tilde{K}/K]$ and deg_K.

Proposition 3.4.6. *For all $a \in K^\times$ we have*

$$[a, \tilde{K}/K] = 1.$$

Proof. Since $[N_{K/k_0}(a), \tilde{k_0}/k_0] = [a, \tilde{K}/K]|_{\tilde{k_0}}$, we may assume that $K = \mathbb{F}_p(T)$. It suffices to prove $[a, \mathbb{F}_{p^n}(T)/\mathbb{F}_p(T)] = 1$. View $\mathbb{F}_{p^n}$ as $\mathbb{F}_p(\zeta_n)$ where ζ_n is a primitive (p^n-1)th root of unity. Then for each place Q of K and $a \in K$

$$(a, K_Q(\zeta_n)/K_Q)\zeta_n = \zeta_n^{p^{v_Q(a)}}$$

by Proposition 3.1.5. So

$$[a, K(\zeta_n)/K]\zeta_n = \prod_P ((a, K_P(\zeta_n)/K_P)\zeta_n) = \prod_P \zeta_n^{p^{\mathrm{Deg}(P)v_Q(a)}} = \zeta_n.$$

□

Therefore $v_K : I_K \to \hat{\mathbb{Z}}$ induces a homomorphism

$$v_K : C_K \to \hat{\mathbb{Z}}.$$

Corollary 3.4.7. $v_K(C_K^0) = 1$. *More precisely ker$(v_K) = C_K^0$.*

Proof. First we prove that $N_{K/k_0} I_K^0 \subset I_{k_0}^0$. Proposition 3.4.2 says for $\alpha \in I_K$, $N_{K/k_0}(\alpha)_P = \prod_{\mathfrak{P}|P} N_{K_{\mathfrak{P}}/(k_0)_P}(\alpha_{\mathfrak{P}})$. So using $\prod_{\mathfrak{P}} |\alpha_{\mathfrak{P}}|_{\mathfrak{P}} = 1$ and $|\alpha_{\mathfrak{P}}|_{\mathfrak{P}} = |N_{K/k_0}\alpha_{\mathfrak{P}}|_P$ we obtain $N_{K/k_0}\alpha \in I_{k_0}^0$.

So as in Proposition 3.4.6 we can assume that $K = \mathbb{F}_p(T)$. Let $\alpha \in I_K^0$. At the end of the proof of Proposition 3.4.6, we obtained that $(\alpha, K(\zeta_n)/K) = 1$ if and only if $\sum_P \mathrm{Deg}(P)v_P(\alpha_P) = 0$. This condition is equivalent to $\alpha \in C_k^0$.

□

Proposition 3.4.8. *The map $v_K : C_K \to \hat{\mathbb{Z}}$ has image $\mathbb{Z}$ and v is a henselian valuation with respect to deg.*

Proof. Recall that in Proposition 3.4.1 we obtained decomposition $C_K = C_K^0 \times T_p^{\mathbb{Z}}$ where $T_P = [\pi_P]_P$ and π_P is a prime element for some rational place P. Using Proposition 3.1.5 we know that $[T_P, \tilde{K}/K] = (\pi_P, \tilde{K}_P/K_P) =$

φ_{K_P}. As P is a rational place $\varphi_{K_P} = \varphi_K$. This automatically implies the first assertion as

$$v_K(C_K) = v_K(\Gamma_K) = v_K(T_p^{\mathbb{Z}}) = deg_K(\varphi_K^{\mathbb{Z}}) = \mathbb{Z} \subset \hat{\mathbb{Z}}.$$

Then the first condition for v_K being the henselian valuation is satisfied since $v_K(C_K) = \mathbb{Z}$. By Proposition 3.4.5, the second condition is also satisfied as for every finite extension L/K we have

$$v_K(N_{L/K}C_L) = deg_K[N_{L/K}I_L, \tilde{K}/K] = f_{L/K}deg_L[I_L, \tilde{L}/L] = f_{L/K}v_L(C_L).$$

Since $v_L(C_L) = \mathbb{Z}$ we obtain the Proposition. □

Recall that the Idele class groups C_K satisfy the class field theory axiom of Neukirch and hence we get the following result.

Theorem 3.4.9. *For every Galois extension L/K of function field over a finite field we have a canonical isomorphism*

$$r_{L/K} : G(L/K)^{ab} \cong C_K/N_{L/K}C_L.$$

Also the Existence Theorem holds.

The inverse of $r_{L/K}$ yields the surjective homomorphism

$$(\ ,L/K) : C_K \to G(L/K)^{ab}.$$

with kernel $N_{L/K}C_L$.

Proposition 3.4.10. *If L/K is an abelian extension and P a prime of K, then the diagram*

$$\begin{tikzcd} K_P^\times \arrow[r, "(\ ,\, L_P/K_P)"] \arrow[d, "[\ ]_P"'] & G(L_P/K_P) \arrow[d] \\ C_K \arrow[r, "(\ ,\, L/K)"] & G(L/K) \end{tikzcd}$$

is commutative.

Proof. Exactly same as number field case. □

As in the abstract case, we can take inverse limit of the global norm residue symbol to obtain map

$$(\ , K^{ab}/K) : C_K \to G(K^{ab}/K).$$

The image of this map is not surjective unlike the number field case (for instance $(\ , \tilde{K}/K)$ is not surjective since the image is isomorphic to $\mathbb{Z}$). Also the kernel of this map is clearly given as the intersection of all $N_{L/K}C_L$ which can be proved to be connected component of 1 ([1]). This is non-trivial for number fields but not for global function fields. So $(\ , K^{ab}/K)$: $C_K \to G(K^{ab}/K)$ is injective.

4

CHARACTERISTIC p HIGHER CLASS FIELD THEORY

In this chapter we generalise Kawada-Satake's approach to higher local fields.

4.1 KAWADA-SATAKE REVISITED

The paper 'Class Formation II' by Kawada and Satake started a new view on class field theory and motivated many people's work. However, since it was written such a long time ago, the notation, style of writing and even some fundamental terminologies are unusual by modern standards. As this paper is of such importance it would make sense to rewrite it.

It turns out that for any characteristic p fields, it is possible to define a class formation using Witt vectors. This class formation deals with the p-extensions of k. Then we can use this to obtain characteristic p Local and Global class field theory restricted to p-extensions only. Here we claim no originality, and unless otherwise stated, the proof in this section was given in [20].

4.1.1 *Class Formation in Characteristic p*

We start by fixing some notations. Let k be any field of characteristic p where $p > 0$. Fix a separable closure k^{sep} of k. We emphasise that we will only be dealing with fields of characteristic p and p-extensions of such fields. So let $k^{sep,p}$ be the maximal separable p-extension of k and $k^{ab,p}$ be the maximal abelian p-extension of k inside k^{sep}. We will also write $G^{ab,p}(k) = G(k^{ab,p}/k)$ which we may shorten to $G^{ab,p}$ if there is no confusion as to which field extension we are referring to. Also by $\otimes$ we shall always mean the tensor product of $\mathbb{Z}$-modules. Consider the group

$$B(k) = (W(k)/\mathscr{P}W(k)) \otimes \mathbb{Z}[p^{-1}]/\mathbb{Z}$$

where $\mathbb{P} := F - id$ is defined earlier in Lemma 2.2.2. All elements of $\mathbb{Z}[p^{-1}]$ can be written uniquely of a/p^n where $a \in \mathbb{Z}$, $n \in \mathbb{Z}_{>0}$. We also observe that $\mathbb{Z}[p^{-1}]/\mathbb{Z} \cong \mathbb{Q}_p/\mathbb{Z}_p$. The subsets $B_n(k) = (W(k)/\mathscr{P}W(k)) \otimes (p^{-n}\mathbb{Z}/\mathbb{Z})$ are submodules of $B(k)$ and $B_i(k) \subset B_{i+1}(k)$ for all $i > 0$. We can also see that $B(k) = \cup_{n=1}^{\infty} B_n(k)$. So $B(k)$ can be seen as the direct limit group of $B_n(k)$ with respect to the canonical injective maps $l_n : B_n(k) \to B_{n+1}(k)$.

On the other hand we also have the isomorphism

$$\phi_n : W_n(k)/\mathscr{P}W_n(k) \cong B_n(k)$$

defined by

$$(x_0, x_1, \ldots, x_{n-1}) + \mathscr{P}W_n(k) \mapsto ((x_0, x_1, \ldots, x_{n-1}, 0, 0, \ldots) + \mathscr{P}W(k)) \otimes (1/p^n).$$

First we prove that that every element in $B_n(k)$ can be uniquely represented by elements of the form

$$((x_0, x_1, \cdots, x_{n-1}, 0, 0, \cdots) + \mathscr{P}W(k) \otimes (a/p^n)$$

where $0 \leq a \leq p^n - 1$.

Let $X = ((x_0, x_1, \cdots) + \mathscr{P}W(k)) \otimes (a/p^n)$. Then

$$\begin{aligned} X = & ((x_0, x_1, \cdots, x_{n-1}, 0, 0, \cdots) + \mathscr{P}W(k)) \otimes (a/p^n) \\ & +(p^n(x_n, x_{n+1}, \cdots) + \mathscr{P}W(k)) \otimes (a/p^n) \end{aligned}$$

by Lemma 2.2.2. Since multiplication by p^n annihilates elements in $B_n(k)$, we obtain that every element can be represented in this way. This representation is unique as multiplication by p^n does not affect the first n components of X. Then it follows that ϕ_n is a well-defined homomorphism and is bijective.

Now we can define $\psi_n : W_n(k)/\mathscr{P}W_n(k) \to W_{n+1}(k)/\mathscr{P}W_{n+1}(k)$ by

$$\psi_n = \phi_{n+1}^{-1} \circ l_n \circ \phi_n$$

for each $n > 0$.

Lemma 4.1.1. *ψ_n coincides with $V : W_n(k)/\mathscr{P}W_n(k) \to W_{n+1}(k)/\mathscr{P}W_{n+1}(k)$.*

Proof. The proof is obtained via the following relatively simple calculations:

$$\psi_n((x_0, x_1, \ldots, x_{n-1}) + \mathscr{P}W_n(k))$$

$$\begin{aligned}
&= \phi_{n+1}^{-1} \circ l_n((x_0, x_1, \ldots, x_{n-1}, 0, \cdots) + \mathscr{P}W_n(k)) \otimes (1/p^n) \\
&= \phi_{n+1}^{-1}((x_0, x_1, \ldots, x_{n-1}, 0, \cdots) + \mathscr{P}W_n(k) \otimes (1/p^n)) \\
&= \phi_{n+1}^{-1}(p(x_0, x_1, \ldots, x_{n-1}, 0, \cdots) + \mathscr{P}W_n(k) \otimes (1/p^{n+1})) \\
&= \phi_{n+1}^{-1}((0, x_0^p, \cdots, x_{n-1}^p, 0, \cdots) + \mathscr{P}W(k) \otimes (1/p^{n+1})) \\
&= (0, x_0^p, \cdots, x_{n-1}^p) + \mathscr{P}W_{n+1}(k)
\end{aligned}$$

and this is exactly same as $V((x_0, x_1, \ldots, x_{n-1}) + \mathscr{P}W_n(k))$ by Lemma 2.2.2.

□

With this we obtain the following commutative diagram:

$$\begin{array}{ccccccccc}
B_1 & \xrightarrow{l_1} & B_2 & \xrightarrow{l_2} & \cdots & \xrightarrow{l_{n-1}} & B_n & \xrightarrow{l_n} & \cdots \\
\uparrow{\scriptstyle \phi_1} & & \uparrow{\scriptstyle \phi_2} & & & & \uparrow{\scriptstyle \phi_n} & & \\
W_1/\mathscr{P}W_1 & \xrightarrow[\psi_1]{} & W_2/\mathscr{P}W_2 & \xrightarrow[\psi_2]{} & \cdots & \xrightarrow[\psi_{n-1}]{} & W_n/\mathscr{P}W_n & \xrightarrow[\psi_n]{} & \cdots
\end{array}$$

Thus, we can consider $B(k)$ as the direct limit group of $(W_n(k)/\mathscr{P}W_n(k))$ with respect to the group homomorphisms ψ_n. Furthermore we denote by ψ_n^* for the injective map $W_n(k)/\mathscr{P}W_n(k) \to B(k)$ which satisfies the relation $\psi_{n+1}^* = \psi_n \circ \psi_n^*$.

Let $\sigma \in G^{sep,p}$. Then σ acts on $W(k^{sep,p})$ by

$$\sigma \cdot (x_0, x_1, \ldots, x_n, \ldots) = (\sigma x_0, \sigma x_1, \ldots, \sigma x_n, \ldots).$$

It is clear that σ commutes with V and $\mathscr{P}$.

We let $R(W(k)) = \{X \in W(k^{ab,p}) : \mathscr{P}X \in W(k)\}$. Our aim is to prove the following:

Proposition 4.1.2. *To each $X \in R(W(k))$ we associate a function*

$$\phi_X : G^{ab,p} \to W(k)$$

defined by $\phi_X(\sigma) = (\sigma - 1)X$. Then in fact, $\phi_X(\sigma) \in W(\mathbb{F}_p)$ for all σ and X. Furthermore the map Θ given by $X \mapsto \phi_X$ induces the isomorphism

$$R(W(k))/W(k) \cong Hom_{cont}(G^{ab,p}, W(\mathbb{F}_p)).$$

The first part of the Proposition follows from the equation $\mathscr{P} \circ (\sigma - 1)X = (\sigma - 1) \circ \mathscr{P}X = 0$. Before proving the rest of the Proposition we state a Theorem of Witt on cohomology. We note that in the original text, this statement was given in several Lemmas.

Theorem 4.1.3. *Let K/k be a finite Galois p-extension. Then the r-cohomology groups of the Galois group $G = G(K/k)$ over $W(K)$ and $W_n(K)$ ($n \in \mathbb{Z}_{>0}$), namely $H^r(G, W(K))$ and $H^r(G, W_n(K))$ are all trivial.*

Proof. By a well known Theorem in cohomology theory (for instance in [3, Ch 6.8]) it suffices to prove the assertion for positive r and for arbitrary subgroup H of G. So let f_r be an r-cocycle of H over $W(K)$. Let v be a vector in $W(K)$ such that $\mathrm{Tr}_H v$ is a unit vector in $W(k)$. Then $f_r = \delta g_{r-1}$ where $g_{r-1}(\sigma_1, \cdots, \sigma_{r-1})$ is defined as

$$(-1)^r((\sum_{\tau \in H} f_r(\sigma_1, \cdots, \cdots, \sigma_{r-1}, \tau) \cdot \sigma_1 \cdots \sigma_{r-1} \cdot \tau v) \cdot (\mathrm{Tr}_H(v)))^{-1}.$$

The result for $W_n(K)$ can be proved in a similar way. □

Proof of Proposition 4.1.2. We first show that ϕ_X is a continuous homomorphism. The map ϕ_X is a group homomorphism as

$$\phi_X(\sigma\tau) = (\sigma\tau - 1)X = \sigma(\tau - 1)X + (\sigma - 1)X = \phi_X(\sigma) + \phi_X(\tau).$$

To show continuity let $X = (x_0, x_1, \dots, x_n, \dots)$ and take a neighbourhood $V^n(W(\mathbb{F}_p))$ of 0 in $W(\mathbb{F}_p)$. Since $\mathscr{P}X \in W(k)$, field $K = k(x_0, x_1, \dots, x_{n-1})$ is a cyclic extension of k of degree at most p^n. Thus, $G(k^{ab,p}/K)$ is an open subgroup of $G^{ab,p}$ and $\phi_X(\sigma) \in V^n(W(\mathbb{F}_p))$ for all $\sigma \in G(k^{ab,p}/K)$. This is exactly the definition of ϕ_X being continuous at $\mathrm{Id}_{G^{ab,p}}$.

Let $f \in Hom_{cont}(G^{ab,p}, W(\mathbb{F}_p))$. Recall that $G^{ab,p}$ acts trivially on $W(\mathbb{F}_p)$. Thus, f is a continuous 1-cocyle over $W(\mathbb{F}_p)$. To prove the existence of an element $X = (x_0, x_1, \dots, x_n, \dots) \in W(k^{ab,p})$ such that $\phi_X = f$, we choose $x_0, x_1, \dots$ recursively as follows. Let h_1 be a canonical homomorphism from $W(\mathbb{F}_p)$ to $W(\mathbb{F}_p)/V(W(\mathbb{F}_p)) \cong \mathbb{F}_p$. Then $f_1 = h_1 \circ f$ is a continuous homomorphism of $G^{ab,p}$ into $\mathbb{F}_p$. The kernel of f_1 is an open subgroup $U^{(1)}$ of $G^{ab,p}$ of finite index. This means that there is a field K_1 with Galois group $G_1 = G^{ab,p}/U^{(1)}$. Using Theorem 4.1.3 we obtain $f_1(\sigma) = (\sigma - 1)x_0$ for $\sigma \in G_1$ for some $x_0 \in k^{ab,p}$. Now put $X_0 = (x_0, 0, \dots)$ and $\phi^{(1)}(\sigma) = (\sigma - 1)X_0$. By the choice of x_0 we have $(f - \phi^{(1)})(\sigma) \in V(W(\mathbb{F}_p))$.

Assume we have chosen $x_0, \dots, x_{n-1}$ such that $(f - \phi^{(n)}) \in V^n(W(\mathbb{F}_p))$ where $\phi^{(n)} = (\sigma - 1)X_n$ and $X_n = (x_0, x_1, \dots, x_{n-1}, 0, 0, \dots)$. Then we put

$$\phi^{(n+1)}(\sigma) = (f - \phi^{(n)})(\sigma) \bmod V^{n+1}(W(\mathbb{F}_p)).$$

Let $U^{(n+1)}$ be the kernel of ϕ_{n+1}. This is an open subgroup of $G^{ab,p}$ of finite index. Then ϕ_{n+1} induces a homomorphism f_{n+1} of a finite group $G^{ab,p}/U^{(n+1)}$ into $\mathbb{F}_p$ by taking its $(n+1)$th component. Just as before there exists $x_n \in k^{ab,p}$ such that $f_{n+1}(\sigma) = (\sigma - 1)x_n$. Then put

$$\phi^{(n+1)}(\sigma) = (\sigma - 1)(x_0, x_1, \cdots, x_n, 0, 0, \cdots).$$

This gives $(f - \phi^{(n+1)})(\sigma) \in V^{n+1}(W(\mathbb{F}_p))$. In this way, we can choose inductively x_0, x_1, $\cdots$ and for $X = (x_0, x_1, \dots, x_n, \dots) \in W(k^{ab,p})$ we have $f(\sigma) = (\sigma - 1)X$. We observe that $X \in R(B(k))$ since $\mathscr{P}(f(\sigma)) = 0$.

Finally we prove that Θ is injective. The kernel of Θ is precisely the subset of X such that $(\sigma - 1)X = 0$ for all $\sigma \in G^{ab,p}$. This condition is equivalent to $X \in W(k)$. $\square$

Proposition 4.1.4. *The map $\mathscr{P} : R(W(k)) \to W(k)$ induces an isomorphism*

$$R(W(k))/W(k) \cong W(k)/\mathscr{P}W(k)$$

Proof. Consider the equation $\mathscr{P}X = r$ for $r \in W(k)$. We observe that there exists $X \in W(k^{ab,p})$ which satisfies the equation so the map is surjective. The inverse image of $\mathscr{P}W(k)$ by $\mathscr{P}$ is $W(k) + \mathscr{P}^{-1}(0) = W(k)$ so $\mathscr{P}$ induces the required isomorphism. □

For $f \in Hom_{\text{cont}}(G^{ab,p}, W(\mathbb{F}_p))$ we define

$$g_n(\sigma) = \eta \circ f(\sigma) (\text{mod } p^n)$$

where η is a canonical isomorphism $W(\mathbb{F}_p) \to \mathbb{Z}_p$. So g_n is a group homomorphism of $G^{ab,p}$ into $\mathbb{Z}/p^n\mathbb{Z}$.

Proposition 4.1.5. *Consider the map $f \otimes (s/p^n) \mapsto \chi$ where $\chi \in Hom_{\text{cont}}(G^{ab,p}, \mathbb{R}/\mathbb{Z})$ is defined by*

$$\chi(\sigma) := s \cdot g_n(\sigma)/p^n (\text{mod } \mathbb{Z}) \ (f \in Hom_{\text{cont}}(G^{ab,p}, W(\mathbb{F}_p)),\ \sigma \in G^{ab,p}).$$

This map is an isomorphism, i.e.

$$Hom_{\text{cont}}(G^{ab,p}, W(\mathbb{F}_p)) \otimes (\mathbb{Q}^{(p)}/\mathbb{Z}) \cong Hom_{\text{cont}}(G^{ab,p}, \mathbb{R}/\mathbb{Z}).$$

Proof. Firsts we suppose that $f \otimes (s/p^n) = f' \otimes (s'/p^m)$ where $n \leq m$ and $f, f' \in Hom_{cont}(G^{ab,p}, W(\mathbb{F}_p))$. Hence, $s \cdot p^{m-n} \cdot f = s' \cdot f'$ in $W(\mathbb{F}_p)$. More specifically, this is same as saying

$$s \cdot p^{m-n} (\eta \circ f)(\sigma) = \eta(s \cdot p^{m-n} \cdot f(\sigma)) = \eta(s' \cdot f'(\sigma)) = s'(\eta \circ f')(\sigma).$$

Therefore we have $(\eta \circ f)(\sigma) \cdot (s/p^n) \equiv (\eta \cdot f')(\sigma) \cdot (s'/p^m)$ (mod $\mathbb{Z}$) and this map is well-defined. By simply working backwards, we obtain the injectivity.

Let $f_1 \otimes (s_1/p^{n_1})$ and $f_2 \otimes (s_2/p^{n_2})$ be elements of $Hom_{cont}(G^{ab,p}, W(\mathbb{F}_p)) \otimes (\mathbb{Q}^{(p)}/\mathbb{Z})$ and let χ_1 and χ_2 be elements in $Hom_{cont}(G^{ab,p}, \mathbb{R}/\mathbb{Z})$ which corresponds to them. Furthermore we assume that $n_1 = n_2 = n$ and we write $g_{i,n} = \eta \circ f_i$. Then $\chi_i(\sigma) = s_i \cdot g_{i,n}(\sigma)/p^n \pmod{\mathbb{Z}}$ so $\chi_1\chi_2 = (s_1 \cdot g_{1,n}(\sigma) + s_2 \cdot g_{2,n}(\sigma))/p^n \pmod{\mathbb{Z}}$. On the other hand

$$(f_1 \otimes (s_1/p^n)) \cdot (f_2 \otimes (s_2/p^n)) = (s_1 f_1 + s_2 f_2 \otimes (1/p^n))$$

so we obtain that the map is a homomorphism.

We now show that χ is continuous. For $f \in Hom_{cont}(G^{ab,p}, W(\mathbb{F}_p))$ we can choose an open subgroup $U^{(n)}$ of $G^{ab,p}$ such that $f(U^{(n)}) \subset V^n(W(\mathbb{F}_p))$. Then $g_n(\sigma) \equiv 0 \pmod{p^n}$ holds for $\sigma \in U^{(n)}$ which implies $\chi(\sigma) = 0$ for $\sigma \in U^{(n)}$ hence we obtain the continuity of χ.

Now we prove that this map is surjective. Any element $\chi_1 \in Hom_{cont}(G^{ab,p}, \mathbb{R}/\mathbb{Z})$ is of a finite order, since $G^{ab,p}$ is totally disconnected. So if χ_1 has order p^m we may write

$$\chi_1(\sigma) = (a_0(\sigma) + a_1(\sigma)p + \cdots + a_{m_1}(\sigma)p^{m-1})/p^m \pmod{\mathbb{Z}}.$$

Then there exists $f_m \in Hom_{cont}(G^{ab,p}, W(\mathbb{F}_p)/V^m(W(\mathbb{F}_p))$ such that $\eta \circ f_m(\sigma) \equiv \chi(\sigma) \pmod{p^m}$. We can extend $f_m(\sigma)$ to $f(\sigma) = \phi_X(\sigma)$
$\in Hom_{cont}(G^{ab,p}, W(\mathbb{F}_p))$ by choosing $X \in k^{ab,p}$ suitably. Hence we get that χ_1 is the image of $f \otimes (1/p^m)$. □

By combining these results we obtain:

Theorem 4.1.6. *The continuous character group $(G^{ab,p})^\wedge$ of the compact abelian group $G^{ab,p}$ is canonically isomorphic to $B(k) = (W(k)/\mathscr{P}W(k)) \otimes (\mathbb{Q}^{(p)}/\mathbb{Z})$:*

$$F_k : (G^{ab,p})^\wedge \cong (W(k)/\mathscr{P}W(k)) \otimes (\mathbb{Q}^{(p)}/\mathbb{Z})$$

where the isomorphism is given for $\chi \in (G^{ab,p})^\wedge$:

$$\begin{cases} \chi(\sigma) = g_m(\sigma)/p^m (mod\ \mathbb{Z}) & \\ g_m(\sigma) \equiv \eta \circ f(\sigma) (mod\ p^m) & f \in Hom_{\mathrm{cont}}(G^{ab,p}, W(\mathbb{F}_p)) \\ f(\sigma) = (\sigma - 1)X, & X \in W(k^{ab,p}) \text{ determined} \\ \mathscr{P}X = r \in W(k) & \text{uniquely up to mod } \mathscr{P}W(k) \end{cases}$$

and

$$F_k : \chi \to (r + \mathscr{P}W(k)) \otimes (1/p^m).$$

Verification of the Class Formation Axioms

Fix a field k_0 of characteristic $p > 0$. Let $t = \{K : k_0 \subset K \subset k_0^{sep,p},\ [K : k_0] < \infty\}$. For each $K \in t$ put $E(K) = B(K)^\wedge$. Let $k \subset K$ with $k, K \in t$. Then the trace map $\mathrm{Tr}_{K/k} : W(K) \to W(k)$ commutes with $\mathscr{P}$ so it induces the map

$$\mathrm{Tr} : W(K)/\mathscr{P}W(K) \to W(k)/\mathscr{P}W(k).$$

Thus we can define the map Tr=Tr$\otimes 1$: $B(K) \to B(k)$ and its adjoint map (cotrace map)

$$\mathrm{Cotr}_{K/k} : E(k) \to E(K).$$

Now we define $\phi_{k,K} = \mathrm{Cotr}_{K/k}$.

We are ready to verify that $E(K)$ satisfies the axioms of Class Formation. The first two axioms consists of four conditions below:
C1. For $k \subset K \subset L$ with k, K, $L \in t$, the chain relation $\phi_{K,L} \circ \phi_{k,K} = \phi_{k,L}$ holds.
C2. $\phi_{k,K}$ is an injective homomorphism of $E(k)$ into $E(K)$.
C3. Let K/k be Galois. Then the Galois group $G = G(K/k)$ acts on $E(K)$ and we have $\phi_{k,K}E(k) = E(K)^G$.
C4. For $k \subset K \subset L$ with k, K, $L \in t$ and also assume that both L/k and K/k be Galois. Then for all $\sigma \in G(L/k)$ and $f \in E(K)$

$$\sigma \cdot \phi_{K,L}(f) = \phi_{K,L}(\sigma|_K \cdot f).$$

Proof. Field trace map is transitive so C1 follows. To verify C2 it suffices to check that Tr$\otimes$1 is surjective. From now on we write $\overline{X} = X + \mathscr{P}W(k)$ for $X \in W(k)$. If $[K : k] = p^r$ and $\overline{X} \in W(k)/\mathscr{P}W(k)$ then

$$(\mathrm{Tr} \otimes 1)(\overline{X} \otimes (1/p^{n+r})) = (p^r\overline{X}) \otimes (1/p^{n+r}) = \overline{X} \otimes (1/p^n).$$

C3 is the trickiest one to prove. By definition for $\chi_K \in B(K)^\wedge$ and for $\overline{X} \in W(K)/\mathscr{P}W(K)$, $(\chi_K)^\sigma(\overline{X} \otimes (1/p^n)) = \chi_K(\sigma^{-1}\overline{X} \otimes (1/p^n))$ where $\sigma \in G$. So the condition that $\chi_K \in E(K)^G$ is equivalent to

$$\chi_K((\sigma - 1) \cdot B(K)) = 0 \text{ for all } \sigma \in G.$$

On the other hand $\chi_K = \phi_{k,K} E(k)$ means that $\chi_K(\overline{X} \otimes (1/p^n)) = \chi_k((\mathrm{Tr}\overline{X}) \otimes (1/p^n))$ for $\overline{X} \in W(K)/\mathscr{P}W(K)$. Now we prove the sequence

$$0 \to \sum_{\sigma \in G} (\sigma - 1) B(K) \xrightarrow{i} B(K) \xrightarrow{j} B(k) \to 0$$

is exact where i is the inclusion map and j=Tr$\otimes$1. If this sequence is exact then we obtain C3. Namely, each $\chi_K \in B(K)^\wedge$ which annihilates $\sum_{\sigma\in G}(\sigma - 1)B(K)$ can be expressed as $\chi_K = \chi_k \circ j$ and conversely, which implies C3 by the above equivalence.

Now we prove the exactness. It is clear that i is injective and $j \circ i = 0$. Surjectivity of j was already proved when proving C2. So it only remains to prove that ker(j)$\subset$im(i). Suppose that $\overline{X} \otimes (1/p^n) \in \ker(j)$. That is to say (Tr $\overline{X}) \otimes (1/p^n) = 0$. Now let $\overline{X} \in W(K)/\mathscr{P}W(K)$. Then Tr $X \in p^n a + \mathscr{P}W(k)$ holds for some $a \in W(k)$. Since $H^0(G, W(K)) = 0$ by Theorem 4.1.3 we can take $X_1 \in W(K)$ such that a =Tr(X_1). So put $X_2 = X - p^n X_1$, then Tr$X_2 \in \mathscr{P}W(k)$. Let $[K : k] = p^r$, then $p^r \cdot X_2 \in -\sum_{\sigma\in G}(\sigma - 1)X_2 + \mathscr{P}W_K$ holds. Take the classes mod $\mathscr{P}W(K)$, i.e. $\overline{X}_2 = X_2 + \mathscr{P}W(K)$, then we have

$$\overline{X} \otimes (1/p^n) = \overline{X}_2 \otimes (1/p^n) = -\sum_{\sigma \in G} (\sigma - 1)\overline{X}_2 \otimes (1/p^{n+r})$$
$$= \sum_{\sigma \in G} (\sigma - 1)(-\overline{X}_2) \otimes (1/p^{n+r}).$$

This exactly says ker(j) $\subset$im(j).

Following from this C4 is easily verified. $\square$

We state the final condition for the set $\{E(k) : k \in t\}$ with respect to the cotrace map being a Class Formation:

Theorem 4.1.7. *Let $k, K \in t$ and K/k be a finite Galois extension of degree p^n. Then the Galois group $G = G(K/k)$ acts on $E(K)$ and we can define the cohomology groups $H^i(G, E(K))$ for all $i \in \mathbb{Z}$. Then*

$$H^i(G, E(K)) \cong H^{i-2}(G, \mathbb{Z}) \text{ for all } i \in \mathbb{Z}$$

where G acts on trivially on $\mathbb{Z}$.

If we assume this then we obtain the following result:

Theorem 4.1.8. *Assume that if* $[K:k] = p^n$ *that* $H^0(G, E(K)) \cong \mathbb{Z}/p^n\mathbb{Z}$ *and* $H^1(G, E(K)) \cong 0$. *Then* $E(K)$ *and* $\phi_{k,K}$ *defined above satisfy all class formation axioms. Consequently* $E(k)/N_G(E(K))$ *is isomorphic to* $G(K/k)^{ab}$.

The proof of Theorem 4.1.7 will take a bit of work. We prove two simple lemmas first. Note that brief proofs were given in the original text([20]) but we give more details here.

Lemma 4.1.9. $W(K)/\mathscr{P}W(K)$ *has no element of finite order* p^n *for each* $n \in \mathbb{Z}_{>0}$.

Proof. From Lemma 2.2.2 it follows that $p^n X \in \mathscr{P}W(K)$ is equivalent to $V^n X = \mathscr{P}Y$ for some $Y \in W(K)$. Let $Y = (y_0, y_1, \cdots, y_{n-1}, \cdots)$ We may rewrite $Y = (y_0, y_1, \cdots, y_{n-1}, 0, 0, \cdots) + V^n Y_1$. Then $\mathscr{P}Y = V^n X$ implies $\mathscr{P}(y_0, y_1, \cdots, y_{n-1}, 0, 0, \cdots) = 0$ and so we have $V^n X = \mathscr{P}Y = \mathscr{P} \circ V^n Y_1 = V^n \circ \mathscr{P}Y_1$. Since V is injective, we obtain the result. □

Lemma 4.1.10. $(W(K)/\mathscr{P}W(K)) \otimes \mathbb{Q}^{(p)}$ *and* $\mathbb{Z}_p/\mathbb{Z}$ *are uniquely* p^n*-divisible for each* n.

Proof. Every element of $(W(K)/\mathscr{P}W(K)) \otimes \mathbb{Q}^{(p)}$ can be represented in the form $\overline{X} \otimes (1/p^r)$ by some $X \in W(K)$ and $r \in \mathbb{Z}_{>0}$. Then $p^n \cdot \overline{X} \otimes (1/p^{n+r}) = \overline{X} \otimes (1/p^r)$ so $(W(K)/\mathscr{P}W(K)) \otimes \mathbb{Q}^{(p)}$ is p-divisible. By Lemma 4.1.9 there is no p^n torsion in $(W(K)/\mathscr{P}W(K)) \otimes \mathbb{Q}^{(p)}$ so it's also uniquely p-divisible.

Let $\alpha = (a_0 + a_1 p + \cdots a_n p^n \cdots) + \mathbb{Z}$ be an element of $\mathbb{Z}_p/\mathbb{Z}$. Then $(a_0 + a_1 p + \cdots a_n p^n + a_{n+1} p^{n+1} + \cdots) + \mathbb{Z} = (a_n p^n + a_{n+1} p^{n+1} + \cdots) + \mathbb{Z}$. Then $\alpha = p^n \beta$ where $\beta = (a_n + a_{n+1} p + \cdots) + \mathbb{Z} \in \mathbb{Z}_p/\mathbb{Z}$. Observing that p^n torsion element does not exist in $\mathbb{Z}_p/\mathbb{Z}$ we obtain the p^n-unique divisibility. □

Now we are ready to prove Theorem 4.1.7.

Proof of Theorem 4.1.7. We first consider four exact sequences:

$$0 \to W(K)/\mathscr{P}W(K) \xrightarrow{i} (W(K)/\mathscr{P}W(K) \otimes \mathbb{Q}^{(p)} \xrightarrow{j} B(K)$$

$$0 \to \mathscr{P}W(K) \xrightarrow{i} W(K) \xrightarrow{j} W(K)/\mathscr{P}W(K) \to 0$$

$$0 \to \mathscr{P}W(\mathbb{F}_p) \xrightarrow{i} W(K) \xrightarrow{\mathscr{P}} \mathscr{P}W(K) \to 0$$

$$0 \to \mathbb{Z} \xrightarrow{i} W(\mathbb{F}_p) \xrightarrow{j} W(\mathbb{F}_p)/\mathbb{Z} \to 0.$$

The exactness of the first sequence follows from Lemma 4.1.9 and the others are obviously exact. By Theorem 4.1.3 and Lemma 4.1.10 we know that

$$H^n(G, W(K)) = H^n(G, (W(K)/\mathscr{P}W(K)) \otimes \mathbb{Q}^{(p)}) = H^n(G, W(\mathbb{F}_p)/\mathbb{Z}) = 0$$

for all $n \in \mathbb{Z}$. So taking cohomology of each sequence, we obtain four isomorphisms

$$H^{n-1}(G, \mathscr{P}W(K) \cong H^n(G, W(\mathbb{F}_p)),\ H^n(G, \mathbb{Z}) \cong H^n(G, W(\mathbb{F}_p)),$$

$$H^{n-2}(G, W(K)/\mathscr{P}W(K) \cong H^{n-1}(G, \mathscr{P}W(K)),$$

$$H^{n-3}(G, B(K)) \cong H^{n-2}(G, W(K)/\mathscr{P}W(K)).$$

From this it follows that

$$H^{n-3}(G, B(K)) \cong H^n(G, \mathbb{Z})$$

for all n. We now recall the Duality Theorem of Tate (See [28, Ch 3.1]) which states

$$H^n(G, M)^\wedge \cong H^{-n-1}(G, M^\wedge)$$

for discrete G-module M and its character group $M^\wedge$. By applying this theorem we get $H^n(G, \mathbb{Z})^\wedge \cong H^{-n-1}(G, \mathbb{Z}^\wedge)$. Again using the exact sequence

$$0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \to 0$$

and noting that $H^n(G, \mathbb{R}) = 0$, we obtain

$$H^{-n-1}(G, \mathbb{R}/\mathbb{Z}) \cong H^{-n}(G, \mathbb{Z}).$$

But $\mathbb{Z}^\wedge \cong \mathbb{R}/\mathbb{Z}$, so combining the two we get $H^n(G, \mathbb{Z})^\wedge \cong H^{-n}(G, \mathbb{Z})$. Therefore,

$$H^n(G, E(K)) = H^n(G, B(K)^\wedge) \cong H^{-n-1}(G, B(K))^\wedge$$

$$\cong H^{-n+2}(G, \mathbb{Z})^\wedge \cong H^{n-2}(G, \mathbb{Z}).$$

$\square$

Existence Theorem

Let $k \in t$ and consider the set

$$\mathscr{M}(E(k)) = \{A(K/k) = \phi_{k/K}^{-1} \circ \mathrm{Tr}_{K/k}E(k) : K \in t\}.$$

We know that this set has one to one correspondence with all abelian p-extensions of k. Now we aim to determine this set explicitly. Let M be a finite subgroup of $B(k)$. Then $M = \{a \otimes (1/p^n) : a \in S\}$ for some positive integer n and a finite subset S of $W(k)/\mathscr{P}W(k)$. Take

$$C_n = \{a + \mathscr{P}W(k)(\text{mod } V^n(W(k)) : a \in S\}$$

which is a submodule of $W_n(k)$ and $C_n/\mathscr{P}W_n(k) \cong M$. Let $K = k(\mathscr{P}^{-1}a_n : a_n \in C_n)$. As mentioned earlier in the remark in Ch 2.2, we know that K/k is abelian and $G(K/k) \cong C_n/\mathscr{P}W_n(k)$.

Lemma 4.1.11. *For the above extension K/k we have*

$$\mathrm{ann}(M) = \phi_{k/K}^{-1}(\mathrm{Tr}_{K/k}E(k)).$$

Proof. For convenience denote the right hand side of the above expression by M^*. By the main Theorem of Class Formation in Ch 2.1, $E(k)/M^* \cong G(K/k)$. However, we also know that $E(k)/ann(M) \cong G(K/k)$. Hence to prove the equality, it suffices to prove the inclusion relation $ann(M) \subset M^*$, i.e. $\mathrm{Tr}_{K/k}E(K) \subset \phi_{k,K}(ann(M))$. Let $\chi = \mathrm{Tr}_{K/k}\chi_1$ for some $\chi_1 \in E(K)$. Then we consider the natural homomorphism

$$\psi_{k/K} : B(k) \to B(K)$$

induced by the injection $W(k) \to W(K)$. Put $\chi_0 = \chi_1 \circ \psi_{k/K}$ (which is an element in $E(k)$). Then we have $\chi = \phi_{k,K}(\chi_0)$. This is because $\phi_{k,K} \circ \chi_0(A) = \phi_{k,K} \circ \chi_1 \circ \psi_{k/K}(A) = \chi_1(\mathrm{Tr}A) = \mathrm{Tr} \circ \chi_1(A) = \chi(A)$ for any $A \in B(K)$. By the construction of K, it follow that $\psi_{k/K}(M) = 0$ and therefore $\chi_0 \in ann(M)$. $\square$

It also follows that $A(K/k) = ann(M)$ is an open subgroup of the compact group $E(k)$ of finite index. Conversely, let A be an open subgroup of $E(k)$ of finite index. Then the annihilator $M = ann(A)$ is a finite submodule of $B(k)$. We construct the abelian p-extension K/k as above and see that $ann(M) = A(K/k)$ is the group A. So we conclude:

Theorem 4.1.12. *$\mathscr{M}(E(k))$ is the set of all open subgroups of $E(k)$ of finite indices p^n for $n \in \mathbb{Z}_{>0}$.*

We have considered the isomorphism $F_K : (G^{ab,p}(K))^\wedge \cong B(K)$ explicitly. Now we want to construct its dual isomorphism

$$F_K^* : G^{ab,p}(K) \cong B(K)^\wedge.$$

Let $\alpha \in G(K^{ab,p}/K)$. Then $F_K^*(\alpha) = \chi \in B(K)^\wedge$ is defined by the relations:

$$\begin{cases} \chi(\mathbf{X}) = \eta_n \circ (\alpha - 1) \circ \mathscr{P}^{-1} X \\ \mathbf{X} = (X + \mathscr{P} W(K)) \otimes (1/p^n) \in B(K), \quad \mathscr{P}^{-1} X \in R(W(K)) \end{cases}$$

where η_n is the canonical injection $W(\mathbb{F}_p) \to \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$.

Let K/k be a finite Galois p-extension with the Galois group $G = G(K/k)$. We recall that $\sigma \in G$ acts on $\chi_K \in B(K)^\wedge$ via the rule $\chi_K^\sigma(\mathbf{X}) = \chi_k(\sigma^{-1}\mathbf{X})$ for $\mathbf{X} \in B(K)$. On the other hand, let $\mathcal{G} = G(K^{ab,p}/k)$. Then we have

$$\mathcal{G} = \cup_{\sigma \in G} G^{ab,p}(K) \cdot u_\sigma, \; u_\sigma \cdot u_\tau = \rho_{\sigma,\tau} u_{\sigma\tau}, \; \rho_{\sigma,\tau} \in G^{ab,p}(K)$$

where u_σ is a representative of the class $\sigma \in G$ by the canonical isomorphism $\mathcal{G}/G^{ab,p}(K) \cong G$. Hence G acts on $G^{ab,p}(K)$ via the rule $\alpha^\sigma = u_\sigma \cdot \alpha \cdot u_\sigma^{-1}$.

Lemma 4.1.13. *F_k^* is a G-isomorphism of $G^{ab,p}$ onto $B(K)^\wedge$.*

Proof. Let $F_K^*(\alpha) = \chi$ and $F_k^*(u_\sigma \cdot \alpha \cdot u_\sigma^{-1}) = \chi_1$. Then

$$\begin{aligned} \chi_1(\mathbf{X}) = \eta_n \circ (u_\sigma(\alpha - 1) \cdot u_\sigma^{-1}) \circ \mathscr{P}^{-1} X &= \eta_n \circ ((\alpha - 1) u_\sigma^{-1}) \circ \mathscr{P}^{-1} X \\ &= \eta_n \circ (\alpha - 1) \circ \mathscr{P}^{-1}(\sigma^{-1} X) = \chi(\sigma^{-1}\mathbf{X}) = (\chi^\sigma)(\mathbf{X}). \end{aligned}$$

$\square$

Now we consider the transfer map $\mathrm{Ver}_{\mathcal{G}/G^{ab,p}(K)}$: $\mathcal{G}/[\mathcal{G},\mathcal{G}] = G^{ab,p}(k) \to G^{ab,p}(K)$. Recall that the transfer map is defined by (in this specific case) $\alpha \bmod [\mathcal{G},\mathcal{G}] \mapsto \prod_{\sigma\in G}\alpha^{\sigma} =: N_G\alpha$.

Lemma 4.1.14. *The transfer map $\mathrm{Ver}_{\mathcal{G}/G^{ab,p}(K)}$ coincides with $(F_K^*)^{-1}\circ\phi_{k,K}\circ F_k^*$.*

Proof. We compare $(F_K^*)\circ\mathrm{Ver}_{\mathcal{G}/G^{ab,p}(K)}$ and $\phi_{k,K}\circ F_K^*$ on $G^{ab,p}(k)$. Let $\alpha \in G^{ab,p}(K)$, $F_K^*(N_G(\alpha)) = \chi \in B(K)^\wedge$ and $\mathbf{X} = (X + \mathscr{P}W(K)) \otimes (1/p^n) \in B(K)$ as before. Then

$$\chi(\mathbf{X}) = \eta_n \circ (N_G\alpha - 1)\mathscr{P}^{-1}X = \eta_n \circ \sum_{\sigma\in G}(\alpha^{\sigma}-1)(\mathscr{P}^{-1}X) = \chi(\sum_{\sigma\in G}\sigma\mathbf{X})$$
$$=: \chi(\mathrm{Tr}_G\mathbf{X})$$

where the last equality follows from the above Lemma.

On the other hand for $\alpha' = \alpha \bmod [\mathcal{G},\mathcal{G}]$, $(\phi_{k,K}\circ F_K^*)(\alpha') = \chi'$ is defined by

$$\chi'(A) = F_K^*(\alpha')(\mathrm{Tr}\ A) = \eta_n(\alpha'-1)\circ\mathscr{P}^{-1}(\mathrm{Tr}X)$$
$$= \eta_n(\alpha-1)\circ\mathrm{Tr}_G\circ\mathscr{P}^{-1}X = \chi(\mathrm{Tr}_G A).$$

This means this Lemma holds on $G^{ab,p}(K)$. Let $\gamma \in \mathcal{G}$. Then there exists some m such that $\gamma^m \in G^{ab,p}(K)$. So we have

$$(F_K^*)\circ\mathrm{Ver}_{\mathcal{G}/G^{ab,p}(K)}(\gamma)^m = (F_K^*)\circ\mathrm{Ver}_{\mathcal{G}/G^{ab,p}(K)}(\gamma^m)$$
$$= \phi_{k,K}\circ F_k^*(\gamma^m) = \phi_{k,K}\circ F_k^*(\gamma)^m.$$

Recall that $G^{ab,p}(K)$ does not have a torsion element so we obtain the result for a general $\gamma \in \mathcal{G}$. □

In Ch 2.1, we defined the norm residue symbol

$$a \to (a,k) \in G^{ab,p}(k) \text{ for } a \in E(k)$$

which is a homomorphism from $E(k)$ to $G^{ab,p}(k)$. We compare this homomorphism with our isomorphism

$$(F_k^*)^{-1} : E(k) = B(k)^\wedge \to G^{ab,p}(k).$$

Let K/k be a finite Galois p-extension with the Galois group G. By the general theory of class formation we know that as the canonical 2-cocyle of G over $E(k)$ we can take

$$f_{K/k}(\sigma, \tau) = F_K^*(\rho_{\sigma,\tau}) \text{ for } \sigma, \tau \in G$$

where $\rho_{\sigma,\tau}$ was defined earlier in this chapter. Then the norm residue symbol is given for $\sigma \in G$ by

$$\begin{aligned}(\sigma, K/k)^{-1} &= \phi_{K/k}^{-1}\Big(\prod_{\rho \in G} f_{K/k}(\rho, \sigma)\Big) \pmod{\phi_{k,K}^{-1}(\mathrm{Tr}_G E(K))} \\ &= \phi_{k/K}^{-1} \circ F_K^*\Big(\prod_{\rho} u_\rho \cdot u_\sigma \cdot u_{\rho\sigma}^{-1}\Big) = \phi_{k/K}^{-1} \circ F_K^* \circ \mathrm{Ver}_{\mathcal{G}/G^{ab,p}(K)}(u_\sigma) \\ &= F_K^*(u_\sigma) \pmod{A(K/k)}\end{aligned}$$

where the last equality follows from Lemma 4.1.14. This value is independent of choice of the representatives u_σ of $\mathcal{G}$ mod $G^{ab,p}(K)$. Now take the inverse limit with respect to all finite abelian p-extensions K of k such that $\cup K = k^{ab,p}$. Then $\cap_K A(K/k) = 0$ by Theorem 4.1.12. Now let $\sigma \in G^{ab,p}(k)$ and $\lambda_K(\sigma)$ its image in $G(K/k)$ then

$$\varprojlim(\lambda_K(\sigma), K/k)^{-1} = F_k^*(\sigma) \in E(k).$$

By the definition of the generalised norm residue symbol we obtain the formula

$$(\chi, k) = (F_k^*)^{-1}(\chi) \in G^{ab,p}(k)$$

for $\chi \in E(k)$.

4.1.2 *Formal Power Series Fields*

Here we consider the case where the ground field k_0 is a formal power series field in one variable over a finite field; i.e. when the k_0 is a local field of finite characteristic p.

Let $t = t(k_0)$ be a set containing the finite subextensions of $k_0^{sep,p}$ as before and let $k \in t$. Then k is also a formal power series field in one variable T over a finite field $\mathbb{F}_q$ where $q = p^f$ for some $f > 0$ by classification of local fields. As usual we denote by $\mathcal{O}_k$ its ring of integers and m_k its unique maximum ideal. We also denote by U_k the group of units of $\mathcal{O}_k$ and $U_{n,k} = 1 + T^n\mathcal{O}_k$ or U_n if the field we are referring to is clear.

Inner product

Let $\{\alpha_1, \ldots, \alpha_f\}$ be a basis of $\mathbb{F}_q$ over $\mathbb{F}_p$ and let $\{\beta_1, \ldots, \beta_f\}$ be a complementary basis of $\{\alpha_1, \ldots, \alpha_f\}$ such that

$$\mathrm{Tr}(\alpha_i\beta_j) = \delta_{ij}$$

for $i, j \in \{1, \ldots, f\}$ (Recall that $\delta_{ij} = 1$ if $i = j$ and 0 otherwise.) In particular we choose $\alpha_1 = 1$ so that $\mathrm{Tr}\beta_1 = 1, \mathrm{Tr}\beta_i = 0$ $(i \neq 1)$ hold.

Note that $(k^\times)^{p^n}$ for each n are closed subgroups of $k^\times$ and the factor groups $\mathbf{k}_n := k^\times/(k^\times)^{p^n}$ are compact groups. The fundamental system of neighbourhoods of unity in $\mathbf{k}_n$ are $\{U_m \cdot (k^\times)^{p^n} : m > 0\}$. Also consider the natural continuous homomorphism $\phi_{n+1} : \mathbf{k}_{n+1} \to \mathbf{k}_n$ for each $n > 0$ defined by

$$a \cdot (k^\times)^{p^{n+1}} \mapsto a \cdot (k^\times)^{p^n}.$$

Then we can define the projective limit group $\mathbf{k}$ with respect to $\{\phi_n\}$. $\mathbf{k}$ is a compact abelian group and we denote by ϕ_n^* the continuous homomorphism $\mathbf{k} \to \mathbf{k}_n$ such that $\phi_n^* = \phi_{n+1} \circ \phi_{n+1}^*$ holds for all $n > 0$.

We deduce that $\cap_{n=1}^{\infty}(k^\times)^{p^n} = \mathbb{F}_q^\times$ using the decomposition $k^\times = T^{\mathbb{Z}} \oplus \mathbb{F}_q^\times \oplus U_1$. Hence we have a natural injection

$$l : k^\times/\mathbb{F}_q^\times \to \mathbf{k}.$$

It is easy to see that l is continuous and the image of l is dense in $\mathbf{k}$. In fact the decomposition of $k^\times$ gives $\mathbf{k}_n \cong (\mathbb{Z}/p^n) \oplus (U_1/U_1^{p^n})$[1]. Since $\cap_{n=1}^{\infty} U_1^{p^n} = 1$ we get $\mathbf{k} \cong \mathbb{Z}_p \oplus U_1$. So we obtain two exact sequences:

$$1 \to \mathbb{F}_q^\times \to k^\times \to k^\times/\mathbb{F}_q^\times \to 1$$

$$1 \to k^\times/\mathbb{F}_q^\times \xrightarrow{l} \mathbf{k} \to \mathbb{Z}_p/\mathbb{Z} \to 1$$

Now we intend to establish group pairings of $\mathbf{k}_n$ and $W_n(k)/\mathscr{P}W_n(k)$ for each $n > 0$ as well as $\mathbf{k}$ and $B(k)$ explicitly.

Let us recall some facts. The multiplicative group $\mathbf{k}_1$ is the infinite product group of a countable set of cyclic groups of order p. According to [14, Ch 1.6] we can choose a minimal set of (topological) generators

$$\begin{cases} t \\ 1 + \alpha_i t^m \quad i = 1, \ldots, f;\ m > 0,\ (m, p) = 1 \end{cases}$$

Furthermore the additive group $k/\mathscr{P}k$[2] is the direct sum of a countable set of cyclic groups of order p. As a set of minimal generators we can choose

$$\begin{cases} \beta_1 \pmod{\mathscr{P}k} \\ \beta_j t^{-m} \pmod{\mathscr{P}k}, \quad j = 1, \ldots, f;\ m > 0,\ (m, p) = 1 \end{cases}.$$

We recall that residue function on $F((T))$ (for a general field F) is defined as the coefficient of T^{-1}. More precisely, if $z = \sum_{i=n}^{\infty} a_i T^i$ with $n \in \mathbb{Z}$ and $a_i \in F$ then $\operatorname{res}(z) = a_{-1}$. We can extend this to differential one form just by defining res(zdT) to be res(z) for $z \in F((T))$. It is well known that the residue on $F((T))$ does not depend on the choice its prime element T.

We state a result which is of fundamental importance to this chapter.

Theorem 4.1.15. *Let res be the residue function in $\mathbb{F}_q((T))$ as defined above and let Tr=$\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}$. Then*

[1] $k^\times \cong \mathbb{Z} \oplus U_1 \otimes \mu_k$ but μ_k is p-divisible.

[2] $k = W_1(k)$

$$\begin{cases} Tr \circ res(\beta_1 dt/t) = 1 & \\ Tr \circ res(\beta_j t^{-m} dt/t) = 0 & m > 0 \\ Tr \circ res(\beta_1 d(1+\alpha_i t^m)/(1+\alpha_i t^m)) = 0 & m > 0 \\ Tr \circ res(\beta_j t^{-m'} d(1+\alpha_i t^m)/(1+\alpha_i t^m)) = \begin{cases} m\delta_{ij} & \text{for } m = m' \\ 0 & \text{for } m' < m \end{cases} & \end{cases}$$

Now consider $k_f = \mathbb{Q}_p(\zeta_{p^f-1})$, the unramified extension field of degree f of $\mathbb{Q}_p$. Then its ring of integers has a unique prime ideal $P = p\mathcal{O}_P$ and the residue field is $\mathbb{F}_q$ ($q = p^f$). Fix the unique homomorphism $\psi : \mathbb{F}_q^\times \to \mathcal{O}_P^\times$ and put $\psi(0) = 0$.

We recall that $W(\mathbb{F}_q)$ is canonically isomorphic to $\mathcal{O}_P$. In fact the isomorphism is given by

$$\Psi : a = (a_0, a_1, \ldots, a_{n-1}, \ldots) \to \sum_{n=0}^{\infty} \psi(a_n)^{p^{-n}} \cdot p^n.$$

So the Frobenius automorphism $F(a) = a^p$ of $W(\mathbb{F}_q)$ induces the automorphism $\sigma = \Psi \circ F \circ \Psi^{-1}$ of $k_f/\mathbb{Q}_p$. So we can show that $\mathrm{Tr}_{k_f/\mathbb{Q}_p} \circ \Psi(a) = \eta \circ \mathrm{Tr}(a)$ for $a \in W(\mathbb{F}_q)$ where $\eta : W(\mathbb{F}_p) \cong \mathbb{Z}_p$ is the canonical isomorphism. This can be proved by simply writing down the explicit expression for both sides and comparing. Recall that Tr commutes with $\mathscr{P}$ and action of p on $W(\mathbb{F}_q)$ and on $W_n(\mathbb{F}_q)$. Recall that for each n we defined $\eta_n : W(\mathbb{F}_p) \to \mathbb{R}/\mathbb{Z}$ given as composition of two natural maps $W(\mathbb{F}_p) \to \mathbb{Z}_p/p^n \cong \mathbb{Z}/p^n$ and $\mathbb{Z}/p^n \to \mathbb{R}/\mathbb{Z}$. Then η_n maps $V^n(W(\mathbb{F}_p))$ to 0 so η_n induces

$$\eta_n : W_n(\mathbb{F}_p) \to \mathbb{R}/\mathbb{Z}.$$

From the definition we can deduce that for $X \in W_n(\mathbb{F}_p)$,

$$\eta_{n-1}(A(X)) = \eta_n(pX),\ \eta_{n+1}(V(X)) = \eta_n(X)^3.$$

[3]Recall that $A : W_n(k) \to W_{n-1}(k)$ is the natural restriction map discarding the last entry.

Furthermore, we introduce the notations:

$$\begin{cases} W_n(\mathcal{O}_k) = \{X^{(n)} = (x_0, x_1, \cdots, x_{n-1}) \in W_n(k) : x_i \in \mathcal{O}_k,\ i \in \{0, 1, \cdots, n-1\}\} \\ W_n(m_k) = \{X^{(n)} = (x_0, x_1, \cdots, x_{n-1}) \in W_n(k) : x_i \in m_k,\ i \in \{0, 1, \cdots, n-1\}\} \\ W_n(m_k)^{(m)} = \{X^{(n)} = (x_0, x_1, \cdots, x_{n-1}) \in W_n(k) : \\ \qquad\qquad v(x_i) \geq m/p^{n-1-i},\ i \in \{0, 1, \cdots, n-1\}\} \end{cases}$$

where v is the discrete valuation of k.

For $a \in k^\times$ and $X \in W_n(k)$ we define a generalisation of the residue map. We recall that every Witt vector $X = (x_0, x_1, \cdots, x_{n-1}) \in W_n(k)$ has the ghost component $(x^{(0)}, x^{(1)}, \cdots, x^{(n-1)})$. To define this ghost component, we need to use characteristic 0 ring. Let $\tilde{k}$ be the field of fractions of $W(\mathbb{F}_q)$, namely the unramified extension of degree l of $\mathbb{Q}_p$ if $q = p^l$. Then $\tilde{k}((t)) = \mathcal{R}$ is an appropriate ring which we can do the process that was mentioned in Ch 2.2. Then $res^{(n)}(Xda/a) \in W_n(\mathbb{F}_q)$ is defined to be the Witt vector which has the ghost component

$$(res(x^{(0)}da/a), res(x^{(1)}da/a), \cdots, res(x^{(n-1)}da/a)).$$

This map was first introduced in [34] but with different notation. To define $res^{(n)}$ more explicitly, let $\tilde{x} \in \tilde{k}((t))$ denote the lift of $x \in k$, namely an element such that $\tilde{x}$ mod $p = x$. Also we define maps $P_n : \mathcal{R}^{n+1} \to \mathcal{R}$ for each n such that $P_n(x^{(0)}, x^{(1)}, \cdots, x^{(n)}) = x_n$. One can prove that such P_n exists and $P_n(X_0, X_1, \cdots, X_n)$ is a polynomial in $\mathbb{Z}[p^{-1}][X_0][X_1]\cdots[X_n]$. In this case

$$res^{(n)}(Xda/a) = (w_0, \cdots, w_{n-1})$$

where w_i is the image of $P_i(res(\tilde{x}^{(0)}da/a), \cdots, res(\tilde{x}^{(i)}da/a))$under the natural map $\tilde{k}(t) \to k$. We now state some properties of $res^{(n)}$.

Theorem 4.1.16. *The symbol introduced above has the following properties: let $a, a_1, a_2 \in K^\times$, $X, X_1, X_2 \in W_n(k)$, then*
*R*1. $res^{(n)}(X\frac{d(a_1 \cdot a_2)}{(a_1 \cdot a_2)}) = res^{(n)}(X\frac{da_1}{a_1}) + res^{(n)}(X\frac{da_2}{a_2})$
*R*2. $res^{(n)}((X_1 + X_2)\frac{da}{a}) = res^{(n)}(X_1\frac{da}{a}) + res^{(n)}(X_2\frac{da}{a})$

*R*3. $res^{(n)}(X\frac{d(\lambda a)}{\lambda a}) = res^{(n)}(X\frac{da}{a})$ *for* $\lambda \in \mathbb{F}_q^\times$
*R*4. $res^{(n)}(YX\frac{da}{a}) = Yres^{(n)}(X\frac{da}{a})$ *for* $Y \in W_n(\mathbb{F}_q)$
*R*5. $res^{(n)}(X^p\frac{da}{a}) = (res^{(n)}(X\frac{da}{a}))^p$
*R*6. $res^{(n)}(\mathscr{P}X\frac{da}{a}) = \mathscr{P}res^{(n)}(X\frac{da}{a})$
*R*7. $res^{(n-1)}(AX\frac{da}{a}) = A \circ res^{(n)}(X\frac{da}{a})$
*R*8. $res^{(n+1)}(VX\frac{da}{a}) = V \circ res^{(n)}(X\frac{da}{a})$
*R*9. $res^{(n)}(X\frac{dt}{t}) = (res(x_0\frac{dt}{t}), res(x_1\frac{dt}{t}), \ldots, res(x_{n-1}\frac{dt}{t}))$, *for* $X = (x_0, x_1, \ldots, x_{n-1})$
in particular,
*R*9′. $res^{(n)}(Y\frac{dt}{t}) = Y$ *for* $Y \in W_n(\mathbb{F}_q)$
*R*9″. $res^{(n)}(X\frac{dt}{t}) = 0$ *for* $X \in W_n(m_k)$
*R*10. $res^{(n)}(X\frac{da}{a}) = 0$ *for* $a \in U_m$ *and* $X \in W_n(m_k)^{(-m)}$

All properties but *R*9 follow easily from the definition. We can obtain *R*9 by checking the ghost component for $(res(x_0\frac{dt}{t}), res(x_1\frac{dt}{t}), \ldots, res(x_{n-1}\frac{dt}{t}))$ is indeed

$$(res(x^{(0)}dt/t), res(x^{(1)}dt/t), \cdots, res(x^{(n-1)}dt/t))$$

as *res* commutes with operations on polynomials.

Consequently we can define an inner product on $k^\times \oplus W_n(k)$ by

$$(a, X]_n = \eta_n \circ \mathrm{Tr}(res^{(n)}(X^{\frac{da}{a}})) \in \mathbb{R}/\mathbb{Z}$$

for $a \in k^\times$, $X \in W_n(k)$ and Tr denotes the trace from $W_n(\mathbb{F}_q)$ to $W_n(\mathbb{F}_p)$. Then the inner product has following properties which are easy consequences of Theorem 4.1.16.

*R**1. $(a_1a_2, X]_n = (a_1, X]_n(a_2, X]_n$
*R**2. $(a, X_1X_2]_n = (a, X_1]_n(a, X_2]_n$
*R**3. $(\lambda a, X]_n = (a, X]_n$ for $\lambda \in \mathbb{F}_q$
*R**4. $(a^{p^n}, X]_n = 0$
*R**5. $(a, \mathscr{P}X]_n = 0$
*R**6. $(a, AX]_{n-1} = (a^p, X]_n$
*R**7. $(a, VX]_{n+1} = (a, X]_n$.

We are ready to prove the following result:

Proposition 4.1.17. *By the inner product $(a,X]_n$ defined on $k^\times \oplus W_n(k)$, $U_m \cdot (k^\times)^{p^n}$ and $W_n(m_k)^{(-m)} + \mathscr{P}W_n(k)$ are mutually annihilators of the others for integers $m = p^n, 2p^n, \cdots$.*

Proof. The fact that each of these subgroups annihilate each other follows from R^*4, 5 and $R10$.

Now we prove by induction on n that $ann(U_m) = W_n(m_k)^{(-m)} + \mathscr{P}W_n(k)$ and $ann(W_n(m_k)^{(-m)}) = U_m \cdot (k^\times)^{p^n}$ for $m = m'p^n$. For $n = 1$, this follows from the explicit description on generators of $k^\times/(k^\times)^p$ and $k/\mathscr{P}k$ and Theorem 4.1.15.

Assume the result holds for n and for all suitable m'. Let $X \in W_{n+1}(k)$ be in $ann(U_m)$. Since $a'^p \in U_m$ for $a' \in U_{m/p}$, we have by R^*6

$$(a', AX]_n = (a'^p, X]_{n+1} = 0.$$

By our assumption $AX \in W_n(m_k)^{(-m/p)} + \mathscr{P}W_n(k)$. Let $AX = X_1 + \mathscr{P}Y_1$ with $X_1 \in W_n(m_k)^{(-m/p)}$ and $Y_1 \in W_n(k)$. Then for $X_1* := (X_1, 0) \in W_{n+1}(m_k)^{(-m)}$, there is the equality

$$X = X_1 + \mathscr{P}(Y_1, 0) + V^n X^*,$$

for some $X^* \in W_1(k)$. So from R^*7 we can deduce that

$$0 = (a, X]_{n+1} = (a, X_1^*]_{n+1} + (a, \mathscr{P}(Y_1, 0)]_{n+1} + (a, V^n x^*]_{n+1}$$

$$= (a, V^n x^*]_{n+1} = (a, x^*]_1$$

Following on from the $n = 1$ case, we have $x* = x' + \mathscr{P}y$ where $x' \in m_k^{-m}$. Moreover we obtain

$$X = (X_1, x') + \mathscr{P}(Y_1, y) \in W_{n+1}(m_k)^{(-m)} + \mathscr{P}W_{n+1}(k).$$

Let $a \in ann(W_{n+1}(m_k)^{(-m)})$. For $X_1^{(n)} \in W_{n+1}(m_k)^{(-m)}$, $VX_1^{(n)} \in W_{n+1}(m_k)^{(-m)}$. From R^*7 it follows that $(a, X_1]_n = 0$. So by our induction assumption we obtain $a = a_1 b_1^{p^n}$ for some $a_1 \in U_m$ and $b_1 \in k^\times$. Put $X_0 = (x_0, 0, \cdots, 0)$

for $x_0 \in m_k^{-m'p}$ and $m = m'p^n$. Then X_0 belongs to $W_{n+1}(m_k)^{(-m)}$. It now follows from $R10$ and R^*6 that

$$0 = (a, X_0]_{n+1} = (a_1, X_0]_{n+1} + (b_1^{p^n}, X_0]_{n+1} = (b_1, x_0]_1$$

By $n = 1$ case $b_1 = a_2 \cdot b_2^p$ for some $a_2 \in U_{m'p}$ and $b_2 \in k^\times$. Hence we finally obtain $a = (a_1 \cdot a_2^{p^n}) \cdot b_2^{p^{n+1}} \in U_m \cdot (k^\times)^{p^{n+1}}$. □

Using R^*4 and R^*5 we can actually define an inner product on $\mathbf{k}_n \oplus W_n(k)/\mathscr{P}W_n(k)$. We shall denote the inner product in the same fashion. For $a \in k^\times$ and $X \in W_n(k)$ we write $\bar{a} = a + (k^\times)^{p^n} \in \mathbf{k}_n$ and $\overline{X} = X + \mathscr{P}W_n(k) \in W_n(k)/\mathscr{P}W_n(k)$.

Theorem 4.1.18. *By the inner product on $\mathbf{k}_n \oplus W_n(k)/\mathscr{P}W_n(k)$ defined above, the compact multiplicative group $\mathbf{k}_n$ and the additive abelian group $W_n(k)/\mathscr{P}W_n(k)$ are orthogonally paired to $\mathbb{R}/\mathbb{Z}$. Hence by Pontryagin Duality, $\mathbf{k}_n$ is the compact character group of the discrete abelian group $W_n(k)/\mathscr{P}W_n(k)$.*

We recall that for topological groups G_1, G_2 and G_3, an inner product $(\cdot,\cdot)$ on $G_1 \oplus G_2$ is orthogonally paired to G_3 if

1. $(\ ,\ )$ is continuous on $G_1 \oplus G_2$
2. $(g_1, G_2) = 0$ if and only if $g_1 = 1_{G_1}$
3. $(G_1, g_2) = 0$ if and only if $g_2 = 1_{G_2}$.

Proof. To prove the continuity of the paring $(\bar{a}, \overline{X}]_n$, it suffices to show that for any given $\overline{X}$ there exists a neighbourhood $\overline{U}$ of the unity in $\mathbf{k}_n$ such that $(\overline{U}, \overline{X}) = 1$. Now for any given $\overline{X}$ there exists a positive integer m such that $\overline{X} \in W_n(m_k)^{(-m)}$. Then for this value m we have $(\overline{U}_m, \overline{X}]_n = 1$ by Theorem 4.1.16 where $\overline{U}_m$ is the image of U_m in $\mathbf{k}_n$.

The second property follows from the observation that $ann(W_n(k))$ in $k^\times$ is equal to $\cap_m U_m \cdot (k^\times)^{p^n} = (k^\times)^{p^n}$. This is true as $W_n(k) = \cup_m W_n(m_k)^{(-m)}$.

For the last property it suffices to prove

$$ann(k^\times) = \mathscr{P}W_n(k) \text{ in } W_n(k).$$

We prove by induction on n. For $n = 1$ this follows from classical results on generators of $\mathbf{k}_1$ and $k/\mathscr{P}k$. So suppose the assertion is true for n. Take $X \in ann(k^\times)$ in $W_{n+1}(k)$. Then for an arbitrary $a' \in k^\times$ we have

$$(a', AX]_n = (a'^p, X]_{n+1} = 0.$$

Hence $AX \in \mathscr{P}W_n(k)$ holds by our assumption. So we write $X = \mathscr{P}(X_1, 0) + V^n x^*$ by some elements $X_1 \in W_n(k)$, $x^* \in k^\times$. Then by the relation

$$(a, x^*]_1 = (a, V^n x^*]_{n+1} = (a, X]_{n+1} = 0 \ \forall a \in k^\times$$

we can conclude that $x^* = \mathscr{P}x$ for some $x \in k^\times$. This means $V^n x^* = \mathscr{P}(0, \cdots, 0, x) \in \mathscr{P}W_n(k)$ so the result holds for $n+1$. $\square$

Now consider the sequence $\{W_n(k)/\mathscr{P}W_n(k)\}$ and its direct limit group with respect to the homomorphism V, and the sequence of compact groups $\{\mathbf{k}_n\}$ and its projective limit group $\mathbf{k}$ with respect to continuous map ϕ_n as shown earlier. Let $\bar{a} \in \mathbf{k}_n$ and $X_n \in W_n(k)$. Now we write $X_{n-1} = AX_n$. Then we have

$$(\bar{a}, V\overline{X}_{n-1}]_n = (\phi_n \bar{a}, \overline{X}_{n-1}]_{n-1}.$$

This follows easily from R^*7 and definitions. Hence we can define a pairing on $\mathbf{k}$ and $B(k)$ as follows. Let $a \in \mathbf{k}$ and $X \in B(k)$. By the definition of $B(k)$ there exists an integer n such that $X = \psi_n^* X_n$ for some $X_n \in W_n(k)/\mathscr{P}W_n(k)$ where $\psi_n^* : W_n(k)/\mathscr{P}W_n(k) \to B(k)$ is the injective map defined earlier. We can then define

$$(a, X]_k = (\phi_n^*(a)), X_n]_n$$

which is independent of choice of n by the above equation. Thus we get:

Theorem 4.1.19. *The groups $\mathbf{k} = \varprojlim \mathbf{k}_n$ and $B(k) = \varinjlim W_n(k)/\mathscr{P}W_n(k)$ are orthogonally paired by the inner product defined above. Hence $\mathbf{k}$ is topologically isomorphic to the compact character group of the discrete additive group $B(k)$.*

Recall that we defined $l : k^\times/\mathbb{F}_q^\times \to \mathbf{k}$ at the beginning of this section. Using this and the inner product $(\cdot,\cdot]_k$ we can define the inner product between $k^\times$ and $B(k)$. It can be verified that this is well-defined and it is a continuous homomorphism of $k^\times \times B(k)$ into $\mathbb{R}/\mathbb{Z}$

Local Class Field Theory

In here, we shall establish classical characteristic p local class field theory under the restriction that we only consider p-extensions.

Let k_0 and t be defined as before and take $E^*(K) = K^\times$ for $K \in t$. For $k \subset K$ with k, K both in t we define $\phi^*_{k,K}$ to be the inclusion map $k^\times = E^*(k) \hookrightarrow E^*(K) = K^\times$. Then it is clear that C_1, C_2, C_3, C_4 in p.56 of this thesis holds.

Now we consider the map $l_k : E(k)^* \to E(k) = B(k)^\wedge$ defined by composing l with the isomorphism $\mathbf{k} \cong B(k)^\wedge$ as in Theorem 4.1.19.

Proposition 4.1.20. *i) The diagram*

$$\begin{array}{ccc} E^*(k) & \xrightarrow{l_k} & E(k) \\ \Big\downarrow \phi^*_{k/K} & & \Big\downarrow \phi_{k,K} \\ E^*(K) & \xrightarrow{l_K} & E(K) \end{array}$$

is commutative for $k \subset K$.

ii) For a Galois extension K/k the map l_K is a $G(K/k)$-homomorphism.

Proof. Recall that residue field of k is $\mathbb{F}_q$ for this proof only we assume that residue field of K is $\mathbb{F}_{q^l}$.

i) Since $\phi_{k,K} = \mathrm{Cotr} K/k$ the assertion is equivalent to $(a, X]_K = (a, \mathrm{Tr}_{K/k} X]_k$ for $a \in k^\times$ and $X \in B(K)$, where $(\cdot,\cdot]_k$ and $(\cdot,\cdot]_K$ are inner product functions defined on k and K respectively. So let

$$X = ((x_0, x_1, \cdots,) + \mathscr{P} W(K) \otimes (1/p^n).$$

Write $X_n = (x_0, \cdots, x_{n-1})$. Then the above equality translates to

$$\eta_n \circ \mathrm{Tr}_{\mathbb{F}_{q^l}/\mathbb{F}_p}(res_K^{(n)}(X\frac{da}{a})) = \eta_n \circ \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(res_k^{(n)} \mathrm{Tr}_{K/k}(X\frac{da}{a})).$$

This equality was proved for $n = 1$ case by Hasse ([16, Ch 5]) and the other cases follow easily by definition.

ii) can be proved in the similar manner.

□

Now we prove the last condition of class formation using the above Proposition. Namely,

Theorem 4.1.21. *$H^n(G, E^*(K)) \cong H^n(G, E(K)) \cong H^{n-2}(G, \mathbb{Z})$ for all $n \in \mathbb{Z}$ where K/k is a Galois extension $(k,\ K \in t)$ with Galois group G.*

Proof. Suppose the constant field of K is $\mathbb{F}_q$. Then we have exact sequences

$$1 \to \mathbb{F}_q^\times \to K^\times \to K^\times/\mathbb{F}_q^\times \to 1$$

$$1 \to K^\times/\mathbb{F}_q^\times \xrightarrow{l} \mathbf{K} \to \mathbb{Z}_p/\mathbb{Z} \to 1$$

where $\mathbf{K} = \varprojlim (K^\times/(K^\times)^m)$. By the previous Proposition, the homomorphisms in these sequences are G-homomorphisms. Recall that $\mathbb{F}_q$ and $\mathbb{Z}_p/\mathbb{Z}$ are both uniquely p^n-divisible for all n. So we get that $H^n(G, \mathbb{F}_q) = H^n(G, \mathbb{Z}_p/\mathbb{Z}) = 0$ for all $n \in \mathbb{Z}$. By taking cohomology in the two exact sequences and using Theorem 4.1.19 we obtain the result. □

Therefore we have proved that for our ground field k_0 and for the maximal Galois p-extension $k_0^{sep,p}$

$$\{E^*(k) = k^\times,\ \phi^*_{k/K} = \text{inclusion}\}$$

defines a Class Formation.

Existence Theorem

Existence Theorem is an important feature of class field theory since it gives one to one correspondence between abelian extensions and appropriate groups depending on fields.

Proposition 4.1.22. *The set*

$$\mathscr{M}(E^*(k)) = \{A^*(K/k) = \phi^{*-1}_{k/K} N_{K/k}(E^*(K)) : K/k \text{ abelian } p\text{-extensions}\}$$

is the set of all open subgroups of $E^*(k)$ of finite indices p^n $(n \in \mathbb{Z}_+)$.

Proof. Each $A^*(K/k)$ is an open subgroup of $E^*(k)$ as ϕ^* is continuous and $N_{K/k}$ is an open map. The index follows from the isomorphism theorem in class formation theory.

By Proposition 4.1.20 we also have

$$l_k A^*(K/k) = l_k \circ \phi^{*-1}_{k/K} \circ N_{K/k} E^*(K) = \phi^{-1}_{k,K} \circ l_K \circ N_{K/k} E^*(K)$$

$$= \phi^{-1}_{k,K} \circ \mathrm{Tr}_{K/k} \circ l_K \circ E^*(K) \subset A(K/k).$$

Moreover, since $l_K E^*(K)$ is dense in $E(K)$ and $\mathrm{Tr}_{K/k}$ and $\phi^{-1}_{k,K}$ are closed maps, we have closure of $l_k A^*(K/k)$ in $E(k)$ is $A(K/k)$. Since $[E(k) : A(K/k)] = [E^*(k) : A^*(K/k)] = [K : k]$, we also have

$$l_k A^*(K/k) = A(K/k) \cap l_k E^*(k).$$

Now let A^* be an open subgroup of $E^*(k)$ of index p^n and A the closure of $l_k A^*$ in $E(k)$. Then we have $[E(k) : A] = [E^*(k) : A^*] = p^n$ and $l_k A^* = A \cap l_k E^*(k)$. From the general theory of existence theorem proved earlier in Ch 4.1, we know that $\mathscr{M}(E(k)) = \{A(K/k) : K/k \text{ abelian } p\text{-extensions } \}$ is the set of all open subgroups of $E(k)$ of finite indices p^n for each $n \in \mathbb{Z}_+$. Hence there exists an abelian p-extension K of k with $[K : k] = p^n$ such that $A(K/k) = A$. Then we have

$$A^*(K/k) = l_k^{-1}(A(K/k) \cap l_k E(k)) = l_k^{-1}(A \cap l_k E^*(k)) = A^*.$$

This proves the Proposition. □

Now we give an explicit description of $A^*(K/k)$ if K is given by

$$K = k(\mathscr{P}^{-1}(C_n))$$

where C_n is a submodule of $W_n(k)$. We recall that if $S = (C_n + \mathscr{P} W_n(k)) / \mathscr{P} W_n(k))$ and $M = L \otimes 1/p^n$ then $G(K/k) \cong M \cong L$. Then we have:

Proposition 4.1.23. *If a finite abelian p-extension K/k is given by above then $A^*(K/k)$ is the set of all a's in $E(k)^*$ such that $(a, C_n]_n = 0$.*

Proof. Follows from Lemma 4.1.11 and Proposition 4.1.20. □

4.1.3 *Algebraic Function Fields*

Kawada and Satake prove the same result for an algebraic function field over a finite field, namely the global field of characteristic p in [20] as well. Since global theory in this perspective is not of much interest in this thesis, we summarise it very briefly.

Let K be a global field of characteristic p and K_P be the completion of K for each place P. Since K_P is a local field, by Theorem 4.1.19 we have that $\mathbf{K_P} := \varprojlim(k_P^\times/(k_P^\times)^{p^i})$ is dual to $B(F_P)$. Using this we obtain that $I_K/I_K^{p^n}$ and $W_n(\mathbb{A}_F)$ are mutually dual as locally compact groups. Then we follow the exactly same pattern as the local case to obtain global class field theory as well.

4.2 HIGHER DIMENSION

Here we extend the approach by Kawada-Satake we presented in the previous section to develop characteristic p higher local class field theory. Parshin made an attempt to do this in [30] and [31] but this contains several errors and we fix those mistakes in this exposition.

We first give topology on higher local fields. We note that one of the problems with Parshin's paper was that he used incorrect topology. The topology defined by Madunts and Zhukov is suitable for most purposes for study of higher local fields but for class field theory we need to use its sequential saturation instead which Parshin overlooked. The definitions and properties of topology on Higher local fields can be found in many texts and papers but we follow the presentation of [13, Ch. 1].

4.2.1 *Topology on Higher Local Fields*

Since higher local fields are complete discrete valuation fields, they have natural rank 1 discrete valuation topology. However, this does not take account of the topology of the residue fields so it is rather unsatisfactory. So we now define a more refined topology on higher local fields F of positive characteristic by introducing a way of 'lifting' topology from the residue fields. Namely we deploy technique to lift topology from K to $F = K((t))$. We note that it is also possible to do this for the mixed characteristic cases but is more complicated. This topology was first introduced in [24].

Let the basic open neighbourhoods of 0 to be

$$\sum_i U_i t^i := \{\sum_i a_i t^i \in F : a_i \in U_i \text{ for all } i\}$$

where $(U_i)_{i=-\infty}^{\infty}$ are open neighbourhoods of 0 in K with the property that $U_i = K$ for $i \gg 0$. Then this defines the topology τ on F and thus for all characteristic p higher local fields. This topology does not depend on the choice of local parameter t. We can now define the topology on $F^\times$ as follows. Recall that $F^\times = t_1^{\mathbb{Z}} \times \cdots \times t_n^{\mathbb{Z}} \times \mathbb{F}_q^\times \times V_F$. Then we put discrete topology on $t_1^{\mathbb{Z}} \times \cdots \times t_n^{\mathbb{Z}} \times \mathbb{F}_q^\times$ and give the topology of V_F induced from F. Then take product topology of the two components. This topology also does not depend on the choice of the sequence of the local parameters $t_1, t_2, \cdots t_n$. See [24] for detailed explanation and proof of the property of this topology.

The topology τ is sufficient for most purposes for study of higher local fields but for class field theory it is not quite correct. Instead we need to take its sequential saturation.

Definition. *Let X be a topological space with topology τ. Define its sequential saturation λ :*
a subset U of X is open with respect to λ if for all $\alpha \in U$ and a sequence (α_i) which converges to α (with respect to τ), almost all α_i belong to U. We say that topology τ is sequentially saturated if it coincides with its sequential saturation.

These two topologies have the same converging sequences. In fact the sequential saturation is the strongest topology which has the same convergent sequences and their limit as the original one.

Let λ be sequential saturation of τ. We note that the product topology of sequential saturated topology is not necessarily sequentially saturated. So the topology we give for $(F^\times)^m$ is the sequential saturation of the product topology of λ.

Proposition 4.2.1. *Topology λ has the following properties:*
1. *A Cauchy sequence with respect to the topology τ converges in $F^\times$.*
2. *Multiplication by a fixed element is continuous.*
3. *Multiplication in F is sequentially continuous but not continuous if (complete discrete) dimension of F is greater than 2. In particular, in general $F^\times$ is not a topological group.*
4. *For any sequence $a_i \in V_F$, the sequence $a_i^{p^i}$ converges to 1.*
5. *V_F and $(F^\times)^m$ are closed subgroups of $F^\times$ for every $m \geq 1$.*

Here we stress that τ coincides with the topology which is defined and used in [30] and this is not equal to λ. To see this let $F = \mathbb{F}_q(t_1))((t_2))$

and $W = F\backslash\{t_2^a t_1^{-c} + t_2^{-a} t_1^c : a, c \geq 1\}$. Then this set is only open in λ. We conclude this section with the following Lemma.

A set $\Omega \subset \mathbb{Z}^n$ is called admissible if for every $1 \subset l \subset n$ and every $j_{l+1}, \cdots, j_n$ there exists $i = (j_{l+1}, \cdots, j_n) \in \mathbb{Z}$ such that

$$(i_1, \cdots, i_n) \in \Omega,\ i_{l+1} = j_{l+1}, \cdots, i_n = j_n \implies i_l \geq i$$

Lemma 4.2.2. *Let $\epsilon \in V_F$. Then ϵ is uniquely expanded in a convergent product*

$$\epsilon = \prod (1 + \theta_{i_n, \cdots, i_1} t_n^{i_n} \cdots t_1^{i_1}),\ \theta_{i_n, \cdots, i_1} \in \mathbb{F}_q$$

such that $\Omega = \{(i_n, \cdots, i_1) : \theta_{i_n, \cdots, i_1} \neq 0\}$ is an admissible set and for all $J \in \Omega$, $J > 0$ (with colexicographic ordering) and $p \not| J$.[4]

The proof involves using induction and elementary tools which were used in one dimensional local field which can be found in [14, Ch 1].

[4] $p \not| J = (i_n, \cdots, i_1)$ if $p \not| gcd(i_n, \cdots, i_1)$

4.2.2 *Topological Milnor K-group*

Milnor K-group is a fundamental object in higher class field theory since we can construct a reciprocity map $\Psi_F : K_n(F) \to G^{ab}(F)$ for all n-dimensional local field F which is functorial with respect to the maps to the residue field (Theorem 4.2.16). So far it is the unique object (along with topological Milnor K-group which we will define) which has this property.

Let F be an n-dimensional local field. Let λ_m be the finest topology on $K_m(F)$ such that subtraction in $K_m(F)$ and the natural symbol map

$$\phi_F : (F^\times)^m \to K_m(F)$$

are sequentially continuous. Then λ_m and its sequential saturation coincide.

Define $\Lambda_m(F) \subset K_m(F)$ to be an intersection of all open neighbourhoods of 0 with respect to λ_m. Then a constant sequence $\alpha_i = \alpha$ converges to 0 iff $\alpha \in \Lambda_m(F)$. Hence $\Lambda_m(F)$ is a subgroup of $K_m(F)$. We also note that Λ_m is closed.

Definition. *Set*

$$K_m^{top}(F) = K_m(F)/\Lambda_m(F)$$

and endow it with the quotient topology of λ_m which we will denote the same way.

This definition was first given by Parshin ([30])[5] and Fesenko clarified some confusion and proved several properties of this group in [9] and [7]. Let $VK_m(F) = \langle \{V_F\} \cdot K_{m-1}(F) \rangle$ and $VK_m^{top}(F)$ its image of $VK_m(F)$ in $K_m^{top}(F)$. We observe that $\Lambda_m(F) \subset VK_m(F)$ since the topology with $VK_m(F)$ and its shifts as a system of fundamental neighbourhoods satisfies the two conditions of the definition. Thus the definition of $VK_m^{top}(F)$ makes sense and for most results which we prove for $VK_m^{top}(F)$, the same holds for $VK_m(F)$.

We have the following decomposition of $K_m^{top}(F)$ (and similarly for $K_m(F)$.) This result is taken from [13, Ch. 6]

[5]As mentioned before, the topology is different but the intersection of all neighbourhoods of 0 with respect to both topology coincide thus obtaining same topological K-group.

Proposition 4.2.3. *We have*

$$K_m^{top}(F) \cong \mathbb{Z}^{a(m)} \times (\mathbb{F}_q^\times)^{b(m)} \times VK_m^{top}(F)$$

where $a(m)$ corresponds to number of different $(t_{j_1}, \cdots, t_{j_m})$ where $1 \leq j_1 < \cdots < j_m \leq n$ and $b(m)$ corresponds to number of different $(t_{j_2}, \cdots, t_{j_m})$ where $1 \leq j_2 < \cdots < j_m \leq n$ and respectively.

Proof. Recall that $F^\times = t_1^{\mathbb{Z}} \times \cdots \times t_n^{\mathbb{Z}} \times \mathbb{F}_q^\times \times V_F$ where $t_1, \cdots, t_n$ is a sequence of local parameters and V_F is the group of principal units. By noting that $K_n(\mathbb{F}_q) = 0$ for $n > 1$ ([14, Ch 9]) and that V_F is $q-1$ divisible we obtain the decomposition. □

Out of the direct summands in the Proposition, clearly the only object which is hard to study is $VK_m^{top}(F)$. We can find topological generators of $VK_m^{top}(F)$ as we did in F.

Proposition 4.2.4. *An element $x \in VK_m^{top}(F)$ is a convergent sum of symbols*

$$\sum_{\Omega} \{1 + \theta_{i_n,\cdots,i_1} t_n^{i_n} \cdots t_1^{i_1}, t_{j_1}, \cdots, t_{j_{m-1}}\},$$

where Ω is defined as in Lemma 4.2.2, $j_1 < \cdots < j_{m-1}$, and $k = k(i_1, \cdots, i_n) \notin (j_1, \cdots, j_{m-1})$ where $k = min(l : p \not| \, i_l)$.

This result is taken from [7]. The proof is given in the same text but we give more details here.

Proof. We observe that $\{1-\alpha, 1-\beta\} = \{\alpha, 1 + \frac{\alpha\beta}{1-\alpha}\} + \{1-\beta, 1 + \frac{\alpha\beta}{1-\alpha}\}$.

Let $\epsilon \in 1 + p_F^{l_1}$ and $\eta \in 1 + p_F^{l_2}$ Then using the above calculation we can show that $\{\epsilon, \eta\} = \{\epsilon', \eta\} \sum_{i=1}^n \{\rho_i, t_i\}$ where $\rho_i \in 1 + p_F^{l_1}$ and $\epsilon' \in 1 + p_F^{l_1+l_2}$. Using topological convergence we obtain the result except for the last condition. For this we consider the equality

$$1 = \{1 + \theta t_n^{i_n} \cdots t_1^{i_1}, -\theta t_n^{i_n} \cdots t_1^{i_1}\} = \sum_{i=1}^n \{1 + \theta t_n^{i_n} \cdots t_1^{i_1}, t_i\} + \{1 + \theta t_n^{i_n} \cdots t_1^{i_1}, -\theta\}$$

$$= \sum_{i=1}^n \{1 + \theta t_n^{i_n} \cdots t_1^{i_1}, t_i\}$$

since V_F is $q-1$ divisible. Then we may rewrite the factor $i = k$ as $\sum_{i \neq k}\{(1+\theta t_n^{i_n} \cdots t_1^{i_1})^{-1}, t_i\}$. By applying the calculation $(1+k)^{-1} = \prod_{i \geq 1}(1+(-1)^i k^{i^2}$ to $(1+\theta t_n^{i_n} \cdots t_1^{i_1})^{-1}$, we have the result. □

Using the two Propositions above we can see that $K_m^{top}(F) = 0$ if $m > n+1$ and $K_{n+1}^{top}(F) \cong \mathbb{F}_q^{\times}$.

Fesenko in [9] proves that Λ_m has an algebraic description which means that $K_n^{top}(F)$ can be defined without using topology. Parshin did not know this and consequently, he believed that the norm map from the original K-group cannot be extended to the topological K-group. We discuss this issue further at the end of this section.

Theorem 4.2.5. *Let F be an n-dimensional local field. Then*

$$\Lambda_m(F) = \bigcap_{l \geq 1} lK_m(F).$$

One inclusion is straightforward. Since $\cap_{l \geq 1} lK_n(F) \in VK_n(F)$ write $x \in \cap_{l \geq 1} lK_n(F)$ as $\sum_J \{\alpha_J, t_{j_1}, \cdots, t_{j_{n-1}}\}$ mod $\Lambda_n(F)$. So

$$p^r x = \sum \{\alpha_J^{p^r}, t_{j_1}, \cdots, t_{j_{n-1}}\} + y_r$$

where $y_r \in \Lambda_n$. Then by Proposition 4.2.1.4, $p^r x \to y$ where $y \in \Lambda(F)$. By the definition of sequential saturated topology, there exists $k \geq 0$ such that $p^k x \in \Lambda(F)$. There must be the least k with such property and this is necessarily 0 by the definition of the topology on $K_n(F)$.

The other inclusion is slightly tricky to prove and for this we need to do some background work for it. Let k be a field of characteristic p and denote

$$\nu = \nu(k) = H^n(k, \mathbb{Z}/p(n)) = ker(\mathscr{P} : \Omega_k^n \to \Omega_k^n / d\Omega_k^{n-1});$$

$$\mathscr{P} : (a\frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_n}{b_n}) \mapsto (a^p - a)\frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_n}{b_n} + d\Omega_k^{n-1}.$$

Then we have the following result.

Theorem 4.2.6 (Bloch-Kato-Gabber)**.** *The differential symbol*

$$d_k : K_m(k)/p \to \nu(k), \qquad \{a_1, \cdots, a_m\} \to \frac{da_1}{a_1} \wedge \cdots \wedge \frac{da_m}{a_m}$$

is an isomorphism.

We do not give proof of this since the tools used in this proof are too different from the other ingredients which are used in this thesis. Kato's proof can be found in [13, A.2].

We state a result originally stated in [30]. As mentioned earlier, he was using different topology so we present a proof which uses 'correct' topology. The idea of the proof is very similar to the original work of Parshin and was proved by Fesenko in [9].

Theorem 4.2.7. *Let $J = \{j_1, \cdots, j_{m-1}\}$ run over all $(m-1)$ elements subsets of $\{1, \cdots, n\}$ where $m \leq n+1$. Let $\mathcal{E}_J$ be the subgroup of V_F generated by $1 + \theta t_n^{i_n} \cdots t_1^{i_1}$ $(\theta \in \mu_{q-1})$ such that $p \not| \, gcd(i_1, \cdots, i_n)$ and $min\{l : p \not| \, i_l\} \in J$. Then the homomorphism*

$$h : \sum_J \mathcal{E}_J \to VK_m^{top}(F),\ (\epsilon_J) \mapsto \sum_{J=\{j_1,\cdots,j_{m-1}\}} \{\epsilon_J, t_{j_1}, \cdots, t_{j_{m-1}}\}$$

is a homeomorphism.

Again the topology on $\sum_J \mathcal{E}_J$ is given as the sequential saturation of the product topology.

Proof. From Proposition 4.2.4 we obtain that h is a surjective homomorphism. Injectivity follows from Proposition 4.2.11 (this result would be proved without using the algebraic description of Λ_m). There exists a sequentially continuous map, say $f : V_F \times (F^\times)^{m-1} \to \prod_J \mathcal{E}_J$ such that its composition with h coincides with the restriction of the symbol map $\phi_F : (F^\times)^m \to K_m^{top}(F)$. So the topology on $\prod_J \mathcal{E}_J$ is weaker than λ_m by the definition of sequentially saturated topology. We suppose that $h^{-1}(U)$ is not open for some open set $U \in K_n^{top}(F)$. Then there exists a sequence $(\alpha_J^{(i)})$ all outside $h^{-1}(U)$ which converges to $\alpha_J \in h^{-1}(U)$. Then the sequence $\phi_F(\alpha_J^{(i)}) \notin U$ converges to an element in U which is a contradiction. □

Proof of Theorem 4.2.5. By Bloch-Kato-Gabber Theorem, $d_F : K_m(F)/p \to \Omega_F^m$ is injective.

Ω_F^m is a finite dimensional vector space over F/F^p, so the intersection of all neighbourhoods of zero in Ω_F^m with respect to the topology induced by λ is trivial. Since d_F is continuous, injectivity implies $\Lambda_m(F) \subset pK_m(F)$.

So now it remains to prove that $\Lambda_m(F)$ is p-divisible since all elements of $VK_n(F)$ is l-divisible for all l which are coprime to p. By Theorem 4.2.7 we know that $VK_m(F)/\Lambda_m(F) \cong \prod \mathcal{E}_J$ does not have a p-torsion. So we obtain $p\Lambda_m(F) = \Lambda_m(F)$ and the Theorem. □

Definition. *A finite separable extension L/F of n-dimensional local fields is called* ***purely ramified*** *if the degree of the extension of L/F equals the degree of the extension of the final residue fields of L and F. In other words, F has a system of local parameters $t_n, \cdots, t_1$ which is also a system of local parameters of L.*
A finite separable extension L/F is called ***tamely ramified*** *if its degree divided by the degree of the last residue field extensions is relatively prime to p, the characteristic of the final residue field of F.*

We note that the definition of tamely ramified given here is different from the usual definition (for instance the one given in [14]) used for a general complete discrete valuation field. Nonetheless this terminology is standard for the study of higher class field theory.

We conclude this chapter by making some remarks about the Norm maps on topological Milnor K-groups. Norm maps on topological K groups can be defined in two different ways. First way would be to use the norm map induced from the original Milnor K-group. One could gather that the norm map from $K_m(F)$ factorises through the quotient $\Lambda_m(F)$ by considering its algebraic description as in Theorem 4.2.5. Namely, the homomorphic image of a divisible element is still divisible.

The second definition of the norm map is originally due to Parshin. Let L/F be a cyclic extension of prime degree. Then $K_m^{top}(L)$ is generated by symbols of the form $\{a, \alpha_2, \cdots, \alpha_m\}$ where $a \in L^\times$ and $\alpha_i \in F^\times$. For a purely unramified extension this is obvious. For other extensions we use the fact that system of local parameters of L could be chosen so that it is the same as F except for one. Then using similar argument as the last part of the proof of Proposition 4.2.4 we obtain the validity of the generators given above. So we can define $N'_{L/F} : K_m^{top}(L) \to K_m^{top}(F)$ by defining $N'_{L/F}\{N_{L/F}a, \alpha_2, \cdots, \alpha_m\} = \{a, \alpha_2, \cdots, \alpha_m\}$ for the generators given above and extend linearly. For a general extension we find a tower of cyclic extensions and take the compositions of these norm maps for each cyclic extensions. To prove the well-definedness, Parshin uses Proposition 4.2.13 of this thesis but the simpler way of deducing this would be to compare the image of $N_{L/F}\{a, \alpha_2, \cdots, \alpha_m\}$ and $N'_{L/F}\{a, \alpha_2, \cdots, \alpha_m\}$. By Theorem 2.3.4, they are equal when L/F is a cyclic extension of prime degree. Recall that the norm map in Milnor K-group is well-defined and they are transitive. Thus, we obtain that it is also true for $N'_{L/F}$ and they are in fact equal.

4.2.3 *Pairing*

We need to make use of several pairings to study the structure of K^{top}-groups. Maximal abelian extension of a higher local field F is compositum of three extensions of F. Namely the maximal purely unramified extension F^{pur}, the maximal totally tamely ramified extension F^{tr}, and a maximal p-extension $F^{(p)}$. We define three maps/pairings which correspond to these extensions.

In this chapter we will always assume that F is a n-dimensional local field with the last residue field $\mathbb{F}_q$.

Definition of pairings

Let $\delta : K_m(F) \to K_{m-1}(\overline{F})$ be the boundary map for Milnor K-group. This map factors through the quotient $\Lambda_m(F)$ by Theorem 4.2.5 so we shall denote the boundary map in the same way for K^{top}-groups. We define

$$v_F : K_n^{top}(F) \to \mathbb{Z}$$

by $v_F(x) = \delta^n(x)$ where this means δ composed n times. This map takes the role of the usual valuation map in the case of one dimensional local field.

The higher tame symbol $T_F : K_{n+1}^{top}(F) \to \mathbb{F}_q^\times$ is defined as $T_F(x) = \delta^n(x)$. We note that the tame symbol can be defined explicitly and this can be found in [13, Ch 6.4]. Let $m > 0$ and l be a positive integer coprime to p. We recall that the primitive lth root of unity is in F if and only if $l|q-1$ so for such l we can define a map $K_{n+1}^{top}(F) \to \mathbb{Z}/l$ by $x \mapsto (T_F(x))^{(q-1)/l}$. Then this gives the Kummer pairing which would be related to $\mathbb{F}_q((\sqrt[q-1]{t_1}))\cdots((\sqrt[q-1]{t_n}))$ which is the maximal tamely ramified extension which is linearly disjoint to F^{pur}.

Proposition 4.2.8. *The map above defines a continuous and non-degenerate pairing*

$$(\cdot,\cdot)_F : K_n^{top}(F)/l \times F^\times/l \to \mathbb{Z}/l.$$

Proof. We start by analysing the generators of $F^\times/l$ and $K_n^{top}(F)/l$. The generators of $F^\times/l$ are $t_1, \cdots, t_n$ and θ where θ is a generator of $\mathbb{F}_q^\times$ as the cyclic group. Since VK_n^{top} is l-divisible, we can see that $K_n^{top}(F)/l$ is generated by $\{t_1, \cdots, t_n\}$ and $\{\theta, t_1, \cdots, \hat{t}_i, \cdots, t_n\}$ for each i. One can check that they are linearly independent by looking at their values under δ, and their compositions up to n times. Thus we obtain that the pairing is non-degenerate. $\square$

We note that this result can easily be generalised for

$$K_m^{top}(F)/l \times K_{n+1-m}^{top}(F)/l.$$

Proposition 4.2.9. *Let L/F be a finite Galois extension. Then $(N_{L/F}(x), y)_F = (x, y)_L$ for $x \in K_n^{top}(L), y \in F^\times$ and $(x, N_{L/F}(y))_F = (x, y)_L$ for $x \in K_n^{top}(F), y \in L^\times$.*

Proof. The Proposition follows immediately if L and F has the same last residue fields, by considering the commutative diagram in Ch 2.3.2. So it suffices to prove result when L/F is purely unramified. Suppose that $[L : F] = n$ and let $\alpha = T_L(x, y)$. Then, again using this diagram, we see that the LHS of the relation is equal to $(N_{\mathbb{F}_q^n/\mathbb{F}_q}\alpha)^{\frac{q-1}{l}}$. On the other hand the RHS of the relation is $\alpha^{\frac{q^n-1}{l}}$ and by the explicit description of norm map of finite fields we have the result. $\square$

Now we define the Artin-Schreier-Witt pairing : $K_n^{top}(F) \times B(F) \to \mathbb{Q}/\mathbb{Z}$ which would enable us to define the p-part reciprocity map $\Psi^{(p)}$. This pairing generalises the pairing $(\cdot, \cdot]_n$ that was used in Ch 4.1. We now use the bold notation $\mathbf{y}$ and $(\mathbf{y_0}, \mathbf{y_1}, \cdots, \mathbf{y_n})$ for Witt vectors to avoid confusion.

Let $W(F)$ be ring of Witt vectors over F and for each $(\mathbf{y_0}, \cdots, \mathbf{y_n}, \cdots)$ in $W(F)$, let $\mathbf{y}^{(\mathbf{i})}$ denote its i-th ghost component. We also recall that $W_m(F) = W(F)/V^m W(F)$ where $V : W_m(F) \to W_m(F)$ is the 'shift' map.

Then we can define the pairing $(\cdot, \cdot]_m : (F^\times)^n \times W_m(F) \to \mathbb{Z}/p^m$ by

$$(x_0, x_1, \cdots, x_n, \mathbf{y_0}, \cdots, \mathbf{y_{m-1}}]_m = \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\mathbf{w_0}, \cdots, \mathbf{w_{m-1}})$$

where the i-th ghost component $\mathbf{w^{(i)}}$ is given by $\mathrm{res}_F(\mathbf{y^{(i)}}x_1^{-1}dx_1 \wedge \cdots \wedge x_n^{-1}dx_n)$. For the details of the definition of residue map on the differential forms, see [22]. When $n = 1$ this is the pairing defined and used in Ch 4.1. Similarly to the usual local field case, we need to do most of the calculations in characteristic 0 fields and for this we use $\tilde{k}((t_1))((t_2))\cdots((t_n))$ where $\tilde{k}$ is the unramified extension of $\mathbb{Q}_p$ of degree l if $q = p^l$.

Then we have the following result (for brevity we denote the argument $x_1, \cdots, x_n$ by x and $\boldsymbol{y_0, \cdots, y_{m-1}}$ by $\boldsymbol{y}$):

Proposition 4.2.10. *The symbol* $(x_1, \cdots, x_n, \mathbf{y_0}, \cdots, \mathbf{y_{m-1}}]_m$ *is independent of the choice of the local parameters of the n-dimensional local field F and has the following properties:*

1. $(x_1, \cdots, x_i x_i', \cdots, x_n, \mathbf{y}]_m = (x_1, \cdots, x_i, \cdots, x_n, \mathbf{y}]_m + (x_1, \cdots, x_i', \cdots, x_n, \mathbf{y}]_m$
2. $(x, \mathbf{y_0}, \cdots, \mathbf{y_i} + \mathbf{y_i'}, \cdots, \mathbf{y_{m-1}}]_m$
$= (x, \mathbf{y_0}, \cdots, \mathbf{y_i}, \cdots, \mathbf{y_{m-1}}]_m + (x, \mathbf{y_0}, \cdots, \mathbf{y_i'}, \cdots, \mathbf{y_{m-1}}]_m$
3. $(x_1, \cdots, x_n, \mathbf{y}]_m = 0$ *if there exists* $i < j$ *such that* $x_i = 1 - x_j$
4. $(x_1, \cdots, x_n, \mathbf{y}]_m = 0$ *if* $x_i \in (F^\times)^{p^m}$ *for some* i
5. $(x, \mathbf{y_0}^p, \cdots, \mathbf{y_{m-1}}^p]_m = (\mathbf{w_0}^p, \cdots, \mathbf{w_{m-1}}^p)$, *if* $(x, \mathbf{y_0}, \cdots, \mathbf{y_{m-1}}]_m$
$= (\mathbf{w_0}, \cdots, \mathbf{w_{m-1}})$.
6. *The pairing is continuous in each argument.*
7. $(x, \mathbf{0}, \mathbf{y_1}, \cdots, \mathbf{y_{m-1}}]_m = (0, (x, \mathbf{y_1}, \cdots, \mathbf{y_{m-1}}]_{m-1})$
8. $(x, \mathbf{y_0}, \cdots, \mathbf{y_{m-2}}]_{m-1} = (\mathbf{w_0}, \cdots, \mathbf{w_{m-1}})$ *if* $(x, \mathbf{y_0}, \cdots, \mathbf{y_{m-1}}]_m = (\mathbf{w_0}, \cdots, \mathbf{w_m})$

Proof. The residue is independent of the choice of $t_1, \cdots, t_n$ and thus so is the pairing.

Properties 1, 2, 3 are obvious. One simply needs to use that $d(x_i x_i') = x_i' dx_i + x_i dx_i'$. Property 7 and 8 follows from definition of ghost component of Witt vectors. We obtain property 4 by considering the calculation $res(\mathbf{y_i}\frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_{i-1}}{x_{i-1}} \wedge \frac{dx_i^{p^m}}{x_i^{p^m}} \wedge \frac{dx_{i+1}}{x_{i+1}} \wedge \cdots \wedge \frac{dx_n}{x_n}) = p^m res(\mathbf{y_i}\frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n})$.

Property 5 can be proved via induction. In [29], Parshin proved the result in the case when $n = 2$ and $m = 1$ and this can be generalised to higher

n. By property 8, we immediately obtain the result for higher m as well by induction.

Continuity of $(\cdot,\cdot]_m$ follows from the fact that this map is essentially a polynomial map with coefficients which comes from x_i. $\square$

Now we can prove the following:

Proposition 4.2.11. *Let F be an n-dimensional local field. Then the symbol*

$$(x_0, x_1, \cdots, x_n, (\mathbf{y_0}, \cdots, \mathbf{y_{m-1}})]_m$$

defines a continuous and non-degenerate pairing

$$(\cdot,\cdot]_m : K_n^{top}(F)/p^m \times W_m(F)/\mathscr{P}W_m(F) \to \mathbb{Z}/p^m.$$

Proof. It remains to prove the non-degeneracy of $(\cdot,\cdot]_m$. We first prove that the pairing is non-degenerate for $m = 1$. To do this we calculate the image under the pairing on the generators of the group $K_n^{top}(F)/p$ and $F/\mathscr{P}F$. The generators of $K_n^{top}(F)/p$ are symbols $\{t_1, \cdots, t_n\}$ and $\{1 + \theta t_1^{i_1} \cdots t_n^{i_n}, t_1, \cdots, \hat{t}_k, \cdots, t_n\}$ (where the latter element is as in Proposition 4.2.4) and the generators of $F/\mathscr{P}F$ are $b_0 \in \mathbb{F}_q$ such that $Tr_{\mathbb{F}_q/\mathbb{F}_p}(b_0) = 1$, and $\theta' t_1^{i'_1} \cdots t_n^{i'_n}$ where $(i'_1, \cdots, i'_n) < 0$ and $gcd(i'_1, \cdots, i'_n) \not| \, p$. We may also assume that θ, θ' which are the bases of $\mathbb{F}_q$ over $\mathbb{F}_p$, are dual to each other with respect to the bilinear form $Tr_{\mathbb{F}_q/\mathbb{F}_p}$ on $\mathbb{F}_q$.

Then we obtain that

$$(\{t_1, \cdots, t_n\}, b_0]_1 = 1,$$

$$(\{t_1, \cdots, t_n\}, bt_1^{b1} \cdots t_n^{b_n}]_1 = 0 = (\{1 + at_1^{a_1} \cdots t_n^{a_n}, t_1, \cdots, \hat{t}_k, \cdots, t_n\}, b_0]_1$$

and

$$(\{1 + at_1^{a_1} \cdots t_n^{a_n}, t_1, \cdots, \hat{t}_k, \cdots, t_n\}, bt_1^{b1} \cdots t_n^{b_n}]_1 = 0$$

unless $(a_1, \cdots, a_n) = (-b_1, \cdots - b_n)$. If $(a_1, \cdots, a_n) = (-b_1, \cdots - b_n)$, then by simple calculation of exterior products we obtain that the value of the pairing is $Tr_{\mathbb{F}_q/\mathbb{F}_p}(aba_k)$. This is non-zero for $a \neq b$ but zero otherwise. Thus the pairing is non-degenerate.

Now let $m > 1$ and suppose that

$$(x, (\mathbf{y_0}, \cdots, \mathbf{y_{m-1}})]_m = 0$$

for any $(\mathbf{y_0}, \cdots, \mathbf{y_{m-1}}) \in W_m(F)$

Let

$$(x, (\mathbf{y_0}, \cdots, \mathbf{y_{m-1}})]_m = 0$$

for any $(\mathbf{y_0}, \cdots, \mathbf{y_{m-1}}) \in W_m(F)$. Then by property 8 of Proposition 4.2.10 and non-degeneracy at $m = 1$ implies that $x = px'$. Then

$$0 = (x, \mathbf{y_0}, \cdots, \mathbf{y_{m-1}}]_m = p(x', \mathbf{y_0}, \cdots, \mathbf{y_{m-1}}]_m = (0, (\mathbf{w_0}^p, \cdots, \mathbf{w_{m-2}}^p)]_{m-1}$$

if $(x', \mathbf{y_0}, \cdots, \mathbf{y_{m-1}}]_m = (\mathbf{w_0}^p, \cdots, \mathbf{w_{m-1}}^p)]_m$. Thus by induction we obtain that $x \in p^m K_n^{top}(F)$.

Now suppose that

$$(x, (\mathbf{y_0}, \cdots, \mathbf{y_{m-1}})]_m = 0$$

for all $x \in K_n^{top}(F)$. By induction principle we can assume that $(\mathbf{y_0}, \cdots, \mathbf{y_{m-2}}) \in \mathscr{P}W_{m-1}(F)$ and $\mathbf{y_m} \in F/\mathscr{P}(F)$. Thus we obtain that $\mathbf{y} \in \mathscr{P}W_m(F)$. $\square$

Now we consider the inductive limit $B(F)$ of the groups $W_m(F)/\mathscr{P}W_m(F)$ with respect to V which sends $(\mathbf{y_0}, \cdots, \mathbf{y_{m-1}})$ to $(0, \mathbf{y_0}, \cdots, \mathbf{y_{m-1}})$.

Kawada and Satake in [20] defines a pairing $F^\times \times B(F) \to G^{ab,p}$ for all characteristic p fields induced by isomorphism

$$G^{ab,p} \cong B(F)^\wedge = Hom_{cts}(B(F), \mathbb{Q}/\mathbb{Z}) := E(F).$$

Hence we obtain p-part reciprocity map $\Psi_F^{(p)} : K_n^{top}(F) \to G^{ab,p}(F)$ as well as $\Psi_{L/F} : K_n^{top}(F) \to G(L/F)^{ab}$ for any Galois p-extension L/F. $\Psi_{L/F}$ is defined by composing Ψ_F with the natural restriction map $G(F^{ab}/F) \to G(L/F)^{ab}$. Now we study the properties of $\Psi_F^{(p)}$.

4.2.4 *Study of p-extension*

The aim of this section is to prove this result:

Theorem 4.2.12. *Let L/F be a cyclic extension of degree p^m. Then we have exact sequence*

$$K_n^{top}(L) \xrightarrow{N} K_n^{top}(F) \xrightarrow{\Psi_{L/F}} G(L/F) \to 1$$

For the proof of this Theorem, the following result is crucial.

Proposition 4.2.13. *Let L/F be a finite Galois extension. Then the following relations are valid:*
1. $(N_{L/F}(x), \mathbf{y}]_{F,m} = (x, j_{F/L}\mathbf{y}]_{L,m}$ *for* $x \in K_n^{top}(L), \mathbf{y} \in W_m(F)$,
2. $(x, Tr(\mathbf{y})]_{L,m} = (x, \mathbf{y}]_{F,m}$ *for* $x \in K_n^{top}(F), \mathbf{y} \in W_m(L)$.

Proof of Theorem 4.2.12. Consider the exact sequence

$$ker(i) \to W_m(F)/\mathscr{P}W_m(F) \xrightarrow{i} W_m(L)/\mathscr{P}W_m(L)$$

where i is induced by the injection $W_m(F) \to W_m(L)$.

Take the dual exact sequence from Proposition 4.2.11 and use results from [19] to obtain

$$K_n^{top}(L) \xrightarrow{j_1} K_n^{top}(F) \xrightarrow{j_2} G(L/F).$$

We note that Pontryagin duality can be valid since $ker(i)$ is a finite group so the quotient group $W_m(F)/\mathscr{P}W_m(F) + ker(i)$ is a closed subgroup of $W_m(L)/\mathscr{P}W_m(L)$ by [2, Ch 3].

Then by Proposition 4.2.13 we obtain that $j_1 = N_{L/F}$ and $j_2 = \Psi_{L/F}$. To put it more precisely we have the commutative diagram below and via diagram chasing we obtain that $j_1(x) = N_{L/F}(x)$ for all $x \in K_n^{top}(L)$. Similar calculation shows that $j_2 = \Psi_{L/F}$. □

The rest of section will be dedicated for the proof of Proposition 4.2.13.

Lemma 4.2.14. *Let L/F be finite separable extension of an n-dimensional local field, l be the last residue fields of L and $\omega \in \Omega^n_{L/l}$. Then*

$$Tr_{l/\mathbb{F}_q}(res_L(\omega)) = res_F(Tr\omega)$$

where Tr : $\Omega^n_{L/l} \to \Omega^n_{F/\mathbb{F}_q}$ *is the trace.*

Proof. The proof of this can be found in [22] and also in the original paper of Parshin ([30]). It suffices to prove this result assuming L/F is a cyclic extension of a prime degree.

Note that we already used this Lemma in the case of $n = 1$ when proving Proposition 4.1.20. Since the residue fields $\overline{L}$ and $\overline{F}$ are subsets of L and F respectively, we can apply induction to prove this lemma in general L/F is a separable extension. We shall leave out the case when L/F is purely inseparable but note that this can be done by explicit calculation. $\square$

Using this we obtain the following:

Lemma 4.2.15. *Let L/F be a finite separable extension. Then the diagram*

$$\begin{tikzcd} K_2(L) \arrow[r, "\zeta"] \arrow[d, "N_{L/F}"'] & \Omega^2_L \arrow[d, "Tr"] \\ K_2(F) \arrow[r, "\zeta"] & \Omega^2_F \end{tikzcd}$$

is commutative where Tate map $\zeta : K_n(F) \to \Omega^n_F$ is defined by

$$\zeta(\{x_1, \cdots, x_n\}) = \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n} \in \Omega^n_F.$$

We note that this result is the reason why we needed to prove the norm map on the Topological K-group coincides with the norm map on the original Milnor K-group.

Proof. Let M/F be a finite separable normal extension containing L. Then one could prove that for any $y \in K_2(L)$,

$$\sum_{\sigma \in Hom_F(L,M)} i_{L/M}(y)^{\sigma'} = i_{F/M}(N_{L/F}(y))$$

where $\sigma' \in G(M/F)$ is any continuation of σ to M.

We write $j_{F/L}$ for the natural embedding $\Omega^2_F \to \Omega^2_L$ and similarly for other fields. Then $\zeta(\sum i_{L/M}(y)^{\sigma'}) = \sum_\sigma \zeta \circ i_{L/M}(y)^{\sigma'} = \sum_\sigma j_{L/M}(\zeta(y)^{\sigma'}) =$

$j_{L/M} \circ j_{F/M}(Tr(\zeta(y)))$. But this is also equal to $\zeta \circ (i_{F/M}(N_{L/F}y)) = j_{L/M} \circ \zeta \circ i_{F/L}(N_{L/F}y) = j_{F/M} \circ \zeta(N_{L/F}y)$. Since $j_{F/M}$ is an embedding we obtain the diagram.

$\square$

We recall that $\mathrm{Tr}(y\omega) = y\mathrm{Tr}(\omega)$ for $y \in F$ and $\omega \in \Omega^n_{L/F}$. Now we have obtained all the ingredients to prove Proposition 4.2.13.

Proof of Proposition 4.2.13. We start by proving that it suffices to show the result in the case of cyclic extensions of prime degree. We suppose that the equality is true for the extensions L/M and M/F. Then

$$(N_{L/F}x, \mathbf{y}]_{F,m} = (N_{L/M}x, i_{F/M}\mathbf{y}]_{M,m} = (x, i_{F/L}y]_{L,m}.$$

We first prove both relations when $m = 1$. The first relation is proved by the following

$(N_{L/F}(x), y]_{F,1} = Tr_{\mathbb{F}_q/\mathbb{F}_p} res_F(y \cdot \zeta(N_{L/F}(x))) = Tr_{\mathbb{F}_q/\mathbb{F}_p} res_F(Tr(y(\zeta(x)))$
$= Tr_{l/\mathbb{F}_p} res_F(y(\zeta(x))) = (x, y]_{L,1}$.

The second relation has been proved for one dimensional local field in Proposition 4.1.20 and this can be easily generalised.

For higher m, both relations follow by induction using 7 and 8 of Proposition 4.2.10.

$\square$

4.2.5 *Higher Local Class Field Theory in Positive Characteristic*

Theorem 4.2.16. *Let F be an n-dimensional local field of positive characteristic. Then there exists a canonical map $\Psi_F : K_n^{top}(F) \to G(F^{ab}/F)$ such that*
1. *For any finite abelian extension L/F the sequence*

$$K_n^{top}(L) \overset{N_{L/F}}{\to} K_n^{top}(F) \overset{\Psi_{L/F}}{\to} G(L/F) \to 1$$

is exact.

2. *For any finite separable extension L/F the natural diagram*

$$\begin{tikzcd} K_n^{top}(L) \arrow[r, "\Psi_L"] \arrow[d, "N_{L/F}"'] & G(L^{ab}/L) \arrow[d] \\ K_n^{top}(F) \arrow[r, "\Psi_F"] & G(F^{ab}/F) \end{tikzcd}$$

commutes.
3. *The diagram*

$$\begin{tikzcd} K_n^{top}(F) \arrow[r, "\Psi_F"] \arrow[d, "\delta"'] & G(F^{ab}/F) \arrow[d] \\ K_{n-1}^{top}(\overline{F}) \arrow[r, "\Psi_{\overline{F}}"] & G(\overline{F}^{ab}/\overline{F}) \end{tikzcd}$$

is commutative.
4. *Ψ_F is injective and its image is dense in $G(F^{ab}/F)$.*

Proof. Let F be a local field of dimension n. Then $F \cong \mathbb{F}_q((t_1))\cdots((t_n))$ where $t_1, \cdots, t_n$ are sequence of local parameters. We consider three subfields of F. Let $L_1 = F \cdot \mathbb{F}_q^{sep}$, $L_2 = F(t_1^{1/(q-1)}, \cdots, t_n^{1/(q-1)})$, and L_3 be the maximal abelian p-extension of F. Let G_1, G_2, and G_3 be the corresponding Galois groups for these fields. Clearly $G_1 \cong \hat{\mathbb{Z}}$ and the Frobenius automorphism *Frob* is the canonical topological generator of G_1. Then the reciprocity map $\Psi_1 : K_n^{top}(F) \to G_1$ is defined by $(x_1, \cdots, x_n) \mapsto Frob^{v_F(x_1,\cdots,x_n)}$.

Reciprocity maps in the latter two cases have been defined earlier and let $\Psi_2 : K_n^{top}(F) \to G_2$ and $\Psi_3 : K_n^{top}(F) \to G_3$ be those maps.

Since elements of G_1, G_2 and G_3 generate whole of $G^{ab} = G(F^{sep}/F)^{ab}$, we can merge the reciprocity maps on these groups to obtain $\Psi_F : K_n(F) \to G^{ab}$ if each reciprocity maps coincide on the intersections. We observe that $L_1 \cap L_2 = L_2 \cap L_3 = F$ and $L_1 \cap L_3$ is the maximal purely unramified p-extension of F. We have that $\Psi_1\{t_1, \cdots, t_n\} = Frob|_{G_1 \cap G_3}$. On the other hand $\Psi_3(t_1, \cdots, t_n) = f \circ \varinjlim_m (t_1, \cdots, t_n, \cdot]_m$ where f is the isomorphism $B(F)^\wedge \cong G^{ab,p}(F)$.[6] We note that $(t_1, \cdots, t_n, \mathbf{x}_0, \mathbf{x}_1, \cdots \mathbf{x}_{m-1}]_m = Tr_{\mathbb{F}_q/\mathbb{F}_p}(res(\mathbf{x}_0), \cdots, res(\mathbf{x}_{m-1}))$. Thus $\Psi_3(t_1, \cdots, t_n)$ coincides with the Frobenius, if we restrict it to $G_1 \cap G_3$. So we obtain that $\Psi_F : K_n^{top}(F) \to G(F^{ab}/F)$. The non-degeneracy of the pairings gives 4.

We now prove the first property of the Theorem. It suffices to prove that the sequence is exact for all cyclic extensions. We suppose that we have exact sequences

$$K_n^{top}(L_1) \overset{N_{L_1/F}}{\to} K_n^{top}(F) \overset{\Psi_F}{\to} G(L_1/F) \to 1$$

$$K_n^{top}(L_2) \overset{N_{L_2/F}}{\to} K_n^{top}(F) \overset{\Psi_F}{\to} G(L_2/F) \to 1$$

where L_1 and L_2 are abelian extension of F such that $L_1 \cap L_2 = F$. Let $L = L1 \cdot L_2$. Then $N_{L_1/F}K_n^{top}(L_1) \cap N_{L_2/F}K_n^{top}(L_2) = N_{L/F}K_n^{top}(L)$ and $G(L/F) = G(L_1/F) \times G(L_2/F)$ by considering them as subgroups of $G(F^{ab}/F)$. So $ker(\Psi_{L/F}) = N_{L/F}K_n^{top}(L)$ and $\Psi_{L/F}$ is surjective.

The exactness of property 1 for purely unramified extensions are obvious and for cyclic p-extensions we have Theorem 4.2.12. So it remains to prove this result for cyclic Kummer extensions. Consider the exact sequence

$$(L^\times)^l \cap F^\times/(F^\times)^l = ker(i) \to F^\times/(F^\times)^l \overset{i}{\to} L^\times/(L^\times)^l.$$

If we take the dual sequence with respect to the pairing in Proposition 4.2.8 we obtain

$$K_n(L) \to K_n(F) \to G(L/F)$$

[6]This is the inverse of the isomorphism $F_F^* : G^{ab,p} \to B(F)^\wedge$ defined in Ch 4.1.

and by Proposition 4.2.9 we obtain that the maps in the sequence is the one required. The same arguments yield property 2.

To prove property 3 we consider an abelian unramified extension L/F and verify the commutativity of the diagram

$$\begin{tikzcd} K_n^{top}(F) \arrow[r, "\Psi_F"] \arrow[d, "\delta"'] & G(L/F) \arrow[d] \\ K_{n-1}^{top}(\overline{F}) \arrow[r, "\Psi_{\overline{F}}"] & G(\overline{L}/\overline{F}). \end{tikzcd}$$

Viewing $\overline{F}$ as a subfield of F and choosing a sequence of local parameters compatibly, we see that Kummer extensions are generated by $\sqrt[l]{a}$, $a \in F^{\times}$, and Artin-Schreir extensions are generated by an element x with $x^p - x = b \in \overline{F}$. The diagram can easily be calculated if we use the explicit form of the duality for extensions L_α such that a or b is a "generator" of the group $\overline{F}^{\times}/l$ or $\overline{F}/\mathscr{P}\overline{F}$, respectively. Since $\cap_\alpha ker[G(L/F) \to G(L_\alpha/F)] = 1$, if L is the maximal abelian extension, we obtain the required property. $\square$

BIBLIOGRAPHY

[1] E. Artin and J.Tate. *Class Field Theory*. W. A. Benjamin, Inc, 1968.

[2] N. Bourbaki. *General Topology: Chapters 1-4*. Elements of mathematics. Springer, 1998.

[3] Kenneth S. Brown. *Cohomology of groups*, volume 87 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1982.

[4] J. W. S. Cassels and A. Frohlich. *Algebraic Number Theory*. London Mathematical Society, 2010.

[5] I. B. Fesenko. Multidimensional local class field theory. *Acad. Scienc. Dokl. Math. 43*, pages 1103–1126, 1991.

[6] I. B. Fesenko. Class field theory of multidimensional local fields of characteristic 0, with the residue field of positive characteristic. *St. Petersburg Math. J. 3*, pages 165–196, 1992.

[7] I. B. Fesenko. Multidimensional local class field theory II. *St. Petersburg Math. J. 3*, pages 674–677, 1992.

[8] I. B. Fesenko. Noncommutative (nonabelian) local reciprocity maps. *In Class Field Theory - Its Centenary and Prospects, Advanced Studies in Pure Math*, 30:63–78, 2001.

[9] I. B. Fesenko. Sequential topologies and quotients of Milnor K-groups of higher local fields. *Algebra i Analiz 13*, pages 485–501, 2001.

[10] I. B. Fesenko. On the image of noncommutative reciprocity map. *Homology, Homotopy and Applications*, 5:53–62, 2005.

[11] I. B. Fesenko. Analysis on arithmetic schemes II. *J. K-theory 5*, pages 437–557, 2010.

[12] I. B. Fesenko. Reciprocity and IUT, 2016. Talk at RIMS workshop on IUT Summit.

[13] I. B. Fesenko and M. Kurihara (eds). Invitation to higher local fields. *Geometry and Topology Monographs 3*, 2000.

[14] I. B. Fesenko and S. V. Vostokov. *Local Fields and their Extensions*. American Mathematical Society, 2nd edition, 2002.

[15] G. Harder. An essay on Witt vectors. *Springer Verlag*, 1998.

[16] Helmut Hasse. Theorie der differentiale in algebraischen funktionenkörpern mit vollkommenem konstantenkörper. *Journal für die reine und angewandte Mathematik*, 172:55–64, 1935.

[17] K. İlhan Ikeda and E. Serbest. Generalized Fesenko reciprocity map. *ArXiv e-prints*, 0805.3431.

[18] Kazuya Kato. A generalization of local class field theory by using K-groups, II. *Proc. Japan Acad. Ser. A Math. Sci.*, 54(8):250–255, 1978.

[19] Y. Kawada. Class formations. *Amer.Math.Soc.Transl.(2)*, 166:165–177, 1995.

[20] Y. Kawada and I. Satake. Class formations. II. *J. Math. Sci. Univ. Sect I 7*, pages 453–490, 1956.

[21] S. Lang. *Algebraic Number Theory*. Springer, 2000.

[22] V. G. Lomadze. On residues in algebraic geometry. *Mathematics of the USSR-Izvestiya*, 19(3):495–520, 1982.

[23] F. Lorenz. *Einführung in die Algebra II*. Spektrum akademischer Verlag, 1997.

[24] A. I. Madunts and I. B. Zhukov. Multidimensional complete fields: Topology and other basic construction. *Amer.Math.Soc.Transl.(2)*, 166, 1995.

[25] M. Morrow. An introduction to higher dimensional local fields and adeles. *ArXiv e-prints*, 1204.0586, April 2012.

[26] J. Neukirch. *Class Field Theory*. Springer-Verlag, 1986.

[27] J. Neukirch. *Algebraic Number Theory*. Springer, 1999.

[28] J. Neukirch, Alexander Schmidt, and Kay Wingberg. *Cohomology of Number Fields*. Springer, 1999.

[29] A. N. Parshin. On the arithmetic of two-dimensional schemes. I. distributions and residues. *Mathematics of the USSR-Izvestiya*, 10(4), 1976.

[30] A. N. Parshin. Local class field theory. *Proc. Steklov Inst. Math*, 165:157–185, 1985.

[31] A. N. Parshin. Galois cohomology and the brauer group of local fields. *Proc. Steklov Inst. Math*, 183:191–201, 1991.

[32] H. Stichtenoth. *Algebraic Function Fields and Codes*. Springer, 2008.

[33] K. Syder. Two-dimensional local-global class field theory in positive characteristic. *PhD Thesis, Nottingham University*, 2014.

[34] E. Witt. Zyklische körper und algebren der charakteristik p vom grad p^n. *J. Reine Angew. Math.*, 176:126–140, 1936.

[35] Seok Ho Yoon. Neukirch's Approach to the Reciprocity Map and Global Class Field theory in Positive Characteristic. *ArXiv e-prints*, 1610.07486, October 2016.