

Scalar Fields and Gravity

by

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Abstract

In this thesis we discuss scalar field theories, and their applications to gravity. We provide a summary of why there is interest in modifying Einstein's General Relativity, and discuss why scalar fields make a good candidate for a modification to make. We demonstrate their effects on the dynamics of matter, and discuss the necessity of screening mechanisms in order for these scalar fields to not be ruled out by current observations. We present discussion on two screening mechanisms in particular, the Chameleon and Vainshtein mechanisms.

We then present work that aims to study the soft behaviour of scattering amplitudes belonging to single scalar field theories. We generalise current techniques in the literature such that the study of a much wider set of theories is possible. We use this technique to perform a detailed study of a particular family of theories, a so called $(1, 2)$ theory, and demonstrate that the DBI symmetry is the unique way to enhance the soft behaviour of the scattering amplitudes of this family. We also identify the special Galileon as the unique way to maximally enhance the soft behaviour within the $(1, 2)$ class, and verify the validity of recursion techniques to calculate scattering amplitudes.

We then move on to studying the Chameleon in more detail. We provide motivation for modifying its high energy behaviour by studying the 'surfer solution', and use this to propose the DBI-Chameleon. We demonstrate that this theory avoids the problems the Chameleon suffers in the early Universe and forms a good effective field theory in this regime.

Finally we present a UV complete theory describing a massive Galileon, and study its dynamics to verify if it exhibits Vainshtein screening. Theories with Vainshtein screening are usually unable to be UV completed in a Wilsonian way. We present our preliminary findings which suggest screening is possible for at least some parameter values.

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CHAPTER 1

1 Introduction

1.1 General Relativity

Gravity is the weakest force we know of in the Universe, with Newton's law of gravitation predicting the force on two protons to be $F_{\text{grav}} \sim 10^{-40} F_{\text{EM}}$, where F_{EM} is the electromagnetic force between them. This is something we understand intuitively, a simple fridge magnet can hold itself aloft against the entire planet pulling it down. Despite its weakness gravity is the dominant force on large scales, and so understanding gravity is key to understanding the evolution of our Universe. For a long time Newton's theory on gravity was the pinnacle of our understanding, but several problems revealed themselves which led the way for Einstein to propose General Relativity as the more complete picture of gravitation. Key to the acceptance of General Relativity were its early experimental successes, famously correctly predicting Mercury's perihelion precession [1] and the correct amount of bending of light by the Sun [2], observed by Eddington soon after [3]. To accentuate this, it also had theoretical advantages over Newton's picture in that it made gravity consistent with Special Relativity by setting a finite propagation speed on gravity and also treating time and space in the same way, all while reducing to Newtonian gravity in appropriate limits, ensuring that it did not violate all previous measurements made in gravitational experiments.

While Newton's law gave no deeper insight into gravitation beyond being predictive, Einstein's General Relativity offers the view of gravity as a geometric phenomena, the outcome of allowing space-time to evolve dynamically as opposed to being a fixed background as is the case for Special Relativity. General Relativity describes space-time as a pseudo-Riemannian

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manifold, whose geodesics govern how freely falling particles evolve. The evolution of this manifold is governed by the Einstein equations,

$$M_{pl}^2 G_{\mu\nu} + g_{\mu\nu} \Lambda_b = T_{\mu\nu} \quad (1.1)$$

where $M_{pl} = (8\pi G_N)^{(-1/2)}$ is some dimensionfull scale related to Newton's constant, $g_{\mu\nu}$ is the metric that describes space-time, $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ is the Einstein tensor which built from the Ricci tensor $R_{\mu\nu}$ and its trace R . $T_{\mu\nu}$ is the stress-energy tensor, which describes the distribution of the matter in the space-time. These equations were arrived at by the requirement that the tensor on the left hand side should satisfy the following; be symmetric in its indices $A_{\mu\nu} = A_{\nu\mu}$, depend on the metric and its first two derivatives only $A = A(g, \partial g, \partial\partial g)$, and finally be divergence free $\nabla_\mu A^\mu = 0$. It has been shown that the most general tensor to satisfy these conditions in 4 dimensions is the linear sum of the Einstein tensor and the metric [4], $A_{\mu\nu} = aG_{\mu\nu} + bg_{\mu\nu}$ for arbitrary a and b . The inclusion of the constant b here has had a tumultuous past [5], with people historically arguing about its inclusion in Einstein's equations, but modern understanding entails its inclusion. These requirements come from Einstein's motivation that his theory should respect the equivalence principle and general covariance, that is that any physical result from a system in free-fall should be independent of velocity and position. The Einstein equations can be derived from an action, the Einstein-Hilbert action [6, 7] given by

$$\mathcal{S}_{EH} = \int d^4x \sqrt{-g} \left[\frac{M_{pl}^2}{2} R - \Lambda_b \right] + \mathcal{S}_m, \quad (1.2)$$

where g is the determinant of the metric and hence $\sqrt{-g}d^4x$ the covariant volume element. $\mathcal{S}_m$ is the matter sector, which gives rise to $T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta \mathcal{S}_m}{\delta g^{\mu\nu}}$. General Relativity is the unique way to construct a four dimensional theory of a massless spin-2 field below the Plank scale, which is relativistic, divergence free, and has field equations that feature no higher than second order derivatives [8], which makes it even more appealing from a theoretical

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view. To see this from a different perspective, one can take a particle theorist approach and begin to construct an effective theory for a spin-2 field by writing down all the operators that could contribute to such an action that have fewer than 2 derivatives, order by order. Imposing Lorentz invariance and locality is enough to leave you with General Relativity [9–15]. The fact that one only includes operators up to two derivatives indicates that GR is only unique at low energies, as mentioned previously.

1.2 Experimental Successes

If we wish to modify Einstein’s theory of gravity, we should understand its successes as they must be replicated in any theory that wishes to supplant it. The uniqueness of General Relativity as the low energy effective theory of a massless spin-2 particle can be considered a theoretical success, but science thrives on experimental support for theories, and this is also a field that has seen much success. Most of the observational evidence to support General Relativity comes from weak field observations, many from solar system tests. Einstein’s field equations are a set of ten partial differential equations, and so can be quite difficult to solve in general. Within the solar system the curvature of space-time can be considered small, and we can be satisfied that predictions made by looking at perturbations around flat Minkowski space are accurate. In fact, linearised gravity can be shown to be accurate enough for observations at scales above the Schwarzschild radius, and so for the solar system holds perfectly well [16]. In this regime, we have measurements on the gravitational bending of light, the precession of the perihelion of Mercury and the Shapiro delay, which is caused by photons travelling through gravitational fields experiencing time dilation [17]. Observations of all of these phenomena agree with the predictions of General Relativity very well, giving strong evidence of its validity. [2,3,17,18]

Precise tests of General Relativity began with gravitational redshift and

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lensing measurements. A photon moving through a gravitational field will be redshifted, and experiments both lab and space based have seen this effect in agreement with General Relativity predictions [19]. Gravitational lensing results have since moved on from looking at light-bending by the Sun, and we can now measure lensing around galaxies instead. Gravitational lensing is now a large part of cosmology, where we can predict the amount of lensing CMB photons undergo as they travel to us, and also use lensing to create profiles of dark matter distribution. [2, 20]. Measurements of the Shapiro time-delay effect also suggest no deviation from General Relativity [21].

Strong field tests of gravity are much harder to perform, as areas with strong gravitational fields are prone to astrophysics that is difficult to model and account for. Nevertheless, binary pulsar systems have been used to test the strong field predictions of General Relativity. These are rapidly spinning neutron stars orbiting each other in a highly relativistic scenario, and the regular emission of radiation from the pulsar helps monitor their motion. Observing these systems one can measure a periastron precession that General Relativity predicts accurately but also binary pulsars provided the first evidence to the existence of gravitational waves. The orbits of observed binary pairs are seen to decay at a rate consistent with energy loss through gravitational radiation predicted by General Relativity [22–24]. Recently, direct detection of gravitational waves has been achieved by Advanced LIGO [25–30], from the merging of black holes and later neutron stars, whose profiles matched with expectations from General Relativity. A recent detection of gravitational waves was also accompanied by an electromagnetic observation, which constrains the speed of gravitational waves to be very close to the speed of light [31], as predicted by General Relativity. This has provided new constraints on many popular models of Modified Gravity which alter the speed of gravitational waves. For a more thorough review of the state of popular theories after this event, we refer the reader to [32–34]. It is worth noting that only one measurement constraining the speed of gravitational waves has been made, and so are liable to change as

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more events are observed.

All of the above tests of General Relativity are somewhat small in scale, cosmologically speaking. Our current model of cosmology, Λ CDM, is based on General Relativity, with cold dark matter and a small cosmological constant. Solving Einstein's equations for a homogeneous and isotropic universe with such content yields predictions about the evolution of our universe, which agree remarkably well with our observations. This approach to cosmology was first popularised when Lemaître and Friedmann's [35, 36] result that our universe is expanding was seen by Hubble [37], and found to be replicated by General Relativity, widely overthrowing the until then knowledge that the universe was static and unchanging. Change does not happen overnight though, and there were of course still theories trying to fit this observation into the steady-state theory of cosmology, such as Hoyle-Narlikar gravity [38], worked on by those who rejected the idea of the Big Bang, a term which was popularised by the aforementioned Hoyle. More recently, predictions of the Cosmic Microwave Background and its features from Λ CDM match with observations [39–41], something that ruled out many of the steady-state models [42].

1.3 Problems and Motivation to Modify

Despite all the successes that General Relativity has enjoyed in the past century, there is much effort in supplanting it as our current description of the Universe. Motivation to do so comes from both the extremely large and small scales. I will begin by discussing the small scale motivation for modifying General Relativity, before moving onto the large scale motivations.

General Relativity at its heart is a classical theory of gravity, a fact that sets it apart from our understanding of every other force which is formulated in a quantum mechanical way in the Standard Model. Often one hears that General Relativity and Quantum Mechanics are incompatible, but one can

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write General Relativity in the language of an effective theory of gravity, where one is free to quantise in a way similar to Yang-Mills theories [43]. Initially General Relativity was thought to be a untenable quantum theory due to the fact that it is non-renormalisable, and thus un-physical. In the years since, we have come to the understanding that even non-renormalisable theories can be used to make predictions, but only in an effective theory context in which it is understood they only remain valid for a specific range of energies. Doing so for General Relativity allows us to make quantum predictions about gravity all the way down to the Planck scale [16, 43], at which point loop perturbation theory breaks down and we understand that our effective theory is no longer effective. Most famously one can calculate the 1-loop correction to the non-relativistic gravitational potential as [44]

$$\frac{\delta V(r)}{V(r)} = \frac{41}{10\pi} \frac{G\hbar}{r^2}, \quad (1.3)$$

where $V(r) = -Gm_1m_2/r$ is the Newtonian potential, and the occurrence of $\hbar$ emphasises the quantum nature of this correction. The main motivation then at short scales is to provide a UV completion to General Relativity, and eventually create a Grand Unified Theory of all forces. There are many areas of research into this, although currently String Theory seems the most promising [45, 46].

Moving on to discussing large scale motivations to modify General Relativity, it may seem counter-productive that we wish to change anything given how accurate Λ CDM seems to be at reproducing observations. Still, the dark energy and dark matter parts of Λ CDM are currently not well understood, any many view them as an indication that gravity works differently to how we expect, in a similar fashion to General Relativity explaining Mercury's odd behaviour instead of the then postulated extra planet that would alter its orbit. As gravity is the dominant force over cosmological scales, it makes sense that we should consider modifying it to explain cosmological phenomena. Dark matter is a postulated non-Standard Model particle that interacts

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directly with Standard Model particles incredibly weakly, and would explain the anomalous behaviour of galactic rotation curves [47–49], gravitational lensing observations [50–53], and large scale structure such as galaxy clusters [54–58]. As no particle candidate has currently been observed, despite much effort to do so, many have attempted to explain the behaviour by modifying General Relativity instead. MOND [59, 60] and TeVeS [61] are the most well known attempts at this, however their overall popularity is small and most prefer the idea of particulate Dark Matter. This is due to the difficulties present in explaining a range of observations, not just the rotation curves they are designed to reproduce. Colliding galaxy cluster observations and the power spectrum of large scale structure, along with Cosmic Microwave Background and Big Bang Nucleosynthesis data are all difficult to explain in these theories, compared to Λ CDM [55, 62–66]. Supersymmetry also provides motivation for the presence of many more particles than those we have currently seen, which makes the idea of a new particle being Dark Matter more palatable.

There is a large body of observations, from Type 1a supernova and CMB measurements, suggesting that our universe is currently under a period of accelerating expansion, driven by so called ‘dark energy’ [67]. A first pass is to associate this dark energy with the cosmological constant present in GR. As Λ_b in General Relativity (1.2) is a constant its energy density does not redshift away as other forms of energy do, and so will dominate the universe’s dynamics at late times, driving the universe to expand exponentially. This is because it pushes space-time to be de Sitter, where the size of the cosmological constant governs the magnitude of the curvature [68]. One can then match this constant Λ_b to give the observed expansion and pat oneself on the back. This is indeed how Λ CDM works, by matching Λ with observations.

This is not the whole story though, as one expects this constant to receive corrections from the vacuum energy of each massive particle present in $\mathcal{S}_m$. The equivalence principle tells us that this vacuum energy will gravitate in

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the same way as all other energy, and thus we have an effective cosmological constant $\Lambda = \Lambda_b + V_{vac}$. Let us consider an example situation where we can calculate V_{vac} . Consider a matter action that consists only of a canonical scalar field ϕ , of mass m ,

$$\mathcal{S}_m = \int d^4x \sqrt{-g} \left[-\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 + \lambda\phi^4 + \dots \right]. \quad (1.4)$$

The vacuum energy of such a system is given by the sum of all vacuum loop diagrams, schematically given by

$$V_{vac} = \text{circle} + \text{two circles} + \dots \quad (1.5)$$

$$V_{vac} = V_{vac,1} + V_{vac,2} + \dots \quad (1.6)$$

Focussing on just the one loop contribution, in a semi-classical regime where we calculate in flat space and then study gravity on the resulting vacuum energy,

$$V_{vac,1} = -\frac{1}{2} \int d^4x \int \frac{d^4p}{(2\pi)^4} \log(p^2 + m^2). \quad (1.7)$$

Using dimensional regularisation and introducing counterterms to cancel the divergence, we finally find that

$$V_{vac,1} \sim \frac{m^4}{(8\pi)^2} \left[\log\left(\frac{m^2}{M^2}\right) - \frac{3}{2} + \gamma_e \right], \quad (1.8)$$

where M is the renormalisation mass scale and γ_e is the Euler-Mascheroni constant. It is worth mentioning here that we can absorb the finite constants here by redefinition of M . The general form of this result holds across different types of matter fields, leading us to believe that a generic matter

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action would contribute a vacuum energy of

$$V_{vac} \sim \sum_i C_i m_i^4, \quad (1.9)$$

where i represents a sum over all particle species and C_i are $\mathcal{O}(1)$ coefficients. Using the content of the Standard Model, this gives us a whopping $V_{vac} \sim (\text{TeV})^4$. This is often cited at the cosmological constant problem, but in the same way we introduce counterterms to cancel the divergence in the calculation above, and in renormalisation in general, one should be perfectly happy to simply choose a value of Λ_b such that $\Lambda = V_{vac} + \Lambda_b \sim (\text{meV})^4$ as observed. The real problem comes about when one then considers higher order loop corrections to V_{vac} . Any tuning applied at one loop order will be completely spoilt if one includes 2 loop corrections, which also come through as terms $\sim \lambda m^4$, with no significant suppression relative to the 1 loop corrections for a theory without finely tuned parameters. One can re-tune again, but the story is the same for the next loop order, and the next. This is radiative instability, one requires a completely new tuning at every loop order if one wishes to cancel the bare cosmological constant against the vacuum energy. This is telling us that the vacuum energy is incredibly sensitive to the UV properties of the theory, and without details of this UV theory we are unable to sum all loop orders. This is analogous to the Higgs mass problem in the Standard Model, where it is expected that the Higgs' mass would pick up corrections from every other massive particle present in a similar way, leading to a mass $m_H \sim \text{TeV}$ which is larger than the observed value of 125 GeV [69]. This itself is not a huge difference, but the problem gets even worse if one assumes heavier undiscovered particles exist and are coupled to the Higgs, which would drag the Higgs' mass up to that new heavy scale. This is in contrast to similar calculations of the loop corrections to say the electron's mass, where one finds that each higher loop order's contribution is suppressed relative to the previous one, meaning one can set the bare mass in the EFT to give the physical mass we observe,

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and not worry about any UV Physics changing it. This protection stems from the fact that massless fermions possess chiral symmetry, ensuring that no mass is generated. Breaking this symmetry by including a mass term means that corrections to that bare mass will now be generated, but will only be proportional to the bare mass and not any higher scale, keeping the corrections small. If one doesn't like the perturbative approach taken here, there is an equivalent argument that involves the Wilson action, and shows that the vacuum energy is unstable against changing the cut-off of the theory. For details of that discussion, see [70, 71], but note that it has the same interpretation that the vacuum energy is highly dependant on the UV physics.

This problem provides a lot of motivation for modifying General Relativity, and there is an abundance of theories that attempt to tackle this. These vary from scalar-tensor theories like Brans-Dicke, theories where the graviton is endowed with a mass such as DGT, to $f(R)$ theories. For a comprehensive review of these theories, we refer the reader to [20, 72].

1.4 Screening Mechanisms

As GR is so well tested, any new theory must look very similar to GR at solar system scales, while still giving rise to some non-trivial effects at others. To create a long distance modification to GR and respect locality, one requires new degrees of freedom be introduced, which will mediate long range forces. There are many examples of this in the literature, with most of these including an additional light scalar field. They are arguably the simplest modification one could make, but further motivation comes from looking at UV completions to gravity as well as dark energy. Scalar Tensor theories in general often appear as effective theories in 4 dimensions of higher dimensional models of gravity after compactification, such as from string theory and Kaluza-Klein models [20], and have gravitational strength couplings.

Let us study what effects these scalar fields will have and how we can get

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them to agree with solar system observations. If we take a massive scalar field which we couple to the trace of the stress-energy tensor in flat-space,

$$\mathcal{S} = \int d^4x \left[-\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 + \frac{g}{M_{pl}}\phi T \right] \quad (1.10)$$

where T is the trace of the stress-energy and g is a dimensionless coupling constant of gravitational strength $g \sim \mathcal{O}(1)$. This coupling between the stress-energy tensor and the scalar field can be understood by considering a modified gravitational sector, with an additional scalar degree of freedom, in which the metric couples to matter fields minimally as in GR. This theory can be transformed into a theory with a standard Einstein-Hilbert gravitational sector and an canonical scalar sector, where one would see the coupling to matter now feature the scalar field as well as the metric. We can make this more concrete by looking at (1.15), which is a theory with a Einstein-Hilbert gravitational term, a scalar field, and a coupling to matter that will give the ϕT term. Here we can see that the coupling is really minimal in the Jordan frame metric.

The scalar field equation of motion is given by

$$\square\phi - m^2\phi + \frac{g}{M_{pl}}T = 0. \quad (1.11)$$

Modelling the stress-energy as being sourced from a non-relativistic point source approximating a planetary body, where M is the source mass, we solve the equation of motion to see that

$$\phi = -\frac{gM}{4\pi M_{pl}} \frac{e^{-mr}}{r}. \quad (1.12)$$

This gives the expected Yukawa potential for a massive scalar boson of mass m . This field will act on the matter coupled to it with a force per unit mass

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given by $f_\phi = -\frac{g}{M_{pl}}\nabla\phi$ [73, 74], meaning

$$f_\phi = -\frac{g^2 M}{4\pi M_{pl}^2} \frac{e^{-mr}}{r^2} (1 + mr). \quad (1.13)$$

If we compare this to the size of the gravitational force exerted by the source, we find that

$$\left| \frac{F_\phi}{F_N} \right| = 2g^2 e^{-mr} (1 + mr). \quad (1.14)$$

Here we have used the Newtonian expression for the gravitational force to compare the magnitude of the forces at solar system scales, at which this choice is valid. Given that $g \sim \mathcal{O}(1)$ generically, and that m has to be small to allow the scalar field to have an effect at cosmological scales, it would seem that we can rule out the theory given it predicts a large deviation away from the observed Newtonian potential. To help avoid this, screening mechanisms are often used.

A screening mechanism is something that dynamically suppresses the behaviour of extra fields at scales where it is unwanted, yet allows them to have a non-trivial effect at other scales. It is for this reason that they come hand in hand with modifications to gravity, where at solar system scales we must suppress their effects, yet at cosmological scales we need them to drive expansion. In this body of work the focus is on two of the most popular screening mechanisms, the Chameleon and Vainshtein mechanisms, which will be briefly introduced in this Section.

1.4.1 The Chameleon

The Chameleon mechanism was proposed by J. Khoury and A. Weltman [75, 76] and allows us to suppress the scalar field's effects in regions of high matter density, ie in the solar system, while allowing it to have cosmological effects. It was originally proposed to tackle the moduli stability problem in string theory [77, 78]. A conformal coupling to the matter sector means that the potential the scalar field feels depends on the local matter density,

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leading to a density dependant mass for the scalar. A larger mass leads to a larger Yukawa suppression, and hence a smaller ranged force, and no detectable effects in dense areas such as the solar system. It is worth pointing out that the Chameleon theory on its own cannot be dark energy [79, 80], but as light scalar fields are a generic prediction of string theory and other modifications of GR it is a fruitful theory to study anyway, as these fields will need to be screened to pass observational tests. The simplest Chameleon theory is one given by

$$\mathcal{S} = \int d^4x \sqrt{-g} \left[\frac{M_{pl}^2}{2} R - \frac{1}{2} (\nabla\phi)^2 - V(\phi) \right] + \mathcal{S}_m[\tilde{g}_{\mu\nu}, \psi] \quad (1.15)$$

where ϕ is the scalar field (dubbed the Chameleon in the following) and $\tilde{g}_{\mu\nu} = A(\phi) g_{\mu\nu}$ is the conformal coupling between the Chameleon field and matter, also known as the Jordan frame metric as its geodesics are those that the matter fields will follow. The ability for the theory to screen local effects depends on the interplay between the conformal coupling and the bare potential $V(\phi)$. To see this explicitly we calculate the Chameleon field's equation of motion on an FLRW background,

$$\ddot{\phi} + 3H\dot{\phi} = -V'(\phi) - \frac{1}{2}A(\phi)A'(\phi)\tilde{\rho}\Sigma \quad (1.16)$$

$$= -V'_{\text{eff}}, \quad (1.17)$$

where H is the Hubble parameter and $\tilde{\rho}$ is the Jordan frame energy density, as we have modelled the matter content as a perfect fluid with $\Sigma = 1 - 3w$. It is worth noting that the Jordan frame energy density is covariantly conserved, since the matter fields couple minimally to the Jordan frame metric, and is given by $\tilde{T}_{\mu\nu} = -\frac{2}{\sqrt{-\tilde{g}}} \frac{\delta \mathcal{S}_m}{\delta \tilde{g}^{\mu\nu}}$. The prime denotes differentiation with respect to ϕ and the dot differentiation with respect to the time co-ordinate. Clearly the effective potential V_{eff} depends on the matter content. If one has suitable $V(\phi)$ and $A(\phi)$, the effective potential will have a minimum at some field value ϕ_{min} . In the literature, it is common to use $A(\phi) = e^{2\beta\phi/M_{pl}}$ [76],

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so if one chooses a potential that drops off monotonically in ϕ then there will be one minimum in the effective potential, as shown in Figure 1.1 which will lead to a Chameleon mass of

$$m_\phi^2 = V''_{\text{eff}}(\phi_{\min}). \quad (1.18)$$

If we choose a potential

$$V(\phi) = \frac{M^{4+n}}{\phi^n} \quad (1.19)$$

where M is some energy scale there for dimensional grounds, we can see that it satisfies the property of dropping off monotonically for $n > 0$. This potential was first used in Chameleon mechanism at its inception [75]. Constraints placed on variations of particle masses from Big Bang Nucleosynthesis until today [78, 81] require the Chameleon only make small excursions in field space, relative to M_{pl} , and so we can approximate the conformal factor as

$$A(\phi) \approx 1 + \frac{2\beta\phi}{M_{pl}}. \quad (1.20)$$

Finding the minimum of the effective potential,

$$\phi_{\min} = \left(\frac{nM^{4+n}M_{pl}}{\beta\rho\Sigma} \right)^{\frac{1}{n+1}} \quad (1.21)$$

where $\rho = A(\phi)\tilde{\rho}$ is the Einstein frame matter density, that is the background matter density which appears in the Einstein equation when found by varying the action with respect to the non-rescaled metric $g_{\mu\nu}$. Using this expression to evaluate the mass of the scalar we find that

$$m_\phi^2 = V''_{\text{eff}}(\phi_{\min}) = n(n+1)M^{-\frac{n+4}{n+1}} \left(\frac{\beta\rho\Sigma}{nM_{pl}} \right)^{\frac{n+2}{n+1}}. \quad (1.22)$$

We can now see explicitly that as long as $n > 0$, a higher background density leads to a higher m_ϕ . This is shown in Figure 1.1, where a change in the background density changes the mass of the Chameleon and hence

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the Compton wavelength. The effective potential is the solid line, while $V(\phi)$ and the conformal contribution are shown in dashed lines. One can see the monotonic drop-off behaviour of the bare potential $V(\phi)$, and the gradient of the conformal contribution given by the background matter density. When the density is small, as on the right, a low gradient means a small second derivative in the effective potential, and hence a small mass for the Chameleon. Raising the background density increases the curve of the effective potential, and hence the mass.

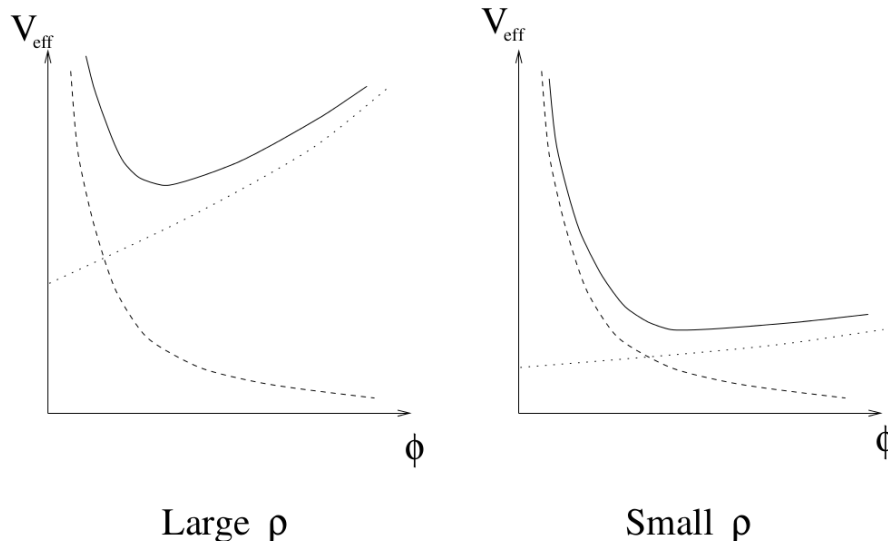


Figure 1.1: Taken from [76], the plot on the left shows the effective potential for a high background density. The plot on the right shows the same bare potential, but a lowered background density. We note the change in the position of the minimum, and in the value of $V''_{\text{eff}}(\phi_{\text{min}})$.

Problems do exist with the Chameleon theory though, which challenge its use as an effective theory of modified gravity. A breakdown in the theory around Big Bang Nucleosynthesis was first studied in [81, 82], where it is noted that the Chameleon must be weakly coupled to matter in order to avoid violating observational bounds, challenging its relevance as a screening mechanism. I will discuss this problem in depth in Section 3, and also detail work that mitigates this problem by modifying the behaviour of the

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Chameleon at high energies while retaining all the desirable properties such as screening at lower energies. For a more thorough review of the mechanics behind the Chameleon mechanism we refer the reader to [20, 72, 75, 76].

1.4.2 The Vainshtein Mechanism

Opposed to the Chameleon mechanism, which relies on the interplay between the coupling to matter and the potential, the Vainshtein mechanism relies on the non-linearity of the kinetic part of the action. It was first utilised in the context of massive gravity, specifically Fierz-Pauli theory [83], where attempts to give the graviton mass lead to predictions that violated observations in the solar system, even in the limit that the graviton mass was taken to zero. This is worrying because one would expect that a massive theory of gravity would recreate GR, a theory of a massless graviton, in its massless limit, at least in solar system scales. This dissonance between the massless limit of Fierz-Pauli theory and GR is known as the vDVZ discontinuity in the literature. This difference in predictions between massive gravity and GR is caused by the presence of a new scalar degree of freedom, which represents the longitudinal mode of the massive graviton, and the fact that even in the zero mass limit it does not decouple from the trace of the stress-energy tensor [83–86]. We can explicitly see this by writing the Fierz-Pauli action down

$$\mathcal{S}_{FP} = \int d^4x \left[\mathcal{L}_{GR,lin} - \frac{m^2}{2} (h_{\mu\nu}h^{\mu\nu} - h^2) + \frac{h_{\mu\nu}T^{\mu\nu}}{M_{pl}} \right] \quad (1.23)$$

where $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, $\mathcal{L}_{GR,lin}$ is the linearised GR Lagrangian and m is the graviton mass. By performing the Stukelberg trick we can explicitly extract the new degrees of freedom, and by diagonalising the result to remove kinetic mixing between the degrees of freedom and taking the $m \rightarrow 0$ massless limit we find the action is

$$\mathcal{S} = \int d^4x \left[\mathcal{L}_{GR,lin} - \frac{1}{2}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial\phi)^2 + \frac{\phi T}{\sqrt{6}M_{pl}} + \frac{h_{\mu\nu}T^{\mu\nu}}{M_{pl}} \right], \quad (1.24)$$

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where the metric now contains the 2 degrees of freedom it does in GR, and we have extracted the new degrees of freedom into a new vector field where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and a new scalar ϕ . It is now clear that the scalar field has a coupling to the trace of the stress-energy in a way identical to (1.10), and as discussed in Section 1.4 this will lead to a gravitational strength force in addition to the gravitational force predicted from GR. The fact that this ϕT coupling does not vanish as $m \rightarrow 0$ is the dVDZ discontinuity.

It was first realised by Vainshtein [87, 88] that if one works with a non-linear theory that reduces to the linearised massive gravity theory, the non-linear interactions became important at a scale that has since become known as the Vainshtein scale. At distances below this scale, the non-linear terms contribute significantly to the dynamics and thus the linearised approximation breaks down. In the case of DGP gravity, the Vainshtein radius is seen to be $r_v = ((GM)/m^4)^{1/5}$ [16, 20], where m is the graviton mass and the source of mass M is modelled as a non-relativistic spherically symmetric body. This shows that as one takes the massless graviton limit, the linear theory becomes invalid at all distance scales and so its predictions should not be trusted, avoiding the vDVZ discontinuity. If one uses a non-linear theory to make predictions about deviations from Newton's potential, one finds

$$\frac{F_\phi}{F_N} \sim \left(\frac{r}{r_v} \right)^{\frac{3}{2}} \quad (1.25)$$

when $r < r_v$ [20]. This shows that these non-linear self interactions cause the scalar force to be suppressed, *screening* the force around the source. It is worth mentioning that although the roots of Vainshtein screening are in massive gravity, it is by no means unique to it, and can be found in many other theories where non-linear interactions exist.

For illustrative purposes, let us consider

$$S = \int d^4x \left[-\frac{1}{2} (\partial\phi)^2 + \frac{c}{\Lambda^4} (\partial\phi)^4 + \frac{g}{M_{pl}} \phi T \right] \quad (1.26)$$

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as an example theory, where $g, c \sim \mathcal{O}(1)$ are dimensionless coupling coefficients, Λ is the cut-off scale and we are working on flat-space. It is worth noting that even though the coupling to matter is linear in ϕ , it does not break the shift symmetry due to the property that $\partial_\mu T^\mu_\nu = 0$. This ensures that the constant shift contributes only a total derivative, and thus can be ignored. If we linearise this action and find the solution for ϕ around a spherically symmetric point source of mass M , we again find (1.12) but with $m \rightarrow 0$,

$$\phi = -\frac{gM}{4\pi M_{pl}} \frac{1}{r}. \quad (1.27)$$

This is not unexpected, as linearising the theory in (1.26) corresponds to just a massless scalar coupled to the trace T . This solution is obviously only an approximation to the full non-linear theory, and will be corrected by the presence of the non-linear operator. We can estimate the Vainshtein radius by asking when that non-linear term becomes important, aka where

$$\left. \frac{(\partial\phi)^2}{\Lambda^4} \right|_\phi \sim \mathcal{O}(1). \quad (1.28)$$

This gives us a Vainshtein radius of

$$r_v \sim \frac{1}{\Lambda} \sqrt{\frac{gM}{4\pi M_{pl}}} \sim \frac{1}{\Lambda} \sqrt{\frac{M}{M_{pl}}}. \quad (1.29)$$

Using dimensional analysis one can say that within the Vainshtein radius the solution is of the form

$$\phi \sim \Lambda^{\frac{4}{3}} \left(\frac{M}{M_{pl}} \right)^{\frac{1}{3}} r^{\frac{1}{3}} \quad (1.30)$$

which means the force applied by the scalar field is

$$\left| \frac{F_\phi}{F_N} \right| \sim \left(\frac{r}{r_v} \right)^{\frac{4}{3}}. \quad (1.31)$$

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We note here that the scalar solution will only take this form if $c < 0$, otherwise the solution will run into a branch-cut singularity [89]. The implications of this point will be discussed in more detail later in this section. The expression in (1.31) makes it clear that instead of (1.14), we now have suppression of the force when $r \ll r_v$. In fact, as we get closer to the source the force is more suppressed.

To make this argument more rigorously, we can instead look at how the fluctuations around a given background solution couple to the matter sector. If we choose some background solution $\tilde{\phi}$ such that $\phi = \tilde{\phi} + \pi$, we can expand the action (1.26) to quadratic order

$$\int d^4x \left[-\frac{1}{2} Z^{\mu\nu} \partial_\mu \pi \partial_\nu \pi + \frac{g}{M_{pl}} \pi T \right] \quad (1.32)$$

where

$$Z^{\mu\nu} = \eta^{\mu\nu} \left(1 - \frac{4c}{\Lambda^4} (\partial \tilde{\phi})^2 \right) - \frac{8c}{\Lambda^4} \partial^\mu \tilde{\phi} \partial^\nu \tilde{\phi} \quad (1.33)$$

Here we can see that as the non-linear terms become important, Z grows large. If we canonically normalise it is clear that the fluctuations couple to the matter source with a coefficient $\frac{g}{M_{pl}\sqrt{Z}}$. This shows us that as the non-linear terms become important, the coupling to the matter sector is dynamically weakened and hence any scalar force acting on the matter sector is weakened as well.

To look closer at the Z factor and screening in solar system like environments, we now specify a spherically symmetric background $\tilde{\phi} = \tilde{\phi}(r)$. We then find that

$$Z^{00} = -1 + \frac{4c}{\Lambda^4} \tilde{\phi}'^2, \quad (1.34)$$

$$Z^{0i} = 0, \quad (1.35)$$

$$Z^{ij} = \delta^{ij} \left(1 - \frac{4c}{\Lambda^4} \tilde{\phi}'^2 \right) - \frac{8c}{\Lambda^4} \partial^i \tilde{\phi} \partial^j \tilde{\phi}, \quad (1.36)$$

where $\tilde{\phi}'$ is the derivative of $\tilde{\phi}$ with respect to r . Physically we want a solu-

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tion in which the gradient of $\tilde{\phi}$ vanishes at infinity, which tells us that our Z factor has one negative and 3 positive eigenvalues. To retain this throughout evolution of the system, we see that we have to demand $c < 0$ [89, 90], as mentioned previously. If this is not satisfied then the fluctuations π become infinitely strongly coupled to the matter due to the $g/\sqrt{Z}$ coupling. This result is more general than for this particular operator [89], and demonstrates that even though theories may have the required non-linear derivative operators, only certain theories will lead to Vainshtein screening occurring. If instead of static, spherically symmetric backgrounds we decide to look at homogeneous and isotropic backgrounds of the form $\tilde{\phi} = \tilde{\phi}(t)$, the Z factor will take on the form

$$Z^{00} = - \left(1 + \frac{12c}{\Lambda^4} \dot{\phi}^2 \right), \quad (1.37)$$

$$Z^{0i} = 0, \quad (1.38)$$

$$Z^{ij} = \delta^{ij} \left(1 + \frac{4c}{\Lambda^4} \dot{\phi}^2 \right), \quad (1.39)$$

where the dot represents derivatives with respect to the time co-ordinate t . Following the same logic, to retain the 3 positive and one negative eigenvalue we require that $c > 0$. Theories satisfying this condition will then exhibit a ‘cosmological Vainshtein’ effect, which will be used in Section 3.2 to help avoid problems in the early Universe behaviour of the Chameleon theory.

The fact that only certain values of the coefficient c lead to theories that can successfully screen the scalar force is not on its own surprising, but is one part of a larger problem with implementing the Vainshtein mechanism. In [91], apparently consistent low energy effective theories were analysed through their scattering properties, and it was found that the signs of coefficients of irrelevant operators play a large role in determining whether or not a theory can be UV completed in a local way and have an analytic S-matrix. These ‘positivity conditions’ therefore must be satisfied by any theory which can be seen in the low energy limit of a local quantum field theory or weakly coupled string theory. More specifically, it was found that in effective theo-

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ries where the leading interactions are described by irrelevant operators, the coefficient of the leading s^2 term in the forward tree level scattering amplitude ($t \rightarrow 0$) of a $2 \rightarrow 2$ scattering process must be positive, where s and t are the Mandelstam variables defined in the usual way of $s = -(p_1 + p_2)^2$, $t = -(p_1 + p_3)^2$ and $u = -(p_1 + p_4)^2$. To demonstrate this, we consider our example theory from (1.26) and present the 4 point tree level scattering amplitude,

$$\mathcal{A}_4 = \frac{2ic}{\Lambda^4} (s^2 + t^2 + u^2). \quad (1.40)$$

We can simplify this by noting that our theory in (1.26) has no mass term and a shift symmetry meaning no mass can be generated. This implies $s + t + u = 0$ and so

$$\mathcal{A}_4 = \frac{4ic}{\Lambda^4} (s^2 + t^2 + st). \quad (1.41)$$

Taking the forward limit we find that the amplitude is $\mathcal{A}_4(t \rightarrow 0) = (4c/\Lambda^4)s^2$, and hence the positivity conditions require that $c > 0$ if (1.26) is a theory that can be UV completed in the standard way. Again, this result is shown to be far more general than for the particular operator we have used in this example [91, 92]. The problem here should now be obvious, that the condition for UV completion and the condition for Vainshtein screening around a spherical mass are opposed to each other, implying one cannot embed a low energy theory that utilises the Vainshtein mechanism to screen fifth forces into a local and Lorentz invariant high energy theory. This is one of the main problems facing the use of the Vainshtein mechanism, although we note that the ‘cosmological Vainshtein’ effect on homogeneous backgrounds satisfies both conditions. This problem provides motivation for work in Section 4 where we show that there does exist a theory that utilises Vainshtein screening around static backgrounds and is valid as an effective theory up to the Plank scale. It is worth noting that theories that exhibit the wrong sign and do not satisfy the conditions required for a Wilsonian UV completion are candidates to undergo Classicalisation. In a standard Wilsonian picture, new degrees of freedom will restore unitarity to scattering amplitudes above

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a given scale, whereas Classicalisation takes the stance that this unitarity breaking is an artefact of perturbative approach one takes when considering Feynman diagrams. In this picture extended classical objects, such as multi-particle states of the low energy quanta, restore unitarity and extend the regime of validity. We refer the reader to [93–95] for more detail on this. To give an example, [93] discusses gravity itself as a good example of how Classicalisation would work. If one considers Plank scale scattering amplitudes in GR, the theory breaks down and one requires a UV completion. Classicalisation argues that at such scales the scattering processes would create black holes (classical objects), preventing the probing of Plankian scales and thus avoiding the loss of unitarity. As of the time of writing, the study of Classicalisation is still in its infancy and there are currently no known theories that exhibit it successfully.

CHAPTER 2

2 Soft Limits of Scalar Effective Theories

2.1 Review

Cosmologists often employ the use of scalar effective field theories in their model building, and recently much use of a distinct class of these effective theories called ‘Galileons’ has been made [20, 96]. A subset of the Horndeski theory, they offer second order equations of motion, and also enjoy the property of screening fifth forces by employing the Vainshtein mechanism, and also are found to emerge in low energy effective descriptions of theories such as DGP Gravity [16, 97]. All of these things make them a natural choice when trying to build modified gravity theories. They are so named because they are unchanged under the transformation $\phi \rightarrow \phi + c + b_\mu x^\mu$, where c is a constant scalar and b_μ a constant vector. This looks very much like simple Galilean invariance, hence the name.

Unrelated to their influence in cosmology, Galileons are also of interest to those studying the S-matrix. The modern S-matrix, built upon the assumptions of unitarity and Lorentz invariance, has revealed structures previously hidden from standard Lagrangian methods. For a review of this, see [98]. Although somewhat unrelated, the use of Galileons in this context was first made by Cachazo, He and Yuan in [99], where they included the so called ‘special Galileon’ theory in their work. Studies of this ‘special Galileon’ and its scattering amplitudes and soft limits [100, 101] highlight the reason behind calling it special, by noting that it is an example of an *exceptional* effective field theory. Exceptional field theories are those whose soft limits are maximally enhanced, DBI being another example [102, 103]. Scalar theories with these enhanced soft limits are of interest, because they admit on-shell recursion relations that mean all tree level scattering amplitudes can be built

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out of scattering amplitudes with fewer external legs [101–105]. Exceptional effective theories are the easiest to work with in this sense, as all couplings depend on a single parameter, simplifying the process. In the case of the special Galileon, it was noted that it had an extra symmetry over ‘normal’ Galileon theories before it was understood to be an exceptional theory [106].

The soft limit of a scattering amplitude is simply given by taking the limit of one of the external momenta in a process to zero, i.e. by expanding around $p = 0$. Well known in pion scattering amplitudes is the Adler zero [107, 108], which states that the amplitude of a process with a single soft pion should vanish. For a derivation of this, see Appendix .2. As explored in [100, 101], it is possible to consider generalisations of Adler’s zero to other theories for which the soft limit of an amplitude scale with the soft momenta as

$$\mathcal{A}(p) = \mathcal{O}(p^\sigma) \tag{2.1}$$

where σ is some positive integer which ensures the soft limit does indeed vanish. For a soft limit to be enhanced, σ must take a value that is above what you would expect from a naive counting of the number of derivatives per field in the Lagrangian. The presence of such an enhancement across all of the scattering amplitudes in a theory hints at a hidden structure related to the existence of a symmetry that may not be obvious in the Lagrangian picture.

In [100, 101] a study of the enhancements in single scalar effective theories was carried out, in which they categorised the theories by the number of derivatives per interaction and the behaviour of the tree level soft limits. We follow this convention here, where to enforce cancellations between different tree level scattering diagrams, required for an enhancement of the soft limit, only Lagrangians of the following schematic form were considered

$$\mathcal{L} = (\partial\phi)^2 \sum_{m,n=0}^{\infty} c_n \partial^m \phi^n, \tag{2.2}$$

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where c_n are coupling coefficients. To ensure Lorentz invariance we will require that the number of derivatives, m , is even in every term in the Lagrangian. To ensure that the theory has a canonical kinetic term we require that the $m = n = 0$ term has coefficient $c_0 = -\frac{1}{2}$. We define ρ by $m = \rho n$, such that ρ will be given by a positive rational number and this will be how we categorise theories. As an example to cement this idea, the theory represented by $\rho = 1$ will have a Lagrangian of the form $\mathcal{L} = c_0(\partial\phi)^2 + c_1(\partial\phi)^4 + \dots$ and represents a k-essence theory. It is also worth noting that theories of this general form are unchanged under a constant shift of the scalar, $\phi \rightarrow \phi + c$, which guarantees they will have soft limits of at least $\sigma = 1$ due to the Alder's zero. The total number of derivatives in a given term is then $m + 2$, which act on $n + 2$ fields. The aim is to calculate a scattering amplitude of this theory, and by imposing that the scattering amplitude has a soft limit of order σ one can find relations between the coupling constants c_n . These relations can be used to then identify new structures and symmetries within families of scalar effective theories.

To give more detail about how a soft limit becomes enhanced, we note that by simply counting the number of derivatives there are per field in the Lagrangian, one can say that a soft limit of order $\sigma_0 \lesssim (m + 2)/(n + 2)$ is guaranteed to occur. All terms contributing at order σ_0 must cancel if higher order terms are to dominate the soft limit, giving an enhancement. For any two Feynman diagrams to cancel in principle, they must have the same number of external legs such that they contribute to the same scattering amplitude. As tree level diagrams may be built out of multiple vertices, this means that for two diagrams to cancel

$$\sum_a n_a = \sum_b n_b, \quad (2.3)$$

where a and b represent summing over each individual vertex in a given diagram. If we now evaluate this amplitude in the soft limit for some given external momenta, there is a choice of c_n that will cause these two diagrams

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to cancel with each other. For this cancellation to be of any relevance though, it must hold for all choices of external momenta, and hence the diagrams must also scale with p in the same way to give cancellations that can lead to enhancements of the soft limit. This means that a further condition

$$\sum_a m_a = \sum_b m_b$$

which we can write as

$$\sum_a \rho_a n_a = \sum_b \rho_b n_b. \quad (2.4)$$

The easiest way to ensure these conditions are satisfied is to enforce a common value of ρ between every term in the Lagrangian. This would mean that every single diagram in a given amplitude would scale the same way with momentum. We note that this is not the unique way to satisfy these constraints though, as it can be done with interactions that do not have the same ρ values. An example of this is a Lagrangian of the schematic form

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 + g_1(\partial\phi)^4 + g_2(\partial\phi)^2\partial^4\phi^2 + g_3(\partial\phi)^4\partial^4\phi^2. \quad (2.5)$$

Clearly every term has a different value of ρ ; 1, 2 and 6/4 in fact. If we look at the contact diagram built with a single g_3 vertex, it scales the same with momentum as a diagram built from a g_1 and g_2 vertex connected by one internal propagator; both scale as p^8 .

In the case of every interaction having the same value of ρ , we call it a ‘single ρ ’ theory. It is this category of theory that was studied in [100, 101], and a summary of the results of their analysis will be provided before moving onto original work that aims to generalise this framework to theories where different terms in the action have different ρ values, such as in (2.5). The motivation to do this comes from asking whether the special Galileon really is as unique as it seems, or part of a larger class of theories with extra symmetries.

2.2 Numerical Method

Before getting too involved in the discussion, we present the motivation for numerical analysis of the soft behaviour of the scattering amplitudes, and detail the method used in [109]. In all derivatively coupled single scalar field theories, the on-shell 3 point amplitude always vanishes, and hence the 4 point amplitude is the first one we need to calculate. The 4 point amplitude in the cases we study is never too complicated and so can be done by hand, and it is ‘simple’ to read off the soft limit degree and any conditions required to enhance it. For larger point amplitudes the number of diagrams and channels one must include to describe the full scattering amplitude increases dramatically, and keeping track of everything by hand becomes infeasible. Also to consistently take the soft limit, one must account for the fact that all of the external momenta are not independent. Momentum conservation allows us to remove one of the momenta in favour of a linear sum of the others, and noting that all external momenta are on-shell means that $\sum_{i,j \neq n} S_{ij} = 0$, where n is the momenta removed via energy conservation. This allows us to remove a further S_{ij} variable. We also note that in 4 space-time dimensions there are only 4 possible linearly independent momentum vectors, which leads to relations between them such that 4 momentum vectors can be used as a basis with which we write all other momenta vectors. These relations are called the Gram-determinant relations, and for more detail on those we refer the reader to [110]. For our system, the relevant information is that a matrix G with elements $G_{ij} = S_{ij}$ must have a determinant equal to zero. For the 5 point amplitude this boils down to momentum conservation, but beyond that it imposes extra non-linear constraints which must be taken into account. So to summarise, calculation the soft limit behaviour of a given amplitude in principle requires that one should: write all possible diagrams and channels down using the Feynman rules of the theory; apply momentum conservation and the fact that all external momenta vectors are on shell; use the Gram-determinant relations to finally have an amplitude written in an independent basis of variables; take

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one of these independent momenta to be soft and expand around zero.

The most obvious way to proceed is perform just that procedure, but computationally is very time consuming, even for a computer. We find that any study of an amplitude above 6 point is infeasible by this procedure. Instead we assign numeric values to a redundant basis of momenta, but making sure that the relevant relations between them are satisfied. This is trivial for the Gram determinant relations, as they are satisfied by the fact we have more than 4 different 4 component vectors. We assign arbitrary values to four of the momenta, hereafter referred to as p_1 to p_4 , such that they are on-shell. We can then define the remaining momenta as a linear sum of these $p_j = \sum_{i=1}^4 \alpha_{ij} p_i$ for $j > 4$. Some of these α are set such that all the momenta are on-shell, or to ensure the system conserves momentum as a whole. The remaining α are randomly assigned values, and one should check that any conclusions about soft behaviour are independent of the choices made here. We do this by calculating the soft amplitudes for multiple different randomised initial momenta, and find that all the relations between coefficients presented in this work indeed hold regardless of those values. As long as one starts with an action, one can define the Feynman rules of the theory, and we put these into the programme such that we can build the diagrams and take all relevant channels algorithmically, eliminating the need to write them all by hand. This then gives us an amplitude for a set of numerical momenta vectors. To take the soft limit we send $p_1 \rightarrow zp_1$ and expand the amplitude for small z . We always take p_1 to be soft, which does not affect our results as the theories we consider have amplitudes that are invariant under exchanges of the external momenta. As a final output we then obtain a function of the coupling coefficients c_n , multiplied by some numerical factors that are dependant on our choice of α and p_1 to p_4 . One can then begin to find enhancements by looking at how the coefficients must be related in order to send the lowest power of z term in the amplitude to zero.

2.3 Single ρ Theories

To review the work of [100, 101], our starting point will be the Lagrangian in (2.2), which we can write as

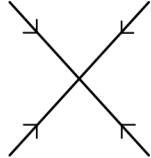
$$\mathcal{L}_\rho = (\partial\phi)^2 \sum_n c_n \partial^{(\rho n)} \phi^n \quad (2.6)$$

as every term has the same ρ value by definition. We will also work in $3+1$ dimensions, unless explicitly stated otherwise. To get a grasp on how the procedure works, we will use the case of $\rho = 1$ as an example. The action of a $\rho = 1$ theory is given by

$$\mathcal{L}_1 = -\frac{1}{2}(\partial\phi)^2 + \sum_{n=2}^{\infty} c_{2n}(\partial\phi)^{2n}, \quad (2.7)$$

and by simple counting of the derivatives we can see that the soft limit is guaranteed to scale as $\sigma = 1$. This means that we are interested in relations between the c_n that change the soft limits such that $\sigma > 1$. The first non-trivial scattering amplitude is the four point $\mathcal{A}_4$, which has only one diagram coming from the $(\partial\phi)^4$ term in the Lagrangian. This means that

$\mathcal{A}_4 =$



(2.8)

$$\mathcal{A}_4 = 2ic_4 (S_{12}^2 + S_{13}^2 + S_{23}^2) \quad (2.9)$$

where $S_{ij} = (p_i + p_j)^2 = 2p_i \cdot p_j$ is the usual Mandelstam variable, and we have eliminated p_4 from the equation using momentum conservation. It is clear then that the amplitude goes like $\mathcal{O}(p^2)$, implying a soft limit of the form $\sigma = 2$. This is higher than the naive estimate from derivative counting, but to truly imply an interesting structure the enhancement must hold across all scattering amplitudes. As it only has one of the coefficients

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in it, apart from setting $c_4 = 0$ there is no way to enhance the soft limit order, so it is indeed good that it is higher than our naive estimate as it leaves room for enhancement to $\sigma = 2$ of any theory that contains a $(\partial\phi)^4$ operator. $\mathcal{A}_5$, the 5 point amplitude, is zero due to the fact that there are no interactions with an odd power of the field, and so the next non-trivial amplitude is the 6 point one. To calculate this by hand is quite an ordeal, and so we use the computational method from [109] that was described in Section 2.2 to numerically evaluate the scattering amplitudes and find any possible enhancements. There are only 2 vertices that can contribute to the tree level scattering, and we can schematically see that only two diagrams will contribute, the contact diagram and two 4-point vertices connected by a single internal propagator. Schematically the amplitude can be written as

$$\mathcal{A}_6 = \text{[Contact Diagram]} + \text{[Exchange Diagram]} \quad (2.10)$$

$$\mathcal{A}_6 = -\frac{i}{2^3} c_6 [S_{12} S_{34} S_{56} + \dots] + i \frac{c_4^2}{2^4 q^2} [S_{12} S_{3q} + \dots] [S_{q4} S_{56} + \dots] + \dots \quad (2.11)$$

where q is the internal momenta, the ellipses in the first term indicate all other momenta contractions, and the ellipses in the second term indicate both permutations of momenta contractions *and* the different ways of distributing the external momenta on the diagram. To clarify this somewhat, consider the term explicitly written in (2.11) which is $[S_{12} S_{3q} + \dots] [S_{q4} S_{56} + \dots]$. These ellipses denote all permutations of contracting the momenta, i.e. in the first bracket the ellipses will represent $S_{13} S_{2q} + S_{1q} S_{23}$ as these are all of the possible permutations. These terms all come from the scattering of three particles with momenta p_1, p_2 and p_3 , and they represent the kind of processes depicted in Figure 2.1. Having now included all processes for a $(p_1, p_2, p_3) \rightarrow (p_4, p_5, p_6)$ scattering event, we can now consider all of the other terms that will contribute to (2.11) which are represented in the el-

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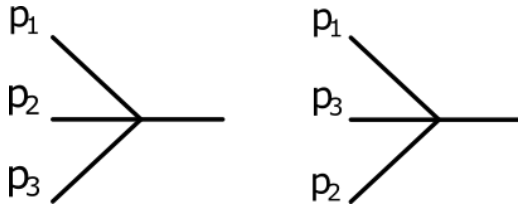


Figure 2.1: These are the left hand side of 3-3 scattering diagrams. They are examples of diagrams that have the momenta distributed in different ways, and thus have different algebraic forms in the amplitude.

lives outside of the brackets, at the end of the formula. These terms will correspond to scattering processes where the momenta are grouped differently, for example $(p_1, p_2, p_6) \rightarrow (p_3, p_4, p_5)$. Once all of the unique ways of splitting the momenta into two groups are accounted for we will then have the full scattering amplitude.

Using the result from (2.11), one finds that the soft amplitude behaves like $\mathcal{O}(z)$, indicating a soft limit of order $\sigma = 1$ which is what we would expect from the naive derivative counting. As mentioned previously, the 6 point amplitude is on the very edge of what is feasible to compute analytically, and in doing so one finds that the $\mathcal{O}(z)$ term has an overall factor of $(c_6 + 4c_4^2)$ multiplying it. This means that by choosing our coefficients such that $c_6 \rightarrow -4c_4^2$, we can enhance the soft amplitude to $\mathcal{O}(z^2)$. As we can calculate the 6 point amplitude exactly it offers us a unique opportunity to test our numerical procedure against a result that is easily checked, and indeed our numerical computation confirms these findings.

This behaviour is beyond that expected by the derivative counting, and looking at higher scattering amplitudes one sees that there is always a consistent set of tunings between coefficients such that the $\mathcal{O}(z)$ piece can be removed from the soft limit. At 8 point we see that by sending $c_8 \rightarrow 20c_4^3$ and $c_6 \rightarrow -4c_4^2$ we again recover a soft limit with $\sigma = 2$, which is consistent with the conditions required to enhance the lower scattering amplitudes. Checking higher amplitudes introduces even more operators into our enhanced theory, and it is seen that at every amplitude order the coefficients

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of these new operators are proportional to c_4 also. Indeed it is shown that the only $\rho = 1$ theory with $\sigma = 2$ soft limits is given by the DBI Lagrangian,

$$\mathcal{L}_{\text{DBI}} = -\frac{1}{8c_4} \sqrt{1 + 8c_4(\partial\phi)^2}. \quad (2.12)$$

By expanding the square root we can see how the structure sets all the coefficients in such a way that the soft limits of all tree level scattering amplitudes are enhanced, as the coefficients match the required tunings at all orders. This is why DBI is the unique enhancement of $\rho = 1$ theories, because it is the unique way to satisfy all of the constraints in a non-trivial way. The symmetry that causes this enhancement is then clearly the DBI symmetry, $\phi \rightarrow \phi + c + v_\mu x^\mu + \phi v^\mu \partial_\mu \phi$, which is unique to this particular Lagrangian amongst all other $\rho = 1$ theories, indicating its uniqueness in possessing enhanced soft limits. To discuss the origin of this symmetry let us consider a 5D Minkowski bulk with a brane localised to $y = \phi(x^\mu)/\Lambda^2$, where y is the coordinate of the extra dimension. To build an effective description of the brane we use the induced metric,

$$\hat{g}_{\mu\nu} = \eta_{\mu\nu} + \frac{\partial_\mu \phi \partial_\nu \phi}{\Lambda^4}. \quad (2.13)$$

The leading contribution to the brane effective action will be given by

$$\mathcal{S} = -\Lambda^4 \int d^4x \sqrt{\hat{g}} = -\Lambda^4 \int d^4x \sqrt{1 + \frac{(\partial\phi)^2}{\Lambda^4}} \quad (2.14)$$

where the brane tension is given by Λ^4 . This is precisely the DBI Lagrangian in (2.12) with $\Lambda = (8c_4)^{-1}$, and so we can see the origin of the symmetry is from higher dimensional Poincare invariance.

As we have identified the unique enhancement to $\sigma = 2$ for the class of $\rho = 1$ theories, one could ask if further enhancements are possible. The answer to that is no, and this is clear without needing to do any further calculation. Looking back to (2.9), the 4 point amplitude only receives con-

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tributions from one interaction term in the Lagrangian, and this contribution has leading $\mathcal{O}(z^2)$ behaviour. As there is only the one diagram contributing there's nothing to tune c_4 against to cancel this leading order piece, and so we can clearly see that the only way to remove the $\mathcal{O}(z^2)$ term is to set $c_4 = 0$. We also already know that to remove the order $\mathcal{O}(z)$ terms from the soft limits of the other scattering amplitudes that we must start with the DBI Lagrangian, where the coefficient of every interaction in the Lagrangian comes with overall factors of c_4 to some power. Combining these two pieces of information simply reduces our $\rho = 1$ theory to the trivial case of a free scalar field theory, which means the DBI Lagrangian is one of maximal enhancement and is hence an *exceptional* effective theory.

Another family of theories that are of interest to Cosmologists is that of $\rho = 2$, as this contains the well known Galileon theory. The $\rho = 2$ Lagrangian schematically looks like

$$\mathcal{L}_2 = -\frac{1}{2}(\partial\phi)^2 + (\partial\phi)^2 \sum_{n=1}^{\infty} F_n (\partial^2\phi). \quad (2.15)$$

In contrast to the $\rho = 1$ theory, there are multiple ways of contracting the derivatives in a $\rho = 2$ theory, and as such multiple terms for a given power of ϕ , each with their own arbitrary coupling coefficients. We demonstrate this with F_n which simply represents all contractions at a given order, for example $F_1(\partial^2\phi) = c_1 \square\phi$ and $F_2(\partial^2\phi) = c_2 (\square\phi)^2 + c_3 (\partial_\mu\partial_\nu\phi)^2$, etc. Again, a naive derivative counting tells us that the first enhancement occurs when we raise all soft limits to $\sigma = 2$. Following the same procedure as previously, it is found [100] that in order to enhance every soft amplitude to $\sigma = 2$ the constraints on the coefficients imply that one must have the Galileon theory. This is not surprising, as the Galileon has an extra symmetry over a general $\rho = 2$ theory. The Lagrangian is then given by

$$\mathcal{L}_{\text{gal}} = -\frac{1}{2}(\partial\phi)^2 + (\partial\phi)^2 \sum_{n=1}^3 c_n \det_n, \quad (2.16)$$

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where we have defined $det_n = n! \partial^{[\mu_1} \partial_{\mu_1} \phi \dots \partial^{\mu_n]} \partial_{\mu_n} \phi$. Writing this in a slightly different way,

$$\mathcal{L}_{\text{gal}} = -\frac{1}{2}(\partial\phi)^2 + (\partial\phi)^2 [\mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3], \quad (2.17)$$

where

$$\mathcal{L}_1 = c_1 \square\phi, \quad (2.18)$$

$$\mathcal{L}_2 = c_2 ((\square\phi)^2 - (\partial_\mu \partial_\nu \phi)^2), \quad (2.19)$$

$$\mathcal{L}_3 = c_3 ((\square\phi)^3 + 2(\partial_\mu \partial_\nu \phi \partial^\mu \partial^\sigma \phi \partial^\nu \partial_\sigma \phi) - 3\square\phi(\partial_\mu \partial_\nu \phi)^2), \quad (2.20)$$

where c_n are the remaining arbitrary coefficients, and we will refer to $\mathcal{L}_1$ as the ‘cubic’ Galileon, $\mathcal{L}_2$ as the ‘quartic’ Galileon and $\mathcal{L}_3$ as the ‘quintic’ Galileon. Even though all three of these interactions are invariant under the Galileon symmetry and also admit enhanced soft limits, thanks to the Galileon duality we can consistently remove the cubic Galileon by setting $c_1 \rightarrow 0$ without actually changing the information contained in the tree level scattering amplitude [111, 112]. Briefly put, the Galileon family of theories enjoy an interesting property that some transformations of the field will preserve the structure of the Lagrangian, and only change the coupling coefficients. It is noted in [111, 113] that every Galileon theory is dual to that where the cubic Galileon term is switched off, and it indeed makes the calculation of our scattering amplitudes easier to remove an unnecessary interaction vertex, and so we will set it to zero. To see a proof that the cubic Galileon can be switched off, see Appendix .1. We also note that there is one other interaction that is invariant under Galileon symmetry, which is simply $c_0 \phi$. We ignore this particular term because it has no impact on tree level scattering amplitudes, which are the main focus of this work.

Similarly to the DBI symmetry, the Galileon symmetry is inherited from a higher dimensional symmetry in a bulk-brane picture, and leaves the theory invariant under

$$\phi \rightarrow \phi + c + v_\mu x^\mu, \quad (2.21)$$

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where c is constant and v^μ some constant vector. Consider a 5 dimensional Minkowski bulk, with a probe brane located at $y = \phi(x^\mu)/\Lambda^2$, one can build the induced metric on the brane as $\hat{g}_{\mu\nu} = \eta_{\mu\nu} + (\partial_\mu \partial_\nu \phi)/\Lambda^4$. Constructing an effective description of the brane, we can write its action as [113–115]

$$\mathcal{S} = -\Lambda^4 \int d^4x \sqrt{-\hat{g}} \left[-\Lambda^4 - \alpha K + \beta R + \gamma L \right], \quad (2.22)$$

where we have defined $K = K_\mu^\mu$ as the trace of the extrinsic curvature tensor, $\hat{g}$ is the determinant of the induced metric, R as the Ricci scalar and L is given by the Lovelock tensor in 5D [114]. The leading order piece of this action, coming from $\sqrt{-g}\Lambda^4$ gives the DBI action as discussed previously. Taking the non-relativistic limit that $(\partial\phi)^2 \ll 1$, this action reduces to the 4D Galileon in (2.17). If one considers an even higher dimensional brane and bulk, one will obtain more terms in the action that respect the same Galileon symmetry. In D dimensions there are in fact $D+1$ terms invariant under Galileon symmetry, but as we are only interested in the 4D case we will refer the interested reader to [116] to see the higher dimensional Galileon terms.

In contrast to the $\rho = 1$ theory, we have two independent coefficients remaining in our enhanced $\rho = 2$ theory, which hints at the possibility that there is still the ability to enhance further. After satisfying the required conditions to ensure that all scattering amplitudes are $\mathcal{O}(z^2)$, one finds that there in fact is still an opportunity to further enhance the soft limit up to order $\sigma = 3$. All of the $\mathcal{O}(z^2)$ terms feature an overall factor of one of the remaining coupling coefficients, c_3 . This means that when we send $c_3 \rightarrow 0$ we remove all $\mathcal{O}(z^2)$ terms from all soft amplitudes. The theory that this corresponds to is the ‘quartic’ galileon, a canonical kinetic term and only one interaction given by $\mathcal{L}_2$ in (2.19). As there is only an even-legged interaction left, all odd numbered tree level scattering amplitudes will trivially be zero, as one cannot construct a diagram with an odd number of external legs from even vertices at tree level. As its soft limits are enhanced over a

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generic galileon theory, we know that it must possess an extra symmetry in comparison. In [106] the symmetry that causes this enhancement was identified as an extension of the galileon symmetry, specifically of the form

$$\phi \rightarrow \phi + s_{\mu\nu} x^\mu x^\nu + 12c_2 s^{\mu\nu} \partial_\mu \phi \partial_\nu \phi, \quad (2.23)$$

where $s_{\mu\nu}$ is some arbitrary symmetric and traceless tensor. This extra symmetry is why the quartic galileon is also called the ‘special’ galileon in the literature. The quartic galileon is an *exceptional* effective theory, as there is only one interaction remaining the only tuning that will remove the $\mathcal{O}(z^3)$ terms in the soft amplitudes is sending the remaining coefficient $c_2 \rightarrow 0$, which reduces us trivially to a free theory.

Although the special galileon was indeed identified as possessing an extended symmetry over a generic galileon theory before this soft amplitude analysis was performed, the power of the method is clear as looking at how to enhance soft amplitudes allows one to search an entire class of scalar effective theories for interesting sub-structure coming from extended global symmetries. As one must start with a Lagrangian to define the Feynman rules which then build the amplitude, one can only look for extensions to symmetries of theories that you know the action of, though. To that end we generalise our method further by writing down a *generic* scattering amplitude, which we can then search for enhanced soft limits, eliminating the need to start with a specific Lagrangian. For the case of a single- ρ theory this was performed in [100, 101], and a general ansatz for an $n+2$ scattering amplitude with $\rho = m/n$ is given by

$$\mathcal{A}_{n+2} = \sum_{\alpha} c_{\alpha}^{(0)} \left(S_{\alpha_1} \dots S_{\alpha_{m/2+1}} \right) + \sum_{\alpha, \beta} c_{\alpha}^{(1)} \frac{\left(S_{\alpha_1} \dots S_{\alpha_{m/2+2}} \right)}{S_{\beta}} + \dots \quad (2.24)$$

where α labels pairs of external legs that enter into numerator factors together, and β label factorisation channels which represent off-shell internal propagators, $S_{\beta} = \sum_{i,j} S_{ij}$. This includes all kinematic objects that will

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enter into the amplitude, with arbitrary coefficients $c_\alpha^{(d)}$, where d indicates the number of internal propagators present in a particular term. We can see that the first term in (2.24) represents a sum of all the channels of a single vertex contribution to the amplitude, as it features no inverse powers of momenta. Moving to the second term, we can see this is a sum of all channels of all diagrams with two vertices connected by a single internal propagator. The fact that this represents all diagrams at a given number of internal propagators is ensured by the sum over all different permutations of momenta contraction, that is the sum over all α and β . The only information we would now need is the value of σ we wish the soft limit to begin at, and the value for ρ . Given these, we can begin to constrain the coefficients c_α , as we will know how many times a given momentum vector can appear in a single term or if a term must vanish to allow σ to be the value required, for example. In this way one can begin to learn about the structure of a theory from this very general scattering amplitude ansatz, without the need to write a Lagrangian in advance that is in some way is more general than the theory you wish to probe, such that you can see how it emerges when one requires particular behaviour of the soft amplitudes.

2.4 Multiple ρ Theories

All of the analysis performed above focussed on theories in which all interactions in the Lagrangian shared a value of ρ , that is that they all had the same number of derivatives per field. This is a rather constrained space of theories, and so we detail a generalisation of this work into the space of theories which possess multiple ρ values. A generic multi- ρ theory will be described by the following schematic Lagrangian,

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 + (\partial\phi)^2 \sum_{n_1+\dots+n_Q \geq 1} c_{n_1,\dots,n_Q} (\partial^{\rho_1}\phi)^{n_1} \dots (\partial^{\rho_Q}\phi)^{n_Q}, \quad (2.25)$$

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where Q describes the number of different ρ values in the theory. By construction, there will be Q sets of interactions that have only a single ρ value, but to generalise beyond simply adding the interactions from Q different single- ρ theories together we also include all interactions that come from mixing different ρ values together in the same term. We will also assume that $\rho_1 < \rho_2 < \dots < \rho_Q$ with no loss of generality, and refer to a theory of this type as a $(\rho_1, \dots, \rho_Q)$ theory. Lorentz invariance demands that we have an even number of derivatives in each term, and so our theories must also ensure that $\sum_{k=1}^Q \rho_k n_k$ is even. We can again provide a naive estimate on the order of the soft limits, but it is more complicated by virtue that not all terms in the Lagrangian have the same ρ value. The naive estimate tells us that an enhanced soft limit is one for which

$$\sigma > \frac{2 + \sum_{k=1}^Q \rho_k n_k}{2 + \sum_{k=1}^Q n_k} \quad \forall n_k. \quad (2.26)$$

This equation adds up the contributions from each different ρ value in a particular term, and thus must be checked against every interaction in the Lagrangian, hence it must be valid for all sets of n_k values. We can maximise the right hand side of this inequality by looking at the interaction which is purely a ρ_Q term, as then every field will be carrying the most derivatives it can. For an N point amplitude, this means that $n_Q = N - 2$ and $n_i = 0$ for $i \neq Q$, and we can simplify (2.26) to

$$\sigma > \rho_Q + \frac{2}{N}(1 - \rho_Q). \quad (2.27)$$

Assuming that $\rho_Q > 1$ then it follows that this relation corresponds to

$$\sigma \geq \rho_Q > \rho_{Q-1} > \dots > \rho_1. \quad (2.28)$$

To reinforce this with an example, consider a toy $(1, 2)$ theory given by

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 + (\partial\phi)^2 [c_1(\partial\phi)^2 + c_2(\partial^2\phi) + c_3(\partial^2\phi)(\partial\phi)^2], \quad (2.29)$$

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where we have included a pure $\rho = 1$ term, a pure $\rho = 2$ term and also one mixed from the two. Looking at the 4-point amplitude, (2.27) tells us that enhanced soft limits will have σ such that $\sigma > 3/2$. As σ can only be an integer number this means that enhancement begins at $\sigma = 2$ and a generic soft amplitude will have $\sigma = 1$. This could be expected from our results in the single- ρ analysis, as we know that the interactions that have $\rho = 1$ or $\rho = 2$ do indeed yield enhanced soft limits for arbitrary coefficients as they are a part of DBI and Galileon respectively and do not mix at 4 point scattering, whereas the other interaction benefits from neither symmetry and so would only offer $\mathcal{O}(z)$ behaviour as expected from a shift symmetric term.

To enhance the order of the soft limit, we still need cancellations to occur between different Feynman diagrams in each scattering amplitude. As mentioned in Section 2.1, the conditions any two diagrams must satisfy to in principle cancel are that they have the same number of exterior legs and that they scale in the same way in momenta. Ensuring a theory shares one common value of ρ is the easiest way to satisfy these conditions, but it can also be achieved in a multiple- ρ theory. We write the conditions from (2.3) and (2.4) again, but here we write them in such a way as to apply to our construction of multiple- ρ theories. We let the first diagram have A vertices where the i^{th} vertex has $\sum_{k=1}^Q n_{i,k} + 2$ legs, where $n_{i,k}$ is the number of legs to the i^{th} vertex carrying $\rho_k n_k$ derivatives and thus $\sum_{k=1}^Q \rho_k n_{i,k} + 2$ powers of momenta. Similarly we let the second diagram have B vertices where the j^{th} vertex has $\sum_{k=1}^Q n_{j,k} + 2$ legs and $\sum_{k=1}^Q \rho_k n_{j,k} + 2$ powers of momenta. Requiring that the number of external legs is the same yields

$$\sum_{i=1}^A \sum_{k=1}^Q n_{i,k} = \sum_{j=1}^B \sum_{k=1}^Q n_{j,k}, \quad (2.30)$$

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and requiring they scale the same way with momenta is equivalent to

$$\sum_{k=1}^Q \left(\rho_k \sum_{i=1}^A n_{i,k} \right) = \sum_{k=1}^Q \left(\rho_k \sum_{j=1}^B n_{j,k} \right). \quad (2.31)$$

As a consistency check one can set $Q = 1$ such that we return to the space of single- ρ theories, in which case our new conditions reduce to those in (2.3) and (2.4). It is much harder to satisfy these when one doesn't assume $Q = 1$ as was done in [100, 101], but it is possible. We will focus here on the simplest extension to multiple ρ theories, by looking at double- ρ theories where $Q = 2$.

2.4.1 Double ρ Theories

In the case where $Q = 2$ the conditions for diagrams to cancel with each other from (2.30) and (2.31) can be simplified to

$$\sum_{i=1}^A (n_{i,1} + n_{i,2}) = \sum_{j=1}^B (n_{j,1} + n_{j,2}) \quad (2.32)$$

and

$$\rho_1 \sum_{i=1}^A n_{i,1} + \rho_2 \sum_{i=1}^A n_{i,2} = \rho_1 \sum_{j=1}^B n_{j,1} + \rho_2 \sum_{j=1}^B n_{j,2}, \quad (2.33)$$

where it is much easier to see these equations do in fact represent matching the number of external legs and derivatives in the diagrams than for the more general case. By definition our (ρ_1, ρ_2) theory satisfies $\rho_1 < \rho_2$ and so we can further simplify these relations by to

$$\sum_{i=1}^A n_{i,1} = \sum_{j=1}^B n_{j,1} \quad (2.34)$$

$$\sum_{i=1}^A n_{i,2} = \sum_{j=1}^B n_{j,2}. \quad (2.35)$$

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There is one notable theory that falls into the double- ρ category which is of interest, the DBI-Galileon. This theory comes from the same bulk-brane picture as both the DBI and Galileon theories, but does not assume the non-relativistic limit like the Galileon theory, nor truncate to first order in the brane action like DBI. In this sense it can be seen as a mix of the two theories as the $\sqrt{\hat{g}}$ that gives the DBI structure will multiply the tensors built from the induced metric $\hat{g}$ that give rise to the Galileon in low velocities. This theory is of interest because of its fundamental connection to the Galileon, it fully describing the theory of a fluctuating hypersurface in higher dimensions, and that despite its non-linear nature it still retains second order field equations meaning we could expect them to arise in the effective descriptions of many different physical phenomena [114].

We begin by generalising the scattering amplitude ansatz from (2.24) to accommodate theories with two ρ values. Considering only contact diagrams in an $n + 2$ point amplitude, we note there can be several different terms that contribute because n_1 and n_2 can vary between different terms. For example, $n_1 = n$ and $n_2 = 0$ is a contact term for an $n + 2$ scattering amplitude, but so is $n_1 = n - 1$ with $n_2 = 1$, even though it will have a different algebraic structure. Any individual one of these terms will carry $\rho_1 n_1 + \rho_2 n_2$ derivatives, and hence that many powers of momenta. Summing over all contributions we can write a generic scattering amplitude from only contact diagrams as

$$\mathcal{A}_{n+2,0} = \sum_{n_1=0}^n \sum_{\alpha} c_{\alpha}^{(0,n_1)} S_{\alpha_1} \dots S_{\alpha_{X(n_1)}}, \quad (2.36)$$

where

$$X(n_1) = \frac{\rho_1 n_1 + \rho_2 (n - n_1)}{2} + 1. \quad (2.37)$$

The sum over α again represents summing over the permutations of all pairs of external legs, and the sum over n_1 ensures that we include all the different contact terms that are possible in a (ρ_1, ρ_2) theory. Moving on to

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now consider diagrams with a single internal propagator, the total number of derivatives and hence powers of momenta is given by $\rho_1 n_1 + \rho_2 n_2 + 4$, where we consider n_1 running from 0 up to n and $n_2 = n - n_1$. Summing over all the different values that n_1 can take, and including the sums over different contractions of momenta we find that the contribution of diagrams built from 2 vertices linked by a single internal line takes the generic form

$$\mathcal{A}_{n+2,1} = \sum_{n_1=0}^n \sum_{\alpha,\beta} c_{\alpha}^{(1,n_1)} \frac{S_{\alpha_1} \dots S_{\alpha_{X(n_1)}}}{S_{\beta}} \quad (2.38)$$

where we have defined

$$X(n_1) = \frac{\rho_1 n_1 + \rho_2 (n - n_1)}{2} + 2. \quad (2.39)$$

The sum over α is the familiar sum over different ways of contracting the momenta, and β labels factorisation channels whose propagators are fixed by energy conservation at each vertex. Following the pattern that emerges as one considers higher numbers of internal lines, diagrams with y internal propagators will contribute

$$\mathcal{A}_{n+2,y} = \sum_{n_1=0}^n \sum_{\alpha,\beta} c_{\alpha}^{(y,n_1)} \frac{S_{\alpha_1} \dots S_{\alpha_{X(n_1)+y}}}{S_{\beta_1} \dots S_{\beta_y}}. \quad (2.40)$$

To then finally construct a generic $n + 2$ scattering amplitude we then only need to sum over all possible numbers of internal lines, leaving us with

$$\mathcal{A}_{n+2} = \sum_{y=0}^{n-1} \sum_{n_1=0}^n \sum_{\alpha,\beta} c_{\alpha}^{(y,n_1)} \frac{S_{\alpha_1} \dots S_{\alpha_{X(n_1)+y}}}{S_{\beta_1} \dots S_{\beta_y}} \quad (2.41)$$

We note here that the coefficients $c_{\alpha}^{(y,n_1)}$ for $y \geq 1$ are related to the coefficients appearing at lower point amplitudes due to on-shell recursion relations. These follow on account of the fact that near a pole in the propagator the amplitude factorises into lower point amplitudes, as detailed further in

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Section 2.5.

2.4.2 (1, 2) Theory

We will now focus on a specific theory, the (1, 2) class of double ρ theories. This particular family is of interest because it contains both DBI and Galileon interactions, as mentioned previously. To study the soft limits of this family we could use the generic amplitude ansatz from (2.41) and constrain the unknown coefficients, but as this theory has a small number of different ρ values and those values themselves are small, it is actually simpler to just write the most generic (1, 2) Lagrangian down and build the amplitude from there. Having larger ρ values means there are more derivatives in the interactions, and hence many more ways to contract them meaning many more interactions to consider.

To fully probe the soft limit behaviour of this family of theories, one should start with the most general (1, 2) theory and begin to constrain the coefficients by requiring enhancements of the soft limit. The most general theory of this kind would include operators built out of $L_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi$, $H_{\mu\nu} = \partial_\mu\partial_\nu\phi$ and traces of these objects, where every operator must contain at least one factor of $L_{\mu\nu}$ otherwise it cannot come from a (1, 2) theory by definition. Up to 5 point interactions, the theory is given by a canonical kinetic term and one pure $\rho = 1$ interaction given by

$$L^2 = (\partial\phi)^4, \quad (2.42)$$

where we have, and will from now on, denote traces of $L_{\mu\nu}$ and $H_{\mu\nu}$ as L and H without indices. Further, there will be a large number of pure $\rho = 2$ interactions given by

$$\begin{aligned} LH &= (\partial\phi)^2 \square\phi, \\ LH^2 &= (\partial\phi)^2 (\square\phi)^2, \\ L(H_{\mu\nu})^2 &= (\partial\phi)^2 (\partial_\mu\partial_\nu\phi)^2, \end{aligned} \quad (2.43)$$

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$$\begin{aligned}
L_{\mu\nu}H^{\mu\nu}H &= \partial_\mu\phi\partial_\nu\phi\partial^\mu\partial^\nu\phi\Box\phi, \\
L_\mu^\nu H_\nu^\alpha H_\alpha^\mu &= \partial_\mu\phi\partial^\mu\partial_\alpha\phi\partial^\alpha\partial_\nu\phi\partial^\nu\phi, \\
LH^3 &= (\partial\phi)^2(\Box\phi)^3, \\
LHH_{\mu\nu}H^{\mu\nu} &= (\partial\phi)^2\Box\phi(\partial_\mu\partial_\nu\phi)^2, \\
LH^3 &= (\partial\phi)^2(\partial_\mu\partial_\nu\phi)^3, \\
L_{\mu\nu}H^{\mu\nu}H^2 &= \partial_\mu\phi\partial_\nu\phi\partial^\mu\partial^\nu\phi(\Box\phi)^2, \\
L_{\mu\nu}H^{\mu\alpha}H_\alpha^\nu H &= \partial_\mu\phi\partial_\nu\phi\partial^\mu\partial^\alpha\phi\partial_\alpha\partial^\nu\phi\Box\phi, \\
L_{\mu\nu}H^{\mu\alpha}H_{\alpha\beta}H^{\beta\nu} &= \partial_\mu\phi\partial_\nu\phi\partial^\mu\partial^\alpha\phi\partial_\alpha\partial_\beta\phi\partial^\beta\partial^\nu\phi,
\end{aligned}$$

and finally a mixed interaction that is made from both $\rho = 1$ and $\rho = 2$ pieces

$$L^2H = (\partial\phi)^4\Box\phi.$$

There is actually one other term that comes from mixing the different ρ values, $(\partial\phi)^2\partial_\mu\phi\partial_\nu\phi\partial^\mu\partial^\nu\phi$, but it is equivalent to the one above under integration by parts and a rescaling of the arbitrary coefficient. Even though some these interactions are not explicitly in the form required by (2.25), that is with an explicit factor of $(\partial\phi)^2$, they can all be written that way using integration by parts. Each of these interactions will have its own arbitrary coefficient which will be constrained by requiring enhancement of the soft limits at each scattering amplitude order.

For this family of $(1, 2)$ theories, the relation in (2.28) tells us that an enhanced soft limit corresponds to $\sigma \geq 2$. An interesting grouping of Feynman diagrams occurs in this multi- ρ theory that was not present in our single- ρ theory, which we can see without need for any calculation. Consider the diagrams built only from $\rho = 2$ interactions, at a given N point amplitude they will scale with the largest power of momenta due to the fact that $\rho_2 > \rho_1$. Diagrams built from purely $\rho = 1$ terms will scale with their own power of momenta too, leaving these two different groups of interactions independent

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of each other in the sense of diagrammatic cancellation at all scales. There will also be several groups of mixed interactions, but working up to only 5 point amplitudes we can see from (2.43) that the only mixed interaction is zero on-shell at $\mathcal{A}_5$ and does not feature in tree level diagrams in amplitudes below that. As these two groups will have to cancel independently, we can use our results from the single- ρ analysis to come to some conclusions about the behaviour of this double- ρ theory. This implies that up to 5-point, for enhancement up to $\sigma = 2$ the interactions in (2.43) must have coefficients such that the amplitude is a combination of DBI and Galileon amplitudes, and that the unique way to enhance this further to $\sigma = 3$ is to have only special Galileon amplitudes contributing. Once the 4 and 5 point amplitudes are determined, we can uniquely derive all of the higher point amplitudes using recursion relations, which are detailed and derived in Section 2.5. This recursion property means that all the higher point amplitudes can be built from lower point amplitudes, and as such any enhancement of the higher order amplitudes will also only occur when DBI and Galileon tunings are present.

All that remains to do is to identify the symmetry that gives rise to this enhancement, and thus the actual form of the theory. Given our knowledge that DBI and Galileons cause enhancement of amplitudes considered above, we begin with the shift symmetric generalised galileons [117],

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 + \sum_{n \geq 1} a_n (\partial\phi)^{2n+2} + \sum_{n \geq 1} (b_n \det_1 + c_n \det_2 + d_n \det_3) (\partial\phi)^{2n}, \quad (2.44)$$

where with $H_{\mu\nu} = \partial_\mu \partial_\nu \phi$ we define

$$\det_1 = H_\mu^\mu \quad (2.45)$$

$$\det_2 = (H_\mu^\mu)^2 - H_{\mu\nu} H^{\mu\nu} \quad (2.46)$$

$$\det_3 = (H_\mu^\mu)^3 + 2H_{\mu\nu} H^{\mu\alpha} H_\alpha^\nu - 3H_\mu^\mu H_{\alpha\beta} H^{\alpha\beta}. \quad (2.47)$$

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We begin with this particular theory as it has been shown that it is the most general scalar theory which has no derivatives higher than second order and has only up to second order field equations [117]. This means it will contain theories that cause enhancements to our general $(1, 2)$ theory, while also retaining the useful physical properties of theories with second order equations of motion. The fact that the generalised shift symmetric galileon does not contain all the interactions of a general $(1, 2)$ theory given in (2.43) does also imply that the interactions that are not Galileon or DBI invariant cause the equations of motion to be higher than second order. We can in fact see many interesting known theories within this generalised galileon; K-essence and $P(X)$ theories all have $b_n = c_n = d_n = 0$, galileons have $a_n = 0 \forall n$ and $b_n = c_n = d_n = 0$ for $n > 1$, and finally the DBI-Galileons can be seen by [114]

$$a_n = -\frac{1}{2} \frac{(2n)!}{n!(n+1)!} \left(-\frac{1}{4\lambda_0} \right)^n, \quad (2.48)$$

$$b_n = \frac{\lambda_1}{2n} \left(-\frac{1}{\lambda_0} \right)^n, \quad (2.49)$$

$$c_n = \lambda_2 \frac{(2n)!}{(n!)^2} \left(-\frac{1}{4\lambda_0} \right)^n, \quad (2.50)$$

$$d_n = \lambda_3 \left(-\frac{1}{\lambda_0} \right)^n, \quad (2.51)$$

where λ_i are arbitrary coupling coefficients. The DBI-Galileon theory can be seen as a relativistic extension of the Galileon theory, where one considers the action on a 4D brane in a Poincare invariant 5D bulk in a non-relativistic limit. Its action was written in (2.22).

Calculating the scattering amplitudes of the shift symmetric generalised galileon theory explicitly is rather involved, so instead we utilise the numerical analysis method that was detailed in Section 2.2. This method reveals that the amplitudes up to 7 point go as follows:

$$\mathcal{A}_4 = a_1 \#_{4,4} z^2 + \mathcal{O}(z^3), \quad (2.52)$$

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$$\mathcal{A}_5 = (4b_1^3 + 6b_1c_1 + 3d_1) \#_{5,8} z^2 + \mathcal{O}(z^3), \quad (2.53)$$

$$\begin{aligned} \mathcal{A}_6 = & [(a_2 + 4a_1^2) \#_{6,6} + (8a_1b_1^2 + 2b_1b_2 + 6a_1c_1 + c_2) \#_{6,8}] z \\ & + [a_1^2 \#'_{6,6} + (2b_1b_2 + c_2) \#'_{6,8} + a_1b_1^2 \#''_{6,8} + a_1c_1 \#'''_{6,8}] z^2 + \mathcal{O}(z^3), \end{aligned} \quad (2.54)$$

$$\begin{aligned} \mathcal{A}_7 = & (2b_2c_1 + 2b_1c_2 + d_2 + 8a_1d_1 + 20a_1b_1c_1 + 16a_1b_1^3 + 4b_1^2b_2) \#_{7,10} z \\ & + [a_1^2b_1 \#_{7,8} + (2c_1b_2 + 2b_1c_2 + 4b_1^2b_2 + d_2) \#'_{7,10} + a_1b_1^3 \#''_{7,10} + a_1b_1c_1 \#'''_{7,10} \\ & + a_1d_1 \#^{(4)}_{7,10} + (6b_1c_1^2 + 3b_1^2d_1 + 3c_1d_1 + 10b_1^3c_1 + 4b_1^5) \#_{7,12}] z^2 + \mathcal{O}(z^3), \end{aligned} \quad (2.55)$$

where z is again the soft parameter, and $\#_{n,m}$, $\#'_{n,m}$ etc denote different distinct kinematic combinations of momentum vectors entering the n point amplitude and scaling overall with m powers of momentum. It is these $\#_{n,m}$ which are dependant upon our choice of numerical values for our momentum, as discussed previously in Section 2.2, but we can extract the interesting information about the coefficients and how they group together by obtaining and comparing the same amplitude across multiple different numerical values of these momenta. We can clearly see here that the first two amplitudes, $\mathcal{A}_4$ and $\mathcal{A}_5$ both have enhancement over the expected $\sigma = 1$ result for a general $(1,2)$ theory, which reinforces our earlier conclusion that DBI and Galileon interactions, both of which are act independently of each other for these 2 amplitudes, cause enhancement to $\sigma = 2$. To enhance the other two amplitudes calculated to $\sigma = 2$ we must enforce the following relations between the coefficients;

$$a_2 = -4a_1^2, \quad (2.56)$$

$$c_2 = -(8a_1b_1^2 + 2b_1b_2 + 6a_1c_1), \quad (2.57)$$

$$d_2 = -(2b_2c_1 + 8a_1d_1 + 8a_1b_1c_1), \quad (2.58)$$

to ensure that the $\mathcal{O}(z)$ pieces vanish for all scattering momenta. By carrying

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out the following field transformation,

$$\phi \rightarrow \phi - (b_2 + 4a_1b_1)(\partial\phi)^4 - \left(b_3 + \frac{8}{3}a_1(b_2 + 4a_1b_1)\right)(\partial\phi)^6 \quad (2.59)$$

on our newly constrained shift symmetric galileon theory, we find the Lagrangian (2.44) keeps the same structure up to 7 point terms, but with $b_2 \rightarrow -4a_1b_1$ and $b_3 \rightarrow \frac{64}{3}a_1^2b_1$, which brings the Lagrangian into the standard form of writing the DBI Galileon theory,

$$\mathcal{L} = -\lambda_0\gamma^{-1} + \lambda_1\ln(\gamma)\det_1 + \lambda_2\gamma\det_2 + \lambda_3\gamma^2\det_3, \quad (2.60)$$

where $\gamma = 1/\sqrt{1 + (\partial\phi)^2/\lambda_0}$ and $\det_n$ are as defined in (2.45)-(2.47). The coefficients λ_i are related to our coefficients a_i , b_i etc by the relations given in (2.48)-(2.51),

$$\lambda_0 = \frac{1}{8a_1}, \quad (2.61)$$

$$\lambda_1 = -\frac{b_1}{4a_1}, \quad (2.62)$$

$$\lambda_2 = -\frac{c_1}{4a_1}, \quad (2.63)$$

$$\lambda_3 = -\frac{d_1}{8a_1}. \quad (2.64)$$

This proves that the unique enhancement of soft limits to a $(1, 2)$ theory, to $\sigma = 2$, is present in the DBI-Galileon theory, showing that the symmetry responsible for the enhancement is the DBI symmetry

$$\phi \rightarrow \phi + c + v_\mu x^\mu + \phi v^\mu \partial_\mu \phi. \quad (2.65)$$

To further enhance our $(1, 2)$ theory to $\sigma = 3$ the conditions we must impose on the coefficients are;

$$a_1 = a_2 = 0, \quad (2.66)$$

$$c_2 = -2b_1b_2, \quad (2.67)$$

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$$d_2 = -2b_2c_1, \quad (2.68)$$

$$4b_1^3 + 6b_1c_1 + 3d_1 = 0, \quad (2.69)$$

where we have also imposed that the conditions to ensure no $\mathcal{O}(z)$ pieces contribute are also satisfied. By carrying out the same field redefinition as before (2.59),

$$\phi \rightarrow \phi - b_2(\partial\phi)^4 - b_3(\partial\phi)^6, \quad (2.70)$$

we can eliminate b_2 and b_3 from our Lagrangian, which up to 7 point interactions leaves us with a purely galileon Lagrangian characterised by 3 coefficients: b_1 , c_1 and

$$d_1 = -\frac{2}{3}b_1(2b_1^2 + 3c_1). \quad (2.71)$$

Using the Galileon duality [111,112], we can map apparently distinct galileon theories to each other. Imposing that the kinetic term takes the canonical form, we can map our theory with coefficients (b_1, c_1, d_1) to a set of equivalent galileon theories with coefficients $(b_1(q), c_1(q), d_1(q))$ where

$$b_1(q) = b_1 - \frac{q}{2}, \quad (2.72)$$

$$c_1(q) = c_1 - b_1q - \frac{q^2}{4}, \quad (2.73)$$

$$d_1(q) = d_1 + c_1q + \frac{1}{2}b_1q^2 - \frac{q^3}{12}. \quad (2.74)$$

The freedom to make these galileon duality transformations without affecting the amplitudes is a good way to check that our numeric method works correctly, and indeed by putting galileon theories in that have different structure but can be shown to be equivalent under the duality relations does indeed yield the same scattering amplitudes as expected. If we identify $b_1 = q/2$ and use (2.71) then the above relations tell us that the (1,2) theory that has $\sigma = 3$ soft limits up to 7 point can be identified as a galileon theory with coefficients $b_1(q) = 0$, $d_1(q) = 0$ and $c_1(q) = c_1 - 3b_1^2$, which has

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a Lagrangian of

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 + c_1(q)(\partial\phi)^2 \det_2. \quad (2.75)$$

In a mirror result to the $\rho = 2$ case in Section 2.3, this is the special galileon theory, meaning that the special galileon symmetry in (2.23) is the unique way to give rise to $\sigma = 3$ enhanced soft limits within in $(1, 2)$ class of theories as well as the $\rho = 2$ class, and is an exceptional theory in both cases as no further enhancement is possible.

2.5 On-Shell Recursion Relations

A tool that proves incredibly useful when studying scattering amplitudes are the on-shell recursion relations, described in [101–105]. Scalar field theories with enhanced soft limits admit these recursion relations, which allow one to construct all scattering amplitudes of a theory using only a small number of scattering amplitudes with few external momenta. They are reminiscent of the BCFW recursion relations originally developed for Yang-Mills theory [118]. In this section I will demonstrate these relations in a language that allows us to adapt them to our multi- ρ theories with minimal hassle.

To begin, consider the N point amplitude of a given theory, $\mathcal{A}_N(p_1, \dots, p_N)$, which has Q different ρ parameters, and perform what is known as an *all but r* rescaling of the external momenta in the literature, such that $p_i \rightarrow p_i(h)$ where h is some complex deformation parameter,

$$p_i(h) = \begin{cases} p_i(1 - h\alpha_i) & \text{for } i = 1, \dots, N_s \\ p_i + hq_i & \text{for } i = N_s + 1, \dots, N \end{cases} \quad (2.76)$$

and $N_s = N - r$. This rescales r of the momenta vectors in the scattering amplitude, and the remaining N_s momenta vectors are then shifted by additional pieces such that the new set of momentum vectors satisfies momentum

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conservation when

$$\sum_{i=1}^{N_s} \alpha_i p_i = \sum_{i=N_s+1}^N q_i. \quad (2.77)$$

We also require that the new momenta are on-shell, and as such

$$q_i^2 = 0 \quad (2.78)$$

and

$$p_i \cdot q_i = 0 \quad (2.79)$$

for $i = N_s + 1, \dots, N$. In d space-time dimensions the set of (α, q) parameters are not entirely free due to these constraints, in fact there will be $N_s + rd$ parameters with a total of $d + 2r$ constraints. There is also some redundancy in our parametrisation, as can be seen by looking at the following,

$$\alpha_i \rightarrow \alpha_i + \lambda \alpha_i, \quad q_i \rightarrow q_i + \lambda q_i \quad (2.80)$$

and

$$\alpha_i \rightarrow \alpha_i + \mu, \quad q_i \rightarrow q_i - \mu p_i, \quad (2.81)$$

which result in either $h \rightarrow (1 + \lambda)h$ or $p_i \rightarrow (1 - \mu h)p_i$ respectively, where α and μ are arbitrary constants. These particular transformations can be viewed as ‘pure gauge’ [101]. This reduces the true number of degrees of freedom by two and so in order to avoid over-constraining the system of all but r rescaling, we require that there are not more constraints than degrees of freedom, implying that

$$N_s + rd - 2 \geq d + 2r. \quad (2.82)$$

It follows from this result that the rescaled momenta can only be used to probe scattering amplitudes where

$$N \geq d + 2 - (d - 3)r \quad (2.83)$$

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where $N \geq r$ by assumption.

Rescaling the momenta will inevitably lead to the scattering amplitude picking up a dependence on the complex deformation parameter h , such that $\mathcal{A}_N \rightarrow \mathcal{A}_N(h)$. The exact form of the scaling in (2.76) is chosen such that $h \rightarrow 1/\alpha_i$ corresponds to taking p_i soft, and such the amplitude will go like $\mathcal{A}_N(z) \sim (1 - h\alpha_1)^\sigma$ in this regime. To derive the recursion relation we now consider the contour integral that encircles all poles at finite h ,

$$\oint_C \frac{A_N(h)}{F_N(h)} \frac{dh}{h} = 0 \quad (2.84)$$

where C is a large contour at infinity and

$$F_N(h) = \prod_{i=1}^{N_s} (1 - h\alpha_i)^\sigma \quad (2.85)$$

with σ being the soft degree of the field under study. As the original amplitude $\mathcal{A}_N$ is given by choosing the contour in integral (2.84) to be an infinitesimal circle centred around $h = 0$, we can use Cauchy's theorem to relate the original amplitude to the sum of the residues. (2.84) is designed such that it is non-singular when $h \rightarrow 1/\alpha_i$, as zeros in the amplitude will cancel any divergence coming from the denominator, and so the soft behaviour does not cause poles. To ensure that there is no divergence as h tends to infinity, we note that the large h behaviour of $F_N(h) \sim h^{\sigma N_s}$ and $A_N(h) \sim h^{m+2}$, where simply counting derivatives gives $m = \sum_{k=1}^Q \rho_k n_k$. To avoid the integral diverging at large h we require that $m + 2 < N_s \sigma$ for all possible n_k combinations, which represents all possible diagrams contributing to the amplitude. The left hand side of this inequality is maximised by a diagram that is built entirely out interactions coming from the largest ρ_k , as this will maximise the number of derivatives it carries, m . As the inequality must hold for all diagrams in the amplitude, it then follows as a sufficient

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and necessary condition that

$$2 + \rho_Q(N - 2) < (N - r)\sigma. \quad (2.86)$$

Note that when $r = 0$ this inequality is identical to (2.27), which is satisfied by a theory with an enhanced soft limit. This means that for all the theories of interest in this work, those with enhanced soft limits, there will always exist a rescaling of the momenta such that the integral in (2.84) vanishes. This is why theories with enhanced soft limits always permit recursion relations. Increasing r will make the condition (2.86) even stronger, and so the rescaling will only cause the integral (2.84) to vanish for theories with an even higher level of soft limit enhancement. All that now remains are poles coming from $\mathcal{A}_N(z)$ which correspond to different factorisation channels contributing to that amplitude. For a Feynman diagram with an internal line with momenta p_I , there are two poles which correspond to the roots of $p_I^2(h) = p_I^2 + 2p_I \cdot q_I h + q_I^2 h^2 = 0$, where $p_I = \sum_{i \in I} p_i$ and $q_I = -\sum_{i \in I} \alpha_i p_i$. Applying the Cauchy theorem then finally yields the recursion relation,

$$\mathcal{A}_N(0) = \sum_I \text{Res}_{h_I^\pm} \left(\frac{\mathcal{A}_N(h)}{h F_N(h)} \right) \quad (2.87)$$

where $h_I^\pm$ are the two roots of $p_I^2(h) = 0$. As these residues correspond to factorisation channels they can be written as the product of two lower point amplitudes, and so we can simplify this relation to

$$\mathcal{A}_N(0) = \sum_I \frac{\mathcal{A}_L(h_I^+) \mathcal{A}_R(h_I^+)}{p_I^2(1 - h_I^+/h_I^-) F_N(h_I^+)} + (h_I^+ \longleftrightarrow h_I^-), \quad (2.88)$$

where $\mathcal{A}_L$ and $\mathcal{A}_R$ are lower point amplitudes than $\mathcal{A}_N$ and the sum over I represents a sum over all factorisation channels.

To quickly summarise the results of this section, as long as a theory has an enhanced soft limits, one can always perform a complex transformation, the all but r transformation, on the momentum vectors in a scattering process.

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By use of Cauchy's theorem one can then show that the scattering amplitude you are interested in can be calculated by looking at the transformed lower order scattering amplitudes, and all associated factorisation channels.

2.6 Recursion and DBI-Galileon Duality

In [113] it was found that the DBI Galileon has a duality relation that is similar to the way that Galileon theories enjoy a duality. Here we will indeed show that this is true from a scattering amplitude perspective, by utilising the recursion relations derived in Section 2.5. This will also serve as an example calculation for how we can use the recursion relation to derive higher order scattering amplitudes from a small number of initial ones. For convenience sake we re-present the DBI-Galileon action here as

$$\mathcal{L} = -\lambda_0 \gamma^{-1} + \lambda_1 \ln(\gamma) \det_1 + \lambda_2 \gamma \det_2 + \lambda_3 \gamma^2 \det_3, \quad (2.89)$$

where the 'det' structures are given by (2.45)-(2.47) and $\gamma = \sqrt{1 + \frac{(\partial\phi)^2}{\lambda_0}}^{-1}$. We can explicitly calculate the first few on-shell scattering amplitudes without too much difficulty, to find that

$$\mathcal{A}_3 = 0, \quad (2.90)$$

$$\mathcal{A}_4 = \frac{1}{4\lambda_0} (S_{12}^2 + S_{23}^2 + S_{13}^2) + \frac{1}{2} \left(\frac{\lambda_1^2}{2\lambda_0^2} - \frac{\lambda_2}{\lambda_0} \right) (S_{12}^3 + S_{23}^3 + S_{13}^3), \quad (2.91)$$

$$\mathcal{A}_5 = 24 \left(\frac{2\lambda_3}{\lambda_0} - \frac{\lambda_1\lambda_2}{\lambda_0^2} + \frac{\lambda_1^3}{3\lambda_0^3} \right) \det(1, 2, 3, 4) \quad (2.92)$$

where we define $\det(1, 2, 3, 4) = n! p_1^{[\mu_1]} \dots p_n^{[\mu_n]} p_{1\mu_1} \dots p_{n\mu_n}$. We can use these as the seed amplitudes required for the on-shell recursion procedure derived in Section 2.5. Our DBI-Galileon theory has a soft limit of order $\sigma = 2$, and if we carry out the momenta shift such that $r = 0$, $p_i \rightarrow p_i(1 - \alpha_i h)$ for all momenta, then we can build the 6-point amplitude from the 10 factorisation channels that contribute to it. There are 10 factorisation

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channels due to the fact that the 3 point amplitude is zero by virtue of energy conservation and on-shell-ness of the momenta, and the transformations retain those properties. This means that all contributions must come from having two 4 point vertices joined by one internal line, as there is no other way of creating diagrams of the right topology, and so the 6 point amplitude can be written as the sum of terms where the internal momenta is given by $p_I = p_1 + p_2 + p_3 = p_{123}$ etc. Using (2.88) we can then write this particular factorisation channel's contribution to the 6 point amplitude as

$$\mathcal{A}_6^{(123)} = \frac{\mathcal{A}_4^{(123)}(h_I^+) \mathcal{A}_4^{(456)}(h_I^+)}{(1 - h_I^-/h_I^+) F(h_I^+)} \frac{1}{p_{123}^2} + (h_I^+ \longleftrightarrow h_I^-) \quad (2.93)$$

where $F(h) = \prod_{i=1}^6 (1 - \alpha_i h)^2$, $h_I^\pm$ are the two roots of $p_{123}^2 = S_{12} + S_{13} + S_{23} = 0$, and $\mathcal{A}_4(h)$ is given by simply transforming the momenta in (2.91), which gives us

$$\mathcal{A}_4^{(123)} = \frac{1}{4\lambda_0} (S_{12}^2(h) + S_{23}^2(h) + S_{13}^2(h)) + \frac{1}{2} \left(\frac{\lambda_1^2}{2\lambda_0^2} - \frac{\lambda_2}{\lambda_0} \right) (S_{12}^3(h) + S_{23}^3(h) + S_{13}^3(h)), \quad (2.94)$$

where $S_{ij}(h) = (1 - \alpha_i h)(1 - \alpha_j h) S_{ij}$. The final 6 point amplitude is then given by the sum over all such factorisation channels,

$$\mathcal{A}_6 = \mathcal{A}_6^{(123)} + \mathcal{A}_6^{(124)} + \dots + \mathcal{A}_6^{(156)} \quad (2.95)$$

which are defined similarly to (2.93) but with the appropriate momenta and $h_I^\pm$ values. It is not obvious at first sight, but one can check that all unphysical parameters related to the shift of the momenta vectors drop out of this expression for the scattering amplitude, leaving us with an expression for the amplitude built out of only the momenta vectors and coupling constants present in the lower point amplitudes. One can continue to use the recursion relations to calculate higher point amplitudes in the same way, by considering how the $\mathcal{A}_6$ and $\mathcal{A}_4$ vertices can combine to make the correct kind of diagrams, but there is no need to do so to show DBI Galileon duality

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invariance, so we will stop here. We could explicitly check that the form presented above agrees with the amplitude one obtains from more traditional Feynman diagram methods, but instead we utilise the numeric method we detailed in Section 2.2 as it is based upon building all possible Feynman diagrams. Indeed, the results of that procedure agree with the result of evaluating the expression for the amplitude coming from the recursion relations for arbitrary values of the non-physical shift parameters h and α_i , showing that the dependence drops out as claimed and that the recursion relations do indeed give the correct amplitudes.

Having shown that the 6 point amplitude is built from the 4 point amplitude explicitly, we understand that all even numbered higher amplitudes will also be built from only the 4 point by the same logic, whereas odd numbered higher amplitudes will also contain some $\mathcal{A}_5$ dependence. To then show that *all* scattering amplitudes of the DBI Galileon theory, and hence the theory itself, are invariant under the duality transformation we need only prove that the 4-point and 5-point amplitude are invariant, and the rest will follow by recursion. To describe the duality transformation we write our DBI Galileon Lagrangian in a different form, as presented in [113], such that

$$\mathcal{L} = \sum_{i=1}^3 \frac{N_i}{\gamma} \det \left[\delta_{\nu}^{\mu} + C_i \gamma \left(\partial^{\mu} \partial_{\nu} \phi - \gamma^2 \partial^{\mu} \phi \left(\partial^{\alpha} \partial_{\nu} \phi \partial_{\alpha} \phi \right) \right) \right], \quad (2.96)$$

where we can identify our original coefficients from (2.89) with

$$\lambda_0 = - \sum_{i=1}^3 N_i, \quad (2.97)$$

$$\lambda_1 = - \frac{\sum_{i=1}^3 N_i C_i}{\sqrt{\lambda_0}}, \quad (2.98)$$

$$\lambda_2 = - \frac{\sum_{i=1}^3 N_i C_i^2}{2\lambda_0}, \quad (2.99)$$

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$$\lambda_3 = -\frac{\sum_{i=1}^3 N_i C_i^3}{12\lambda_0^{3/2}}, \quad (2.100)$$

after some integration by parts. Written in this form, the duality transformation is easily written as a constant shift $C_i \rightarrow C_i + \beta$, which when using (2.97)-(2.100) translates through to

$$\lambda_0 \rightarrow \lambda_0, \quad (2.101)$$

$$\lambda_1 \rightarrow \lambda_1 + \beta\sqrt{\lambda_0}, \quad (2.102)$$

$$\lambda_2 \rightarrow \lambda_2 + \beta\frac{\lambda_1}{\sqrt{\lambda_0}} + \frac{\beta^2}{2}, \quad (2.103)$$

$$\lambda_3 \rightarrow \lambda_3 + \frac{\beta}{2}\frac{\lambda_2}{\sqrt{\lambda_0}} + \frac{\beta^2}{4}\frac{\lambda_1}{\lambda_0} + \frac{\beta^3}{12}\frac{1}{\sqrt{\lambda_0}} \quad (2.104)$$

in our original coefficients. Performing these transformations on the 4 and 5 point amplitudes in (2.91) and (2.92) we verify that indeed they are invariant, and thus show that indeed the entire theory enjoys this duality relation between seemingly different DBI-Galileon theories.

2.7 Summary

In this chapter we have reviewed the work of [100, 101], in which a study of the soft behaviour of scattering amplitudes of scalar effective field theories with a single derivative power counting parameter, ρ , was carried out. Theories of this type have the schematic Lagrangian

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 + (\partial\phi)^2 \sum_n \partial^{\rho n} \phi^n. \quad (2.105)$$

This work shows that the soft behaviour of the amplitude $\mathcal{A} \sim \mathcal{O}(p^\sigma)$ can be enhanced by the presence of symmetries in the Lagrangian, increasing σ . It is then shown that of the family of $\rho = 1$ theories that the DBI scalar effective theory is the unique way to enhance the soft behaviour, thanks to the

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presence of the DBI symmetry, and that no further enhancement is possible. Of the $\rho = 2$ family of theories, the Galileon family of theories is the unique way to enhance the soft behaviour to $\sigma = 2$, but a further enhancement is also possible and identified as the so called ‘special galileon’ which causes $\sigma = 3$ enhancement thanks to the presence of an extra symmetry over the generic Galileon. These results highlight this approach as a novel way to identify highly symmetric theories within a larger class.

We then present a generalisation of this technique to theories with more than a single derivative power counting parameter, allowing us to study the soft behaviour of a much wider class of theories. These theories we call ‘multi- ρ theories’, and they are described schematically by

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 + (\partial\phi)^2 \sum_{n_1+\dots+n_Q \geq 1} c_{n_1,\dots,n_Q} (\partial^{\rho_1}\phi)^{n_1} \dots (\partial^{\rho_Q}\phi)^{n_Q} \quad (2.106)$$

where Q is the number of unique power counting parameters $\rho_1 < \rho_2 < \dots < \rho_Q$. Our results show that generically for a theory with multiple ρ values one recovers the results of a single ρ theory where we identify the highest power counting parameter, ρ_Q with that in the single power counting parameter case. This is illustrated by the fact that in a single ρ theory an enhanced soft limit is given by $\sigma \geq \rho$, and in the multi- ρ case by $\sigma \geq \rho_Q$. In the single ρ theory space all the diagrams contributing to a scattering amplitude scale with momentum in the same way, meaning that all diagrams can be cancelled against each other. This is not the case in a multi- ρ theory, where different interactions have different ρ values and so different diagrams scale differently, leading to many distinct groups of diagrams which must cancel amongst themselves. Using the enhanced behaviour of these multi- ρ theories we show that the recursion formula generalise to multi- ρ theories easily, allowing us to construct all tree level scattering amplitudes from a small number of initial seed amplitudes.

We pay particular attention to the $(1, 2)$ class of multi- ρ theories, those with $\rho_1 = 1$ and $\rho_2 = 2$, because it contains the generalised galileons, which

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are also known as Horndeski theory. By studying the soft behaviour of this class of theories we identify the non-trivial effective theories with enhanced soft behaviour. For minimal enhancement to $\sigma = 2$, we identify the DBI symmetry as the unique cause for enhancement which singles out the DBI-Galileon theory from the Horndeski class. Further enhancement to $\sigma = 3$ is possible, and is found to correspond to the special galileon, showing it is indeed special in a wider context than just galileon theories. We also calculated the 4 and 5 point amplitudes of the DBI Galileon theory explicitly to use as seed amplitudes in the recursion relations, which we used to calculate the 6 point amplitude and verify that it matches with the usual Feynman diagram method of calculating the amplitude. We use this result to show that as our seed amplitudes are invariant under the DBI-Galileon duality transformations from [113], recursion implies all tree level scattering amplitudes are also invariant, and hence the entire theory.

At the time of this work a study of the special Galileon was conducted in [119], which found that the special Galileon is not entirely unique but a part of a larger family of ‘special Galileon’ theories which all poses the same soft behaviour and respect the same symmetry. This family is equivalent under a new ‘hidden Galileon duality’, which maps from one such theory to another. One might worry that this result contradicts our own which declares the uniqueness of the special Galileon. We can see that the two results are consistent by noting that in a similar way to the cubic Galileon operator being removed under the Galileon duality transformation, one can remove the first Galileon operator, ϕ , via the new ‘hidden duality’. Doing so singles out the form of the special Galileon found in this work and the literature. It is then clear why this family of special Galileons was not apparent in our work, as we did not allow the ϕ operator by construction. This indicates that the special Galileon of the form we found is indeed unique in the class of $(1, 2)$ theories.

CHAPTER 3

3 Kicking Chameleon

3.1 The Surfer

As mentioned briefly in Section 1.4.1, the Chameleon suffers from a breakdown around Big Bang Nucleosynthesis, as discussed in [81, 82]. In this section we will present a more detailed review of this phenomena, and also a new way in which to see how the breakdown occurs. This motivates a high energy modification to the Chameleon mechanism, as first proposed here [120], and how this helps avoid the problem present in the original Chameleon mechanism will be discussed. We also show that it retains all the screening properties of the Chameleon at low energies and so can still be used effectively as a screening mechanism at late times.

The Chameleon action is given by

$$\mathcal{S} = \int d^4x \sqrt{-g} \left[\frac{M_{pl}^2}{2} R - \frac{1}{2} (\partial\phi)^2 - V(\phi) \right] + \mathcal{S}_m[\tilde{g}_{\mu\nu}, \psi] \quad (3.1)$$

with $\tilde{g}_{\mu\nu} = e^{2\beta\phi/M_{pl}} g_{\mu\nu}$. On an FLRW background, the equation of motion is

$$\ddot{\phi} + 3H\dot{\phi} = -V'(\phi) - \frac{\beta}{M_{pl}} \rho \Sigma, \quad (3.2)$$

where we have defined ρ and p as the Einstein frame density and pressure, and $\Sigma = \frac{\rho-3p}{\rho}$. The Einstein density and pressure can be related to their Jordan frame counterparts by using the fact that

$$T_{\mu\nu} = e^{2\beta\phi/M_{pl}} \tilde{T}_{\mu\nu} \implies \rho - 3p = e^{\frac{4\beta\phi}{M_{pl}}} (\tilde{\rho} - 3\tilde{p}). \quad (3.3)$$

In a universe consisting solely of relativistic material, the quantity $\Sigma = \frac{\rho-3p}{\rho}$ is exactly zero and the scalar will evolve in a way dictated purely by its

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bare potential $V(\phi)$. In the early universe we expect all massive particle species to be relativistic, as they will be in thermal equilibrium with the photon bath, and as such do not contribute to Σ . As the universe expands and the radiation cools, eventually the massive particles will become non-relativistic and begin contributing to Σ . The radiation will continue to cool and eventually these newly non-relativistic particles will become Boltzmann suppressed, which will cause their contribution to Σ to drop off again. All of this means that Σ is zero for most of the time, apart from when the Jordan frame temperature of the photon bath becomes comparable with the mass of the particles present in the universe. This was first discussed in [78], and later [81,82] where this effect was analysed using Standard Model particles. It was found that the contribution to Σ can be $\mathcal{O}(0.1)$, and so can significantly affect the dynamics of the Chameleon field. Because of the highly localised nature of these contributions in time, we will refer to Σ as the *kick* function, as it allows the particles in the matter sector to *kick* the Chameleon field in the early universe.

It was found that these kicks drive the scalar field towards the minimum of the effective potential with a large velocity, causing it to climb up the steep part until it reaches the reflection point at which it has zero kinetic energy, and then fall back down. For a reminder on the generic shape of the effective potential, see Figure 1.1. This process rapidly varies the Chameleon particle's effective mass over a short time scale, and thus excites highly energetic modes. Energy stored in the fluctuations becomes comparable to the energy stored in the background, thus invalidating the classical treatment of the Chameleon. This implies that the EFT of the Chameleon breaks down at Big Bang Nucleosynthesis, according to [81,82]. The study of this led to the discovery of a solution dubbed the *surfer*, which causes the large field velocity that drives the scalar up the steep part of the potential, characterised by a constant Jordan frame temperature. It was found that as long as $\beta > 1.82$ then the surfer will exist for generic initial conditions. If $0.42 < \beta < 1.82$ one will avoid the surfer solution but still reach the

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effective potential minimum with too high a velocity and encounter the same problems, assuming BBN bounds on the field value prior to the last kick. This leads us to believe that the only way to avoid this breakdown of the theory is to weakly couple the Chameleon to matter, avoiding the kicks but reducing the effectiveness of the Chameleon to screen fifth forces in the late universe. Coupling the Chameleon to matter weakly enough to avoid the kicks challenges the relevance of lab experiments for fifth forces, as discussed in [76, 121–123]. One could always take the view that the Chameleon EFT is not valid as far back as BBN, in which case one doesn't trust any of these predictions and need not worry. In this work we take the view that any alternative to General Relativity should remain valid at BBN, such that one can make predictions in a regime that is experimentally testable.

We will not reproduce the original derivation of the surfer solution here, and instead refer the reader to [81, 82] to see that. Here we present a different way of seeing its existence using fixed point analysis, as done in [120]. An important point we take from the original derivation though is that the surfer solution is a solution upon which the Jordan frame temperature is held constant.

For a flat FLRW background, the Chameleon action in (3.1) yields the following equations of motion,

$$3M_{pl}^2 H^2 = \rho_\phi + \rho, \quad (3.4)$$

$$M_{pl}^2 \left(2\dot{H} + 3H^2 \right) = -(p_\phi + p), \quad (3.5)$$

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = -\frac{\beta}{M_{pl}}\rho\Sigma, \quad (3.6)$$

where $H = \frac{\dot{a}}{a}$ is the Hubble parameter, ρ and p are the energy density and pressure of matter, and $\rho_\phi = \frac{\dot{\phi}^2}{2} + V$ and $p_\phi = \frac{\dot{\phi}^2}{2} - V$ are the energy density and pressure associated with the scalar field. From here on in this analysis we will ignore the bare potential $V(\phi)$, as we assume that in the early universe the matter fields are the dominant contribution to the chameleon's

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dynamics. This is a reasonable assumption to make because the scalar field is generically away from its potential minimum in the early universe, and the kicks are truly important on so called ‘undershoot’ solutions [78] where we indeed have the conformal coupling dominate over the bare potential. Obviously this assumption will break down as the scalar field is pushed towards the potential minimum, as dictated by the surfer solution. We can simplify the three equations of motion by eliminating ρ to leave us with

$$M_{pl}^2 H^2 (4 - \Sigma) + 2M_{pl}^2 \dot{H} + \frac{2 + \Sigma}{3} \rho_\phi = 0, \quad (3.7)$$

$$\ddot{\phi} + 3H\dot{\phi} + 3M_{pl}\beta\Sigma H^2 \left(1 - \frac{\rho_\phi}{3M_{pl}^2 H^2}\right) = 0. \quad (3.8)$$

As we have ignored the bare potential, we can approximate $\rho_\phi = p_\phi = \frac{\dot{\phi}^2}{2}$. It is here that we can begin to look for fixed points in this autonomous dynamical system. To this end we define the following variables,

$$x = \beta\phi + M_{pl}\ln(a), \quad (3.9)$$

$$y = \frac{\rho_\phi}{3M_{pl}^2 H^2}, \quad (3.10)$$

$$z = \beta\dot{\phi} + M_{pl}H. \quad (3.11)$$

We have defined x such that a constant x , that is $\dot{x} = 0$ corresponds to a constant Jordan frame temperature $T_J \propto e^{-\beta\phi/M_{pl}}/a$. Our choice of y is a standard one for fixed point analysis in cosmology, as it is the ratio of the energy density in ϕ and the critical energy density. Our autonomous system of equations written in these variables is now

$$\dot{x} = z, \quad (3.12)$$

$$\dot{y} = \frac{1}{H_y} \left[H_z \left(3Hz - \frac{1}{2} M_{pl} H^2 (1 - y) (2 + \Sigma(1 - 6\beta^2)) \right) + \frac{H^2}{2} (\Sigma - 4 - y(\Sigma + 2)) \right], \quad (3.13)$$

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$$\dot{z} = -3Hz + \frac{1}{2}M_{pl}H^2(1-y)(2+\Sigma(1-6\beta^2)), \quad (3.14)$$

where $H_y = \frac{\partial H}{\partial y}$, $H_z = \frac{\partial H}{\partial z}$. By manipulating (3.10) and using the fact we have ignored the bare potential, we also find an expression for $H(y, z)$,

$$6M_{pl}^2\beta^2H^2y = (z - M_{pl}H)^2, \quad (3.15)$$

which we can solve to find that

$$H(y, z) = \frac{z}{M_{pl} \left(1 \pm \sqrt{6\beta^2 y}\right)}. \quad (3.16)$$

We assume the conformal factor decreases over time, in which case we take the negative sign root in the above equation. By setting $\dot{y} = z = 0$ we can begin to look for the fixed points in our system. (3.16) then tells us that, as long as $y \neq 1/(6\beta^2)$, our fixed point corresponds to $H = 0$. This is a very uninteresting cosmology as it corresponds to an empty universe, where $\dot{\phi} = 0$ and the universe is static. Our only way to avoid this trivial solution is to set $y = 1/(6\beta^2)$, and note that (3.16) no longer constrains $H(y, z)$. This is an important point that will come up again when studying the fixed points of the modified chameleon mechanism, that non-trivial fixed points do not constrain the Hubble parameter. Setting $y = 1/(6\beta^2)$ in (3.13) and (3.14) and applying the fixed point conditions that $z = \dot{y} = 0$ we find that we must satisfy $\Sigma = 2/(6\beta^2 - 1)$ to be consistent. To summarise, when Σ reaches this critical value suddenly a new *line* of fixed points exist for all H , and so can exist in a Universe at any scale, and is characterised by a fixed Jordan frame temperature. This is precisely the surfer solution. In previous work [81,82] it was stated that the surfer existed for $\Sigma = 1/(3\beta^2)$, obviously in contention with the value that we have derived here. We note that it agrees with our result for large values of β , and that the difference is merely a result of their approximation. They suppose that Σ is small and ignore *most* terms including it, but not all. If one reintroduces these ignored terms in their expressions it agrees with our result.

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We can solve (3.9)-(3.11) numerically for a large array of initial conditions, for various values of Σ . By imposing $\dot{\Sigma} = 0$ we can create 2D phase plots of the system in the (H, z) plane.

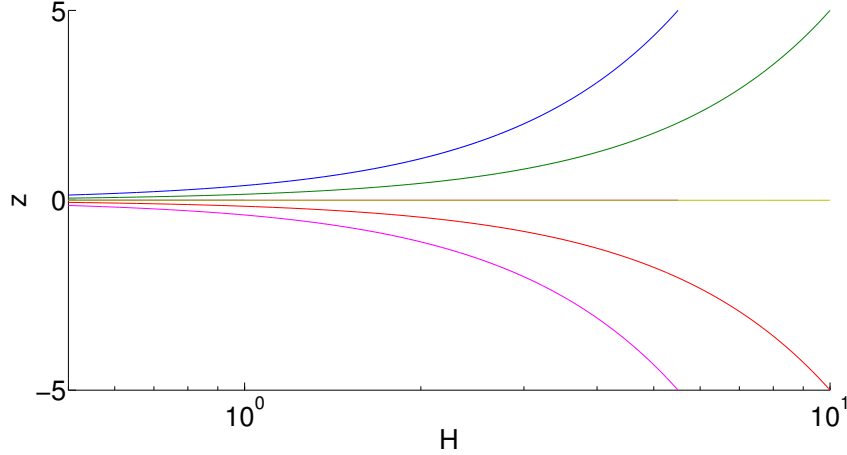


Figure 3.1: Plot of z against H for a range of initial conditions, indicated by different colours. $\Sigma = \frac{2}{6\beta^2-1}$. We numerically set $M_{pl} = 10^4$ and $\beta = 1$.

Figure 3.1 shows us that for a wide range of initial conditions, the solutions are attracted to the $z = 0$ line. This implies that the surfer is an attractor, as long as Σ takes on the required value for it to exist. We can see this analytically by expanding the dynamical system equations around the surfer's trajectory, $x \rightarrow x_0 + \delta x$, $y \rightarrow \frac{1}{6\beta^2} + \delta y$ and $z \rightarrow \delta z$. (3.12) - (3.15) then yield

$$\delta \dot{x} \approx \delta z \quad (3.17)$$

$$\delta \dot{y} \approx \frac{1}{3M_{pl}\beta^2} \delta z \quad (3.18)$$

$$\delta \dot{z} \approx -3H\delta z, \quad (3.19)$$

and our expression for $H(y, z)$ yields

$$\delta z = -3M_{pl}\beta^2 H \delta y. \quad (3.20)$$

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Solving these equations tells us that $\delta x \propto \delta y \propto \delta z \propto \exp(-3Ht)$, demonstrating that the surfer is indeed an attractor at late times, in an expanding Einstein frame. Although this result is interesting, the set-up of a constant Σ at the critical value is highly un-physical, and so a more natural exercise is to instead study the system in an expansion around the surfer *and* the critical value of Σ . In a more physical scenario Σ will vary with the Jordan frame temperature as discussed previously, where $T_J = T_{J0}\exp(-x/M_{pl})$. Close to the surfer one can then approximate $\Sigma(x) \approx \frac{2}{6\beta^2-1} + \Sigma'(x_0)\delta x$, which leads the dynamical system at leading order to

$$\delta \dot{x} \approx \delta z, \quad (3.21)$$

$$\delta \dot{y} \approx \frac{1}{3M_{pl}\beta^2}\delta z + H \left(\frac{1}{6\beta^2} - 1 \right)^2 \Sigma'(x_0)\delta x, \quad (3.22)$$

$$\delta \dot{z} \approx -3H\delta z - 3\beta^2 M_{pl} H^2 \left(\frac{1}{6\beta^2} - 1 \right)^2 \Sigma'(x_0)\delta x. \quad (3.23)$$

The solution to these equations are all of the form $\mathcal{C}_+ e^{m_+ t} + \mathcal{C}_- e^{m_- t}$ where

$$m_{\pm} = -\frac{3}{2}H \left(1 \pm \sqrt{1 - \frac{4}{3}\beta^2 M_{pl} \left(\frac{1}{6\beta^2} - 1 \right)^2 \Sigma'(x_0)} \right). \quad (3.24)$$

This shows us that the stability of the surfer is dependant on the exact form of the kick. As the Jordan frame temperature cools Σ will pass through the critical value, peak, and then fall back through it again. By using the fact that $T_J \approx T_{J0}\exp(-x/M_{pl})$ we can rewrite $\Sigma'(x_0) = -\frac{T_J}{M_{pl}} \frac{d\Sigma}{dT_J}$. As the kick function peaks and starts to drop, again passing through the critical value, it will have $\frac{d\Sigma}{dT_J} > 0$. This means that m_+ will be real and negative, and m_- real and positive. This positive root does signal an instability of the surfer solution. If we use physically realistic kicks we have that $M_{pl}|\Sigma'(x_0)| \sim \Sigma(x_0) = \frac{2}{6\beta^2-1}$ [81, 82], hence $m_- \lesssim H$. This behaves like a constant during a Hubble time, and so can be absorbed into a redefinition of x_0 . This agrees with numerical results in [81, 82], where this attractor behaviour is seen for

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physically relevant kick functions. Even if the other side of the kick function is such that the surfer is not an attractor, the fact that there is at least one point at which the surfer is attractive is enough to make it a problem for chameleon cosmology.

To conclude the discussion of the surfer in Chameleon theory, we will summarise the results of this section. As the early universe cools, the matter content of the universe will kick the chameleon. The attractive nature of the surfer solution means that the chameleon field will then undergo large field excursions. As the surfer is a solution of constant Jordan frame temperature, once surfing the field will stay there, as Σ becomes stuck in the area around the critical value. At some point the bare potential we have neglected in this analysis will become important and will destroy the surfing behaviour, but the chameleon is left with a large velocity given by the surfer to be $\sim \frac{M_{pl}H}{\beta}$. This means the chameleon continues past its potential minimum and climbs back up the other side of the potential, which is much steeper, leading to production of energetic fluctuations. This means the classical description of the theory is lost and with it any predictability the theory had around BBN.

3.2 DBI Chameleon

As the surfer is an early universe problem, it suggests we should look to modify the UV behaviour of the theory such that we can retain the low energy properties we find useful, such as screening, while fixing the problem. Here I will present the DBI-Chameleon model, proposed in [120], and show that it does exactly that, for a technically natural choice of the scale of the modifications.

The action of the DBI Chameleon is given by

$$\mathcal{S} = \int d^4x \left[\frac{M_{pl}^2}{2} R + \Lambda^4 - \Lambda^4 \sqrt{1 + \frac{(\partial\phi)^2}{\Lambda^4}} - V(\phi) \right] + \mathcal{S}_m[\tilde{g}_{\mu\nu}, \psi], \quad (3.25)$$

where $\tilde{g}_{\mu\nu}$ is the same as it was in the Chameleon theory, and Λ is some

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energy scale associated with the modifications. If one expands this action in the limit $(\partial\phi)^2 \ll \Lambda^4$, physically corresponding to low energy, then the Chameleon action is recovered. This demonstrates that at low energies the properties of the Chameleon mechanism are retained and that the non-linearities introduced in our modification are not needed to screen any scalar forces locally. The addition of higher derivative non-linear terms was made such that the kinetic term describing fluctuations of the scalar field comes with a large Z factor for non-trivial backgrounds. This means that the coupling between fluctuation of the scalar and the matter sector goes like $\beta/(M_{pl}\sqrt{Z})$ when canonically normalised, and so is dynamically weakened when the derivative interactions dominate over the behaviour of the kinetic term. There is also the added benefit that the DBI structure imposes an upper limit on the velocity of the scalar, which helps suppress the worryingly large field excursions that cause problems in the original Chameleon. If we consider a Minkowski space-time and a homogeneous background solution $\phi(t)$ this Z factor is given by

$$Z^{tt} = \left(1 - \frac{\dot{\phi}^2}{\Lambda^4}\right)^{-\frac{3}{2}}, \quad Z^{ti} = 0, \quad Z^{ij} = \left(1 - \frac{\dot{\phi}^2}{\Lambda^4}\right)^{-\frac{1}{2}} \delta^{ij} \quad (3.26)$$

Looking at this, it is clear that as $(\partial\phi)^2 = \dot{\phi}^2 \rightarrow \Lambda^4$ the Z factor becomes large, causing the dynamic weakening of the coupling. This is in essence the Vainshtein mechanism as discussed in Section 1.4.2, where we also provide more detail on the Z function and its meaning. As mentioned in that section, there are problems with the Vainshtein mechanism when it is employed on static backgrounds such that the theory cannot admit a Lorentz invariant UV completion [91]. Here we avoid this problem as we are interested in cosmology and homogeneous backgrounds, instead of static ones, and thus satisfy the positivity condition ensuring that all of the operators in our action could come from a Lorentz invariant UV theory, and hence form a good effective description. It is important that our modifications do form a healthy effective theory as the main motivation of this work is to fix the

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problem of break down in the standard Chameleon model.

One might wonder why we have chosen a DBI structure for our non-linear interactions, instead of a more generic correction of the form

$$\mathcal{L} \ni c_n \frac{(\partial\phi)^{2n}}{\Lambda^{4n-4}}. \quad (3.27)$$

We note that corrections of this generic form are problematic. As soon as the first of these irrelevant operators becomes important, they all become important, and so the dynamics would be dominated by the $n \rightarrow \infty$ terms. As all of the coefficients are arbitrary any ability to calculate is lost, and so this is a poor replacement for the Chameleon as it suffers a similar calculability breakdown. We choose the DBI structure because it allows us to keep control of the non-linearities due to the extra symmetry that comes with it. This extra symmetry leaves the theory invariant under

$$\phi \rightarrow \phi + c + v_\mu x^\mu + \phi v^\mu \partial_\mu \phi, \quad (3.28)$$

where v^μ is a constant vector. This symmetry protects the structure, meaning we can trust our description even when the non-linearities become important, unlike a theory with a more generic set of non-linear terms such as those in (3.27), which represent K-essence type theories. The symmetry that selects the DBI structure can also be seen as setting the coefficients c_n in (3.27), if one expands the square root of the DBI term one can see it gives the same operators but with the coefficients set.

We discussed the origin of this symmetry in Section 2.3, but we will briefly present it again here for convenience. Let us consider a 5D Minkowski bulk with a brane localised to $y = \phi(x^\mu)/\Lambda^2$, where y is the coordinate of the extra dimension. To build an effective description of the brane we use the induced metric,

$$\hat{g}_{\mu\nu} = \eta_{\mu\nu} + \frac{\partial_\mu \phi \partial_\nu \phi}{\Lambda^4}. \quad (3.29)$$

The leading contribution to the brane effective action was presented in

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(2.14), and was given by

$$\mathcal{S} = -\Lambda^4 \int d^4x \sqrt{\hat{g}} = -\Lambda^4 \int d^4x \sqrt{1 + \frac{(\partial\phi)^2}{\Lambda^4}} \quad (3.30)$$

where the brane tension is given by Λ^4 . This is simply the DBI term that we have included in our modified Chameleon action. It becomes clear now that although the DBI term respects a non-linear symmetry, its origin is simply Poincare invariance in the 5 dimensional bulk. Understanding this allows us to consider other operators that respect the same symmetry. One would expect there to be more terms in the brane action as we only included leading order terms, and would also expect these higher order terms to appear in our modified Chameleon action as well. These operators will be built from powers of the brane's extrinsic curvature K_ν^μ , encoding derivatives of the induced metric and hence higher derivatives of ϕ . We expect these operators to appear in our modified Chameleon action because they are guaranteed to respect the same symmetry as the DBI term, and hence could be generated by quantum corrections, at least in principle. The next to leading order contribution to the brane action will be linear in the extrinsic curvature, and so the only term is simply K , the trace of $K_{\mu\nu}$. The next order contains quadratic terms, and so will include K^2 and $K_{\mu\nu}K^{\mu\nu}$, and so on. In order to exclude these higher order operators from our modified Chameleon theory we need to make sure they are suppressed relative to the zero-th order term we wish to keep. Again, I will emphasise this is important, as we want our modified Chameleon to be a valid effective theory to avoid the problems of the original Chameleon. Asking that the higher order operators are suppressed is equivalent to asking that

$$K_\nu^\mu = \frac{\gamma}{\Lambda^2} \nabla^\mu \nabla_\nu \phi - \frac{\gamma^3}{\Lambda^6} \nabla^\sigma \phi \nabla^\mu \phi \nabla_\nu \nabla_\sigma \phi \ll \Lambda, \quad (3.31)$$

where we have defined $\gamma = \left(1 + \frac{(\partial\phi)^2}{\Lambda^4}\right)^{-1/2}$. Evaluating this on a cosmological background and requiring that both terms must be independently small,

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we obtain the following conditions

$$\frac{\gamma H \dot{\phi}}{\Lambda^3} \ll 1, \quad \frac{\gamma^3 \ddot{\phi}}{\Lambda^3} \ll 1 \quad (3.32)$$

which must be satisfied at all times in order for us to consistently ignore these higher order derivatives. As we will see this is indeed the case even where the DBI term becomes important.

All of the above analysis is done with just the brane action, but in our DBI Chameleon theory this symmetry is broken by the potential $V(\phi)$ and by finite M_{pl} . Corrections coming from finite M_{pl} will be Plank suppressed, and hence can be ignored consistently as long as we impose $\Lambda \ll M_{pl}$. The corrections coming from the potential breaking the symmetry will depend on $V(\phi)$ and its exact form, and so here we will suppose that the potential decays for large values of ϕ as is done with the original Chameleon mechanism as detailed in Section 1.4.1. This means any mass scale associated with the potential must feature with positive powers, for example in (1.19). The symmetry is broken softly, and associated corrections easily ignored, as long as $M \ll \Lambda$. It is worth noting that this condition is naturally satisfied as Λ is some energy scale controlled by early universe physics, whereas M is controlled by late universe physics. Combining the conditions such that our DBI Chameleon model is stable against all corrections, we find that we must choose our new energy scale Λ such that $M \ll \Lambda \ll M_{pl}$.

3.3 To Surf, or not to Surf?

Having now motivated our DBI Chameleon action, we can perform the same fixed point analysis as done to the Chameleon action, to see if the surfer solution still exists. It is worth pointing out that even if the surfer itself no longer exists, the problem of large field excursions could still occur and hence the underlying problem not fixed. This is discussed further in Section 3.3.2.

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As a reminder, we define the action of our DBI Chameleon theory as

$$\mathcal{S} = \int d^4x \left[\frac{M_{pl}^2}{2} R + \Lambda^4 - \Lambda^4 \sqrt{1 + \frac{(\partial\phi)^2}{\Lambda^4}} - V(\phi) \right] + \mathcal{S}_m[\tilde{g}_{\mu\nu}, \psi], \quad (3.33)$$

where Λ is some new energy scale and $\tilde{g}_{\mu\nu} = e^{2\beta\phi/M_{pl}} g_{\mu\nu}$ is the Jordan frame metric as for the standard Chameleon model. For convenience we define the following variable $X = -(\partial\phi)^2/2$ such that we can write the DBI term in (3.33) as $P(X) = \Lambda^4 - \Lambda^4 \sqrt{1 - \frac{2X}{\Lambda^4}}$. Calculating the equations of motion, again on a FLRW background, gives us

$$3M_{pl}^2 H^2 = \rho_\phi + \rho, \quad (3.34)$$

$$M_{pl}^2 (2\dot{H} + 3H^2) = -p_\phi - p, \quad (3.35)$$

$$(P_{,X} + 2XP_{,XX}) \ddot{\phi} + 3HP_{,X} \dot{\phi} + V'(\phi) = -\frac{\beta}{M_{pl}} \rho \Sigma, \quad (3.36)$$

where we have again defined ρ and p as the energy density and pressure of the matter sector with $\Sigma = \frac{\rho - 3p}{\rho}$, H as the Hubble parameter, $\rho_\phi = 2XP_{,X} - P(X)$ is the energy density and $p_\phi = P(X)$ is the pressure of the scalar field. We have again ignored the scalar potential $V(\phi)$ for the same reasons as before. If one takes the $\Lambda \rightarrow \infty$ limit of the above equations of motion, the standard Chameleon equations of motion are recovered. This reinforces our point that the high energy modifications we have made to the theory are not needed for the screening behaviour to occur at low energies. Again eliminating ρ from these equations leaves us with

$$M_{pl}^2 H^2 (4 - \Sigma) + 2M_{pl}^2 \dot{H} + p_\phi + \frac{\Sigma - 1}{3} \rho_\phi = 0, \quad (3.37)$$

$$(P_{,X} + 2XP_{,XX}) \ddot{\phi} + 3HP_{,X} \dot{\phi} + 3M_{pl} \beta \Sigma H^2 \left(1 - \frac{\rho_\phi}{3M_{pl}^2 H^2} \right) = 0. \quad (3.38)$$

We now introduce the same variables from (3.9)-(3.11), for convenience we

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repeat their definitions

$$x = \beta\phi + M_{pl}\ln(a)$$

$$y = \frac{\rho_\phi}{3M_{pl}^2 H^2},$$

$$z = \beta\dot{\phi} + M_{pl}H.$$

The equations of motion in these variables are

$$\dot{x} = z, \tag{3.39}$$

$$\begin{aligned} \dot{y} = \frac{1}{H_y} & \left[H_z \left(3Hs^2z - \frac{1}{2}M_{Pl}H^2 [(1-y)(2+\Sigma(1-6\beta^2s^3)) + 3y(1-s) - 6(1-s^2)] \right) \right. \\ & \left. + \frac{H^2}{2} (\Sigma - 4 - y(\Sigma + 2) + 3y(1-s)) \right], \end{aligned} \tag{3.40}$$

$$\dot{z} = -3Hs^2z + \frac{1}{2}M_{Pl}H^2 [(1-y)(2+\Sigma(1-6\beta^2s^3)) + 3y(1-s) - 6(1-s^2)], \tag{3.41}$$

where $s = \sqrt{1 - \frac{(z - M_{pl}H)^2}{\beta^2\Lambda^4}}$, and we have again defined $H_y = \frac{\partial H}{\partial y}$ and $H_z = \frac{\partial H}{\partial z}$. In a similar fashion to before, massaging (3.40) allows us to form a relation that implicitly gives $H(y, z)$, which is

$$\left(\Lambda^4 - \frac{(z - M_{pl}H)^2}{\beta^2} \right) \left(1 + \frac{3M_{pl}^2 H^2 y}{\Lambda^4} \right)^2 = \Lambda^4. \tag{3.42}$$

If now look at the fixed points of the system by setting $\dot{x} = \dot{y} = \dot{z} = 0$, the above relation tells us that

$$\left(\frac{9y^2}{\beta^2\Lambda^8} \right) (M_{pl}H)^6 + \left(\frac{6y}{\beta^2\Lambda^4} - \frac{9y^2}{\Lambda^4} \right) (M_{pl}H)^4 + \left(\frac{1}{\beta^2} - 6y \right) (M_{pl}H)^2 = 0. \tag{3.43}$$

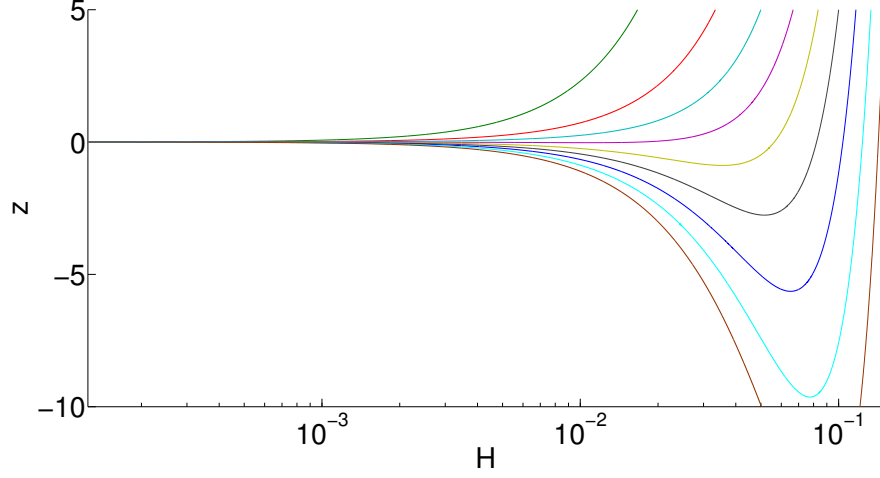
In Section 3.1 we used the analogous relation and imposed that it did not constrain H . We did this because all other fixed points corresponded to a universe where $H = 0$, which was uninteresting. The same can be seen here, where for arbitrary y the only solution to this relation is that $H = 0$, which then trivially satisfies the other equations in the dynamical system (3.39)-

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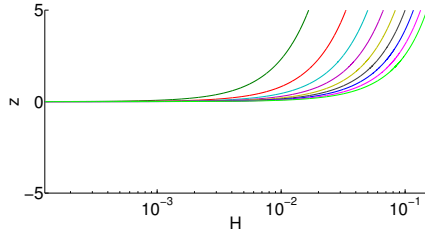
(3.41). To look for interesting fixed point solutions, we must again look for situations where H is unconstrained. Looking at the form of (3.42), we can see that this will only happen if the coefficient of each $(M_{pl}H)$ term vanishes. There are two ways in which we can do this, firstly by sending $\Lambda \rightarrow \infty$ and sending $y \rightarrow 1/(6\beta^2)$ simultaneously. This is simply the Chameleon limit of our DBI theory, where by sending Λ to infinity we have decoupled the interesting scalar self-interaction terms we introduce with our modification and leave only the standard Chameleon mechanism relevant to dynamics. It is unsurprising then, that the Surfer will still exist in this limit. The second way we can leave H unconstrained is by sending $\beta \rightarrow \infty$ and setting $y = 0$ as well. This corresponds to the limit where the scalar field is infinitely strongly coupled to the matter sector. It is again unsurprising that the surfer appears in this situation, because by strongly coupling the scalar field to matter the kicks coming from the matter sector will overpower any scalar self interactions coming from the DBI term. More generally, we expect that the DBI terms will become important when $\dot{\phi} \sim \Lambda^2$, and so we expect any surfer behaviour to be prevented as long as $\Lambda^2 \lesssim M_{pl}H/\beta$. This result comes from the fact that on the surfer solution, $\dot{\phi} = -M_{pl}H/\beta$.

These conclusions can be reinforced using numerical analysis on the dynamical system for a range of initial conditions. If we fix $\Sigma = 2/(6\beta^2 - 1)$ such that the surfer will appear if it is able to, we can vary β and Λ and see how the surfer emerges. Figure 3.2b shows β being set to a large number, and Figure 3.2c shows Λ being large instead. We can see the presence of a stable and attractive $z = 0$ line, paralleling Figure 3.1 such that we can see the surfer emerge. Figure 3.2a shows us the DBI Chameleon evolution for a more ‘normal’ set of parameters, where we set our initial H values such that $\Lambda^2 \lesssim M_{pl}H$. As explained above, it is for this range of parameter space that we expect the DBI interactions to become important and spoil the surfer behaviour. We can see from Figure 3.2a that at high scales there is no stable $z = 0$ behaviour, until H drops such that it is below Λ^2/M_{pl} and the DBI interactions begin to give way to the standard Chameleon dynamics,

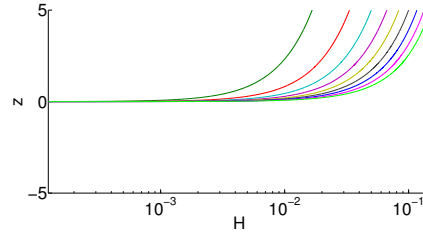
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(a) $\Lambda = 0.005M_p$ and $\beta = 2$.



(b) $\beta = 20$



(c) $\Lambda = 0.5M_p$

Figure 3.2: Plots of z against H for a range of initial conditions, indicated by different colours. We have set $\Sigma = \frac{2}{6\beta^2 - 1}$ for all graphs, and vary β and Λ .

which for our selected value of Σ is the surfer. I will point out that the above scenario is again un-physical, as we have imposed $\dot{\Sigma} = 0$ such that the scalar field is always being kicked in just the right way to surf. In a more plausible scenario the kick will evolve with Jordan frame temperature, and so we need only make sure that when Σ is at it's critical value the DBI interactions are able to mask the surfer, not at all scales.

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3.3.1 Non-constant Σ

If we allow Σ to evolve, $\Sigma = \Sigma(T_J)$, we can take a measure of how ‘close’ Σ is to this critical value by looking at the induced change in T_J , given by

$$\frac{\Delta T_J}{T_J} = - \int_{t_i}^{t_i + \Delta t} \frac{z}{M_p} dt. \quad (3.44)$$

If we require $\Delta T_J/T_J \lesssim 1$ in absolute value, we can find Δt which represents the time for which Σ is approximately constant. We note that in this time Δt one can still see the behaviour shown in Figure 3.2, demonstrating that the DBI interactions can indeed mask the surfer solution.

While we see that the scalar does not surf in the DBI theory, unless one takes Λ or β to infinity; we have not yet discussed if it actually avoids the problem of forcing large field excursions in the scalar, and thus leaving it with large velocity near its potential minimum. It was this result that gave rise to the fast fluctuation in the scalar’s mass, and hence a breakdown in the EFT. To compute the field excursions, we note that

$$\Delta\phi = \int_{t_i}^{t_i + \Delta t} \dot{\phi} dt, \quad (3.45)$$

where Δt is the same as in (3.44). The maximum field excursions will occur if the field is initially at $z = 0$, such that it is on the surfer solution. Obviously we wish to see that our modifications cause this to not be a stable solution, and that the system will evolve away from this point. Using the fact that at $z = 0$ the scalar field goes like $\dot{\phi} = -M_{pl}H/\beta$ we can say that while Σ is close the critical value the field excursions will be

$$\frac{\Delta\phi}{M_{pl}} \approx -\frac{H\Delta t}{\beta}. \quad (3.46)$$

Similarly simplifying (3.44) by noting that $z(t_i) \approx \dot{z}(t_i)(t - t_i)$ we can say

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that

$$\frac{\Delta T_J}{T_J} \approx -\frac{\dot{z}(t_i)}{M_{pl}} \frac{\Delta t^2}{2}. \quad (3.47)$$

We can extract an expression for $\dot{z}(t_i)$ from (3.41) by again saying the field is initially at $z = 0$, choosing $\Sigma = 2/(6\beta^2 - 1)$ and making $y \sim \mathcal{O}(1)$. These choices are made to try and make the field excursions as large as possible. Doing so yields that $\dot{z}(t_i) \sim \mathcal{O}(1)M_{pl}^3 H^4/\beta^2 \Lambda^4$ and so combining this with (3.46) and (3.47) tells us that

$$\frac{\Delta\phi}{M_{pl}} \approx \mathcal{O}(1) \frac{\Lambda^2}{M_{pl}H} \sqrt{\left|\frac{\Delta T_J}{T_J}\right|}. \quad (3.48)$$

As we are worried about the field excursions caused by the kicks, which are localised in T_J we can say that $\Delta T_J < T_J$ in the region we are interested in. This finally leads us to the result that for a scalar field that starts on the surfer solution, with a kick at exactly the critical value, the field excursions will be

$$\frac{\Delta\phi}{M_{pl}} \lesssim \frac{\Lambda^2}{M_{pl}H}, \quad (3.49)$$

where the H present is the scale at which the kick occurs. Here we see the desired effect of the modification to the Chameleon theory, that the introduction of the DBI operators suppresses the size of the field excursions caused by the kicks in the early universe. We can see that taking $\Lambda \rightarrow \infty$ gives us an infinite field excursion, in line with the picture that the Chameleon once surfing will stay that way, being driven to ever lower field values. It is the bare potential eventually becoming relevant that causes this calculation to break down at some point. Given that the last kick from Standard Model particles will come from the electron becoming non-relativistic in the early universe, we can say that it will occur on scales $M_{pl}H \sim (\text{MeV})^2$. We can also impose an upper bound on how large the field excursions can be by using constraints on variation of particle masses since Big Bang Nucleosynthesis, which are that $|\Delta\phi|/M_{pl} \lesssim 0.1/\beta$ [124]. Combining all of this together, we find that to protect the Chameleon from

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field excursions that rule the theory out observationally we must satisfy $\Lambda \lesssim \text{MeV}/\sqrt{\beta}$.

3.3.2 Numerics with $\Sigma(T_J)$

In this section we confirm these results by performing a numerical analysis of the dynamical system, with realistic initial conditions and a simplified kick function. We follow the example from [81, 82] and make a change of variables suited to numerical analysis. This involves a simple rescaling of the scalar field ϕ by M_{pl} and by changing our time coordinate to Einstein frame e-folds, N , such that $\varphi = \phi/M_{pl}$ and $N = \ln(a/a_i)$ where a is the Einstein frame scale factor and a_i is the scale factor at some initial time. Re-writing the Chameleon equations of motion in these variables gives us

$$2\frac{H'}{H} + (4 - \Sigma) + \left(\frac{2 + \Sigma}{3}\right) \frac{\varphi'^2}{2} = 0, \quad (3.50)$$

$$\varphi'' + \left(\frac{H'}{H} + 3\right) \varphi' + 3\beta\Sigma \left(1 - \frac{\varphi'^2}{6}\right) = 0, \quad (3.51)$$

where the prime denotes differentiation with respect to N from now on. The equations of motion for our DBI Chameleon theory are given by

$$2\frac{H'}{H} + (4 - \Sigma) + \left(p_\phi - \rho_\phi + \left(\frac{2 + \Sigma}{3}\right) \rho_\phi\right) \frac{1}{M_{pl}^2 H^2} = 0, \quad (3.52)$$

$$(P_{,X} + P_{,XX} M_{pl}^2 H'^2 \varphi'^2) \left(\varphi'' + \left(\frac{H'}{H}\right) \varphi'\right) + 3P_{,X} \varphi' + 3\beta\Sigma \left(1 - \frac{\rho_\phi}{3M_{pl}^2 H^2}\right) = 0. \quad (3.53)$$

To model the kick, we use a simple Gaussian in Jordan frame temperature, in order to include the localised nature of the kick function in T_J . As such, we use

$$\Sigma(T_J) = A \exp \left[-\frac{(\log(T_J) - \log(T_{peak}))^2}{\sigma^2} \right], \quad (3.54)$$

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where the parameters A , σ and T_{peak} will characterise the kick function. We choose these parameters to accurately model the kick that comes from the electron-positron kick in the Standard Model, as this dominates Σ_{SM} as seen in Figure 2 from [81]. We can also model only one kick, instead of many, as the electron-positron will be the last kick to impact the scalar. This means that if our theory can mask the effects of the electron-positron kick, it will definitely be able to do so for the others which are both weaker and occur at higher energies, where the DBI interactions will be even more relevant. To best match the electron-positron kick we will use $A = 0.1$, $T_{peak} = 2 \times 10^{-4} \text{ GeV}$ and $\sigma = 0.3$. We can also write the Jordan frame temperature as a function of N by remembering that $T_J \propto \exp[-\beta\varphi]/a$, which implies that

$$T_J(N) = T_{J,i} \exp[-N - \beta\Delta\varphi(N)], \quad (3.55)$$

where $T_{J,i}$ is the Jordan frame temperature at $N = 0$ and $\Delta\varphi(N) = \varphi(N) - \varphi_i$ is the field excursion after N Einstein frame e-folds of evolution. It is interesting to note that if φ decreases faster than N increases then T_J does not decrease, leading to the unusual situation where the Einstein frame is expanding while the Jordan frame contracts. The rate at which φ must grow relative to N to achieve this is actually bound by the surfer solution, at which point the Jordan frame freezes and neither expands or contracts as previously noted.

We solve the equation of motion for both the Chameleon, (3.50)-(3.51) and DBI Chameleon, (3.52)-(3.53), with the Gaussian kick (3.54). We will also vary the value that Λ takes in the DBI Chameleon theory to emphasise the effect that it has on the dynamics. To compare the Chameleon and DBI Chameleon theories consistently we will match the initial conditions of ϕ and $\dot{\phi}$ between them. This is equivalent to setting the initial energy densities to be similar, as long as $\gamma \sim \mathcal{O}(1)$, which is true throughout our numerics. To close the system of equations we note that our initial energy density is

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dominated by radiation as most matter is still relativistic, giving us

$$\rho_i = \frac{\pi^2}{30} g_*(T_{J,i}) T_{J,i}^4 e^{4\beta\varphi_i}, \quad (3.56)$$

with $g_*(T_{J,i}) = 10.75$ [81]. From these initial conditions, that is ϕ_i , $\dot{\phi}_i$ and ρ_i , we can infer values of φ_i , φ'_i and H_i to numerically evolve the system of equations. We set $\beta = 3$ and that the initial Jordan frame temperature be $T_{J,i} = 10^{-2}$ GeV, which is approximately the start of the electron-positron kick. Our initial conditions in $\dot{\phi}$ will be related to scale of the kick by

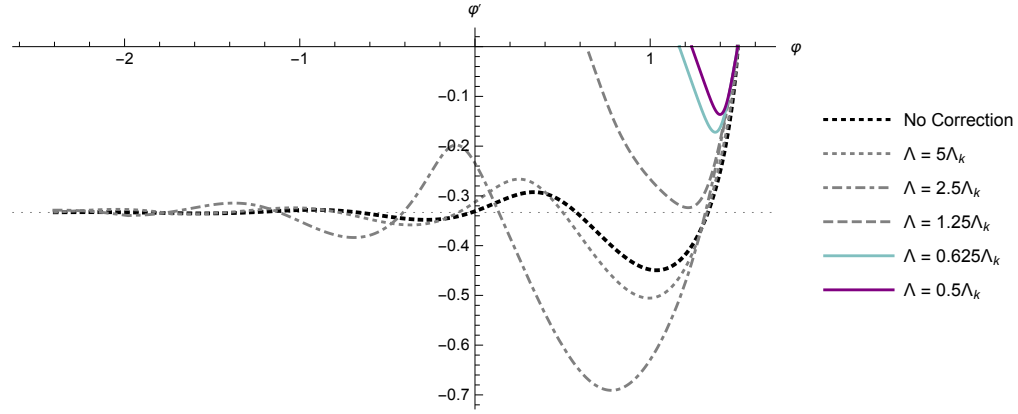
$$\dot{\phi}_i = 0.8\Lambda_k^2 \quad (3.57)$$

where $\Lambda_k = 2 \times 10^{-3}$ GeV. I point out that this easily satisfies the condition that $\Lambda \ll M_{pl}$ that we derived from requiring that corrections to our action from the broken DBI symmetry can be ignored. It is worth noting that although the numeric results we will present here are for this initial condition, we also performed the analysis for $\dot{\phi}_i = -0.8\Lambda_k^2$, but the results were so similar we decided to remove them from the figures for clarity's sake. When $\Lambda \approx \Lambda_k$, $\gamma \approx 1.7$ and so the scalar field is in the strong DBI regime where we expect the surfer solution to not be present.

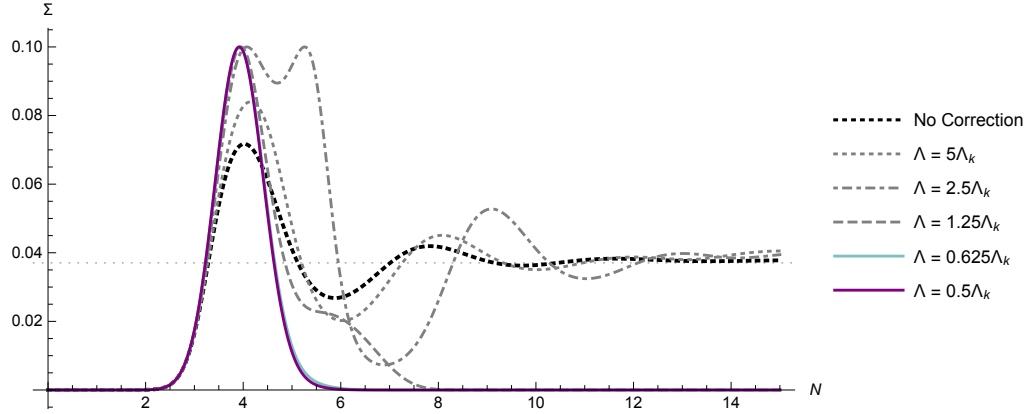
In Figure 3.3a we plot the trajectories in the φ - φ' phase plane, where we have included a horizontal grey line that represents the surfer solution. We can see that for values of Λ such that $\Lambda \lesssim \Lambda_k$ the trajectory never reaches the surfer solution, and instead the field velocity goes to zero. This again demonstrates that the DBI terms can help fix the problem of the scalar field having a large velocity. As the scale of the DBI interactions is raised we can see that the trajectory begins to be pulled towards the surfer solution, with the trajectory of highest Λ being very close to the pure Chameleon theory trajectory. This behaviour matches exactly what we expect from our previous discussion and analytic analysis.

Figure 3.3b shows a plot of $\Sigma(N)$, the kick function, against the Einstein

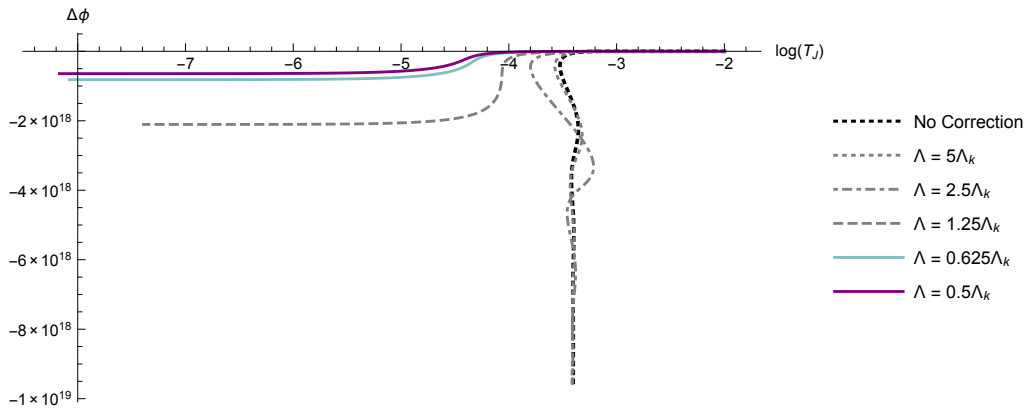
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(a) Phase diagram for φ' vs. φ for a range of Λ . Dotted horizontal grey line indicates the surfing solution at $-1/\beta$, $\beta = 3$ and evolution was for $N = 15$ efolds.



(b) Plot of $\Sigma(N)$. Dotted horizontal grey line indicates $\Sigma(N) = 2/(6\beta^2 - 1)$. $\beta = 3$ and a range of Λ values are plotted.



(c) Field excursions in GeV against Jordan frame temperature. $\beta = 3$ and a range of Λ values are used.

Figure 3.3: Above figures produced by Anthony Walters and Emma Platts for [120].

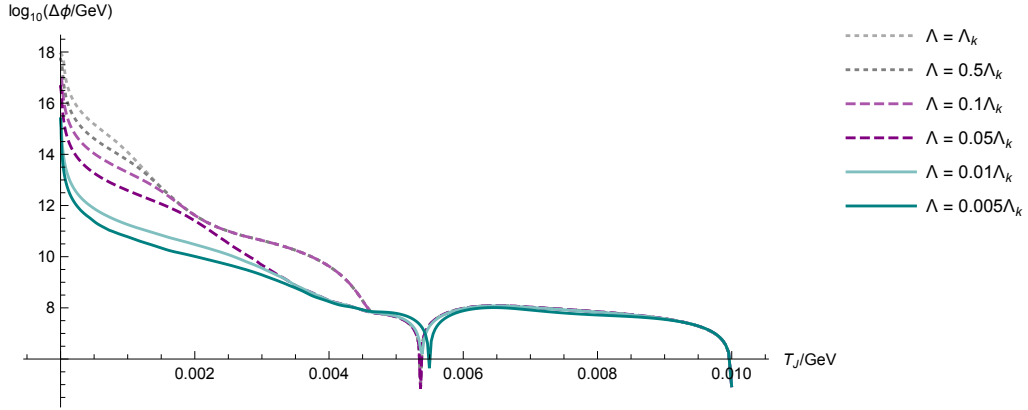
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frame efolds. We can see here the standard Chameleon theory getting ‘stuck’ at the critical value of Σ as it begins to surf. As discussed previously, the surfer solution causes the Jordan frame temperature to stop evolving, and hence keeps the kick function at the critical value. We can also see that if the scale of the DBI interactions is too high such that the theory still surfs, the kick function still converges on this critical value for large N , showing approximate surfing behaviour. As the DBI interactions become more important as we lower Λ we can see that the kick the scalar sees is more like what we expect, and no longer sticks to the critical value and instead drops back to zero. Unsurprisingly the runs that did not approach the surfer trajectory in Figure 3.3a are the ones in which the kick function does not asymptote to the critical value. We can also see that only one of the two places the kick crosses the critical value cause a problem, the one at larger N . This matches the conclusions we came to with our perturbative analysis in Section 3.1, where we saw that only the side of the kick with the lower Jordan frame temperature, equivalent to higher N , has an attractive surfer solution.

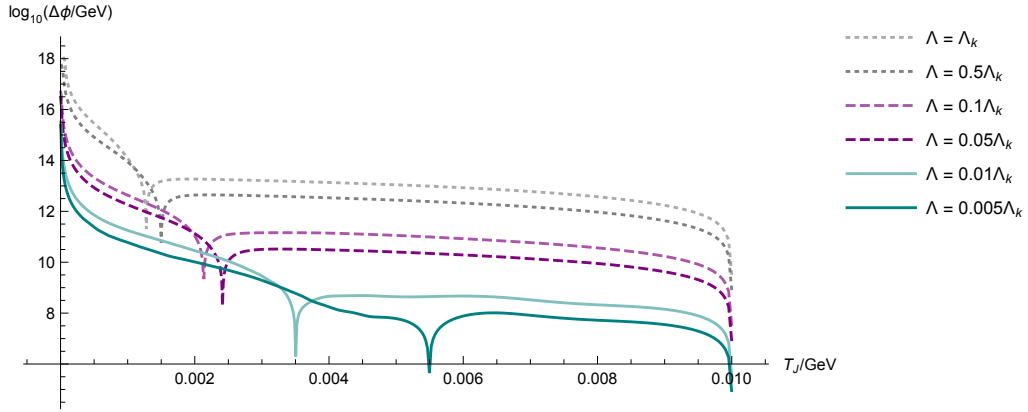
Finally, Figure 3.3c shows a plot of the field excursion against the Jordan frame temperature. We can see the standard Chameleon gets stuck at the Jordan frame temperature that corresponds to the critical value of the kick, and the field excursion keeps growing. Theories where Λ is large enough to still allow the surfer to exist also exhibits the same behaviour. Of more interest are the trajectories with small Λ , as they show clear suppression of the field excursions. Although this is the effect we want, one should still worry about the magnitude of these field excursions, as even the suppressed ones are $\Delta\phi \sim \mathcal{O}(M_{pl})$ in Figure 3.3c. This would still lead to problems as the Big Bang Nucleosynthesis bound requires that just before the electron kick we have $\phi < 0.1M_{pl}/\beta$ [124] which guarantees that the scalar field will still be pushed all the way to its potential minimum. Luckily though, these plots all represent a *worst case scenario* where the scale of the DBI interactions is comparable to the scale of the kick, and the scalar field’s initial

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velocity close to the upper speed limit imposed by the DBI term. Either reducing the initial velocity or lowering the scale of the DBI interactions would reduce the field excursions to acceptable levels.



(a) Field excursions against Jordan frame temperature. $\beta = 3$. Each run has the same $\dot{\phi}_i$ with Λ decreased.



(b) Field excursions vs Jordan frame temperature. $\beta = 3$. Each run has the same γ , achieved by lowering $\dot{\phi}_i$ and Λ simultaneously.

Figure 3.4: Figures produced by Emma Platts and Anthony Walters for [120].

This is shown in Figures 3.4a and 3.4b. Figure 3.4a shows the field excursion plotted against Jordan frame temperature for fixed initial conditions, but each trajectory has a different Λ value where $\Lambda < \Lambda_k$. It is clear that

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reducing Λ by just an order of magnitude is enough to reduce the field excursions to values that are safe in the context of the Big Bang Nucleosynthesis bounds. Figure 3.4b shows a plot of field excursion against Jordan frame temperature also, but we change both Λ and $\dot{\phi}_i$ between different runs such that every trajectory has the same initial value of γ . We do this so that the DBI term's impact on the dynamics, which is governed by γ , is the same over each run. This allows us to investigate the effects of changing the initial velocity, while keeping the influence of the DBI structure constant throughout. This also leads to a suppression of the field excursions, with values that are acceptable with regards to the Big Bang Nucleosynthesis bounds being easily reached.

To truly believe the numeric analysis just presented, we should convince ourselves that all of the corrections to our action can be ignored. As we have picked a value of Λ for our numerics such that $\Lambda \ll M_{pl}$, all of the finite M_{pl} corrections are suppressed and thus can be ignored. The other corrections come in the form of higher order operators, originating in the brane action as seen in Section 3.2. We showed that these corrections can be consistently ignored if the conditions presented in (3.32) are satisfied. When $\gamma \sim \mathcal{O}(1)$, and $\dot{\phi} \sim \Lambda^2$ initially, these conditions can be simplified to $H \ll \Lambda$, where we have assumed that $\ddot{\phi} \sim H\dot{\phi}$. If we impose that $\Lambda \lesssim \sqrt{M_{pl}H/\beta}$ to keep the field excursions at acceptably small levels, and also impose the hierarchy $\Lambda \ll M_{pl}$ such that other corrections can be ignored, we can satisfy $H \ll \Lambda$ ensuring that higher order operators are also suppressed. Doing so requires that $\Lambda \gg (\text{MeV})^2/M_{pl}$ for the Standard Model.

3.4 Summary

In this section we have proposed a UV modification to the Chameleon theory, which was motivated by the so called surfer solution. In the early universe, matter becoming non-relativistic ‘kicks’ the chameleon field due to the conformal coupling between the two, driving the chameleon rapidly towards the

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minimum of its potential with a large velocity. This leads to quick variations in the effective mass of the chameleon, exciting highly energetic modes and thus causing a breakdown of calculability in the effective description of the theory. This problem could be avoided in a certain region of parameter space, but instead we propose a dynamical way to avoid this problem, avoiding the need to tune parameters and thus keeping the Chameleon relevant as a screening mechanism.

The DBI Chameleon theory includes derivative self interactions, which dynamically weaken the effective coupling between the scalar and matter in the early Universe, in a process akin to Vainshtein screening. This suppression can eliminate the surfer solution and bring down the field excursions caused by the kicks, avoiding the problem entirely. To do so requires the scale of the self interactions to be $\Lambda \lesssim \sqrt{M_{pl}H/\beta}$, where $M_{pl}H$ is the scale of the kick. If we look at the effects the Standard Model has on our theory, the most significant kick is due to the electron-positron, and hence $\Lambda \lesssim \text{MeV}/\sqrt{\beta}$. The reason we chose a DBI structure was because the symmetry in the DBI structure ensures we do not lose calculability as the non-linear terms become important, which would happen for generic derivative, self-interacting terms of the form $(\partial\phi)^{2n}$. We also note that the DBI modification has no impact on the low energy dynamics of the Chameleon, and so the theory will still exhibit screening of fifth forces in the solar system.

As the Chameleon theory ceases to be classical due to the large velocity at the minimum of the potential, our proposal to fix this problem should be stable against quantum corrections too. We showed that indeed the DBI Chameleon structure is stable against corrections as long as $M \ll \Lambda \ll M_{pl}$ and $H \ll \Lambda$. The hierarchy required between the 3 scales of the theory is naturally satisfied, and given the electron-positron kick occurs at $M_{pl}H \sim (\text{MeV})^2$ we must satisfy $\Lambda \gg (\text{MeV})^2/M_{pl}$. Combining both constraints on

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Λ to properly screen Standard Model kicks, we require that

$$\frac{(\text{MeV})^2}{M_{pl}} \ll \Lambda \lesssim \frac{\text{MeV}}{\sqrt{\beta}}. \quad (3.58)$$

CHAPTER 4

4 Vainshtein Screening and UV Galileons

4.1 Introduction

Galileon theories are often used in modified theories of gravity, due to the fact they offer second order field equations and also enjoy a screening of fifth forces courtesy of the Vainshtein mechanism. They often emerge in low energy descriptions of massive gravity theories or higher dimensional bulk-brane pictures. Their defining characteristic is that they are unchanged under the transformation $\phi \rightarrow \phi + b_\mu x^\mu + c$, where b_μ and c are constant. Currently Galileon theories used to explain dark energy are in tension with recent gravitational wave detections from LIGO [32–34], due to the predicted speed that gravitational waves propagate differing from GR and observation. Nevertheless they are an interesting topic of study, and more gravitation wave data is needed to conclusively rule them out at this parameter range as there is currently only one measurement. Theories that utilise the Vainshtein mechanism often suffer from a limited range of validity compared to GR, which poses a problem when one wishes to use these theories to replace GR.

This would not be a problem if one were to find a theory that had a much larger range of validity, which gave the Vainshtein mechanism in the low energy limit. Unfortunately though, there is often a problem with finding UV completions to theories which enjoy the Vainshtein mechanism. Any low energy effective theory that admits a Lorentz invariant, local, Wilsonian UV completion must satisfy positivity conditions, which were discussed in detail in Section 1.4.2, along with the fact that to get Vainshtein screening one must often violate this condition, indicating that screening of this type cannot come from a UV complete theory. It is well known that the Galileon theories can not satisfy these positivity conditions, which can be seen directly

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by calculating the $2 \rightarrow 2$ tree level scattering amplitude for the galileons and seeing that it goes like $\mathcal{A}_4 \propto s^3$, where s is the usual Mandelstam variable. The positivity conditions require that the coefficient of the s^2 part is greater than zero, something which here cannot be satisfied as it is exactly zero. This indicates that Galileons cannot be UV completed into a local QFT or string theory. There are arguments that even though Galileons come from massive gravity, these results do not extend fully to massive gravity and that there is a parameter space for dRGT to be consistent with these constraints [125], but this statement is under some debate. Recent work in [126] derives a generalised positivity constraint, which goes beyond considering analyticity and unitarity as the traditional positivity constraints do, and finds that dRGT cannot satisfy these newer constraints without being ruled out by observation. Regardless of the status of dRGT, Galileons themselves are worthy of independent study and so we focus on them in this work.

In [127] a mass term is added to the Galileon action, and its status with respect to the positivity conditions is re-examined. They argue that this is the correct way to study the positivity conditions as they are only well-defined for massive theories. One might worry that the addition of this new term in the action breaks the Galileon symmetry, but it is found that the mass term does not introduce any new operators, nor break the Galileon non-renormalisation theorem, which states that loops of the Galileon field do not renormalise the Galileon interactions at any loop order [128]. These two results make it very natural to consider this extension to the Galileons as it does not ruin any of the desirable properties the theory has. It was also found that this minimal extension to the Galileon theory does pass the positivity constraints, indicating that the massive Galileon can be UV completed.

Further to showing that the massive Galileons pass the positivity conditions, they propose a simple UV completion which is found by introducing a single new scalar field, H . The action of this UV complete massive galileon

theory is given by

$$\mathcal{S}_{UV}[\phi, H] = \int d^4x \left[-\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}(\partial H)^2 - \alpha H \square\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{2}M_H^2 H^2 - \frac{\lambda}{4!}H^4 \right] \quad (4.1)$$

where ϕ is the galileon field, H is some heavy field of mass $M_H > m$ and λ, α are dimensionless coupling coefficients. We must impose $|\alpha| < 1$ in order to avoid ghost instabilities [127]. One can see that this action will transform in the same way as the standard Galileon with a mass term added under the Galileon transformation of $\phi \rightarrow \phi + b_\mu x^\mu + c$, verifying it is indeed a theory of a massive Galileon.

Having found a Galileon theory which is UV complete, we now wish to ask whether this theory demonstrates Vainshtein screening in the same way standard Galileons do.

4.2 UV Complete Massive Galileon

4.2.1 Low Energy Effective Theory

We begin our analysis of this theory by integrating out the heavy field H , to construct a low energy effective theory of the galileon field. The Wilsonian effective action, $\mathcal{S}_W[\phi]$, is defined using the path integral

$$e^{i\mathcal{S}_W[\phi]} = \int \mathcal{D}H \left[e^{i\mathcal{S}_{UV}[\phi, H]} \right]. \quad (4.2)$$

We calculate the effective action as an expansion in loops of the heavy field, and so start at tree-level. The equation of motion of the heavy field H is given by

$$\square H = \alpha \square\phi + M_H^2 H + \frac{\lambda}{3!}H^3. \quad (4.3)$$

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We can use this recursively to define H as a function of the galileon field ϕ , and find that

$$H = -\frac{1}{1 - \frac{\square}{M_H^2}} \left(\frac{\alpha}{M_H^2} \square \phi - \frac{\lambda \alpha^3}{3! M_H^8 \left(1 - \frac{\square}{M_H^2}\right)^3} (\square \phi)^3 \right) + \mathcal{O}\left(\frac{1}{M_H^{10}}\right). \quad (4.4)$$

At this point we will ignore the higher order contributions, as we wish to only consider terms in our effective action up to a certain power in inverse M_H . We hope this will be a sensible thing to do because our aim is to construct an effective theory valid at scales less than M_H , and all the higher order operators will be suppressed by higher inverse powers of this scale. We will check that this is a consistent thing to do later in this section. Using (4.4) to integrate out the H field we find our tree level contribution to the effective action to be

$$\begin{aligned} \mathcal{S}_W^{(0)}[\phi] = \int d^4x \left[-\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 + \frac{\alpha^2}{2} \left(\frac{\phi \square^2 \phi}{M_H^2} + \frac{\phi \square^3 \phi}{M_H^4} \right. \right. \\ \left. \left. + \frac{\phi \square^4 \phi}{M_H^6} + \frac{\phi \square^5 \phi}{M_H^8} \right) - \frac{\lambda \alpha^4}{4! M_H^8} (\square \phi)^4 \right] + \mathcal{O}(M_H^{-10}). \end{aligned} \quad (4.5)$$

Here we can see explicitly that all of the interactions are invariant under the Galileon transformation, with only the mass term breaking the symmetry as discussed previously.

If we now consider the quantum corrections to (4.5), the one heavy field loop contributions will feature as [127, 129]

$$\mathcal{S}_W^{(1)}[\phi] = \frac{i\hbar}{2} \text{Tr} \ln \left[-\square + M_H^2 + \frac{\lambda}{2} H^2 \right]. \quad (4.6)$$

If we expand this to the same order in inverse M_H as our tree level action, we find that M_H and λ are corrected and a constant term introduced, but no new operators are generated. If one goes beyond M_H^{-8} then new operators such as $\phi^2 \square^n \phi^2 / M_H^{8+2n}$ will be generated. We note that this also matches

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the form of the one loop corrected effective action found in [127]. This is true of all higher loop corrections too, no new operators can be generated below order M_H^{-10} due to the galileon symmetry and quartic self-coupling of the heavier field.

As we are interested in whether this particular theory can screen fifth forces, we will now consider an additional interaction that couples the galileon field ϕ to the trace of the stress energy tensor, in the usual way, by the interaction $\mathcal{L}_W[\phi] \supset \phi T/M_{pl}$. This does not break the shift symmetry but does break the Galileon symmetry. Due to the gravitational strength coupling though, any corrections will be Plank suppressed, which can be ignored as $M_H \ll M_{pl}$. Doing so results in the following equation of motion

$$\square\phi = m^2\phi - \alpha^2 \left(\frac{\square^2\phi}{M_H^2} + \dots + \frac{\square^5\phi}{M_H^8} \right) + \frac{\lambda\alpha^4}{3!M_H^8} \square(\square\phi)^3 - \frac{T}{M_{pl}}. \quad (4.7)$$

To obtain an estimate of if and at what scale screening will occur, we follow the procedure described in Section 1.4.2 and find the scale at which the linearised theory breaks down around a static, spherical body of mass M_S . Linearising the equation of motion and requiring the scalar field asymptotically vanishes, we find that it takes on the expected form from (1.12),

$$\phi_{\text{lin}}(r) = -\frac{M_S}{4\pi M_{pl}} \frac{e^{-mr}}{r}. \quad (4.8)$$

We note that there are other solutions to the linearised version of (4.7), but we have ignored them here as they are tachyonic and also exist at the cut-off of our theory, M_H , giving $\phi_{\text{lin}} \sim \exp[iM_H r]/r$. This solution is un-physical and is simply telling us that our theory requires the introduction of new physics at this scale, something we already knew as this is the low energy effective theory of a more complete theory given by (4.1). Evaluating the size of the higher order linear terms on this linear solution ϕ_{lin} relative to

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the kinetic term,

$$\frac{\alpha^2}{M_H^2} \frac{\Box^2 \phi}{\Box \phi} \Big|_{\phi=\phi_{\text{lin}}} = \frac{\alpha^2 m^2}{M_H^2}. \quad (4.9)$$

This is always very small given that $\alpha \sim \mathcal{O}(1)$ and $m \ll M_H$, and as it has no coordinate dependence it tells us that the higher order linear interactions will not cause a Vainshtein radius to exist as they will always be dominated by the simple free field behaviour. The other terms of the form $\Box^n \phi$ will yield the same results also, as the derivatives come with inverse powers of M_H , ensuring the ratio of terms always comes as $(m/M_H)^n$. We can now check the non-linear term,

$$\begin{aligned} \frac{\lambda \alpha^4}{3! M_H^8} \frac{\Box(\Box \phi)^3}{\Box \phi} \Big|_{\phi=\phi_{\text{lin}}} &= \frac{\lambda \alpha^4 m^4}{3! M_H^8} \frac{\Box(\phi^3)}{\phi} \\ &\sim \lambda \alpha^4 \frac{m^4}{M^8} \frac{M_S^2}{16\pi^2 M_{pl}^2} r^{-4}, \end{aligned} \quad (4.10)$$

where we have assumed that $mr < 1$. The fact that this expression has some coordinate dependence means that there is some radius, the Vainshtein radius, at which the non-linear interactions become significant to the dynamics of the galileon field ϕ . While this does not prove that the theory screens, it does indicate that it is possible as some non-trivial self interactions will dominate over the free-field behaviour which gives rise to a fifth force as shown in Section 1.4. The expression in (4.10) indicates that the Vainshtein radius is given by

$$r_V \sim \lambda^{\frac{1}{4}} \alpha \frac{m}{M_H^2} \sqrt{\frac{M_S}{4\pi M_{pl}}}. \quad (4.11)$$

It is at this point that we will now consider whether ignoring the operators that come at higher order than $\mathcal{O}(M_H^{-8})$ is a consistent thing to do at these scales. Raising the truncation to find the next operator, we find that

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a single new non-linear interaction is now included in (4.5),

$$\mathcal{L}_W[\phi] \supset -\frac{\lambda\alpha^4}{3!M_H^{10}}\Box^2\phi(\Box\phi)^3, \quad (4.12)$$

along with a higher derivative linear term. This will change the equation of motion of the scalar field to

$$\begin{aligned} \Box\phi = m^2\phi - \alpha^2 \left(\frac{\Box^2\phi}{M_H^2} + \dots + \frac{\Box^6\phi}{M_H^{10}} \right) + \frac{\lambda\alpha^4}{3!M_H^8}\Box(\Box\phi)^3 \\ + \frac{\lambda\alpha^4}{2M_H^{10}}\Box(\Box^2\phi(\Box\phi)^2) + \frac{\lambda\alpha^4}{3!M_H^{10}}\Box^2(\Box\phi)^3 - \frac{T}{M_{pl}}. \end{aligned} \quad (4.13)$$

We already know the scale at which the first non-linear operator becomes important, from (4.11), so to ensure that ignoring the higher order interactions is a consistent thing to do we should make sure that they are less significant to the dynamics of the scalar field at that scale. Performing a similar calculation to before, we can check the size of the higher order operators relative to the non-linear interaction we studied earlier. The new terms in our equation of motion give

$$\frac{\Box^2(\Box\phi)^3}{M_H^2\Box(\Box\phi)^3} \Big|_{\phi=\phi_{\text{lin}}} \sim \frac{1}{(M_H r)^2}, \quad (4.14)$$

and

$$\frac{3\Box(\Box^2\phi(\Box\phi)^2)}{M_H^2\Box(\Box\phi)^3} \Big|_{\phi=\phi_{\text{lin}}} \sim \left(\frac{m}{M_H} \right)^2. \quad (4.15)$$

We have not bothered here to calculate the size of the higher order linear operator, as we previously argued that it will never become dominant over the linear behaviour. These results look promising, as (4.14) suggests that the higher order term is small as long as $r > M_H$ which is the range of r we expect this low energy theory to be valid for, and (4.15) is small for all values of r , implying that we can indeed ignore these higher order operators. Sadly though, this property is not generic to all of the higher order operators. One such example can be found by looking at interactions that are higher order

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in λ . At $\mathcal{O}(\lambda^3)$, one finds that the operator

$$\frac{\lambda^3 \alpha^{10}}{9M_H^{24}} (\Box \phi)^2 (\Box (\Box \phi)^3)^2 \quad (4.16)$$

is present in the action. This features at a much higher inverse power of M_H , so we hope that it is very suppressed relative to the operator at $\mathcal{O}(M_H^{-8})$ at $r \sim r_V$. Calculating its contribution to the equations of motion and then comparing its size to the non-linear operator from our truncated action,

$$\frac{4\lambda^2 \alpha^6}{3M_H^{16}} \frac{\Box \left[\Box \phi (\Box (\Box \phi)^3)^2 \right]}{\Box (\Box \phi)^3} \Big|_{\phi=\phi_{\text{lin}}} \sim 880 \lambda^2 \alpha^6 \left(\frac{m}{M_H^2} \right)^8 \left(\frac{M_S}{4\pi M_{pl}} \right)^4 r^{-8}. \quad (4.17)$$

The radius at which this ratio becomes $\mathcal{O}(1)$, and thus the scale at which the higher order operator becomes comparable to the non-linear operator in (4.5), is then

$$r \sim (\lambda \alpha^6)^{\frac{1}{8}} \frac{m}{M_H^2} \sqrt{\frac{M_S}{4\pi M_{pl}}}. \quad (4.18)$$

Given that λ and α are both $\mathcal{O}(1)$ parameters, this expression is the same as (4.11), the scale at which the M_H^{-8} operator becomes important. This tells us that we cannot truncate our low energy action, as once the first non-linear interaction becomes important to the field dynamics, higher order operators will also become important. As integrating out the heavy field will lead to an infinite number of operators, and operators at all orders will generically become important when entering the non-linear regime, we cannot use this method to make any predictions about whether the scalar field will screen. The silver lining is that there is definitely a distance scale at which the non-linear interactions will take over from the linear behaviour of the scalar, and so it is possible that screening could occur because of this.

4.2.2 Full Theory

Having found that the low energy theory cannot give us predictions on the behaviour of the scalar field in the non-linear regime, we now move back to the full description given by the action in (4.1). The equations of motion of the full theory are given by

$$\square H = \alpha \square \phi + M_H^2 H + \frac{\lambda}{3!} H^3, \quad (4.19)$$

$$\square \phi = \alpha \square H + m^2 \phi. \quad (4.20)$$

We cannot solve the equations (4.19) and (4.20) fully, and so to obtain an analytic form for the solution to this system of equations, we solve perturbatively in λ , such that $H = H_0 + \lambda H_1 + \dots$ and $\phi = \phi_0 + \lambda \phi_1 + \dots$. Linearising this system leaves us with the following equations

$$\square H_0 - M_H^2 H_0 - \alpha \square \phi_0 = 0, \quad (4.21)$$

$$\square \phi_0 - m^2 \phi_0 - \alpha \square H_0 = \frac{\rho}{M_{pl}}, \quad (4.22)$$

where we have set $T = -\rho$ as we have modelled our source as a spherically symmetric, non-relativistic body of density ρ and mass M_S . Solving these equations of motion we find that for a source of the form $\rho(r) = \rho_0 \theta(w - r)$, where $\theta(r)$ is the Heaviside step function and w the radius of the source, the fields take the form

$$\begin{aligned} \phi_0(r)_{r < w} = & -\frac{\rho_0}{m^2 M_{pl}} + \frac{\rho_0}{\tilde{m}^2 M_{pl}} \frac{e^{-\mu_+ w} (1 + \mu_+ w)}{\mu_+} \frac{(m^2 - M^2 + \tilde{m}^2)}{2m^2} \frac{\sinh(\mu_+ r)}{r} - \\ & \frac{\rho_0}{\tilde{m}^2 M_{pl}} \frac{e^{-\mu_- w} (1 + \mu_- w)}{\mu_-} \frac{(m^2 - M^2 - \tilde{m}^2)}{2m^2} \frac{\sinh(\mu_- r)}{r}, \end{aligned} \quad (4.23)$$

$$\begin{aligned} \phi_0(r)_{r>w} = & -\frac{m^2 - M^2 + \tilde{m}^2}{2m^2\alpha} \left(\frac{\rho_0\alpha}{\tilde{m}^2 M_{pl}} \right) \left(w \cosh(\mu_+ w) - \frac{\sinh(\mu_+ w)}{\mu_+} \right) \frac{e^{-\mu_+ r}}{r} + \\ & \frac{m^2 - M^2 - \tilde{m}^2}{2m^2\alpha} \left(\frac{\rho_0\alpha}{\tilde{m}^2 M_{pl}} \right) \left(w \cosh(\mu_- w) - \frac{\sinh(\mu_- w)}{\mu_-} \right) \frac{e^{-\mu_- r}}{r}, \end{aligned} \quad (4.24)$$

$$H_0(r)_{r<w} = \frac{\rho_0\alpha}{\tilde{m}^2 M_{pl}} \left(\frac{e^{-\mu_+ w}(1 + \mu_+ w)}{\mu_+} \frac{\sinh(\mu_+ r)}{r} - \frac{e^{-\mu_- w}(1 + \mu_- w)}{\mu_-} \frac{\sinh(\mu_- r)}{r} \right), \quad (4.25)$$

$$\begin{aligned} H_0(r)_{r>w} = & -\frac{\rho_0\alpha}{\tilde{m}^2 M_{pl}} \left(w \cosh(\mu_+ w) - \frac{\sinh(\mu_+ w)}{\mu_+} \right) \frac{e^{-\mu_+ r}}{r} + \\ & \frac{\rho_0\alpha}{\tilde{m}^2 M_{pl}} \left(w \cosh(\mu_- w) - \frac{\sinh(\mu_- w)}{\mu_-} \right) \frac{e^{-\mu_- r}}{r}, \end{aligned} \quad (4.26)$$

where $\tilde{m}^2 = \sqrt{m^4 + M^4 + 2m^2 M^2(2\alpha^2 - 1)}$ and $\mu_{\pm} = \sqrt{\frac{m^2 + M^2}{2(1 - \alpha^2)} \pm \frac{\tilde{m}^2}{2(1 - \alpha^2)}}$.

To solve the equations we imposed that the fields asymptotically vanish and that the first derivatives of the fields at $r = 0$ also vanish. We can see that exterior to the source, both fields have the characteristic $\exp[mr]/r$ form, giving rise to extra forces that we expect from the introduction of scalar fields coupled to matter.

We can use these solutions to the linear system and the same boundary conditions to find the next order in λ contribution by now solving

$$\square H_1 - M_H^2 H_1 - \alpha \square \phi_1 - \frac{\lambda}{6} H_0^3 = 0, \quad (4.27)$$

$$\square \phi_1 - m^2 \phi_1 - \alpha \square H_1 = 0. \quad (4.28)$$

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The solutions on the exterior of the source, such that $r > w$, are given by

$$\begin{aligned} \phi_1(r)_{r>w} = & \frac{m^2 - M^2 + \tilde{m}^2}{2m^2\alpha} \frac{\lambda}{12} \frac{(M^2 - m^2(1 - 2\alpha^2) + \tilde{m}^2)}{(\alpha^2 - 1)\tilde{m}^2\mu_+} [\text{int}_2(r, \mu_+)] + \\ & \frac{m^2 - M^2 - \tilde{m}^2}{2m^2\alpha} \frac{\lambda}{12} \frac{(m^2(1 - 2\alpha^2) - M^2 + \tilde{m}^2)}{(\alpha^2 - 1)\tilde{m}^2\mu_-} [\text{int}_2(r, \mu_-)], \end{aligned} \quad (4.29)$$

$$\begin{aligned} H_1(r)_{r>w} = & \frac{\lambda}{12} \frac{(M^2 - m^2(1 - 2\alpha^2) + \tilde{m}^2)}{(\alpha^2 - 1)\tilde{m}^2\mu_+} [\text{int}_2(r, \mu_+)] + \\ & \frac{\lambda}{12} \frac{(m^2(1 - 2\alpha^2) - M^2 + \tilde{m}^2)}{(\alpha^2 - 1)\tilde{m}^2\mu_-} [\text{int}_2(r, \mu_-)], \end{aligned} \quad (4.30)$$

where

$$\begin{aligned} \text{int}_2(r, \mu_0) = & \frac{e^{-\mu_0 r}}{r} \left(\int_0^w s * \sinh(\mu_0 s) \left(A \frac{\sinh(\mu_+ s)}{s} + B \frac{\sinh(\mu_- s)}{s} \right)^3 ds + \right. \\ & \left. \int_w^r s * \sinh(\mu_0 s) \left(C \frac{e^{-\mu_+ s}}{s} + D \frac{e^{-\mu_- s}}{s} \right)^3 ds \right) + \\ & \frac{\sinh(\mu_0 r)}{r} \int_r^\infty s e^{-\mu_0 s} \left(C \frac{e^{-\mu_+ s}}{s} + D \frac{e^{-\mu_- s}}{s} \right)^3 ds \end{aligned} \quad (4.31)$$

and we have defined the following constants

$$A = \frac{\rho_0 \alpha}{\tilde{m}^2 M_{pl}} \frac{e^{-\mu_+ w} (1 + \mu_+ w)}{\mu_+} \quad (4.32)$$

$$B = -\frac{\rho_0 \alpha}{\tilde{m}^2 M_{pl}} \frac{e^{-\mu_- w} (1 + \mu_- w)}{\mu_-} \quad (4.33)$$

$$C = -\frac{\rho_0 \alpha}{\tilde{m}^2 M_{pl}} \left(w \cosh(\mu_+ w) - \frac{\sinh(\mu_+ w)}{\mu_+} \right) \quad (4.34)$$

$$D = \frac{\rho_0 \alpha}{\tilde{m}^2 M_{pl}} \left(w \cosh(\mu_- w) - \frac{\sinh(\mu_- w)}{\mu_-} \right). \quad (4.35)$$

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Inside the source, we find

$$\begin{aligned} \phi(r)_{r < w}^{(1)} = & \frac{m^2 - M^2 + \tilde{m}^2}{2m^2\alpha} \frac{\lambda}{12} \frac{(M^2 - m^2(1 - 2\alpha^2) + \tilde{m}^2)}{(\alpha^2 - 1)\tilde{m}^2\mu_+} [\text{int}_1(r, \mu_+)] + \\ & \frac{m^2 - M^2 - \tilde{m}^2}{2m^2\alpha} \frac{\lambda}{12} \frac{(m^2(1 - 2\alpha^2) - M^2 + \tilde{m}^2)}{(\alpha^2 - 1)\tilde{m}^2\mu_-} [\text{int}_1(r, \mu_-)], \end{aligned} \quad (4.36)$$

$$\begin{aligned} H(r)_{r < w}^{(1)} = & \frac{\lambda}{12} \frac{(M^2 - m^2(1 - 2\alpha^2) + \tilde{m}^2)}{(\alpha^2 - 1)\tilde{m}^2\mu_+} [\text{int}_1(r, \mu_+)] + \\ & \frac{\lambda}{12} \frac{(m^2(1 - 2\alpha^2) - M^2 + \tilde{m}^2)}{(\alpha^2 - 1)\tilde{m}^2\mu_-} [\text{int}_1(r, \mu_-)], \end{aligned} \quad (4.37)$$

where

$$\begin{aligned} \text{int}_1(r, \mu_0) = & \frac{e^{-\mu_0 r}}{r} \int_0^r s * \sinh(\mu_0 s) \left(A \frac{\sinh(\mu_+ s)}{s} + B \frac{\sinh(\mu_- s)}{s} \right)^3 ds + \\ & \frac{\sinh(\mu_0 r)}{r} \left(\int_r^w s e^{-\mu_0 s} \left(A \frac{\sinh(\mu_+ s)}{s} + B \frac{\sinh(\mu_- s)}{s} \right)^3 ds + \right. \\ & \left. \int_r^\infty s e^{-\mu_0 s} \left(C \frac{e^{-\mu_+ s}}{s} + D \frac{e^{-\mu_- s}}{s} \right)^3 ds \right) \end{aligned} \quad (4.38)$$

and A, B, C and D have the same definitions as above. These are clearly much more complicated expressions than the ones at linear order in λ , and indicate that solving to even higher orders by hand will be a difficult task. Instead, we will take a numerical approach to solve the full non-linear equations of motion, and use the perturbative solutions in λ to ensure that our numerical method is accurate in the small λ regime. This will be discussed in Section 4.3, before that though we can glean some useful information from looking at these perturbative solutions.

Figure 4.1 shows a plot of the perturbative solution for the light field outside of the source, as shown in (4.24) and (4.29). Parameter values were

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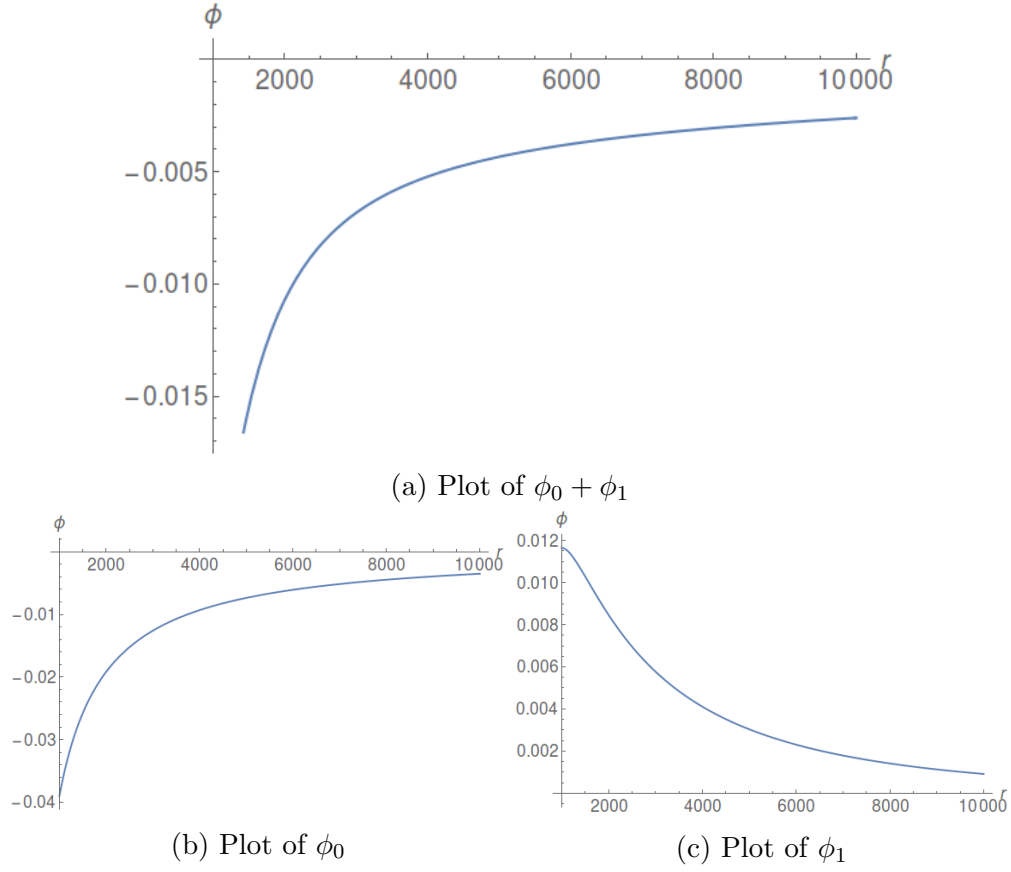


Figure 4.1: This is a plot of (4.24) and (4.29). Numerical parameters: $M_{pl} = 1$, $M_H = 10^{-4}$, $m = 10^{-6}$, $\alpha = 0.4$, $\lambda = 0.1$, $\rho_0 = 10^{-7}$, $w = 10^3$.

selected such that higher order in λ contributions are small and thus can be ignored. This of course means that the field is in a close-to-linear regime, which can clearly be seen by the fact that the $\mathcal{O}(\lambda)$ contribution is much smaller than the ϕ_0 one. This perturbative approach will break down if the field is in a highly non-linear regime as each higher order in λ contribution would be of the same order, requiring one to calculate to all orders. As screening of the scalar force is something that would require the nonlinearities to play a significant role, we will not see it using this perturbative approach, but nonetheless there are promising signs here. Figure 4.1c shows us that ϕ_1 has the opposite sign to ϕ_0 , which indicates that it will suppress

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the force the scalar exerts even if only a little bit for these particular parameters. One would hope that in the full non-linear solution that this property remains but its magnitude is increased, leading to significant screening.

We should be careful to make sure that the mass of the light field does not receive any corrections from loops; if it receives large corrections from the heavy field then it loses all usability as a theory that could screen fifth forces while providing a cosmological scale effect. It is clear to see that this is not the case, when integrating out the heavy field in (4.5) we can see that the theory respects Galileon symmetry if $m = 0$. This is also the case if one includes every operator, instead of truncating to some power in inverse M_H as was done in the previous section. This implies that any corrections to the mass will be proportional to m , and hence it is a technically natural parameter. We can see this explicitly via calculation with the full theory of two scalars too. We redefine our fields to diagonalise the kinetic and mass terms, such that the action is now written as

$$\mathcal{S}_d = \int d^4x \left[-\frac{1}{2} (\partial \tilde{\phi})^2 - \frac{1}{2} (\partial \tilde{H})^2 - \frac{1}{2} \tilde{m}^2 \tilde{\phi}^2 - \frac{1}{2} \tilde{M}^2 \tilde{H}^2 - \frac{\tilde{\lambda}}{4!} (\tilde{H} + \beta \tilde{\phi})^4 \right], \quad (4.39)$$

where $\tilde{m} \sim m$, $\tilde{M}^2 \sim M_H^2/(1 - \alpha)$, $\tilde{\phi} = \phi - \alpha H$, $\tilde{H} \sim \sqrt{1 - \alpha^2} H$, and $\beta \sim -\alpha \sqrt{1 - \alpha^2} m^2 / M_H^2$. For full details of the transformations used to write the action in this way, we refer the reader to the Appendix in [127]. Calculating the 1-loop contribution to the action in a similar way as in (4.6), we use the fact that the one-loop contributions are given by

$$\mathcal{S}_{d,1} = \frac{i\hbar}{2} \text{Tr} \ln \left[-\frac{\delta^2 \mathcal{S}_d}{\delta \Sigma^2} \right], \quad (4.40)$$

where $\Sigma = (\tilde{\phi}, \tilde{H})$. Manipulating the above expression, one finds that the correction to the mass of the light field is $\Delta \tilde{m}^2 \propto \tilde{m}^4 / \tilde{M}^2$. This again demonstrates that the corrections to the mass are $\mathcal{O}(\tilde{m})$, but also shows that the corrections are multiplied by $\tilde{m}/\tilde{M}$ which is small itself. It is not

particularly surprising that these corrections to the light field's mass are small, consider the inter-field coupling in (4.39). The coefficient is very small as it is proportional to m^2/M_H^2 , and it is this weak coupling between fields that means the mass corrections are small.

4.3 Numerical Analysis

To solve the equations of motion from the full theory shown in (4.19) and (4.20) we resort to numerical methods. The coupled non-linear differential equations of (4.19) and (4.20) are difficult to solve, and so we employ a numerical method which iteratively solves a linear problem to find the solution to the non-linear problem. To give an overview of our method, this is achieved by re-writing our pair of differential equations in the following way,

$$\square H = \alpha \square \phi + M_H^2 H + \frac{\lambda}{3!} H_k^3, \quad (4.41)$$

$$\square \phi = \alpha \square H + m^2 \phi, \quad (4.42)$$

where we have replaced the non-linear H^3 term with H_k^3 , which we initially set to zero. This leaves the problem identical to the linear problem, which is easily solved. We then set H_k to that linear solution, and re-solve the equations to find an approximation to the full non-linear solution. We repeat this procedure of using our previous solution as H_k and re-solving the equations until the solution converges and no-longer changes significantly between iterations. To check that what we are left with is indeed a solution to the full non-linear differential equations we then evaluate the equations on our solution, making sure they are satisfied down to small numerical error. This is identical in methodology to solving the equations of motion order by order in λ , as was done analytically to first order.

This is currently still a work in progress, and so we cannot yet make any definitive statements about the final results of our numeric analysis. Instead we present our current preliminary findings. Figures 4.2 and 4.3 show the

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solutions to the full equations of motion given the boundary conditions that both fields asymptotically vanish and their first derivatives at $r = 0$ vanish, for different values of λ . We calculate the $\lambda = 0$ solution so that we can see it matches precisely with the analytic form of the linear solution shown in (4.23) - (4.26).

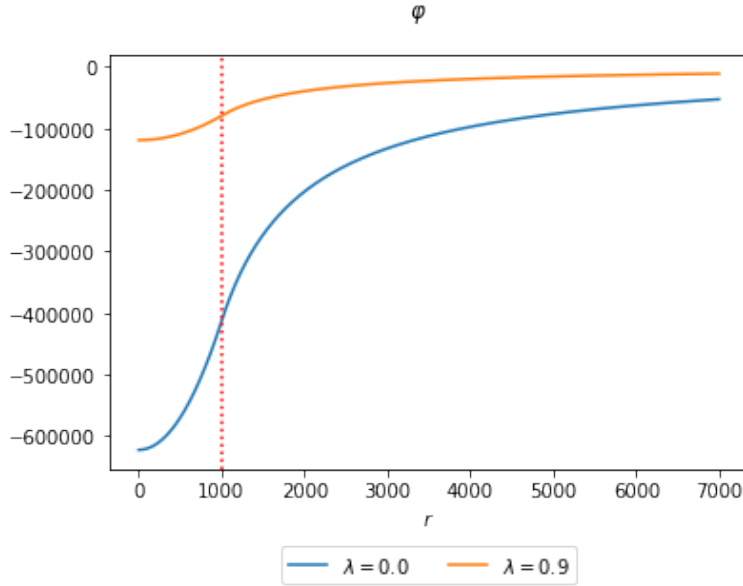


Figure 4.2: A plot of ϕ against r , for different λ values. Dashed red line indicates boundary of source. Numerical parameters: $M_{pl} = 1$, $M_H = 10^{-5}$, $m = 10^{-8}$, $\alpha = 0.9$, $\rho_0 = 1$, $w = 10^3$. Figure by Daniela Saadeh.

As we are interested in screening of the force associated with the lighter field, Figure 4.2 is of particular interest to us. It is apparent by looking at the two curves that the solution for $\lambda = 0.9$, a value for which the non-linearities will play a significant role, causes a ‘flattening’ of the profile. This will mean that the derivative of ϕ has been reduced by the non-linear interactions being switched on, and hence the force it will impart reduced. Indeed, Figure 4.2 would indicate that the fifth force is being successfully screened for these parameter values. It is interesting to note that for this parameter selection that causes seemingly effective screening for the ϕ field,

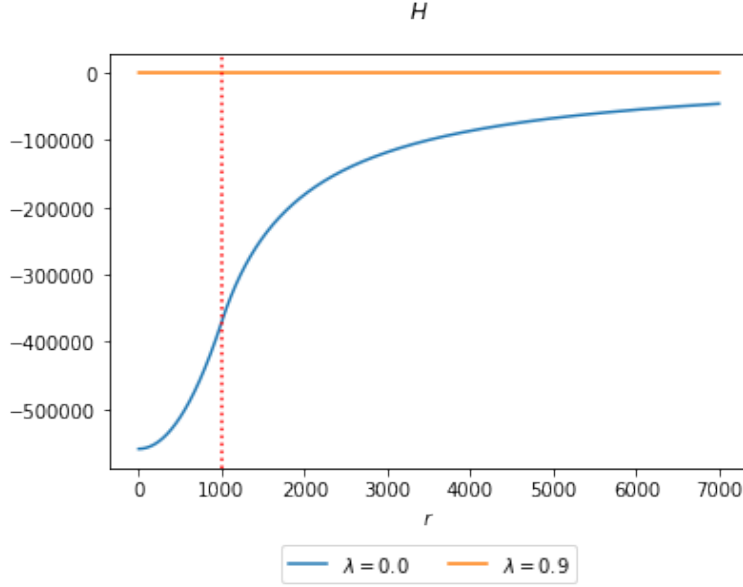


Figure 4.3: A plot of H against r , for different λ values. Dashed red line indicates boundary of source. Numerical parameters: $M_{pl} = 1$, $M_H = 10^{-5}$, $m = 10^{-8}$, $\alpha = 0.9$, $\rho_0 = 1$, $w = 10^3$. Figure by Daniela Saadeh.

the heavy field H remains effectively constant and close to zero for the entire range of r values.

Obviously demonstrating that the force can be screened for one particular choice of parameters which are not entirely physically relevant is not the final objective of this work. By improving the numerical process we hope to be able to solve the system for more physically relevant parameters to see if the force can be screened on solar system scales with a source that looks like the Sun, which will conclusively tell us if this screening mechanism can work effectively for cosmological purposes.

4.4 Summary

In this chapter we have presented our initial findings in a study to see if a recently proposed UV complete massive Galileon theory exhibits screening of fifth forces on solar system scales. A typical Galileon theory can screen,

but resists UV completion due to the fact it cannot satisfy the positivity constraints required of a theory to have a Lorentz invariant and local Wilsonian completion. By adding a mass term to the Galileon theory, one does not generate any new operators or renormalise any existing ones, meaning that it is a very natural extension to consider. Doing so allows the theory to satisfy the positivity bounds, indicating that UV completion is possible.

One particular UV complete massive Galileon theory can be obtained by the addition of an extra, heavy scalar field coupled to the Galileon, proposed in [127]. Coupling this Galileon field to matter in a minimal way, to the trace of the stress-energy tensor, we investigate whether this theory can screen the extra force mediated by the Galileon, thus providing an example of a theory which can screen fifth forces while retaining a large domain of predictability.

We provide some analytic work that suggests that screening is possible in this theory, and reinforce this with numerical solutions to the equations of motion which demonstrate a clear reduction in the magnitude of the gradient of the Galileon field, and hence the force it mediates, for some parameter values.

As this work is still currently ongoing we cannot comment on the ability of the theory to screen the force over solar system scales, but our main goal is to answer this question. To this end our numerical method is being refined such that we can obtain results for physically relevant parameters, allowing us to make statements about solar system scale behaviours of the theory.

CHAPTER 5

5 Final Words

5.1 Summary

The work in this thesis focuses on the behaviour of scalar field theories, and their use in a modified gravity context. We begin this by discussing the motivations behind modifying GR, and why scalar fields are often used in this context. Introducing light scalar fields also introduces new long range forces with gravitational strength which we do not observe locally, and so in Section 1.4 we discuss popular screening mechanisms which mask these forces at certain scales. These are employed frequently to hide extra forces in solar system scales but allow the scalar to have a non-trivial effect on cosmological scales.

Moving on to Chapter 2, we begin by reviewing work in the literature that categorises a family of scalar field theories that have a shift symmetry by their scattering amplitudes' soft behaviour. This family of theories consists of only those whose interactions all have the same number of derivatives per field, schematically this means all the interactions go like $\partial^m \phi^n$ where m/n is the same across the board. Soft limits of shift symmetric scalar fields depend on the soft momentum as $\mathcal{A} = \mathcal{O}(p)$, but some theories benefit from an enhancement of this relation to higher powers, such as DBI having soft limits of $\mathcal{A} = \mathcal{O}(p^2)$. These enhancements to the soft behaviour of the amplitudes are caused by extra symmetries present in the theory, and so looking for theories with enhanced soft limits proves to be a new and interesting technique to find scalar field theories that are highly symmetric. We then present our own original work, which generalises this idea to a larger class of theories where interactions can now have different values of m/n . Having built the machinery we then focus in particular on a theory with two

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values for m/n , one which contains many physically interesting subsets such as the DBI, Galileon and Horndeksi theories, with the aim of asking if the so called special Galileon theory is unique in that it has a symmetry more extensive than the general Galileon theory itself. We show that the special Galileon is indeed the unique theory that has maximal enhancement to its soft amplitudes, demonstrating its uniqueness in the family of Horndeski theories.

In Chapter 3 we begin by reviewing the Chameleon mechanism and the so called surfer solution, a problematic behaviour the Chameleon demonstrates around Big Bang Nucleosynthesis for a large range of parameter space. In the early universe all matter is relativistic, and as the universe expands the matter will cool. Eventually this matter will become non-relativistic, and when this happens the Chameleon field will experience a ‘kick’ from these matter fields. This drives the Chameleon field towards the minimum of its potential very quickly, causing a rapid change in the effective mass of the chameleon and hence excites highly energetic modes, breaking the classical treatment of the Chameleon. We present a new way of seeing the emergence of this behaviour from the Chameleon theory, and also demonstrate its attractive nature, indicating this is a problem that requires an active solution. We then present a modification to the Chameleon theory called the DBI Chameleon. This modification leaves the low energy properties of the Chameleon theory untouched, retaining the desirable screening behaviour, while modifying the Chameleon’s dynamics at high energy scales. We demonstrate that this theory only permits the surfer solution in particular limits, which decouple the modifications we introduce or strongly couple matter to the Chameleon. We then demonstrate that the modification also reduces the field space the Chameleon will move over during the kicks, and find that there is a natural choice of parameters that remove the problem of large field excursions for Standard Model matter. We also show that this theory forms a good effective theory in this regime by noting that all corrections remain small during the evolution of the system from BBN.

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In Chapter 4 we begin by noting that theories which demonstrate Vainshtein screening often struggle with low cutoffs and an inability to be UV completed in a standard Wilsonian way, by failing to satisfy the positivity conditions. Galileons are one such example of a popular theory which screens, but cannot satisfy these conditions. We briefly review work carried out in which a mass term was added to the Galileon theory, which does break the Galileon symmetry but does not introduce any new operators nor break the Galileon non-renormalisation theorem. This small modification to the Galileon allows it to satisfy the positivity conditions, indicating UV completion is possible. A simple UV complete Galileon theory is then proposed, which involves the Galileon field and one new heavy scalar field. We discuss our work, in which we study the dynamics of this theory with the aim to study whether it screens fifth forces or not. We begin by looking at a low energy effective theory obtained by integrating out the heavy field, and note that it has a Vainshtein radius and exhibits screening of the force. We then move on to studying the full theory, and find a perturbative solution to the equations of motion. We use this result to validate a numerical method that solves the equations of motion, which demonstrates that screening of the force from the Galileon field is possible for some parameters. Work is still ongoing at the time of writing to improve our numerical procedure such that we can study whether the theory screens the forces for solar system scales and physical parameters.

5.2 Future Directions

The work presented in this thesis presents new and interesting questions, which would be prime for further study. Here we will discuss some of these questions.

In Chapter 2 we have generalised the machinery to study soft limits of scalar field theories to include a much wider family of theories, but focused on the $(1, 2)$ family of theories as it contained physically interesting theories such as the Galileons and DBI. We discussed the origin of these theories

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in the context of a Minkowski bulk with a brane embedded in it, so it seems fruitful to look at what analogue comes from considering different bulk geometries. Indeed, the Conformal Galileon and warped DBI theories come in analogy to their Minkowski relatives, but from an AdS bulk space-time. A question that is currently unanswered, which our machinery would be able to tackle, is asking if there is a special Galileon analogue to the Conformal Galileon theory; that is a subset which displays enhanced soft limits and an extended symmetry, and if DBI and Galileon equivalents are the unique way of enhancing the soft limits. The warped DBI and Conformal Galileon are examples of $(0, 1)$ and $(0, 1, 2)$ theories, which leads into another interesting question to look at; families of theories that don't have shift symmetry.

All of the work in this thesis concentrated on starting with at least a shift symmetry, which through Adler's zero ensures soft limits of $\mathcal{O}(p)$, but removing this requirement opens up the study to a wider class of theories. Without the Adler's zero one would generically expect a non-zero piece left over in the soft limit, independent of the soft momenta, but it is possible for some tunings of coefficients to give rise to special structures and enhancements. An example of this would be a theory with conformal symmetry; one would expect this to give rise to interesting structures in the soft amplitude. As mentioned previously, the special Galileon is a part of a wider family of dual theories. Extending the reach of our machinery in this way would allow these to be seen too. Exactly what the soft theorem associated with these special structures would be, and if there are other theories that demonstrate them, is currently an open question.

In Chapter 4 we study whether a UV complete theory of a massive Galileon can screen via the Vainshtein mechanism. Although the work is incomplete, preliminary results indicate it can screen for at least some range of parameter space. A further avenue of work here is to identify what causes this theory to demonstrate screening behaviour, and try to generalise it in some way to discover more theories that are UV complete and also screen fifth forces. Devising precision tests that will be able to rule out or confirm

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the presence of the fifth force is also a direction that would benefit from study. These tests range from Astrophysical tests such as so called Galaxy Cluster tests, in which the mass of a cluster is calculated through both weak lensing and X-ray observations, to lab based tests such as Atom Interferometry. For a review of many tests proposed to look for fifth forces, although presented in context of looking for the Chameleon mechanism, we refer the reader to [130, 131].

Appendices

.1 Appendix A

In this appendix we will show that any Galileon theory is equivalent to a different Galileon theory where the cubic vertex is removed. We will demonstrate this by following the argument of [113].

We begin by writing the Galileon action in the following form,

$$\mathcal{S} = \int d^4x (\partial\phi)^2 \det[\mathbf{1} + c\mathbf{\Pi}], \quad (.1)$$

where $\mathbf{\Pi} = \Pi_\mu^\nu = \partial_\mu \partial^\nu \phi$. We can go from the determinant structure in the action above to the form of the Galileon action more commonly presented in this thesis by noting that

$$\det \mathbf{M} = \frac{1}{4!} M_{\mu_1}^{\nu_1} M_{\mu_2}^{\nu_2} M_{\mu_3}^{\nu_3} M_{\mu_4}^{\nu_4} \epsilon_{\nu_1 \nu_2 \nu_3 \nu_4} \epsilon^{\mu_1 \mu_2 \mu_3 \mu_4} \quad (.2)$$

where ϵ is the antisymmetric Levi Civita symbol. Using this relation leaves us with the action

$$\mathcal{S} = \mathcal{N} \int d^4x \left[(\partial\phi)^2 \left(1 + c\mathcal{L}_1 + \frac{c^2}{2}\mathcal{L}_2 + \frac{c^3}{6}\mathcal{L}_3 \right) \right] \quad (.3)$$

where $\mathcal{N}$ is an overall numerical factor and the $\mathcal{L}_i$ are defined as

$$\mathcal{L}_1 = \square\phi, \quad (.4)$$

$$\mathcal{L}_2 = ((\square\phi)^2 - (\partial_\mu \partial_\nu \phi)^2), \quad (.5)$$

$$\mathcal{L}_3 = ((\square\phi)^3 + 2(\partial_\mu \partial_\nu \phi \partial^\mu \partial^\sigma \phi \partial^\nu \partial_\sigma \phi) - 3\square\phi(\partial_\mu \partial_\nu \phi)^2). \quad (.6)$$

There is also a total derivative term, which we have discarded. These three terms are the cubic, quartic and quintic Galileons respectively. This indeed demonstrates that the action written in (.1) is the Galileon action with each

term's coefficient related to the same parameter c . To obtain a Galileon action with 3 arbitrary coefficients we can simply sum three sets of the action together with a different c , as noted in [113]. Doing so will leave you with a Galileon theory with coefficients given by

$$d_1 = \mathcal{N} (c_1 + c_2 + c_3), \quad (.7)$$

$$d_2 = \mathcal{N} (c_1^2 + c_2^2 + c_3^2), \quad (.8)$$

$$d_3 = \mathcal{N} (c_1^3 + c_2^3 + c_3^3), \quad (.9)$$

where the subscript corresponds to which $\mathcal{L}_i$ in (.3) the coefficient will multiply.

The Galileon duality is based around transforming co-ordinates by the following

$$x^\mu \mapsto \tilde{x}^\mu = x^\mu + \theta \partial^\mu \phi \quad (.10)$$

where θ is an arbitrary coefficient, while also transforming the field as

$$\phi \mapsto \tilde{\phi} = \phi + \frac{1}{2} \theta (\partial \phi)^2. \quad (.11)$$

These transformations are undone by the following

$$\tilde{x}^\mu \mapsto x^\mu = \tilde{x}^\mu - \theta \tilde{\partial}^\mu \tilde{\phi}, \quad (.12)$$

$$\tilde{\phi} \mapsto \phi = \tilde{\phi} - \frac{1}{2} \theta \left(\tilde{\partial} \tilde{\phi} \right)^2. \quad (.13)$$

By using the duality transformations we can show that the derivative of

the scalar field maps to itself as follows,

$$\begin{aligned}
\partial_\mu \phi &= \frac{\partial \tilde{x}^\sigma}{\partial x^\mu} \tilde{\partial}_\sigma \left(\tilde{\phi} - \frac{1}{2} \theta \left(\tilde{\partial} \tilde{\phi} \right)^2 \right) \\
&= \frac{\partial \tilde{x}^\sigma}{\partial x^\mu} \tilde{\partial}_\beta \tilde{\phi} \frac{\partial x^\beta}{\partial \tilde{x}^\sigma} \\
&= \tilde{\partial}_\mu \tilde{\phi}.
\end{aligned}$$

We can use this result to also see that

$$\begin{aligned}
\tilde{\Pi}_\mu^\nu &= \tilde{\partial}_\mu \tilde{\partial}^\nu \tilde{\phi} \\
&= \tilde{\partial}_\mu \partial^\nu \phi \\
&= \Pi_\alpha^\nu \left(\frac{\partial \tilde{x}^\mu}{\partial x^\alpha} \right)^{-1} \\
&= \Pi_\alpha^\nu (\delta_\alpha^\mu + \theta \Pi_\alpha^\mu)^{-1}.
\end{aligned}$$

Now we can finally determine how the action changes under the duality,

$$\mathcal{S} \mapsto \tilde{\mathcal{S}} = \int d^4 \tilde{x} \left(\tilde{\partial} \tilde{\phi} \right)^2 \det \left[\delta_\mu^\nu + c \tilde{\Pi}_\mu^\nu \right] \quad (.14)$$

$$= \int d^4 x (\partial \phi)^2 \det \left[(\delta_\nu^\beta + \theta \Pi_\nu^\beta) (\delta_\mu^\nu + c \Pi_\sigma^\nu (\delta_\sigma^\mu + \theta \Pi_\sigma^\mu)^{-1}) \right] \quad (.15)$$

$$= \int d^4 x (\partial \phi)^2 \det \left[\delta_\mu^\beta + (\theta + c) \Pi_\mu^\beta \right]. \quad (.16)$$

We can now see that the duality transformations map a Galileon theory to another Galileon theory, but one where the value of c is changed to $c + \theta$. Summing three versions of the action as mentioned previously to obtain a general Galileon theory, and using the relation in (.16), we can see that the coefficients will be mapped to

$$d_1 = \mathcal{N} (c_1 + c_2 + c_3 + \theta), \quad (.17)$$

$$d_2 = \mathcal{N} \left((c_1 + \theta)^2 + c_2^2 + c_3^2 \right), \quad (.18)$$

$$d_3 = \mathcal{N} \left((c_1 + \theta)^3 + c_2^3 + c_3^3 \right), \quad (.19)$$

It is then trivial to select a value of θ such that $d_1 = 0$, and the quartic and quintic Galileon terms are retained.

.2 Appendix B

In this appendix we will demonstrate the derivation of the Adler's zero. We begin by supposing we have an action of a massless scalar field π given by $\mathcal{S}(\pi)$ which is invariant under the symmetry $\pi(x) \rightarrow \pi(x) + a(x)$. This symmetry ensures there will be a conserved current, given by

$$J_\mu = \frac{\delta \mathcal{L}}{\delta (\partial^\mu \pi)}. \quad (.20)$$

The Ward identity corresponding to this Noether current then tells us that

$$\langle \partial^\mu J_\mu(x) \pi(y) \rangle = -ia(y) \delta(x - y). \quad (.21)$$

We can now use the LSZ reduction formula to calculate the matrix element, which leaves us with

$$\langle 0 | J_\mu(x) | p \rangle = ip_\mu F e^{-ip \cdot x} \quad (.22)$$

where we have defined $|p\rangle$ as a state of π with momentum p_μ , which is on-shell and so satisfies $p^2 = 0$. F is a dimensionful parameter that acts as a decay constant [101, 132], and its exact form is not relevant to the derivation. This result means that the current has a non-vanishing matrix element for the transition between vacuum state and a one particle state $|p\rangle$, which further implies that there will be a pole at $p^2 = 0$ in the corresponding correlation functions.

We now turn our attention to the object $\langle 0 | \tilde{J}^\mu(p) | p_1, p_2, \dots, p_n \rangle$, where the tilde indicates a Fourier transformation, and we are not to take the momentum p associated with the current to be on-shell. We can break this down into its constituent parts and organise those into three groups: those where J_μ turns into π , where J_μ attaches to an external leg, and where J_μ attaches to an internal leg. These different contributions are shown diagrammatically in Figure .1.

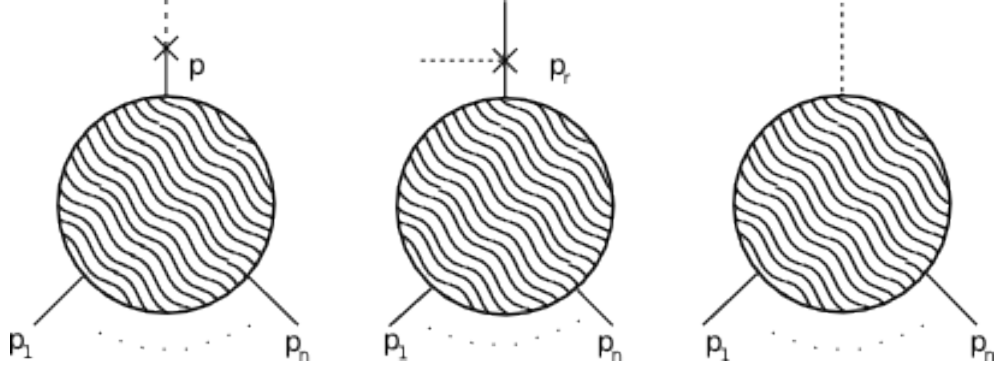


Figure 1: Schematic Feynman diagrams showing how the current can attach to a scattering process. The circle with wavy lines represents the specific structure of the scattering event, such that we do not show any internal lines. Solid external legs are π , and the dashed line represents the current. Ellipses indicate the presence of other external legs we omit for clarity.

The processes which are represented by diagrams where the current turns into π contribute as $(i/p^2)\langle 0|\tilde{J}_\mu(p)|p\rangle\langle p|S|p_1, \dots, p_n\rangle$, where S is the matrix element describing the scattering process. Similarly we can write the contribution from diagrams where the current attaches to an external leg with momentum p_i as $\mathcal{O}_\mu(p, p_i)\langle 0|S|p_1, \dots, p_n\rangle$, where $\mathcal{O}_\mu$ is some operator that describes the effect of the current attaching to a leg. This operator is assumed to be a smooth function of p , due to the PCAC hypothesis [133]. The final category of diagrams, where the current attaches to an internal line, will not contribute poles and so we will neglect their contribution from here on as we are interested in the $p^2 \rightarrow 0$ limit in which they will vanish. Combining all of this, we find that

$$\langle 0|-ip^\mu \tilde{J}_\mu(0)|p_1, \dots, p_n\rangle = \lim_{p \rightarrow 0} \left(\langle 0|p^\mu \tilde{J}_\mu(p)|p\rangle\langle p|S|p_1, \dots, p_n\rangle - ip^\mu \mathcal{O}_\mu(p, p_i)\langle 0|S|p_1, \dots, p_n\rangle \right). \quad (.23)$$

We can simplify this using (.21) to see that

$$\langle 0|p^\mu \tilde{J}_\mu(0)|p_1, \dots, p_n\rangle = \lim_{p \rightarrow 0} [\tilde{a}(p)\langle p|S|p_1, \dots, p_n\rangle + p^\mu \mathcal{O}_\mu(p, p_i)\langle 0|S|p_1, \dots, p_n\rangle]. \quad (.24)$$

One can use the Ward identity for $n = 2$ and the LSZ formula to find that $\langle f | \partial^\mu J_\mu | i \rangle = 0$, where $|i\rangle$ and $|f\rangle$ are initial and final states respectively. This is simply current conservation, and so we do not present its derivation here. We can however Fourier transform this and substitute into (.24) to find that

$$\lim_{p \rightarrow 0} [\tilde{a}(p) \langle p | S | p_1, \dots, p_n \rangle + p^\mu \mathcal{O}_\mu(p, p_i) \langle 0 | S | p_1, \dots, p_n \rangle] = 0. \quad (.25)$$

As we have assumed that $\mathcal{O}_\mu(p, p_i)$ is a smooth function of p we know that it will vanish in the limit $p \rightarrow 0$, and hence

$$\lim_{p \rightarrow 0} \tilde{a}(p) \langle p | S | p_1, \dots, p_n \rangle = 0. \quad (.26)$$

In the context of Adler's zero, we wish to show that a shift symmetry implies that all scattering amplitudes vanish in the limit that one of the external momenta are taken to zero, hence showing that soft pion scattering amplitudes vanish. A shift symmetry has $a(x) = a$, and so (.26) shows that

$$\lim_{p \rightarrow 0} \langle p | S | p_1, \dots, p_n \rangle = 0, \quad (.27)$$

which indeed shows that scattering amplitudes of one soft particle to n particles vanishes.

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