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New perspectives in gravity beyond General Relativity: from fundamental physics to strongly gravitating systems

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To my parents

for all their support

Abstract

General Relativity (GR) has been extremely successful experimentally. However there are several reasons to consider that GR is not the complete theory of gravity. Although directly probing quantum gravity in the near future seems highly unlikely observations of the non-perturbative regime of classical gravity is improving, so any classical modification of gravity may well be testable.

In this thesis we will focus on two interesting depatures from GR inspired by quantum gravity: violations of Lorentz symmetry and the weak equivalence principle.

Lorentz violating (LV) gravity theories are interesting for various reasons. For example: they may have improved UV behaviour as quantum theories. Additionally testing principles which are fundamental to our understanding of nature is extremely important.

We demonstrate how the causality of certain types of LV theories is realised. We then show how the relationship between two different LV theories could be used to find whether or not black holes can truly form from collapse, at least in spherical symmetry.

One may be worried about whether or not LV in gravity can spoil the Lorentz invariance of matter. We generalise some known results about this problem in Hořava gravity and then consider the viability of one proposal to correct this problem.

Weak equivalence principle violations alter the structure of matter. By developing a simple model we demonstrate that one can maintain the weak equivalence principle in the solar system, while breaking it in high-curvature regimes. We go on to demonstrate the efficacy of the mechanism.

This thesis is largely aimed at tackling questions related to the strong field regime in alternative theories of gravity: a topic of increasing interest.

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Conventions

What follows is a “cheat sheet" detailing the conventions used throughout this text.

The metric signature is chosen to be $(-,+,+,+)$.

Spacetime indices will be denoted with Greek letters.

Indices of objects restricted to spacelike hypersurfaces will be denoted with Latin letters.

The spacetime metric will be written as $g_{\mu\nu}$. $\Gamma^{\alpha}_{\mu\nu}$ will be the corresponding Christoffel symbols, and ∇_{μ} the corresponding covariant derivative.

The Riemann tensor is defined such that $\nabla_{\alpha}\nabla_{\beta}\omega_{\mu} - \nabla_{\beta}\nabla_{\alpha}\omega_{\mu} = R^{\nu}_{\alpha\beta\mu}\omega_{\nu}$.

The Ricci tensor is defined as $R_{\alpha\beta} = R^{\mu}_{\alpha\mu\beta}$.

The projector onto a spacelike hypersurface will be written as $p_{\alpha\beta}$.

The induced metric on a spacelike hypersurface (the restriction of the above) will be written as γ_{ij}. The Christoffel symbols will be written as Γ^{i}_{jk} and the corresponding covariant derivative will be written D_i.

The Ricci tensor and scalar associated to the above covariant derivative will have the symbol $\mathcal{R}$ When g or γ appear without indices this will refer to the determinants of the respective metric.

In other circumstances objects missing their indices stand for the whole object.

The full stress energy tensor will be written as $T_{\alpha\beta}$.

u, n and $\hat{n}$ will always be unit timelike vectors.

We will denote the combination $c_1 g^{\alpha\mu}g^{\beta\nu} + c_2 g^{\alpha\beta}g^{\mu\nu} + c_3 g^{\alpha\nu}g^{\beta\mu} - c_4 u^{\alpha}u^{\mu}g^{\beta\nu}$ as $Z^{\alpha\mu\beta\nu}$.

Units will be chosen such that $c = 1$ (in the infra-red), though some stand-in may be used where it is illustrative.

Where it is used [anti-](symmetrisation) will be denoted by [square brackets] or (parenthesis) around [anti-](symmetrised) indices.

Publications

The original work in this thesis is largely based on the following works produced during my studies:

- Chapter 3: Coates et al. 2017b, Coates et al. 2016,
- Chapter 4: Bhattacharyya et al. 2016a and Bhattacharyya et al. 2016b,
- Chapter 5: Coates et al. 2017a and Franchini et al. 2017.

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1

INTRODUCTION

Before starting it should be noted that much of this Chapter is based upon the books (Wald 1984; Carroll 2004; Hawking et al. 1973; Weinberg 2013), a set of online notes (Gourgoulhon 2007) and my undergraduate gravitation course (Walet 2013). Wald 1984; Carroll 2004 are used throughout, Hawking et al. 1973 is used only in the discussion on exact solutions, Weinberg 2013 informs much of the quantum gravity section, Gourgoulhon 2007 is the basis of much of the discussion of the $3+1$ and ADM formalism, and Walet 2013 is the inspiration for the following motivation of GR.

As the title suggests this thesis is concerned with alternative theories of gravity, i.e theories of gravity which differ from General Relativity (GR) (Einstein 1915; Einstein 1916). In this chapter there will be a review of GR with a focus on the issues which motivate such modifications. After all GR is incredibly successful from a phenomenological point of view and still has many interesting open problems in and of itself. During this chapter it may be useful to keep in mind two of the reasons that led to the replacement of Newtonian gravity (Newton 1687) with GR, it having been extremely successful itself.

The first of these reasons is that the perihelion precession of Mercury was difficult to explain in Newtonian gravity. Indeed it could be accounted for with the existence of a hypothetical planet, dubbed Vulcan. In particular it was possible that Newtonian gravity was correct, if there was the correct matter content in the solar system. The second, more theoretical, motivation

is that Newtonian gravity does not respect Special Relativity. This means there was some physical principle that was expected “should" be true of physics which was not reflected in the theory of gravity.

1.1 GENERAL RELATIVITY

There are many ways to introduce GR, here I will follow a line of reasoning inspired by the above, and taking some statement of the equivalence principle for granted. The statement which will be most useful is along the lines of

> “Experiments performed in a sufficiently small patch, local to an observer in free fall, behave as in special relativity."

One of the central objects in special relativity is the Minkowski metric, $\eta_{\mu\nu}$. It allows the discussion of causality and its geodesics are the paths of free particles. As one expects gravitational influence to deflect the paths of particles and any metric $\mathbf{g}$ can always be transformed to $\eta_{\mu\nu}$ in a local patch it may be fruitful to go beyond special relativity by allowing a general metric to take the place of $\eta_{\mu\nu}$. A simple way to then satisfy the equivalence principle is to assume that particles which are influenced only by gravity follow the geodesics of this general metric. Of course, now that it is completely free, we need some theory to determine $\mathbf{g}$. To determine a candidate theory we will consider some of the most useful tools in physics, particularly conservation laws. In special relativity (in the usual Cartesian coordinates) we have,

$$\partial_\nu T^{\mu\nu} = 0, \tag{1.1.1}$$

where T is the stress-energy tensor of the matter. Due to our rough statement of the equivalence principle we should demand that equation (1.1.1) be true local to any observer in free-fall. Some differential geometry tells us that this is true if and only if,

$$\nabla_\nu T^{\mu\nu} = 0, \tag{1.1.2}$$

generically for any spacetime in any coordinate system. So, we wish to find some equations which can impose this for us and which will determine a spacetime. If we were to attempt using,

$$g_{\mu\nu} = kT_{\mu\nu}, \tag{1.1.3}$$

we will have a problem immediately. An algebraic equation of this form will not reduce to a Poisson equation in any limit. This is a problem because we expect Newtonian gravity to be recovered in some limit. To see that this will cause a problem consider a rough statement of Newton's first law,

"A FREE OBJECT HAS CONSTANT VELOCITY."

Of course this is a coordinate dependent statement and so we need to find a covariant statement with a similar flavour. Without introducing any background structure an obvious candidate is that particles follow geodesic motion, *i.e.* for a particle with 4-velocity u,

$$u^\mu \nabla_\mu u_\nu = 0. \tag{1.1.4}$$

That is the derivative of the 4-velocity, along the trajectory of the particle, vanishes. From this equation we can see that the gravitational force will have to come from derivatives of $\mathbf{g}$. In particular $\mathbf{g}$ should have some relation to the Newtonian potential in the Newtonian limit. For this reason we must find some tensor built out of $\mathbf{g}$ which is second order in its derivatives and ensures this "conservation" of stress-energy. Differential geometry again provides for us an object which fits the requirements. In four dimensions the unique object is the Einstein tensor (Lovelock 1971),

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R, \tag{1.1.5}$$

which satisfies,

$$\nabla_\nu G^{\mu\nu} = 0 \tag{1.1.6}$$

identically. This leaves us with the Einstein equations,

$$G_{\mu\nu} = \kappa T_{\mu\nu}. \tag{1.1.7}$$

κ is then fixed by calculating the trajectory of a slowly moving test particle in the exterior of a spherically symmetric mass distribution in the linearized limit and matching it to the Newtonian result. That is, we expect that Newtonian gravity should be recovered when special relativistic effects are negligible and gravity is weak. This gives

$$\kappa = \frac{8\pi G}{c^4}, \quad \text{or} \quad 8\pi G, \tag{1.1.8}$$

in the units we will be using for most of this thesis. We could also choose to include a form proportional to $g_{\mu\nu}$, such a term is referred to as a cosmological constant and is important in some contexts. Here we note that it should have no observable impact in the Solar system and so we shall ignore it for now.

1.1.1 *Formulations of GR*

An important aspect of modern physics is using action principles to describe theories. This is required for the standard quantisation of field theories, via path integrals. Also, having an action principle is useful for understanding various aspects of the theory at a glance or with minimal work. Using the fact that the volume element defined by the Lorentzian metric $\mathbf{g}$ is $\sqrt{-g}\mathrm{d}^4x$ and Jacobi's formula for the derivative of a determinant one could guess that the action for vacuum GR is,

$$\frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{-g}\,[R - 2\Lambda]\,, \tag{1.1.9}$$

we shall drop the Λ term from here but it is included for completeness (it is the cosmological constant term). To include matter we take the action of the matter in Minkowski space and make the replacements,

$$\int \mathrm{d}^4x[\mathcal{L}] \to \int \mathrm{d}^4x\sqrt{-g}\,[\mathcal{L}]\,, \quad \eta \to \mathbf{g}, \text{ and } \quad \partial \to \nabla. \tag{1.1.10}$$

From this one can show

$$T_{\mu\nu} = \mathcal{L}g_{\mu\nu} - 2\frac{\delta\mathcal{L}}{\delta g^{\mu\nu}}. \tag{1.1.11}$$

If we wish, as we often do, to predict something that happens in the future we first have to define what we mean by now, or any other instant. Just as in special relativity where there is a relativity of simultaneity due to the Lorentz transformations, the diffeomorphism invariance also gives such an ambiguity, though much greater. If we define an instant as a fixed value of some time coordinate (which must therefore be a good time coordinate everywhere we are interested in) then this is equivalent to the instants of time being level sets of a scalar field, ϕ, with a time-like gradient ($\partial_\mu\phi\partial^\mu\phi$ negative). Thus an instant of time is a 3-D space-like surface in spacetime, though these are highly non-unique. This is not always achievable in a general Lorentzian manifold and so in discussing an initial value formulation of GR we are removing some of its solutions from the outset. However, these solutions often have pathologies not expected to exist in nature. For example, closed-time-like-curves (CTCs) are possible if such a scalar cannot be found. It should be noted that this scalar field need only exist in some region, but that region is the only part of the full solution which will be found using these methods.

Now assume we have space-like hypersurfaces of co-dimension 1 (and hence dimension 3) we can characterise them with a unit normal,

$$n_\mu = -N\partial_\mu\phi, \tag{1.1.12}$$

where,

$$N = \frac{1}{\sqrt{-\partial_\mu\phi\partial^\mu\phi}}. \tag{1.1.13}$$

The projector onto the hypersurface (orthogonal to n) is then,

$$p^\alpha_\beta = \delta^\alpha_\beta + n^\alpha n_\beta. \tag{1.1.14}$$

This induces a metric on the hypersurfaces,

$$\gamma_{ij} = g_{ij}, \tag{1.1.15}$$

where i, j are indices which run from 1 to 3, and x^i are coordinates on the hypersurface. From here we can define an intrinsic and an extrinsic curvature for the hypersurface. The intrinsic curvature is simply the Riemann

tensor associated to γ_{ij} (in the usual way using the Levi-Civita connection associated to γ_{ij}), which in 3D can be given in terms of only the Ricci tensor (and the metric),

$$\mathcal{R}_{ijmn} = \mathcal{R}_{jn}\gamma_{im} - \mathcal{R}_{in}\gamma_{jm} + \mathcal{R}_{im}\gamma_{jn} - \mathcal{R}_{jm}\gamma_{in} - \frac{1}{2}\mathcal{R}\left(\gamma_{im}\gamma_{jn} - \gamma_{jm}\gamma_{in}\right). \tag{1.1.16}$$

The extrinsic curvature is defined as,

$$K_{\mu\nu} = p^{\alpha}_{\mu}\nabla_{\alpha}n_{\nu}, \tag{1.1.17}$$

i.e. (roughly) the derivative of the normal vector as you move around the surface. This gives an intuitive picture for why this is a feature of the embedding of the hypersurface into a higher dimensional manifold. Also note that $K_{[\mu\nu]} = 0$ follows from Frobenius' theorem (see *e.g.* Wald 1984).

We can now also write the intrinsic curvature in terms of quantities on the whole manifold and the extrinsic curvature in terms of quantities intrinsic to the (family of) hypersurfaces. To do so we will need the identity,

$$D_{\mu}T^{\alpha_1\ldots\alpha_r}_{\beta_1\ldots\beta_s} = p^{\alpha_1}_{\rho_1}\ldots p^{\alpha_r}_{\rho_r}p^{\sigma_1}_{\beta_1}\ldots p^{\sigma_r}_{\beta_s}p^{\nu}_{\mu}\nabla_{\nu}T^{\rho_1\ldots\rho_r}_{\sigma_1\ldots\sigma_s}, \tag{1.1.18}$$

where T is the push-forward of a tensor on the family of hypersurfaces and D is the push-forward of the Levi-Civita connection associated to γ_{ij}. Applying equation (1.1.18) to the definition of the Riemann tensor of the hypersurfaces, in terms of derivatives of vectors one arrives at the Gauss relation. This can be written as,

$$\mathcal{R}_{\alpha\beta\mu\nu} = p^{\rho}_{\alpha}p^{\sigma}_{\beta}p^{\gamma}_{\mu}p^{\delta}_{\nu}R_{\rho\sigma\gamma\delta} + K_{\alpha\nu}K_{\beta\mu} - K_{\alpha\mu}K_{\beta\nu}. \tag{1.1.19}$$

If we define $u = Nn$ then one can also show that,

$$K_{\alpha\beta} = -\frac{1}{2N}\mathcal{L}_m p_{\alpha\beta}. \tag{1.1.20}$$

As $n^{\alpha}K_{\alpha\beta} = 0$ the pull-back of this tensor onto the hypersurface will contain all the same information as the tensor. Note that this can also be written in terms of a derivative with respect to the scalar field which labels the hypersurfaces, ϕ,

$$K_{ij} = -\frac{1}{2N}\left[\mathcal{L}_{\partial_\phi}\gamma_{ij} - \mathcal{L}_{\mathbf{N}}\gamma_{ij}\right], \tag{1.1.21}$$

where **N** is an arbitrarily chosen vector in the tangent space of the hypersurface (it is related to the identification of points on nearby leaves of the foliation) and is called the shift. It is important to note that this definition can only be used if one is considering a family of hypersurfaces, else the derivative with respect to ϕ is not defined. Using similar methods as for the Gauss relation one can relate other projections of the spacetime Riemann tensor to $\mathcal{R}$ and K, these relations are called the Codazzi relation and the Ricci equation. Combining these together one can then write,

$$R = \mathcal{R} + K^2 + K_{ij}K^{ij} - \frac{2}{N}\mathcal{L}_m K - \frac{2}{N}D_i D^i N. \tag{1.1.22}$$

Having set up the foliation and using coordinates adapated to that foliation the metric, and its inverse, can be written as,

$$g_{\mu\nu} = \begin{bmatrix} -N^2 + N_i N^i & N_j \\ N_i & \gamma_{ij} \end{bmatrix}, \quad g^{\mu\nu} = \begin{bmatrix} -1/N^2 & N_j/N^2 \\ N_i/N^2 & \gamma^{ij} - N^i N^j/N^2 \end{bmatrix}. \tag{1.1.23}$$

Some simple matrix algebra reveals that,

$$g = -N^2\gamma \tag{1.1.24}$$

and so, combining Eqs. (1.1.22) and (1.1.24) with Eq. (1.1.9) (*i.e.* inserting them into the Einstein-Hilbert action), one arrives at,

$$\frac{1}{16\pi G}\int \mathrm{d}^3x\mathrm{d}tN\sqrt{\gamma}\left[\mathcal{R} + K^2 + K_{ij}K^{ij} - \frac{2}{N}\mathcal{L}_m K - \frac{2}{N}D_i D^i N\right], \tag{1.1.25}$$

where $t = t(\phi)$, *i.e.* a time coordinate adapted to the foliation. Expanding $\mathcal{L}_m K$ and integrating by parts, one arrives at,

$$\frac{1}{16\pi G}\int \mathrm{d}^3x\mathrm{d}tN\sqrt{\gamma}\left[\mathcal{R} - K^2 + K_{ij}K^{ij} + \text{total derivatives}\right], \tag{1.1.26}$$

and we discard the total derivative part. The Einstein equations are then found by varying action (1.1.26) with respect to (N, N^i, γ^{ij}). One can also put the Einstein equations into the form of evolution equations, with the equations from the variation with γ^{ij} giving evolution equations for K_{ij},

and the definition of K_{ij} determining g_{ij}. The other equations act as constraints on initial data. To see this we can reformulate GR in a Hamiltonian description, known as the ADM formalism in this context (Arnowitt et al. 2008).

To rewrite in a Hamiltonian formulation we first need to identify the momenta. Given some coordinate q and velocity $\dot{q}$, of which the Lagrangian L is a functional, the momentum associated with q is,

$$p \equiv \frac{\partial L}{\partial \dot{q}}. \tag{1.1.27}$$

The Hamiltonian is then,

$$H = p\dot{q} - L. \tag{1.1.28}$$

As such the momenta associated to N, N^i, γ_{ij} are,

$$\tilde{\pi} = \frac{\partial \mathcal{L}}{\partial \dot{N}}, \quad \pi_i = \frac{\partial \mathcal{L}}{\partial \dot{N}^i}, \quad \pi^{ij} = \frac{\partial \mathcal{L}}{\partial \dot{\gamma}_{ij}}. \tag{1.1.29}$$

A quick calculation then shows that the only one of these that is non-zero is,

$$\pi^{ij} = \sqrt{\gamma}\left(K\gamma^{ij} - K^{ij}\right). \tag{1.1.30}$$

The Hamiltonian density is then,

$$\mathcal{H} = \dot{\gamma}_{ij}\pi^{ij} - \mathcal{L}. \tag{1.1.31}$$

Rewriting $\dot{\gamma}_{ij}$ by using Eq. (1.1.21), integrating by parts the terms with spatial derivatives of N^i and discarding the boundary term, one arrives at,

$$\mathcal{H} = \sqrt{\gamma}\left[N\mathcal{C}_0\left(\gamma_{ij}, \pi^{ij}\right) - 2N^i\mathcal{C}_i\left(\gamma_{ij}, \pi^{ij}\right)\right], \tag{1.1.32}$$

where,

$$\mathcal{C}_0 = K_{ij}K^{ij} - K^2 - \mathcal{R} = \frac{1}{\gamma}\left[\pi^{ij}\pi_{ij} - \frac{1}{2}\left(\pi^{ij}\gamma_{ij}\right)^2 - \mathcal{R}\right] \tag{1.1.33}$$

$$\mathcal{C}_i = D_j\left(K_i^j - K\delta_i^j\right) \quad = -\gamma_{ik}D_j\left(\frac{\pi^{jk}}{\sqrt{\gamma}}\right), \tag{1.1.34}$$

form the constraint equations,

$$\mathcal{C}_\mu = 0, \tag{1.1.35}$$

which means these equations simply relate initial data. This can be seen from the fact that they are algebraic in the momenta, coordinates and the spatial derivatives thereof, given that they are preserved in time. This is guaranteed by the fact that the Poisson bracket of $\mathcal{C}_\mu$ with $\mathcal{H}$ vanishes.

Next we will look at a useful technique to arrive at the constraint equations and their preservation in time directly from the Lagrangian (see for example Barausse et al. 2011; Jacobson 2011). First however, we can demonstrate quickly that GR has two degrees of freedom (or four Hamiltonian degrees of freedom, two coordinates and their momenta). In principle there could be up to twenty Hamiltonian degrees of freedom, 2×6 from the three metric and its momenta, 2×1 from the lapse and its momentum, and 2×3 from the shift and its momenta. However, we know that the momenta of the lapse and shift vanish, and thus we are down to sixteen. The constraint equations remove another four. At this stage we have twelve remaining but also a large degree of symmetry, the diffeomorphisms. This symmetry is essentially the freedom to pick a coordinate system, so we can introduce four more conditions, usually called coordinate conditions, which can in principle be written as,

$$q^\mu \left(\gamma_{ij}, \pi^{ij}\right) = x^\mu. \tag{1.1.36}$$

Importantly - to truly fix the coordinate system for the entirety of the part of the manifold for which the coordinates are good ones - we must also impose conditions on the time derivative of this relation, *i.e.*,

$$\partial_t q^\mu = \frac{\partial q^\mu}{\partial \gamma_{ij}} \partial_t \gamma_{ij} + \frac{\partial q^\mu}{\partial \pi^{ij}} \partial_t \pi^{ij} = \delta^\mu_0. \tag{1.1.37}$$

This can be turned into a relation between initial data by trading out $\partial_t \gamma_{ij}$ and $\partial_t \pi^{ij}$ using Hamilton's equations. Thus the diffeomorphisms impose eight more relations between the initial data, leaving four and thus showing that there are two physical degrees of freedom.

As previously mentioned this next part will demonstrate a useful technique for extracting the constraint equations from the action, without having to go via the Hamiltonian (again see Barausse et al. 2011; Jacobson 2011).

This technique will be used heavily in 4. To extract the constraint equations we can take the Einstein-Hilbert action [Eq. (1.1.9)] and consider its variation under an infinitesimal diffeomorphism, *i.e.*,

$$x^\mu \to x^\mu + \xi^\mu. \tag{1.1.38}$$

Under such a transformation a tensor,

$$\mathbf{T} \to \mathbf{T} + \mathcal{L}_\xi \mathbf{T}, \tag{1.1.39}$$

in particular,

$$g^{\mu\nu} \to g^{\mu\nu} + \mathcal{L}_\xi g^{\mu\nu} = g^{\mu\nu} - 2\nabla^{(\mu}\xi^{\nu)}. \tag{1.1.40}$$

Therefore,

$$\delta S = -2\int \mathrm{d}^4x\sqrt{-g}\left[\frac{\delta S}{\delta g^{\mu\nu}}\nabla^{(\mu}\xi^{\nu)}\right], \tag{1.1.41}$$

and so, integrating by parts and taking into account the symmetries of the variational derivative,

$$\delta S = 2\int \mathrm{d}^4x\sqrt{-g}\left[\xi^\nu\nabla^\mu\frac{\delta S}{\delta g^{\mu\nu}}\right]. \tag{1.1.42}$$

To extract the constraint equations we need now only use the fact that this must vanish identically and inspect,

$$\nabla^\mu\frac{\delta S}{\delta g^{\mu\nu}} \equiv \nabla^\mu\mathcal{E}_{\mu\nu} = 0. \tag{1.1.43}$$

As this must be identically 0 we see that either $\mathcal{E}_{0\nu} = 0$ identically (which it is not) or else it has less time derivatives than, at least one, other component of $\mathcal{E}$. Indeed by expanding this out into partial derivatives and Christoffel symbols one can immediately see that if all of the equations are solved at one instant, $t = t_0$, then $\mathcal{E}_{0\nu} = 0$ will be preserved in time. These facts are enough to establish that $\mathcal{E}_{0\nu} = 0$ constitute constraint equations and indeed, upon doing a 3+1 decomposition of $\mathcal{E}_{0\nu}$ one finds that,

$$\mathcal{E}_{0\nu} = \mathcal{C}_\nu. \tag{1.1.44}$$

1.1.2 *Exact solutions*

While we have discussed some of the aspects of the theory of GR we have not yet touched on any of its solutions. Here we will endeavour to talk about some exact solutions, their success when applied to reality and the reasons that despite this success one begins to suspect that GR is not the final picture.

Cosmology

I decided to present the cosmological solutions of GR first for several reasons. First of all GR is, in some sense, unreasonably successful in this field. To paraphrase R. M. Wald (from Wald 1984): we have access to data about only a small portion of the Universe and thus to make cosmological predictions we have to insert assumptions that *a priori* are simply prejudices. Yet GR with these assumptions predicts (for example) the cosmic microwave background (CMB) whose power spectrum can be matched incredibly well. Of course, as mentioned earlier, Newtonian gravity could explain the perihelion precession of Mercury, if the necessary matter content was in the Solar system. Here there is a similar situation which leaves room for the possibility of an alternative theory of gravity.

To arrive at these solutions (the Friedmann-Lemaître-Robertson-Walker [FLRW] solutions) we assume that the universe is homogeneous and isotropic. Roughly stated these mean that the universe is the same everywhere and the same in every direction. Of course one can immediately see that this is not true, look away from the page or go almost anywhere else and you can no longer see these words (and what a relief that would be). Therefore it is important to state that this should be true when averaging over sufficiently large regions of the universe and that the scale of these regions has always been small compared to the horizon size.

More formally we say that a spacetime is homogeneous if there is a foliation by spacelike hypersurfaces which has the following property: given

any two points (p, q) on any one leaf Σ_t of the foliation there exists a diffeomorphism which takes $p \to q$ and leaves the form of the metric unchanged, *i.e.* there exists an isometry which takes any point on the hypersurface into any other. The formal statement of isotropy is that there exists a timelike curve (which is the integral curve of some vector field u) such that for each point p on the curve there exists an isometry that takes any vector at p, orthogonal to u, and transforms it into any other vector orthogonal to u, leaving p and u unchanged.

One can show that these two requirements are enough to enforce that the spacetime be an FLRW spacetime, *i.e.* the metric can be written in the form,

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a\left(t\right)^2 \left[\mathrm{d}\psi^2 + \chi\left(k, \psi\right)^2 \mathrm{d}\Omega^2\right] \tag{1.1.45}$$

where, $\mathrm{d}\Omega^2$ is the standard metric on a unit 2-sphere,

$$\chi(k, \psi) = \begin{cases} \sin\psi, & \text{if } k = 1 \\ \psi, & \text{if } k = 0 \\ \sinh\psi, & \text{if } k = -1, \end{cases} \tag{1.1.46}$$

k is a parameter which is 1 if the Σ_t are positively curved, 0 if they are flat and -1 if they are negatively curved. In particular one can see from Eqs. (1.1.45) & (1.1.46) that in each case Σ_t is a 3-sphere, the $3D$ Euclidean plane or a 3-hyperboloid respectively. In the first and last case $a(t)$ can be seen as the radius of curvature of these spaces, which we have to determine from the Einstein equations. Assuming matter has the form of a perfect fluid whose flow follows the congruence of observers for whom space looks isotropic, *i.e.* that,

$$T_{\mu\nu} = \rho u_\mu u_\nu + p\left(g_{\mu\nu} + u_\mu u_\nu\right), \tag{1.1.47}$$

and allowing for a non-zero cosmological constant, Λ, the independent parts of the Einstein equations can be put into the form,

$$3\left(\frac{\dot{a}}{a}\right)^2 = \quad 8\pi G\rho + \Lambda - 3\frac{k}{a^2}, \tag{1.1.48}$$

$$3\frac{\ddot{a}}{a} = -4\pi G\left(\rho + 3p\right) + \Lambda. \tag{1.1.49}$$

It is often useful to define an effective curvature density,

$$\rho_k = \frac{1}{8\pi G}\frac{-3k}{a^2} \tag{1.1.50}$$

Given some equation of state, $p = p(\rho)$, or $p_I = p_I(\rho_I)$ in a multiple component universe where $\rho = \sum_I \rho_I$, one can solve Eqs. (1.1.48) & (1.1.49). Performing perturbation theory upon this background has proved a very successful model of cosmology. In particular it predicts the existence of the CMB and, if one allows for an inflatonary period in the early unvierse, it predicts the power spectrum of the CMB to high accuracy, along with fitting various structure formation studies and other tests. However, to fit these data we need a positive cosmological constant which must be the dominant term on the right of (1.1.48) when evaluated today. We must also have $\rho_k \sim 0$, and ρ (again evaluated today) must be dominated by a type of matter that has not yet been detected in the lab. The first is a problem as quantum field theory predicts that this value should be much larger than it is, and tunings are unstable (see Padilla 2015 for a review) this is a puzzle with no good solution that I know of. The second is a problem as $\rho_k = 0$ is an unstable configuration, similar to balancing a pencil on its tip, it would be remarkable if it didn't fall over almost immediately, never mind not have any discernible deviation from standing straight up after the lifetime of the universe. This problem is usually tackled using inflationary models, which can alleviate this problem, though it is worth noting that inflation happens at an energy scale which has never been probed directly and so there is also room for alternative gravity here. Finally the domination of the matter density by a yet to be isolated form of matter is not a problem *per se* however it remains to be seen whether this is an example analogous to the prediction

of the existence of Neptune or that of Vulcan. In other words it is not clear if this will turn out to be a valid prediction (as of Neptune) or simply a sign of the failure of our current theory of gravity (as Vulcan was for Newtonian gravity).

There is one more issue: in cosmologies which are good models for our universe there is a singularity, where $a(t) \to 0$ (Hawking et al. 1970). Before discussing this it will be useful to discuss some other exact solutions of the Einstein equations.

Black holes and Singularities

A very important case for gravity is, of course, the gravitational field outside of a spherically symmetric body. In Newtonian gravity it acts as a very useful simple model for a wide range of scenarios. In GR it can be shown that there is a unique spherically symmetric vacuum solution and it is static, this is known as Birkhoff's theorem. That solution is the Schwarzschild metric, which in Schwarzschild coordinates is written,

$$\mathrm{d}s^2 = -\left(1 - \frac{r_s}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{r_s}{r}\right)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right), \quad (1.1.51)$$

where, for now, r_s is a fiducial radius, *i.e.* it has no immediate physical interpretation. Also note that a spacelike coordinate, x, which allows one to write part of the metric as,

$$x^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right), \quad (1.1.52)$$

will be called an "areal radius" in the rest of this thesis. If we now consider the geodesics of time-like particles with a small coordinate velocity in the limit $r \gg r_s$, one can see that matching the Newtonian prediction requires,

$$r_s = \frac{2GM}{c^2}, \quad (1.1.53)$$

Where M is the "mass" of the object. As the metric appears to become singular at $r = r_s$ it is a great comfort that $r_s \ll R$ for all objects in the Solar system (for example) and thus this part of the metric will not be included in the metric of a spacetime with a spherically symmetric massive body.

However, it turns out that this value of r is only problematic for the coordinate system, not the spacetime, *i.e.* it is a coordinate singularity. This can be seen by computing invariants and showing that they are perfectly regular at $r = r_s$, though $r = 0$ is a true singularity. This singularity is actually potentially worrisome as, once neutron stars reach the Tolman-Oppenheimer-Volkoff limit, there is no known mechanism which could prevent them from collapsing completely.

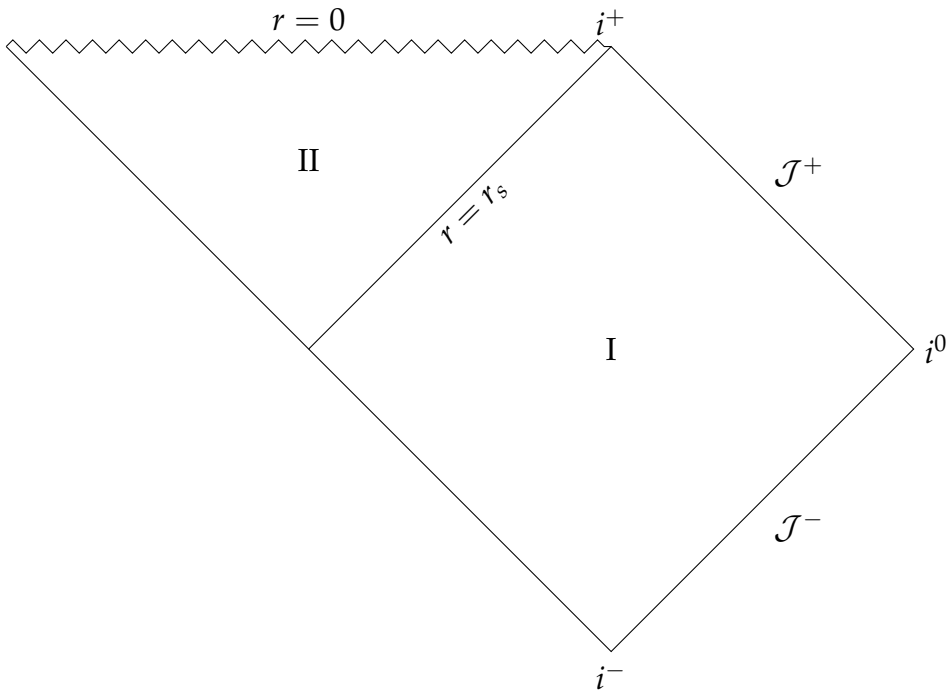


Figure 1.1: Penrose diagram of Schwarzschild spacetime

To discuss the causal properties of this solution which lead to it being called a black hole it is useful to consider its Penrose diagram. To generate such a diagram one generally expresses the metric in terms of some null coordinates which (together with the angular coordinates) cover the whole solution with a finite range of coordinate values. One then introduces a conformally transformed metric,

$$d\tilde{s}^2 = \Omega^2 ds^2, \tag{1.1.54}$$

where Ω is such that the new metric remains finite for all of the aforementioned coordinate values. The diagram is of a fixed θ and ϕ slice of the

spacetime, with constant values of the null coordinates corresponding to lines at 45 degrees to the vertical. The diagram is drawn in such a way that the forward light cones point up the page. In Figure 1.1, and wherever they are used subsequently, $i^{+,0,-}$ mean future timelike, spacelike and past timelike infinity respectively and $\mathcal{J}^{+,-}$ indicates future and past null infinity respectively. We can also use this figure to remind ourselves that the formal definition of an event horizon is that it is the intersection of the boundary of the causal past of $\mathcal{J}^+$ with the manifold $\mathcal{M}$. That is, denoting the causal past of some collection of points (S) by $J^-(S)$

$$\partial J^- \left(\mathcal{J}^+\right) \cap \mathcal{M}. \tag{1.1.55}$$

Recalling that null paths are at 45 degrees to the vertical one can see that the surface $r = r_s$ fulfils this definition, it is also useful to note that on this surface ∂_t is a null vector and thus this is a Killing horizon. We can also see why this is a sensible definition, any signal which is not superluminal (*i.e.* any causal signal) cannot leave region II and must end up at the singularity at $r = 0$. The proper time along any trajectory from $r = r_s$ to $r = 0$ is also finite. This means that after a finite amount of proper time we will no longer be able to predict the future of the collapsing matter, even in principle, using GR. This leaves several options:

- classical GR is only an approximation to some more accurate theory,
- the ability of physics to predict things simply breaks down in black holes or,
- this singularity is an artefact of the high symmetry and this solution is not representative of the features of a real collapsed object.

The second option demands a rather defeatist attitude and seems a bad working assumption and so we shall give it little attention. Now, having taken the time to remind ourselves of some properties of black holes, let us now turn to the third option, *i.e.* that this is an artefact of symmetry. The

first thing to note is that the Kerr family exists, *i.e.* black holes that are not spherically symmetric. Also note that it cannot be put down to the vacuum nature of the solutions. This can be seen from the existence of the Kerr-Newman family of solutions (including the Reissner-Nordström solution) which have non-zero electromagnetic fields.

If we want to go beyond stationarity we have to decide what we mean by a black hole. In stationarity one can rely on the existence of a Killing vector which is timelike in some region outside of the black hole to find the event horizon. However, if such a vector does not exist, one requires knowledge of the whole spacetime to determine where an event horizon may be. This clearly constitutes a problem for dynamical spacetimes as little can be done without numerics, and one cannot generate the entire future of the spacetime numerically. Observations also share a similar problem in that one cannot observe the infinite future.

To get around this problem we must use some local condition. A particularly useful one is the notion of a trapping horizon. This is a surface where the tangents to outgoing null-rays, outgoing meaning directed away from the object in question, remain within the surface. Of course this cannot be just any surface, it should be of co-dimension 2 and there should be larger surfaces which enclose it.

Given some technical conditions, which essentially restrict the mathematics to avoid counter-examples which most would not consider physically important (and also some energy conditions) one can prove that a trapping horizon will always be strictly inside an event horizon. Further one can show that if a trapping horizon forms there will be some singularity inside (Penrose 1965). In this case a singularity is defined as some geodesic incompleteness and thus GR fails to be predictive inside black holes, or else predicts that the inside of black holes are unpredictable. Similar proofs exist to also show that the beginning of a physically reasonable cosmology in GR will include a singularity. The message here is that one cannot say that the singularities arise from the stationarity assumption or any other symmetry

imposed on the exact black hole solutions. These theorems show that GR predicts singularities extremely generically.

An important question that I skipped over in describing the proof of these theorems was the meaning of the energy conditions used. The important part of these restrictions is that on the Ricci tensor which, via the Einstein equations, can also be interpreted as a restriction on the matter content of the spacetime. In terms of the Ricci tensor the restrictions are about how geodesics behave, specifically they minimally require that some form of geodesics (timelike or null) are focused by the curvature, *i.e.* that gravity is always attractive. These translate to some positivity conditions on the stress-energy tensor. The weakest is equivalent to the statement that the matter should not violate causality. In the context of perfect fluids, stronger conditions can also require that the local energy density is positive to all observers and that the local pressure is not too negative in units of the energy density; how negative it is allowed to be depends on the exact energy condition.

The intuitive understanding of the black hole singularity theorems is this: if we have a trapping horizon all causal matter inside it is restricted to converge on the centre of the object. Therefore, unless gravity becomes repulsive, we will eventually form some type of singularity.

To summarize, the existence of singularities in GR leaves three distinct possibilities,

- the ability of physics to predict things breaks down in black holes,
- singularities are non-generic,
- classical GR is only an approximation to some more accurate theory.

The first of these is possibly true, but is not something to accept without exhausting all other options. The second is shown to be false by the singularity theorems and thus we are left with the final option. The final option can be interpreted in two broad ways, either GR is not complete or the singularities will be resolved in quantum GR. This second option is what we

will discuss in the next section. However, to do so it will first be useful to review some points of perturbative GR, and while we are on the topic it will be a good opportunity to discuss various related points which will be used elsewhere.

1.1.3 *Perturbation theory*

To obtain the field equations in the linearized regime it is sufficient to expand $G_{\mu\nu}$ to first order in some perturbation, however it can be useful to instead expand the action to second order, *i.e.* obtain the "quadratic action". The quadratic Einstein-Hilbert action, expanded around Minkowski is,

$$S^{(2)}_{\text{EH}} = \frac{1}{16\pi G}\int \mathrm{d}^4x \frac{1}{4}\left[h\left(2h_{\alpha\beta,}{}^{\alpha\beta} - h_{,\alpha}{}^{\alpha}\right) + h^{\mu\nu}\left(h_{\mu\nu,\alpha}{}^{\alpha} - 2h_{\alpha\mu,\nu}{}^{\alpha}\right)\right], \tag{1.1.56}$$

where,

$$h_{\alpha\beta} = g_{\alpha\beta} - \eta_{\alpha\beta}, \quad h = h_{\alpha\beta}\eta^{\alpha\beta}, \quad A_{,\alpha_1\ldots\alpha_n} = \partial_{\alpha_1}\ldots\partial_{\alpha_n}A, \tag{1.1.57}$$

and all indices are raised and lowered with η. Recall that, either by variation of Equation (1.1.57) or by direct expansion, the linearized Einstein equations are,

$$h_{\beta\mu,\nu}{}^{\beta} + h_{\beta\nu,\mu}{}^{\beta} - h_{\mu\nu,\beta}{}^{\beta} - h_{,\mu\nu} + \eta_{\mu\nu}\left(h_{,\alpha}{}^{\alpha} - h_{\alpha\beta,}{}^{\alpha\beta}\right) = 16\pi G T_{\mu\nu}. \tag{1.1.58}$$

Also recall that this can be written as,

$$\bar{h}_{\beta\mu,\nu}{}^{\beta} + \bar{h}_{\beta\nu,\mu}{}^{\beta} - \bar{h}_{\mu\nu,\beta}{}^{\beta} - \eta_{\mu\nu}\bar{h}_{\alpha\beta,}{}^{\alpha\beta} = 16\pi G T_{\mu\nu}, \tag{1.1.59}$$

where,

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h. \tag{1.1.60}$$

In the form of Equation (1.1.59) one can reduce the Einstein equations to a standard wave equation, by choosing the gauge condition,

$$\bar{h}_{\mu\nu,}{}^{\mu} = 0. \tag{1.1.61}$$

Thus GR predicts gravitational waves, whose existence have been recently verified (see Abbott 2016a; Abbott 2016b; Abbott 2017 and companions). Note that the gauge condition can always be satisfied as if $\overline{h}_{\mu\nu}$ doesn't satisfy this condition the gauge transformation,

$$x^\mu \to x^\mu + \xi^\mu, \tag{1.1.62}$$

where,

$$\Box\xi_\mu = \overline{h}_{\mu\nu,}{}^{\nu}, \tag{1.1.63}$$

will take us to the desired gauge. The statement that this is always possible is simply that the wave equation in Minkowski space has a solution (four times).

Using this linearization one can also get to the Newtonian limit. By considering the geodesic equation for timelike paths with small coordinate speed (compared to the speed of light) in coordinates where the source of the gravitational field is at rest, one finds that the effective potential is dominated by the contribution from Γ^i_{00}. Expanding this and using the Einstein equations, the Newtonian equations for the trajectory of a particle and the gravitational field are recovered. Of course using GR we would like to go beyond Newtonian physics and here we will take the opportunity to review some aspects of the parametrised post-Newtonian (PPN) expansion, which is also useful for describing deviations from GR (Will 1971, see Will 1993 for a more complete view).

It is important to note that we were very loose with some notions in the quick discussion of the Newtonian limit here. GR is such (obeying the strong equivalence principle) that doing so does not lead to the wrong answer. However, as the PPN expansion has more general application one needs to be much more careful. In particular, in theories with additional fields the external environment can affect the isolated bodies we are usually interested in when using the PPN expansion.

An example which is of great relevance to this thesis is a non-minimally coupled scalar field (or scalar fields). Additional scalar fields generically

lead to dependence on the value scalar field in the external environment (as a boundary condition). Another (again relevant) example is that of additional (time-like) vector fields can lead to dependence on the motion of the system of interest relative to the vector field in the environment. To first post-Newtonian order there are ten parameters, conventionally written,

$$(\gamma, \beta, \xi, \alpha_1, \alpha_2, \alpha_3, \xi_1, \xi_2, \xi_3, \xi_4). \tag{1.1.64}$$

These have various heuristic interpretations, with γ quantifying how much spatial curvature is produced by unit rest mass, β quantifying some of the non-linearity, α_i describing preferred frame effects and ξ_i & α_3 quantifying the failure of conservation of momentum. In GR the values are $(1, 1, 0, 0, 0, 0, 0, 0, 0, 0)$. The metric itself is constructed from various potentials which are constructed as integrals of the local matter distribution.

1.1.4 *Quantum gravity*

As matter obeys the laws of quantum, rather than classical physics, it seems inevitable that gravity must be fundamentally quantum too (though non-locality is technically an option for both). One could then hope that quantising GR would lead to a theory without singularities (at least generically). However, as is well known, GR is power-counting non-renormalizable. We can remind ourselves of this results like so: consider a Feynman diagram with n incoming particles, m outgoing particles, which contains: L loops, P propagators and V_i^j vertices of degree i involving j derivatives with coupling constant q_i^j. Then,

$$4L - 2P + \sum_i \sum_j \left(\left[q_i^j \right] + j \right) V_i^j = \left[\begin{array}{ccc} & i_1 \quad o_1 & \\ \vdots & \otimes & \vdots \\ & i_n \quad o_m & \end{array} \right] = \left[q_{n+m}^p \right] + p, \tag{1.1.65}$$

where $[f]$ is the mass dimension of f and p is the number of derivatives on the external lines. Further, recalling that the superficial degree of divergence,

$$D = 4L - 2P + \sum_i \sum_j j V_i^j - p, \tag{1.1.66}$$

we then have that,

$$D = \left[q_{n+m}^p\right] - \sum_i \sum_j \left[q_i^j\right] V_i^j, \tag{1.1.67}$$

and thus if any of the q_i^j has negative mass dimension, say q_l^k, then D can be made arbitrarily large by adding vertices of degree l and involving k derivatives within the "loopy" part of the diagram. In the above discussion we assumed that the fields involved are canonically normalized. However, given the quadratic Einstein-Hilbert action,

$$S_{\rm EH}^{(2)} = \frac{1}{16\pi G} \int \mathrm{d}^4x \frac{1}{4} \left[h \left(2h_{\alpha\beta,}{}^{\alpha\beta} - h_{,\alpha}{}^{\alpha} \right) + h^{\mu\nu} \left(h_{\mu\nu,\alpha}{}^{\alpha} - 2h_{\alpha\mu,\nu}{}^{\alpha} \right) \right], \tag{1.1.68}$$

we see that canonical normalization requires a field redefinition,

$$\mathbf{h} \to \tilde{\mathbf{h}}, \tag{1.1.69}$$

where,

$$[\mathbf{h}] = \left[\sqrt{G}\tilde{\mathbf{h}}\right] = \left[\tilde{\mathbf{h}}\right] - 1. \tag{1.1.70}$$

The important point here is that the interaction terms (coming from a higher order expansion of the Einstein-Hilbert action) come with no further dimensionful constants in terms of $\mathbf{h}$. This means that all the interaction terms have a coupling constant with a negative canonical mass dimension, and thus there are diagrams which diverge as much as you (don't) want. Therefore, unless there is some hidden symmetry, GR is not perturbatively renormalizable.

There remains the scenario of asymptotic safety, one can see this as a generalization of the idea of asymptotic freedom, which appears to be the last refuge of those attempting to get away without modifying GR (Weinberg 1979). To understand the idea of asymptotic safety we first have to recall

why the non-renormalizability of GR is a problem. To cancel divergences in the quantum corrections we have to introduce counter-terms, which bring along with them free coupling constants which must then be fixed by experiment. In a non-renormalizable theory there are an infinite number of such counter-terms and thus (in any regime where the counter-terms are important, $\sim$ Planck scale in GR) the theory ceases to be predictive. If there is some fixed point in the parameter space which is UV attractive for some finite dimensional subspace then the requirement that the theory be predictive leads to a finite number of independent parameters (still to be determined by experiment) (Percacci 2007). For more see Percacci 2007; Percacci 2011; Reuter et al. 2013; Reuter et al. 2012

Otherwise there is a large marketplace of ideas for quantum gravity. Two of the most common features of these proposals are additional degrees of freedom and difficulty in giving any meaning to, or else simply preserving, the weak equivalence principle. As the natural scale for quantum gravity effects is Riemann $\sim \mathcal{O}\left(l_{\mathrm{pl}}^{-2}\right)$ it seems unlikely such effects will ever be directly observable, definitely not soon. However, one could hope that this naïve estimate is not quite correct. Perhaps some extra degrees of freedom survive into the classical regime and can be modelled with some modified gravity theory, and/or perhaps quantum gravity deforms away from GR smoothly as the curvature scale increases and precision experiments could, in principle, detect deviations from GR at any scale.

Having spent quite some time discussing GR and why there is little hope for those wishing it to be the final word [1], we will now move on to reviewing some of the background upon which the work of myself and my collaborators has been based.

1 It should be noted that although GR uniqueness theorems exist they are only as good as their assumptions. To my knowledge there is no uniqueness theorem which states that gravity must minimally couple to other fields. Also the symmetry assumptions, although no deviation has ever been detected, can never be completely verified.

2

ALTERNATIVE THEORIES OF GRAVITY

Before starting to construct alternative theories of gravity one should recall Lovelock's theorem (Lovelock 1971), which states that the Einstein field equations (1.1.7) are unique in four dimensions, given some assumptions. These assumptions are that:

- there is an action principle for gravity
- the gravitational action involves only a metric and its derivatives,
- the Lagrangian contains only local operators,
- the field equations contain only up to second order derivatives of the metric,
- and the action is invariant under diffeomorphisms.

If we want to go beyond GR, as we do, we must break one of these assumptions.

2.1 SCALAR-TENSOR THEORIES

In this section we will review alternative theories of gravity which avoid Lovelock's theorem in a minimal way, by introducing a scalar field which

couples non-minimally to the metric (Fujii et al. 2007; Faraoni 2004). That is, schematically,

$$\int \mathrm{d}^4x\sqrt{-g}\left[\mathcal{L}_g\left[g_{\mu\nu}\right]+\mathcal{L}_m\right] \rightarrow \int \mathrm{d}^4x\sqrt{-g}\left[\mathcal{L}'_g\left[g_{\mu\nu},\Phi\right]+\mathcal{L}_m\right]. \quad (2.1.1)$$

The simplest of these theories is Brans-Dicke theory (Brans et al. 1961; Jordan 1959), and it serves as a prototype for a whole class of interesting alternative theories of gravity.

2.1.1 *Brans-Dicke theory*

The original motivation behind Brans-Dicke theory was to introduce an alternative theory of gravity where Mach's principle held true. For our purposes however it is enough to say that we want to promote Newton's constant to a dynamical field. The Brans-Dicke action is then,

$$\frac{1}{16\pi}\int \mathrm{d}^4x\sqrt{-g}\left[\Phi R-\frac{\omega_0}{\Phi}\partial_\mu\Phi\partial^\mu\Phi\right]+S_m[\mathbf{g},\psi_m], \quad (2.1.2)$$

where S_m is the matter action which is independent of Φ. The choice of coefficient for the kinetic term *i.e.*, ω_0/Φ was originally made simply to have ω_0 be dimensionless, though it has the fortunate property of forbidding $\Phi = 0$.

One of the main uses of Brans-Dicke theory in recent times is to demonstrate the power of weak-field test, particularly in the Solar system. By calculating the predictions of Brans-Dicke theory in the PPN framework and comparing this to precision tests in the Solar system one can put extreme bounds of $\omega_0 > 40000$ (Bertotti et al. 2003). This makes having a non-trivial scalar field very costly in the variational sense, and thus the theory is forced to behave almost identically to GR. In other words all deviations from GR are parameterized by ω_0 and so any tests which constrain ω_0 put equally strong constraints on the theory in all regimes. To have an interesting alternative to GR we of course demand that it has significantly different behaviour. To do so we will have to consider more complicated theories,

where Solar system tests do not completely rule out significant deviations from GR in all known and postulated astrophysical phenomena.

Before moving on it does not seem proper to ignore the existence of $f(R)$ gravity and its relation to Brans-Dicke theory. For a review see Sotiriou et al. 2010

2.1.2 *"General" scalar-tensor theories*

When discussing scalar-tensor theories one is usually talking about theories which can be written in the form,

$$\frac{1}{16\pi}\int \mathrm{d}^4x\sqrt{-\tilde{g}}\left[\Phi\tilde{R}-\frac{\omega(\Phi)}{\Phi}\partial_\mu\Phi\partial^\mu\Phi-U(\Phi)\right]+S_m[\tilde{\mathbf{g}},\psi_m], \tag{2.1.3}$$

in subsection 2.1.3 we will see that this is not the most general theory one can write using a metric and a scalar field, but they are an interesting and well studied subset. Note that this is not because we did not allow for the coefficient of $\tilde{R}$ to be any function, $F(\Phi)$. One can see this by considering the field redefinition $\Phi \to F(\Phi)$ which results in an action of the form of (2.1.3), with different functional forms for ω and U.

These theories can also, via field redefinitions and a conformal transformation, be recast in the Einstein frame, rather than the Jordan frame representation above. In the Einstein frame all theories of this type can be written as,

$$\frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{-g}\left[R-2\partial_\mu\phi\partial^\mu\phi-V(\phi)\right]+S_m[A^2(\phi)\mathbf{g},\psi_m], \tag{2.1.4}$$

where,

$$\tilde{\mathbf{g}}=A^2(\phi)\mathbf{g},\quad A^2(\phi)=\frac{1}{G\Phi(\phi)},\quad V(\phi)=\frac{U(\Phi(\phi))A^4(\phi)}{G}, \tag{2.1.5}$$

and,

$$\mathrm{d}\phi=\frac{\sqrt{3+2\omega}}{\Phi}\mathrm{d}\Phi. \tag{2.1.6}$$

This formulation will be particularly useful in Chapter 5. The equations of motion in the Einstein frame are,

$$G_{\mu\nu} = 8\pi G \left(T_{\mu\nu} + T^{(\phi)}_{\mu\nu} \right) \tag{2.1.7}$$

$$\Box\phi - \frac{1}{4}V'(\phi) = -4\pi G T \alpha(\phi), \tag{2.1.8}$$

where,

$$T^{(\phi)}_{\mu\nu} = \frac{1}{4\pi G}\left(\partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\left(\partial_\alpha\phi\,\partial^\alpha\phi + \frac{1}{2}V(\phi) \right) \right), \tag{2.1.9}$$

$$V'(\phi) = \frac{\mathrm{d}V}{\mathrm{d}\phi}, \quad \alpha(\phi) = \frac{\mathrm{d}\log A}{\mathrm{d}\phi}, \text{ and } \quad T = g^{\mu\nu}T_{\mu\nu}. \tag{2.1.10}$$

From Equation (2.1.8) one can see that, generically, in the presence of matter the scalar field will be non-trivial. Therefore one may expect this class of theories to have the same problems as Brans-Dicke theory. This does not have to be the case, as we shall see in subsection 2.1.4. We mention here that the above class of scalar-tensor theories can be can be generalized quite simply to multiple and/or complex scalar fields (Damour et al. 1992 or Horbatsch et al. 2015 for a more comprehensive picture). Before moving on to discussing how one can avoid the Brans-Dicke problem, we note that one can further generalize the scalar-tensor theories by allowing for derivative interactions.

2.1.3 *Horndeski theory*

To arrive at Horndeski theory one must allow for the action to contain a metric, a scalar field (here denoted φ) and their derivatives, while demanding second order equations of motion (Horndeski 1974). This approach is essentially a generalization of Lovelock's theorem which adds a scalar field into the allowed field content. Before giving a formulation of the action it will be useful to introduce the notation,

$$X = -\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi, \quad f_X(X,\varphi) = \frac{\partial f}{\partial X}. \tag{2.1.11}$$

With this notation one of the most common formulations of the Horndeski action is,

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5\right], \tag{2.1.12}$$

where,

$$\mathcal{L}_2 = f_2, \tag{2.1.13}$$

$$\mathcal{L}_3 = -f_3\Box\varphi, \tag{2.1.14}$$

$$\mathcal{L}_4 = f_4 R + f_{4X}\left[(\Box\varphi)^2 - \left(\nabla_\mu\nabla_\nu\varphi\right)^2\right], \tag{2.1.15}$$

$$\mathcal{L}_5 = f_5 G_{\mu\nu}\nabla^\mu\nabla^\nu\varphi - \frac{1}{6}f_{5X}\left[(\Box\varphi)^3 - 3\Box\varphi\left(\nabla_\mu\nabla_\nu\varphi\right)^2 + 2\left(\nabla_\mu\nabla_\nu\varphi\right)^3\right], \tag{2.1.16}$$

where the f_n are functions of φ and X,

$$\left(\nabla_\mu\nabla_\nu\varphi\right)^2 = \left(\nabla_\mu\nabla_\nu\varphi\right)\left(\nabla^\mu\nabla^\nu\varphi\right), \tag{2.1.17}$$

and,

$$\left(\nabla_\mu\nabla_\nu\varphi\right)^3 = \left(\nabla_\mu\nabla_\nu\varphi\right)\left(\nabla^\nu\nabla^\rho\varphi\right)\left(\nabla_\rho\nabla^\mu\varphi\right). \tag{2.1.18}$$

The scalar-tensor theories of the form of the action (2.1.3) correspond to,

$$f_3 = f_5 = 0, \tag{2.1.19}$$

$$f_2 = \frac{1}{16\pi}\left(2\frac{\omega(\varphi)}{\varphi}X - U(\varphi)\right), \tag{2.1.20}$$

$$f_4 = \frac{1}{16\pi}\varphi, \tag{2.1.21}$$

and, $\varphi = \Phi$. Finally we mention that one further generalization can be made by allowing for a degenerate Lagrangian, see Gleyzes et al. 2015.

2.1.4 *Screening mechanisms*

In general we are mostly interested in corners of Horndeski theory which avoid the problem of Brans-Dicke theory. That is, we want to consider specific versions in which weak-field constraints don't significantly constrain

deviations from GR outside of the weak field regime. This is the requirement that interesting physical models should predict observable phenomena which are not predicted by the current best models we have. Motivation from theory for any such models will be specific to that model.

To have a theory of the type we are interested in, there must be some feature of the theory which forces it to be close to GR in the solar system but which "switches off" in some other interesting regime. Such a feature is usually called a screening mechanism. There are two broad classes of screening mechanisms, which can be roughly stated as those that screen above a certain curvature scale and those that screen below a certain curvature scale. Theories which have no screening on small curvature scales can be interesting cosmologically and those with no screening on large curvature scales are perhaps more interesting from a "quantum gravity phenomenology" point of view. Of course the designation of a scale as large or small will be in reference to some characteristic solar system scale, precisely which scale is meant is model dependent though it is usually some curvature scale.

From Equation (2.1.9), and taking the trace-reverse, it is clear that if we wish to have agreement with GR we will require that,

$$2\left(\partial_\mu\phi_0\right)\left(\partial_\nu\phi_0\right)+\frac{1}{2}g_{\mu\nu}V(\phi_0)=\lambda g_{\mu\nu}, \tag{2.1.22}$$

where λ acts as a shift to the bare cosmological constant. For generic $g_{\mu\nu}$, it is not possible to arrange $g_{\mu\nu}\propto\partial_\mu\phi_0\partial_\nu\phi_0$ everywhere, as such ϕ_0 must be a constant. Equation (2.1.8) then implies that a constant ϕ is only a solution regardless of the presence of matter if $V'(\phi_0)=\alpha(\phi_0)=0$. The screening mechanism of most relevance to this thesis is spontaneous scalarization (Damour et al. 1993; Damour et al. 1996). From the name perhaps it is clear that it is usually spoken of in the inverse sense to the above. That is, instead of being considered as a screening mechanism in the solar system it is often discussed as enhancing deviations around highly compact objects. Of course this is just a matter of perspective. Spontaneous scalarization relies on the fact that the GR solution, though it exists, needn't be unique. We

must then find a restriction on $\alpha(\phi)$ that leads to the $\phi = \phi_0$ solution being preferred in the solar system but a non-trivial solution being preferred in, say, the vicinity of a neutron star.

Expanding Equation (2.1.8) to linear order in $\phi - \phi_0$ then (for convenience) defining G in such a way that $A^2(\phi_0) = 1$, and relabelling $\phi - \phi_0 \to \phi$, some rearrangement then gives,

$$\Box\phi = \left(m_\phi^2 - 4\pi G T\beta\right)\phi, \tag{2.1.23}$$

where,

$$m_\phi^2 = \frac{1}{4}V''(0), \quad \beta = \alpha'(0). \tag{2.1.24}$$

If we take a non-relativistic perfect fluid, then,

$$T = -\rho = -A^4(\phi)\tilde{\rho}, \tag{2.1.25}$$

where $\tilde{\rho}$ is the energy density of the fluid in the Jordan frame, *i.e.* where the weak equivalence principle is manifest. If we now take $\beta < 0$, and use the fact that $\rho = \tilde{\rho}(1 + \mathcal{O}(\phi))$ and assume we are dealing with an object much larger than its Schwarzschild radius then,

$$m_{\text{eff}}^2 = m_\phi^2 - 4\pi G\tilde{\rho}|\beta|. \tag{2.1.26}$$

For this reason we expect an instability (when the metric is Minkowski + corrections) in the Fourier modes of wavelength $\sim \lambda$, when,

$$r_\rho^2 \gtrsim \lambda^2 > \frac{\pi}{G|\beta|\tilde{\rho}\left(1 - \frac{m_\phi^2}{4\pi G|\beta|\tilde{\rho}}\right)}, \tag{2.1.27}$$

where r_ρ is the linear scale of the matter distribution (radius of a sphere for example). The upper bound comes from requiring that the unstable mode "fits" inside the matter. For vanishing scalar field mass this gives,

$$s = \frac{GM}{r_\rho} \sim \frac{1}{|\beta|}, \tag{2.1.28}$$

as the compactness scale for which spontaneous scalarization will occur. This crude estimate for the compactness scale at which an instability arises

does not tell us the end state of this process, and a healthy theory requires that the non-linear terms quench this instability after some finite growth of the scalar field. However, there is an important thing to stress: so long as,

$$|\beta| \ll \frac{R_\odot}{GM_\odot} \sim 10^6, \tag{2.1.29}$$

we have a theory which completely coincides with GR in the solar system yet can have large deviations in more compact environments (ignoring the possible complications from matching to cosmology). This means that the weak-field constraints have essentially no bearing on the strong-field behaviour of this restricted class of theories, and hence the Brans-Dicke problem does not exist for them. It should be noted that in order for this mechanism to actually be effective, and not be ruled out by binary pulsar and cosmological tests, then a non-zero (though small) mass for the scalar field is all but required. Without any mass there is a window where spontaneous scalarization can occur $-5 \lesssim \beta \lesssim -4.35$ (Freire et al. 2012; Harada 1998) [or $\lesssim -3.9$ for certain classes of rotating neutron stars Doneva et al. 2013]. If a mass is included the constraints on β are much looser, $\beta \gtrsim -10^3$ appears to be consistent with observations, for a mass in the range $10^{-16}\,\text{eV} \ll m_\phi \lesssim 10^{-9}\,\text{eV}$ (Ramazanolu et al. 2016).

Other screening mechanisms in scalar-tensor theories can be constructed with different choices for the potential $V(\phi)$ and the conformal factor $A^2(\phi)$. Examples include the chameleon and symmetron models (Khoury et al. 2004 & Hinterbichler et al. 2010 respectively). These both have the scalar field being important cosmologically but screened in the solar system (and regions of higher matter density than that). Despite the similar effect of the screening, the mechanisms are quite different. If we define an effective potential,

$$V_{\text{eff}} = \frac{1}{4}V(\phi) - 4\pi G T \log A(\phi), \tag{2.1.30}$$

so that equation (2.1.8) reads,

$$\Box\phi = \frac{\partial V_{\text{eff}}}{\partial \phi}, \tag{2.1.31}$$

then chameleon models correspond to choosing V_{eff} such that the mass monotonically increases with the energy density (if matter is described by a perfect fluid) in such a way that the scalar field can have a significant impact on cosmology while being too massive to have been detected in solar system tests. In symmetron models the action is constructed to be invariant under $\phi \to -\phi$, the vacuum potential is selected such that the solutions break this symmetry, and the conformal factor is chosen such that the symmetry is restored above some threshold density. The construction of the symmetron model also guarantees that perturbations around the minimum of the potential couple with a strength proportional to the value of the scalar field at this minimum. An example effective potential for the chameleon is (Khoury 2013),

$$V_{\text{eff}} = \frac{C_n}{\phi^n} - \xi T\phi, \tag{2.1.32}$$

and for the symmetron (Hinterbichler et al. 2011),

$$V_{\text{eff}} = \frac{1}{2}\left(C^2 T - \mu^2\right)\phi^2 + \frac{1}{4}\lambda\phi^4. \tag{2.1.33}$$

So far we have touched only on screening mechanisms for scalar-tensor theories of the form of action (2.1.4). For the more general Horndeski theories there is another mechanism which exists for a large subset of them, the Vainshtein mechanism (Vainshtein 1972 or Babichev et al. 2013 for a more modern view). This mechanism relies on derivative interactions which modify the effective kinetic term for the scalar field. If the coefficient of this kinetic term gets large in the presence of significant matter densities then, upon canonical normalization, the scalar field will then be weakly coupled to matter.

To see how this works we can consider a toy model. Consider a scalar field, φ, with a Lagrangian,

$$\mathcal{L} = \frac{a^2}{2}\partial_\mu\varphi\partial^\mu\varphi + \mathcal{L}_{\text{int}}(\varphi, \psi), \tag{2.1.34}$$

where $\mathcal{L}_{\text{int}}$ contains some interactions between φ and some other field, ψ, and the coupling constants are not too large. Then we introduce a field redefinition to canonical normalize the kinetic term for φ,

$$\varphi \to \frac{\varphi}{a}, \quad \mathcal{L} \to \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + \mathcal{L}_{\text{int}}. \tag{2.1.35}$$

If we had

$$\mathcal{L}_{\text{int}} \supset q\varphi^n f(\psi), \tag{2.1.36}$$

then $q \to q/a^n$. Thus, if a is large enough, the scalar field effectively decouples from ψ (as the coupling constant becomes very small). This is what happens in the Vainshtein mechanism, but the coefficient a is actually dependent on φ and its derivatives, so the screening only occurs in certain regimes. In particular those in which $a(\varphi, \partial\varphi)$ gets large.

2.2 LORENTZ-VIOLATING THEORIES OF GRAVITY

Lorentz invariance and general covariance are extremely important ingredients in the standard model of particle physics and GR respectively. Due to this fundamental importance it is vital that these principles be tested as stringently as possible. Constraints on Lorentz violations in the matter sector are already very stringent (see Mattingly 2005; Liberati 2013; Kostelecky et al. 2011 and references therein). The situation in the gravity sector is much less well constrained; hence Lorentz violating theories of gravity are interesting as a means of producing testable predictions to guide experiments.

We shall also see in Section 2.2.2 that Lorentz violating theories can also be interesting from the theoretical perspective.

2.2.1 *Einstein-æther theory*

We begin by reviewing some aspects of Einstein-æther theory (Jacobson et al. 2001 also, for example, Gasperini 1987). Einstein-æther theory is a

phenomenological model of Lorentz violations in gravity. It is a vector-tensor theory supplemented by the constraint that the vector must always be unit timelike, thus defining a special frame. The formulation used in this thesis is,

$$\frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{-g}\left[R - Z^{\alpha\mu\beta\nu}\left(\nabla_\mu u_\nu\right)\left(\nabla_\alpha u_\beta\right) + \left(\lambda\left(u_\alpha u^\alpha + 1\right)\right)\right] \tag{2.2.1}$$

where,

$$Z^{\alpha\mu\beta\nu} = c_1 g^{\alpha\mu}g^{\beta\nu} + c_2 g^{\alpha\beta}g^{\mu\nu} + c_3 g^{\alpha\nu}g^{\beta\mu} - c_4 u^\alpha u^\mu g^{\beta\nu} \tag{2.2.2}$$

and λ is a Lagrange multiplier. The Lagrange multiplier will occasionally be omitted, and the constraint will be imposed at the level of the variations by requiring,

$$\delta u^\mu = -\frac{1}{2}u^\mu\left(u_\alpha u_\beta \delta g^{\alpha\beta}\right) + p^\mu_\nu \delta u^\nu, \tag{2.2.3}$$

where p^μ_ν is the projector orthogonal to u, *i.e.* $p^\mu_\nu = \delta^\mu_\nu + u^\mu u_\nu$. This is derived from demanding that the variation of the constraint,

$$u_\alpha u_\beta g^{\alpha\beta} = -1, \tag{2.2.4}$$

vanishes.

Around flat space the theory has a spin-2 mode, a spin-1 mode and a spin-0 mode, with squared sound speeds (Jacobson et al. 2004),

$$\begin{aligned} s_2^2 &= \frac{1}{1-c_{13}}, \quad s_1^2 = \frac{c_1 + c_1^2/2 + c_3^2/2}{c_{14}\left(1-c_{13}\right)}, \\ s_0^2 &= \frac{c_{123}\left(2-c_{14}\right)}{c_{14}\left(1-c_{13}\right)\left(2+c_{13}+3c_2\right)}, \end{aligned} \tag{2.2.5}$$

where, s_i^2 is the squared sound speed of the spin-i mode, and we have used the notation,

$$c_{13} = c_1 + c_3, \quad c_{123} = c_1 + c_2 + c_3, \textit{ etc.} \tag{2.2.6}$$

We also note here these all of these modes are massless. In Lorentz violating theories one of the important questions, especially for black hole physics, is how one defines a notion of causality. Einstein-æther theory has what is referred to as quasi-relativistic causality. To see what this means note that if

we assume that matter is minimally coupled to $\mathbf{g}$ and is not coupled to the æther then there are up to four relevant speeds for causality. That is 1 and the s_i. To avoid gravitational Čerenkov radiation (Elliott et al. 2005)[1], and for stability of perturbations the s_i are usually taken to be $s_i > 1$ (Jacobson 2007). Taking this constraint, the direct causal influence of the matter is bounded by the null cones of the metric, as usual. For the spin-i mode, its influence is bounded instead by the null cones of the metric,

$$g^{(i)}_{\mu\nu} = g_{\mu\nu} - \left(s_i^2 - 1\right) u_\mu u_\nu. \tag{2.2.7}$$

As such we have that all causal influence is bounded by the null cones of the metric $\mathbf{g}^{(j)}$, where s_j^2 is the largest of the speeds (Eling et al. 2006). This null-cone structure is what leads us to call this theory quasi-relativistic, as opposed to simply non-relativistic.

For the purposes of solar system tests, the effective value of Newton's constant is (Carroll et al. 2004),

$$G_{\mathrm{N}} = \frac{G}{1 - c_{14}/2}. \tag{2.2.8}$$

This means that there is always a Newtonian limit for $c_{14} < 2$, however for $c_{14} > 2$ gravity becomes repulsive. The post-Newtonian parameters for Einstein-æther theory all agree with GR, except for two of the preferred frame parameters α_1, and α_2 (recall that α_3 controls momentum non-conservation, not just Lorentz violations) (Foster 2005; Graesser et al. 2005; Will 2014). Their values are,

$$\alpha_1 = \frac{-8\left(c_3^2 + c_1 c_4\right)}{2c_1 - c_1^2 + c_3^2} \tag{2.2.9}$$

$$\alpha_2 = \frac{\alpha_1}{2} - \frac{\left(2c_{13} - c_{14}\right)\left(2c_{12} + c_{234}\right)}{c_{123}\left(2 - c_{14}\right)} \tag{2.2.10}$$

Which were first found in Foster 2005; Graesser et al. 2005.

1 If particles move faster than some gravitational degree of freedom (which is massless) then they can decay.

2.2.2 *Hořava gravity*

Along with the phenomenological considerations which motivate Einstein-æther theory, Lorentz violating theories also have interesting properties from a theoretical stand point. As previously stated: one of the main difficulties with GR is that it is not perturbatively renormalizable as a quantum field theory. Therefore the fact that Lorentz violations allow the introduction of higher order derivatives without generating ghosts makes them quite attractive (see Visser 2009 whose arguments we shall follow here). That higher order derivatives improve UV behaviour can be seen from the contribution of the propagator to the superficial degree of divergence (see Equation (1.1.66) for the relativistic case with second order derivatives). In particular a relativistic propagator of the form,

$$\frac{1}{\sum_{n=0}^{N} \alpha_n^2 M^{2-2n} \left(\omega^2 - k^2\right)^n}, \tag{2.2.11}$$

contributes $-2N$ to the superficial degree of divergence. However, for $N > 1$ these theories are non-unitary. For example with $N = 2$ one can have a renormalizable but non-unitary theory of gravity (Stelle 1977). In the classical regime the higher order time-derivatives generically lead to Ostragradsky ghosts. In the Lorentz violating case we can allow for such suppression whilst maintaining unitarity by adding terms with higher order 3-momenta (spatial derivatives) only, *i.e.* by constructing theories which lead to propagators of the form,

$$\mathcal{P}_z = \frac{1}{\omega^2 - m^2 - \sum_{n=1}^{z} \alpha_n^2 M^{2-2n} k^{2n}}. \tag{2.2.12}$$

Note that in the deep UV we have the pole of the propagator at,

$$\omega \sim \alpha_z M^{1-z} k^z. \tag{2.2.13}$$

Therefore when introducing a momentum cutoff, Λ, the corresponding cutoff in energy should be,

$$\Omega = \alpha_z M^{1-z} \Lambda^z. \tag{2.2.14}$$

This gives, for the most basic loop integral,

$$\int \mathrm{d}\omega \mathrm{d}^d k \, \mathcal{P}_z \sim \Omega \Lambda^d \alpha_z^{-2} M^{2z-2} \Lambda^{-2z} = \alpha_z^{-1} M^{z-1} \Lambda^{d-z}. \tag{2.2.15}$$

Which means, for any diagram which does not include derivative interactions, the superficial degree of divergence,

$$D \leq (d - z) \, L, \tag{2.2.16}$$

where L is the number of loops. Therefore such a theory is power-counting renormalizable (has a bounded superficial degree of divergence) if $z \geq d$. If we include derivative interactions up to order $2z + \delta$, the maximum degree of divergence is,

$$D \leq (z + d) \, L - 2zP + (2z + \delta) \, V. \tag{2.2.17}$$

Using the Euler's formula for connected planar graphs,

$$L - P + V = 1, \tag{2.2.18}$$

and thus,

$$D \leq (d - z) \, L + 2z + \delta V. \tag{2.2.19}$$

All together this means that power-counting renormalizability requires $z \geq d$ and $\delta \leq 0$. This final requirement, particularly the case $\delta = 0$, can be enforced freely.

If one is interested in constructing a theory of gravity with the above properties, one needs to find the symmetries which allow the hierarchy $\omega \sim k^z$ in the UV to be maintained. In particular this means, if we expect our symmetries to be some subset of diffeomorphisms, that under our symmetry transformations spatial derivatives should remain purely spatial. If, in some coordinate system, we have a spatial derivative ∂_i then we are interested in its expression in terms of another coordinate system,

$$\partial_i = \frac{\partial \tilde{x}^j}{\partial x^i} \partial_{\tilde{j}} + \frac{\partial \tilde{x}^0}{\partial x^i} \partial_{\tilde{0}}. \tag{2.2.20}$$

So we must require that the symmetry transformations are some subset of transformations which take,

$$t \to \tilde{t}(t), \quad x \to \tilde{x}(t, x). \tag{2.2.21}$$

This means that Lorentz violations are also potentially interesting from a quantum gravity perspective.

Taking the symmetries (2.2.21) for a theory of a lapse, shift and 3-metric (*c.f.* the decomposition of the Einstein-Hilbert action leading to Equation (1.1.26)), and demanding the anisotropic scaling $\omega \sim k^z = k^d$, in the UV one arrives at Hořava gravity (Horava 2009). This is a Lorentz violating quantum gravity candidate which is also useful from the phenomenological perspective. There are several different versions of this theory (see for example Sotiriou 2011 for an overview). The projectable version has even been shown to be renormalizable (in full, not just power-counting) (Barvinsky et al. 2016).

It is also worth noting that the symmetries (2.2.21) are often called foliation-preserving diffeomorphisms ($\mathrm{Diff}_{\mathcal{F}}(\mathcal{M})$). This is because they are the subset of diffeomorphisms which leave a given foliation of spacetime unchanged. One can see this as the coordinates x are coordinates on the leaves of this foliation, and so $x \to x'(t, x)$ amounts to performing a transformation of the coordinates on each separate leaf, whereas $t \to t'(t)$ is a relabelling of the scalar field which defines the leaves of the foliation. Leaves of the foliation will be denoted by Σ.

Due to the lesser symmetry, compared to GR, there is a choice to be made about the field content before starting to construct the theory. Namely the symmetries preserve the choice $N = N(t)$, which corresponds to imposing the restriction that the congruence orthogonal to the foliation be geodesic. Thus Hořava gravity immediately splits into two categories: projectable Hořava gravity and non-projectable Hořava gravity where N is restricted to be a function of time alone or unrestricted respectively.

Projectable Hořava gravity

The most general projectable theory with the scaling $z = 3$ in 3 spatial dimensions (and one 1 temporal) is (Sotiriou et al. 2009a; Sotiriou et al. 2009b),

$$S_{\text{proj}} = \frac{1}{16\pi G}\int \mathrm{d}^3x\mathrm{d}tN\sqrt{\gamma}\left[K_{ij}K^{ij} - \lambda K^2\right] + S_{V,\,\text{proj}}, \tag{2.2.22}$$

where,

$$S_{V,\,\text{proj}} = \frac{1}{16\pi G}\int \mathrm{d}^3x\mathrm{d}tN\sqrt{\gamma}\left[\mathcal{L}_1 + \frac{1}{M_*^2}\mathcal{L}_2 + \frac{1}{M_*^4}\mathcal{L}_3\right], \tag{2.2.23}$$

with,

$$\mathcal{L}_1 = \beta\mathcal{R}, \tag{2.2.24}$$

$$\mathcal{L}_2 = \beta_1\mathcal{R}_{ij}\mathcal{R}^{ij} + \beta_2\mathcal{R}^2, \tag{2.2.25}$$

$$\mathcal{L}_3 = \beta_3 D_i\mathcal{R}_{jk}D^i\mathcal{R}^{jk} + \beta_4 D_i\mathcal{R}D^i\mathcal{R} + \beta_5\mathcal{R}_j^i\mathcal{R}_k^j\mathcal{R}_i^k + \beta_6\mathcal{R}\mathcal{R}_{ij}\mathcal{R}^{ij} + \beta_7\mathcal{R}^3. \tag{2.2.26}$$

One can also include a cosmological constant.

It is important to mention that for the projectable theory the field equation which corresponds to the Hamiltonian constraint is truly a constraint on the Hamiltonian, and not its density. That is, instead of having an equation of the form,

$$\mathcal{H} = 0, \tag{2.2.27}$$

in the projectable case one obtains,

$$H = \int_\Sigma \sqrt{\gamma}\mathcal{H} = 0. \tag{2.2.28}$$

This is because the variation with respect to N gives,

$$\sim \int \mathrm{d}^3x\mathrm{d}tN\sqrt{\gamma}\left[\mathcal{H}\delta N(t)/N\right], \tag{2.2.29}$$

and so to guarantee this vanishes for any variation $\delta N(t)$, which depends only on time, we need to impose that the integral over each leaf vanishes $\forall t$ rather than the integrand vanishing at every point.

Non-projectable Hořava gravity

In the non-projectable case we again have an action of the form,

$$S_{\text{np}} = \frac{1}{16\pi G}\int \mathrm{d}^3x\mathrm{d}tN\sqrt{\gamma}\left[K_{ij}K^{ij} - \lambda K^2\right] + S_{V,\,\text{proj}}, \tag{2.2.30}$$

where the potential terms have the form,

$$S_{V,\,\text{np}} = \frac{1}{16\pi G}\int \mathrm{d}^3x\mathrm{d}tN\sqrt{\gamma}\left[\mathcal{L}_1 + \frac{1}{M_*^2}\mathcal{L}_2 + \frac{1}{M_*^4}\mathcal{L}_3\right]. \tag{2.2.31}$$

However, there are many more possible terms in $S_{V,\,\text{np}}$. There are two reasons for this. First of all, instead of the potential terms being constructed out of all combinations (that are not equivalent-up-to-boundary-terms) of $\mathcal{R}_{ij}, D_i, \gamma^{ij}$ we must also add $a_i = D_i \log N$ to the list, a_i being the proper acceleration of the unit normal vector (Blas et al. 2010b). Secondly, due to the fact that N is position dependent, terms which could previously have been considered boundary terms cease being so. Overall this means there are an order of magnitude more terms in non-projectable Hořava gravity than in the projectable case. The other consequence of this second point is that the Hamiltonian constraint in non-projectable Hořava gravity is a local one.

This version will be the one of interest for Chapter 4 and an extension of it will also be studied in Section 3.2.

Detailed Balance

A principle, dubbed the detailed balance condition, has been used to reduce the number of free parameters (Horava 2009). This restriction amounts to the requirement that,

$$\mathcal{L}_V = E^{ij}G_{ijkl}E^{kl}, \tag{2.2.32}$$

where,

$$E^{ij} = \frac{1}{\sqrt{\gamma}}\frac{\delta W}{\delta \gamma_{ij}}, \tag{2.2.33}$$

W is a superpotential, with all terms up to z-th order in derivatives and,

$$G_{ijkl} = \frac{1}{2}\left(\gamma_{ik}\gamma_{jl} + \gamma_{il}\gamma_{jk}\right) + \frac{\lambda}{1-3\lambda}\gamma_{ij}\gamma_{kl}, \tag{2.2.34}$$

which is the inverse of the generalized DeWitt metric,

$$\mathcal{G}^{ijkl} = \frac{1}{2}\left(\gamma^{ik}\gamma^{jl} + \gamma^{il}\gamma^{jk}\right) - \lambda\gamma^{ij}\gamma^{kl}. \tag{2.2.35}$$

It is unclear whether or not this restriction will be respected by quantum corrections, though apparently it has some links to condensed matter theory. In any case the most general superpotential for arbitrary z is,

$$W = W^{(z)}_{\text{proj}} + W^{(z)}_{\text{extra}}, \tag{2.2.36}$$

where $W^{(z)}_{\text{proj}}$ is the most general superpotential for the projectable version and $W^{(z)}_{\text{extra}}$ are the additional terms allowed when taking the non-projectable theory (Vernieri et al. 2012; Vernieri et al. 2013). For $z = 3$ we have,

$$W^{(3)}_{\text{proj}} = \frac{1}{w^2}\int_\Sigma \omega_3(\Gamma) + \mu\int \mathrm{d}^3x\sqrt{\gamma}\left(\mathcal{R} - 2\Lambda_W\right), \tag{2.2.37}$$

$$W^{(3)}_{\text{extra}} = \mu\eta\int \mathrm{d}^3x\sqrt{\gamma}a_i a^i, \tag{2.2.38}$$

where,

$$\omega_3(\Gamma) = \epsilon^{ijk}\left(\Gamma^m_{il}\partial_j\Gamma^l_{km} + \frac{2}{3}\Gamma^n_{il}\Gamma^l_{jm}\Gamma^m_{kn}\right)\mathrm{d}^3x \tag{2.2.39}$$

is the gravitational Chern-Simons term. These superpotentials lead to the actions,

$$\begin{aligned} S^{\text{DB}}_{\text{proj}} = \frac{1}{16\pi G}\int \mathrm{d}^3x\mathrm{d}tN\sqrt{\gamma}\Bigg[&K_{ij}K^{ij} - \lambda K^2 + \beta\mathcal{R} - 2\Lambda \\ &+ \frac{M^2_{\text{pl}}}{M^2_4}\left(\frac{1-4\lambda}{4\left(1-4\lambda\right)}\mathcal{R}^2 - \mathcal{R}_{ij}\mathcal{R}^{ij}\right) \\ &+ \frac{M^2_{\text{pl}}}{M^2_6}\left(\frac{2M_{\text{pl}}}{M_4}\epsilon^{ijk}\mathcal{R}_{il}D_j\mathcal{R}^l_k - \frac{M^2_{\text{pl}}}{M^2_6}C_{ij}C^{ij}\right)\Bigg], \end{aligned} \tag{2.2.40}$$

$$\begin{aligned} S^{\text{DB}}_{\text{np}} = S^{\text{DB}}_{\text{proj}} + \frac{1}{16\pi G}\int \mathrm{d}^3x\mathrm{d}tN\sqrt{\gamma}\Bigg[&\alpha a^i a_i + \frac{M^2_{\text{pl}}}{M^2_6}\frac{2\alpha M_{\text{pl}}}{\beta M_4}C_{ij}a^i a^j \\ &+ \frac{M^2_{\text{pl}}}{M^2_4}\left\{\left(\frac{1-4\lambda}{4\left(1-3\lambda\right)}\mathcal{R}\gamma_{ij} - \mathcal{R}_{ij}\right)\frac{2\alpha}{\beta}a^i a^j - \frac{\alpha^2}{4\beta^2}\frac{3-8\lambda}{1-3\lambda}\left(a_i a^i\right)^2\right\}\Bigg]. \end{aligned} \tag{2.2.41}$$

In the above, the following definitions have been used,

$$\frac{M_6^2}{M_{\rm pl}^2} = \frac{w^2}{2}, \quad \frac{M_4^2}{M_{\rm pl}^2} = \frac{1}{\mu^2}, \quad \beta = \frac{\Lambda_W \mu^2}{(1-3\lambda)}, \tag{2.2.42}$$

$$\alpha = \frac{\eta\beta}{\mu}, \quad \Lambda = \frac{3}{2\mu^2}\beta^2\,(1-3\lambda)\,. \tag{2.2.43}$$

The theory with detailed balance has several criticisms in the literature (Sotiriou et al. 2009a; Charmousis et al. 2009; Lu et al. 2009), beyond the question of whether or not the principle is respected by quantum corrections. These are:

1. There are parity violating terms, which are inevitable for $z = 3$.

2. The scalar graviton does not gain the $z = 3$ dispersion relation, thus the power-counting is spoiled.

3. The (bare) cosmological constant is both negative and large.

Additionally the projectable version has the bad infrared behaviour that is generic to the projectable model (as will be discussed in detail in subsection 2.2.2). Most of these problems can actually be solved within detailed balance, simply by constructing the theory with $z = 4$ instead of $z = 3$ and restoring parity invariance (Vernieri et al. 2012). The problem posed by the cosmological constant is an interesting one. Some have proposed that it may be a mechanism to cancel off the vacuum energy of matter (Appignani et al. 2010), though how exactly this would tune itself is not clear. There are also questions about how the detailed balance principle might be modified in the non-projectable case, as there is an additional field one could vary the superpotential with (again see Vernieri et al. 2012 for a more complete discussion).

In any case, one important thing to note is that in the infra-red (*i.e.* where there $z = 1$ terms dominate) the only difference between a theory with or without detailed balance is the fixing of the cosmological constant. There-

fore, for considerations below the energy scale $\sim M_*$ (or M_4) it is sufficient to consider the actions,

$$S^{\rm IR}_{\rm proj} = \frac{1}{16\pi G}\int {\rm d}^3x{\rm d}tN\sqrt{\gamma}\left[K_{ij}K^{ij} - \lambda K^2 + \beta\mathcal{R} - 2\Lambda\right], \tag{2.2.44}$$

$$S^{\rm IR}_{\rm np} = \frac{1}{16\pi G}\int {\rm d}^3x{\rm d}tN\sqrt{\gamma}\left[K_{ij}K^{ij} - \lambda K^2 + \beta\mathcal{R} - 2\Lambda + \alpha a^i a_i\right], \tag{2.2.45}$$

and treat all the constants $(\lambda, \beta, \Lambda, \alpha)$ as free parameters. It is also worth noting that, in vacuum, one can freely set $\beta = 1$ by scaling the coordinates.

Consistency and strong coupling

Hopefully it is clear from Equations (2.2.44) &(2.2.45) that a GR limit, if it exists, would correspond to $\lambda \to 1$, $\beta \to 1$ and $\alpha \to 0$. We will shortly see that, in additional to the spin-2 graviton, there is a propagating scalar mode in Hořava gravity. The main question about the existence of a corner of the theory which behaves like GR at low energies is thus how the scalar mode behaves in this limit (see Sotiriou et al. 2009a; Lu et al. 2009; Henneaux et al. 2010; Li et al. 2009; Charmousis et al. 2009; Koyama et al. 2010; Blas et al. 2009; Blas et al. 2010b; Papazoglou et al. 2010; Blas et al. 2010a; Kimpton et al. 2010 *etc.* for original works in this direction). To study this it will be instructive to expand the actions (2.2.44) &(2.2.45) to quadratic order around Minkowski space (having set $\Lambda = 0$). Since we are expanding around flat space it will be useful to use the decomposition:

$$N = 1 + A, \quad N^i = B^i + \partial^i B, \tag{2.2.46}$$

$$\gamma_{ij} = \delta_{ij}\left(1 + 2\psi\right) + \left(\partial_i\partial_j - \frac{\delta_{ij}}{3}\Delta\right)E + \partial_{(i}E_{j)} + h_{ij}, \tag{2.2.47}$$

where, $\partial^i h_{ij} = h^i_i = 0 = \partial^i B_i = \partial^i E_i$. For the tensor and vector sectors the quadratic actions coincide and are given by (in terms of Fourier modes) (this follows the presentation style of Coates et al. 2016),

$$S^{(2)}_{\rm tensor} = \frac{1}{16\pi G}\int {\rm d}t{\rm d}^3k\frac{1}{4}\left(|\dot{h}_{ij}|^2 - \beta k^2|h_{ij}|^2\right) \tag{2.2.48}$$

$$S^{(2)}_{\rm vector} = \frac{1}{16\pi G}\int {\rm d}t{\rm d}^3k\frac{1}{2}\left|B^i - \frac{\dot{E}^i}{2}\right|^2. \tag{2.2.49}$$

Solving for B^i and replacing it in the action shows that there are no dynamical vector modes, and the tensor sector is stable so long as,

$$\beta > 0. \tag{2.2.50}$$

For the scalar sector the projectable and non-projectable cases are inequivalent. This is because $N = N(t)$ in the projectable case and so we can freely set A to 0 there. Thus we will write only the action for the scalar sector of the non-projectable theory, and extract the criteria for the projectable theory by setting $A \to 0$. The relevant action is,

$$\begin{aligned} S^{(2)}_{\text{scalar}} = \frac{1}{16\pi G}\int \mathrm{d}t\mathrm{d}^3k \Bigg[& 3\left(1-3\lambda\right)\left|\dot{\psi}+\frac{k^2}{6}\dot{E}\right|^2 + 2\beta k^2\left|\psi+\frac{k^2}{6}E\right|^2 \\ & + k^4\left(1-\lambda\right)\left|B-\frac{\dot{E}}{2}\right|^2 + k^2\left(1-3\lambda\right)\left\{\left(B-\frac{\dot{E}}{2}\right)^*\left(\dot{\psi}+\frac{k^2}{6}\dot{E}\right)+\text{c.c.}\right\} \\ & + \alpha k^2|A|^2 + 2\beta k^2\left\{A^*\left(\psi+\frac{k^2}{6}E\right)+\text{c.c.}\right\}\Bigg]. \end{aligned} \tag{2.2.51}$$

From here we define,

$$\mathcal{B} = B - \frac{\dot{E}}{2}. \tag{2.2.52}$$

Varying the action w.r.t $\mathcal{B}$ and replacing the solution,

$$\mathcal{B} = -\frac{1}{k^2}\frac{1-3\lambda}{1-\lambda}\left(\dot{\psi}+\frac{k^2}{6}\dot{E}\right), \tag{2.2.53}$$

in the action one arrives at,

$$\begin{aligned} S^{(2)}_{\text{scalar}} = \frac{1}{16\pi G}\int \mathrm{d}t\mathrm{d}^3k \Bigg[& \left(6+\frac{4}{\lambda-1}\right)\left|\dot{\psi}+\frac{k^2}{6}\dot{E}\right|^2 + 2\beta k^2\left|\psi+\frac{k^2}{6}E\right|^2 \\ & + \alpha k^2|A|^2 + 2\beta k^2\left\{A^*\left(\psi+\frac{k^2}{6}E\right)+\text{c.c.}\right\}\Bigg]. \end{aligned} \tag{2.2.54}$$

We can now immediately read off that the projectable theory does not have consistent stability criteria. Unless one turns the scalar mode ($\psi + \frac{k^2}{6}E \equiv \Psi$) into a ghost, the scalar sector requires $\beta < 0$ to avoid a gradient instability, and thus there are no regions of parameter space where both the scalar

mode and tensor modes are stable. One could attempt to arrange the parameters such that the time scale of the instability (in the scalar sector) is longer than the age of the universe. To do so requires $|\lambda - 1| \lesssim 10^{-61}$. Doing so causes strong coupling to be present at very low energy scales. To see the heart of the problem, imagine one has a Lagrangian with an auxiliary scalar field and a dynamical scalar field such that, in terms of perturbations, the part of the Lagrangian which depends on the auxiliary scalar field can be written as,

$$\mathcal{L}_b = \left(ab^2 + b\phi\right)\left(1 + \mathcal{O}(\phi)\right), \tag{2.2.55}$$

then the variation with respect to b gives, to lowest order,

$$b = \frac{1}{a}\phi. \tag{2.2.56}$$

Replacing this into the action we have,

$$\mathcal{L}_b = \frac{2}{a}\left(\phi^2\right)\left(1 + \mathcal{O}(\phi)\right). \tag{2.2.57}$$

It is clear that the cubic terms of $\mathcal{L}_b$ will dominate over the quadratic part of rest of the Lagrangian (

textiti.e. the part independent of b) if a is smaller than 2ϕ, thus the perturbation theory breaks down. In the case of the low energy limit of Hořava gravity the role of a is played by $\lambda - 1$ and one can associate an energy scale with this loss of perturbativity. Because of this, one could state that this is simply the scale at which the higher order terms (in derivatives) must become important. However, the strong coupling scale goes like $(\lambda - 1)^{3/4} M_{\rm pl}$ (see Sotiriou 2011 and references therein), and the requirement that the instability has not yet shown up makes this scale unacceptably small. It has been previously argued that the strong coupling may not pose a problem for classical phenomenology as there are indications that a Vainshtein like mechanism comes into effect (Mukohyama 2010; Gumrukcuoglu et al. 2012). One can see this from looking at the form of the kinetic term for Ψ, in the limit $\lambda \to 1$; the kinetic term blows up, so although there is a break down of the perturbation structure this is perhaps better termed weak-coupling than strong-coupling (in terms of canonically normalised fields).

For the non-projectable case we must also integrate out A, before reading off any stability criteria. Doing so we arrive at,

$$S^{(2)}_{\text{scalar}} = \frac{1}{16\pi G}\int \mathrm{d}t\mathrm{d}^3k \left[\left(6+\frac{4}{\lambda-1}\right)\left|\dot{\Psi}\right|^2 + 2\beta k^2\left(1-\frac{2\beta}{\alpha}\right)|\Psi|^2\right]. \tag{2.2.58}$$

Thus all sectors can be stable in the non-projectable case iff,

$$2\frac{3\lambda-1}{\lambda-1} > 0, \quad \beta > 0, \quad 2\beta > \alpha > 0. \tag{2.2.59}$$

The first condition can be rewritten as,

$$\lambda < \frac{1}{3} \text{ or } > 1. \tag{2.2.60}$$

$\lambda > 1$ is usually considered to be the interesting branch, due to the possibility of being close to GR. As there need not be any instability, $|\lambda - 1|$ needn't be so small. There are also multiple possible strong coupling scales in this case, as integrating out A when α is small follows the same pattern (Papazoglou et al. 2010). Once taking into account Čerenkov constraints (Elliott et al. 2005) one can find that the strong coupling scale is $M_{\text{sc}} \sim \sqrt{\lambda-1}M_{\text{pl}}$ (Papazoglou et al. 2010; Kimpton et al. 2010) and thus, if one wishes to avoid strong coupling, constraints on $\lambda - 1$ also lead to an upper-bound on the scale at which the higher order derivatives become important. This means that one has a double sided limit on the scale at which these terms can become important and the theory (without strong coupling) is actually falsifiable, which is a surprisingly unusual feature of alternative theories of gravity (and candidate quantum gravity theories in particular).

Recent experimental constraints on the low energy parameters can be found in (Yagi et al. 2014). Though note that the conventions used are such that, if we denote their values with a subscript Y,

$$\frac{1}{1-\beta_Y} = \beta, \quad \frac{\alpha_Y}{1-\beta_Y} = \alpha, \quad \frac{1+\lambda_Y}{1-\beta_Y} = \lambda. \tag{2.2.61}$$

These low energy parameters, as mentioned, in the non-projectable case can also be translated into an upper bound on the scale at which the higher order derivative terms become important. Taking all of these into account

the current bounds do not yet rule out Hořava gravity, though the allowed parameter space is not large and one must hope that the GR values of these constants are attractive fixed points in the IR which are rapidly converged upon.

Finally for this section let us stress the importance of requiring that $\alpha \neq 0$. We have already seen that its presence renders the non-projectable theory sufficiently different to the projectable case so as to avoid the complications therein. However, the requirement that $\alpha \neq 0$ is much stronger than avoiding complicated IR dynamics, the theory is dynamically inconsistent if $\alpha = 0$. For the full details see Henneaux et al. 2010; Li et al. 2009; Blas et al. 2009; Blas et al. 2010b.

Defining black holes

Earlier we mentioned that, in Einstein-æther theory, causality is "quasi-relativistic", meaning that there is some metric whose null cones determine causality but this metric is not that which matter couples to [if one is to avoid Čerenkov radiation (Elliott et al. 2005)]. This allows the use of all of the usual definitions in GR for causality and black holes, but they should be applied to whichever metric defines the causality. However, in Hořava gravity this is no longer the case. There are two issues, the most obvious issue is that the higher order derivatives lead to arbitrarily high propagation speeds. One might hope that in the infra-red one can avoid this issue by neglecting the higher order terms, however, in the non-projectable case there is an instantanteous mode at all energy scales (Blas et al. 2011b). This can be seen from the fact that A solves an elliptic equation, from the previous section. One might wonder why this is not also the case for $\mathcal{B}$. The difference is essentially because its equation is the scalar part of the momentum constraint, and thus only needs to be imposed as a relation between initial data. For more details see sections 2.2.3 & 4.3.

The result of the above is that Hořava gravity is truly non-relativistic, rather than quasi-relativistic. This means that the usual definitions of GR

are not applicable. However, one can define some, rather intuitive, notions of causality and black holes in Hořava gravity however (Blas et al. 2011a for many of the initial ideas and Bhattacharyya et al. 2016c for a more rigorous approach). A quick and dirty summary is that: the future (past) of a point is the same as the future (past) of the leaf of the foliation to which it belongs, causal curves are ones which pass through any given leaf at most once, and a horizon is a leaf which does not have spacelike infinity in its future (or as part of the leaf). A quicker and dirtier summary is that the causality is essentially the same as in pre-relativity physics, and black hole horizons are instants of time whose future is “inwards". To give an example of such a horizon, one can take the Schwarzschild solution and, in a suitable decoupling limit, construct a preferred foliation on it (see Blas et al. 2011a for the full details). Figure 2.1 shows the spacetime diagram of this situation (Barausse et al. 2013a). The horizons in this case are referred to as Universal Horizons (UHs). Of course there remain questions which will be partially explored in 4.3, in particular whether or not these UHs can actually form from collapse. There are also other questions about the thermodynamics and stability of these beyond spherical symmetry which we shall not explore, see Blas et al. 2011a; Liberati et al. 2017.

2.2.3 *Relationship between Einstein-æther theory and Hořava gravity*

One of the more useful tools for Chapter 4 is covariantization. Using a technique reminiscent of the Stückelberg trick in massive electrodynamics, one can write the action of non-projectable Hořava in a covariant form (Germani et al. 2009; Jacobson 2010 also see Blas et al. 2009; Kimpton et al. 2010 for useful presentations). To do so one promotes the time coordinate t to a bona fide scalar field, or equivalently constructs the terms in the Hořava action using a Lorentzian metric and a unit timelike vector, *c.f.* Section 1.1.1. The terms which are second order in derivatives in Hořava gravity become

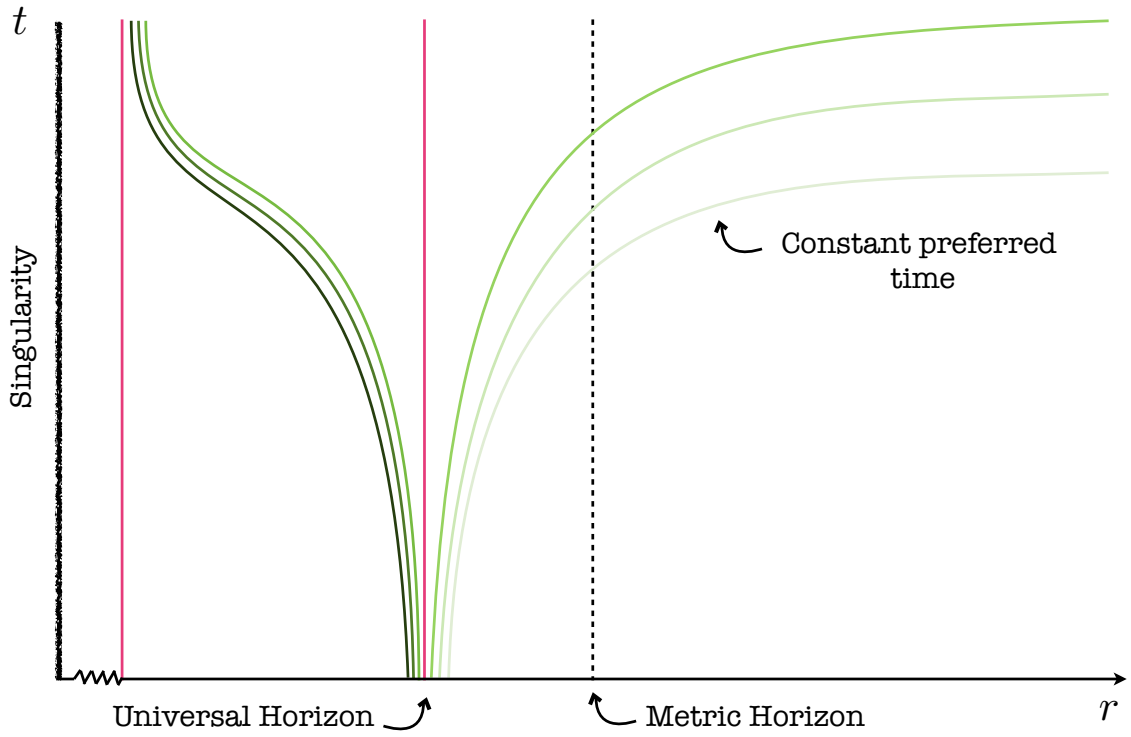


Figure 2.1: Spacetime diagram of the preferred foliation and Universal Horizon in decoupling limit on a Schwarzschild background. Republished with permission of IOP Publishing, from Barausse et al. 2013a; permission conveyed through Copyright Clearance Center, Inc.

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terms which are formally identical to the Einstein-æther Lagrangian, with the coefficients related by,

$$\beta = \frac{1}{1-c_{13}}, \quad \lambda = \frac{1+c_2}{1-c_{13}}, \quad \alpha = \frac{c_{14}}{1-c_{13}}. \tag{2.2.62}$$

However, the fundamental fields in the two cases are distinct. In Einstein-æther theory one has $(\mathbf{g}, u)$ whereas in the covariant formulation of Hořava the fundamental fields are $(\mathbf{g}, \phi)$. To make the actions formally identical one must define,

$$u_\mu = -\frac{\partial_\mu \phi}{\sqrt{-\partial_\alpha \phi \partial^\alpha \phi}}, \tag{2.2.63}$$

c.f. Equations (1.1.12)&(1.1.13). When one truncates the action to include only these lowest order terms (in derivatives), ϕ is referred to as the Khronon,

and the theory is often called the Khronometric theory (coined in Blas et al. 2011b). The covariantization is of the form,

$$S_H = \frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{-g}\left[R - Z^{\alpha\mu\beta\nu}\nabla_\alpha u_\beta \nabla_\mu u_\nu + \Big(\lambda(u^2+1)\Big) + \mathcal{O}\left(\partial^4, u^2\right)\right], \tag{2.2.64}$$

where $u_\mu = -N\partial_\mu\phi$. Now let us look more carefully at how the equations of the Khronometric theory can be derived from this action. It will be useful to first perform the variation as though the æther was the fundamental field. One has to choose how to proceed, either performing general variations with the Lagrange multiplier term $\lambda(u^2+1)$, or else manually restricting the variations to maintain the norm of the æther. Additionally one has to choose between treating u^α or u_α as the fundamental field. As the constraint on u is algebraic, all of the possible choices lead to equivalent equations and we will choose to treat u_α as the fundamental field.

Performing the unrestricted variation of the æther action one arrives at,

$$\delta S_{\text{æ}}[g^{\alpha\beta}, u_\alpha] = \frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{-g}\left[\left(G_{\alpha\beta} - \hat{t}_{\alpha\beta} + \lambda u_\alpha u_\beta\right)\delta g^{\alpha\beta} + \left(2\hat{\text{Æ}}^\alpha + \lambda u^\alpha\right)\delta u_\alpha\right],$$

The restricted variations version of this involves requiring that,

$$\delta(u^2) \equiv \delta(u_\alpha u_\beta g^{\alpha\beta}) \equiv u_\alpha u_\beta \delta g^{\alpha\beta} + 2\delta u_{(\alpha}u_{\beta)}g^{\alpha\beta} = 0, \tag{2.2.65}$$

i.e.

$$u^\alpha \delta u_\alpha = \frac{1}{2}u^\alpha u_\alpha u_\mu u_\nu \delta g^{\mu\nu}, \tag{2.2.66}$$

so that finally,

$$\delta u_\alpha = \frac{1}{2}u_\alpha u_\mu u_\nu \delta g^{\mu\nu} + p^\mu_\alpha \delta u_\mu, \tag{2.2.67}$$

where p is the projector orthogonal to u. By inspection one can see that this will eliminate the Lagrange multiplier from the variation, and project the æther equation orthogonal to u. We will write the resulting equation for the æther as $\vec{\text{Æ}}^\alpha = 0$.

Now we turn to Khronometric theory by making the replacement, $u_\mu \to -N\partial_\mu\phi$, where $N = \sqrt{-\left(\partial_\mu\phi\partial^\mu\phi\right)^{-1}}$ (see equation (2.2.63). The generic variation of the æther becomes,

$$\delta u_\mu = -\delta(N\partial_\mu\phi) = -\left(N\partial_\mu\delta\phi - \frac{\delta N}{N}u_\mu\right) = \frac{1}{2}u_\alpha u_\beta u_\mu \delta g^{\alpha\beta} - N\left(p^\alpha_\mu\partial_\alpha\delta\phi\right). \tag{2.2.68}$$

Note that this is identical to the restricted form of the variation with respect to the æther, therefore any choice of how to proceed with the æther variation will lead to equivalent equations of motion for ϕ. We will denote the variation of the Khronometric action as,

$$\delta S_\chi\left[g^{\alpha\beta},\phi\right] = \frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{-g}\left[\mathcal{E}_{\alpha\beta}\delta g^{\alpha\beta} + 2\nabla_\alpha\left(N\vec{\text{Æ}}^\alpha\right)\delta\phi\right]. \tag{2.2.69}$$

The field equations have fourth order time derivatives of ϕ, and thus appears to be a ghost. This occurs except when one picks the foliation whose leaves are level sets of ϕ. It is also worth noting that, in this covariant picture, there does not seem to be any reason to discount terms which would contribute as (for example) K^4 in the preferred foliation when going beyond the IR truncation. Further, the truncation itself looks inconsistent, with the metric having only second order terms, in derivatives, while the scalar field has higher order ones.

Before moving onto my original work, it is worth mentioning that certain corners of Horndeski theory can be Lorentz violating. We shall see exactly how this can come about in a manifestly covariant theory later in the context of the Cuscuton Afshordi et al. 2007. Its action is given by the choices,

$$f_2 = \mu^2\sqrt{2X} - V(\varphi), \quad f_3 = f_4 = f_5 = 0. \tag{2.2.70}$$

One can become suspicious by noticing that the speed of sound for the scalar field diverges (Afshordi et al. 2007).

3

HOŘAVA GRAVITY AND NATURALNESS IN THE MATTER SECTOR

It has been discussed in Section 2.2.2 that non-projectable Hořava gravity has regions of parameter space for which it exhibits viable phenomenology in the infra-red, classical regime. However, despite this and its attractive features from a theoretical point of view, there are still open problems. The two main areas which are of interest to us are some aspects of its quantum phenomenology, specifically its impact on the matter sector, and questions related to black hole formation. The first of these will be studied in this chapter, the other will studied in Chapter 4.

3.1 VERIFICATION AND EXTENSION OF THE NATURALNESS PROBLEM

The following is largely based on Coates et al. 2017b.

As one would expect, once Hořava gravity is coupled to it, quantum effects will generate Lorentz violations in matter via loop contributions. Therefore an important question is whether or not these Lorentz violations can be sufficiently small to satisfy the very stringent experimental constraints in the matter sector (Mattingly 2005; Liberati 2013; Kostelecky et al. 2011). Before moving on to study some approaches to this question, it is worth mentioning two mechanisms that won't be considered below.

The first of these is supersymmetry. The mechanism here relies on the fact that an exactly supersymmetric theory disallows Lorentz violating cor-

rections to the sound speeds (Groot Nibbelink et al. 2005; Xue 2010; Pujolas et al. 2012; Pospelov et al. 2014). Therefore, once there is some susy breaking, any LV corrections to the sound speeds can be suppressed by $(m_{\text{soft}}/M)^2$, where m_{soft} is the soft SUSY breaking scale and M is some Lorentz violating scale, M_* in Hořava gravity. However, there is no known supersymmetric version of Hořava gravity. As a proposal without a concrete realisation we will consider it no further here.

The second is that strong coupling above a certain scale could be a viable suppression mechanism (Chadha et al. 1983; Anber et al. 2011; Bednik et al. 2013). As one of the main attractions of Hořava gravity is its possibility to be a perturbatively renormalizable theory of quantum gravity, this non-perturbative behaviour is not so desirable.

One suggested approach was based on the fact that the modified dispersion relations of a Lifschitz scalar naturally leads to a suppression of Lorentz violating corrections to any fields coupled to it, if those couplings are suppressed by some scale M (Pospelov et al. 2012). Instead of generically having corrections which diverge as Λ^2/M^2, where Λ is the UV cutoff, one instead finds that the corrections scale as M_*^2/M^2 where M_* is the scale at which the higher order terms in the dispersion relation start to dominate over the relativistic part, *i.e.*

$$\omega^2 \sim k^2 + \mathcal{O}(1)\frac{k^6}{M_*^4}. \tag{3.1.1}$$

Therefore, if $M_* \ll M$ these corrections can be heavily suppressed. Naïvely speaking one could expect this to be true of Hořava gravity, where $M = M_{\text{pl}}$ is the high energy scale, and the requirement that the modified dispersion relation becomes important before the break down of the perturbative expansion of Hořava gravity (as discussed in Section 2.2.2 already requires $M_* < M_{\text{pl}}$.

Unfortunately the naïve expectation that this suppression occurs in Hořava gravity is exactly that, and it turns out that this feature of the Lifschitz scalar

does not carry over to Hořava gravity (Pospelov et al. 2012). The reason for this is essentially that the shift, N^i, shows up only in the kinetic term,

$$K_{ij}K^{ij} - \lambda K^2. \tag{3.1.2}$$

Because of this it does not enjoy the $1/k^6$ suppression in its propagators and one recovers the quadratic divergence (in $3+1$ dimensions). To see this more explicitly, we will first review some of the set up used to prove the renormalizability of projectable Hořava gravity (Barvinsky et al. 2016) and then calculate the one loop corrections to the difference in squared sound speeds between a scalar and $\mathrm{U}(1)$ gauge field coupled to projectable Hořava gravity. To demonstrate that there is likely no possibility of removing this problem by some compactification of a higher dimensional theory, we shall extend the results of Pospelov et al. 2012 to $d+1$ dimensions with $z = d$. The generalisation to $d+1$ dimensions is, to my knowledge, novel and the use of the guage-fixing terms which lead to regular propagators makes the result much harder to doubt.

The issues of strong coupling and instabilities are problems in the infrared only, and thus we can extract the UV behaviour of the theory by considering only the kinetic terms and terms of order $2d$ in spatial derivatives. That is, we require the anisotropic scaling $z = d$ and keep only the leading UV terms (those of highest momentum dimension). For calculational purposes we will consider the Wick rotated theory with,

$$t \to \tau = it, \quad N^J \to N^j_{\mathrm{E}} = -iN^j, \tag{3.1.3}$$

though we will drop the subscript E. We will perform the calculation perturbatively around the background of Euclidean space with

$$N^i = 0 + N^i, \quad \gamma_{ij} = \delta_{ij} + h_{ij}. \tag{3.1.4}$$

It is also worth noting that choosing a gauge-fixing term which contains only fields and local operators greatly increases the difficulty of performing calculations. By allowing for non-local gauge-fixing terms (*i.e.* allowing

inverse Laplacians) one can decouple the shift and 3-metric, whilst generating an effective kinetic term for the shift and still respecting the anisotropic scaling. To do so one includes the gauge-fixing term,

$$\mathcal{L}_{\text{g.f.}} = \frac{1}{16\pi G}\sigma F^i O_{ij} F^j, \tag{3.1.5}$$

where,

$$O_{ij} = -\Delta^{-d+2}\left[\delta_{ij}\Delta + \xi\partial_i\partial_j\right], \tag{3.1.6}$$

$$F^i = \dot{N}^i + \frac{1}{2\sigma}O^{-1}_{ij}\partial_k h_{jk} - \frac{\lambda}{2\sigma}O^{-1}_{ij}\partial_j h. \tag{3.1.7}$$

One can then check (most easily by decomposing the shift into a scalar and vector part) that the Fourier space propagator for the shift is,

$$G^{ij}_N(P) = a_1\left(\delta_{ij} - \hat{p}_i\hat{p}_j\right)\mathcal{P}_1 + a_2\hat{p}_i\hat{p}_j\mathcal{P}_2, \tag{3.1.8}$$

where,

$$\mathcal{P}_I = \frac{p^{2(d-1)}}{\omega^2 + \alpha_I^2 p^{2d}}, \quad \hat{p}_i = \frac{p_i}{p}, \quad a_1 = \frac{1}{\sigma}, \tag{3.1.9}$$

$$\alpha_1^2 = \frac{1}{2\sigma}, \quad a_2 = \frac{1+\xi}{\sigma}, \quad \alpha_2^2\frac{(1-\lambda)(1+\xi)}{\sigma}. \tag{3.1.10}$$

As the graviton propagator depends on the kinetic terms, the gauge-fixing terms, and the terms of order $2d$ in spatial derivatives, the general propagator is not so easy to write down. However, it is worth noting that the non-local terms are restricted to only the shift and so, unless the spatial derivative terms are finely tuned, the graviton propagator will always be of the form,

$$G_h^{ijkl}\sum_I M_I^{ijkl}\frac{a'_I}{\omega^2 + \alpha'^2_I p^{2d}}, \tag{3.1.11}$$

where I labels the helicity of the modes in $\mathbf{h}$, and M_I^{ijkl} is the relevant combination of δ_{mn} and $\hat{p}_m$ for a spin-I mode in d dimensions. Now taking the fact that we are going to be considering minimally coupled relativistic fields, and that we are interested only in the contributions to their relative

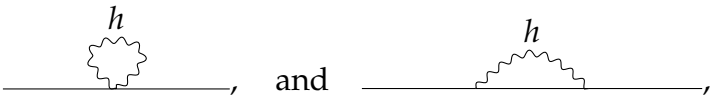
squared sound speeds one can see that all of the most divergent graviton contributions must be a linear combination of integrals of the form,

$$a^{-1}\mathcal{J}_4 = \int \frac{\mathrm{d}\omega \mathrm{d}^d k}{(2\pi)^{d+1}} \frac{1}{\omega^2 + \alpha^2 k^{2d}}, \tag{3.1.12}$$

$$a^{-1}\mathcal{J}_5 = \int \frac{\mathrm{d}\omega \mathrm{d}^d k}{(2\pi)^{d+1}} \frac{1}{\omega^2 + \alpha^2 k^{2d}} \frac{\omega^2}{\omega^2 + \beta^2 k^2}, \tag{3.1.13}$$

$$a^{-1}\mathcal{J}_6 = \int \frac{\mathrm{d}\omega \mathrm{d}^d k}{(2\pi)^{d+1}} \frac{1}{\omega^2 + \alpha^2 k^{2d}} \frac{k^2}{\omega^2 + \beta^2 k^2}. \tag{3.1.14}$$

This is essentially the statement that the relevant diagrams are of the form,



and, (3.1.15)

and the metric determinant in the action guarantees that there are the derivative interactions necessary for $\mathcal{J}_5$ (it also will contribute some part as $\mathcal{J}_6$). To calculate these integrals we introduce a UV cutoff for k, Λ and an infra-red cutoff (where required), m. As it doesn't have an effect on the qualitative results and simplifies the expressions significantly, we will simply integrate over all ω (from $-\infty$ to ∞) as a first step. In doing so one finds that,

$$a^{-1}\mathcal{J}_4 \propto \log\frac{\Lambda}{m}, \quad a^{-1}\mathcal{J}_5 \propto \log\Lambda + \mathcal{O}(1), \quad a^{-1}\mathcal{J}_6 \propto \log m + \mathcal{O}(1). \tag{3.1.16}$$

In particular no divergence is worse than logarithmic and $\mathcal{J}_6$ is even UV finite. As the gauge-fixing term decouples the shift and 3- metric at quadratic order, and we are interested in verifying power-law divergences in the UV we shall, from here, ignore any possible graviton contributions.

It will also be useful to define,

$$a^{-1}\mathcal{J}_1 = \int \frac{\mathrm{d}\omega \mathrm{d}^d k}{(2\pi)^{d+1}} \mathcal{P}, \tag{3.1.17}$$

$$a^{-1}\mathcal{J}_2 = \int \frac{\mathrm{d}\omega \mathrm{d}^d k}{(2\pi)^{d+1}} \frac{\omega^2}{\omega^2 + \beta^2 k^2} \mathcal{P}, \tag{3.1.18}$$

$$a^{-1}\mathcal{J}_3 = \int \frac{\mathrm{d}\omega \mathrm{d}^d k}{(2\pi)^{d+1}} \frac{k^2}{\omega^2 + \beta^2 k^2} \mathcal{P}. \tag{3.1.19}$$

Now, as we have the form of the shift propagator we need only specify the matter action, including its interactions with the shift. Introducing β as the bare speed of light for book-keeping purposes we have for a relativistic scalar field,

$$S = \frac{1}{2}\int \mathrm{d}^{d+1}x \left[\left(\left(\partial_t - N^i\partial_i\right)\phi\right)^2 + \beta^2\left(\partial_i\phi\partial_i\phi\right)\right]. \tag{3.1.20}$$

As we wish to find the effective action for ϕ we calculate,

$$\Gamma_1(\varphi) = \frac{1}{2}\mathrm{tr}\left(\log S^{(2)}(\varphi) - \log S^{(2)}(0)\right), \tag{3.1.21}$$

where $S^{(2)}(\varphi)$ is the operator matrix obtained by taking all possible second order variational derivatives of the action and setting $N^i = 0, \phi = \varphi$. For example, introducing the test fields f, u^i, v^j,

$$\int \mathrm{d}^{d+1}y\, S^{(2)}_{\phi\phi} f(y) = -\left[\partial_0^2 f(x) + \beta^2\Delta f(x)\right], \tag{3.1.22}$$

$$\int \mathrm{d}^{d+1}y\, v(x)^i S^{(2)}_{N^iN^j} u(y)^j = v^i\left[\left(\partial_i\varphi\right)\left(\partial_j\varphi\right)\right]u^j. \tag{3.1.23}$$

Note that $S_{\phi\phi}$ happens to coincide with the free scalar kinetic operator. If we define the propagator matrix G and write,

$$G.S^{(2)}(\varphi) = G.S^{(2)}(0) + G.U(\varphi) = 1 + G.U(\varphi), \tag{3.1.24}$$

then one finds that U has components which satisfy,

$$\begin{aligned}
\int \mathrm{d}^{d+1}x\mathrm{d}^{d+1}y\, f(x)U_{\phi\phi}(\varphi)g(y) &= 0,\\
\int \mathrm{d}^{d+1}x\mathrm{d}^{d+1}y\, v^i(x)U_{N^iN^j}(\varphi)u^j(y) &= \int \mathrm{d}^{d+1}x\, v^i(x)u^j(x)\left[\left(\partial_i\varphi(x)\right)\left(\partial_j\varphi(x)\right)\right],\\
\int \mathrm{d}^{d+1}x\mathrm{d}^{d+1}y\, f(x)U_{\phi N^i}(\varphi)v^i(y) &= \int \mathrm{d}^{d+1}x f(x)\left[\partial_0\left(v^i(x)\partial_i\varphi(x)\right)\right.\\
&\qquad \left. +\partial_i\left(v^i(x)\partial_0\varphi(x)\right)\right],
\end{aligned} \tag{3.1.25}$$

where $U_{fg} = U_{f(x)g(y)}$. Note that $U_{N\phi} = U_{\phi N}$. This means that the quadratic part of Γ_1 (since $G_{\phi N} = 0$),

$$\Gamma_1^{(2)} = \frac{1}{2}\mathrm{tr}\left(G_N^{ij}U_{N^iN^j}\right) - \frac{1}{2}\mathrm{tr}\left(G_\phi U_{\phi N^i}G_N^{ij}U_{N^j\phi}\right). \tag{3.1.26}$$

In a more explicit form the first trace is,

$$\begin{aligned}\frac{1}{2}\mathrm{tr}\left(G_N^{ij}U_{N^iN^j}\right) &= \int \mathrm{d}^{d+1}x\mathrm{d}^{d+1}y\, U_{N^i(x)N^j(y)}G_N^{ij}(x,y)\\ &= \int \mathrm{d}^{d+1}x\ \left(\partial_i\varphi(x)\right)\left(\partial_j\varphi(x)\right)G_N^{ij}(x,x).\end{aligned}$$

Writing $G_N^{ij}(x,x)$ in terms of its Fourier transform and noting that in this integral we can replace $k_ik_j \to \delta_{ij}/d$ (by rotational invariance), the coefficient of $(\partial_i\varphi)^2$ in this trace reduces to,

$$\zeta_3 = \sum_I A_I \int \frac{\mathrm{d}\omega\mathrm{d}^dk}{(2\pi)^{d+1}}\frac{a_Ik^{2(d-1)}}{\omega^2+\alpha_I^2k^{2d}} \equiv \sum_I A_I\mathcal{J}_1^{(I)}, \tag{3.1.27}$$

where,

$$A_1 = 1-\frac{1}{d},\quad A_2 = \frac{1}{d}. \tag{3.1.28}$$

The second trace can be written as,

$$\begin{aligned}\frac{1}{2}\int \mathrm{d}^{d+1}x\mathrm{d}^{d+1}y\Big[&\left(\partial_i\varphi(x)\right)\left(\partial_j\varphi(x)\right)\left(\partial_{x^0}G_N^{ij}(x,y)\right)\left(\partial_{y^0}G_\phi(y,x)\right)\\ &\dot\varphi^2(x)\left(\partial_{x^i}G_N^{ij}(x,y)\right)\left(\partial_{y^j}G_\phi(y,x)\right)+\mathcal{O}(\varphi^3,\partial^3)\Big].\end{aligned} \tag{3.1.29}$$

Writing the terms we are interested in as,

$$\frac{1}{2}\int \mathrm{d}^{d+1}x\left[\zeta_1\dot\varphi^2+\zeta_2\left(\partial_i\varphi\right)^2\right], \tag{3.1.30}$$

we again calculate these coefficients in terms of their Fourier transform. This gives,

$$\begin{aligned}\zeta_1 &= -a_2\int\frac{\mathrm{d}\omega\mathrm{d}^dk}{(2\pi)^{d+1)}}\left[\frac{k^{2(d-1)}}{\omega^2+\alpha_2k^{2d}}\frac{k^2}{\omega^2+\beta^2k^2}\right] = -\mathcal{J}_3^{(2)},\\ \zeta_2 &= -\sum_I A_I\int\frac{\mathrm{d}\omega\mathrm{d}^dk}{(2\pi)^{d+1}}\frac{a_Ik^{2(d-1)}}{\omega^2+\alpha_I^2k^{2d}}\frac{\omega^2}{\omega^2+\beta^2k^2} \equiv -\sum_I A_I\mathcal{J}_2^{(I)}.\end{aligned} \tag{3.1.31}$$

Introducing counter-terms and temporarily reviving $\hbar$, we see that the effective action is,

$$\begin{aligned}S_{\text{eff}} = \frac{1}{2}\int \mathrm{d}^{d+1}x\Big[&\left(1+\hbar\left(\zeta_1+\delta Z_\varphi\right)+\mathcal{O}(\hbar^2)\right)\dot\varphi^2+\\ &\left(\beta^2+\hbar\left(\zeta_2+\zeta_3+\beta^2\delta Z_\varphi\right)+\mathcal{O}(\hbar^2)\right)\left(\partial_i\varphi\right)^2\Big].\end{aligned} \tag{3.1.32}$$

We then find (using the wavefunction renormalization to make φ canonical and dropping $\hbar$ again) that to one-loop order,

$$\delta c_\phi^2 = \beta^2 \left[\left(1 - \frac{1}{d}\right) \mathcal{J}_3^{(1)} + \left(1 + \frac{1}{d}\right) \mathcal{J}_3^{(2)} \right], \tag{3.1.33}$$

where we have used the fact that,

$$\mathcal{J}_2 + \beta^2 \mathcal{J}_3 = \mathcal{J}_1. \tag{3.1.34}$$

For the U(1) gauge field we have,

$$S = \int \mathrm{d}^{d+1}x \left[\frac{1}{2} \left(E_i - N^k F_{ki} \right)^2 + \frac{\beta^2}{4} F_{ij} F_{ij} \right], \tag{3.1.35}$$

where, defining the operators D^E and D^F, we have,

$$D^{E\mu}_{\ \ i} = \partial_0 \delta_i^\mu - \partial_i \delta_0^\mu \implies E_i = D^{E\mu}_{\ \ i} A_\mu, \tag{3.1.36}$$

$$D^{F\mu}_{\ \ ij} = \partial_i \delta_j^\mu - \partial_j \delta_i^\mu \implies F_{ij} = D^{F\mu}_{\ \ ij} A_\mu. \tag{3.1.37}$$

Given this the operator U, defined in analogy to the scalar field case and introducing the test fields X_μ, Y_μ, satisfies,

$$\begin{aligned}
&\int \mathrm{d}^{d+1}x \mathrm{d}^{d+1}y \, X_\mu(x) U_{A_\mu(x) A_\nu(y)}(A) Y_\nu(y) = 0, \\
&\int \mathrm{d}^{d+1}x \mathrm{d}^{d+1}y \, X_\mu(x) U_{A_\mu(x) N^j(y)}(A) v^j(y) = \int \mathrm{d}^{d+1}x \, X_\mu(x) \Big[E_k D^{F\mu}_{\ \ jk} v^j \\
&\qquad\qquad + F_{jk} D^{E\mu}_{\ \ k} v^j + 2 \left(2 D^{E\mu}_{\ \ k} F_{jk} \right) v^j \Big], \\
&\int \mathrm{d}^{d+1}x \mathrm{d}^{d+1}y \, u^i(x) U_{N^i(x) A_\mu(y)}(A) Y^\mu(y) = - \int \mathrm{d}^{d+1}x \, v^i(x) \Big[E_k D^{F\mu}_{\ \ ik} \\
&\qquad\qquad + F_{ik} D^{E\mu}_{\ \ k} \Big] Y_\mu(x), \\
&\int \mathrm{d}^{d+1}x \mathrm{d}^{d+1}y \, u^i(x) U_{N^i(x) N^j(y)}(A) v^j(y) = \int \mathrm{d}^{d+1}x \, u^i(x) v^j(x) F_{ik} F_{jk}.
\end{aligned} \tag{3.1.38}$$

Thus the first trace,

$$\frac{1}{2} \mathrm{tr} \left(G.U(A) \right) = \frac{1}{2} \int \mathrm{d}^{d+1}x \, G^{ij}(x,x) \left(F_{ik} F_{jk} \right)(x). \tag{3.1.39}$$

Using the rotational invariance again we find that,

$$\frac{1}{2} \mathrm{tr} \left(G.U(A) \right) = \frac{1}{4} \zeta_3^A \int \mathrm{d}^{d+1}x \left(F_{ij} F_{ij} \right)(x), \tag{3.1.40}$$

where,

$$\zeta_3^A = 2\zeta_3^\phi. \tag{3.1.41}$$

As before there are two important contributions from the other trace,

$$\frac{1}{2}\int \mathrm{d}^{d+1}x \left[\zeta_1^A E_i E^i + \frac{\zeta_2^A}{4} F_{ij}F_{ij}\right]. \tag{3.1.42}$$

The ζ_1^A comes from the term,

$$\frac{1}{2}\int \mathrm{d}^{d+1}x \int \mathrm{d}^{d+1}y \left(E_l D^{F\mu}_{\ jl}\right)^{(x)} G^A_{\mu\nu}(x,y) \left(E_k D^{F\mu}_{\ ik}\right)^{(y)} G_N^{ij}(y,x), \tag{3.1.43}$$

Following the same procedure as in the scalar field case one can then show that the dominant piece is,

$$\begin{aligned}\zeta_1^A &= \frac{1}{d}\int \mathrm{d}^{d+1}y\, D^{F\mu\ (x)}_{\ jk} G^A_{\mu\nu}(x,y) D^{F\nu\ (y)}_{\ ik} G_N^{ij}(y,x) \\ &= -\left(1-\frac{1}{d}\right)\sum_I \mathcal{J}_3^{(I)}.\end{aligned} \tag{3.1.44}$$

For the F^2 term, on top of the usual $p_i p_j \to p^2\delta_{ij}/d$, we can also use the substitution $p_i p_j p_k p_l \to 0$ due to the structure of $F_{ik}F_{jl}$. Again (due to the canonical factor for F^2 being $1/4$) we pick up a factor of 2 and the dominant piece comes from,

$$2\int \mathrm{d}^{d+1}y\,, D^{E\mu\ (x)}_{\ l} G^A_{\mu\nu}(x,y) D^{E\nu\ (y)}_{\ k} G_N^{ij}(y,x) = \zeta_2^A \delta_{ij}\delta_{kl} \tag{3.1.45}$$

and therefore,

$$\zeta_2^A = -2\left[\left(1-\frac{1}{d}\right)\mathcal{J}_1^{(1)} - \beta^2\left(1-\frac{2}{d}\right)\mathcal{J}_3^{(1)} + \frac{1}{d}\mathcal{J}_2^{(2)}\right]. \tag{3.1.46}$$

Following the analogous wavefunction renormalization scheme (*i.e.* setting the coefficient of E^2 to $1/2$) and using the relations (3.1.34), one finds,

$$\delta c_A^2 = \left(\zeta_2^A + \zeta_3^A\right) - \beta^2\zeta_1^A = \beta^2\left[\left(3-\frac{5}{d}\right)\mathcal{J}_3^{(1)} + \left(1+\frac{1}{d}\right)\mathcal{J}_3^{(2)}\right]. \tag{3.1.47}$$

Finally we have,

$$\delta c_A^2 - \delta c_\phi^2 = \frac{2\,(d-2)}{d}\mathcal{J}_3^{(1)}, \tag{3.1.48}$$

This is valid for all $d > 1$. As, setting $\beta = 1$

$$a^{-1}\mathcal{J}_3 = \frac{1}{2^d\Gamma(d/2)\pi^{d/2}\,(d-1)}\left(\frac{\Lambda^{d-1}}{\alpha^2} + \mathcal{O}(\log\Lambda)\right), \tag{3.1.49}$$

and, for $I = 1$, $a_I/\alpha_I^2 = 2$, we have a naturalness problem whenever $d \neq 2, 1$. This is because these divergences signal a radiative instability which lead to large differences in relative sound speeds at unacceptably low energy scales. Explicitly we have,

$$\delta c_A^2 - \delta c_\phi^2 = \frac{4\,(d-2)}{2^d \Gamma(d/2)\pi^{d/2} d\,(d-1)} \left(8\pi G \Lambda\right)^{d-1} \tag{3.1.50}$$

Having established the existence of the problem we now move on to studying one proposed solution, the mixed derivative extension of Hořava gravity (Pospelov et al. 2012).

3.2 FULL MIXED DERIVATIVE EXTENSION

The following is based on Coates et al. 2016.

For this section we shall focus on $d = 3$ and choose units in which,

$$\frac{1}{16\pi G} \to \frac{M_p^2}{2}. \tag{3.2.1}$$

When the naturalness problem was initially found it was suggested that the problem could be remedied by including so called mixed derivative terms (Pospelov et al. 2012). In particular it was noted that adding the term,

$$\frac{2M_p^2}{M_*^2} D^i K_{ij} D_k K^{kj}, \tag{3.2.2}$$

to the bare action turns this power-law divergence into a purely logarithmic one, with the suppression M_*^2/M_p^2. However, one can ask what other terms are allowed with the same dimensions, and whether they spoil the power-counting renormalizability of Hořava gravity. The latter is especially important as including such terms leads (in the absence of fine tuning) to the dispersion relation,

$$\omega^2 \propto k^4, \tag{3.2.3}$$

in the UV. Following the presentation in Section 2.2 (also see Colombo et al. 2015a; Colombo et al. 2015b for some additional details), we note that in-

cluding an analogous term (with an arbitrary number of spatial derivatives, y) in the Lifschitz scalar leads to a propagator of the form,

$$\mathcal{P}_{z,y} = \frac{1}{\omega^2 - m^2 - \sum_{n=1}^{z-y} \alpha_n^2 M^{2(1-n)} k^{2n}} \frac{1}{k^{2y}}, \tag{3.2.4}$$

in terms of Equation (2.2.12),

$$\mathcal{P}_{z,y} = \frac{1}{k^{2y}} \mathcal{P}_{z-y}. \tag{3.2.5}$$

Therefore the pole of the propagator in the UV is at,

$$\omega \sim \alpha_{z-y} M^{1+y-z} k^{z-y}, \tag{3.2.6}$$

and the energy cutoff corresponding to a momentum cutoff Λ is,

$$\Omega = \alpha_{z-y} M^{1+y-z} \Lambda^{z-y}. \tag{3.2.7}$$

Hence,

$$\int \mathrm{d}\omega \mathrm{d}^d k \, \mathcal{P}_{z,y} = \Omega \Lambda^d \alpha_{z-y}^{-2} M^{2(z-y)-2} \Lambda^{2y-2z} = \alpha_{z-y}^{-1} M^{z-y-1} \Lambda^{d-y-z}. \tag{3.2.8}$$

More generally we have $(z-y+d)L - 2zP$ from loops involving no derivative interactions. Allowing derivative interactions up to order $2(z+\delta)$ gives a maximum degree of divergence,

$$D \leq (z-y+d)L - 2zP + 2(z+\delta)V. \tag{3.2.9}$$

Then, using Euler's formula we have,

$$D \leq 2z + (d-y-z)\, L + 2\delta V. \tag{3.2.10}$$

Therefore, power-counting renormalizability requires $z \geq d-y$ and $\delta \leq 0$. We can however go further, a time derivative scales as m and so to maintain unitarity we require that $4m > 2z$ (three time derivatives can be ruled out by time reversal symmetry). This implies the additional condition that,

$$m > y. \tag{3.2.11}$$

Bringing these together we have, for $y = 1$,

$$m \geq 2, \quad z \geq 3, \tag{3.2.12}$$

thus allowing the proposed terms. From here we shall consider the building blocks that are required for the most general extension to the (non-projectable) Hořava action with this scaling and then consider its perturbations around Minkowski. We will denote the part of the action including mixed derivatives as,

$$\frac{M_p^2}{2M_*^2}\int \mathrm{d}^3x\mathrm{d}tN\sqrt{\gamma}\left[\mathcal{L}_\times\right]. \tag{3.2.13}$$

One can see that on top of $D_i, \mathcal{R}_{ij}, a_i$, and K_{ij} one can also include terms involving,

$$\mathcal{A}_i \equiv \frac{1}{2N}\left(\dot{a}_i - N^j D_j a_i - a_j D_i N^j\right), \text{ and} \tag{3.2.14}$$

$$r_{ij} \equiv \frac{1}{2N}\left(\dot{\mathcal{R}}_{ij} - N^k D_k \mathcal{R}_{ij} - \mathcal{R}_{ik} D_j N^k - \mathcal{R}_{jk} D_i N^k\right), \tag{3.2.15}$$

in the action. That these are invariant under $\mathrm{Diff}_{\mathcal{F}}(\mathcal{M})$ is more apparent from their fully covariant forms,

$$\mathcal{A}_\nu \equiv -\frac{p_\nu^\mu}{2}\mathcal{L}_u a_\mu, \quad r_{\alpha\beta} \equiv -\frac{p_\alpha^\mu p_\beta^\nu}{2}\mathcal{L}_u \mathcal{R}_{\mu\nu}. \tag{3.2.16}$$

As these are covariant but depend on u, the expressions (3.2.14) & (3.2.15) depend only on the choice of foliation (the one defined by u gives that particular expression). When expanding around Minkowski space, in its standard Cartesian coordinates, to quadratic order (in the action) many of the possible terms, both in the standard non-projectable Hořava gravity and the mixed derivatives terms, become redundant or vanish. For example r_{ij} is entirely redundant at this order. To see this, expand $\dot{\mathcal{R}}_{ij}$ to first order in perturbations (*i.e.* $\gamma_{ij} = \delta_{ij} + h_{ij}$), one should obtain,

$$r_{ij}^{(1)} = \frac{1}{2}\left[\partial_k\partial_j\dot{h}_{ki} + \partial_k\partial_i\dot{h}_{kj} - \partial_k\partial_k\dot{h}_{ij} - \partial_i\partial_j\dot{h}\right] + \mathcal{O}(\text{perturbations}^2). \tag{3.2.17}$$

If one then expands $D_k D_l K_{ij}$ to first order and replaces all of the $\dot{\mathbf{h}}$ terms with the extrinsic curvature, one finds that all the shift terms cancel and you are left with,

$$r_{ij} = \frac{1}{2}\left[\partial_k\partial_j K_{ki} + \partial_k\partial_i K_{kj} - \partial_k\partial_k K_{ij} - \partial_i\partial_j K\right] + \mathcal{O}(\text{perturbations}^2). \tag{3.2.18}$$

Therefore including r_{ij} terms is equivalent to changing the coefficients of terms involving $D_k D_l K_{ij}$ at this order in perturbation theory. Similarly, taking into account that $a_i, \mathcal{R}_{ij}, K_{ij}$ all vanish in the background, one can also discard any terms which are cubic or higher order in these. Also, as derivatives must be hitting perturbation order quantities and discounting total derivatives, any terms related by commutation of derivatives are redundant at this order too. Finally, some terms are related by an integration by parts plus terms cubic in perturbations. For example $ND_i\left(\mathcal{R}a^i\right)$ which is equivalent to $-N\mathcal{R}D_i a^i$ up to a total derivative and $\mathcal{R}a_i a^i$.

Taking into account all of the above (one way to write) the most general action for quadratic perturbations around Minkowski is,

$$S = \frac{M_p^2}{2}\int \mathrm{d}^3x\mathrm{d}t N\sqrt{\gamma}\left[K_{ij}K^{ij} - \lambda K^2 + \mathcal{L}_1 + \frac{\mathcal{L}_2}{M_*^2} + \frac{\mathcal{L}_3}{M_*^4} + \frac{\mathcal{L}_\times}{M_*^2}\right], \quad (3.2.19)$$

where the terms already present in standard non-projectable Hořava gravity are,

$$\mathcal{L}_1 = \alpha a_i a^i + \beta\mathcal{R}, \quad (3.2.20)$$

$$\mathcal{L}_2 = \alpha_1 + \mathcal{R}D_i a^i + \alpha_2 D_i a_j D^i a^j + \beta_1 \mathcal{R}_{ij}\mathcal{R}^{ij} + \beta_2\mathcal{R}^2, \quad (3.2.21)$$

$$\mathcal{L}_3 = \alpha_3 D_i D^i\mathcal{R}D_j a^j + \alpha_4 D^k D_k a_i D_j D^j a^i + \beta_3 D_i\mathcal{R}_{jk}D^i\mathcal{R}^{jk} + \beta_4 D_i\mathcal{R}D^i\mathcal{R}, \quad (3.2.22)$$

and the relevant mixed derivative terms are,

$$\mathcal{L}_\times = M^{ijklmn}D_i K_{jk}D_l K_{mn} + 2\mathcal{A}_i\left(\sigma_1\mathcal{A}^i + \sigma_2 D^i K + \sigma_3 D_j K^{ij}\right). \quad (3.2.23)$$

In the above we have used,

$$M^{ijklmn} = g_1\gamma^{ij}\gamma^{lm}\gamma^{kn} + g_2\gamma^{il}\gamma^{jm}\gamma^{kn} + g_3\gamma^{il}\gamma^{jk}\gamma^{mn} + g_4\gamma^{ij}\gamma^{kl}\gamma^{mn}. \quad (3.2.24)$$

Along with providing UV deformations of the kinetic terms of the tensor and scalar modes, the new terms (with coefficients σ_i) lead to a new scalar degree of freedom. Thus this theory has two tensor and two scalar modes.

To study the perturbations around Minkowski space we shall use the decomposition,

$$N = 1 + A, \quad N^i = B^i + \partial^i B, \tag{3.2.25}$$

$$\gamma_{ij} = \delta_{ij}\left(1 + 2\psi\right) + \left(\partial_i\partial_j - \frac{\delta_{ij}}{3}\Delta\right)E + \partial_{(i}E_{j)} + h_{ij}, \tag{3.2.26}$$

where we impose $\partial_i h_{ij} = h = 0 = \partial_i B_i = \partial_i E_i$. As h_{ij} is a symmetric 3×3 tensor and we have 4 conditions, that leaves 2 tensor degrees of freedom, 4 vector degrees of freedom, and there are 4 scalars. This covers the 10 degrees of freedom which one usually expects from a gravitational theory. In the following we shall use the Fourier transform of all these functions,

$$Q(t, x) = \frac{1}{(2\pi)^{3/2}}\int \mathrm{d}^3k\, Q_k(t)e^{ikx}, \tag{3.2.27}$$

where Q is any perturbation, and we will drop the subscript k. It should be noted that this imposes boundary conditions in a non-trivial way (Colombo et al. 2015a). Expanding the action in terms of these Fourier modes, we get an action which is the sum of three decoupled actions. One for the tensor sector, one for the vector sector, and one for the scalar sector (see Colombo et al. 2015a for the original derivations for the tensor and vector sectors). The quadratic action for the tensor perturbations is,

$$\frac{M_p^2}{8}\int \mathrm{d}t\mathrm{d}^3k\left(1 + g_2\mu^2\right)\left(|\dot{h}_{ij}|^2 - \omega_T^2|h_{ij}|^2\right), \tag{3.2.28}$$

where,

$$\omega_T^2 = k^2\frac{\beta - \beta_1\mu^2 - \beta_3\mu^4}{1 + g_2\mu^2}. \tag{3.2.29}$$

In the above we have defined $\mu \equiv k/M_*$ and this will be used heavily in the following. From this we can read off the stability criteria for the tensor modes as,

$$\gamma_2 > 0, \quad \beta - \beta_1\mu^2 - \beta_3\mu^4 > 0. \tag{3.2.30}$$

Taking the IR and UV limits of this expression we can further impose,

$$\beta > 0, \quad \beta_3 < 0. \tag{3.2.31}$$

As we have seen the vector modes, despite being either gauge or non-dynamical, contribute to loop diagrams. This is analogous to the Coulomb field in electrodynamics. The gauge invariant combination of vector modes is $\mathcal{B}_i = B_i - \dot{E}_i/2$. The quadratic action for these vector perturbations is,

$$\frac{M_p^2}{4}\int \mathrm{d}t\mathrm{d}^3k\, k^2\left[1+\frac{\mu^2}{2}\left(g_1+2g_2\right)\right]\left|B^i-\frac{\dot{E}^i}{2}\right|^2. \tag{3.2.32}$$

The equation of motion can be solved by,

$$B^i = \frac{1}{2}\dot{E}^i, \tag{3.2.33}$$

and therefore the action vanishes (up to the choice of boundary conditions). This means there are no dynamical vector modes but their "propagator" will now have $1/k^4$ falloff in the deep UV.

The most important part for this analysis is the scalar sector. This is where all of most interesting features of (classical) standard Hořava occur and it is perhaps not surprising that this remains the case in this extension. The quadratic action for the scalar sector is,

$$\begin{aligned}
&\frac{M_p^2}{2}\int \mathrm{d}t\mathrm{d}^3k\left\{\left[3\left(1-3\lambda\right)+\left(g_1+3g_2+9g_3+3g_4\right)\mu^2\right]\left|\dot{\psi}+\frac{k^2}{6}\dot{E}\right|^2\right.\\
&+\frac{\sigma_1\mu^2}{2}\left|\dot{A}\right|^2+2k^2\left[\beta+\left(3\beta_1+8\beta_2\right)\mu^2+\left(3\beta_3+8\beta_4\right)\mu^4\right]\left|\psi+\frac{k^2}{6}E\right|^2\\
&+k^2\left(\alpha+\alpha_2\mu^2+\alpha_4\mu^4\right)|A|^2+2k^2\left(\beta-\alpha_1\mu^2+\alpha_3\mu^4\right)\left[A^*\left(\psi+\frac{k^2}{6}E\right)+\text{c.c.}\right]\\
&+\frac{\mu^2\left(3\sigma_2+\sigma_3\right)}{2}\left[A^*\left(\dot{\psi}+\frac{k^2}{6}\dot{E}\right)+\text{c.c.}\right]+\frac{k^2\mu^2\left(\sigma_2+\sigma_3\right)}{2}\left[A^*\left(B-\frac{\dot{E}}{2}\right)+\text{c.c.}\right]\\
&+k^2\left[1-3\lambda+\left(g_1+g_2+3g_3+2g_4\right)\mu^2\right]\left[\left(B-\frac{\dot{E}}{2}\right)^*\left(\dot{\psi}+\frac{k^2}{6}\dot{E}\right)+\text{c.c.}\right]\\
&\left.+k^4\left[1-\lambda+\left(g_1+g_2+g_3+g_4\right)\mu^2\right]\left|B-\frac{\dot{E}}{2}\right|^2\right\}.
\end{aligned} \tag{3.2.34}$$

At linear order in perturbations,

$$\Psi \equiv \psi+\frac{k^2}{6}E,\quad \mathcal{B}\equiv B-\frac{\dot{E}}{2}, \text{and}\quad kA, \tag{3.2.35}$$

are invariant under the linearized $\mathrm{Diff}_{\mathcal{F}}(\mathcal{M})$, and thus the quadratic action for the scalar sector is manifestly gauge invariant. In particular, it is worth

noting that A is not gauge invariant as $A \to A + f'(t)$ when $t \to t + f(t)$, but its spatial derivative (and hence kA) is. So we have three independent scalar modes, two of which are dynamical. Varying with $\mathcal{B}$ (which gives the scalar part of the linearized momentum constraint) gives us an equation with which we can replace $\mathcal{B}$. Doing so we obtain,

$$\mathcal{B} = -\frac{1}{k^2}\frac{1}{1-\lambda+\sum_i g_i\mu^2}\left\{\left[1-3\lambda+(g_1+g_2+3g_3+2g_4)\,\mu^2\right]\left(\dot{\psi}+\frac{k^2}{6}\dot{E}\right)\right.$$
$$\left.+\frac{\sigma_2+\sigma_3}{2}\mu^2\dot{A}\right\}. \tag{3.2.36}$$

Unlike in Section 2.2.2 we cannot perform any further eliminations due to the fact that A carries time derivatives in the action. We therefore have a scalar action with two dynamical degrees of freedom, $\Upsilon = (\Psi, A)$. Introducing the symmetric 2×2 matrices $\mathcal{K}$ and M we can write the action as,

$$\frac{M_p^2}{2}\int \mathrm{d}t\mathrm{d}^3k\,\left(\dot{\Upsilon}^\dagger\mathcal{K}\dot{\Upsilon}-\Upsilon^\dagger M\Upsilon\right). \tag{3.2.37}$$

This coincides with our scalar sector (with $\mathcal{B}$ eliminated) when,

$$\mathcal{K}_{11} = 6+(4g_1+6g_2)\,\mu^2+\frac{4+\left[8\,(g_1+g_2)+4g_4\right]\mu^2+\left[2(g_1+g_2)+g_4\right]^2\mu^4}{\lambda-1-\sum_i g_i\mu^2},$$
$$\mathcal{K}_{12} = -\sigma_3\mu^2-\frac{\sigma_2+\sigma_3}{2}\frac{2\mu^2+\left[2(g_1+g_2)+g_4\right]\mu^4}{\lambda-1\sum_i g_i\mu^2},$$
$$\mathcal{K}_{22} = \frac{\sigma_1\mu^2}{2}+\frac{(\sigma_2+\sigma_3)^2\mu^4}{4\left[\lambda-1\sum_i g_i\mu^2\right]}, \tag{3.2.38}$$

and

$$M_{11} = -2k^2\left[\beta+(3\beta_1+8\beta_2)\,\mu^2+(3\beta_3+8\beta_4)\,\mu^4\right],$$
$$M_{12} = -2k^2\left[\beta-\alpha_1\mu^2+\alpha_3\mu^4\right],$$
$$M_{22} = -k^2\left[\alpha+\alpha_2\mu^2+\alpha_4\mu^4\right]. \tag{3.2.39}$$

To diagonalize the kinetic matrix we introduce a new field basis,

$$Z = R^{-1}\Upsilon, \tag{3.2.40}$$

where,

$$R = \begin{pmatrix} 1 & -\frac{\mathcal{K}_{12}}{\mathcal{K}_{11}} \\ 0 & 1 \end{pmatrix}. \tag{3.2.41}$$

This transformation is highly non-unique but provides a very simple basis for extracting the IR and UV stability criteria. In this new field basis, $R^T K R = \text{diag}\left(\bar{\mathcal{K}}_{11}, \bar{\mathcal{K}}_{22}\right)$ where,

$$\bar{\mathcal{K}}_{11} = \mathcal{K}_{11}, \quad \bar{\mathcal{K}}_{22} = \frac{\det \mathcal{K}}{\mathcal{K}_{11}}. \tag{3.2.42}$$

One can see that $\bar{\mathcal{K}}_{11}$ is independent of the σ_i, and $\bar{\mathcal{K}}_{22}$ vanishes when they do. For this reason, we consider the first mode as the scalar graviton of standard Hořava gravity and the second mode as the new one. To avoid ghosts in the IR we must ensure $\bar{\mathcal{K}}_{11}$, $\bar{\mathcal{K}}_{22}$ are both positive in the limit $\mu \to 0$. Taylor expanding in μ we find,

$$\bar{\mathcal{K}}_{11} = \frac{2(3\lambda - 1)}{\lambda - 1} + \mathcal{O}(\mu^2), \quad \bar{\mathcal{K}}_{22} = \frac{\sigma_1 \mu^2}{2} + \mathcal{O}(\mu^4). \tag{3.2.43}$$

In the UV there are far more parameters, and so the ghost free condition there is not so restrictive. So the important conditions are,

$$\frac{3\lambda - 1}{\lambda - 1}, \quad \sigma_1 > 0. \tag{3.2.44}$$

To find the dispersion relations for the modes we note that if we write,

$$Y = Y_0 e^{-i\omega t}, \tag{3.2.45}$$

at the level of the equations then we can find the dispersion relations by finding the two solutions to,

$$\det\left[(-i\omega)^2 \mathcal{K} + M\right] = 0. \tag{3.2.46}$$

The exact form of the solutions are not particularly informative, however if we again Taylor expand in terms of μ then we find, up to second order,

$$\frac{\omega_1^2}{M_*^2} = 2\frac{\beta\left(\beta - \alpha/2\right)(\lambda - 1)}{\alpha\left(3\lambda - 1\right)}\mu^2, \tag{3.2.47}$$

$$\frac{\omega_2^2}{M_*^2} = -\frac{2\alpha}{\sigma_1} + \left[\frac{4\left[\beta\sigma_1 - \alpha/\left(3\sigma_2 + \sigma_3\right)/2\right]^2}{\alpha\sigma_1^2\left(3\lambda - 1\right)} - \frac{2\left(\beta\sigma_1 + \alpha\sigma_3\right)^2}{\alpha\sigma_1^2} - \frac{6\alpha_2}{\sigma_1}\right]\frac{\mu^2}{3}, \tag{3.2.48}$$

all corrections are $\mathcal{O}(\mu^4)$. Again the first expression coincides with the IR dispersion relation from standard Hořava gravity . If we impose the stability

of the tensor modes, and the no ghost conditions in the scalar sector, stability of the first mode then requires,

$$\beta > \alpha/2 > 0. \tag{3.2.49}$$

However, with $\alpha > 0$ and $\sigma_1 > 0$ (which is one of the no ghost conditions) the second mode has a tachyonic instability. The time scale for this instability is given by,

$$t_{\rm ins} \sim \frac{\sqrt{\sigma_1}}{M_* \sqrt{\alpha}} \sim 10^{-30} s \left(\frac{M_*}{10^9\,{\rm GeV}}\right)^{-1} \left(\frac{\alpha}{10^{-7}}\right)^{-1/2} \sigma_1. \tag{3.2.50}$$

The fiducial values of α and M_* are chosen to (approximately) saturate experimental bounds and thus, for the instability to have not appeared in the lifetime of the universe we get a very strong constraint that,

$$\frac{\sigma_1^2}{\alpha} \sim 10^{103} \left(\frac{M_*}{10^9\,{\rm GeV}}\right)^2. \tag{3.2.51}$$

As M_* is the scale at which dispersion relations become dominated by the higher order terms, 10^{103} can be taken as an effective lower bound. One of the important things to note here is that the mixed derivative terms are not all irrelevant to the low energy effective theory. In particular, the σ_1 term generates a new degree of freedom and its value controls the IR behaviour of the theory. Note that the acceleration term $a_i a^i$ provides an effective mass term for the new degree of freedom due to the existence of the gradients in the kinetic term.

Due to its control over the IR dynamics, the σ_1 term should be included in an IR truncation of the theory. Doing so allows one to see the choice between the new mode being a ghost or a tachyon much clearer (assuming the stability of the other degrees of freedom is maintained). So the IR action is,

$$\int {\rm d}^3x {\rm d}t N\sqrt{\gamma}\left[K_{ij}K^{ij} - \lambda K^2 + \alpha a_i a^i + \beta \mathcal{R} + \frac{2}{M_*^2}\sigma_1 \mathcal{A}_i \mathcal{A}^i\right]. \tag{3.2.52}$$

The Lagrangian quadratic in perturbations on a Minkowski background for the scalar sector is then,

$$\mathcal{L}^{(2)}_{\text{IR, scalar}} = -3\left(3\lambda-1\right)\dot{\Psi}^2 + \frac{\sigma_1}{2M_*^2}\partial_i\dot{A}\partial_i\dot{A} + 2\beta\partial_i\Psi\partial_i\Psi + \alpha\partial_iA\partial_iA + 4\beta\partial_iA\partial_i\Psi$$
$$-\left(\lambda-1\right)\left(\Delta\mathcal{B}\right)^2 + 2\left(3\lambda-1\right)\Delta\mathcal{B}\dot{\Psi}. \tag{3.2.53}$$

Again we integrate out $\mathcal{B}$ and the reduced Lagrangian is,

$$\frac{2(3\lambda-1)}{\lambda-1}\dot{\Psi}^2 + \frac{\sigma_1}{2M_*^2}\partial_i\dot{A}\partial_i\dot{A} + 2\beta\partial_i\Psi\partial_i\Psi + \alpha\partial_iA\partial_iA + 4\beta\partial_iA\partial_i\Psi. \tag{3.2.54}$$

As there is no kinetic mixing we can immediately read off that the no-ghost conditions are (as before),

$$\frac{3\lambda-1}{\lambda-1} > 0, \quad \sigma_1 > 0. \tag{3.2.55}$$

As the canonically normalized field is,

$$\frac{\sqrt{\sigma}_1}{M_*}\partial_iA, \tag{3.2.56}$$

we can also read off the lowest order part of its dispersion relation. That is its mass is,

$$m^2 = -2\frac{M_*^2\alpha}{\sigma_1}. \tag{3.2.57}$$

Next we shall consider an alternative. As this new mode has an effective mass term it is tempting to follow a standard approach of effective field theories. If we allow the field, kA, to be a ghost but with a large mass, it is conceivable that the ghost is never generated at low energies and the UV completion rescues the theory. The natural cutoff for this low energy theory is the strong coupling scale discussed in 2.2.2 and thus we will require,

$$m^2 > M_{\text{sc}}^2 \quad \Longrightarrow \quad \frac{2\alpha}{|\sigma_1|} > 1, \tag{3.2.58}$$

with $\sigma_1 < 0$. As we know the UV completion of this theory, we can investigate whether the standard expectation is realised and the theory is healthy. As the ghost is a real degree of freedom which continues to propagate in the UV, it is not a true artefact of the truncation, and so we must then require that the kinetic term changes sign at some intermediate momentum.

However, this is not sufficient for stability, we will also require that the sum of the mass and gradient terms changes sign at the same momentum. This will clearly require fine-tuning, if such a tuning even exists. Due to the complexity of the full dispersion relation, it does not seem likely that any such tuning would coincide with a symmetry and thus does not seem a viable option for a bona fide quantum theory of gravity.

One alternative is to notice that the problematic terms involve the acceleration, a_i, and thus the projectable theory will avoid this particular problem. The projectability criterion affects only the scalar sector, and as such the stability of the tensor modes require the same inequalities as before and the vector modes still gain the same improved UV behaviour. In the scalar sector, however, the lapse perturbation is trivialised and the kinetic term for the remaining scalar field (after integrating out $\mathcal{B}$) is,

$$\mathcal{K}_{\text{proj}} = \mathcal{K}_{11}, \tag{3.2.59}$$

and so in the IR we have the same no ghost condition as in the non-projectable theory,

$$\frac{3\lambda - 1}{\lambda - 1} > 0. \tag{3.2.60}$$

However, the dispersion relation is significantly different,

$$\begin{aligned}\omega^2_{\text{proj}} &= \frac{-2\mu^2\left[\beta + (3\beta_1 + 8\beta_2)\,\mu^2 + (3\beta_3 + 8\beta_4)\,\mu^4\right]}{\mathcal{K}_{\text{proj}}} \\ &= -\frac{\beta\,(\lambda - 1)}{3\lambda - 1}k^2\left[1 + \mathcal{O}(\mu^2)\right].\end{aligned} \tag{3.2.61}$$

Therefore the stability of the scalar mode in the IR is incompatible with the no ghost condition for the tensor modes, *i.e.* the scalar must either be a ghost or have a gradient instability (for a discussion of the projectable theory with a ghost see Ramazanov et al. 2016). In this case, the timescale of this instability can be quite long and exists in standard projectable Hořava (see Section 2.2.2 and references therein). The non-perturbativity that accompanies this in standard Hořava gravity appears to be removed in this extension of the theory. The important aspect of the break down of

the perturbation theory in the standard case is that, in the limit $\lambda \to 1$, $\mathcal{B}$ diverges as $1/(\lambda - 1)$ (again see Section 2.2.2. The action is always quadratic in $\mathcal{B}$, but terms of the form $\Psi^n k^4 \mathcal{B}^2$ which occur at higher orders may be dominant over the quadratic action as, introducing the expansion parameter ϵ,

$$\Psi^n k^4 \mathcal{B}^2 \propto \frac{\epsilon^{n+2}}{(\lambda - 1)^2}. \tag{3.2.62}$$

However, in this extended theory we have,

$$|\mathcal{B}|_{\lambda \to 1} = \frac{1}{k^2} \frac{2 - (g_1 + g_2 + 3g_3 + 2g_4)\,\mu^2}{\sum_i g_i \mu^2} \dot{\Psi}, \tag{3.2.63}$$

and thus,

$$\Psi^n k^4 \mathcal{B}^2 \propto \frac{\epsilon^{n+2}}{\sum_i g_i \mu^2}. \tag{3.2.64}$$

This means that there is no immediate indication that the perturbative expansion breaks down, though a more thorough study is warranted. However, this means that the gradient instability may not be screened by any non-perturbative behaviour and must be taken seriously. The final note to make about the projectable theory with this extension is that the usual IR dominant term vanishes in the limit $\lambda \to 1$ and thus,

$$\omega^2_{\text{proj}} \propto k^2 \mu^2, \tag{3.2.65}$$

i.e. the dispersion relation is non-linear even in the IR limit.

3.3 SUMMARY

The message of this Chapter is that there is a serious naturalness problem for the universality of the speed of light in the matter sector when coupled to standard Hořava gravity. Further, the mixed derivative extension of Hořava gravity has serious issues which need to be addressed if one wishes to consider it a viable solution of the naturalness problem. In particular the extended theory is plagued by instabilities which seem to require some break down of perturbation theory to resolve. Whether this is an acceptable

choice is not fully resolved. As this is the only known mechanism for resolving the naturalness problem, without making the theory non-perturbative or introducing additional fields, any viable resolution likely cannot be so conservative. Whether or not this is really the case is the subject of some ongoing work. Though it is possible that the mixed derivative extension of the projectable theory is viable. To address this the complicated IR dynamics would need to be better understood. Other potential solutions involve additional field or strong coupling.

4

BLACK HOLES AND PREFERRED FOLIATIONS

The following is heavily based on Bhattacharyya et al. 2016a; Bhattacharyya et al. 2016b.

4.1 PREFERRED FOLIATIONS

Having reviewed the covariantization of Hořava gravity in Section 2.2.3 we can use it to neatly demonstrate more concretely the full meaning behind the statement that the foliation in Hořava gravity is “preferred". It is important to note that here we are distinguishing between the usual definition of a preferred foliation (or frame), which we shall refer to as a special foliation, and what we are calling a “preferred" foliation. A special foliation (or frame) is one which is detectable. For example one could have non-zero preferred frame parameters in the PPN expansion. While this is also true for Hořava gravity (see Blas et al. 2011b; Blas et al. 2011c), here we will establish that the preferred nature of this foliation goes much further than simply being detectable.

4.1.1 *The preferred foliation of Hořava gravity*

For full Hořava gravity, or any theory with a modified dispersion relation of the form,

$$\omega^2 = c^2k^2 + \alpha\mathcal{O}(k^4), \tag{4.1.1}$$

it is clear that the sound speeds are unbounded from above and thus the only hope is that there is some foliation by spacelike hypersurfaces that does bound the propagation. Indeed writing dispersion relations in this form requires specifying a foliation (see for example Sotiriou 2017). This is because under a change of foliation there is a mixing of k and ω, so that the dispersion relation will contain higher powers of ω too.

One may be puzzled about what occurs in the limit that the higher order terms disappear ($\alpha \to 0$, as there seems to be a discontinuity in the causal structure. That is, for $\alpha \neq 0$, there are arbitrarily fast modes and thus the preferred foliation is required to define causality. Then, when $\alpha \to 0$, the speed of all modes becomes c. In Hořava this puzzle is resolved by the fact that there is an elliptic equation to solve on every slice. This elliptic equation means that there is instantaneous propagation of data, along the preferred foliation, even in the IR limit and thus the foliation also defines causality when the higher order dispersion relations can be ignored (Blas et al. 2011b; Blas et al. 2011a).

To see the existence of the instantaneous mode we take the covariant form for the action of Hořava gravity (2.2.64) and consider only the lowest order terms. As it is covariant the variation of the action under an infinitesimal diffeomorphism ($x^\mu \to x^\mu + \xi^\mu$) should be identically 0, even when the fields are off-shell. This is the basis of a standard method for identifying the constraint equations (as mentioned in the introduction, also see Barausse et

al. 2011; Jacobson 2011). We can extract the constraint equations by making the replacements,

$$\begin{aligned} \delta g^{\alpha\beta} = \mathcal{L}_{\tilde{\xi}} g^{\alpha\beta} &= -2\nabla^{(\alpha}\xi^{\beta)}, \\ \delta\phi = \mathcal{L}_{\tilde{\xi}}\phi &= \xi^{\mu}\partial_{\mu}\phi, \end{aligned} \tag{4.1.2}$$

in (2.2.69) and integrating by parts where necessary. This gives us,

$$\delta_{\tilde{\xi}} S_{\chi}\left[g^{\alpha\beta},\phi\right] \propto \int \mathrm{d}^4x\sqrt{-g}\left[\xi^{\beta}\left(\nabla_{\alpha}\mathcal{E}^{\alpha}_{\beta} - N^{-1}\nabla_{\alpha}\left(N\vec{\text{Æ}}^{\alpha}\right)u_{\beta}\right)\right] = 0. \tag{4.1.3}$$

As one expects from a scalar tensor theory, once the Einstein equations are solved the generalised Bianchi identities guarantee the scalar field is also on shell. Some simple manipulations allow us to rewrite this generalised Bianchi identity as,

$$\nabla_{\alpha}\left(\mathcal{E}^{\alpha}_{\beta} - \vec{\text{Æ}}^{\alpha}u_{\beta}\right) + \vec{\text{Æ}}^{\alpha}\left(\nabla_{\beta}u_{\alpha}\right) = 0, \tag{4.1.4}$$

which is formally identical to that of Einstein-æther theory. In Einstein-æther theory we can then say that,

$$\hat{n}_{\alpha}\left(\mathcal{E}^{\alpha}_{\beta} - \vec{\text{Æ}}^{\alpha}u_{\beta}\right) = 0, \tag{4.1.5}$$

are the constraint equations, where $\hat{n}$ is the unit (timelike) normal to some foliation. To see this: decompose the divergence in the first term in the generalised Bianchi identity (4.1.4) into its part along and orthogonal to $\hat{n}$, one then finds that,

$$\partial_n(\mathcal{E}^{n}_{\mu} - \vec{\text{Æ}}^{n}u_{\mu}) = F_{\mu}, \tag{4.1.6}$$

where F_{μ} is a sum of terms which are each proportional to a field equation of Einstein-aether theory and a subscript n means the index has been contracted with $\hat{n}$. Therefore, if we have some initial leaf of the foliation on which we prescribe initial data which solves the equations on that slice, then the (candidate) constraints (4.1.5) are preserved in time. We can also determine, by the fact that Equation (4.1.4) is an identity which holds regardless of any field equations, that the combination (4.1.5), is no more than first order in time derivatives.

In the Khronometric theory, however, this is argument does not hold. First of all, this combination is non-zero in general, as $\vec{\text{Æ}}^n u_\mu \neq 0$. Secondly, it is second-order in time derivatives of the Khronon. Finally, it is not preserved in time, again by $\vec{\text{Æ}} \neq 0$. However, a sensible set of constraint equations can (only) be obtained by setting $\hat{n} = u$, *i.e.* the constraint equations are,

$$u_\alpha \mathcal{E}^\alpha_\beta = 0, \tag{4.1.7}$$

and so one way in which this foliation is preferred is for the construction of an initial value problem. One subtlety to note here is that the "Hamiltonian constraint",

$$\mathcal{E}^u_u = 0, \tag{4.1.8}$$

is only a true constraint if one solves the scalar's equation of motion directly, otherwise it must be imposed on every slice. By inspecting the Hamiltonian constraint (or the scalar's equation) one can see that in a generic foliation the scalar field solves a higher order equation of motion and so one expects an Ostragradsky ghost. This occurs except when one goes to the preferred foliation, in which case any higher order derivatives become purely spatial. Thus, beyond its simple existence, the preferred foliation of Hořava gravity is preferred also in the sense of it defining causality, it being required to set up a sensible initial value formulation and from the point of view of boundedness of the Hamiltonian (although these are all linked in this case). The additional spatial derivatives are a manifestation of the instantaneous mode. Finally we can also note that in action (2.2.64) the higher order terms (in derivatives) appear fine tuned and the truncation to the IR limit seems to be inconsistent, having truncated at different orders in derivatives for the metric and scalar field. This means that the foliation is also preferred for the purposes of formulating the theory consistently.

4.1.2 *Preferred foliations without higher order derivatives*

We can also have theories with less obviously, but still actually, preferred foliations. To see this it will be instructive to first consider a toy model with the Lagrangian,

$$\mathcal{L} = \frac{1}{2}\left(\nabla_u \phi\right)^2 - \frac{1}{2} h^{\mu\nu} \nabla_\mu \psi \nabla_\nu \psi - \lambda \phi \psi. \tag{4.1.9}$$

We have a pair of scalar fields, ϕ and ψ, a background metric, $g_{\mu\nu}$, and a unit timelike, hypersurface orthogonal vector, u_μ (note $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$). We will take $g_{\mu\nu} = \eta_{\mu\nu}$, and u_μ as a constant vector pointing along one of the Cartesian time coordinates of Minkowski space. Including dynamics for them would not change the conclusions so long as u_μ remained unit timelike and hypersurface orthogonal. The equations of motion for the scalar fields in this case are,

$$\nabla_\mu \left[u^\mu u^\nu \nabla_\nu \phi\right] = -\lambda\psi, \quad \nabla_\mu \left[h^{\mu\nu} \nabla_\nu \psi\right] = \lambda\phi. \tag{4.1.10}$$

In coordinates adapted to the foliation defined by u^μ, the equations of motion reduce to,

$$\ddot{\phi} = -\lambda\psi, \quad \Delta\psi = \lambda\phi. \tag{4.1.11}$$

If we now introduce a function $\pi = \dot{\phi}$ we can cast this system into first order form,

$$\dot{\pi} = -\lambda\psi, \quad \dot{\phi} = \pi, \quad \Delta\psi = \lambda\phi. \tag{4.1.12}$$

Given some initial data for π, ϕ and ψ, one can integrate the first two equations over some time step, ϵ. However, to integrate to the next time step we need to supply an updated value for ψ and so we must solve the equation for ψ again. This means that the equation for ψ must be solved on every slice and does not simply relate initial data. This is in contrast to GR where any elliptic equations only relate initial data and so constitute constraints in the usual relativist's sense, which differs from the Dirac classification

of constraints [1]. In the terminology of Dirac this would be a second class constraint.

To see that this distinction is meaningful contrast the above procedure for integrating Equations (4.1.12) to the GR case. There one could be given only the dynamical equations and some initial data (which must solve the constraints) and produce the full solutions without even knowing the constraint equations exist.

If we now consider performing the analysis in a frame which is boosted with respect to the vector u_μ then,

$$u^\mu = \left\{\cosh\eta,\, \sinh\eta\Omega^i\right\}_{i=1}^{3}, \tag{4.1.13}$$

in this new frame, where Ω^i is a constant unit (spacelike) vector. If we denote a time derivative in the new frame by $'$, the Lagrangian then reads,

$$\begin{aligned}\mathcal{L} = &\frac{1}{2}\cosh^2\eta\,\phi'^2 + \frac{1}{2}\sinh^2\eta\,\left(\Omega^i\partial_i\phi\right)^2 + \sinh\eta\,\cosh\eta\,\left(\phi'\partial_i\phi - \psi'\partial\psi_i\right)\Omega^i \\ &- \frac{1}{2}\sinh^2\eta\,\psi'^2 - \frac{1}{2}\left(\delta^{ij} + \sinh^2\eta\Omega^i\Omega^j\right)\partial_i\psi\partial_j\psi - \lambda\phi\psi.\end{aligned} \tag{4.1.14}$$

Therefore the momentum conjugate to ϕ and ψ are,

$$\pi_\phi = \cosh\eta\left(\cosh\eta\,\phi' + \sinh\eta\,\Omega^i\partial_i\phi\right), \tag{4.1.15}$$

$$\pi_\psi = -\sinh\eta\left(\sinh\eta\,\psi' + \cosh\eta\,\Omega^i\partial_i\psi\right). \tag{4.1.16}$$

As the term quadratic in ψ' of $\psi'\pi_\psi$ is negative this means the Hamiltonian will not be bounded from below, unless $\eta = 0$. In the $\eta = 0$ foliation the dynamics are sensible (though non-relativistic) whereas in any other foliation ψ is a ghost. Hence the choice $\eta = 0$ is singled out beyond simply existing, and the foliation defined by u_μ is preferred. One could, of course, have integrated out the scalar field ψ. Doing so, however, leads to an explicitly non-local theory (integrating out ψ requires introducing an inverse Laplacian directly into the action) and data will still propagate instantaneously along the foliation defined by u. Thus it is a strong signal of a preferred

1 There are exceptions to this such as when one imposes a foliation by CMCs in GR, we will discuss this later.

foliation if there is a foliation in which an equation becomes elliptic and it is not a first class constraint. If an arbitrary transformation away from that foliation leads to a ghost appearing in the theory we consider that somewha conclusive.

It should be noted that this is a stronger statement of preference than the theory simply being difficult to study in an alternative foliation. When information propagates instantaneously along the leaves of some foliation this prevents the construction of any causal evolution in another foliation. To see this we can again look to special relativity. If one has two observers moving relative to each other part of the simultaneity as seen by one observer will be in the future (and another part in the past) of the simultaneity of the other. Therefore information which propagates along an “instant" in one foliation partly comes from the future in another.

Interestingly there is also a gravity theory which is very analogous to this above example, as we will establish in the next section, that is the Cuscuton.

4.2 CUSCUTON

Another motivation to study this theory is that it has been claimed in the literature that the Cuscuton (see equation (2.2.70) and surrounding text) is equivalent to a corner of the IR limit of Hořava gravity (Afshordi 2009). In this section we shall establish the validity (or otherwise) of this statement and also show that the foliation defined by the Cuscuton field is preferred (this constitutes the main content of Bhattacharyya et al. 2016b). The Cuscuton action is usually written as,

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{16\pi G}R + \mu^2\sqrt{2X} - V(\phi)\right], \tag{4.2.1}$$

where ϕ is referred to as the Cuscuton field,

$$X = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi, \tag{4.2.2}$$

and $V(\phi)$ is the Cuscuton's potential. The square root in the kinetic term means that X must be non-negative. Also the variation leads to a contribu-

tion to the Cuscuton's equation of motion of the form $1/\sqrt{X}$, and thus X must be strictly positive. This means the cusucton always has a time like gradient, and thus there is a special (though not necessarily "preferred") foliation defined by the Cuscuton. One could replace $X \to |X|$, however, due to the singularity in the field equations at $X = 0$ (and even the variation being undefined at that point in this case) one expects that $X > 0$ and $X < 0$ correspond to two disconnected branches of solutions. Therefore, as the universe can only be in one branch, we can safely simply pick one of the branches and study it as its own theory. The final point to note here is that the Cuscuton is known to propagate with an infinite speed of sound (Afshordi et al. 2007). This, along with the special foliation, suggest some similarity with Hořava gravity. One puzzling feature is that the equations of motion in the Khronometric theory have up to fourth order time derivatives of the Khronon, signalling the presence of a ghost, which goes away when the Khronon's foliation is picked. As the Cuscuton's equation of motion is only second order in derivatives, if the Cuscuton is to have a preferred foliation, the scenario must be more similar to the theory defined in (4.1.9). To begin, we shall review a reformulation of the Cuscuton as a scalar-vector-tensor theory. As part of our goal is to establish in what way the Cuscuton does (or does not) violate Lorentz invariance this will prove particularly useful. Consider the following action,

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{16\pi G}\left[R + \sigma\left(u^2 + 1\right)\right] + \mu^2\left(u^\mu\nabla_\mu\phi\right) - V(\phi)\right], \quad (4.2.3)$$

here σ is a Lagrange multiplier, enforcing that u_μ is a unit time-like vector. The equations of motion derived from this action are,

$$E_{\mu\nu} \equiv G_{\mu\nu} - 8\pi G T_{\mu\nu} = 0, \quad (4.2.4)$$

$$F_\mu \equiv \frac{1}{8\pi G}\sigma u_\mu + \mu^2\nabla_\mu\phi = 0, \quad (4.2.5)$$

$$H \equiv -\mu^2\theta - V'(\phi) = 0, \quad (4.2.6)$$

where we have defined $\theta = \nabla_\mu u^\mu$, and the stress-energy tensor is defined as usual, it reads,

$$T_{\mu\nu} = \left[\mu^2 u^\alpha \nabla_\alpha \phi - V(\phi)\right] g_{\mu\nu} + \frac{1}{8\pi G}\sigma u_\mu u_\nu. \tag{4.2.7}$$

The vector field in the above action can be thought of as an auxiliary field as its equation of motion is algebraic in u_μ. Using this equation to integrate it out and taking the unit constraint into account, one arrives at the Cuscuton action, provided $\sigma > 0$. It is also worth noting that the equations of motion then force,

$$X = \left(\frac{1}{8\pi G}\right)^2 \frac{\sigma^2}{2\mu^4}, \tag{4.2.8}$$

and so its positivity is guaranteed. When $\sigma < 0$ this corresponds to $\mu^2 < 0$. There does not seem to be a clear reason to require $\mu^2 > 0$ as what it multiplies is not a standard kinetic term, more on this later.

By integrating the term $(u^\mu \nabla_\mu \phi)$ by parts we can trade the interpretation of u_μ as an auxiliary field for treating ϕ as the auxiliary field. If we assume that $V(\phi)$ is neither constant nor linear in ϕ, and additionally assuming its derivative can be inverted one has,

$$\theta = -\frac{1}{\mu^2} V'(\phi) \implies \phi = \frac{1}{16\pi G}\frac{1}{\mu^2}\Phi'(\theta). \tag{4.2.9}$$

Φ has been defined so that,

$$\mu^2\theta + V(\phi) = \frac{1}{16\pi G}\Phi(\theta). \tag{4.2.10}$$

Using this definition, eliminating ϕ from the action leads to the action,

$$S = \frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{-g}\left[R - \Phi(\theta) + \sigma\left(u^2 + 1\right)\right], \tag{4.2.11}$$

which is of the form of a "generalized Einstein-æther theory" (Zlosnik et al. 2007), with the addition of some tuning to include only the expansion θ^2.

The equation of motion derived from this æther theory by varying with the vector is,

$$2\sigma u_\mu = -\Phi''(\theta)\nabla_\mu\theta. \tag{4.2.12}$$

2 For $V(\phi)$ of higher than quadratic order this equivalence may breakdown at the quantum level though this is not a problem for the present discussion.

Thus we can see that u_μ is hypersurface orthogonal on-shell, so long as $\sigma \neq 0$. It is worth noting that when $\sigma = 0$ the theory does not correspond to the Cuscuton theory, as these solutions correspond to $X = 0$. However, these $\sigma = 0$ solutions are interesting. To study them let us first decompose equation (4.2.12) into its projection along and orthogonal to the æther,

$$\Phi''(\theta) u^\mu \nabla_\mu \theta = 2\sigma, \quad \Phi''(\theta) p^\mu_\nu \nabla_\mu \theta = 0. \tag{4.2.13}$$

From equation (4.2.12) we know that if $\sigma \neq 0$ then the æther is hypersurface orthogonal and thus defines a foliation. The second part of equation (4.2.13) then implies that this is a foliation made up of constant mean curvature (CMC) surfaces, with mean curvature θ. The value of θ on each leaf will depend on the boundary conditions, which can be freely imposed. How one chooses these boundary conditions to change in time simply fixes σ. In the case when $\sigma = 0$ we have two types of solutions, one where u_μ is not hypersurface orthogonal and one where the hypersurface orthogonality occurs "accidentally". In the first case we do not have enough equations to fix u_μ completely and further it seems unlikely that such solutions are dynamically connected to solutions where u_μ is hypersurface orthogonal. In the second case u_μ defines a foliation by CMCs throughout the solution, even when $\sigma \to 0$.

One may wonder whether this scenario can actually occur. However, in stationary solutions σ is forced to be 0. If such a stationary solution is the end-point of a ($\sigma \neq 0$) dynamical process then continuity will ensure that the æther will remain hypersurface orthogonal. To show that stationarity implies $\sigma = 0$, let χ^μ be a Killing vector which is timelike in some region, and assume that u_μ respects the corresponding symmetry. In this case we have,

$$\mathcal{L}_\chi \theta = \chi^\mu \nabla_\mu \theta = 0 \implies \sigma \chi^\mu u_\mu = 0, \tag{4.2.14}$$

where we have simply applied the first equalities to equation (4.2.12). As both χ and u are timelike in some region, σ must vanish in this region.

This casts doubts on whether or not the Cuscuton admits stationary solutions, due to the fact that when $\sigma = 0$ the equivalence fails. If we inspect the Cuscuton's equation of motion,

$$\nabla_\mu \left(\frac{\nabla^\mu \phi}{\sqrt{2X}} \right) = \frac{1}{\mu^2} V'(\phi), \tag{4.2.15}$$

we see that the gradient of ϕ has to be timelike. Therefore, in a region where we have a timelike Killing vector, ϕ cannot respect the stationarity as $\chi^\mu \nabla_\mu \phi \neq 0$. Since ϕ appears in the stress-energy tensor without derivatives (in the potential), there can not generically be stationary solutions. The exception is in the case $V(\phi) = V_0$ *i.e.* the potential is a constant. In the constant potential case, one can have a stationary metric but the Cuscuton itself cannot respect the symmetry.

We remark here that in the case where the potential is quadratic in ϕ, say $V(\phi) = \alpha\phi^2$, the Cuscuton is equivalent to some corner of standard Einstein-æther theory. In particular it corresponds to the case $c_1 = c_3 = c_4 = 0$ while $c_2 = -4\pi G\mu^4/\alpha$. One can see that this is not a representative corner of Einstein-æther theory from the discussion above. In particular the fact that the æther is either hypersurface orthogonal or underdetermined is not a generic feature of Einstein-æther theory. On top of this if one recalls equation (2.2.5),

$$s_2^2 = \frac{1}{1-c_{13}}, \quad s_1^2 = \frac{c_1 + c_1^2/2 + c_3^2/2}{c_{14}\,(1-c_{13})}, \quad s_0^2 = \frac{c_{123}\,(2-c_{14})}{c_{14}\,(1-c_{13})\,(2+c_{13}+3c_2)},$$

one can see that the spin-1 speed is indeterminate and the spin-0 speed diverges for this choice of the c_i. Indeed, the second part of equations (4.2.13) is a boundary value problem and will, in principle, involve the instantaneous transfer of information.

The final specific case we shall consider has some qualitative differences from the generic potential. In the case of a linear potential the derivative of the potential is not invertible and so one cannot cast this into the form of a generalized æther theory. If we write the linear potential as,

$$V(\phi) = -\mu^2 K_0 \phi + \frac{1}{8\pi G}\Lambda, \tag{4.2.16}$$

then, integrating the $\nabla_u \phi$ term in action (4.2.3) by parts one arrives at,

$$S = \frac{1}{16\pi G} \int \mathrm{d}^4x \sqrt{-g} \left[R - 2\Lambda - 16\pi G \mu^2 \phi \left(\theta - K_0\right) + \sigma \left(u^2 + 1\right) \right]. \tag{4.2.17}$$

Thus, ϕ acts as a second Lagrange multiplier which imposes the constraint $\theta = K_0$, while variation with respect to u still implies it is hypersurface orthogonal when $\sigma \neq 0$, *i.e.* when the equivalence with the Cuscuton is maintained. As u is hypersurface orthogonal and satisfies the unit constraint it is fully determined by $\theta = K_0$ and thus one can determine ϕ from the equation obtained by said variation with u. To determine σ we must then use the contracted Bianchi identity, which reads,

$$p^{\mu\nu} \nabla_\mu \sigma + \sigma u^\mu \nabla_\mu u^\nu = 0. \tag{4.2.18}$$

Interestingly this means that, along with u defining a foliation by spacelike CMCs, the CMCs must all have the same curvature and one can further foliate each of these leaves with level surfaces of σ, so long as it is not constant on a leaf. If σ is a constant on each leaf this means that the integral curves of u are geodesic.

Constraint equations

We will now tackle the question of whether or not the Cuscuton's foliation is preferred. At the perturbative level this is already quite clear. Expanding the kinetic term of the Cuscuton to quadratic order around some background one finds,

$$\sqrt{2X_{(2)}} = -\frac{1}{2} \left(2X_{(0)}\right)^{-1/2} p^{00}_{(0)} \left(\partial_t \delta\phi\right)^2 + ... \tag{4.2.19}$$

where we have only picked out the part leading to a second time derivative of $\delta\phi$ in the equations of motion, $p^{\mu\nu} = g^{\mu\nu} + (2X)^{-1} \nabla^\mu \phi \nabla^\nu \phi$ and we use its background value, $p^{\mu\nu}_{(0)}$. For $mu^2 > 0$ this means that the Cuscuton's perturbation is a ghost around every background, unless one studies the theory in the Cuscuton's foliation. As $\mu^2 > 0$ in the original definition of the Cuscuton (Afshordi et al. 2007) we consider this enough to conclude

that the Cuscuton has a preferred foliation. There are, however, two caveats to mention: in the case when $\mu^2 < 0$ the Hamiltonian is always unbounded from below as the offending terms are the gradient terms which are not removed by a choice of foliation. One could also, coming from an EFT perspective, treat the Cuscuton as a spurious degree of freedom, which has its origin in the truncation of a UV complete theory of quantum gravity. However, doing so effectively trivialised the Cuscuton and is against the "spirit" of the original proposal. To summarize, if one wishes to treat the Cuscuton as a true degree of freedom then the Hamiltonian can only be bounded from below when the theory is studied in the Cuscuton's foliation.

Now let us consider the initial value formulation of the Cuscuton in general. Using the same technique to extract the constraint equations as we have already demonstrated several times, we will first consider the variation of the action (4.2.3) under an infinitesimal diffeomorphism. Doing so we find the generalised, contracted Bianchi identity,

$$\nabla_\mu \left(\frac{1}{8\pi G} E^\mu_\nu + u^\mu F_\nu \right) + \left(\nabla_\nu u^\mu \right) F_\mu + H \nabla_\nu \phi = 0, \tag{4.2.20}$$

now recalling that ϕ is never constant, this means that the scalar's equation of motion, $H = 0$, is redundant. We can also see from this that the constraint equations can be written as,

$$\hat{n}_\mu \left(\frac{1}{8\pi G} E^\mu_\nu + u^\mu F_\nu \right) = 0, \tag{4.2.21}$$

where $\hat{n}$ is a timelike unit vector, which defines a foliation. Due to the perturbative argument, we must pick $\hat{n} = u$, *i.e.* the Cuscuton's foliation is preferred. Picking this foliation will automatically satisfy the vector's equation of motion (so long as we set $\phi = \phi(t)$) and thus the constraint equations reduce to,

$$u_\mu E^\mu_\nu = 0. \tag{4.2.22}$$

Projecting the free index onto and orthogonal to u gives the Hamiltonian and momentum constraints respectively. The Hamiltonian constraint is,

$$\mathcal{R} - K_{ij} K^{ij} + K^2 - 16\pi G V(\phi) = 0, \tag{4.2.23}$$

and the momentum constraint is,

$$D_i K^i_j - D_j K = 0. \tag{4.2.24}$$

Note that, in these coordinates, $\theta = K$ and thus $K = K(t) = K(\phi)$. Using this fact we can simplify the momentum constraint to,

$$D_i K^i_j = 0. \tag{4.2.25}$$

To complete the system of equations we need the full projection of $E_{\mu\nu}$ orthogonal to u. Doing so we arrive at the dynamical Einstein equations,

$$\mathcal{L}_u K_{ij} = -K K_{ij} + 2K^k_i K_{kj} + 8\pi G \left[V(\phi) - \frac{\mu^2\sqrt{2X}}{2} \right] \gamma_{ij} + a_i a_j + D_i a_j - \mathcal{R}_{ij}. \tag{4.2.26}$$

If one uses either the Hamiltonian constraint or the scalar's equation of motion to fix K the trace of the Einstein equation can be used to determine X. If we make the substitution $\zeta = 1/\sqrt{X}$ then this trace reads,

$$D_i D^i \zeta = \frac{\sqrt{2}}{\mu^2} \left[12\pi G \mu^4 - V''(\phi) \right] - 8\pi G \zeta V(\phi) + \zeta K_{ij} K^{ij}. \tag{4.2.27}$$

This equation is not a constraint of the type we find in GR and must be solved on every slice of the foliation. Hence X (or rather ζ) is an elliptic mode, choosing a suitable gauge it is easily related to the lapse.

One should note that such an equation can be found in vacuum GR, if one imposes that the solution be foliated by CMCs (as mentioned in an earlier footnote). The interpretation here is different, in the Cuscuton this equation exists and is elliptic whenever the theory is well defined whereas in GR this is a result of the fact that picking a CMC foliation is stronger than a gauge choice. In particular one can interpret this equation in GR as an existence criterion for the solution to have a foliation by CMCs.

4.3 BLACK HOLE FORMATION

The universal horizons we previously discussed in Section 2.2.2 have been found in static, spherically symmetric spacetimes (Blas et al. 2011a; Barausse

et al. 2011; Berglund et al. 2012), in slowly rotating black hole solutions (Barausse et al. 2013b), and in lower-dimensional rotating black holes (Sotiriou et al. 2014). However, whether or not they form from collapse remains an open question. One of the reasons this is hard to uncover is that the universal horizon has a different topology to the leaves in its past (Bhattacharyya et al. 2016c), making it hard to deal with numerically. However, it is known that spherically symmetric, static and asymptotically flat solutions of Einstein-æther theory and the Khronometric theory are the same (Barausse et al. 2012) [though this breaks down once rotation is introduced (Barausse et al. 2013b)]. In this section we shall demonstrate that the equivalence does not actually require staticity, and so spherically symmetric collapsing geometries can be studied in both settings. This is very useful in terms of performing numerics, as one can study the spherically symmetric solutions of Einstein-æther language and choose a more convenient foliation to uncover whether or not a universal horizon forms.

To begin with we shall discuss the initial value formulation of both the Khronometric theory and Einstein-æther theory. In the second case, to uncover some more subtle differences between the theories, we shall enforce that the æther be hypersurface orthogonal at the level of the equations, essentially as an ansatz [3]. Doing so makes the theories resemble each other much more closely, the comparison is especially illuminating when cast in terms of the foliation defined by the “æther" in both cases. This allows us to demonstrate the difference in the status of the foliation in the different theories, giving an interesting contrasting case to build upon Sections 4.1 & 4.2.

Once we have discussed the formulation of these theories, we shall then specialise to spherically symmetric and asymptotically flat spacetimes. Fol-

3 There are known hypersurface orthogonal solutions of Einstein-æther theory (any spherically symmetric solution, see, for example Berglund et al. 2012) and so this really is simply an ansatz that restricts which solutions we can find, without rendering the equations inconsistent.

lowing this will be a discussion of some simulations in Einstein-æther theory which were performed prior to the development of the notion of a universal horizon (see Garfinkle et al. 2007 for these). Finally we discuss general spherical collapse simulations in Einstein-æther theory and the implications for Hořava gravity. Along with this will be a brief discussion of where the Cuscuton fits into this picture.

Recalling our results from Section 2.2.3, the variation of the action of Khronometric theory (Equation (2.2.69)) gives (up to boundary terms),

$$\frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{-g}\left[\mathcal{E}^{(\chi)}_{\alpha\beta}\delta g^{\alpha\beta}+2\left(\nabla_\alpha\left[N\vec{\text{Æ}}^\alpha\right]\right)\delta\phi\right], \tag{4.3.1}$$

where,

$$\mathcal{E}^{(\chi)}_{\alpha\beta}=G_{\alpha\beta}-T^{(\chi)}_{\alpha\beta}, \tag{4.3.2}$$

$G_{\alpha\beta}$ is the Einstein tensor and $T^{(\chi)}_{\alpha\beta}$ is the stress-energy tensor of the Khronon. We also have defined Æ as the unrestricted variation of the Einstein-æther Lagrangian with respect to the æther, and thus,

$$\vec{\text{Æ}}^\alpha=p^\alpha_\beta\text{Æ}^\beta. \tag{4.3.3}$$

The variation of the Einstein-æther action gives,

$$\frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{-g}\left[\mathcal{E}^{(\text{æ})}_{\alpha\beta}\delta g^{\alpha\beta}+2\vec{\text{Æ}}^\alpha\delta u_\alpha\right], \tag{4.3.4}$$

where ,

$$\mathcal{E}^{(\text{æ})}_{\alpha\beta}=G_{\alpha\beta}-T^{(\text{æ})}_{\alpha\beta}. \tag{4.3.5}$$

It is worth noting that, written in terms of u_μ, $T^{(\text{æ})}_{\alpha\beta}$ and $T^{(\chi)}_{\alpha\beta}$ are formally identical. This means, taking the hypersurface orthogonal ansatz for the æther, that the "Einstein equations" are the same in both theories. Furthermore, as the æther's equation implies that the Khronon's equation is satisfied but not vice versa, any hypersurface orthogonal solution of the Einstein-æther system is a solution of the Khronometric system, but the converse need not be true. Full equivalence has been shown for spherically symmetric solutions in the following three cases:

- asymptotically flat solutions where all leaves of the foliation reach the centre and that centre is regular (Blas et al. 2011b),

- static solutions with asymptotic flatness (Barausse et al. 2012), and

- static solutions with more general asymptotics but with a regular horizon (Bhattacharyya et al. 2014).

Using the hypersurface orthogonal ansatz, the covariant derivative of the æther can be written as,

$$\nabla_\alpha u_\beta = -u_\alpha a_\beta + K_{\alpha\beta}, \tag{4.3.6}$$

where $K_{\alpha\beta}$ is the extrinsic curvature of the leaves of the foliation defined by u,

$$K_{\alpha\beta} = \frac{1}{2}\mathcal{L}_u p_{\alpha\beta}, \tag{4.3.7}$$

and a_α is the acceleration of the æther congruence,

$$a^\alpha = \nabla_u u^\alpha \iff a_\alpha = N^{-1}\vec{\nabla}_\alpha N. \tag{4.3.8}$$

Both the acceleration and the extrinsic curvature are orthogonal to the æther. Although it is not necessary to decompose the æther equations in terms of the foliation defined by the æther, doing so will facilitate the comparison of Einstein æther theory with the Khronometric theory, and so we shall do so. Once we impose that the æther is hypersurface orthogonal, the Bianchi identities in this foliation are formally identical for the two theories and can be written as,

$$\partial_t \mathcal{E}^t_i = 0, \quad \partial_t \mathcal{E}^t_t + \left(\sqrt{-g}\right)^{-1} \partial_i \left[\sqrt{-g} N \vec{\text{Æ}}^i\right] = 0, \tag{4.3.9}$$

once the Einstein equations are taken to be solved on some given leaf of the foliation. The restriction that the æther be hypersurface orthogonal is stronger than imposing this on initial data as vorticity may be generated

by time evolution (). From the above, we can see that in this foliation (and treating the one-form u_μ as the fundamental field) that the constraints do not involve the æther's equation of motion and simply read $\mathcal{E}^t_t = \mathcal{E}^t_i = 0$. The preservation of the Hamiltonian constraint is guaranteed once the æther (or Khronon in the Khronometric case) is on-shell.

The Hamiltonian constraint in this foliation is explicitly given by,

$$(1 - c_{13})\, K_{\alpha\beta}K^{\alpha\beta} - (1 + c_2)\, K^2 + c_{14}\left[2\left(\vec{\nabla}_\alpha a^\alpha\right) + a^2\right] - \mathcal{R} = 0. \tag{4.3.10}$$

The momentum constraint takes the explicit form,

$$(1 - c_{13})\, \vec{\nabla}_\beta K^\beta_\alpha = (1 + c_2)\, \vec{\nabla}_\alpha K. \tag{4.3.11}$$

The remaining Einstein equations (*i.e.* the projection of both indices of the Einstein equations onto the foliation) can be decomposed into an evolution equation for the mean curvature,

$$c_l \nabla_u K = -\,(1 - c_{13})\, K_{\alpha\beta}K^{\alpha\beta} - \frac{c_{123}}{2}K^2 + \left[1 - \frac{c_{14}}{2}\right]\left[\left(\vec{\nabla}_\alpha a^\alpha\right) + a^2\right], \tag{4.3.12}$$

and an evolution equation for the traceless part of the extrinsic curvature, denoted by $[K]_{\alpha\beta} = K_{\alpha\beta} - (K/3)p_{\alpha\beta}$,

$$\begin{aligned}\mathcal{L}_u\, [K]_{\alpha\beta} = {} & 2\, [K]_{\alpha\gamma}\, [K]^\gamma_\beta - \frac{K}{3}\, [K]_{\alpha\beta} + \frac{1 - c_{14}}{1 - c_{13}}\left[a_\alpha a_\beta - \frac{a^2}{3}p_{\alpha\beta}\right] \\ & + \frac{1}{1 - c_{13}}\left[\vec{\nabla}_\alpha a_\beta - \frac{\vec{\nabla}_\gamma a^\gamma}{3}p_{\alpha\beta} - \mathcal{R}_{\alpha\beta} + \frac{\mathcal{R}}{3}p_{\alpha\beta}\right].\end{aligned} \tag{4.3.13}$$

Here we introduced the notation $c_l = 1 + (1/2)c_{13} + (3/2)c_2$. Taken together, the above equations comprise the entire set of Einstein equations for both theories. To turn these into first-order partial differential equations one would introduce some coordinate system on the foliation, but the current forms suffice for our purposes. To see where the difference lies in the dynamics of the two theories we must now look at the æther and Khronon equations.

One can write,

$$\vec{\text{Æ}}_\alpha = \frac{c_{123}}{1 - c_{13}}\vec{\nabla}_\alpha K - c_{14}\left[K a_\alpha + \mathcal{L}_u a_\alpha - 2K_{\alpha\beta}a^\beta\right]. \tag{4.3.14}$$

Rearranging the æther's equation, $\vec{Æ}_\alpha = 0$, one can interpret this as an evolution equation for the acceleration, *i.e.* it can be written in the form,

$$\mathcal{L}_u a_\alpha = 2K_{\alpha\beta}a^\beta - Ka_\alpha + \frac{c_{123}}{c_{14}(1-c_{13})}\vec{\nabla}_\alpha K. \tag{4.3.15}$$

In the Khronometric theory we instead have,

$$\nabla_\alpha \left[N\vec{Æ}^\alpha\right] = 0 \iff \vec{\nabla}_\alpha \left[N^2\vec{Æ}^\alpha\right] = 0. \tag{4.3.16}$$

The difference between the two theories thus lies in the difference between Equations (4.3.14) & (4.3.16). In spherical symmetry Equation (4.3.16) is easily integrated and the consequence of this difference becomes apparent. Further, in spherical symmetry the hypersurface orthogonality of the æther is guaranteed geometrically.

To set up a coordinate system which makes the spherical symmetry manifest and puts the equations into a useful form, we first mention that, in any coordinate system adapted to the foliation, the time coordinate may be identified with the Khronon, ϕ. Secondly, that the unit spacelike vector along the acceleration, s_α, must be a "radial vector", due to the imposed symmetry of the æther. If we introduce a radial coordinate, ρ, we then have,

$$u^\alpha\partial_\alpha = N^{-1}\partial_\phi - N^{-1}N^\rho\partial_\rho, \quad s^\alpha\partial_\alpha = S^{-1}\partial_\rho, \quad s_\alpha = SN^\rho\nabla_\alpha\phi + S\nabla_\alpha\rho, \tag{4.3.17}$$

where the functions $N = N(\phi, \rho)$, $S = S(\phi,\rho)$, and $N^\rho = N^\rho(\phi,\rho)$ describe the æther configuration completely, while making its spherical symmetry manifest. In this choice of coordinates the shift vector $N^\alpha\partial_\alpha = N^\rho\partial_\rho$. We can also write the projector as,

$$p_{\alpha\beta} = s_\alpha s_\beta + \hat{g}_{\alpha\beta}, \tag{4.3.18}$$

where $\hat{g}_{\alpha\beta}$ is the projector onto the spheres defined by the symmetry. Introducing the areal radius, r, we have,

$$r = \sqrt{\frac{\text{Area of two-sphere}}{4\pi}}, \quad \hat{\mathbf{g}} = r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right), \tag{4.3.19}$$

and θ and ϕ are the standard polar coordinates on the unit two-sphere. We shall use ρ, rather than r, as the radial coordinate and write $r = r(\rho, \phi)$. In this coordinate system we have,

$$\mathrm{d}s^2 = -N^2\mathrm{d}\phi^2 + S^2\left(N^\rho \mathrm{d}\phi + \mathrm{d}\rho\right)^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right). \tag{4.3.20}$$

To fully specify the solution one needs to find the four functions, N, N^ρ, S and r in terms of ρ and ϕ. In this coordinate system the Khronon equation reads,

$$\partial_\rho\left[r^2 S N^2 \vec{Æ}^\rho\right] = 0 \iff \left(s.\vec{Æ}\right) = \frac{f_{\mathrm{IM}}(\phi)}{r^2 N^2}, \tag{4.3.21}$$

where $f_{\mathrm{IM}}(\phi)$ is the constant (in ρ) of integration. Using (4.3.14) one arrives at the following evolution equation for the acceleration,

$$\partial_\phi(a.s) = N^\rho\partial_\rho(a.s) - N\hat{K}(a.s) + \frac{c_{123}N\partial_\rho K}{c_{14}(1-c_{13})S} - \frac{f_{\mathrm{IM}}(\phi)}{c_{14}r^2 N}, \tag{4.3.22}$$

where we have introduced $\hat{K} = \hat{g}^{\alpha\beta}K_{\alpha\beta}$. The æther's equation of motion is equivalent to the above with $f_{\mathrm{IM}}(\phi) = 0$, for all ϕ. As the rest of the equations are identical, this means that the difference between Einstein-æther theory and the Khronometric theory in spherical symmetry is fully characterized by $f_{\mathrm{IM}}(\phi)$, solutions with $f_{\mathrm{IM}} \neq 0$ are those that are not also solutions of Einstein-æther theory.

Having solved Equation (4.3.22) for $(a.s)$, one can then find the lapse by inverting the equation,

$$\partial_\rho \log N = (a.s)S, \tag{4.3.23}$$

to obtain,

$$\log N(\phi, \rho) = \log N(\phi, \infty) + \int_\infty^\rho \mathrm{d}x (a.s)S(\phi, x). \tag{4.3.24}$$

The residual gauge freedom $\phi \to t(\phi)$ allows one to freely add any function of time to the above, for example we can pick,

$$\log N(t, \infty) = 0. \tag{4.3.25}$$

Note that the divergence we integrated requires boundary data on every slice and the equation is not automatically preserved by time evolution. This

is a concrete manifestation of the instantaneous mode of Hořava gravity (and Khronometric theory) which has previously been discussed mostly in a perturbative setting (see Blas et al. 2011a; Blas et al. 2011b). We can also look at the form of equation (4.3.22) and see why the conditions allowing a proof of the equivalence between Khronometric theory and Einstein-æther worked. If $f_{\rm IM}(\phi) \neq 0$ the fact that it is multiplied by $1/r^2$ means that the solution will not be regular at the centre, and so imposing regularity at the centre will force $f_{\rm IM} = 0$ and thus make the theories equivalent. In the second case one can use asymptotic flatness and count powers of $1/r$, the $f_{\rm IM}$ term falls off as $1/r^2$ whereas all the other terms fall off as $1/r^3$ and thus asymptotic flatness requires $f_{\rm IM} = 0$. Finally, in a coordinate system where the lapse remains finite outside of a universal horizon one expects $N \to 0$ on the horizon, thus the factor of $1/N$ means that regularity on the horizon will imply that $f_{\rm IM} = 0$ on the horizon.

Although this is a nice demonstration there is a puzzle one may have spotted. Due to the redundancy of the scalar's field equation in a scalar-tensor theory of gravity, one may wonder how this picture emerges if one chooses to use only the Einstein equations for the Khronometric theory; as the Einstein equations are formally identical for the Einstein-æther and Khronometric theories. The key difference is that the æther's equation is not redundant, and so equation (4.3.9) tells us that in the æther case the Hamiltonian constraint is a true constraint which will be preserved by time evolution. However, if we discard the scalar's equation, in the scalar case it must be imposed on each slice and represents part of the dynamical content of the theory. If we define $v^2 = N$ then the Hamiltonian equation can be written as a Poisson equation,

$$\Delta v = \frac{v}{4c_{14}}\left[\mathcal{R} - (1-c_{13})K_{\alpha\beta}K^{\alpha\beta} + (1+c_2)K^2\right]. \tag{4.3.26}$$

Once more it is important to stress that in the æther case (and for similar equations in GR for example) this equation must only be imposed on initial

data, but in the Khronometric theory (and in Hořava gravity) it is a true "dynamical" equation[4].

Let us now give a brief summary of what we have achieved here. We have found that that for any black hole spacetime in spherical symmetry standard regularity requirements (at the centre during the collapsing phase) make the Khronometric theory and Einstein-æther theory equivalent. For this reason, performing spherical collapse simulations in Einstein-æther theory, choosing a different foliation to the aether's, could definitively answer whether or not a universal horizon forms in such a collapse.

It should be noted that a universal horizon in a static and spherically symmetric spacetime is a leaf of the preferred foliation that is also a constant r hypersurface. This means that the lapse of the preferred foliation will vanish on this horizon, given a finite asymptotic value. Furthermore, as a universal horizon is also a Cauchy horizon, it would naturally be the end (or in the asymptotic future if $N \to 0$ smoothly) of any simulation (Bhattacharyya et al. 2016c).

Given these pieces of information one can already offer a possible alternative interpretation of the results of previous spherical collapse simulations in Einstein-aether theory. Garfinkle et al. 2007 studied the spherically symmetric collapse of a minimally coupled scalar field, ψ, in the æther's foliation. They chose, in the first case, to investigate a region of parameter space just inside and just outside of the range of parameters where static black hole solutions had been found at the time (Eling et al. 2006). As few details were provided in Garfinkle et al. 2007, and some of the results in this case have been disputed (Barausse et al. 2011), we shall instead focus on the second case.

Here the parameters were chosen such that the speed of the spin-0 mode was $\sqrt{2}$. With suitable initial data, they (Garfinkle et al. 2007) found a regu-

4 Again one can impose additional restrictions on GR which lead to a similar equation but this has an interpretation as an existence criterion for your imposition, rather than a true dynamical equation.

lar spin-0 horizon inside the metric horizon. At 'late times' they found that the geometry outside of the spin-0 horizon settled down to the static solution. Interestingly, for us, they found that the preferred time lapse 'is driven to zero as the singularity is approached.' This could also be interpreted as the possible formation of a universal horizon. To briefly summarize the important points which could lead to this conclusion: the behaviour of the lapse is consistent with such a formation, the formation of the universal horizon would be asymptotic in time, and the exterior geometry had settled down to the (essentially unique) static solution. As the notion of a universal horizon had not yet been introduced, it is natural that such an interpretation was not present in Garfinkle et al. 2007.

To uncover whether or not this is the correct interpretation would require performing simulations of the same situation, but adopting a foliation which penetrates the surface which may be the universal horizon. It seems eminently plausible that this should be possible.

It should be noted, if a universal horizon is indeed forming, that the fact that it is asymptotic in time and that the foliation is truly preferred can be seen as a confirmation of its status as a Cauchy horizon, *i.e.* evolution truly 'ends' at the universal horizon.

4.3.1 *Black holes in the Cuscuton theory*

In the previous section we showed that the Cuscuton and Hořava gravity are not equivalent, however they do share some similarities. Of particular interest here is that the Cuscuton has an instantaneous (elliptic) mode and a preferred foliation. This means that a black hole in the Cuscuton theory should also be defined with a universal horizon.

We wish to make some comments about the work of Saravani et al. 2014, in which some preliminary studies into the formation of universal horizons in the Cuscuton theory were carried out. Saravani et al. 2014 worked in a decoupling limit, solving for the Cuscuton's configuration on a GR back-

ground, essentially the Oppenheimer-Snyder solution in the limit of a thin dust shell.

With the further approximation that the value of the potential is constant a universal horizon was shown to form. Although this result is encouraging, the number and severity of the approximations involved mean that one should be cautious of saying this is anything more than an indication.

One particular example of a cause for concern is that the decoupling limit allows asymptotically flat solutions, whereas (for $V \neq 0$) these solutions do not exist in the full Cuscuton theory (see Section 4.2). One can argue that these asymptotics may not be important at the small scales where the horizon forms but horizons are defined globally and the theory has instantaneous propagation so this is not clear.

An interesting side note is that the solution found in Saravani et al. 2014 appears to be approaching the static solution of Einstein-æther when only $c_2 \neq 0$ (Berglund et al. 2012; Bhattacharyya et al. 2014). For details see Bhattacharyya et al. 2016b.

4.4 SUMMARY

There are several main results in this chapter. We have clarified a long standing confusion in the literature about how the Cuscuton relates to Hořava gravity and Einstein-æther theory. In particular we have shown that, so long as the potential meets some fairly basic criteria, the Cuscuton is equivalent to a (rather unrepresentative) corner of the generalised Einstein-æther theory of Zlosnik et al. 2007 at the classical level. We have also shown that in spherical symmetry regularity is enough to make Hořava gravity and Einstein-æther theory equivalent and that the Cuscuton is not generally equivalent to Hořava gravity. This is actually a positive for the Cuscuton as the version of Hořava gravity to which it was argued in Afshordi 2009 that it was equivalent is unhealthy (see Li et al. 2009; Henneaux et al. 2010). The link between Einstein-æther theory and Hořava gravity in spherical

symmetry also suggests the possibility of uncovering whether spherical collapse leads to the formation of a universal horizon, by studying spherical collapse in Einstein-æther theory outside of the æther's foliation. One may worry about whether the results of such a study would depend heavily on how the system is aligned with the preferred foliation in its environment. Although it is hard to give a definitve answer one could expect that it should not be so important. The heuristic argument goes as follows:

- The preferred foliation is a dynamical object and one would expect that, at least sufficiently local to the object of interest, the foliation would be "dragged along" with the object,
- In terms of the preferred foliation, any formation of a Universal horizon can only be asymptotic in time (*i.e.* it would never actually happen) and so the global question of whether the surface is truly a causal boundary is not so interesting.
- Analytic results have suggested that the horizon is an unstable feature (Blas et al. 2011a) and so the real question is whether or not the collapse is approaching something that looks like a universal horizon or a finite area singularity.
- Given that the question is thus whether or not the geometry is becoming singular, this seems to only be able to be changed if the environment is having a strong impact on the system of interest.
- Unless the Laplacian is significantly different in character to that in flat space one can have a notion of an "effectively isolated system", *c.f.* Newtonian physics.

Taken together the above seem to suggest that the important qualtitative results of such a simulation should not be changed by environmental factors, though one would expect small numerical differences and a greater difficulty in performing the computation.

We have clarified some situations in which a special foliation can be preferred from the perspective of the initial value problem. The crux of this, at least in these case, is that the theories we considered have a Hamiltonian which is bounded from below only when this is constructed in the preferred foliation. We have also shown that the Cuscuton (and thus the corresponding corner of generalized Einstein-æther theory) has a preferred foliation which defines causality. One consequence of this is that the appropriate notion of a horizon in these theories is also a Universal Horizon.

5

NEUTRON STARS, SCALAR-FIELDS AND THE WEAK EQUIVALENCE PRINCIPLE

The most basic part of an equivalence principle (in the context of gravitation) is the statement that all bodies (at least in some approximation) follow the same path when in the same gravitational field with the same initial conditions for their trajectory (Will 2014). There are three main equivalence principles that are of interest to us here. The weak equivalence principle (WEP) is the statement that this is true for test bodies, *i.e.* bodies whose own gravitational influence can be neglected. The Einstein equivalence principle is the WEP supplemented by the requirement of local Poincaré invariance. The strong equivalence principle (SEP) is the Einstein equivalence principle but without taking the approximation that the bodies self-gravity can be neglected.

The only known, phenomenologically viable, theory which generically obeys the SEP is GR. It is often considered that violations of the strong equivalence principle are related to having additional fields in the gravitational sector, though there is no general proof that extra gravitational fields lead to such violations. Either way it seems unlikely that a counter example exists which does not involve some fine-tuning and so one may expect that quantum gravity will violate the SEP (one can make a simple argument for additional fields based on Lovelock's theorem and the power-counting non-renormalizability of GR).

Even the WEP can be attacked in a similar way. If one expects additional fields to exist in quantum gravity, what mechanism prevents the generation of WEP violating terms? There is no obvious symmetry associated with the WEP, and couplings between matter and additional gravitational fields needn't be non-renormalizable. This challenge even persists in metric only theories. One reason to avoid such couplings in models of gravity is that one would need to introduce additional parameters into the theory, and experimental constraints imply that any violation of the WEP (in the solar system) must be tiny. For these reasons, WEP violations are usually considered irrelevant to modelling of astrophysical objects and thus largely uninteresting. For discussions related to this see Damour 2009; Damour 2012.

It should be noted here that similarly there are strong constraints on SEP violations in the solar system, but these can be avoided by having some screening mechanism which trivializes the additional fields in solar system like environments (Will 2014). One interesting example is the spontaneous scalarization mechanism (Damour et al. 1993; Damour et al. 1996). As discussed in Section 2.1.4 if one has a scalar-tensor theory one can arrange for the scalar field to be trivial in situations where matter is not sufficiently compact, but become significant once some threshold compactness is reached. At this threshold the scalar field undergoes a phase transition (Harada 1997; Harada 1998; Salgado et al. 1998). A seemingly natural question is then whether or not the mechanism can apply to the WEP violating theory.

Using binary pulsar observations, one can significantly constrain violations of the SEP due to the interaction of the scalar field profile of one neutron star with the other (Damour et al. 1996). It should be noted, however, that if the scalar field profile is sufficiently localised (say there is even a small mass for the scalar field) these constraints essentially disappear (Antoniadis 2013; Ramazanolu et al. 2016). This means that to constrain violations of any of the equivalence principles within the neutron stars we need to understand their structure as predicted by the standard model of particle physics and GR. Therefore, as the exact equation of state for the matter in

neutron stars is notoriously difficult to pin down, if one could construct a WEP breaking scalar-tensor theory which maintains the spontaneous scalarization mechanism it is possible that it could lead to an acceptable equation of state, given current constraints.

It is important to note that this is just an example of how the WEP could be violated in a way which is astrophysically (or cosmologically) relevant while avoiding all current tests. This is interesting because one could imagine that such violations could lead to either smoking gun phenomena or suggest new experiments to constrain these violations more generally. It also gives another possible detection method for additional gravitational degrees of freedom.

It is our eventual goal to generate a well motivated equation of state for the nuclear matter which depends on the scalar field profile while not spoiling the spontaneous scalarization. It will also be interesting to see how this could be generalized to theories which have other screening mechanisms. However, our work is still preliminary and what follows is the construction and some analysis of a toy model, this will be based on the original pieces Coates et al. 2017a; Franchini et al. 2017. The purpose of the toy model is to show that WEP violations don't necessarily spoil the screening mechanism and to demonstrate how drastic the WEP violations can be within scalarized stars.

5.1 GRAVITATIONAL HIGGS MECHANISM

Let us first recall a few things about scalar-tensor theories and spontaneous scalarization. First of all note that if the WEP is violated, there will be no such thing as a Jordan frame, thus we take a scalar-tensor theory written in the Einstein frame as our starting point ,

$$S = \frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{-g}\left[R - 2g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right] + S_{\mathrm{m}}\left[\Psi^A, A^2(\phi)g_{\mu\nu}\right], \quad (5.1.1)$$

where Ψ^A stand for the matter fields, and one could (and should for most phenomenologically interesting models) include a potential for $V(\phi)$. Next, recall that the type of theories we are interested in are such that,

$$\alpha(\phi_0) = \frac{\mathrm{d}}{\mathrm{d}\phi} \log A(\phi)\bigg|_{\phi=\phi_0} = 0, \tag{5.1.2}$$

for some constant ϕ_0, and $\alpha'(\phi_0) = \beta < 0$. This allows GR solutions to exist but makes the coupling of matter to ϕ contribute to the equation for ϕ as a tachyonic mass around these solutions (for $T > 0$). This will cause an instability in modes with wavelengths above some threshold and below (around) the linear scale of the matter distribution (see Section 2.1.4 and references within). To allow this range to be non-empty for neutron stars requires $\beta \lesssim -4.35$ (Damour et al. 1993; Damour et al. 1996; Chiba et al. 1997).

So if we generalize the matter action to allow more general coupling between matter and ϕ one can have WEP violations in highly compact regions while avoiding WEP violations in the solar system. This assumes there is some generalized coupling between matter and the scalar field which does not spoil the screening mechanism. To demonstrate that such a thing can exist in principle, we look at the rather provocative example of generating a photon mass (or a mass for any other U(1) field). We shall also neglect to include a potential for the scalar field and assume that $\phi_0 = 0$, though these will turn out to be irrelevant to the demonstration that one can have non-minimal coupling without spoiling the spontaneous scalarization mechanism.

Let us consider taking,

$$\begin{aligned} S = &\frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{-g}\left[R - 2g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right] \\ &- \frac{1}{4}\int \mathrm{d}^4x\sqrt{-g}\left[F_{\mu\nu}F^{\mu\nu} + 2\left(m\phi\right)^2 A_\mu A^\mu\right] + \cdots . \end{aligned} \tag{5.1.3}$$

The term $\left(m\phi\right)^2 A_\mu A^\mu$ generates a mass for the photon which is strictly non-negative (for real m). Note for unscalarized solutions there is no effective mass for the photon. Despite being potentially interesting this model is

not perturbative around $\phi = 0$. This casts doubt on the possibility of the scalarized and unscalarized solutions being dynamically connected.

To see the issue clearly let us consider the standard Proca field,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu. \tag{5.1.4}$$

Due to the vanishing longitudinal mode in the limit $m \to 0$ one would naïvely expect some discontinuity, that is if one takes this only at face value. We can explore this futher by introducing a Stueckelberg field ψ to reinstate the $\mathrm{U}(1)$ gauge symmetry,

$$A_\mu \to A_\mu + \partial_\mu \lambda, \quad \psi \to \psi - m\lambda. \tag{5.1.5}$$

Then we have,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\left(mA_\mu + \partial_\mu \psi\right)^2, \tag{5.1.6}$$

which is gauge invariant and equivalent to the Proca Lagrangian (one can see this by using the gauge symmetry to set $\psi \to 0$). Then, in the limit $m \to 0$, the longitudinal mode (ψ) decouples. One can see this by comparing the ratio of the interaction term to the kinetic terms (for A and ψ). Thus the original problem is essentially just a consequence of choosing a bad gauge to address the $m \to 0$ limit.

Returning to the model of interest one might wonder whether the problem is a similar artefact, we will now show that this is not the case. If we introduce the Stueckelberg field, ψ, in this case and keep the same gauge transformations, the mass term becomes,

$$\frac{\phi^2}{2}\left(mA_\mu + \partial_\mu \psi\right)^2. \tag{5.1.7}$$

We can see from this that the $m \to 0$ limit is a proper decoupling limit but there is a different issue. When we expand around ϕ, ψ, $A = 0$ we see that there is a discontinuity in the number of degrees of freedom when increasing the order of perturbation theory. This suggests that this is not a valid background for perturbation theory. We could instead attempt to

include ϕ in the gauge transformation. If we treat $m\phi$ rather than m as the "mass parameter",that is write,

$$A_\mu \to A_\mu + \partial_\mu \lambda, \quad \psi \to \psi - m\phi\lambda, \tag{5.1.8}$$

we can construct a gauge invariant mass term, which has the form,

$$\left(m\phi A_\mu - \psi\partial_\mu \log\phi + \partial_\mu\psi\right)^2. \tag{5.1.9}$$

Due to the logarithmic dependence on ϕ one cannot perturb around $\phi = 0$ directly. However, it is possible to have $\phi = 0$ asymptotically if we have (for example) any power-law decay, exponential decay or Yukawa decay. This case is actually related by field redefinitions to the previous case. If we introduce $\chi = \psi/\phi$, then the term (5.1.9) becomes (5.1.7) with $\psi \to \chi$. This issue of non-perturbativity around $\phi = 0$ can be removed by introducing a bare mass for the photon, however small. Generically one can maintain the perturbative instability of the ϕ_0 configuration in dense enough environments if one introduces a mass term of the form,

$$m^2\left(1 + f(\phi)\right) A_\mu A^\mu, \tag{5.1.10}$$

so long as $f(\phi_0) = 0$ (for any other constant value, that is not -1, one can simply absorb it into the definition of m), and $f'(\phi_0) = 0$. For details see (Coates et al. 2017a).

We can construct a much cleaner model, where one has no free function to tune, which also avoids the above issues. To do so we promote the real scalar field to a complex one which is charged under $\mathrm{U}(1)$. It is known that the spontaneous scalarization picture is not spoiled by having multiple and/or complex scalar fields (Horbatsch et al. 2015), so only the coupling to the gauge field can potentially ruin the mechanism. More explicitly, we are interested in the action,

$$\begin{aligned} S = &\frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{-g}\left[R - 2g^{\mu\nu}\overline{\mathcal{D}_\mu\phi}\mathcal{D}_\nu\phi\right] \\ &- \frac{1}{4}\int \mathrm{d}^4x\sqrt{-g}\left[F_{\mu\nu}F^{\mu\nu}\right] + S_{\mathrm{m}}\left[A^2(|\phi|^2)g_{\mu\nu}, A_\mu, \Psi^A\right], \end{aligned} \tag{5.1.11}$$

where $\mathcal{D}_\mu\phi = \partial_\mu\phi - ieA_\mu\phi$ is the gauge covariant derivative, and e is the electric charge of the scalar field [1]. When $e \to 0$ we can recover a Jordan frame, for lack of a better term we will refer to this frame as the Jordan frame even when $e \neq 0$. As we have taken the conformal factor to depend only on the modulus of the scalar field, the action is thus invariant under the U(1) transformation,

$$\phi \to \phi \exp(ie\lambda), \quad A_\mu \to A_\mu + \partial_\mu\lambda. \tag{5.1.12}$$

Upon spontaneous symmetry breaking (by spontaneous scalarization in this case) the photon acquires a mass (restoring all units to SI),

$$m_\gamma^2(\bar{\phi}\phi) = \frac{\mu_0 e^2 c^2}{4\pi G}\bar{\phi}\phi. \tag{5.1.13}$$

This model also doesn't require any bare photon mass. We will see later that in the centre of neutron stars this mass will be only an order of magnitude or two below the Planck mass if e is of order the electron charge. First, let us write the field equations for this theory, splitting the scalar field into its real and imaginary parts, $\phi = \phi_1 + i\phi_2$. Varying the action with ϕ_1, ϕ_2, A^ν and $g^{\mu\nu}$ leads to,

$$\left(\Box - e^2 A_\mu A^\mu\right)\phi_1 + \left(2eA^\mu\partial_\mu + e\nabla_\mu A^\mu\right)\phi_2 = -T\phi_1\frac{\mathrm{d}}{\mathrm{d}\phi}\log A(|\phi|^2), \tag{5.1.14}$$

$$\left(\Box - e^2 A_\mu A^\mu\right)\phi_2 - \left(2eA^\mu\partial_\mu + e\nabla_\mu A^\mu\right)\phi_1 = -T\phi_2\frac{\mathrm{d}}{\mathrm{d}\phi}\log A(|\phi|^2), \tag{5.1.15}$$

$$\nabla^\mu F_{\mu\nu} = J_\nu + J_\nu^{(\phi)} + m_\gamma^2 A_\nu,\ G_{\mu\nu} = 8\pi G\left(T_{\mu\nu} + T_{\mu\nu}^{(\phi)} + T_{\mu\nu}^{(A)} + T_{\mu\nu}^{(\phi A)}\right) \tag{5.1.16}$$

1 Of course, these kinds of models have been studied in condensed matter physics under the name Ginzburg-Landau theory (see Ginzburg et al. 1965 for an English translation of their original work) and in particle physics as the U(1) Higgs mechanism (as studied in Higgs 1964). The novelty here is the mechanism by which the non-trivial scalar field is generated.

where,

$$J_\mu = -\frac{1}{\sqrt{-g}}\frac{\delta S_m}{\delta A^\mu}, \quad J_\mu^{(\phi)} = \frac{e}{4\pi G}\left(\phi_2\phi_{1,\mu} - \phi_1\phi_{2,\mu}\right), \tag{5.1.17}$$

$$T_{\mu\nu}^{(\phi)} = \frac{1}{4\pi G}\left[\phi_{1,\mu}\phi_{1,\nu} + \phi_{2,\mu}\phi_{2,\nu} - \frac{1}{2}g_{\mu\nu}\left(\phi_{1,\alpha}\phi_{1,}{}^{\alpha} + \phi_{2,\alpha}\phi_{2,}{}^{\alpha}\right)\right], \tag{5.1.18}$$

$$T_{\mu\nu}^{(A)} = F_{\mu\alpha}F_{\nu}{}^{\alpha} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} + m_\gamma^2\left(A_\mu A_\nu - \frac{1}{2}g_{\mu\nu}A_\alpha A^\alpha\right) \tag{5.1.19}$$

$$T_{\mu\nu}^{(\phi A)} = J_\mu^{(\phi)}A_\nu + J_\nu^{(\phi)}A_\mu - g_{\mu\nu}J_\alpha^{(\phi)}A^\alpha. \tag{5.1.20}$$

We will now specialise to spherical symmetry, taking the following ansätze for the metric,

$$\mathrm{d}s^2 = -\exp\nu(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1-2\mu(r)/r} + r^2\mathrm{d}\Omega^2, \tag{5.1.21}$$

the electromagnetic potential,

$$A^\mu = \left(A^t(r),\, A^r(r),\, 0,\, 0\right), \tag{5.1.22}$$

and the scalar fields,

$$\phi_1 = \phi_1(r), \quad \phi_2 = \phi_2(r). \tag{5.1.23}$$

We will model all other matter as a perfect fluid with Jordan frame stress-energy,

$$\tilde{T}^{\mu\nu} = (\epsilon + p)\,u^\mu u^\nu + p\tilde{g}^{\mu\nu}. \tag{5.1.24}$$

The Einstein frame stress energy tensor is then,

$$T^{\mu\nu} = A^6(|\phi|^2)\tilde{T}^{\mu\nu}. \tag{5.1.25}$$

From here we shall adopt units where $4\pi G = 1$ to simplify the expressions. In the more standard spontaneous scalarization case $e \to 0$, $A_\mu \to 0$, and

there is only a single real scalar field. If we take (as it is a minimal example) $A(\phi) = \exp \beta\phi^2/2$, then the system of equations become,

$$\mu' = r^2 \exp\left(2\beta\phi^2\right)\epsilon + \frac{1}{2} r\left(r - 2\mu\right)\psi^2, \tag{5.1.26}$$

$$\nu' = \frac{2r^2 \exp\left(2\beta\phi^2\right) p}{r - 2\mu} + r\psi^2 + \frac{2\mu}{r\left(r - 2\mu\right)}, \tag{5.1.27}$$

$$p' = -\left(\epsilon + p\right)\left(\frac{\nu'}{2} + \beta\phi\psi\right), \tag{5.1.28}$$

$$\phi' = \psi, \tag{5.1.29}$$

$$\psi' = \frac{r \exp 2\beta\phi^2}{r - 2\mu}\left[\beta\phi\left(\epsilon - 3p\right) + r\psi\left(\epsilon - p\right)\right] - \frac{2\left(r - \mu\right)}{r\left(r - 2\mu\right)}\psi. \tag{5.1.30}$$

If we now look at the r component of the Maxwell equations in our system of equations we find a constraint,

$$eA_r(r)|\phi|^2 = \phi_1\phi_2' - \phi_2\phi_1' = |\phi|\left(\arg\phi\right)'. \tag{5.1.31}$$

Substituting this into Equation (5.1.14) or (5.1.15), one arrives at the same equation, which is a consequence of the U(1) gauge invariance. Also, as $\arg\phi$ is the part that is transformed under U(1) (*i.e.* the transformation on ϕ is fully specified by $\arg\phi \to \arg\phi + e\lambda$), this means that A_r and ϕ_2 can be set to 0 by a gauge choice simultaneously. This then implies $J_\mu^{(\phi)} = T_{\mu\nu}^{(A\phi)} = 0$, and we can cast our system of equations in spherical symmetry as,

$$\mu' = \mu'_{\mathrm{DEF}} + \frac{r \exp -\nu}{2}\left[e^2 r A_t^2\phi^2 + \left(r - 2\mu\right) f^2\right], \tag{5.1.32}$$

$$\nu' = \nu'_{\mathrm{DEF}} + \frac{r \exp -\nu}{r - 2\mu}\left[e^2 r A_t^2\phi^2 - \left(r - 2\mu\right) f^2\right], \tag{5.1.33}$$

$$p' = p'_{\mathrm{DEF}} - 2e^2 A_t\phi^2 f \exp -\left(\nu + 2\beta\phi^2\right), \tag{5.1.34}$$

$$\phi' = \psi, \quad A_0' = f, \tag{5.1.35}$$

$$\psi' = \psi'_{\mathrm{DEF}} - \frac{r \exp -\nu}{r - 2\mu}\left[e^2 r A_t^2\phi - \left(r - 2\mu\right) f^2\psi\right], \tag{5.1.36}$$

$$\begin{aligned} f' = {} & f\frac{\left(r - 2\mu\right)\left(r^2\psi^2 - 2\right) + r^3\left(\epsilon + p\right)\exp 2\beta\phi^2}{r\left(r - 2\mu\right)} \\ & - e^2 r A_t\phi^2\frac{1 - rA_t f \exp -\nu}{r - 2\mu}, \end{aligned} \tag{5.1.37}$$

where the subscript DEF refers to the right hand side of the appropriate equation in (5.1.26) to (5.1.30). To close the system one then needs to provide some equation of state to relate ϵ and p.

It is worth noting that the equations coincide with the DEF case when $A_0 = f = 0$, and thus solutions of the original system are also solutions of ours. What is more, given regularity and asymptotic flatness these solutions are actually unique. Here we will present the non-perturbative proof, but a proof at the perturbative level can be found in Coates et al. 2017a. First note that we can always pick a gauge where $J^{(\phi)}$ vanishes and in this gauge the Maxwell equation reads,

$$\nabla^\mu F_{\mu\nu} = m_\gamma^2 A_\nu. \tag{5.1.38}$$

If we now contract with A^ν and integrate over a spacetime volume, $\mathcal{V}$, we obtain,

$$\int_{\mathcal{V}} \mathrm{d}^4x \sqrt{-g} \left[A^\nu \nabla^\mu F_{\mu\nu} - m_\gamma^2 A^\nu A_\nu \right] = 0. \tag{5.1.39}$$

Using the product rule for differentiation and the divergence theorem we have,

$$\int_{\mathcal{V}} \mathrm{d}^4x \sqrt{-g} \left[\frac{1}{2} F^{\mu\nu} F_{\mu\nu} + m_\gamma^2 A^\nu A_\nu \right] = \int_{\partial\mathcal{V}} \mathrm{d}^3\sigma \, n^\mu A^\nu F_{\mu\nu}, \tag{5.1.40}$$

where $\partial\mathcal{V}$ is the boundary of $\mathcal{V}$. Assuming regularity, we can choose this boundary to be a surface of constant r and take the limit $r \to \infty$. If we denote,

$$\mathcal{I}_1 = \frac{1}{2} F^{\mu\nu} F_{\mu\nu} + m_\gamma^2 A^\nu A_\nu, \quad \mathcal{I}_2 = n^\nu A^\mu F_{\mu\nu}, \tag{5.1.41}$$

and then apply our ansätze (as these are scalars, we can evaluate them in any coordinate system) we have,

$$\mathcal{I}_1 = g^{tt} \left[g^{rr} \left(\partial_r A_t \right)^2 + m_\gamma^2 A_t^2 \right], \quad \mathcal{I}_2 = g^{tt} A_t \partial_r A_t. \tag{5.1.42}$$

Now taking the limit $r \to \infty$, asymptotic flatness requires $A_t \propto 1/r$ and $g^{tt} \to -1$. We also have that the volume element scales as r^2. Combining these means that $\mathcal{I}_2 \propto 1/r^3$ and the right hand side of (5.1.40) vanishes. Thus, so long as neither g^{tt} nor g^{rr} change sign, $\mathcal{I}_1$ is non-negative. For the

integral to vanish we must therefore have that $A_t = 0$. This proof follows the no-hair proof for black holes in the Einstein-Proca system, first presented in Bekenstein 1972. It should be stressed that this proof does not rely on the lack of a potential for ϕ or on the specific choice of $A^2(|\phi|^2)$.

We have shown that the regular, asymptotically flat, spherically symmetric solutions of the model we are interested in are the same as in the standard DEF case. This makes calculating the mass-profile of the photon very simple. Here we shall demonstrate the effectiveness of the mechanism using $A^2(|\phi|^2) = \exp(\beta\phi^2)$ and $V(|\phi|) = 0$ and effectively rederiving some known solutions. Though, as mentioned one could use any other choices.

Using the system of equations (5.1.32)-(5.1.37) we have 7 first order differential equations, and thus we require 7 initial conditions. We integrate outwards from $r = 0$ and, following Damour et al. 1993, we take initial conditions, $\mu(0) = 0$, $\nu(0) = 0$, $p(0) = p_c$, $\phi(0) = \phi_c$, and $\psi(0) = 0$ and then, following the proof, $A_t(0) = 0$ and $f(0) = 0$. As mentioned, we must also supplement these equations with an equation of state, relating p and ϵ. To do so we introduce a number density n and use it to define an implicit relationship between p and ϵ, the calculations proceed using only n.

To perform the integration we split the solution into three regions: the core, the crust and the exterior. The core is the region where the number density satisfies $n > 8.3 \times 10^{-5} n_c$, and in the core we use one of five polytropic equations of state with different stiffnesses. In particular, we use the equations of state named 2H, H, HB, B and 2B in Read et al. 2009. The crust is the region between the core and the surface (which is defined by $p = 0$), this surface defines the neutron star's (areal) radius r_s. Following Read et al. 2009 we use the same EOS in the crust for all solutions. In the exterior of the star we set $n = 0$. Although more realistic equations of state exist, this is more than sufficient for demonstrating the mechanism.

To extract the ADM mass, $M_{\rm ADM}$, and the dimensionless asymptotic scalar charge, $\mathfrak{q} = Q/M_{\rm ADM}$, we use the known exact solutions in the exterior which can be used to relate these to quantities evaluated on the surface

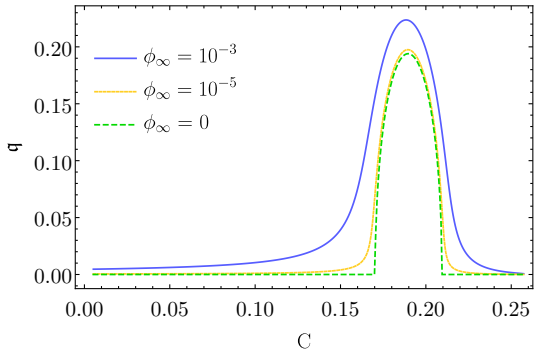


Figure 5.1: Scalar charge α versus compactness C for various values of the scalar field at infinity, for $\beta = -4.5$ and HB equation of state.

of the star (Damour et al. 1992; Damour et al. 1993). If we denote such quantities with a subscript s we have,

$$\mathfrak{q} = \frac{2\psi_s}{\nu_s'}, \quad M_{\rm ADM} = 2\pi r_s^2 \nu_s' \left(1 - \frac{2\mu_s}{r_s}\right)^{1/2} \exp \mathcal{A}, \tag{5.1.43}$$

where,

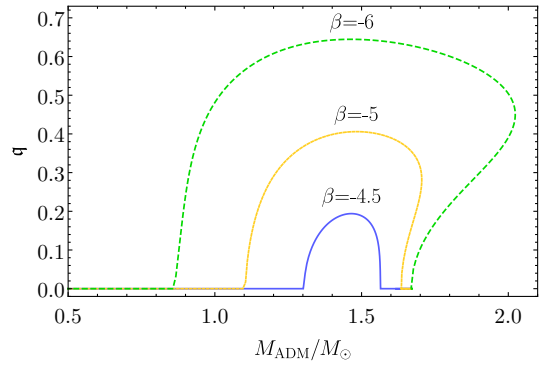
$$\mathcal{A} = -\frac{\nu_s'}{(\nu_s'^2 + 4\psi_s^2)^{1/2}} \tanh^{-1}\left(\frac{(\nu_s' + 4\psi_s^2)^{1/2}}{\nu_s' + 2/r_s}\right). \tag{5.1.44}$$

In spherical symmetry the definitions of the ADM mass and scalar charge can be put in terms of the asymptotic expansions of g_{tt} and ϕ,

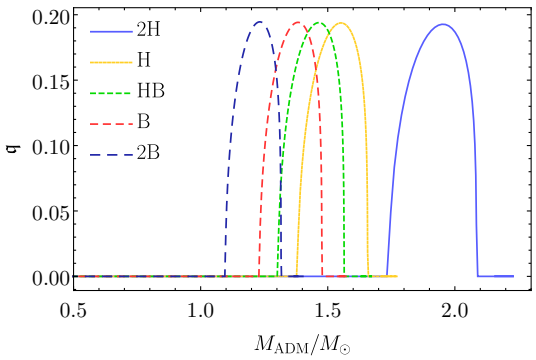
$$g_{tt} = -1 + \frac{M_{\rm ADM}}{2\pi r} + \mathcal{O}\left(\frac{1}{r^2}\right), \quad \phi(r) = \phi_\infty + \frac{Q}{4\pi r} + \mathcal{O}\left(\frac{1}{r^2}\right). \tag{5.1.45}$$

We generate solutions by choosing some central pressure, p_c, and then use a shooting method to fix ϕ_c so as to obtain some ϕ_∞. This way we generate Fig. 5.1 which shows how the dimensionless scalar charge varies with compactness, $C = M_{\rm ADM}/r_s$, for some values of ϕ_∞. When $\phi_\infty = 0$ the scalarization transition is very sharp. For these solutions we fixed $\beta = -4.5$ and the EOS to HB. It is worth noting that the compactness is related to the curvature quite directly and so this can be seen as a curvature threshold also.

For the rest of our plots we set $\phi_\infty = 0$, so that the asymptotic photon mass vanishes. In Figs. 5.2a & 5.2b we show the standard results that the

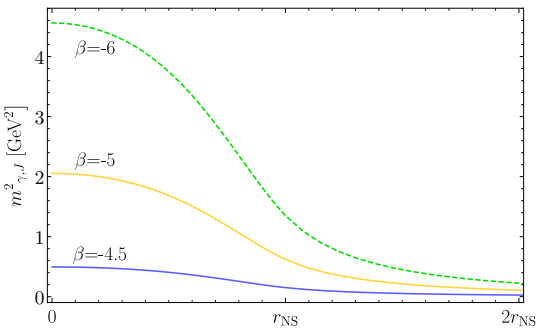


(a) Scalar charge α versus ADM mass for different values of β with HB equation of state.

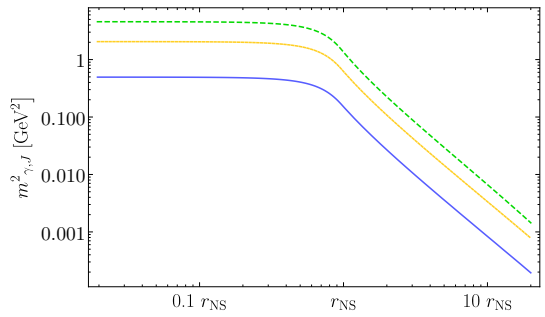


(b) Scalar charge α versus ADM mass for different equations of state with $\beta = -4.5$.

Figure 5.2: Properties of scalarization for different solutions



(a) Photon mass profile in the Jordan frame as a function of the areal radius.



(b) Logarithmic profile of the photon mass in the Jordan frame as a function of the areal radius. The style of the curves is the same as in the left panel.

Figure 5.3: Radial profile of the photon mass in the Jordan frame for different values of β, the HB equation of state, and $e = 10^{-36}$ C.

scalarization is more effective for larger $|\beta|$ and that the equation of state has little effect on the scalar charge, just shifting the range of allowed masses. For previous simulations see *e.g.* Damour et al. 1993.

Using these solutions we can now calculate the mass profile for the photon, seen in Figs. 5.3a & 5.3b. We plot the Jordan frame photon mass and again fix the equation of state to HB. As the absolute value of ϕ is $\sim 10^{-2}$

in the centre of the star (and only gets smaller from there), the difference between the Jordan and Einstein frame masses are not particularly large. We also set $e = 10^{-36}$C, which is 17 orders of magnitude smaller than the electron charge, even so the photon mass in the centre of the star reaches the GeV scale, and the photon mass scales linearly with e.

5.2 SUMMARY

The message of this Chapter is that solar system tests of the WEP needn't constrain WEP violations in other regimes. The model we mainly considered is a simple toy model which demonstrates that even very small, non-universal, couplings between extra gravitational degrees of freedom and matter can lead to very large effects in high curvature regimes. Of course for the particular model we studied the slow fall off ($\propto 1/r$) outside of a scalarised neutron star could rule out such effects very easily. However, if one considers a more general model [even introducing a very small mass for the scalar field (Ramazanolu et al. 2016)] these effects can remain similarly large but be more localised.

One of the exciting directions one could take this is to consider how the structure of neutron stars could differ if, for example, one constructed a model where the scalar field has Yukawa couplings to (some of) the fermions of the standard model. As the neutron star equation of state is poorly constrained currently this is actually a potentially viable modification which could also give testable predictions for the near future. Of course getting from the standard model to an equation of state for neutron star matter is not trivial at all. For this reason the most obvious next step in our minds is to come up with an equation of state which has dependence on the scalar field in a manner that can at least be motivated from expectations of what modification of certain couplings in the standard model could do.

6

CONCLUSIONS

Although the subjects of this thesis have been quite broad impetus for their study comes from the same source. Our main motivation for studying alternatives to GR is that, excluding the possibility of asymptotic safety, it appears that GR is not a complete theory of gravity. However, due to the extreme scales required to directly probe quantum gravity, it seems unlikely that any experiments will directly access that regime in the near future. Therefore, the best hope of finding quantum gravity effects by studying the gravity sector alone is through residual effects at the classical level. Candidates for such residual effects include: extra degrees of freedom, equivalence principle violations and Lorentz symmetry violations, and these considerations led to the work contained in this thesis. For the more detailed summaries of the work in this thesis refer to the summary sections at the end of each chapter (see Sections 3.3, 4.4, and 5.2).

Although we have focused mainly on the theoretical motivations for the work in the main body of the thesis it is worth commenting on the timeliness of this work in terms of observations. Indeed we have mainly focused on questions of relevance to the strong gravity regime and this was done intentionally. This intention was justified by the first direct detection of gravitational waves, which along with the discovery of binary black systems and their mergers (Abbott 2016a) which occurred coincident with this work. This means that understanding dynamical black holes in alternative theories of gravity is of great relevance to experiment and we have made

some steps towards this, for Lorentz violating theories, in Chapter 4. In the theories we studied black holes are rather different than in GR due to the fact that they allow for instantaneous propagation. Of course if one is interested in taking Lorentz violating theories of gravity seriously it is important that they do not spoil the Lorentz invariance of matter to a level excluded by experiments. That this is a serious concern with no easy solution was demonstrated in Chapter 3.

With the efficacy of gravitational wave detectors having been established (Abbott 2016a; Abbott 2016b; Abbott 2017) it seems to be only a matter of time before mergers between neutron star binaries, and black hole - neutron star systems, are detected. In Chapter 5 we have demonstrated that solar system tests do not rightly constrain WEP violations in strong gravity regimes. In the context of gravitational wave experiments this is important as WEP violations could drastically change the structure of neutron stars from what might otherwise be expected. This is particularly relevant as the constraints on this structure are rather weak and yet such possibilities have been ignored.

Much of this work aims to develop theoretical understanding which can connect to observations but it is still preliminary. There is interesting work to be done in bridging the remaining gaps. In particular one of the main directions of interest to me is to develop a model which has the properties of the model discussed in Chapter 5 but which would be more relevant for the structure of neutron stars, and then find the observable predictions of such a model.

Another concrete example of an area of interest which naturally continues the work shown in this thesis is understanding dynamical black holes beyond spherical symmetry in Hořava gravity. This is important for the modelling of astrophysical black holes as they generically have non-zero spin.

More generally the gravitational wave detections are the beginning of a more thorough understanding of the phenomenology of the strong gravity

regime. To best constrain, or to have the highest chance of detecting, deviations from GR in the future the understanding of alternative theories of gravity needs to vastly improve. This is particularly true of causality, the initial value problem and numerics. For GR developing these to the point of being able to simulate black hole mergers took almost a century (Pretorius 2005). For alternative theories, however, there is hope that many of the lessons of GR can be imported in a way which is much less gruelling than their original development.

In any case, though especially if such challenges are met, this is an exciting time to be working on gravity and I am glad to have made some contribution to the field, however small.

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