

Endomorphisms of commutative unital Banach algebras

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Abstract

This thesis is a collection of theorems which say something about the following question:

if we know that a bounded operator on a commutative unital Banach algebra is a unital endomorphism, what can we say about its other properties?

More specifically, the majority of results say something about how the *spectrum* of a commutative unital Banach algebra endomorphism is dependent upon the properties of the algebra on which it acts. The main result of the thesis (the subject of Chapter 3) reveals that *primary ideals* (that is, ideals with single point hulls) can sometimes be particularly important in questions of this type.

The thesis also contains some contributions to the Fredholm theory for bounded operators on an arbitrary complex Banach space. The second major result of the thesis is in this direction, and concerns the relationship between the essential spectrum of a bounded operator on a Banach space and those of its restrictions and quotients - ‘to’ and ‘by’ - closed invariant subspaces.

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Foreword

Given the nature of this document, I will assume that the reader has at least some experience in the following three areas:

- commutative Banach algebras;
- operator theory; and
- the theory of functions of several real variables.

Good references in these areas are the books, [13], [1], and [2] respectively. The reader unfamiliar with multi-index notation can also look to page 2 of [2] for a helpful summary.

On the subject of notation: I strongly suggest that the reader consults my list of standard symbols *before* reading Chapter 1. More specialised symbols will be introduced as we go along.

CHAPTER 1

Endomorphisms of commutative unital Banach algebras: an introduction

§1.1. Overview

Let S be a compact Hausdorff space, and let A be a point-separating unital Banach algebra of continuous complex valued functions on S such that every nonzero multiplicative linear functional $\delta : A \longrightarrow \mathbb{C}$ satisfies

$$\delta(f) = f(x) \quad (f \in A) \tag{1.1.1}$$

for some $x \in S$; the functional δ for which (1.1.1) holds will be denoted using the symbol δ_x .

In this document, a *unital endomorphism* of A is a bounded linear operator mapping A into itself which is multiplicative and preserves the multiplicative identity; such operators respect the structure of A both as a Banach space, and as a unital algebra. This thesis is about how the need to respect this second structure affects the properties of an endomorphism as a bounded linear operator.

The traditional way to investigate matters of this type is through the observation that we have an injective unital semigroup (anti-)homomorphism

$$\varrho : \text{End}(A) \longrightarrow \text{CSM}(S) \tag{1.1.2}$$

mapping the semigroup $\text{End}(A)$ of all unital endomorphisms of A into the semigroup $\text{CSM}(S)$ of all continuous self-maps of S .

To see this, we have to note only that (for any $T \in \text{End}(A)$ and $x \in S$), the composition $T^*\delta_x = \delta_x \circ T$ is a nonzero multiplicative linear functional on A . Having assumed that all of these are of the form δ_y for some $y \in S$, we must therefore have

$$T^*\delta_x = \delta_{\phi(x)}$$

for some $\phi(x) \in S$. The self-map $\phi : S \longrightarrow S$ obtained in this way is easily seen to be continuous, and we clearly have

$$(Tf)(x) = f(\phi(x))$$

for each $f \in A$ and $x \in S$.

With this observation in mind, we are immediately presented with the following possible lines of enquiry; the reader is reminded that a Banach space operator is *meromorphic* if its nonzero spectrum consists entirely of poles of the resolvent.

- (1) For a particular choice of A , identify $\varrho(\text{End}(A))$, the set of all maps $\phi \in \text{CSM}(S)$ which correspond to endomorphisms of A . If this has been done, try to describe the subsets of $\varrho(\text{End}(A))$ which arise as self maps of unital endomorphisms of A with particular operator theoretic properties.
- (2) For some choice of A , try to describe the spectra of the meromorphic elements of $\text{End}(A)$ in terms of their self-maps.

Investigative efforts in these two directions are quite common (see, to give just a small sample, any of [33], [29], [34], [37], [50], [42], [43], [26] or [28]), and a great deal is known in particular cases. It is known, for example, that:

- $\sigma(T) = \{0, 1\} \cup \{\lambda^n : n \in \mathbb{N}\}$ for some $\lambda \in \mathbb{C}$ when T is a meromorphic unital endomorphism of any one among several Banach algebras of infinitely-differentiable functions;
- unital Banach algebras functions admitting coarser behaviour (such as $C(S)$, $C^1([0, 1])$ and some algebras of Lipschitz functions) generally do not have compact unital endomorphisms besides those of the form $f \longmapsto \sum_{i=1}^n f(x_i)p_i$, where $(p_i)_{i=1}^n$ is some system of idempotents;

- meromorphic unital endomorphisms of algebras in this second category tend to have very simple spectra (typically, $\sigma(T) = \{0, 1\}$);
- for endomorphisms of many algebras of differentiable functions, a great deal of useful information can be recovered by examining the behaviour of self-maps near any of their fixed points; this is particularly true when the iterates of ϕ shrink S down to a single point.

We will focus mainly on the first, third and fourth of these phenomena over the coming pages (although I am hopeful that parts of the argument used in Section 5.5 may one day shed some light on the second).

Two important differences between this tract and the majority of other works in this area are as follows. First, I place much greater emphasis on the idea that the study of Banach algebra endomorphisms is a branch of operator theory; although self-map arguments still play an important rôle in some of our discussion, these are used to *complement* (rather than replace) a more conventional arsenal of operator theoretic tools.

The second significant difference between this and the majority of related works is the level of generality of many of the results. With the exception of Chapter 5 (I have a favourite algebra) very little of what we discuss is limited to any single function algebra. Instead, I have (in most cases) made an attempt to identify exactly which ingredients are needed to make certain arguments work, and have chosen hypotheses for theorems accordingly.

The following Chapter summaries will hopefully give the reader a rough idea of what is to follow.

- Chapter 2 begins with some preliminary results, but quickly moves on to discuss, *inter alia*, the implications of the fact that unital Banach algebra homomorphisms out of $C(S)$ always have closed range. The two sections which follow contain some general observations concerning the spectra of commutative unital Banach algebras with particular properties, and it is here that the reader will find the chapter's most interesting results. The chapter concludes with a brief exploration of the relationship between an endomorphism of a function algebra and its extension to an associated Lipschitz algebra.
- Chapter 3 is loosely based on my paper, [46], and explores a hitherto unnoticed link between the outer spectrum of a unital endomorphism of semi-prime commutative unital Banach algebra and the primary ideal structure of the algebra on which it acts.

The main result of this Chapter (and arguably of the whole thesis) is Theorem 3.3.3. The remainder of the chapter is devoted to examining a selection of its more immediate implications.

- Chapter 4 contains the other main result of the thesis. Namely, that the polynomially convex hull of the essential spectrum of a Banach space operator *always* contains the essential spectra of all its restrictions to closed invariant subspaces. This result is then generalised considerably, and some of its consequences for the study of function algebra endomorphisms are briefly explored.
- Chapter 5 contains results about $C^m(\bar{\Omega})$, where Ω is a bounded domain in $\mathbb{R}^d$. The primary focus here is the interaction between unital endomorphisms of $C^m(\bar{\Omega})$ and the operators they induce on certain spaces of multilinear-map-valued functions on $\bar{\Omega}$. The main results of this chapter can be found in sections 5.4 and 5.5, after which can be found a short exploration of the idea that $C^1(\bar{\Omega})$ can be realised as a quotient of a uniform algebra.

In the interests of presenting something resembling a cohesive narrative, I have been deliberately quite selective in the choice of material presented. Omitted, for example, is any discussion of my joint work with J.F. Feinstein on quasi-compact unital endomorphisms on non-commutative Banach algebras, or anything approaching the subject of my conference proceedings paper, [45].

§1.2. Conventions

A few conventions, standard results, and pieces of notation are described below. A list of standard symbols can also be found on page 117.

Quotients of normed spaces. Let X be a normed space, let F be a closed subspace of X , and let $Q_F : X \rightarrow X/F$ be the quotient map. Unless explicitly indicated otherwise, the quotient space X/F will *always* be equipped with the norm which satisfies $\|Q_F u\| = \inf\{\|v\| : u - v \in F\}$ for $u \in X$, even in situations where other natural norms are available. For $u \in X$, the symbol $\|u + F\|$ will always mean $\|Q_F u\|$. When X is a normed space of functions and X contains the constant functions, the symbol $X/\mathbb{C}$ is used to denote “ X modulo the constants”. In this situation, $\|u + \mathbb{C}\|$ denotes the quantity $\inf\{\|v\| : u - v \text{ is constant}\}$ for each $u \in X$.

Quotients of operators. Let E and F be closed subspaces of normed spaces X and Y , and let $T : X \rightarrow Y$ be a bounded operator with $T(E) \subseteq F$. Writing $Q_E : X \rightarrow X/E$ and $Q_F : Y \rightarrow Y/F$ for the appropriate quotient maps, we write $T_{/E,F}$ for the (uniquely determined) bounded operator from X/E into Y/F for which

$$Q_F T = (T_{/E,F}) Q_E.$$

The symbol T/F will be used in place of $T_{/F,F}$ when $X = Y$ and $E = F$. In this situation, T/F will be referred to as the quotient of T by F .

Restrictions and birestrictions. Let X be a set, let $F \subset X$, and let i_F be the inclusion map sending F into X . For a function T with domain X , the restriction of T to F is clearly just the composition $T \circ i_F$.

There is another natural notion of ‘restriction’ available when T maps F into a proper subset G of the codomain of T . The map obtained by ‘considering T as a map from F into G ’ will generally be referred to as the *birestriction* (of T to F and G) in this text, in acknowledgement of the fact that its codomain is not the same as that of T . The birestriction of a function T to an invariant set F is denoted $T|F$. For consistency with the literature we shall often refer to $T|F$ simply as the ‘restriction’ of T to F when T is a linear operator and F is a closed invariant subspace, even though it is really a birestriction.

Spectra and norms of restrictions and quotients. Let F be a closed subspace of a Banach space X , and let T be a bounded linear operator on X which preserves F . It is known (and proved in [30], Section 3.11) that the invertibility of any two of $\{T, T|F, T/F\}$ implies that of the third. This leads to the standard inclusions

$$\begin{aligned} \sigma(T) &\subseteq \sigma(T/F) \cup \sigma(T|F) \quad , \quad \sigma(T|F) \subseteq \sigma(T) \cup \sigma(T/F) \\ \text{and } \sigma(T/F) &\subseteq \sigma(T) \cup \sigma(T|F). \end{aligned}$$

Banach algebras. In this document, a Banach algebra is a Banach space $(B, \|\cdot\|)$ equipped with a continuous bilinear multiplication; we do not insist that

$$\|ab\| \leq k\|a\|\|b\| \quad (a, b \in B)$$

holds with $k = 1$, only that it does so for some $k > 0$.

Function algebras. Let A be a commutative unital Banach algebra, and let $a \in A$. The Gel'fand transform $\hat{a}$ of a is, as usual, the function on Φ_A (the character space of A) given by

$$\hat{a}(x) = x(a) \quad (x \in \Phi_A).$$

The mapping

$$A \ni a \longmapsto \hat{a} \in \mathbb{C}^{\Phi_A} \tag{1.2.1}$$

gives rise to an algebra $\hat{A} = \{\hat{a} : a \in A\}$ of functions on Φ_A . This will always be equipped with the norm

$$f \longmapsto \inf\{\|a\| : f(x) = \hat{a}(x) \ (x \in \Phi_A)\},$$

so is always isomorphic as a unital Banach algebra to the quotient $A/\text{Rad}(A)$ of A by its Jacobson radical. It is standard that $\hat{A}$ is contained in both $C(\Phi_A)$, and the algebra of all functions which are Lipschitz continuous with respect to the ‘Gleason’ metric given by

$$\rho_A(x, x') = \|x - x'\| = \sup\{|\hat{a}(x) - \hat{a}(x')| : a \in A, \|a\| = 1\}$$

for $x, x' \in \Phi_A$. When A is semisimple, we shall omit all ‘hats’ on Gel'fand transforms, and make no distinction between A and $\hat{A}$. Semi-simple commutative unital Banach algebras will be called *function algebras*; function algebras which are closed under uniform convergence will be called *uniform algebras*. The reader is warned that some authors also use the term ‘function algebras’ to refer to what we call uniform algebras.

Notation for character space maps. As we will be working mostly in the category of unital Banach algebras and bounded unital algebra homomorphisms, it is convenient to set

$$\text{Hom}(A, B) = \{T \in L(A, B) : T(1) = 1 \text{ and } T(ab) = T(a)T(b)\}$$

for each pair (A, B) of unital Banach algebras. Let A and B be unital Banach algebras for which Φ_A and Φ_B are non-empty, and let $T \in \text{Hom}(A, B)$. In this situation, the adjoint T^* of T descends to give a continuous map

$$T^\dagger : \Phi_B \longrightarrow \Phi_A, \quad x \longmapsto T^*x;$$

this is the *character space map* associated with T .

With this notation in place, one evidently has

$$\widehat{Ta}(x) = \widehat{a}(T^\dagger(x))$$

for each $a \in A$ and $x \in \Phi_B$.

The category Ban_1 . The symbol Ban_1 is very occasionally used in the text to denote the category of complex Banach spaces and bounded operators of norm ≤ 1 .

Two special subspace types. Let X be a Banach space, and let S be a (not necessarily w^* -closed) subset of X^* . In this situation $M(S) = M_X(S)$ denotes the closed subspace

$$\{f \in X : \mu(f) = 0 \ \forall \mu \in S\}.$$

This will be shortened to $M(\mu)$ when $S = \{\mu\}$ for some $\mu \in X^*$.

Let A be commutative unital Banach algebra, let $S \subseteq \Phi_A$, and let $\mathfrak{V}(S)$ be the set of all Gel'fand open subsets of Φ_A which contain S . In this document, $J_0(S) = J_{0,A}(S)$ is the smallest subspace of A containing every $M_A(U)$ with $U \in \mathfrak{V}(S)$; its norm closure is denoted $J(S)$ or $J_A(S)$. The reader is warned that some authors (c.f. [13]), prefer to write $J(S)$ for the ideal we call $J_0(S)$.

Hulls. Let A be a commutative unital Banach algebra, and let I be a closed ideal in A . Then the *hull* of I is, as usual, the closed subset

$$\mathfrak{h}_A(I) = \bigcap_{a \in I} \{x \in \Phi_A : \widehat{a}(x) = 0\}$$

of Φ_A . The subscript A will be omitted when the algebra is clear. It is standard that $S \mapsto \mathfrak{h}_A(M(S))$ is a closure operator on Φ_A . The algebra A is said to be *Shilov regular* (or just *regular*) if the induced topology (the *hull-kernel topology*) on Φ_A is the same as the Gel'fand (relativised w^*) topology. For us, I is *primary* as soon as $\mathfrak{h}_A(I)$ reduces to a single point.

Iterates of self-maps. Let S be a set. For a self-map ϕ of S and $n \in \mathbb{N}$, let

$$\phi^{on} = \underbrace{\phi \circ \phi \circ \cdots \circ \phi}_{n \text{ times}}.$$

We shall refer to ϕ^{on} as the n -th iterate of ϕ .

The set

$$\bigcap_{n=1}^{\infty} \phi^{\circ n}(S)$$

will play an important rôle in some of our discussion, and will be denoted $\phi^{\circ\infty}(S)$. It is common to refer to $\phi^{\circ\infty}(S)$ as the *fixed set* of ϕ .

Standard isomorphisms. Let X be a Banach space, let F be a closed subspace, and let T be a bounded operator on X which preserves F . The standard isometric isomorphism between $(X/F)^*$ and $F^\perp$ induces an isometric isomorphism

$$L((X/F)^*) \longrightarrow L(F^\perp)$$

which sends $(T/F)^*$ to $T^*|F^\perp$. This means, among other things, that

$$\|T/F\| = \|(T/F)^*\| = \|T^*|F^\perp\|.$$

Now suppose that X is a Banach space of functions on a set S , and let $E \subseteq S$. In this situation, it is standard that the quotient space

$$X/M(E) = X/\{f \in X : f = 0 \text{ on } E\}$$

can be regarded as the space of restrictions to E of functions belonging to X . Given the remarks above regarding norm conventions, the norm we take on $X/M(E)$ is always that given by

$$\|f\| = \inf\{\|u\| : u \in X, u = f \text{ on } E\}$$

for $u \in X$.

Complements. We will only ever be interested in *topological* complements of subspaces of normed spaces; the word ‘complement’ should henceforth be read as ‘topological complement’.

Semi-prime algebras. Since we will only deal with commutative algebras in this thesis, *semi-prime* can simply be read as ‘without nonzero nilpotent elements’.

CHAPTER 2

Preparatory results

§2.1. Introduction

The way in which a unital endomorphism responds to the algebraic structure of its environment can sometimes tell us quite a lot about its properties as an operator. My aim in this chapter is simply to point out a few circumstances in which this is particularly true, and to assemble some of the tools we will need for the more specialised chapters which follow.

After a smattering of introductory material and a short note about some properties of homomorphisms out of $C(S)$, the first main set of results focusses on two topics: ‘endomorphisms and involutions’ and ‘endomorphisms and point derivations’. This is followed by a collection of theorems describing aspects of the relationship between a unital endomorphism and its so-called *fixed set*. To finish, I have included a short section examining the relationship between an endomorphism of a function algebra and the Lipschitz algebra of its underlying Gleason metric space.

§2.2. Some simple observations

This first section is largely introductory in nature, and is intended to equip the reader with the fundamental examples and observations needed in order to derive the fullest enjoyment from the subject matter of later sections. Nearly all the material in this section will be known to specialists in one form or another. However, this may be the first time all the information has appeared together in one place, and I am not sure whether the few proofs that I give are the usual ones.

It is probably sensible to begin by making a note of some of the more immediate observations one can make about the interaction between a unital Banach algebra *homomorphism* and the various algebraic structures of its domain and codomain. It is, for example, easy to see that for any unital Banach algebras A and B , any $T \in \text{Hom}(A, B)$ and any $a \in A$,

$$\sigma(Ta) \subseteq \sigma(a). \quad (2.2.1)$$

The following results for the commutative case are also well known; the proofs are routine, and can mostly be recovered using standard theory and results from Rickart's book, [51].

Lemma 2.2.1. Let A and B be commutative unital Banach algebras, let $T \in \text{Hom}(A, B)$, and let $\phi = T^\dagger$. Then

- (a) $\sigma(Ta) = \sigma(a)$ for every $a \in A$ if ϕ is surjective;
- (b) $T^{-1}(\text{Rad}(B)) = M(\phi(\Phi_B))$;
- (c) the inclusion $T^{-1}(\text{Rad}(B)) \subseteq \text{Rad}(A)$ holds precisely when $\phi(\Phi_B)$ is hull-kernel dense in Φ_A ;
- (d) T is surjective only if ϕ is injective;
- (e) T is bijective only if ϕ is a homeomorphism.

In the special case where A and B are semisimple and A is regular, we also have

- (f) T is injective if and only if ϕ is surjective.

The following standard consequence of (2.2.1) will be needed later, and is an important ingredient in the proof (not given here) of Theorem 3.1.2; I am grateful to Joel Feinstein for pointing this out to me.

Lemma 2.2.2. Let A be a unital Banach algebra, and let $T : A \longrightarrow A$ be a bounded unital endomorphism. Then, subject to the natural identification of $\mathbb{C}$ with the set of all multiples of the identity,

$$\text{Im}(1 - T) \cap \mathbb{C} = \{0\}.$$

Proof. Let $a \in \text{Im}(1 - T) \cap \mathbb{C}$. Then there is a $z \in \mathbb{C}$ and some $a_0 \in A$ for which $a_0 - Ta_0 = a = z \cdot 1$. This leads to the equation

$$a_0 - z = Ta_0,$$

which in turn implies that we have $a_0 - nz = T^n a_0$ for all $n \in \mathbb{N}$. Choose any $\lambda_0 \in \sigma(a_0)$. Then since

$$\{\lambda - nz : \lambda \in \sigma(a_0)\} = \sigma(a_0 - nz) = \sigma(T^n a_0) \subseteq \sigma(a_0),$$

every point of the (usually unbounded) sequence $(\lambda_0 - nz)_{n \in \mathbb{N}}$ belongs to $\sigma(a_0)$; unless $z = 0$, this contradicts the fact that $\sigma(a_0)$ is bounded. $\square$

Given Banach spaces X and Y and closed subspaces $M \subseteq X$ and $N \subseteq Y$, a bounded operator $T : X \rightarrow Y$ maps M into N if and only if

$$T^*(N^\perp) \subseteq M^\perp. \quad (2.2.2)$$

The following comments concern some analogues of this fact which are available in our more specific setting. Letting A and B be commutative unital Banach algebras, choosing some $T \in \text{Hom}(A, B)$ and letting $\phi = T^\dagger$, it is clear that for closed subsets $E \subseteq \Phi_A$ and $F \subseteq \Phi_B$,

$$\phi F \subseteq E \quad \Rightarrow \quad \begin{cases} T(M_A(E)) \subseteq M_B(F); \text{ and} \\ T(J_A(E)) \subseteq J_B(F). \end{cases} \quad (2.2.3)$$

The following result handles the converse situation.

Proposition 2.2.3. Let A and B be commutative unital Banach algebras, let $T \in \text{Hom}(A, B)$, let $\phi = T^\dagger$, and let $I \subseteq A$ and $J \subseteq B$ be arbitrary closed ideals. Suppose that $T(I) \subseteq J$. Then:

- (i) $T_{/I, J} \in \text{Hom}(A/I, B/J)$;
- (ii) $\phi(\mathfrak{h}_B(J)) \subseteq \mathfrak{h}_A(I)$; and
- (iii) $(T_{/I, J})^\dagger$ ‘is’ the birestriction of ϕ to $\mathfrak{h}_B(I)$ and $\mathfrak{h}_A(I)$, subject to the standard identification of $\Phi_{A/I}$ with $\mathfrak{h}_A(I)$ and $\Phi_{B/J}$ with $\mathfrak{h}_B(J)$.

Proof. Statement (i) is obvious. Before tackling (ii), let us first note that a function $f : X \rightarrow Y$ between sets X and Y satisfies $fS_1 \subseteq S'_1$ and $fS_2 \subseteq S'_2$ for subsets $S_1, S_2 \subseteq X$ and $S'_1, S'_2 \subseteq Y$ only if

$$f(S_1 \cap S_2) \subseteq S'_1 \cap S'_2. \quad (2.2.4)$$

In our situation, the fact that $T^*(J^\perp) \subseteq I^\perp$ is clear. Since we also have $T^*(\Phi_B) \subseteq \Phi_A$, (ii) follows simply by setting $f = \phi$, $S_1 = \Phi_B$, $S'_1 = \Phi_A$, $S_2 = J^\perp$ and $S'_2 = I^\perp$ in (2.2.4).

To prove (iii), we need to appreciate that what we are really doing by ‘identifying’ $\mathfrak{h}_A(I)$ with $\Phi_{A/I}$ is pretending that the (injective) character space map $\Phi_{A/I} \hookrightarrow \Phi_A$ induced by the quotient homomorphism (sending A onto A/I) is just a straightforward inclusion. Provided we are willing to accept this slight aberration, (iii) follows from combining the equation

$$Q_J T = (T_{I,J}) Q_I$$

with the fact that $(T_1 T_2)^\dagger = T_2^\dagger \circ T_1^\dagger$ for any compositionally compatible homomorphisms T_1 and T_2 $\square$

Corollary 2.2.4. Let A and B be commutative unital Banach algebras, and let $T \in \text{Hom}(A, B)$. Let $E \subseteq \Phi_A$, $F \subseteq \Phi_B$ be closed subsets and suppose that A is regular. Then $T(M_A(E)) \subseteq M_B(F)$ if and only if $T^\dagger(F) \subseteq E$.

§2.3. Homomorphisms out of $C(S)$

This section is a short selection of results and examples which illustrate how homomorphisms out of $C(S)$ algebras differ from homomorphisms out of other function algebras. Many of the special properties of these operators occur in connection with the following theorem of Kaplansky, which appears below alongside one of its immediate corollaries.

Theorem 2.3.1 (Kaplansky’s theorem, c.f. [13], Theorem 4.2.3). Let S be a compact Hausdorff space, and let $\|\cdot\|$ be a submultiplicative norm on $C(S)$. Then $|f|_S \leq \|f\|$ for every $f \in C(S)$.

Corollary 2.3.2 (c.f. [13], Corollary 4.2.4). Let S be a compact Hausdorff space, let T be a unital Banach algebra homomorphism mapping $C(S)$ into another unital Banach algebra. Then T has closed range.

Corollary 2.3.2 implies, in particular, that every *endomorphism* of $C(S)$ has closed range. As the following example illustrates, the same is not true of every endomorphism of $C^1([0, 1])$.

Example 2.3.3. Consider the endomorphism T of $C^1([0, 1])$ given by setting

$$Tf(x) = f(x^2)$$

for each $f \in C^1([0, 1])$ and $x \in [0, 1]$. Now consider the (C^1) functions $(f_\nu)_{\nu \in (0, 1)}$ given by setting

$$f_\nu(x) = \begin{cases} \frac{1}{2} \left(\sqrt{\nu} + \frac{x}{\sqrt{\nu}} \right) & \text{for } x \in [0, \nu); \text{ and} \\ \sqrt{x} & \text{for } x \in [\nu, 1] \end{cases}$$

Using the standard norm $\|\cdot\| : f \mapsto \|f\|_\infty + \|f'\|_\infty$ on $C^1([0, 1])$, straightforward calculations give

$$\|f_\nu\| = 1 + \frac{1}{2\sqrt{\nu}} \quad \text{and} \quad \|Tf_\nu\| = 2$$

for each $\nu \in (0, 1)$. Considering the difference quotients

$$\frac{\|Tf_\nu\|}{\|f_\nu\|} = \frac{4\sqrt{\nu}}{1 + 2\sqrt{\nu}},$$

we see that T is not bounded below. Since T is injective, this implies that T cannot have closed range.

The endomorphism induced on $C^1([0, 1])$ by the map $\phi : x \mapsto x^2$ also provides us with an example of an endomorphism of a semisimple commutative unital Banach algebra for which ϕ is injective but T is not surjective. One of the consequences of Corollary 2.3.2 is that this situation can never occur for bounded unital homomorphisms between $C(S)$ spaces; the following result does not seem to be in the literature, but must be known to specialists.

Proposition 2.3.4. Let S and S' be compact Hausdorff spaces, and let ϕ be a continuous map sending S' into S . Then ϕ is injective if and only if the homomorphism $T : C(S) \rightarrow C(S') , f \mapsto f \circ \phi$ is surjective.

Proof. That ϕ is injective whenever T is surjective is clear from part (d) of Lemma 2.2.1. The converse statement is proved as follows.

Suppose that ϕ is injective, and let $A = \text{Im}(T)$. It is clear that A is self-adjoint and contains the constant functions. The fact that ϕ is injective means that A also separates points of S' , so it follows from the Stone Weierstrass theorem that A is dense in $C(S')$. Applying Corollary 2.3.2, we see that A is closed, hence that $A = C(S')$, since a proper subspace cannot be both closed and dense. $\square$

In his 1980 paper, [35], Kamowitz observed that when S is a connected compact Hausdorff space, the only compact endomorphisms of $C(S)$ are those of the form

$$P_x : C(S) \longrightarrow C(S) \ , \ f \longmapsto f(x) \cdot 1.$$

However, the following more general result seems to have been overlooked.

Proposition 2.3.5. Let S be a connected compact Hausdorff space, and let B be a unital Banach algebra. Then every compact unital algebra homomorphism $T : C(S) \longrightarrow B$ has finite rank. If, in addition, B has no nonzero nilpotents and no nonzero idempotents besides 1, then there is some $x \in S$ for which

$$Tf = f(x) \cdot 1$$

for every $f \in C(S)$, where 1 is the multiplicative identity in B .

Proof. The first part of the proof is completed at a single stroke: as a compact operator with closed range, T has finite rank. Now suppose that B has no nonzero nilpotents and no nonzero idempotents besides 1. Then $\text{Im}(T)$ is a commutative unital Banach algebra with no nonzero nilpotents and no nonzero idempotents besides 1. Since $\text{Im}(T)$ is finite dimensional, its lack of nilpotents makes it semisimple. It follows that $\text{Im}(T)$ is, up to isomorphism, just $\mathbb{C}^n$ (with entry-wise multiplication) for some $n \in \mathbb{N}$; the lack of non-trivial idempotents then forces $n = 1$, and the result follows (since every character on $C(S)$ is given by evaluation at a point of S). $\square$

This result also fails to hold for function algebras more generally. To see this, we have only to consider the inclusion map of $C^1([0, 1])$ into $C([0, 1])$; this is compact, but is certainly not of finite rank.

Corollary 2.3.2 also has an interesting application in connection with determining whether or not endomorphisms of certain function algebras have closed range. The following theorem is a small step in this direction.

Theorem 2.3.6. Let (S, ρ) be a compact metric space, and let ϕ be a Lipschitz continuous self-map of S . Suppose that there is some $\gamma > 0$ with the following property: given any two distinct points $y, y' \in \phi(S)$, one can select points $x \in \phi^{-1}(\{y\})$ and $x' \in \phi^{-1}(\{y'\})$ for which

$$\rho(x, x') \leq \gamma \cdot \rho(y, y').$$

Then $B = \{f \circ \phi : f \in \text{Lip}(S, \rho)\}$ is a closed subspace of $\text{Lip}(S, \rho)$.

Proof. Let $g \in \overline{B}$, and choose a sequence $(f_n)_{n \in \mathbb{N}}$ in $\text{Lip}(S, \rho)$ for which

$$\lim_{n \rightarrow \infty} \|f_n \circ \phi - g\|_{\text{Lip}(S, \rho)} = 0. \quad (2.3.1)$$

Writing T for the endomorphism $f \mapsto f \circ \phi$ of $C(S)$, (2.3.1) certainly implies that

$$\lim_{n \rightarrow \infty} \|Tf_n - g\|_{\infty} = 0.$$

Invoking Corollary 2.3.2, we see that there is some $f \in C(S)$ such that $Tf = g$. Our next move is to show that f is Lipschitz continuous on $\phi(S)$; if we can prove this, we can take a $u \in \text{Lip}(S, \rho)$ which agrees with f on $\phi(S)$ and have $u \circ \phi = f \circ \phi = g$. Fix any $y, y' \in \phi(S)$, and let M be a Lipschitz constant for g . Taking any $x \in \phi^{-1}(\{y\})$ and $x' \in \phi^{-1}(\{y'\})$, we have

$$\begin{aligned} |f(y) - f(y')| &= |f(\phi(x)) - f(\phi(x'))| \\ &= |g(x) - g(x')| \\ &\leq M\rho(x, x'). \end{aligned}$$

Choosing, as we may, points x and x' for which $\rho(x, x') \leq \gamma \cdot \rho(y, y')$, we are therefore guaranteed that

$$|f(y) - f(y')| \leq M \cdot \gamma \cdot \rho(y, y').$$

Since we can argue similarly for every pair of distinct points $y, y' \in \phi(S)$, we see that f is Lipschitz continuous on $\phi(S)$.

It is, of course, not necessarily the case that f is Lipschitz continuous on all of S . However, the existence of a bounded linear extension operator $\text{Lip}(\phi(S), \rho) \rightarrow \text{Lip}(S, \rho)$ ([55], Theorem 1.5.6) means we can modify f outside $\phi(S)$ to recover some $u \in \text{Lip}(S, \rho)$ which agrees with f on $\phi(S)$. $\square$

§2.4. The spectrum of an endomorphism.

Let A be a unital Banach algebra, and let T be a bounded linear operator mapping A into itself. In this section, we examine what the condition

$$T \in \text{End}(A) = \text{Hom}(A, A)$$

means for the the spectrum $\sigma(T)$ of T in $L(A)$ when A has various properties. For a linear operator T mapping a vector space X into itself, let us agree that $n(T)$ and $d(T)$ will denote, respectively, the dimensions of $\text{Ker}(T)$ and $X/\text{Im}(T)$. We can then consider the sets

$$\begin{aligned} \sigma_p(T) &= \{\lambda \in \mathbb{C} : n(\lambda - T) \neq 0\}, \\ \sigma_{\text{su}}(T) &= \{\lambda \in \mathbb{C} : d(\lambda - T) \neq 0\}, \text{ and} \\ \sigma_e(T) &= \{\lambda \in \mathbb{C} : \text{at least one of } n(\lambda - T) \text{ or } d(\lambda - T) \text{ is not finite}\} \end{aligned}$$

of parameter values λ for which $\lambda - T$ fails to be bijective in each of the ways considered. These sets are referred to as the *point spectrum*, *surjectivity spectrum* and *essential spectrum* respectively. If X is equipped with a norm, one can also consider the *approximate point spectrum*

$$\sigma_{\text{ap}}(T) = \{\lambda \in \mathbb{C} : \lambda - T \text{ is not bounded below}\}.$$

We write

$$r_e(T) = \inf\{r > 0 : |\lambda| \leq r \text{ for all } \lambda \in \sigma_e(T)\}$$

for the ‘radius’ of $\sigma_e(T)$. When X is an infinite dimensional Banach space, the punctured neighbourhood theorem (c.f. [47], page 171, Theorem 7) implies that

$$\sigma(T) \cap \{z : |z| > r_e(T)\} \subseteq \sigma_p(T) \tag{2.4.1}$$

for every $T \in L(X)$.

In this setting, any $\lambda \in \sigma(T) \setminus \sigma_{\text{su}}(T)$ is necessarily the centre of an open disk $D \subseteq \sigma_p(T)$ (see Theorem 6.20 of [1]). These two observations will be useful later. For the time being, it seems best to begin with the following well-known result; its consequences will be with us throughout this thesis.

Proposition 2.4.1. Let A be a unital Banach algebra with no nonzero nilpotent elements. Let $T \in L(A)$ be an operator for which $T(a^2) = T(a)^2$ for every $a \in A$. Then

$$\{\lambda^n : n \in \mathbb{N}, \lambda \in \sigma_p(T)\} \subseteq \sigma_p(T).$$

In particular, $\sigma_p(T)$ is a subset of the closed unit disk. If, in addition, A is an integral domain (so that the product of any two nonzero elements is nonzero), $\sigma_p(T)$ is a unital semigroup.

A proof can be found in, for example, Gamelin's paper, [24]. Although this result is well known, the same cannot be said for the following corollary; this makes use of the remark immediately preceding Proposition 2.4.1.

Corollary 2.4.2. Let A be an infinite dimensional unital Banach algebra with no nonzero nilpotent elements, and let $T \in \text{End}(A)$. Then $\sigma(T) \setminus \sigma_{\text{su}}(T)$ is a subset of the *open* unit disk.

The following corollary is also interesting.

Corollary 2.4.3. Let A be an infinite dimensional unital Banach algebra with no nonzero nilpotent elements, and let $T \in \text{End}(A)$ be an endomorphism for which $r_e(T) \geq 1$. Then $r(T) = r_e(T)$. In particular, we have

$$r(T) = r(T/F)$$

for every finite dimensional T -invariant subspace F of A .

Proof. That $r(T) = r_e(T)$ follows from combining Proposition 2.4.1 with inclusion (2.4.1). Thus $r(T/F) \leq r(T) = r_e(T) = r_e(T/F) \leq r(T/F)$ for each T -invariant finite dimensional subspace F of A . Here, I have used the fact (c.f. [41], Proposition 3.7.1) that $\sigma_e(T) = \sigma_e(T/F)$ when F is a finite dimensional T -invariant subspace. $\square$

The remaining results in this section all pertain to endomorphisms of commutative unital Banach algebras with particular special properties. The following observation should perhaps be attributed to F. V. Atkinson (author of the eponymous theorem concerning Fredholm operators), who noticed something similar as part of his work on so-called *Baxter operators*; this result is obviously quite similar in structure to Proposition 2.4.1.

Proposition 2.4.4. Let A be a unital uniform algebra, and let $T \in \text{End}(A)$. Then we have

$$\{\lambda^n : n \in \mathbb{N}, \lambda \in \sigma_{\text{ap}}(T)\} \subseteq \sigma_{\text{ap}}(T), \quad (2.4.2)$$

and $\sigma(T)$ is a subset of the closed unit disk.

Proof. Since A is a uniform algebra, we may assume that the norm in A is just the uniform norm $\|\cdot\| = \|\cdot\|_\infty$. Inclusion (2.4.2) is established by a simple argument using the fact that $\|f^2\| = \|f\|^2$ for every $f \in A$. The second assertion is obvious, since

$$\|Tf\| = \sup_{x \in \Phi_A} |f(\phi(x))| \leq \sup_{x \in \Phi_A} |f(x)| = \|f\|$$

for every $f \in A$ (where, as usual $\phi = T^\dagger$). □

The following result is established by similarly elementary means, but appears to be new; the reader will recall that an algebra of functions A is *self-adjoint* if, given any element $f \in A$, the function $\tau(f) : x \mapsto \overline{f(x)}$ is also an element of A .

Proposition 2.4.5. Let A be a semisimple commutative unital Banach algebra, regarded as a subalgebra of $C(\Phi_A)$ in the usual way. Suppose that A is self-adjoint, and let $T \in \text{End}(A)$. Then the sets

$$\sigma_{\text{p}}(T), \sigma_{\text{su}}(T), \sigma_{\text{e}}(T) \text{ and } \sigma(T) \text{ are symmetric about the real axis.}$$

If, in addition, $\|\tau(f)\| = \|f\|$ for every $f \in A$, then the approximate point spectrum $\sigma_{\text{ap}}(T)$ is symmetric about the real axis also.

Proof. Our first step is to show that T is necessarily a $*$ -homomorphism, in the sense that we have $T(\tau(f)) = \tau(Tf)$ for every $f \in A$. This is actually rather straightforward. Let $\phi = T^\dagger$. Then for $f \in A$ and $x \in \Phi_A$,

$$\tau(Tf)(x) = \overline{f(\phi(x))} = (\tau f)(\phi(x)) = T(\tau(f))(x),$$

so

$$\tau(\lambda - T) = (\bar{\lambda} - T)\tau$$

for $\lambda \in \mathbb{C}$. Using this, it is easy to see that τ maps $\text{Ker}(\lambda - T)$ into $\text{Ker}(\bar{\lambda} - T)$ and $\text{Im}(\lambda - T)$ into $\text{Im}(\bar{\lambda} - T)$. Combining this with the fact that elements $u_1, \dots, u_n$ of A are linearly independent only if this is true of $\tau u_1, \dots, \tau u_n$, we must clearly have

$$n(\bar{\lambda} - T) = n(\lambda - T)$$

$$\text{and } d(\bar{\lambda} - T) = d(\lambda - T)$$

whenever the right hand sides are finite. This is enough for us to be able to conclude that $\sigma_p(T)$, $\sigma_{su}(T)$ and $\sigma_e(T)$ are all symmetric about the real axis. It remains only to show that the same is true of $\sigma_{ap}(T)$ when τ is isometric. To see this, first choose any $\lambda \in \mathbb{C}$ for which $\lambda - T$ is bounded below, and take some $t > 0$ for which

$$t\|v\| \leq \|(\lambda - T)v\|$$

for every $v \in A$. Now fix any $u \in A$. Then, setting $v = \tau u$, we have

$$\begin{aligned} t\|u\| &= t\|v\| \leq \|(\lambda - T)v\| = \|\tau(\lambda - T)v\| \\ &= \|(\bar{\lambda} - T)u\|. \end{aligned}$$

Concluding from this inequality that $\bar{\lambda} - T$ is bounded below whenever $\lambda - T$ is bounded below, we see that $\bar{z}(\sigma_{ap}(T)) \subseteq \sigma_{ap}(T)$, as asserted. $\square$

Similar arguments to those employed above work when $\tau : A \rightarrow A$ is an arbitrary involution on A satisfying the conditions on page 56 of [25]; any such involution extends to give us an involution $T \mapsto T_\tau = \tau T \tau$ on $L(A)$ which preserves $\text{End}(A)$. In the simple case considered above, we evidently have $T_\tau = T$ for each $T \in \text{End}(A)$.

Example 2.4.6. Let A be the disk algebra and suppose that, instead of being simple complex conjugation, we let $f \mapsto \tau(f)$ be the self-map of A given by setting

$$\tau(f) = \bar{f} \circ \bar{z}$$

for every $f \in A$. Just as before, τ is involutive, additive and preserves linear independence. However, it is clearly no longer the case that $T_\tau = T$ for each $T \in \text{End}(A)$. Fixing $T \in \text{End}(A)$, and letting ϕ be the associated self-map of $\mathbb{D}$, we actually have

$$(T_\tau f)(\zeta) = (f \circ \bar{z} \circ \phi \circ \bar{z})(\zeta)$$

for $\zeta \in \mathbb{D}$ and $f \in A$. Combining the fact that τ is bijective with the identity

$$\tau(\lambda - T)\tau = \bar{\lambda} - T_\tau,$$

it is clear that we have $\sigma(T) = \bar{z}(\sigma(T_\tau))$ for each $T \in \text{End}(A)$.

Having seen that sets belonging to the collection

$$\mathcal{C}(A) = \{\sigma(T) : T \in \text{End}(A)\}$$

have to be symmetric about the real axis when A is self-adjoint, it is now time to see what additional constraints are placed on $\mathcal{C}(A)$ when A has non-trivial bounded point derivations. Our focus, for the remainder of this section, will be an abstraction of the popular ‘repeated’ differentiation trick used by Kamowitz (any many others since) to prove statements about the point spectra of endomorphisms of algebras of highly differentiable functions. This trick essentially just relies on the fact that, when f and ϕ are sufficiently smooth functions of a single variable and x_0 is a fixed point of ϕ , we have

$$(f \circ \phi)^{(k)}(x_0) = \phi'(x_0)^k f^{(k)}(x_0) + \left(\begin{array}{c} \text{terms containing a factor of} \\ f^{(j)}(x_0) \text{ for some } j < k \end{array} \right) \quad (2.4.3)$$

for each $k \in \mathbb{N}$. Our first move will be to rephrase this observation in terms of (higher order, bounded) point derivations.

Remark 2.4.7. The reader is reminded at this point that a $(k+1)$ -tuple $(d_i)_{i=0}^k$ of linear functionals on a commutative unital algebra A constitutes a k -th order point derivation if d_0 is a character and we have

$$d_j(ab) = \sum_{i=0}^j d_i(a)d_{j-i}(b)$$

for each $a, b \in A$ and $j \leq k$. When $k = 1$, this definition means that we have a product rule $d_1(ab) = d_0(a)d_1(b) + d_1(a)d_0(b)$ for $a, b \in A$.

The following proposition is an abstraction of the so-called *Faa-di-Bruno* formula for computing higher derivatives of composite functions; it will be crucial in transferring the ‘repeated differentiation trick’ to a more abstract setting. The following notation will be helpful; for a multi-index $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{Z}_+^k$, we shall write

$$\alpha! = \prod_{i=1}^k \alpha_i! \quad , \quad |\alpha| = \sum_{i=1}^k \alpha_i \quad \text{and} \quad \langle \alpha \rangle = \sum_{i=1}^k i\alpha_i.$$

Proposition 2.4.8. Let A be a commutative unital Banach algebra, let $x \in \Phi_A$, and suppose that there are linearly independent bounded linear functionals $d_0 = x, d_1, d_2, \dots, d_k$ on A for which $(x, d_1, d_2, \dots, d_k)$ is a k -th order point derivation. Let

$$I = M(x) \cap \text{Ker}(d_1) \cap \dots \cap \text{Ker}(d_k),$$

and let T be a bounded unital endomorphism of A such that $T(I) \subseteq I$. Then there are complex numbers $\Gamma_1, \dots, \Gamma_r$ for which the equation

$$d_j(Ta) = \sum \left\{ \frac{|\alpha|!}{\alpha!} d_{|\alpha|}(a) \prod_{r=1}^k \Gamma_r^{\alpha_r} : \alpha \in \mathbb{Z}_+^k, \langle \alpha \rangle = j \right\} \quad (2.4.4)$$

holds for every $a \in A$ and $j \in \mathbb{N}$ with $j \leq k$.

Proof. Our linear independence assumption implies that there is a $g_0 \in A$ such that $d_1(g_0) = 1$ and $d_i(g_0) = 0$ for $i \neq 1$. This ensures, in particular, that the homomorphism $\theta_0 : A \longrightarrow \mathbb{C}[z]_{\leq k}$, $a \longmapsto \sum_{i=0}^k d_i(a)z^i$ is surjective (here and throughout, $\mathbb{C}[z]_{\leq k}$ is the algebra of all polynomials in a single indeterminate z , with multiplication carried out modulo z^{k+1}).

2.4 The spectrum of an endomorphism.

Letting $d'_1, \dots, d'_k$ be the functionals on A/I for which $d_i(a) = d'_i(a + I)$ for $a \in A$, the map $\theta : A/I \rightarrow \mathbb{C}[z]_{\leq k}$ given for $u \in A/I$ by $\theta(u) = \sum_{i=0}^k d'_i(u) z^i$ is clearly an isomorphism with $\theta(g_0 + I) = z$. Setting $g = g_0 + I$, we therefore have $u = \sum_{i=0}^k d'_i(u) g^i$ for each $u \in A/I$. As a homomorphic image of z in $\mathbb{C}[z]_{\leq k}$, g is nilpotent, so the same is clearly true of $\phi = (T/I)g$. This means, in particular, that $d'_0(\phi) = 0$, so

$$(T/I)u = \sum_{i=0}^k d'_i(u) \phi^i = \sum_{i=0}^k d'_i(u) \left(\sum_{j=1}^k d'_j(\phi) g^j \right)^i.$$

Using the multinomial formula (see, for example, [18], page 42), this simplifies to

$$\begin{aligned} (T/I)u &= \sum_{i=0}^k d'_i(u) \sum \left\{ \frac{i!}{\alpha!} \prod_{r=1}^k d'_r(\phi)^{\alpha_r} g^{r\alpha_r} : \alpha \in \mathbb{Z}_+^k, |\alpha| = i \right\} \\ &= \sum \left\{ \frac{|\alpha|!}{\alpha!} d'_{|\alpha|}(u) \prod_{r=1}^k d'_r(\phi)^{\alpha_r} g^{r\alpha_r} : \alpha \in \mathbb{Z}_+^k, \langle \alpha \rangle \leq k \right\}. \end{aligned}$$

Let $j \in \mathbb{N}$ with $j \leq k$. Then, as the coefficient of g^j in the expansion above, $d'_j((T/I)u)$ is given by the formula

$$\sum \left\{ \frac{|\alpha|!}{\alpha!} d'_{|\alpha|}(u) \prod_{r=1}^k d'_r(\phi)^{\alpha_r} : \alpha \in \mathbb{Z}_+^k, \langle \alpha \rangle = j \right\}.$$

Let $a \in A$. Then since $d_j(Ta) = d'_j(Ta + I) = d'_j((T/I)(a + I))$, we have

$$d_j(Ta) = \sum \left\{ \frac{|\alpha|!}{\alpha!} d'_{|\alpha|}(a) \prod_{r=1}^k d'_r(\phi)^{\alpha_r} : \alpha \in \mathbb{Z}_+^k, \langle \alpha \rangle = j \right\}.$$

Setting $\Gamma_r = d'_r(\phi)$ for $r \in \{1, \dots, k\}$, the proof is complete. $\square$

It is probably worth making explicit the fact that the numbers $\Gamma_1, \dots, \Gamma_k$ appearing in this result depend both on $(d_i)_{i=0}^k$ and T and (loosely speaking) encode the interaction between these two entities. With this result at our disposal, we now have all we need to demonstrate the use of the ‘repeated differentiation trick’ in a very general setting.

Proposition 2.4.9. Let A be a commutative unital Banach algebra, and suppose that there is a sequence $(d_i)_{i=0}^\infty$ of linearly independent elements of A^* such that $(d_i)_{i=0}^k$ is a point derivation of order k for each fixed $k \in \mathbb{N}$. Let T be a unital endomorphism of A such that $T(\text{Ker}(d_1)) \subseteq \text{Ker}(d_1)$ and

$$T\left(\bigcap_{i=0}^k \text{Ker}(d_i)\right) \subseteq \bigcap_{i=0}^k \text{Ker}(d_i)$$

for each $k \in \mathbb{Z}_+$. Suppose that $\bigcap_{i=0}^\infty \text{Ker}(d_i) = \{0\}$. Then

$$\sigma_p(T) \subseteq \{1\} \cup \{\Gamma_1^n : n \in \mathbb{N}\},$$

where Γ_1 is uniquely specified by the equation $T^*d_1 = \Gamma_1 d_1$.

Proof. Since T maps $\text{Ker}(d_i)$ into itself for $i \in \{0, 1\}$, there are unique complex numbers Γ_0 and Γ_1 for which

$$T^*d_0 = \Gamma_0 d_0 \quad \text{and} \quad T^*d_1 = \Gamma_1 d_1.$$

The requirement that T^*d_0 be a character forces $\Gamma_0 = 1$, but there is no corresponding restriction on Γ_1 . Let λ be a complex number which is neither 1 nor any positive integer power of Γ_1 , and let $a \in \text{Ker}(\lambda - T)$. Then since $\Gamma_0 = 1$,

$$\begin{aligned} 0 &= d_0(\lambda a - Ta) = (\lambda - 1)d_0(a) \\ \text{and } 0 &= d_1(\lambda a - Ta) = (\lambda - \Gamma_1)d_1(a). \end{aligned}$$

Our assumption about λ therefore forces $d_0(a) = 0 = d_1(a)$. Now take $k \in \mathbb{N}$ such that $d_j(a) = 0$ for all $j < k$. Then, by the previous result, there are complex numbers $\Gamma_2, \dots, \Gamma_k$ such that (2.4.4) holds for every $j \in \mathbb{N}$ with $j \leq k$. In particular,

$$d_k(Ta) = \sum \left\{ \frac{|\alpha|!}{\alpha!} d_{|\alpha|}(a) \prod_{r=1}^k \Gamma_r^{\alpha_r} : \alpha \in \mathbb{Z}_+^k, \langle \alpha \rangle = k \right\}.$$

Our induction hypothesis ensures that $d_{|\alpha|}(a) = 0$ for $|\alpha| < k$, so we have only to consider terms for which $|\alpha| = k$. Fortunately, it is easy to see that the condition $|\alpha| = k = \langle \alpha \rangle$ actually forces $\alpha_1 = k$ and $\alpha_i = 0$ for $i > 1$. Our induction hypothesis therefore actually means that $d_k(Ta) = d_k(a)\Gamma_1^k$.

It follows that

$$d_k(\lambda a - Ta) = (\lambda - \Gamma_1^k) d_k(a),$$

so $d_k(a) = 0$ (since $\lambda \neq \Gamma_1^k$). By induction, $d_i(a) = 0$ for every $i \in \mathbb{Z}_+$. Having assumed that 0 is the only element of A with this property, $a = 0$. The result follows. $\square$

Corollary 2.4.10. Assume that, in addition to the hypotheses of the previous result, A is infinite dimensional, T is compact, and that A has no nonzero nilpotents. Then

$$\sigma(T) = \{0, 1\} \cup \{\Gamma_1^n : n \in \mathbb{N}\}.$$

Proof. Since a compact operator on an infinite dimensional Banach space is never invertible, $0 \in \sigma(T)$. That $1 \in \sigma(T)$ is also immediate from the fact that T is unital. Given that $T^*d_1 = \Gamma_1 d_1$, we certainly have $\Gamma_1 \in \sigma(T)$. As T is compact, this means that either $\Gamma_1 = 0$ or $\Gamma_1 \in \sigma_p(T)$. In the latter case, the absence of nonzero nilpotent elements means that $\Gamma_1^n \in \sigma(T)$ for every $n \in \mathbb{N}$. Having shown that

$$\{0, 1\} \cup \{\Gamma_1^n : n \in \mathbb{N}\} \subseteq \sigma(T),$$

it remains to establish the reverse inclusion. Since T is a compact operator,

$$\sigma(T) = \{0\} \cup \sigma_p(T).$$

Proposition 2.4.9 does the rest. $\square$

The word *compact* can be replaced by the words *meromorphic and non-invertible* in this result without affecting its validity, since we only need the fact that $0 \in \sigma(T)$ and that nonzero points of the spectrum are all eigenvalues. The reader may wish to observe that the subspaces

$$\bigcap_{i=0}^k \text{Ker}(d_i)$$

appearing in Proposition (2.4.9) are closed primary ideals of finite codimension in A ; we will see later that (in a suitably restricted setting) the availability of such ideals is crucial for Riesz endomorphisms to have spectra larger than $\{0, 1\}$.

§2.5. Spectra and fixed sets

The results of this section all concern the set

$$\bigcup \{E \subseteq \Phi_A : T^*E = E\} \quad (2.5.1)$$

when A is a commutative unital Banach algebra, and $T \in \text{End}(A)$. The strongest of these (Proposition 2.5.5) goes some way to explaining some of the links observed in the literature between $\sigma(T)$ and the behaviour of $T^\dagger$ near its fixed points. The following lemma provides us with an important alternative description of the set in (2.5.1), and explains why (from now on) we shall denote it using the symbol $T^{*\infty}\Phi_A$.

Lemma 2.5.1. Let S be a compact Hausdorff space, and let ϕ be a continuous self-map of S . Then

$$\bigcup \{E \subseteq S : \phi(E) = E\} = \phi^{\circ\infty}(S).$$

Proof. That the left hand side is contained in the right is obvious, and holds independently of any topological considerations. A standard finite-intersection-property argument involving sets of the form

$$E_{n,x} = \phi^{-1}(\{x\}) \cap \phi^{\circ n}(S)$$

shows that ϕ maps $\bigcap_{n=1}^{\infty} \phi^{\circ n}(S)$ onto itself; this is clearly enough for the reverse inclusion. $\square$

The following theorem tells us something rather interesting about the point spectrum of $T/M(T^{*\infty}\Phi_A)$. I suspect from the proof of Lemma 1.4 of [35] that Kamowitz was aware of this result, but I have included a proof anyway.

Proposition 2.5.2. Let A be a commutative unital Banach algebra, and let $T \in \text{End}(A)$. Then

$$\bigcup \{\text{Ker}(\lambda - T) : \lambda \in \mathbb{C} \setminus \mathbb{T}\} \subseteq M(T^{*\infty}\Phi_A).$$

In particular,

$$\sigma_p(T/M(T^{*\infty}\Phi_A)) \subseteq \mathbb{T}. \quad (2.5.2)$$

Proof. It is clearly enough to show that (2.5.2) holds. To do this, take $\lambda \in \mathbb{C} \setminus \mathbb{T}$ and $a \in A$ such that $\lambda a - Ta \in M(T^{*\infty}\Phi_A)$.

Then, letting $\phi = T^\dagger$, the functions $\lambda\hat{a}$ and $\hat{a} \circ \phi$ agree on $T^{*\infty}\Phi_A$. Since ϕ maps $T^{*\infty}\Phi_A$ onto itself, we must therefore have

$$|\lambda| \sup_{x \in T^{*\infty}\Phi_A} |\hat{a}(x)| = \sup_{x \in T^{*\infty}\Phi_A} |\hat{a}(\phi(x))| = \sup_{x \in T^{*\infty}\Phi_A} |\hat{a}(x)|.$$

As $|\lambda| \neq 1$, this forces $\hat{a}$ to vanish on $T^{*\infty}\Phi_A$. This puts a in $M(T^{*\infty}\Phi_A)$, and the result follows. $\square$

Corollary 2.5.3. Let A be a regular semisimple commutative unital Banach algebra, and let $T \in \text{End}(A)$ be injective. Then $\sigma_p(T) \subseteq \mathbb{T}$.

Corollary 2.5.4. Let A be a commutative unital Banach algebra, and let $T \in \text{End}(A)$. Then $\sigma_p(T) \cap \mathbb{D} = \sigma_p(T|M(T^{*\infty}\Phi_A)) \cap \mathbb{D}$.

Although the three results considered so far speak of a link between $T^{*\infty}\Phi_A$ and the eigenvalues and eigenvectors of a unital endomorphism, I have yet to make the case that the same is true of the spectrum more generally. The next result is an important step in this direction. Loosely speaking, it says that all the important information about $\sigma(T)$ can be recovered by looking what happens near $T^{*\infty}\Phi_A$.

Proposition 2.5.5. Let A be a semisimple commutative unital Banach algebra, let $T \in \text{End}(A)$, and let V be any open T^* -invariant (Gel'fand) neighbourhood of $T^{*\infty}\Phi_A$ in Φ_A . Then

$$\sigma(T) \cup \{0\} = \sigma(T|M(V)) \cup \{0\}.$$

This is also true when $V = T^{*n}\Phi_A$ for some $n \in \mathbb{N}$.

Proof. Let V be either $T^{*n}\Phi_A$ for some $n \in \mathbb{N}$ or some T^* -invariant (Gel'fand) neighbourhood of $T^{*\infty}\Phi_A$. In either case, $M(V)$ is a T -invariant closed subspace of A . The standard 3-space spectral inclusions now give

$$\begin{aligned} \sigma(T) &\subseteq \sigma(T|M(V)) \cup \sigma(T/M(V)) \\ \text{and } \sigma(T/M(V)) &\subseteq \sigma(T) \cup \sigma(T|M(V)). \end{aligned}$$

The result will clearly follow if we can demonstrate that $\sigma(T|M(V)) = \{0\}$ when $M(V) \neq \{0\}$.

Assume that $M(V) \neq \{0\}$. Let us show that T is actually nilpotent on $M(V)$. This is immediate when $V = T^{*n}\Phi_A$, since

$$\text{Ker}(T^n) = M(T^{*n}\Phi_A).$$

Supposing, instead, that V is a T^* -invariant open neighbourhood of $T^{*\infty}\Phi_A$, a standard compactness argument tells us that there is necessarily some $m \in \mathbb{N}$ for which

$$T^{*\infty}\Phi_A \subseteq T^{*m}\Phi_A \subseteq V.$$

Since this gives $M(V) \subseteq M(T^{*m}\Phi_A) = \text{Ker}(T^m)$, the result follows. $\square$

Whilst $T/M(V)$ and T have the same nonzero spectrum when V is any T^* -invariant *neighbourhood* of $T^{*\infty}\Phi_A$ in Φ_A , the same is not always true when V is $T^{*\infty}\Phi_A$ itself. To see this, it suffices to consider the case where T is a compact endomorphism of the disc algebra; in this case, $T^{*\infty}\Phi_A$ is always a singleton, but $\sigma(T)$ need not be equal to $\{0, 1\}$.

Corollary 2.5.6. For $i \in \{1, 2\}$, let B_i be a semisimple commutative unital Banach algebra, let $T_i \in \text{End}(B_i)$, let $\phi_i = T_i^\dagger$, and let V_i be a closed ϕ_i -invariant neighbourhood of $\phi_i^{\circ\infty}(\Phi_{B_i})$. Suppose that there is a bijection $\psi : V_1 \longrightarrow V_2$ such that $\phi_2 \circ \psi = \psi \circ \phi_1$ on V_1 , and:

- (i) for every $g \in B_2$, $g \circ \psi$ is the restriction to V_1 of some $f \in B_1$;
- (ii) for every $f \in B_1$, $f \circ \psi^{-1}$ is the restriction to V_2 of some $g \in B_2$.

Then $\sigma(T_1) \cup \{0\} = \sigma(T_2) \cup \{0\}$.

Proof. Let $\Psi : B_1/M(V_1) \longrightarrow B_2/M(V_2)$ be the homomorphism given by $\Psi f = f \circ \psi^{-1}$ for each $f \in B_1/M(V_1)$ (identified with the algebra of restrictions to V_1 of functions in B_1). It is clear that Ψ is an isomorphism. Since our assumption that $\phi_2 \circ \psi = \psi \circ \phi_1$ on V_1 gives

$$\Psi(T_1/M(V_1)) = (T_2/M(V_2))\Psi,$$

this means the operators $T_1/M(V_1)$ and $T_2/M(V_2)$ are similar; the result now follows from the previous one. $\square$

The next result records an interesting link between of $T^{*\infty}\Phi_A$ and the spectra of elements of the algebra $\text{Ker}(1 - T)$.

Theorem 2.5.7. Let A be a commutative unital Banach algebra, and let $T \in \text{End}(A)$. Then:

- (i) $B = \text{Ker}(I - T)$ is a closed inverse-closed unital subalgebra of A , so $\sigma_A(b) = \sigma_B(b)$ for $b \in B$;
- (ii) for $b \in B$, $\sigma_A(b) = \widehat{b}(T^{*\infty}\Phi_A)$;
- (iii) if $T^{*\infty}\Phi_A$ is a singleton and A is semisimple then $B = \mathbb{C} \cdot 1$ and $1 - T$ has ascent 1 (in the sense that the kernel of $(1 - T)^2$ is the same as that of $1 - T$).

Proof. Item (i) is immediate. To prove (ii), first let $b \in B$ and choose any $\lambda \in \sigma_A(b)$. Since $\lambda \in \sigma_A(b)$, there is some $x \in \Phi_A$ such that $x(b) = \lambda$. Since $\lambda - b = T(\lambda - b)$, we then have

$$0 = x(\lambda - b) = x(T^n(\lambda - b)) = (\lambda - \widehat{b})(T^{*n}x)$$

for every $n \in \mathbb{N}$. This implies, in particular, that the sets

$$Z_n = \{y \in \Phi_A : y(b) = \lambda\} \cap T^{*n}\Phi_A \quad (n \in \mathbb{N})$$

are all nonempty. It follows that the collection $(Z_n)_{n=1}^{\infty}$ has the finite intersection property (since every finite intersection of Z_n s is another Z_n). As every Z_n is closed in Φ_A , compactness gives us

$$\{y \in \Phi_A : y(b) = \lambda\} \cap T^{*\infty}\Phi_A = \bigcap_{n=1}^{\infty} Z_n \neq \emptyset,$$

from which (ii) obviously follows.

The assumption that A be semisimple for part (iii) is actually slightly more than we need; its rôle is simply to ensure that any element of A with a one point spectrum is a multiple of the identity. With this restriction in place, part (ii) ensures that $B = \text{Ker}(1 - T) = \mathbb{C} \cdot 1$ when $T^{*\infty}\Phi_A$ has only one point. Recalling Lemma 2.2.2, we then have

$$\text{Im}(1 - T) \cap \text{Ker}(1 - T) = \text{Im}(1 - T) \cap (\mathbb{C} \cdot 1) = \{0\}.$$

That $1 - T$ has ascent 1 now follows from the (standard) fact that

$$\text{Ker}(\Lambda^2)/\text{Ker}(\Lambda) \cong \text{Im}(\Lambda) \cap \text{Ker}(\Lambda),$$

for any linear self-map Λ of a vector space. $\square$

Our final result of this section shows that ‘fixed sets’ of the type we have been considering are also important for weighted composition operators. The simple argument used in its proof will be exploited again (in a slightly more complicated setting) in Chapter 5. It would surprise me if the following proposition were not known to specialists, but I have not managed to locate it in the literature.

Proposition 2.5.8. Let S be a compact Hausdorff space, let $w \in C(S)$, and let ϕ be a continuous self-map of S . Let T be the operator on $C(S)$ given by

$$Tf(x) = w(x)f(\phi(x))$$

for each $f \in C(S)$ and $x \in S$. Then $r(T)$ is no greater than the maximum value attained by $|w|$ on $\phi^{\circ\infty}(S)$.

Proof. For each $n \in \mathbb{Z}_+$, let $|\cdot|_n$ be the seminorm on $C(S)$ given by

$$|f|_n = \sup\{|f(x)| : x \in \phi^{\circ n}(S)\}$$

for each $f \in C(S)$, with the convention that $\phi^{\circ 0}$ is the identity on S . An easy computation gives

$$\|T^n\| \leq \prod_{m=0}^{n-1} |w|_m \tag{2.5.3}$$

for each n . Since T is clearly nilpotent if some $|w|_n$ is 0, we need only deal with the case where $|w|_n > 0$ for every n . However, we clearly have

$$\limsup_{n \rightarrow \infty} \left(\prod_{m=0}^{n-1} |w|_m \right)^{1/n} = \lim_{n \rightarrow \infty} |w|_n = \sup\{|w(x)| : x \in \phi^{\circ\infty}(S)\}$$

by, for example, the Stoltz-Cesaro theorem (c.f. [48], page 87). The result now follows from (2.5.3) and the spectral radius formula. $\square$

§2.6. Lipschitz estimates

Let A be a semisimple commutative unital Banach algebra, let $T \in \text{End}(A)$, and let $\phi = T^\dagger$. It is easy to see that

$$C(\Phi_A) \ni f \longmapsto f \circ \phi \in C(\Phi_A)$$

is an endomorphism of $C(\Phi_A)$, and that this endomorphism may be regarded as an extension of T . In general, it is rather difficult to say anything particularly interesting about any relationship which may exist between T and its extension to all of $C(\Phi_A)$. In this section, we consider an extension of T with slightly more favourable properties.

Letting ρ_A be the Gleason metric on Φ_A and noting that

$$\rho_A(\phi(x), \phi(y)) = \|T^*(x - y)\| \leq \|T\| \|x - y\| = \|T\| \rho_A(x, y),$$

it is clear that ϕ is actually a Lipschitz continuous self-map of the metric space (Φ_A, ρ_A) . This means, in particular, that in setting

$$T^{\text{Lip}} f = f \circ \phi$$

for each $f \in \text{Lip}(\Phi_A, \rho_A)$, we recover an endomorphism T^{Lip} of $\text{Lip}(\Phi_A, \rho_A)$. Our aim, for much of this section, will be to show that

$$\|T^{\text{Lip}}/\mathbb{C}\| \leq 3\|T/\mathbb{C}\|,$$

where $T/\mathbb{C}$ and $T^{\text{Lip}}/\mathbb{C}$ are the operators induced by T and T^{Lip} on the quotient spaces $A/\mathbb{C}$ and $\text{Lip}(\Phi_A, \rho_A)/\mathbb{C}$ (these are both spaces of functions modulo the constants). For convenience, we shall assume (in this section only) that we have equipped A with a submultiplicative norm. This imposes no essential restriction, but explains where the factor of 3 comes from in some inequalities; with this assumption in place, it is well known that the Gleason metric space (Φ_A, ρ_A) has diameter ≤ 2 .

We begin by introducing some notation. For any Lipschitz continuous map $f : S \longrightarrow S'$ between metric spaces (S, ρ) and (S', ρ') , let us write $L(f)$ for the infimum of the set of positive numbers c for which

$$\rho'(f(x), f(y)) \leq c\rho(x, y)$$

for every $x, y \in S$.

It is standard that when $f : S \longrightarrow S'$ and $g : S' \longrightarrow S''$ are Lipschitz continuous maps between the metric spaces (S, ρ) , (S', ρ') and (S'', ρ'') , we have the estimate

$$L(g \circ f) \leq L(g)L(f).$$

In this thesis, we will always equip $\text{Lip}(S, \rho)$ with the norm which satisfies

$$\|f\| = \|f\|_\infty + L(f)$$

for $f \in \text{Lip}(S, \rho)$, and define dual and operator norms accordingly. However, there is another norm (or rather, a family of norms) on $\text{Lip}(S, \rho)$ which is useful when (S, ρ) is a metric space of finite diameter. Let (S, ρ) be such a metric space. Then for each $p \in S$, the norm $\|\cdot\|_{(p)}$ given by

$$\|f\|_{(p)} = |f(p)| + L(f)$$

for each $f \in \text{Lip}(S, \rho)$ is equivalent to the standard norm. Writing c_S for the quantity $1 + \text{diam}_\rho(S)$, we have

$$c_S^{-1}\|f\| \leq \|f\|_{(p)} \leq \|f\|$$

for $f \in \text{Lip}(S, \rho)$. Since it is easy to see that

$$L(f) = \inf\{\|f + h\|_{(p)} : h \in \mathbb{C}^S \text{ is a constant function}\},$$

for $f \in \text{Lip}(S, \rho)$ and each $p \in S$, we have

$$c_S^{-1}\|f + \mathbb{C}\| \leq L(f) \leq \|f + \mathbb{C}\|,$$

where $\|f + \mathbb{C}\| = \inf\{\|f + h\| : h \in \mathbb{C}^S \text{ is a constant function}\}$. The following lemma is now elementary.

Lemma 2.6.1. Let (S, ρ) be a metric space of finite diameter, and let ϕ be a Lipschitz continuous self-map of S . Let T_ϕ be the operator on $\text{Lip}(S, \rho)$ given by setting $T_\phi f = f \circ \phi$ for every $f \in \text{Lip}(S, \rho)$. Then

$$\|T_\phi/\mathbb{C}\| \leq c_S \cdot L(\phi).$$

Proof. Let $f \in \text{Lip}(S, \rho)$. Then

$$c_S^{-1} \|T_\phi f + \mathbb{C}\| \leq L(f \circ \phi) \leq L(f)L(\phi) \leq L(\phi) \|f + \mathbb{C}\|.$$

Since $T_\phi f + \mathbb{C} = (T_\phi/\mathbb{C})(f + \mathbb{C})$, this is all we needed to show. $\square$

This lemma implies that we have

$$\|T^{\text{Lip}}/\mathbb{C}\| \leq 3L(\phi)$$

whenever $T \in \text{End}(A)$, $\phi = T^\dagger$, and A is any commutative unital Banach algebra. Our next step is to show that, in the same circumstances, we also have $L(\phi) \leq \|T/\mathbb{C}\|$.

Lemma 2.6.2. Let A be a commutative unital Banach algebra, let T be an element of $\text{End}(A)$, and let $\phi = T^\dagger$. Then

$$L(\phi) \leq \|T/\mathbb{C}\|,$$

where $L(\phi)$ is the Lipschitz constant with respect to the Gleason metric ρ_A .

Proof. We will need the following two standard facts:

- (i) the norm of a Banach space operator is the same as that of its adjoint;
- (ii) for a closed subspace F of a Banach space X , $(X/F)^*$ is isometrically isomorphic to $F^\perp$.

Combining these two facts, we see that $\|T/\mathbb{C}\| = \|T^*|Y\|$, where

$$Y = \{\mu \in A^* : \mu \text{ annihilates multiples of the identity}\}.$$

Keeping this in mind, let us choose any two points $x, y \in \Phi_A$. Then, clearly, $x - y \in Y$. We therefore have

$$\rho_A(\phi(x), \phi(y)) = \|T^*(x - y)\| \leq \|T|Y\| \|x - y\| \leq \|T/\mathbb{C}\| \rho_A(x, y),$$

from which the desired inequality follows. $\square$

Corollary 2.6.3. Let A be a commutative unital Banach algebra (not necessarily semisimple), and let $T \in \text{End}(A)$. Then $\|T^{\text{Lip}}/\mathbb{C}\| \leq 3\|T/\mathbb{C}\|$.

Lemma 2.6.2 also gives us a lower bound for $r(T/\mathbb{C})$ in terms of the iterates of ϕ ; this is one among several occasions on which the spectral radius formula serves to ‘purify’ an estimate by neutralising an unsightly multiplicative constant.

Corollary 2.6.4. Let A be a commutative unital Banach algebra, and let T be an element of $\text{End}(A)$. Let $\phi = T^\dagger$. Then

$$r(T^{\text{Lip}}/\mathbb{C}) \leq \limsup_{n \rightarrow \infty} L(\phi^{\circ n})^{1/n} \leq r(T/\mathbb{C}).$$

As an example of how this estimate can be used, let us again consider the endomorphism T of $C^1([0, 1])$ induced by the map $\phi : x \mapsto x^2$. Obviously,

$$\phi^{\circ n}(x) = x^{2^n}$$

for each $n \in \mathbb{N}$ and $x \in [0, 1]$. Since each $\phi^{\circ n}$ belongs to $C^1([0, 1])$, we have

$$L(\phi^{\circ n}) = \|(\phi^{\circ n})'\|_\infty = 2^n$$

so that $L(\phi^{\circ n})^{1/n} = 2$ for each $n \in \mathbb{N}$. Corollary 2.6.4 now tells us that $r(T/\mathbb{C}) \geq 2$. For this example, one can show that $r(T/\mathbb{C}) = 2$ by combining Corollary 2.4.3 with the results of Section 5.5.

Remarks.

- The reader may have noticed that the word ‘semisimple’ has disappeared from our discussion, even though it was definitely present at the beginning of this section. Actually, none of the estimates considered in this section requires A to be semisimple; this additional assumption is only required if we are to speak of T^{Lip} as an *extension* of T . Without semisimplicity, T^{Lip} is better regarded as an extension of $T/\text{Rad}(A)$ than of T itself.

- The condition considered in Theorem 2.3.6 is similar to that considered by Tougeron, Allan, Kakiko, O'Farrell and Watson in connection with the study of finitely generated algebras of C^∞ functions. See, for example, [4].

Quasi-compact endomorphisms and primary ideals

§3.1. Introduction

Let B be a semiprime commutative unital Banach algebra with connected character space Φ_B . For each $x \in \Phi_B$, let $\pi_B(x)$ be the collection of all closed primary ideals contained in the maximal ideal $M(x) = x^{-1}(0)$. The main aim of this chapter is to illustrate how knowledge of the collection $\pi_B(x)$ at each $x \in \Phi_B$ can be used in describing the outer spectrum of certain endomorphisms of B .

Let X be a complex Banach space. In this chapter, the *essential spectrum* $\sigma_e(T)$ of a bounded operator $T : X \rightarrow X$ is the set of all complex numbers λ for which the difference $\lambda - T$ is not a Fredholm operator (this is consistent with our use of the symbol $\sigma_e(T)$ in Section 2.4). With this understood, we shall say that T is *quasi-compact* if

$$\sigma_e(T) \subseteq \{\lambda : |\lambda| < 1\},$$

and *Riesz* if $\sigma_e(T) \subseteq \{0\}$. It is perhaps worth pointing out that all compact, power compact and strictly singular operators are Riesz, while all Riesz operators are quasi-compact.

Quasi-compact operators have always been primarily studied in the Ban_1 setting, and it is here that the most interesting results concerning them can be found. Even so, it is well known that if X is a Banach space and $T \in L(X)$ then T is quasi-compact if and only if it satisfies one of the following two equivalent conditions:

- (1) $T = S + V$ for some $V \in \mathfrak{F}(X)$ and $S \in L(X)$ with $r(S) < 1$;
- (2) there is a sequence $(K_n)_{n=1}^\infty$ in $K(X)$ such that $\lim_{n \rightarrow \infty} \|T^n - K_n\| = 0$.

Characterisation (1) was discovered by Brunel and Revuz in 1974; (2), which follows more immediately from the definition, is proved in [40] (page 88). It appears from the literature (c.f. [52], page 8) that some authors working with quasi-compact operators have also realised that (1) can be strengthened to

- (1') there are closed T -invariant subspaces $Y, Z \subseteq X$ with Y finite dimensional, $X = Y \oplus Z$, and $r(T|Z) < 1$.

Returning to characterisation (2), the sequence $(K_n)_{n=1}^\infty$ need not generally be particularly well-behaved, and will generally not even be norm bounded. The situation is slightly better for *power bounded* quasi-compact operators, since here we have the famous uniform ergodic theorem of Yoshida and Kakutani.

Theorem 3.1.1 (Yoshida & Kakutani, 1941). Let X be a Banach space, and let $T \in L(X)$ be power-bounded and quasi-compact. Then $\Lambda = \sigma(T) \cap \mathbb{T}$ is finite, and there are projections $(P^\lambda)_{\lambda \in \Lambda}$ with the following properties:

- (i) $\text{Im}(P^\lambda) = \text{Ker}(\lambda - T)$ for every $\lambda \in \Lambda$;
- (ii) each of the projections P^λ commutes with T ; and
- (iii) there is some $S \in L(X)$ with $r(S) < 1$ such that $T^n = \sum_{\lambda \in \Lambda} \lambda^n P^\lambda + S^n$ for every $n \in \mathbb{N}$.

As Feinstein and Kamowitz have observed, the situation is even more favourable for quasi-compact endomorphisms of certain commutative Banach algebras.

Theorem 3.1.2 (Feinstein & Kamowitz, 2005). Let B be a semi-prime commutative unital Banach algebra with connected character space Φ_B . Let T be a quasi-compact endomorphism of B . Then:

- (i) $\sigma(T) \subseteq \mathbb{D} \cup \{1\}$;
- (ii) the eigenvalue 1 has (algebraic) multiplicity 1 and eigenspace $\mathbb{C} \cdot 1$;
- (iii) there is a character $x_0 \in \Phi_B$ such that the sequence $(T^n)_{n=1}^\infty$ converges in norm to the rank 1 projection $b \mapsto x_0(b) \cdot 1$.

With this as our starting point, one of our main aims will be to prove the following result.

Theorem 3.1.3. Let B be a semi-prime commutative unital Banach algebra with connected character space Φ_B . Let T be a quasi-compact endomorphism of B , and suppose that $T^*x_0 = x_0$ for some $x_0 \in \Phi_B$. Then there is a family $\mathcal{J}$ of T -invariant closed primary ideals of finite codimension in B and hull $\{x_0\}$, for which

$$\sigma(T) \cap \{\lambda : |\lambda| > r_e(T)\} = \bigcup_{I \in \mathcal{J}} \sigma(T/I). \quad (3.1.1)$$

The maximal ideal $M(x_0) = \{b \in B : x_0(b) = 0\}$ always belongs to $\mathcal{J}$.

The reader will recall that an ideal in a commutative Banach algebra is said to be *primary* if it is simultaneously modular and contained in only one maximal ideal. In the unital case, an ideal is primary if and only if its hull reduces to a single point. We will later exploit the fact that, for many algebras of differentiable functions, primary ideals are often of a very particular form. For example, if Ω is a convex bounded domain in $\mathbb{R}^d$, every closed primary ideal in $C^k(\bar{\Omega})$ contains an ideal of the form

$$\{u \in C^k(\bar{\Omega}) : (D^\alpha u)(x) = 0 \text{ for all } |\alpha| \leq k\}$$

for some $x \in \bar{\Omega}$. We will see later that this implies that every Riesz endomorphism T of $C^k(\bar{\Omega})$ satisfies $\sigma(T) = \{0, 1\}$. This conclusion will, in fact, be shown to apply to a large class of regular function algebras on the closed unit ball in $\mathbb{R}^d$.

It should be stressed that Theorem 3.1.3 cannot generally be used to predict which ideals will occur in the collection $\mathcal{J}$ associated with a particular quasi-compact endomorphism. Therefore, it is typically (though not exclusively) in algebras where there is a paucity of closed primary ideals of finite codimension that the theorem is most informative.

Remark 3.1.4. Quasi-compact endomorphisms of $C^k(\bar{\Omega})$ are studied in more detail in Chapter 5; in this setting, one can combine Theorem 3.1.3 with direct methods to recover a description of $\sigma(T) \cap \{\lambda : |\lambda| > r_e(T)\}$ (see Theorem 5.3.8).

Theorem 3.1.3 is also particularly effective when applied to Riesz endomorphisms of certain algebras of infinitely-differentiable functions (even though such algebras generally have an abundance of suitable primary ideals). In this direction, Theorem 3.1.3 can be used to reproduce a large proportion of the results in the literature concerning the spectra of Riesz endomorphisms on algebras of holomorphic functions on domains in $\mathbb{C}$. Among these is the following (now well-known) theorem of Kamowitz. Below, $\mathbb{D}$ is the open unit disk, and $z : \overline{\mathbb{D}} \rightarrow \mathbb{C}$ is the associated inclusion map.

Theorem 3.1.5 ([33], Theorem 9). Let T be a Riesz endomorphism of the disk algebra $A(\overline{\mathbb{D}})$, and suppose that the function $\phi = Tz$ fixes a point $p = \phi(p)$ in the open unit disk. Then the spectrum of T is given by

$$\sigma(T) = \{0, 1\} \cup \{\phi'(p)^k : k \in \mathbb{N}\}.$$

As has been (rather comprehensively) established in the literature, a similar conclusion applies for a large number of other algebras of holomorphic functions on $\overline{\mathbb{D}}$. Theorem 3.1.3 will allow us to prove a significant generalisation of Theorem 3.1.5, one which subsumes a large number of existing results in this area. In particular, we will show that the same conclusion applies in every unital Banach algebra obtained by completing the algebra of polynomials in z with respect to a norm for which the multiplication operators $f \mapsto (z - p)f$ are bounded below (for $p \in \mathbb{D}$).

The material in this chapter is organised as follows. After this brief introduction, we begin with a review of some preliminary material. All of this is standard, and is included mainly to provide the reader with an opportunity to remind himself of a few operator theoretic preliminaries. The machinery described is then used to prove a version of Theorem 3.1.3.

Recovery of this central result is followed by a brief examination of some of its implications for the spectra of Riesz endomorphisms of algebras possessing particular primary ideal structures. The subsequent material is devoted to examining some of the implications of our results for Riesz endomorphisms in two rather different classes of function algebra: one class contains only regular algebras, and the other consists solely of algebras of functions which are holomorphic in the unit disk.

Remark 3.1.6. The results of this chapter are largely drawn from my paper ([46]) of the same title. This was submitted to the Journal of Functional analysis in July 2014, accepted subject to minor revisions in November 2015, and published in February 2016.

The final section of this chapter marks a slight departure from the other material, and examines (among other things) the conditions under which an endomorphism of a commutative unital Banach can be simultaneously quasi-compact and invertible.

§3.2. Preliminaries

Let X be a non-zero complex Banach space, and let $T \in L(X)$. We continue to use the notation

$$\begin{aligned}\rho(T) &= \{\lambda \in \mathbb{C} : \lambda - T \text{ is invertible in } L(X)\} & , \quad \sigma(T) = \mathbb{C} \setminus \rho(T), \\ \rho_e(T) &= \{\lambda \in \mathbb{C} : \lambda - T \text{ is a Fredholm operator}\} & , \quad \sigma_e(T) = \mathbb{C} \setminus \rho_e(T).\end{aligned}$$

These are referred to as the resolvent, spectrum, essential resolvent, and essential spectrum of T (respectively).

Suppose that X is infinite dimensional and, as above, let $K(X)$ be the closed, proper ideal in $L(X)$ consisting of all $T \in L(X)$ which are compact. It follows from Atkinson's theorem (c.f. [1], Theorem 4.46, page 161) that, for each $T \in L(X)$, $\sigma_e(T)$ is precisely the spectrum of $T + K(X)$ in the Calkin algebra $\mathfrak{C}(X) = L(X)/K(X)$. As such, $\sigma_e(T)$ is a non-empty, compact subset of $\mathbb{C}$ with $\sigma_e(T) \subseteq \sigma(T)$. An important connection between the essential and actual spectra of an operator is provided by the so-called *punctured neighbourhood theorem*, which we now quickly recall.

Theorem 3.2.1. [47, Section 19, Theorem 4, page 197]. Let X be a complex Banach space, let $T \in L(X)$ and let U be a component of $\rho_e(T)$. Then either $U \subseteq \sigma(T)$ or $U \cap \sigma(T)$ is at most countable and each $\lambda \in U \cap \sigma(T)$ is isolated in $\sigma(T)$.

Now let σ be a non-empty compact subset of $\mathbb{C}$. We write $\mathcal{O}(\sigma)$ for the algebra of germs of holomorphic functions over σ , equipped with the inductive compact open topology (c.f. [13], page 795). We use the symbol z_σ for the germ over σ of the complex co-ordinate functional z on $\mathbb{C}$. Given disjoint compact subsets σ and σ' of $\mathbb{C}$, we denote by $1_{\sigma, \sigma'}$, the germ over $\sigma \cup \sigma'$ obtained in the following manner. Let U, U' be disjoint open neighbourhoods of σ and σ' respectively. Now let h be the holomorphic function on $U \cup U'$ which is 1 on U and 0 on U' and let $1_{\sigma, \sigma'}$ be the germ of h over $\sigma \cup \sigma'$.

It is clear that $1_{\sigma, \sigma'}$ is an idempotent in $\mathcal{O}(\sigma \cup \sigma')$ which does not depend on the particular choice of U and U' .

For an element $b \in B$ of a unital Banach algebra B , the main single-variable holomorphic functional calculus theorem (c.f. [13], Theorem 2.4.4) asserts that there is a unique continuous unital algebra homomorphism $\Theta_b : \mathcal{O}(\sigma(b)) \rightarrow B$ satisfying the condition $\Theta_b(z_{\sigma(b)}) = b$. It is standard that elements of $\text{Im}(\Theta_b)$ commute with b . If $\sigma \subseteq \sigma(b)$ and both σ and its complement $\sigma' = \sigma(b) \setminus \sigma$ are compact, we define $P_b(\sigma) = \Theta_b(1_{\sigma, \sigma'})$, and refer to this as the spectral projection of b over σ . Each $1_{\sigma, \sigma'}$ is an idempotent in $\mathcal{O}(\sigma(b))$ and Θ_b is multiplicative, so each spectral projection $P_b(\sigma)$ is an idempotent in B which commutes with b .

The next lemma summarises some pertinent properties of these idempotents when $B = L(X)$ for a Banach space X . The three assertions made therein can be more or less directly lifted from the results of Chapters 6 and 7 of [1].

Lemma 3.2.2 (c.f. [1], Theorems 6.34 and 7.44 and exercise 14 on page 270). Let X be a non-zero complex Banach space and let $T \in L(X)$. Let $\sigma \subseteq \sigma(T)$ and suppose that both σ and $\sigma(T) \setminus \sigma$ are compact when considered as subsets of $\mathbb{C}$. Then $P_T(\sigma)$ has the following three properties:

- (i) The subspaces $\text{Im}(P_T(\sigma))$ and $\text{Ker}(P_T(\sigma))$ are T -invariant closed subspaces of X such that $X = \text{Im}(P_T(\sigma)) \oplus \text{Ker}(P_T(\sigma))$. If σ is a non-empty proper subset of $\sigma(T)$ then the spectra of the restrictions of T to $\text{Im}(P_T(\sigma))$ and $\text{Ker}(P_T(\sigma))$ are σ and $\sigma(T) \setminus \sigma$ respectively. The projection $P_T(\sigma)$ is zero if and only if σ is empty and is the identity operator if and only if $\sigma = \sigma(T)$.
- (ii) If, for some $t > 0$, σ and $\sigma(T) \setminus \sigma$ lie respectively in the unbounded and bounded components of the set $\{\lambda \in \mathbb{C} : |\lambda| \neq t\}$, $\text{Ker}(P_T(\sigma))$ coincides with the subspace $\{x \in X : \|T^n x\|/t^n \rightarrow 0 \text{ as } n \rightarrow \infty\}$.
- (iii) If σ consists of a single isolated point λ of $\sigma(T)$ and $\lambda - T$ is Fredholm then $P_T(\sigma)$ is a finite rank operator. In this case, there is a non-negative integer k for which $\text{Im}(P_T(\sigma)) = \text{Ker}((\lambda - T)^k)$.

It follows from (iii) that if σ is a finite set of isolated points of $\sigma(T)$ belonging to $\rho_e(T)$ then $P_T(\sigma)$ is a finite rank operator; it is simply the sum of the finite rank operators $P_T(\{\lambda\})$ for $\lambda \in \sigma$.

Assertion (ii) appears as an exercise in [1], and relies on considerations approaching so-called *local* spectral theory. The explicit description of $\text{Ker}(P_T(\sigma))$ provided to us by (ii) in the circumstances described will be very useful later.

The following lemma is crucial for much of what follows, and records a further interesting property of $T^{*\infty}\Phi_B$.

Lemma 3.2.3. Let B be a commutative unital Banach algebra, and choose any $T \in \text{End}(B)$. Let $I \subseteq B$ be any T -invariant closed ideal for which T/I is invertible. Then

$$\mathfrak{h}(I) \subseteq T^{*\infty}\Phi_B. \quad (3.2.1)$$

Assuming, in addition, that $T^{*\infty}\Phi_B$ reduces to a point $x_0 \in \Phi_B$, the ideals with this property all belong to $\pi_B(x_0)$.

Proof. Recalling part (iii) of Proposition 2.2.3, we know that $(T/I)^\dagger$ is, for practical purposes, just T^* considered as a map of $\mathfrak{h}(I)$ into itself. Using part (e) of Lemma 2.2.1, we then see that T^* necessarily maps $\mathfrak{h}(I)$ *onto* itself. Inclusion (3.2.1) now easily follows from the fact that

$$T^{*\infty}\Phi_B = \bigcup \{E \subseteq \Phi_B : T^*E = E\}.$$

With (3.2.1) established, the final assertion is obvious. $\square$

Theorem 3.1.3 gives us a set of circumstances in which $T^{*\infty}\Phi_B$ certainly reduces to a point, and leads us to the following lemma.

Lemma 3.2.4. Let B be a semi-prime commutative unital Banach algebra with connected character space Φ_B . Let T be a quasi-compact endomorphism of B , and suppose that $T^*x_0 = x_0$ for some $x_0 \in \Phi_B$. Let I be any closed T -invariant ideal for which T/I is invertible. Then $I \in \pi_B(x_0)$.

§3.3. From Spectral projection to primary ideals

The following lemma reveals the source of the ideals mentioned in Theorem 3.1.3; they will be the kernels of certain spectral projections.

Lemma 3.3.1. Let B be a semi-prime commutative unital Banach algebra with connected character space, and let T be a quasi-compact endomorphism of B . Let

$$\sigma_r = \sigma(T) \cap \{\lambda : |\lambda| \geq r\}$$

for any $r_e(T) < r \leq 1$. Then σ_r is finite, and $I_r = \text{Ker}(P_T(\sigma_r))$ is a closed T -invariant ideal of finite codimension in B .

Proof. It is clear that σ_r is a compact subset of $\mathbb{C}$. Applying Theorem 3.2.1 with U equal to the unbounded component of $\rho_e(T)$, we observe that, in addition, every $\lambda \in \sigma_r$ is isolated in $\sigma(T)$. As a compact subset of $\mathbb{C}$ with no accumulation points, σ_r is finite. It is clear from Lemma 3.2.2 (and the remarks immediately following it) that $I_r = \text{Ker}(P_T(\sigma_r))$ is a closed T -invariant *subspace* of finite codimension in B .

We have not yet established that I_r is an ideal. To do this, we first observe that since $\sigma(T)$ is compact and every point of σ_r is isolated in $\sigma(T)$, the complement $\sigma(T) \setminus \sigma_r$ is also a compact subset of $\mathbb{C}$. This implies, in particular, that there is a $0 < t(r) < r$ for which

$$\begin{aligned} \sigma_r &= \sigma(T) \cap \{\lambda : |\lambda| > t(r)\} \\ \text{and } \sigma(T) \setminus \sigma_r &= \sigma(T) \cap \{\lambda : |\lambda| < t(r)\}. \end{aligned}$$

Invoking part (ii) of Lemma 3.2.2, we see that

$$I_r = \{b \in B : \|T^n b\|/t(r)^n \rightarrow 0 \text{ as } n \rightarrow \infty\}. \quad (3.3.1)$$

The fact that I_r is an ideal now follows from Theorem 3.1.2 which ensures, of course, that T is power bounded. $\square$

It is clear from part (i) of Theorem 3.1.2 that

$$\text{Im}(P_T(\sigma_1)) = \text{Im}(P_T(\{1\})).$$

By part (ii) of the same theorem, this subspace has dimension 1. It follows that $I_1 = \text{Ker}(P_T(\sigma_1))$ is a maximal ideal. It is already clear that the ideal I_r is contained in the maximal ideal I_1 for each $r_e(T) < r \leq 1$. With Lemma 3.2.4 at our disposal, the following observation helps us show (among other things) that I_1 is the *only* maximal ideal with this property.

Lemma 3.3.2. Let B and T be as in Lemma 3.3.1 and define σ_r and I_r as before for $r_e(T) < r \leq 1$. Then $\sigma(T/I_r) = \sigma_r$ for all such r .

Proof. Let $r_e(T) < r \leq 1$, and let

$$i_r : \text{Im}(P_T(\sigma_r)) \longrightarrow B \text{ and } Q_r : B \longrightarrow B/I_r$$

be the appropriate inclusion and quotient maps. Composing these in the obvious fashion, we obtain a Banach space isomorphism

$$U_r = Q_r i_r : \text{Im}(P_T(\sigma_r)) \longrightarrow B/I_r.$$

Letting T_r be the restriction of T to $\text{Im}(P_T(\sigma_r))$, we have the equations

$$\begin{aligned} i_r T_r &= T i_r \\ \text{and } Q_r T &= (T/I_r) Q_r. \end{aligned}$$

Taken together, these lead to the intertwining relation $(T/I_r)U_r = U_r T_r$. Since this implies that T_r and T/I_r have the same spectrum, the result now follows from part (i) of Lemma 3.2.2. $\square$

We have now assembled everything we need in order to prove Theorem 3.1.3. However, before we do so, it is perhaps worth making explicit the respective rôles of Lemmas 3.2.4, 3.3.1 and 3.3.2. Under the hypotheses of Theorem 3.1.3, Lemma 3.3.1 provides us with a family of closed T -invariant ideals of finite codimension in B ; these are simply the kernels of the spectral projections $P_T(\sigma_r)$ for $r_e(T) < r \leq 1$. Lemma 3.3.2 does two things. Unsurprisingly, it tells us that

$$\sigma(T) \cap \{\lambda : |\lambda| > r_e(T)\} = \bigcup_{r_e(T) < r \leq 1} \sigma(T/I_r).$$

However, this is not its only rôle. At this stage in the proof, we do not yet know anything about the hulls $\mathfrak{h}(I_r)$, except that they all contain x_0 . It is here that Lemma 3.3.2 really comes to our rescue; the fact that $0 \notin \sigma_r$ means that each of the endomorphisms T/I_r is *invertible*, and it is this information which (via Lemma 3.2.4) allows us to show that each I_r belongs to $\pi_B(x_0)$.

We now prove the following strong form of Theorem 3.1.3.

Theorem 3.3.3. Let B be a semi-prime commutative unital Banach algebra with connected character space Φ_B , and let T be a quasi-compact endomorphism of B . Let $x_0 \in \Phi_B$ satisfy $T^*x_0 = x_0$. Then there is a family $\mathcal{J}$ of closed T -invariant ideals with the following properties:

- (i) $\sigma(T) \cap \{\lambda : |\lambda| > r_e(T)\} = \bigcup_{I \in \mathcal{J}} \sigma(T/I)$;
- (ii) $\mathcal{J} \subseteq \pi_B(x_0)$;
- (iii) the maximal ideal $M(x_0)$ always belongs to $\mathcal{J}$;
- (iv) each $I \in \mathcal{J}$ is the kernel of a finite rank spectral projection associated with T ;
- (v) $\mathcal{J}$ is at-most-countable, and its elements form a chain, in the sense that if $I, I' \in \mathcal{J}$ then either $I \subseteq I'$ or $I' \subseteq I$;
- (vi) for $r_e(T) < r \leq 1$, there is some $I \in \mathcal{J}$ for which $\sigma(T|I)$ is contained in the disk $\{\lambda : |\lambda| < r\}$.

Proof. Invoking Lemma 3.3.1, and setting

$$\mathcal{J} = \{\text{Ker}(P_T(\sigma_r)) : r_e(T) < r \leq 1\}, \quad (3.3.2)$$

we have immediately that $\mathcal{J}$ is a family of T -invariant closed ideals of finite codimension in B . Assertions (i)-(vi) are now proved as follows.

- (i) Since $\sigma(T) \cap \{\lambda : |\lambda| > r_e(T)\}$ is the union of the sets

$$\sigma_r = \sigma(T) \cap \{\lambda : |\lambda| \geq r\}$$

for $r_e(T) < r \leq 1$, this part is clear from Lemma 3.3.2.

- (ii) Making a second appeal to Lemma 3.3.2, we observe that for each ideal $I \in \mathcal{J}$, the spectrum $\sigma(T/I)$ consists solely of points of modulus strictly larger than 0. This means, in particular, that each of the operators $(T/I)_{I \in \mathcal{J}}$ is invertible. Lemma 3.2.4 therefore implies that $\mathcal{J} \subseteq \pi_B(x_0)$.

- (iii) Showing that $I_1 = \text{Ker}(P_T(\sigma_1))$ is a maximal ideal is achieved using the argument described in the comments before Lemma 3.3.2. Explicitly, a combination of Theorem 3.1.2 and Lemma 3.2.2 gives us

$$B = \text{Im}(P_T(\{1\})) \oplus I_1.$$

Using part (ii) of Theorem 3.1.2, this implies that I_1 has codimension 1 in B . Having already shown that I_1 is an ideal, this gives us what we need.

- (iv) This is immediate from definition (3.3.2).
- (v) The fact that $\mathcal{J}$ is at-most-countable is a straightforward consequence of the punctured neighbourhood theorem (Theorem 3.2.1). That $\mathcal{J}$ is a chain in the sense indicated is easily proved using description (3.3.1) from the proof of Lemma 3.3.1.
- (vi) Let $r_e(T) < r \leq 1$. Setting $I = \text{Ker}(P_T(\sigma_r))$, Lemma 3.2.2 tells us that

$$\sigma(T|I) = \sigma(T) \setminus \sigma_r = \sigma(T) \cap \{\lambda : |\lambda| < r\}.$$

The proof is now complete. □

§3.4. Some useful corollaries

In this section (as above) a *function algebra* is a semisimple commutative unital Banach algebra, considered as an algebra of continuous functions on its character space. Let B be such an algebra. Then, as the reader will recall, B is said to be:

- *regular* if for each closed subset $F \subseteq \Phi_B$ and each point $x \in \Phi_B \setminus F$, there is some $f \in B$ satisfying $f(x) = 1$ and $f(F) \subseteq \{0\}$; and
- *strongly regular* if $\pi_B(x) = \{M(x)\}$ for every $x \in \Phi_B$.

When B is regular and $x \in \Phi_B$, we write $J(x)$ for the closure of the ideal

$$J_0(x) = \{f \in B : f^{-1}(0) \text{ is a neighbourhood of } x\}.$$

A celebrated result of Shilov ensures that when B is regular, $J(x)$ is the intersection of all the closed ideals $I \in \pi_B(x)$. We exploit this to establish the following consequence of Theorem 3.3.3.

Theorem 3.4.1. Let B be a regular function algebra with connected character space Φ_B . Let T be a Riesz endomorphism of B , and suppose that $T^*x_0 = x_0$. Then $J(x_0)$ is a T -invariant closed ideal, and

$$\{0\} \cup \sigma(T) = \{0\} \cup \sigma(T/J(x_0)).$$

Proof. Let $\phi = T^\dagger$. Showing that $J(x_0)$ is T -invariant is straightforward, and can be achieved by using the fact that $Tf = f \circ \phi$ for each $f \in B$. Let $\mathcal{J} \subseteq \pi_B(x_0)$ be the family of ideals supplied by Theorem 3.3.3, and fix any $\epsilon > 0$. Then, by part (vi), there is some $I \in \mathcal{J}$ such that

$$\sigma(T|I) \subseteq \{\lambda : |\lambda| < \epsilon\}.$$

As the restriction of a Riesz operator to a closed T -invariant subspace, $T|I$ is also a Riesz operator. This implies, in particular, that $\rho(T|I)$ is connected, hence that $\sigma(T|N) \subseteq \sigma(T|I)$ for every $(T|I)$ -invariant closed subspace N of I . Using Shilov's result, we therefore see that

$$\sigma(T|J(x_0)) \subseteq \{\lambda : |\lambda| < \epsilon\}.$$

Since an identical conclusion is available for every $\epsilon > 0$, $T|J(x_0)$ is quasinilpotent. The result now follows from the standard spectral inclusions $\sigma(T) \subseteq \sigma(T|N) \cup \sigma(T/N)$ and $\sigma(T/N) \subseteq \sigma(T) \cup \sigma(T|N)$, which hold for any closed T -invariant subspace N of B . $\square$

The next result indicates, among other things, that a Riesz endomorphism of a strongly regular function algebra can never have a nontrivial spectrum. Its conclusions also apply to algebras such as $C^k(\bar{\Omega})$ when $k \in \mathbb{N}$ and Ω is a convex bounded open subset of $\mathbb{R}^d$.

Corollary 3.4.2. Let B be a regular function algebra with connected character space, and let T be a Riesz endomorphism of B . Let x_0 be the element of Φ_B for which $T^*x_0 = x_0$, and suppose that $J(x_0)$ has finite codimension in B . Then $\sigma(T) = \{0, 1\}$.

Proof. By the previous result, the nonzero spectrum of T coincides with that of $T/J(x_0)$ (an operator on a finite dimensional space). This means that $\sigma(T)$ is finite. Now choose any nonzero point λ of $\sigma(T)$. Since T is a Riesz operator, λ is necessarily an eigenvalue. It follows that $\lambda^n \in \sigma(T)$ for every $n \in \mathbb{N}$, which leads to a contradiction unless $\lambda = 1$ (this uses part (i) of Theorem 3.1.2). $\square$

Although the class of algebras to which Corollary 3.4.2 applies is very large, quite a number of standard regular function algebras lie entirely beyond its reach. Examples include the ‘big’ Lipschitz algebras $\text{Lip}(S, \rho)$ over compact metric spaces. However, even these algebras are not immune to the following theorem.

Theorem 3.4.3. Let B be a semi-prime commutative unital Banach algebra with connected character space Φ_B . Let T be a quasi-compact endomorphism of B , and let x_0 be a character for which $T^*x_0 = x_0$. Suppose that

$$\sigma(T) \cap \{\lambda : |\lambda| > r_e(T)\}$$

contains a point other than 1. Then there is a bounded point derivation d at x_0 such that $T^*d \neq 0$.

Proof. Our hypothesis on $\sigma(T)$ means that there is some $r > 0$ for which the set $\sigma_r = \sigma(T) \cap \{\lambda : |\lambda| \geq r\}$ contains at least two points. The ideal $I_r = \text{Ker}(P_T(\sigma_r))$ therefore has codimension at least 2. As a non-trivial finite dimensional commutative unital Banach algebra with exactly one maximal ideal, the quotient algebra B/I_r supports at least one nonzero bounded point derivation; call this d_0 . Letting $Q : B \rightarrow B/I_r$ be the quotient map, set $d = Q^*d_0$; it is straightforward to verify that this is a bounded point derivation on B at x_0 with

$$T^*d = Q^*(T/I_r)^*d_0.$$

The result follows, since Q^* is injective and (by Lemma 3.3.2), $(T/I)^*$ is invertible. $\square$

This result complements an existing result of Udo Klein (c.f. [24], Theorem 10), who proved an analogous assertion for compact endomorphisms of uniform algebras by more direct methods.

§3.5. Applications for some concrete function algebras

It goes without saying that the results of Section 3.4 have some serious consequences for Riesz endomorphisms of strongly regular algebras such as $C(S_0)$ and $\text{lip}(S, \rho)$ (where S_0 is any connected compact Hausdorff space and (S, ρ) is any connected compact metric space); it is immediate from any of the three previous results that such operators all have spectra equal to $\{0, 1\}$. Our aim, at least for the first half of this section, is to describe another large class of function algebras with this property. For simplicity, we work with function algebras defined on (the closure of) the open ball Ω in $\mathbb{R}^d$. However much of what we have to say applies equally to algebras on $\mathbb{T}^d$ and other connected compact smooth manifolds.

Following [2], we write $C^k(\bar{\Omega})$ for the algebra of k -times continuously-differentiable functions $u : \Omega \rightarrow \mathbb{C}$ which, together with their partial derivatives $D^\alpha u$ of orders $|\alpha| \leq k$, extend to be continuous on $\bar{\Omega}$. This is a regular function algebra under the norm given by

$$\|u\|_{k,\infty,\Omega} = \sum_{|\alpha| \leq k} \frac{1}{\alpha!} \sup_{x \in \Omega} |(D^\alpha u)(x)|$$

for every $u \in C^k(\bar{\Omega})$. Setting $C^\infty(\bar{\Omega}) = \bigcap_{k \geq 1} C^k(\bar{\Omega})$, we now recall a famous result of Shilov; a particularly straightforward proof of this theorem can be found on page 58 of Mirkil's treatise, [44].

Theorem 3.5.1 (Shilov, 1950). Let B be a Banach space of functions on $\bar{\Omega}$ (or any other compact smooth manifold), equipped with pointwise addition and a topology stronger than that of pointwise convergence on $\bar{\Omega}$. Then $C^\infty(\bar{\Omega}) \subset B$ if and only if $C^k(\bar{\Omega}) \subseteq B$ for some $k \in \mathbb{N}$.

The following result is also due to Shilov, and can be found in the book, [25], of Gel'fand, Raikov, and Shilov.

Theorem 3.5.2 ([25], Chapter VI, Theorem 39.1). Let B' and B'' be regular function algebras with the same space X of maximal ideals; furthermore, let $B' \subseteq B''$ with B' dense in B'' . If $J' \subset B'$ and $J'' \subset B''$ are the minimal primary ideals corresponding to the same point $x_0 \in X$ and the quotient algebra B'/J' is finite dimensional then B''/J'' is also finite dimensional, and its dimension is no greater than that of B'/J' .

With these two results at our disposal, we can give the following application of Corollary 3.4.2. The regularity of the domain Ω means that the next result applies, in particular, when B is one of the Sobolev algebras

$$W^{s,p}(\Omega) = \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \text{ for } |\alpha| \leq s\}$$

with $1 \leq p, s < +\infty$ and $sp > d$.

Theorem 3.5.3. Let B be a Banach algebra of continuous functions on $\bar{\Omega}$ for which $C^\infty(\bar{\Omega})$ is dense in B . Let T be a Riesz endomorphism of B . Then $\sigma(T) = \{0, 1\}$.

Proof. It is clear from Theorem 3.5.1 that, under our hypotheses, there is some $k \in \mathbb{N}$ for which $C^k(\bar{\Omega})$ is dense in B . Since the inclusion maps $C^k(\bar{\Omega}) \rightarrow B$ and $B \rightarrow C(\bar{\Omega})$ are both continuous, there are constants C_1 and C_2 such that

$$\|f\|_{0,\infty,\Omega} \leq C_1 \|f\|_B \leq C_2 \|f\|_{k,\infty,\Omega}$$

for every $f \in C^k(\bar{\Omega})$. Since the spectral radius of any element of $C^k(\bar{\Omega})$ is therefore the same with respect to B as it is with respect to $C^k(\bar{\Omega})$, it follows from Theorem 1 of [12] that these two algebras have ‘the same’ space of maximal ideals. This latest assertion is, of course, to be understood in the sense that the restriction map $x \mapsto x|_{C^k(\bar{\Omega})}$ is a homeomorphism of Φ_B onto $\Phi_{C^k(\bar{\Omega})}$. Recalling that every minimal primary ideal in $C^k(\bar{\Omega})$ is of the form

$$J' = \{u \in C^k(\bar{\Omega}) : (D^\alpha u)(x) = 0 \text{ for } |\alpha| \leq k\},$$

for some $x \in \bar{\Omega}$, an application of Theorem 3.5.2 (with $B' = C^k(\bar{\Omega})$ and $B'' = B$) indicates that $B/J(x)$ is finite dimensional for every $x \in \Phi_B$. The result is now an obvious consequence of Corollary 3.4.2. $\square$

As the next result makes clear, double duality can sometimes come to our rescue when dealing with algebras which are not immediately susceptible to the results of Section 3.4. The following theorem has been obtained by Golbaharan and Mahyar in [27] by more concrete methods.

Theorem 3.5.4. Let $0 < \alpha < 1$, and let (S, ρ^α) be a connected compact Hölder metric space. Let T be a Riesz endomorphism of $\text{Lip}(S, \rho^\alpha)$. Then $\sigma(T) = \{0, 1\}$.

Proof. For each $x \in S$, let δ_x denote the functional on $\text{lip}(S, \rho^\alpha)$ given by evaluation at x . Except for an appeal to Corollary 3.4.2, the proof of Theorem 3.5.4 is a straightforward combination of the following standard facts:

- (a) there is a Banach space isomorphism $\Theta : \text{lip}(S, \rho^\alpha)^{**} \longrightarrow \text{Lip}(S, \rho^\alpha)$ which satisfies

$$(\Theta\omega)(x) = \omega(\delta_x) \quad (3.5.1)$$

for every $\omega \in \text{lip}(S, \rho^\alpha)^{**}$ and $x \in S$;

- (b) $\text{lip}(S, \rho^\alpha)$ is strongly regular;
- (c) every endomorphism of $\text{Lip}(S, \rho^\alpha)$ is given by composition with a Lipschitz continuous self-map of S ; such operators always have $\text{lip}(S, \rho^\alpha)$ among their closed invariant subspaces;
- (d) the spectrum of a bounded operator on a Banach space is the same as the spectrum of its adjoint (and double adjoint).

Assertions (a) and (b) are proved in [13], (c) appears in [37], and (d) can be found in any good textbook on operator theory. In light of part (c), we are free to set $T_0 = T|_{\text{lip}(S, \rho^\alpha)}$; there is then a self-map ϕ of S for which

$$Tu = u \circ \phi \quad \text{and} \quad T_0v = v \circ \phi$$

for every $u \in \text{Lip}(S, \rho^\alpha)$ and $v \in \text{lip}(S, \rho^\alpha)$. Combining these two equations with (3.5.1), we easily obtain the intertwining relation

$$\Theta T_0^{**} = T\Theta.$$

As Θ is an isomorphism, this implies that $\sigma(T) = \sigma(T_0^{**})$ which, by fact (d), implies that $\sigma(T) = \sigma(T_0)$. As the restriction of a Riesz operator to a closed invariant subspace, T_0 is a Riesz endomorphism of $\text{lip}(S, \rho^\alpha)$. That $\sigma(T_0) = \{0, 1\}$ is clear from (b) and Corollary 3.4.2. $\square$

The isomorphism $\text{lip}(S, \rho^\alpha)^{**} = \text{Lip}(S, \rho^\alpha)$ is unfortunately not always available when $\alpha = 1$ (that is, when we have a compact metric space which is not a Hölder metric space). For more general compact metric spaces, we must content ourselves with the following (weaker) result, derived from Theorem 3.4.3.

Theorem 3.5.5. Let (S, ρ) be a connected compact metric space, and let ϕ be a Lipschitz continuous self-map of S for which $T : f \mapsto f \circ \phi$ is a Riesz endomorphism of $\text{Lip}(S, \rho)$. Let x_0 be the unique fixed-point of ϕ , and suppose that

$$\inf_{\delta > 0} L(\phi|_{\text{Ball}(x_0, \delta)}) = 0. \quad (3.5.2)$$

Then $\sigma(T) = \{0, 1\}$.

Proof. With Theorem 3.4.3 at our disposal, it suffices to show that T^* annihilates every bounded point derivation on $\text{Lip}(S, \rho)$ at x_0 . For each $x, y \in S$ with $x \neq y$, let $\psi_{x,y}$ be the functional on $\text{Lip}(S, \rho)$ given by setting

$$\psi_{x,y}(f) = \frac{f(x) - f(y)}{\rho(x, y)}$$

for each $f \in \text{Lip}(S, \rho)$. Let Ψ_{x_0} be the set of all $d \in \text{Lip}(S, \rho)^*$ which can be realised as w^* -cluster points of sequences of the form $(\psi_{y_n, z_n})_{n=1}^\infty$, where $y_n \rightarrow x_0, z_n \rightarrow x_0$ and $y_n \neq z_n$ for every $n \in \mathbb{N}$; it is known (c.f. [55], Theorem 4.7.3) that the linear span of Ψ_{x_0} is w^* -dense in the space of all bounded point derivations on $\text{Lip}(S, \rho)$ at x_0 . To prove our theorem, it therefore suffices to prove that T^* annihilates Ψ_{x_0} .

Let $d \in \Psi_{x_0}$ and fix any $\epsilon > 0$. Take a sequence $((y_n, z_n))_{n=1}^\infty$ for which d is a w^* -cluster point of $(\psi_{y_n, z_n})_{n=1}^\infty$, $y_n \rightarrow x_0, z_n \rightarrow x_0$ and $y_n \neq z_n$ for all n . We are obviously free to take a subnet, $(\psi_{y_\gamma, z_\gamma})_{\gamma \in D}$ of $(\psi_{y_n, z_n})_{n=1}^\infty$ which converges w^* to d . As subnets of nets which converge to x_0 , both $(y_\gamma)_{\gamma \in D}$ and $(z_\gamma)_{\gamma \in D}$ converge to x_0 . Granted (3.5.2), we can take a $\delta > 0$ for which $L(\phi|_{\text{Ball}(x_0, \delta)}) \leq \epsilon$. Taking some $\gamma_0 \in D$ for which $y_\gamma, z_\gamma \in \text{Ball}(x_0, \delta)$ whenever $\gamma \geq \gamma_0$, we have

$$|\psi_{y_\gamma, z_\gamma}(Tf)| \leq L(\phi|_{\text{Ball}(x_0, \delta)})L(f)$$

for $\gamma \geq \gamma_0$ and $f \in \text{Lip}(S, \rho)$.

In the limit, we therefore have

$$|(T^*d)(f)| \leq L(\phi|_{\text{Ball}(x_0, \delta)})L(f) \leq \epsilon\|f\|,$$

so $\|T^*d\| \leq \epsilon$; the result follows. $\square$

Since a supercontraction ϕ of a compact metric space (S, ρ) satisfies

$$\inf_{\delta > 0} L(\phi|_{\text{Ball}(x, \delta)}) = 0$$

for every $x \in S$, our latest result actually subsumes the Kamowitz and Scheinberg's observation in [37] that $\sigma(T) = \{0, 1\}$ for every compact endomorphism of $\text{Lip}(S, \rho)$. Although a great deal of attention has been paid to endomorphisms of Lipschitz algebras, it does not appear to be known whether (3.5.2) is guaranteed when T is a Riesz endomorphism of $\text{Lip}(S, \rho)$.

Our final application demonstrates a little of what can be achieved when we appeal to Theorem 3.3.3 more directly. For the remainder of this section, B is a unital Banach algebra of functions which are continuous on the closed unit disk and holomorphic on its interior. We will also assume that B contains the inclusion map $z : \overline{\mathbb{D}} \rightarrow \mathbb{C}$. The only other restriction we will impose is that

$$(z - p)^k B = \{f \in B : f(p) = f'(p) = \cdots = f^{(k-1)}(p) = 0\} \quad (3.5.3)$$

for each $p \in \mathbb{D}$ (the *open* disk) and each $k \in \mathbb{N}$. We note that this condition is automatically satisfied when the polynomials (in z) are dense in B and there is a $c_p > 0$ for each $p \in \mathbb{D}$ such that

$$\|f\| \leq c_p \|(z - p)f\|$$

for each $f \in B$. To see this, suppose that $f \in B$ vanishes to order $k - 1$ at some $p \in \mathbb{D}$, and take polynomials $(f_n)_{n=1}^\infty$ such that $f_n \rightarrow f$ as $n \rightarrow \infty$. Since the right hand side of (3.5.3) is topologically complemented in B , we are free to assume that each f_n also vanishes to order $k - 1$ at p . This, combined with the fact that each f_n is a polynomial, means that there is some $g_n \in B$ for each f_n such that $f_n = (z - p)^k g_n$. Since our assumption makes the operator $g \mapsto (z - p)^k g$ bounded below, the rest is clear.

Our insistence that (3.5.3) holds is purely to give us access to the following theorem of Domar from 1982.

Theorem 3.5.6 ([16], Theorem 4). Let A be a commutative unital Banach algebra, let $f \in A$, and suppose that, for each $n \in \mathbb{N}$, the principal ideal $M_n = f^n A$ has codimension n in A . Then the M_n are the *only* closed primary ideals of finite-codimension with $M_n \subseteq M_1$.

In light of our assumptions on B , this implies that the only closed primary ideals at p of finite codimension are of the form (3.5.3). We can now prove the following theorem; given the looseness of our assumptions on B , it subsumes a large number of existing algebra-specific results already in the literature.

Theorem 3.5.7. Let ϕ be a continuous self-map of $\overline{\mathbb{D}}$ for which $f \circ \phi \in B$ for every $f \in B$. Suppose that the operator defined by

$$Tf = f \circ \phi \quad (f \in B)$$

is Riesz, and that ϕ has a fixed point p in $\mathbb{D}$, the open unit disk. Then

$$\sigma(T) = \{0, 1\} \cup \{\phi'(p)^k : k \in \mathbb{N}\}.$$

Proof. Combining Domar's theorem with assumption (3.5.3), we have

$$\pi_B(p) = \{M_k : k \in \mathbb{N}\}, \tag{3.5.4}$$

where $M_k = \{f \in B : f(p) = f'(p) = \cdots = f^{(k-1)}(p) = 0\}$ for each $k \in \mathbb{N}$. It is clear that B cannot have any idempotents other than the functions $f = 0$ and $f = 1$; an appeal to the Shilov idempotent theorem therefore tells us that Φ_B is connected. Invoking Theorem 3.3.3, there is a family $\mathcal{J} \subseteq \pi_B(p)$ of closed, primary ideals of finite codimension in B such that

$$\sigma(T) \setminus \{0\} = \bigcup_{I \in \mathcal{J}} \sigma(T/I).$$

We know, of course, that each $I \in \mathcal{J}$ is equal to M_k for some $k \in \mathbb{N}$, and this makes the problem of determining $\sigma(T)$ particularly tractable.

For $k \in \mathbb{N}$, let us consider the homomorphisms $\theta_k \in \text{Hom}(B, \mathbb{C}[\xi]_{\leq k})$ and $T_k \in \text{End}(\mathbb{C}[\xi]_{\leq k})$ defined by the equations

$$\theta_k(z - p) = \xi \quad \text{and} \quad T_k(\xi) = \theta_k(\phi - p).$$

Clearly, $T_k \theta_k = \theta_k T$ and $M_k = \text{Ker}(\theta_{k-1})$ for each k . Using either formula (2.4.4) or induction, it is straightforward to show that the matrix of T_k with respect to the basis $(\xi^i)_{i=0}^k$ is lower triangular with $(\phi'(p)^i)_{i=0}^k$ along the diagonal. This implies, in particular, that

$$\sigma(T_k) = \{\phi'(p)^i : i = 0, \dots, k\}.$$

for each k . Since every $I \in \mathcal{J}$ is of the form $M_k = \text{Ker}(\theta_{k-1})$ for some $k \in \mathbb{N}$, the intertwining relations $T_k \theta_k = \theta_k T$ ($k \in \mathbb{N}$) ensure that each T/I ($I \in \mathcal{J}$) is similar to T_k for some k , hence that

$$\sigma(T) \subseteq \{0, 1\} \cup \{\phi'(p)^k : k \in \mathbb{N}\}.$$

That $\{0, 1\} \subseteq \sigma(T)$ is obvious, so it only remains to show that $\phi'(p)^k \in \sigma(T)$ for every $k \in \mathbb{N}$. If it were guaranteed that $\mathcal{J} = \pi_B(p)$, this would already be clear. However, this is not what Theorem 3.3.3 tells us. To complete the proof, we consider the bounded linear functional on B given by

$$\delta'_p(f) = f'(p) \quad (f \in B).$$

The chain rule now gives $T^* \delta'_p = \phi'(p) \delta'_p$. As T is Riesz, this means that either $\phi'(p) = 0$, or $\phi'(p)$ is an eigenvalue of T . The result now follows from the fact that the set of eigenvalues of T is closed under powers. $\square$

The reader may wish to note that we have said nothing about the situation when $p \in \Phi_B \setminus \mathbb{D}$. Here, the result is much more dependent on the algebra under consideration. However, progress can still be made with the help of Theorem 3.4.3; the situation is particularly straightforward when B has no non-zero bounded point derivations at points of $\Phi_B \setminus \mathbb{D}$.

Remark. The argument used to prove Theorem 3.5.7 is also available in quotients of the form $L^1(\mathbb{R}, \omega) / \{f : \hat{f}(\xi) = 0 \text{ for } \xi \in [0, 1]\}$, where the weight ω is given by $\omega(x) = e^{|x|^\alpha}$ ($x \in \mathbb{R}$) for some $0 < \alpha < 1$, and $f \mapsto \hat{f}$ is the usual Fourier transform.

In [15], Domar proves that these algebras have a similar ideal structure to those considered above. However, there is a danger (in light of the existence of theorems of Beurling-Helson type) that many of these algebras may not support any non-trivial Riesz endomorphisms at all; this matter does not seem to have been entirely settled.

§3.6. Quasi-compact automorphisms

In this section, we examine the circumstances under which an infinite-dimensional commutative unital Banach algebra can support a quasi-compact *automorphism*. Our goal will be to prove the following theorem.

Theorem 3.6.1. Let B be a commutative unital Banach algebra, and suppose that B supports a quasi-compact automorphism. Then $\text{Rad}(B)$ has finite codimension in B .

Since the full proof of this result would take us too far afield, we shall rely on the following lemma; this (and more) will be proved in the next chapter.

Lemma 3.6.2. Let X be a Banach space, let $T \in L(X)$, and let F be any closed T -invariant subspace of X . Then $r_e(T/F) \leq r_e(T)$.

Theorem 3.6.1 is now easily obtained by combining this result with the Kamowitz-Scheinberg automorphism theorem; nothing more is required.

Proof of Theorem 3.6.1. Suppose that some $T \in \text{Aut}(B)$ is quasi-compact. Since both T and T^{-1} are necessarily endomorphisms of B , both preserve $\text{Rad}(B)$. It follows that $T_1 = T/\text{Rad}(B)$ is an automorphism of $B/\text{Rad}(B)$. As an automorphism of a semi-simple commutative unital Banach algebra, T_1 is vulnerable to the Kamowitz-Scheinberg automorphism theorem (c.f. [36], Theorem 3); this asserts that an automorphism of a semi-simple commutative Banach algebra is either periodic (in the sense that one of its powers is the identity), or its spectrum contains the entire unit circle. We consider these two possibilities separately; in light of Lemma 3.6.2, T_1 is quasi-compact.

Assuming that $\sigma(T_1) \supset \mathbb{T}$ leads to an immediate contradiction of the punctured-neighbourhood theorem; since $r_e(T_1) < 1$, every point of the set $\sigma(T_1) \cap \{\lambda : |\lambda| \geq 1\}$ must be isolated in $\sigma(T_1)$.

With this first possibility excluded, there must be some $n \in \mathbb{N}$ for which T_1^n is the identity. Now, since any power of a quasi-compact operator is also quasi-compact, the identity of $B/\text{Rad}(B)$ is quasi-compact. This implies that $B/\text{Rad}(B)$ is finite dimensional. $\square$

Remarks.

- Lemma 3.6.2 can also be used in combination with Theorem 3.1.3 to show that $T^{*\infty}\Phi_B$ is a singleton when T is a quasi-compact endomorphism of *any* (not necessarily semi-prime) commutative unital Banach algebra with connected character space.
- The conclusion of Theorem 3.5.4 also applies to Riesz endomorphisms of algebras of the form $C^{k,\gamma}(\bar{\Omega})$, where $k \in \mathbb{Z}_+$, $0 < \gamma < 1$ and Ω is a bounded domain in $\mathbb{R}^d$ (and can be established in a similar way). To prove this, one should consider the closure $\lambda^{k,\gamma}(\bar{\Omega})$ of $C^\infty(\bar{\Omega})$ in $C^{k,\gamma}(\bar{\Omega})$ in place of the little-Lipschitz space; it is shown in [23] that defining Θ as in Theorem 3.5.4 gives an isomorphism of $\lambda^{k,\gamma}(\bar{\Omega})^{**}$ onto $C^{k,\gamma}(\bar{\Omega})$, and Theorem 3.5.3 does the rest.
- It appears that versions of Theorems 3.5.4 and 3.5.5 may also be obtainable using the lower bound for $r_e(T)$ obtained by Behrouzi in [8] using more direct methods.

Some contributions to operator theory

§4.1. Introduction

Our aim for the first part of this chapter is to prove Lemma 3.6.2, the ingredient we need to round off our proof of Theorem 3.6.1. This will require us to examine the relationship between the essential spectrum of an operator and the essential spectra of its quotients and restrictions to closed invariant subspaces. Despite its obvious connections with much older and more established theories, the literature on this subject is surprisingly limited, and the majority of the most useful results in this direction have been discovered only in the last few years. It will make our discussion considerably more straightforward if we introduce some more notation. For a closed subspace F of a Banach space X , we write

$$\begin{aligned} K_F(X) &= L_F(X) \cap K(X) \quad , \quad \mathfrak{K}_F(X) = L_F(X) \cap \mathfrak{K}(X) \\ \mathfrak{C}_F(X) &= L_F(X)/K_F(X) \quad , \quad \mathfrak{C}_F^{\mathfrak{K}}(X) = L_F(X)/\mathfrak{K}_F(X). \end{aligned}$$

The algebra $\mathfrak{C}_F^{\mathfrak{K}}(X)$ will play only the most minor rôle in proceedings, so we need not be particularly troubled by the fact that it doesn't have a 'canonical' topology (except when X is finite dimensional). Given any complex unital algebra A and any element $T \in A$, we will write

$$\sigma(T; A) = \{\lambda \in \mathbb{C} : \lambda - T \text{ has no inverse in } A\},$$

for the spectrum of T relative to A ; we shall return to our previous notation only when the algebra with respect to which the spectrum is being taken is unambiguous. The polynomially convex hull of a compact subset σ of $\mathbb{C}$ will be denoted $\widehat{\sigma}$.

We will be concerned with relationships between the sets

$$\begin{aligned} \sigma_e(T) \quad , \quad \sigma_e(T|F) \quad , \quad \sigma_e(T/F) \quad , \quad \sigma(T + K_F(X); \mathfrak{C}_F(X)) \\ \text{and} \quad \sigma(T + \mathfrak{F}_F(X); \mathfrak{C}_F^{\mathfrak{F}}(X)). \end{aligned}$$

It is already known that:

- (a) for any permutation (T_1, T_2, T_3) of $(T, T|F, T/F)$, we have the inclusion $\sigma_e(T_1) \subseteq \sigma_e(T_2) \cup \sigma_e(T_3)$; and
- (b) $\sigma(T + \mathfrak{F}_F(X); \mathfrak{C}_F^{\mathfrak{F}}(X)) = \sigma_e(T) \cup \sigma_e(T|F)$

whenever F is a closed subspace of a Banach space, and $T \in L_F(X)$.

The first result can be found in the paper [14], of Djordjević and Duggal, which builds on some of the results obtained by Barnes in [6]. Equation (b) is proved in the monograph, [7], of Barnes, Murphy, Smyth and West. All three of the references mentioned contain various improvements of (a) and (b) in the presence of additional hypotheses on X , F and T . For example:

- (c) $\sigma_e(T) = \sigma_e(T|F) \cup \sigma_e(T/F)$ if F admits a T -invariant complement in X .

It is also easy to show that $\sigma_e(T) = \sigma_e(T|F)$ when F is of finite codimension in X . The following example (suggested to me by J.R. Partington) illustrates that the inclusion $\sigma_e(T|F) \subseteq \sigma_e(T)$ does not hold in general.

Example 4.1.1. Let T be the operator on $X = L^2(\mathbb{R})$ given by setting

$$(Tu)(x) = u(x - 1), \quad (u \in C_c(\mathbb{R}), x \in \mathbb{R})$$

and let F be the image of $L^2(\mathbb{R}_+)$ under the natural embedding of $L^2(\mathbb{R}_+)$ into $L^2(\mathbb{R})$. Since T is an invertible isometry, it is clear that

$$\sigma_e(T) \subseteq \sigma(T) \subseteq \mathbb{T}.$$

However, it is also easy to see that $0 \in \sigma_e(T|F)$, since $(Tu|v) = 0$ for every $u \in F$ and every $v \in C_c((0, 1))$.

One important feature of this example is that $0 \in \sigma_e(T|F)$ is still contained in $\widehat{\sigma_e(T)}$. By the end of this chapter, we will have shown that points of $\sigma_e(T|F)$ *always* lie in $\widehat{\sigma_e(T)}$; our main result is the following theorem.

Theorem 4.1.2. Let X be a complex Banach space, and let F be an arbitrary closed subspace of X . Then

$$\sigma_e(T|F) \cup \sigma_e(T/F) \subseteq \widehat{\sigma_e(T)}$$

for every $T \in L_F(X)$. In particular, we always have

$$\max\{r_e(T|F), r_e(T/F)\} \leq r_e(T)$$

in this situation.

This result generalises naturally to diagrams of the form

$$\begin{array}{ccccccc} 0 & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & X_3 \longrightarrow 0 \\ & & \downarrow T_1 & & \downarrow T_2 & & \downarrow T_3 \\ 0 & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & X_3 \longrightarrow 0, \end{array} \quad (4.1.1)$$

where:

- (I) X_1, X_2 and X_3 are Banach spaces, and the arrows are all bounded linear operators;
- (II) the upper and lower rows are the same Fredholm complex;
- (III) the entire diagram commutes modulo the compact operators.

The reader is reminded that a *Fredholm complex* is a sequence

$$\dots \longrightarrow X_i \xrightarrow{\Lambda_i} X_{i+1} \xrightarrow{\Lambda_{i+1}} X_{i+2} \longrightarrow \dots$$

of Banach spaces and bounded linear operators for which the image of each arrow has finite codimension in the kernel of the next.

Diagrams of the form (4.1.1) occur naturally when one begins to explore the connections between weighted composition operators and endomorphisms of algebras of smooth functions, and some interesting results can be obtained (relatively cheaply) by developing the associated Fredholm theory.

The material in this chapter is organised as follows. After arming ourselves with the appropriate tools in Section 4.2, we begin Section 4.3 with a proof of Theorem 4.1.2. This result is then extended to diagrams of the form (4.1.1), along with a few other results concerning operator ideals and essential spectra. Section 4.4 contains two examples in which the theory developed in this Chapter is applicable to the study of endomorphisms and weighed composition operators on certain algebras of smooth functions.

§4.2. Preliminaries

Our first preliminary result is an easy consequence of Riesz's lemma; it will be very important in the next section.

Lemma 4.2.1. Let X and Y be Banach spaces, let F be a closed subspace of X , and let $T \in L(X/F, Y)$. Then TQ_F is compact (or weakly compact) if and only if T is compact (resp. weakly compact).

Proof. By Riesz's lemma, the image under TQ_F of the open unit ball of X is the same as the image under T of the open unit ball of X/F . The result is therefore immediate. $\square$

Some of the following discussion will require the reader to be familiar with the notion of an *operator ideal*. This is a rule $\mathfrak{A}$ which assigns a subspace

$$\mathfrak{F}(X, Y) \subseteq \mathfrak{A}(X, Y) \subseteq L(X, Y)$$

to each pair of Banach spaces (X, Y) in such a way that if

$$X_1 \xrightarrow{S} X_2 \xrightarrow{T} X_3 \xrightarrow{U} X_4$$

is a sequence of Banach spaces and bounded operators, then $UTS \in \mathfrak{A}(X_1, X_4)$ as soon as $S \in \mathfrak{A}(X_1, X_2)$, $T \in \mathfrak{A}(X_2, X_3)$ or $U \in \mathfrak{A}(X_3, X_4)$. We shall say that an operator ideal $\mathfrak{A}$ is *closed* if $\mathfrak{A}(X, Y)$ is a norm-closed subspace of $L(X, Y)$ for each pair of Banach spaces X and Y .

Although the notion of an operator ideal is clearly more general, each closed operator ideal does give us a way to assign a closed ideal $\mathfrak{A}(X) \subseteq L(X)$ to each Banach space X . The following proposition gives us, among other things, a sufficient condition for two algebras of the form $L(X)/\mathfrak{A}(X)$ to be isomorphic (as unital Banach algebras); this result does not appear to be in the literature, but I imagine that it is known to specialists.

Lemma 4.2.2. Let $\mathfrak{A}$ be a closed operator ideal, and let $A : X \longrightarrow Y$ be a Fredholm operator between Banach spaces X and Y . Then there is a unital Banach algebra isomorphism

$$\theta_A : L(X)/\mathfrak{A}(X) \longrightarrow L(Y)/\mathfrak{A}(Y)$$

such that $\theta_A(S + \mathfrak{A}(X)) = T + \mathfrak{A}(Y)$ for all $(S, T) \in L(X) \times L(Y)$ with $AS - TA \in \mathfrak{A}(X, Y)$.

Proof. Since A is Fredholm, there is an operator $A' \in L(Y, X)$ for which $1 - AA'$ and $1 - A'A$ are finite rank operators. Setting

$$\theta(S) = ASA'$$

for $S \in L(X)$ gives us a bounded operator $\theta : L(X) \longrightarrow L(Y)$. Since θ maps $\mathfrak{A}(X)$ into $\mathfrak{A}(Y)$, it descends to give a bounded operator

$$\theta_A : L(X)/\mathfrak{A}(X) \longrightarrow L(Y)/\mathfrak{A}(Y)$$

such that $\theta_A(S + \mathfrak{A}(X)) = \theta(S) + \mathfrak{A}(Y)$ for each $S \in L(X)$. To show that θ_A is multiplicative, we take $S_1, S_2 \in L(X)$ and note that

$$\theta(S_1)\theta(S_2) = \theta(S_1S_2) + AS_1(A'A - 1)S_2A',$$

so $\theta(S_1)\theta(S_2) - \theta(S_1S_2) \in \mathfrak{F}(Y) \subseteq \mathfrak{A}(Y)$. Showing that θ_A is unital is trivial, since

$$\theta(1) = AA' = 1 + (AA' - 1)$$

and $AA' - 1 \in \mathfrak{F}(Y)$. That θ_A is a unital homomorphism is now clear.

Letting $\theta'(T) = A'TA$ for $T \in L(Y)$, two easy calculations show that

$$(\theta'\theta)(S) - S \in \mathfrak{A}(X) \text{ and } (\theta\theta')(T) - T \in \mathfrak{A}(Y),$$

so θ' descends to $L(L(Y)/\mathfrak{A}(Y), L(X)/\mathfrak{A}(X))$ as an inverse for θ_A . To complete the proof, take any $(S, T) \in L(X) \times L(Y)$, and suppose that $C = AS - TA$ belongs to $\mathfrak{A}(X, Y)$. Then

$$\theta(S) = ASA' = (TA + C)A' = T + T(AA' - 1) + CA'$$

so $\theta(S)$ and T belong to the same coset in $L(Y)/\mathfrak{A}(Y)$. $\square$

Corollary 4.2.3. Let X and Y be Banach spaces for which $L(X, Y)$ contains a Fredholm operator $A : X \rightarrow Y$. Let $(S, T) \in L(X) \times L(Y)$ be a pair of operators for which $AS - TA \in \mathfrak{A}(X, Y)$. Then

$$\sigma(S + \mathfrak{A}(X); L(X)/\mathfrak{A}(X)) = \sigma(T + \mathfrak{A}(Y); L(Y)/\mathfrak{A}(Y))$$

and $S \in \mathfrak{A}(X)$ if and only if $T \in \mathfrak{A}(Y)$.

The final ingredients needed to prove Theorem 4.1.2 are two standard facts about spectral projections. Firstly:

Lemma 4.2.4. Let X be a Banach space, let $T \in L(X)$, and let λ be an isolated point of $\sigma(T)$. Then

$$2\pi i P_T(\lambda) = \int_{\partial D} (z - T)^{-1} dz \quad (4.2.1)$$

for any closed disk D containing λ and no other point of $\sigma(T)$.

Secondly:

Theorem 4.2.5. Let X be a Banach space, let $T \in L(X)$, and let λ be either an isolated point of $\sigma(T)$ or a point of $\mathbb{C} \setminus \sigma(T)$. Then $P_T(\lambda)$ has finite rank if and only if $\lambda - T$ is a Fredholm operator.

Remark 4.2.6. We have already used the ‘if’ part of Theorem 4.2.5 in Chapter 3; our work in this chapter uses the ‘only if’ implication as well.

§4.3. Main results

Our first order of business is the proof of Theorem 4.1.2. The reader is reminded that, for a compact subset K of $\mathbb{C}$, the set $\mathbb{C} \setminus \widehat{K}$ is just the unbounded component of $\mathbb{C} \setminus K$. It is standard that when F is a closed subspace of a Banach space X , the inclusion $\sigma(T|F) \subseteq \widehat{\sigma(T)}$ holds for each $T \in L_F(X)$; we will now show that the same is true with σ replaced by σ_e .

Proof of Theorem 4.1.2. Fix any $\lambda \in \mathbb{C} \setminus \widehat{\sigma_e(T)}$. By the punctured neighbourhood theorem, there is a closed disk D containing λ for which $D \setminus \{\lambda\}$ belongs to $\mathbb{C} \setminus \sigma(T)$.

Let us assume, for the moment, that $D \setminus \{\lambda\} \subseteq \mathbb{C} \setminus \widehat{\sigma(T)}$; this is actually always the case, but the proof can proceed more naturally if we leave verifying this until we have finished everything else. The fact that $D \setminus \{\lambda\} \subseteq \mathbb{C} \setminus \widehat{\sigma(T)}$ is important for two reasons. Firstly (remembering that $\sigma(T|F) \subseteq \widehat{\sigma(T)}$), it guarantees that both $z - T$ and $z - T|F$ are invertible for $z \in D \setminus \{\lambda\}$; this gives us access to the spectral projections

$$P_T(\lambda) = \frac{1}{2\pi i} \int_{\partial D} (z - T)^{-1} dz \quad \text{and} \quad P_{T|F}(\lambda) = \frac{1}{2\pi i} \int_{\partial D} (z - T|F)^{-1} dz.$$

However, it is not at all obvious at this point that these two operators have any particular special relationship. In particular, it is not clear that $P_T(\lambda)$ necessarily preserves F or, even if it does, that $P_T(\lambda)|F$ is the same as $P_{T|F}(\lambda)$. This is the second reason why it is so important that points of $D \setminus \{\lambda\}$ belong to the unbounded component of $\mathbb{C} \setminus \sigma(T)$.

To see how this helps, let $i_F \in L(F, X)$ be the inclusion map, and note that for $|z| > r(T)$, a Laurent series expansion gives

$$i_F(z - T|F)^{-1} = \sum_{n=0}^{\infty} \frac{i_F(T|F)^n}{z^{n+1}} = \sum_{n=0}^{\infty} \frac{T^n i_F}{z^{n+1}} = (z - T)^{-1} i_F.$$

The identity principle now ensures that $i_F(z - T|F)^{-1} = (z - T)^{-1} i_F$ holds for all $z \in \mathbb{C} \setminus \widehat{\sigma(T)}$.

We therefore have

$$\begin{aligned} i_F P_{T|F}(\lambda) &= \frac{1}{2\pi i} \int_{\partial D} i_F (z - T|F)^{-1} dz \\ &= \frac{1}{2\pi i} \int_{\partial D} (z - T)^{-1} i_F dz = P_T(\lambda) i_F. \end{aligned}$$

Letting $Q_F : X \rightarrow X/F$ again be the quotient map, this gives

$$Q_F P_T(\lambda) i_F = Q_F i_F P_{T|F}(\lambda) = 0,$$

which is equivalent to saying that $P_T(\lambda)F \subseteq F$. With this observed, the fact that $i_F P_{T|F}(\lambda) = P_T(\lambda) i_F$ is enough for us to conclude that $P_{T|F}(\lambda)$ is the restriction to F of $P_T(\lambda)$. Now we use Theorem 4.2.5: since $\lambda \notin \sigma_e(T)$, $P_T(\lambda)$ is a finite rank operator; the fact that $P_{T|F}(\lambda)$ is a restriction of $P_T(\lambda)$ means the same is therefore true of $P_{T|F}(\lambda)$; however, this implies that $\lambda \notin \sigma_e(T|F)$.

It now remains only to show that the punctured disk $D \setminus \{\lambda\}$ can be chosen so that $D \setminus \{\lambda\}$ is contained in the unbounded component of $\mathbb{C} \setminus \sigma(T)$. One way to do this is as follows. Since λ lies in the unbounded component of $\mathbb{C} \setminus \sigma_e(T)$, there is a continuous curve $c : [0, 1] \rightarrow \mathbb{C} \setminus \sigma_e(T)$ connecting λ to $\{z : |z| > r(T)\}$; let $\Gamma = c([0, 1])$. As the numbers

$$\{\text{dist}(z, \widehat{\sigma_e(T)}) : z \in \Gamma\},$$

are all positive, the punctured neighbourhood theorem ensures that $\sigma(T) \cap \Gamma$ is either empty or consists solely of isolated points of $\sigma(T)$; combined with the manifest compactness of Γ , this ensures that $\sigma(T) \cap \Gamma$ is finite. Assuming (as we may) that c is injective, let $t_1, \dots, t_n \in [0, 1]$ be the points for which $c(t_i) \in (\sigma(T) \cap \Gamma) \setminus \{\lambda\}$. Then, choosing $b_1, \dots, b_n \in C([0, 1])$ with $\|b_i\|_\infty$ small, $b_i(t_i) \neq 0$ and sufficiently small supports, the curve

$$t \mapsto c(t) + b_1(t) + b_2(t) + \dots + b_n(t)$$

connects λ to $\{z : |z| > r(T)\}$ without hitting $\sigma(T) \setminus \{\lambda\}$. It is now clear that λ belongs to the unbounded component of $\widehat{\rho(T) \cup \{\lambda\}}$. With our claim verified, we have now proved that $\sigma_e(T|F) \subseteq \widehat{\sigma_e(T)}$. The corresponding inclusion for $\sigma_e(T/F)$ follows trivially by duality. $\square$

Remark 4.3.1. In the last line of the preceding proof, we have tacitly used the fact that $\sigma_e(T/F) = \sigma_e(T^*|F^\perp)$.

Although we have ostensibly established only that

$$\sigma_e(T_1) \cup \sigma_e(T_3) \subseteq \widehat{\sigma_e(T_2)} \quad (4.3.1)$$

holds for diagrams of the form

$$\begin{array}{ccccccc} 0 & \longrightarrow & X_1 & \xrightarrow{\Lambda_1} & X_2 & \xrightarrow{\Lambda_2} & X_3 \longrightarrow 0 \\ & & \downarrow T_1 & & \downarrow T_2 & & \downarrow T_3 \\ 0 & \longrightarrow & X_1 & \xrightarrow{\Lambda_1} & X_2 & \xrightarrow{\Lambda_2} & X_3 \longrightarrow 0, \end{array} \quad (4.3.2)$$

when the diagram commutes and Λ_1 and Λ_2 are (respectively) the inclusion and quotient maps, this result extends immediately to the case where

$$0 \longrightarrow X_1 \xrightarrow{\Lambda_1} X_2 \xrightarrow{\Lambda_2} X_3 \longrightarrow 0$$

is an arbitrary short exact sequence of Banach spaces and bounded operators (still assuming that the diagram commutes). To see this, set $F = \text{Im}(\Lambda_1)$ and let $\hat{\Lambda}_1 : X_1 \longrightarrow F$ and $\hat{\Lambda}_2 : X/F \longrightarrow X_3$ be the (uniquely determined, bounded) operators which satisfy

$$\Lambda_1 = i_F \hat{\Lambda}_1 \quad \text{and} \quad \Lambda_2 = \hat{\Lambda}_2 Q_F;$$

it is clear that both are Banach space isomorphisms. We have

$$\begin{aligned} Q_F T_2 i_F &= (\hat{\Lambda}_2)^{-1} \Lambda_2 T_2 i_F && \text{(by definition of } \hat{\Lambda}_2) \\ &= (\hat{\Lambda}_2)^{-1} T_3 \Lambda_2 i_F && \text{(commutativity of the diagram)} \\ &= (\hat{\Lambda}_2)^{-1} T_3 \hat{\Lambda}_2 Q_F i_F = 0 \end{aligned}$$

so $T_2(F) \subseteq F$. Similarly trivial computations give $(T_2|F)\hat{\Lambda}_1 = \hat{\Lambda}_1 T_1$ and $T_3 \hat{\Lambda}_2 = \hat{\Lambda}_2(T_2/F)$, so T_1 and T_3 are similar to $T_2|F$ and T_2/F respectively.

This means, in particular, that

$$\sigma_e(T_1) = \sigma_e(T_2|F) \text{ and } \sigma_e(T_3) = \sigma_e(T_2/F).$$

The situation is slightly more complicated (though still eminently tractable) if we assume that

$$0 \longrightarrow X_1 \xrightarrow{\Lambda_1} X_2 \xrightarrow{\Lambda_2} X_3 \longrightarrow 0 \quad (4.3.3)$$

is a (possibly non-exact) Fredholm complex. We proceed with the help of the following lemma.

Lemma 4.3.2. Let Λ_1 and Λ_2 be bounded operators for which (4.3.3) is a Fredholm complex, and let $F = \text{Im}(\Lambda_1)$. Then there are Fredholm operators $A : F \longrightarrow X_1$ and $B : X_2/F \longrightarrow X_3$ such that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F & \xrightarrow{i_F} & X_2 & \xrightarrow{Q_F} & X_2/F & \longrightarrow & 0 \\ & & \downarrow A & & \downarrow \text{id} & & \downarrow B & & \\ 0 & \longrightarrow & X_1 & \xrightarrow{\Lambda_1} & X_2 & \xrightarrow{\Lambda_2} & X_3 & \longrightarrow & 0 \end{array} \quad (4.3.4)$$

commutes modulo the finite rank operators.

Proof. As an operator range of finite codimension in the Banach space $\text{Ker}(\Lambda_2)$, F is clearly a closed subspace of X_2 . Let $\hat{\Lambda}_1$ again be the operator from X_1 onto F such that $\Lambda_1 = i_F \hat{\Lambda}_1$. This (surjective) operator clearly has the same (finite dimensional) kernel as Λ_1 . Since $\hat{\Lambda}_1$ is therefore a Fredholm operator, there is a bounded operator $A : F \longrightarrow X_1$ such that $1 - A\hat{\Lambda}_1$ and $1 - \hat{\Lambda}_1 A$ are finite rank operators.

Now let $\hat{\Lambda}_2 : X_2/\text{Ker}(\Lambda_2) \longrightarrow X_3$ be the injective operator which satisfies

$$\Lambda_2 = \hat{\Lambda}_2 Q_{\text{Ker}(\Lambda_2)}.$$

Since $\hat{\Lambda}_2$ is injective and has the same (finite codimensional) range as Λ_2 , $\hat{\Lambda}_2$ is also a Fredholm operator. We will shortly define B to be $\hat{\Lambda}_2 \alpha$ for a suitable Fredholm operator $\alpha : X_2/F \longrightarrow X_2/\text{Ker}(\Lambda_2)$.

Letting α be the (necessarily surjective) operator which satisfies

$$\alpha Q_F = Q_{\text{Ker}(\Lambda_2)},$$

applying the snake lemma to the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F & \xrightarrow{i_F} & X_2 & \xrightarrow{Q_F} & X_2/F & \longrightarrow & 0 \\ & & \downarrow \text{inc.} & & \downarrow \text{id} & & \downarrow \alpha & & \\ 0 & \longrightarrow & \text{Ker}(\Lambda_2) & \xrightarrow{i_{\text{Ker}(\Lambda_2)}} & X_2 & \xrightarrow{Q_{\text{Ker}(\Lambda_2)}} & X_2/\text{Ker}(\Lambda_2) & \longrightarrow & 0, \end{array} \quad (4.3.5)$$

gives us a linear bijection

$$\delta : \text{Ker}(\alpha) \longrightarrow \text{Ker}(\Lambda_2)/F = \text{Ker}(\Lambda_2)/\text{Im}(\Lambda_1).$$

This implies, in particular, that $\text{Ker}(\alpha)$ is finite dimensional. Since α is surjective, we see that (as a composition of two Fredholm operators),

$$B = \hat{\Lambda}_2 \alpha$$

is a Fredholm operator. The rest is straightforward. $\square$

Combining this with Lemma 4.2.1, we have the following theorem.

Theorem 4.3.3. Let X_1, X_2 and X_3 be Banach spaces, and suppose that T_1, T_2, T_3, Λ_1 and Λ_2 are bounded linear operators for which:

- (i) $0 \longrightarrow X_1 \xrightarrow{\Lambda_1} X_2 \xrightarrow{\Lambda_2} X_3 \longrightarrow 0$ is a Fredholm complex;
- (ii) diagram (4.3.2) commutes modulo the compact operators; and
- (iii) $T_2(\text{Im}(\Lambda_1)) \subseteq \text{Im}(\Lambda_1)$.

Then there are Fredholm operators A and B for which $T_1 A - A(T|_{\text{Im}(\Lambda_1)})$ and $T_3 B - B(T_2/\text{Im}(\Lambda_1))$ are both compact.

Proof. Letting $F = \text{Im}(\Lambda_1)$, Lemma 4.3.2 supplies us with Fredholm operators A and B for which the diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & F & \xrightarrow{i_F} & X_2 & \xrightarrow{Q_F} & X_2/F \longrightarrow 0 \\
 & & \downarrow A & & \downarrow \text{id} & & \downarrow B \\
 0 & \longrightarrow & X_1 & \xrightarrow{\Lambda_1} & X_2 & \xrightarrow{\Lambda_2} & X_3 \longrightarrow 0 \\
 & & \downarrow T_1 & & \downarrow T_2 & & \downarrow T_3 \\
 0 & \longrightarrow & X_1 & \xrightarrow{\Lambda_1} & X_2 & \xrightarrow{\Lambda_2} & X_3 \longrightarrow 0 \\
 & & \uparrow A & & \uparrow \text{id} & & \uparrow B \\
 0 & \longrightarrow & F & \xrightarrow{i_F} & X_2 & \xrightarrow{Q_F} & X_2/F \longrightarrow 0,
 \end{array} \tag{4.3.6}$$

commutes modulo the compact operators. The operator

$$BQ_F T_2 - T_3 B Q_F = [B(T_2/F) - T_3 B] Q_F$$

is therefore compact. This implies, via Lemma 4.2.1, that $B(T_2/F) - T_3 B$ is compact. Let A' be an inverse for A modulo the compact (or finite rank) operators. Then

$$i_F A' T_1 A - T_2 i_F = i_F [A' T_1 A - (T_2|F)]$$

is compact. Letting U be the standard isomorphism of $X^*/F^\perp$ onto F^* , we have $U Q_{F^\perp} = i_F^*$, and

$$[A' T_1 A - (T_2|F)]^* i_F^* = [A' T_1 A - (T_2|F)]^* U Q_{F^\perp}.$$

Lemma 4.2.1 now implies that $[A' T_1 A - (T_2|F)]^* U$ is compact, hence that $A' T_1 A - (T_2|F)$ is compact. The result follows. $\square$

This result allows us (in the situation described in Theorem 4.3.3) to consider T_1 as a restriction of T_2 , and T_3 as a quotient of T_2 when considering the relationships these operators have with closed operator ideals. The following theorem makes this idea a little more precise; its proof is a straightforward combination of Theorem 4.3.3 and Lemma 4.2.2.

Theorem 4.3.4. Let X_1, X_2 and X_3 be Banach spaces, and suppose that $\Lambda_1, \Lambda_2, T_1, T_2$ and T_3 are bounded linear operators for which

$$\begin{array}{ccccccc}
 0 & \longrightarrow & X_1 & \xrightarrow{\Lambda_1} & X_2 & \xrightarrow{\Lambda_2} & X_3 \longrightarrow 0 \\
 & & \downarrow T_1 & & \downarrow T_2 & & \downarrow T_3 \\
 0 & \longrightarrow & X_1 & \xrightarrow{\Lambda_1} & X_2 & \xrightarrow{\Lambda_2} & X_3 \longrightarrow 0
 \end{array} \tag{4.3.7}$$

commutes modulo the compact operators. Suppose, in addition, that T_2 preserves $F = \text{Im}(\Lambda_1)$, and that the top row of the diagram (and hence also the second) is a Fredholm complex. Then, given any closed operator ideal $\mathfrak{A}$ containing the compact operators:

- (i) $T_1 \in \mathfrak{A}(X_1)$ if and only if $T_2|F \in \mathfrak{A}(F)$;
- (ii) $T_3 \in \mathfrak{A}(X_3)$ if and only if $T_2/F \in \mathfrak{A}(X_2/F)$;
- (iii) the spectrum of $T_1 + \mathfrak{A}(X_1)$ in $L(X_1)/\mathfrak{A}(X_1)$ is the same as that of $T_2|F$ in $L(F)/\mathfrak{A}(F)$; and
- (iv) the spectrum of $T_3 + \mathfrak{A}(X_3)$ in $L(X_3)/\mathfrak{A}(X_3)$ is the same as that of T_2/F in $L(X_2/F)/\mathfrak{A}(X_2/F)$.

Specialising to the case where $\mathfrak{A}$ is the compact operators this implies (thanks to Theorem 4.1.2) that

$$\sigma_e(T_1) \cup \sigma_e(T_3) \subseteq \widehat{\sigma_e(T_2)}$$

and $r_e(T_2) = \max\{r_e(T_1), r_e(T_3)\}$.

§4.4. Chain rules

In this section, we examine two (non-obvious) situations in which the ideas developed in this chapter are applicable in investigations involving composition operators and their weighted counterparts (another will be covered in more detail in the next Chapter). Both examples concern endomorphisms of Hölder algebras which preserve kernels of particular differential operators. For what follows, Ω is the open unit disk in $\mathbb{R}^2$, and $0 < \gamma < 1$.

Our first example concerns the sequence

$$0 \longrightarrow \text{Ker}(\Lambda) \xrightarrow{\text{inc.}} C^{2,\gamma}(\bar{\Omega}) \xrightarrow{\Lambda} C^{0,\gamma}(\bar{\Omega}) \longrightarrow 0,$$

where Λ is the Laplace operator given by

$$\Lambda u = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) u \quad (u \in C^{2,\gamma}(\bar{\Omega})).$$

By Kellogg's theorem, Λ is surjective. Letting J be the $\mathbb{R}$ -linear involution of $\mathbb{R}^2$ which satisfies

$$Je_1 = e_2 \quad \text{and} \quad Je_2 = -e_1,$$

let $\phi = (\phi_1, \phi_2)$ be a self-map of $\bar{\Omega}$ for which $\phi_1, \phi_2 \in C^{2,\gamma}(\bar{\Omega})$ and

$$D\phi(x)J = JD\phi(x) \quad (x \in \Omega).$$

In this situation, we have a 'chain rule' for Λ , in the sense that

$$\Lambda(u \circ \phi) = \det(D\phi)(\Lambda u) \circ \phi$$

for each $u \in C^2(\Omega)$. For each $u \in C^{2,\gamma}(\bar{\Omega})$ and $f \in C^{0,\gamma}(\bar{\Omega})$, let

$$T_2 u = u \circ \phi \quad \text{and} \quad T_3 f = \det(D\phi) f \circ \phi.$$

It is easy to see that these formulae define operators $T_2 \in L(C^{2,\gamma}(\bar{\Omega}))$ and $T_3 \in L(C^{0,\gamma}(\bar{\Omega}))$. Letting T_1 be the restriction of T_2 to $\text{Ker}(\Lambda)$, we obtain a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Ker}(\Lambda) & \xrightarrow{\text{inc.}} & C^{2,\gamma}(\bar{\Omega}) & \xrightarrow{\Lambda} & C^{0,\gamma}(\bar{\Omega}) \longrightarrow 0 \\ & & T_1 \downarrow & & T_2 \downarrow & & T_3 \downarrow \\ 0 & \longrightarrow & \text{Ker}(\Lambda) & \xrightarrow{\text{inc.}} & C^{2,\gamma}(\bar{\Omega}) & \xrightarrow{\Lambda} & C^{0,\gamma}(\bar{\Omega}) \longrightarrow 0 \end{array} \quad (4.4.1)$$

suitable for an application of Theorem 4.3.4.

In particular, this ‘reduces’ the problem of determining $r_e(T_2)$ to that of finding $r_e(T_1)$ and $r_e(T_3)$. This observation would be more helpful if we had more information about the properties of: (i) composition operators on spaces of harmonic functions; (ii) weighted composition operators on Hölder spaces. If the current fashion for this topic persists, it is only a matter of time before such information becomes available in the literature (this is the main reason I have stayed away from this topic myself).

Now for a slightly more satisfying example.

This time, let Λ be the d-bar operator from $C^{1,\gamma}(\bar{\Omega})$ into $C^{0,\gamma}(\bar{\Omega})$, so that

$$\Lambda u = \frac{1}{2} \left(\frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2} \right) u \quad (u \in C^{1,\gamma}(\bar{\Omega})).$$

It is clear that $\text{Ker}(\Lambda) = A^{1,\gamma}(\bar{\Omega}) = \{u \in C^{1,\gamma}(\bar{\Omega}) : u|_{\Omega} \in \mathcal{O}(\Omega)\}$ is a unital Banach algebra. Letting $\phi = (\phi_1, \phi_2)$ be a self-map of $\bar{\Omega}$ for which $\phi_1, \phi_2 \in C^{1,\gamma}(\bar{\Omega})$ and $D\phi(x)J = JD\phi(x)$ for $x \in \Omega$, we have

$$\Lambda(u \circ \phi) = \left(\frac{\partial \phi_1}{\partial x_1} - i \frac{\partial \phi_2}{\partial x_1} \right) (\Lambda u) \circ \phi$$

for each $u \in C^{1,\gamma}(\bar{\Omega})$. Setting

$$T_1 h = h \circ \phi, \quad T_2 u = u \circ \phi \quad \text{and} \quad T_3 f = \left(\frac{\partial \phi_1}{\partial x_1} - i \frac{\partial \phi_2}{\partial x_1} \right) f \circ \phi,$$

for each $h \in A^{1,\gamma}(\bar{\Omega})$, $u \in C^{1,\gamma}(\bar{\Omega})$ and $f \in C^{0,\gamma}(\bar{\Omega})$, we obtain another commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & A^{1,\gamma}(\bar{\Omega}) & \xrightarrow{\text{inc.}} & C^{1,\gamma}(\bar{\Omega}) & \xrightarrow{\Lambda} & C^{0,\gamma}(\bar{\Omega}) \longrightarrow 0 \\ & & T_1 \downarrow & & T_2 \downarrow & & T_3 \downarrow \\ 0 & \longrightarrow & A^{1,\gamma}(\bar{\Omega}) & \xrightarrow{\text{inc.}} & C^{1,\gamma}(\bar{\Omega}) & \xrightarrow{\Lambda} & C^{0,\gamma}(\bar{\Omega}) \longrightarrow 0. \end{array} \quad (4.4.2)$$

Since the surjectivity of the Laplace operator (as considered in the previous example) implies that of Λ , we again reach a situation in which we can profitably apply Theorem 4.3.4.

We will see that this is especially true when ϕ maps $\bar{\Omega}$ into Ω . The following helpful lemma was pointed out to me by J.F. Feinstein; its proof is a nice application of Montel's theorem (combined with some standard properties of the holomorphic functional calculus).

Lemma 4.4.1. Let S be a compact subset of $\mathbb{C}$, let $A \subseteq C(S)$ be a unital Banach algebra, and suppose that there is an open set $U \subseteq S$ in which every element of A is holomorphic. Suppose that $\Phi_A = S$ (in the sense that every character is given by evaluation at a point of S), and let $\phi \in A$ be a self-map of S which maps S into U . Then

$$A \ni f \longmapsto f \circ \phi$$

is a compact endomorphism of A .

The immediate impediment to applying this result with $A = A^{1,\gamma}(\bar{\Omega})$ is removed by the following (possibly standard) proposition.

Proposition 4.4.2. Every character on $A^{1,\gamma}(\bar{\Omega})$ is of the form $f \longmapsto f(z)$ for some $z \in \bar{\Omega}$.

Proof. It is standard that the assertion of the theorem is equivalent to the following one: for every $f_1, \dots, f_N \in A^{1,\gamma}(\bar{\Omega})$ with $f_1^{-1}(0) \cap \dots \cap f_N^{-1}(0) = \emptyset$, there exist $g_1, \dots, g_N \in A^{1,\gamma}(\bar{\Omega})$ for which $f_1 g_1 + \dots + f_N g_N = 1$. This can be established using the following standard ('corona') argument, borrowed from [22].

Choose $f_1, \dots, f_N \in A^{1,\gamma}(\bar{\Omega})$ with no common zeros, and set

$$h_i = \bar{f}_i(|f_1|^2 + \dots + |f_N|^2)^{-1}$$

Since it is easy to see that $C^{1,\gamma}(\bar{\Omega})$ is inverse closed and self-adjoint, each h_i belongs to $C^{1,\gamma}(\bar{\Omega})$. This means, in particular, that each of the functions $h_i \Lambda h_j$ belongs to $C^{0,\gamma}(\bar{\Omega})$. Using the surjectivity of Λ , we can therefore find a matrix $\omega \in C^{1,\gamma}(\bar{\Omega}, M_N(\mathbb{C}))$ such that

$$\Lambda \omega_{ij} = h_i \Lambda h_j \quad (1 \leq i, j \leq N).$$

Now set $g_i = h_i + \sum_k (\omega_{ki} - \omega_{ik}) f_k$.

It is immediately clear that each g_i belongs to $C^{1,\gamma}(\bar{\Omega})$, and

$$\sum_{i=1}^N f_i g_i = \sum_{i=1}^N f_i h_i = 1.$$

To finish the proof, it therefore remains to show that $\Lambda g_i = 0$ for each i . However, this is a routine (and entirely standard) calculation. $\square$

We can now prove the following theorem.

Theorem 4.4.3. Let $\phi = (\phi_1, \phi_2)$ be a self-map of $\bar{\Omega}$ for which ϕ_1 and ϕ_2 both belong to $C^{1,\gamma}(\bar{\Omega})$, and let T be the associated endomorphism of $C^{1,\gamma}(\bar{\Omega})$. Let

$$(T'f)(x) = \left(\frac{\partial \phi_1}{\partial x_1} - i \frac{\partial \phi_2}{\partial x_1} \right) (x) f(\phi(x))$$

for every $f \in C^{0,\gamma}(\bar{\Omega})$ and $x \in \Omega$, and suppose that

$$\phi(\bar{\Omega}) \subset \Omega \text{ and } D\phi(x)J = JD\phi(x) \text{ } (x \in \Omega).$$

Then $\sigma_e(T) \cup \{0\} = \sigma_e(T') \cup \{0\}$.

Proof. By one of the essential spectral inclusion results mentioned earlier (one which pre-dates this work) we have

$$\begin{aligned} \sigma_e(T) &\subseteq \sigma_e(T|A^{1,\gamma}(\bar{\Omega})) \cup \sigma_e(T/A^{1,\gamma}(\bar{\Omega})) \\ \text{and } \sigma_e(T/A^{1,\gamma}(\bar{\Omega})) &\subseteq \sigma_e(T) \cup \sigma_e(T|A^{1,\gamma}(\bar{\Omega})). \end{aligned}$$

Since, by Lemma 4.4.1, $T|A^{1,\gamma}(\bar{\Omega})$ is compact, this reduces to

$$\sigma_e(T) \cup \{0\} = \sigma_e(T/A^{1,\gamma}(\bar{\Omega})) \cup \{0\}.$$

Since Λ is surjective, the identity $\Lambda T = T' \Lambda$ means that T' is (up to similarity) just $T/A^{1,\gamma}(\bar{\Omega})$ in disguise. The result follows. $\square$

Remarks.

- It appears that the inequality $r_e(T|F) \leq r_e(T)$ can also be obtained by considering so-called *measures of non-compactness*; see, for example, the papers [3] and [31]. This approach does not appear to be capable of returning information about essential spectral inclusions.
- The following problem appears to be open: can a spectral projection *ever* fail to preserve a closed invariant subspace of its parent operator? Any counterexample must clearly involve an operator on an infinite dimensional Banach space.

Endomorphisms of algebras of C^m functions

§5.1. Introduction.

This chapter marks something of a return to the theme established in Chapters 2 and 3, as we explore another aspect of the relationship between the operator theoretic properties of a commutative unital Banach algebra endomorphism and its interactions with various ambient algebraic structures. Where self-maps, fixed sets, and primary ideals were the focus in Chapters 2 and 3, here it is *derivations* (of various kinds) which receive the lion's share of our attention.

Our primary objects of study will be the endomorphisms of $C^m(\bar{\Omega})$, where m is a positive integer, and Ω is a bounded domain in $\mathbb{R}^d$. For us, this is the algebra of all *complex-valued* continuous functions on Ω which, together with their partial derivatives of orders $\leq m$, extend to be continuous functions on $\bar{\Omega}$. Although, as a semi-simple commutative unital Banach algebra, $C^m(\bar{\Omega})$ is certainly vulnerable to many of our existing results, the ambient m -th order differentiable structure offers us some interesting alternative opportunities.

We will see, in particular, that it is possible to regard each endomorphism $T : C^m(\bar{\Omega}) \longrightarrow C^m(\bar{\Omega})$ as a restriction of a corresponding operator on a product space of the form

$$C(\bar{\Omega}, E_0) \times C(\bar{\Omega}, E_1) \times \cdots \times C(\bar{\Omega}, E_m),$$

where each E_k is a finite dimensional Banach space of symmetric multilinear forms. Embedding $C^m(\bar{\Omega})$ into spaces of this form is a relatively standard practice (see, for example, [11]).

Our program for this chapter is as follows. After an extended preliminary section (covering, among other things, some standard material on symmetric powers) we begin, in Section 5.3, with some low level results concerning the interaction between an endomorphism of $C^m(\bar{\Omega})$ and some related operators on various symmetric powers of $\mathbb{C}^d$. This is followed by a selection of results concerning essential norms and essential spectral radii of endomorphisms of $C^m(\bar{\Omega})$; my efforts in this direction can be found in Sections 5.4 and 5.5.

The final section (Section 5.6) contains some supplementary results concerning the relationship between certain endomorphisms and a construction through which $C^1(\bar{\Omega})$ can be realised as a quotient of a uniform algebra.

§5.2. Preliminaries

For the entirety of this chapter, d, m and q are positive integers, Ω is a bounded domain in $\mathbb{R}^d$, and E is $\mathbb{C}^d$ with the norm $X \mapsto \max_{i=1}^d |X_i|$. The standard basis for $\mathbb{R}^d$ is denoted $\hat{e}_1, \dots, \hat{e}_d$, the letters $e_1, \dots, e_d$ are reserved for their complex counterparts, and we shall write $x \mapsto x_{\mathbf{c}}$ for the inclusion map of $\mathbb{R}$ into $\mathbb{C}$. Given any multi-linear map $\mu : (\mathbb{R}^d)^k \rightarrow \mathbb{R}^q$, the complexification $\mu_{\mathbf{c}} : E^k \rightarrow \mathbb{C}^q$ of μ is obtained as follows: set

$$\mu_{j_1, \dots, j_k}^i = \hat{e}_i \cdot \mu(\hat{e}_{j_1}, \dots, \hat{e}_{j_k})$$

$$\text{and then insist that } \mu_{\mathbf{c}}(e_{j_1}, \dots, e_{j_k}) = \sum_{i=1}^q (\mu_{j_1, \dots, j_k}^i)_{\mathbf{c}} e_i$$

for $i \in \{1, \dots, q\}$ and $j_1, \dots, j_k \in \{1, \dots, d\}$. Although any $\mu_{\mathbf{c}}$ obtained in this way is a complex multi-linear map, it is clearly determined entirely by the behaviour of the corresponding μ .

Given any sufficiently smooth map $\phi : \Omega \rightarrow \Omega_0 \subseteq \mathbb{R}^q$, it is standard that the k -th Fréchet derivative $\phi^{(k)}(x)$ at a point $x \in \Omega$ can be regarded as a k -linear map

$$\phi^{(k)}(x) : (\mathbb{R}^d)^k \rightarrow \mathbb{R}^q.$$

In this situation, we write $D^k \phi(x) = \phi^{(k)}(x)_{\mathbf{c}}$ for the complexification of $\phi^{(k)}(x)$. A similar notation will be used for complex valued functions: given any $u \in C^m(\bar{\Omega})$, we set $D^k u(x) = (\mathcal{R}u)^{(k)}(x)_{\mathbf{c}} + i(\mathcal{I}u)^{(k)}(x)_{\mathbf{c}}$ for $x \in \Omega$.

This notation will be used alongside the standard D^α notation for partial derivatives but this should not cause any confusion: multi-indices will always be denoted by Greek letters, integers by the usual Latin ones.

Although our Fréchet derivatives $D^k u(x)$ are all complex multi-linear maps $E^k \rightarrow \mathbb{C}$, it will actually be more convenient to regard them as linear functionals on the k -th symmetric power $S^k E$ of $E = \mathbb{C}^d$. The treatment of symmetric powers given below is based on the one in Federer's opus, [18], but has some adjustments to notation and presentation in certain places. In the following discussion: i, j and k are always non-negative integers, $q \in \mathbb{N}$, and $F = \mathbb{C}^q$.

For us, the k th symmetric power $S^k E$ of E is the dual of the space $S^k E^*$ of all symmetric k -linear maps $\mu : E^k \rightarrow \mathbb{C}$. Since E is finite dimensional, we are clearly free to regard $S^k E^*$ as the dual of $S^k E$ rather than the other way around. Similarly, $S^k E^* \otimes F$ can be thought of either as the space of all F valued symmetric forms on E^k or as the space $L(S^k E, F)$ of all *linear* maps of $S^k E$ into F ; we will generally prefer the second interpretation.

Our next task is to describe the natural product

$$\vee : S^i E^* \otimes F \times S^j E^* \otimes F \longrightarrow S^{i+j} E^* \otimes S^2 F$$

used to synthesise an element $\theta \vee \psi \in S^{i+j} E^* \otimes S^2 F$ from two symmetric forms $\theta \in S^i E^* \otimes F$ and $\psi \in S^j E^* \otimes F$. For $X, Y \in F$, let $X * Y$ denote the element of $S^2 F$ which sends any symmetric bilinear form $\mu : F^2 \rightarrow \mathbb{C}$ to the complex number $\mu(X, Y)$. The map $(X, Y) \mapsto X * Y$ is, of course, really just the natural bilinear map from F^2 into $S^2 F$.

Following Federer, we define the symmetric product of $\theta \in S^i E \otimes F$ and $\psi \in S^j E \otimes F$ as the element $\theta \vee \psi$ of $S^{i+j} E^* \otimes S^2 F$ given by

$$\begin{aligned} (\theta \vee \psi)(X_1, \dots, X_{i+j}) &= \\ &\sum_{\pi \in \text{Sh}(i, j)} \theta(X_{\pi(1)}, \dots, X_{\pi(i)}) * \psi(X_{\pi(i+1)}, \dots, X_{\pi(i+j)}) \end{aligned}$$

for every $X_1, \dots, X_{i+j} \in E$, where $\text{Sh}(i, j)$ is the set of all permutations of the numbers 1 to $i + j$ which are increasing on the intervals $\{1, \dots, i\}$ and $\{i + 1, \dots, i + j\}$ (these are the so-called (i, j) -*shuffles*). It is standard that the 'product' defined by this formula is associative and commutative. The reader may wish to check that the symmetric product above is $(i + j)!/(i!j!)$ times that obtained by 'averaging over all permutations'.

For $\theta \in S^i E^* \otimes F$ and $k \in \mathbb{N}$, we also write

$$S^k(\theta) = \underbrace{\theta \vee \cdots \vee \theta}_{k \text{ times}}$$

for the k -th symmetric power of θ ; this is an element of $S^{ik} E^* \otimes S^k F$.

We will shortly need some additional notation for dealing with multi-indices. As the reader may recall from Chapter 2, we have elected to set

$$\langle \alpha \rangle = \sum_{i=1}^m i \alpha_i.$$

for each multi-index $\alpha \in \mathbb{Z}_+^m$. It is clear that $|\alpha| \leq \langle \alpha \rangle$, with equality if and only if $\alpha_1 = |\alpha|$ and $\alpha_i = 0$ for $i > 1$; we have used this fact already in Chapter 2.

The primary use of this ‘bra-ket’ notation is tied up with the following definition. Let α be an m -fold multi-index, let

$$\theta = (\theta_1, \dots, \theta_m) \in \bigoplus_{k=1}^m S^k E^* \otimes F,$$

and set $\theta^\alpha = S^{\alpha_1}(\theta_1) \vee \cdots \vee S^{\alpha_m}(\theta_m)$. As a quick degree count reveals, θ^α is an element of $S^{\langle \alpha \rangle} E^* \otimes S^{|\alpha|} F$ and, as such, may be regarded as a linear operator from $S^{\langle \alpha \rangle} E$ into $S^{|\alpha|} F$. This notation does not appear to be in the literature, but does make Federer’s multivariable Faa-di-Bruno formula (c.f. [18], page 222) considerably less cumbersome.

Let Ω_1 and Ω_2 be open subsets of $\mathbb{R}^d$ and $\mathbb{R}^q$, and let $\phi : \Omega_1 \longrightarrow \Omega_2$ and $u : \Omega_2 \longrightarrow \mathbb{C}$ be C^m maps. Using our notation in Federer’s magic formula, we have

$$D^k(u \circ \phi)(x) = \sum_{\langle \alpha \rangle = k} \frac{1}{\alpha!} D^{|\alpha|} u(\phi(x)) \phi^{[k]}(x)^\alpha,$$

for $x \in \Omega_1$ and $k \leq m$, where $\phi^{[k]} : \Omega_1 \longrightarrow \bigoplus_{i=1}^k S^i E^* \otimes F$ is given by

$$\phi^{[k]}(x) = (D\phi(x), D^2\phi(x), \dots, D^k\phi(x))$$

for each $x \in \Omega_1$. The map $\phi^{[k]}$ is similar to what is sometimes referred to in the literature as the k -jet prolongation of ϕ .

We will be particularly interested in the case where $q = 1$, and this is a setting in which we can make some important simplifications. Firstly, under the natural identification of each $S^k\mathbb{C}$ ($k \in \mathbb{N}$) with $\mathbb{C}$, each $D^k u(\phi(x))$ is really just the multiplication functional

$$z \mapsto u^{(k)}(\phi(x))z,$$

where $u^{(k)} : (\Omega_2 \subseteq \mathbb{R}) \rightarrow \mathbb{C}$ is the ordinary k th derivative of u . In identifying $S^k\mathbb{C}$ with $\mathbb{C}$, we also have

$$\frac{1}{k!} S^k(D\phi(x))(X_1, \dots, X_k) = \prod_{j=1}^k D\phi(x)X_j$$

whenever $X_1, \dots, X_k \in E$. The reader will also do well to note that, for arbitrary $q \in \mathbb{N}$, the difference

$$D^m(u \circ \phi)(x) - \frac{1}{m!} D^m u(\phi(x)) S^m(D\phi(x))$$

is a sum of terms of the form

$$\frac{1}{\alpha!} D^{|\alpha|} u(\phi(x)) \phi^{[m]}(x)^\alpha,$$

with $|\alpha| < m$; this fact can sometimes be very helpful, and is at the heart of the ‘repeated differentiation’ arguments which permeate the literature in this area.

Turning our attention to the subject of norms, let us start by equipping $S^k E^*$ with the norm which satisfies

$$\|\mu\| = \sup\{|\mu(X_1, \dots, X_m)| : X_i \in E \text{ and } \|X_i\| = 1 \text{ for all } i\}$$

for each $\mu \in S^m E^*$. Since we are regarding $S^k E^*$ as the dual of $S^k E$, we can now norm $S^k E$ by setting

$$\|\zeta\| = \sup\{|\mu(\zeta)| : \mu \in S^k E^*, \|\mu\| = 1\} \quad (\zeta \in S^k E),$$

and then give $S^k E^* \otimes F \cong L(S^k E, F)$ the induced operator norm.

With these norm conventions, it is known that

$$\frac{1}{k!} \|S^k(\mu)\| = \|\mu\|^k$$

whenever $k \in \mathbb{N}$ and $\mu \in E^*$ (see [18], page 47).

Whilst on the subject of norms, it is perhaps wise to say a little about norms on $C^m(\bar{\Omega})$. In general, we shall prefer the norms $\|\cdot\|_{m,\infty,\Omega}$ and $\|\cdot\|'_{m,\infty,\Omega}$ given by

$$\|u\|_{m,\infty,\Omega} = \sum_{|\alpha| \leq m} \frac{1}{\alpha!} \sup_{x \in \Omega} |D^\alpha u(x)|$$

$$\text{and } \|u\|'_{m,\infty,\Omega} = \sum_{k=0}^m \frac{1}{k!} \sup_{x \in \Omega} \|D^k u(x)\|$$

for $u \in C^m(\bar{\Omega})$. It is standard that $\|\cdot\|_{m,\infty,\Omega}$ is a complete norm on $C^m(\bar{\Omega})$, and it is straightforward to show that

$$\|u\|'_{m,\infty,\Omega} \leq \|u\|_{m,\infty,\Omega} \leq \beta(m) \|u\|'_{m,\infty,\Omega}$$

for $u \in C^m(\bar{\Omega})$ when $\beta(m) = \max_{k=0}^m \sum_{|\alpha|=k} k!/\alpha!$.

Before we discuss *endomorphisms* of $C^m(\bar{\Omega})$, let us pause briefly to introduce some more notation: the symbols $\chi_1, \dots, \chi_d$ are henceforth reserved for denoting the functions

$$\chi_i : \bar{\Omega} \longrightarrow \mathbb{C} \quad , \quad x \longmapsto (\hat{e}_i \cdot x)_{\mathbf{c}}.$$

These functions clearly belong to $C^m(\bar{\Omega})$ for every m , and each $D\chi_i$ is constant: its value $D\chi_i(x)$ at any $x \in \Omega$ is the i th coordinate functional on E . It is clear that

$$\begin{aligned} \|D\chi_i(x)\| &= \sup \{ |D\chi_i(x)X| : X \in E, \|X\| = 1 \} \\ &= \sup \{ |X_i| : X \in E, \max_{j=1}^d |X_j| = 1 \} = 1 \end{aligned}$$

and $D^k\chi_i = 0$ for $k > 1$. A necessary condition for a composition operator $u \longmapsto u \circ \phi$ to preserve $C^m(\bar{\Omega})$ is therefore that each of the component functions $\phi_i = \chi_i \circ \phi$ belongs to $C^m(\bar{\Omega})$.

It is easy to see that this condition is also sufficient: given any self-map $\phi : \bar{\Omega} \longrightarrow \bar{\Omega}$ for which $\phi_1, \dots, \phi_d \in C^m(\bar{\Omega})$, we clearly have $u \circ \phi \in C^m(\bar{\Omega})$ for every $u \in C^m(\bar{\Omega})$. It is also not difficult to show that every unital endomorphism of $C^m(\bar{\Omega})$ is given by composition with a self-map of $\bar{\Omega}$; this is actually the main reason we insist that Ω be a *bounded* domain (this is enough to guarantee that every character on $C^m(\bar{\Omega})$ is given by evaluation at a point of $\bar{\Omega}$).

It would be unwise of me to proceed any further without first including a reminder of the effects which certain geometric properties of Ω can have on the properties of $C^m(\bar{\Omega})$. We recall that:

- $\bar{\Omega}$ is *Whitney regular* (or *uniformly regular*) if there is some $k > 0$ with the property that any two points $p, p' \in \bar{\Omega}$ can be joined by a rectifiable path in $\bar{\Omega}$ of length at most k times the Euclidean distance $|p - p'|$;
- $\bar{\Omega}$ is *locally regular* if every point has a neighbourhood which is Whitney regular;
- $\bar{\Omega}$ has the *strong local Lipschitz condition* if $\partial\Omega$ can be everywhere locally represented as the graph of a Lipschitz function (the precise definition for unbounded domains can be found on page 181 of Stein's wonderful book, [53]).

It is straightforward to show Whitney regularity leads to the operators

$$D^k : C^m(\bar{\Omega}) \longrightarrow C(\bar{\Omega}, S^k E^*)$$

being compact for $k < m$. Local regularity is typically assumed when it is important that

$$J(p) = \{u \in C^m(\bar{\Omega}) : D^k u(p) = 0 \text{ for all } k \leq m\}$$

holds even when $p \in \partial\Omega$ (this notion is considered in Neumann and Laursen's book, [41]). Here, as in Chapter 3, $J(p)$ is the closure of the ideal $J_0(p)$ consisting of all $u \in C^m(\bar{\Omega})$ for which $u^{-1}(0)$ is a neighbourhood of p .

The strong local Lipschitz condition is important when one wishes to ensure that the restriction map $C^m(\mathbb{R}^n) \longrightarrow C^m(\bar{\Omega})$ is surjective: the relevant extension theorem (which is entirely distinct from Whitney's extension theorem) is discussed at length in Section 6.3 of Stein's book, [53].

§5.3. Basic results

Having assembled the necessary machinery, it is now time to see what we can do with it. As well as being something of a ‘warm up’ for Section 5.4, this section also presents us with an opportunity to avail ourselves of some low hanging fruit. Our starting point is the following recent theorem of Tagaki, Miura and Takahasi.

Theorem 5.3.1 ([54], Theorem 1). Let S be a compact Hausdorff space, let $f \in C(S)$, and let ϕ be a continuous self-map of S . Let

$$U_\epsilon(f) = \{x \in S : |f(x)| \geq \epsilon\}$$

for each $\epsilon > 0$. Then $\|fC_\phi\|_e = \inf\{\epsilon > 0 : \phi(U_\epsilon(f)) \text{ is finite or empty}\}$.

Let T be a unital endomorphism of $C^m([0, 1])$ with self-map ϕ . Regarding D^m as the usual m -th derivative operator from $C^m([0, 1])$ into $C([0, 1])$, it is not difficult to show that D^m is Fredholm, and that the difference $D^mT - (\phi')^mC_\phi D^m$ is compact. It then follows from Theorem 4.3.4 that

$$r_e(T) = r_e((\phi')^mC_\phi),$$

regarding $(\phi')^mC_\phi$ as an element of $L(C([0, 1]))$. Although the formula for $r_e((\phi')^mC_\phi)$ one obtains from Theorem 5.3.1 is often not very helpful, there is one case in which it does tell us something rather interesting. Suppose that ϕ is non-constant and affine, so that

$$\phi(x) = p + \lambda(x - p) \quad (x \in [0, 1])$$

for some $p \in [0, 1]$ and $\lambda \in \mathbb{R} \setminus \{0\}$ with $(1 - \lambda)p, (1 - \lambda)p + \lambda \in [0, 1]$. It is clear in this simple case that $((\phi')^mC_\phi)^n = \lambda^{mn}C_{\phi \circ n}$ for every $n \in \mathbb{N}$, and that

$$\phi(U_\epsilon(f)) = \begin{cases} \phi([0, 1]) & \text{for } \epsilon \leq |\lambda|^{mn}; \text{ and} \\ \emptyset & \text{for } \epsilon > |\lambda|^{mn} \end{cases}$$

when f is the constant function $x \mapsto \lambda^{mn}$. Since $\phi([0, 1])$ is always infinite, Theorem 5.3.1 now tells us that $\|((\phi')^mC_\phi)^n\|_e^{1/n} = |\lambda|^m$ for every $n \in \mathbb{N}$.

Applying the spectral radius formula, we conclude that

$$r_e(T) = |\lambda|^m = |\phi'(p)|^m,$$

for the endomorphism T of $C^m([0, 1])$ induced by ϕ . We can actually do slightly better than this. Feinstein and Kamowitz have already proved a stronger result than the one below in the $m = 1$ case (c.f. [20], Theorem 3.2).

Theorem 5.3.2. Let T be a unital endomorphism of $C^m([0, 1])$, and let ϕ be the associated self-map of $[0, 1]$. Suppose that $\phi^{\circ\infty}([0, 1]) = \{p\}$ for some $p \in [0, 1]$, and that ϕ is non-constant and affine in some ϕ -invariant neighbourhood of p . Then $r_e(T) = |\phi'(p)|^m$.

Proof. By hypothesis, there are numbers $a, b \in [0, 1]$ with $a < p < b$ for which the interval $I = [a, b]$ is ϕ -invariant, and

$$\phi(x) = p + \phi'(p)(x - p)$$

for all $x \in I$. Invoking Proposition 2.5.5, we are immediately granted that $T/M(I)$ has the same nonzero spectrum as T ; this means, in particular, that it has the same essential spectral radius. The fact that the restriction map $C^m([0, 1]) \rightarrow C^m(I)$ is trivially surjective in this setting now means that $C^m([0, 1])/M(I)$ is isomorphic to $C^m(I)$ and $T/M(I)$ is similar to the endomorphism T_I of $C^m(I)$ which satisfies

$$(T_I f)(x) = f(p + \phi'(p)(x - p))$$

for $x \in I$ and $f \in C^m(I)$; the result then follows by our existing argument for non-constant affine self-maps. $\square$

We will see, in the next section, that it is possible to achieve the same conclusion even in the absence of the assumption that ϕ is affine near its fixed set. However, this is proved using a completely different (and much longer) method. The use of Proposition 2.5.5 as a localising tool will still be helpful when we move to the higher dimensional setting.

Moving on, let us now consider the relationship between an endomorphism $T : u \mapsto u \circ \phi$ of $C^m(\bar{\Omega})$ and the induced operators of the form

$$\frac{1}{k!} S^k(D\phi(p)) : S^k E \longrightarrow S^k E$$

for $p \in \Omega$ and $1 \leq k \leq m$. It is actually slightly easier to specify what these do via their adjoints; for each symmetric k -linear map $\mu : E^k \longrightarrow \mathbb{C}$, we have

$$\left(\frac{1}{k!} S^k(D\phi(p))^* \mu \right) (X_1, \dots, X_k) = \mu(D\phi(p)X_1, \dots, D\phi(p)X_k)$$

for $X_1, \dots, X_k \in E$. Another way to see these operators is as follows. For $k \in \mathbb{N}$, let $\mu \mapsto \text{Pol}_k(\mu)$ be the usual isomorphism of $S^k E^*$ onto the space $H_k(E)$ of all k -homogeneous polynomials on E . Then, observing that

$$\text{Pol}_k \left(\frac{1}{k!} S^k(D\phi(p))^* \mu \right) (X) = \text{Pol}_k(\mu)(D\phi(p)X)$$

holds for each $X \in E$ and $\mu \in S^k E^*$, we see that $(k!)^{-1} S^k(D\phi(p))^*$ really just effects a ‘linear change of variable’ on $H_k(E)$.

The main non-obvious property of these operators we will use is the following (apparently standard) fact: if $\lambda_1, \dots, \lambda_d$ are the eigenvalues of $D\phi(p)$ then

$$\sigma \left(\frac{1}{k!} S^k(D\phi(p)) \right) = \{ \lambda^\alpha : |\alpha| = k \}. \quad (5.3.1)$$

This is proved (under a different symmetric product convention) in the paper, [10], of Blockhuis and Seidel, and is easy to check directly when $D\phi(p)$ is diagonalisable (which is actually already enough).

Leaving the more complicated operators aside for a moment, let us first focus our attention on the operators

$$(D\phi(p) : E \longrightarrow E)_{p \in \Omega}$$

when ϕ is the self-map of some endomorphism T of $C^m(\bar{\Omega})$.

For $p \in \Omega$, and $X \in E$, let $\kappa_p X$ be the element of $C^m(\bar{\Omega})^*$ which satisfies

$$(\kappa_p X)u = Du(p)X$$

for every $u \in C^m(\bar{\Omega})$. Then the operators

$$(\kappa_p : E \longrightarrow C^m(\bar{\Omega})^*)_{p \in \Omega}$$

provide us with a natural way to inject copies of E into $C^m(\bar{\Omega})^*$. That

$$T^* \kappa_p = \kappa_{\phi(p)} D\phi(p) \tag{5.3.2}$$

is immediate from the chain rule so, since each κ_p is bounded below, $D\phi(p)$ is ‘really’ just the restriction of T^* to the closed subspaces $\kappa_p E$ and $\kappa_{\phi(p)} E$. In particular, $T^*|_{\kappa_p E}$ is similar to $D\phi(p)$ whenever $\phi(p) = p$. The following proposition is now obvious.

Proposition 5.3.3. Let T be a unital endomorphism of $C^m(\bar{\Omega})$, and let ϕ be the associated self-map of $\bar{\Omega}$. Then $\sigma(D\phi(p)) \subseteq \sigma(T/\mathbb{C})$ for every interior fixed point $p = \phi(p) \in \Omega$.

When $\bar{\Omega}$ is Whitney regular, it is actually possible to show that the operators $(\kappa_p : E \longrightarrow C^1(\bar{\Omega})^*)_{p \in \Omega}$ are ‘uniformly’ bounded below, in the sense that there is some $\gamma > 0$ for which

$$\gamma \|X\| \leq \|\kappa_p X\|$$

holds for all $p \in \Omega$. Combining this information with (5.3.2), one can begin to obtain useful information about the norms of endomorphisms of $C^1(\bar{\Omega})$. Whilst this is not a topic on which I will dwell, I would like to point out one result in this direction; it is probably ‘folklore’, but the first allusion to it I have found is Kitover’s 1979 paper, [39], which contains the associated equation for the spectral radius.

Proposition 5.3.4 (folklore). Let T be a unital endomorphism of $C^1(\bar{\Omega})$, and let ϕ be the associated self-map of $\bar{\Omega}$. Assume that $\bar{\Omega}$ is Whitney regular. Then there are numbers $B, C > 0$ for which

$$\sup_{x \in \Omega} \|D\phi(x)\| \leq B \|T/\mathbb{C}\| \leq C \sup_{x \in \Omega} \|D\phi(x)\|.$$

The technique used to prove our next result also comes from the 1970s and is a several variable version of Kamowitz's 1973 'repeated differentiation argument' (see, for example, [33], Lemma 3). This version of the argument is similar to that used by Aron, Galindo and Lindström in [5].

Proposition 5.3.5. Let T be a unital endomorphism of $C^m(\bar{\Omega})$, and let ϕ be the associated self-map of $\bar{\Omega}$. Suppose that ϕ has an interior fixed point $p = \phi(p) \in \Omega$, and that there is some $\lambda \in \sigma_p(T) \cap \{\lambda : \lambda^n \neq 1 \ \forall n \in \mathbb{Z}_+\}$ for which the operators

$$\lambda - \frac{1}{k!} S^k(D\phi(p)) : S^k E \longrightarrow S^k E \quad (1 \leq k \leq m)$$

are all invertible. Then

$$\text{Ker}(\lambda - T) \subseteq \{u \in C^m(\bar{\Omega}) : D^k u(p) = 0 \text{ for } k \leq m\}. \quad (5.3.3)$$

Proof. Let $u \in \text{Ker}(\lambda - T)$. Then since

$$u(p) = u(\phi^{\circ n}(p)) = (T^n u)(p) = \lambda^n u(p)$$

for every $n \in \mathbb{N}$, the fact that λ is not a root of unity implies that $u(p) = 0$. We also have

$$0 = D(\lambda u - Tu)(p) = Du(p)(\lambda - D\phi(p))$$

so our assumption that $\lambda - D\phi(p)$ is invertible also gives $Du(p) = 0$. Fixing some $2 \leq k \leq m$, suppose that we have already shown that $D^j u(p) = 0$ for every $j \leq k - 1$. Then since

$$D^k(u \circ \phi)(p) - \frac{1}{k!} D^k u(p) S^k(D\phi(p))$$

is a sum of terms containing a $D^j u(p)$ with $j \leq k - 1$ and these are all zero,

$$0 = D^k(\lambda u - Tu)(p) = D^k u(p)[\lambda - (k!)^{-1} S^k(D\phi(p))].$$

Having assumed that $\lambda - (k!)^{-1} S^k(D\phi(p))$ is invertible, we therefore actually have $D^j u(p) = 0$ for every $j \leq k$.

Letting

$$\mathfrak{P}_k = \{v \in \text{Ker}(\lambda - T) : D^j v(p) = 0 \text{ for } j \leq k\},$$

we have now established that $\text{Ker}(\lambda - T) = \mathfrak{P}_1$ and $\mathfrak{P}_{k-1} = \mathfrak{P}_k$ for any $2 \leq k \leq m$. We therefore have

$$\text{Ker}(\lambda - T) = \mathfrak{P}_1 = \mathfrak{P}_2 = \cdots = \mathfrak{P}_m,$$

which is what we needed to show. $\square$

The following corollary is really just the previous result ‘dressed up’ a little, but I have included a proof for the sake of transparency.

Corollary 5.3.6. Let T be a unital endomorphism of $C^m(\bar{\Omega})$, let ϕ be the associated self-map of $\bar{\Omega}$, and suppose that ϕ has an interior fixed point $p = \phi(p) \in \Omega$. Then every point of

$$\sigma_p(T) \setminus \sigma_p(T|J(p))$$

is either a root of unity or a monomial of order $\leq m$ in the eigenvalues $\lambda_1, \dots, \lambda_d$ of $D\phi(p)$.

Proof. Let $\lambda \in \sigma_p(T) \setminus \sigma_p(T|J(p))$. Then, by the previous result, either $\text{Ker}(\lambda - T) \subseteq J(p)$, λ is a root of unity, or there is some $1 \leq k \leq m$ for which

$$\lambda \in \sigma \left(\frac{1}{k!} S^k(D\phi(p)) \right).$$

In the first case, λ would be an eigenvalue of $T|J(p)$. Having explicitly assumed that this is not the case, the result follows from equation (5.3.1). $\square$

Our main interest in this result will be in connection with what it can tell us about the outer spectra of *quasi-compact* endomorphisms of $C^m(\bar{\Omega})$ whose self-maps have interior fixed points. We have already (several times) used the fact that, when T is a bounded operator on a Banach space, the set $\sigma(T) \cap \{\lambda : |\lambda| > r_e(T)\}$ consists entirely of eigenvalues of T .

The next result should arguably have been included in Chapter 3, but I think it fits in just as well here.

Proposition 5.3.7. Let B be a regular semisimple commutative unital Banach algebra with no non-zero idempotents besides 1, and let T be a quasi-compact bounded unital endomorphism of B . Let x_0 be the fixed point of T^* in Φ_B . Then $r_e(T) \geq r(T|J(x_0))$, with equality if $B/J(x_0)$ is finite dimensional.

Proof. Let $r_e(T) < r \leq 1$. Then by part (vi) of Theorem 3.3.3, there is a closed T -invariant primary ideal with hull $\{x_0\}$ for which

$$\sigma(T) \cap \{\lambda : |\lambda| < r\} = \sigma(T|I).$$

Since $J(p) \subseteq I$, this gives us $r(T|J(x_0)) \leq r(T|I) \leq r$. With an identical conclusion available whenever $r_e(T) < r \leq 1$, the result follows. $\square$

Combining this with Theorem 5.3.6, we arrive at the following theorem.

Theorem 5.3.8. Let T be a quasi-compact unital endomorphism of $C^m(\bar{\Omega})$ for which ϕ , the associated self-map of $\bar{\Omega}$, has an interior fixed point $p \in \Omega$. Then $\sigma(T) \cap \{\lambda : |\lambda| > r_e(T)\}$ consists of monomials of order $\leq m$ in the eigenvalues of $D\phi(p)$.

Although this latest result gives us some idea of what goes on beyond the essential spectral radius of a quasi-compact (unital) endomorphism of $C^m(\bar{\Omega})$, this is of very little use unless we know what this radius is. In light of the results with which I began this section, it does not seem unreasonable to make the following conjecture.

Conjecture 5.3.9. Let T be a unital endomorphism of $C^m(\bar{\Omega})$, and let ϕ be the associated self-map of $\bar{\Omega}$. Suppose that the fixed set

$$\phi^{\circ\infty}(\bar{\Omega}) = \bigcap_{n=1}^{\infty} \phi^{\circ n}(\bar{\Omega})$$

is a singleton $\{p\} \subset \Omega$. Then $r_e(T) = r(D\phi(p))^m$.

Remark 5.3.10. I have concentrated on *interior* fixed points in this section simply so that we are not distracted by too many unnecessary technical considerations involving the shape of Ω and its boundary. The reader can verify that Theorem 5.3.8 also holds when p is a boundary point, provided that $\bar{\Omega}$ is locally regular. Thanks to Proposition 2.5.5, some of the results presented here will also generalise to the case where Ω is a compact C^m manifold.

§5.4. Lower bounds for $\|T\|_e$ and $r_e(T)$

In this section, we examine how some particularly unruly bounded sequences in $C^m(\bar{\Omega})$ can be used to establish lower bounds for the essential norms and essential spectral radii of its unital endomorphisms. The approach used is broadly similar to some of the arguments by Linxin Zheng in [56] and Feinstein and Kamowitz in [20]; we have to work slightly harder than these authors to get our special sequences, but are rewarded handsomely for our efforts.

Lemma 5.4.1. Let $I \subset \mathbb{R}$ be a non-degenerate compact interval, and let $(y_n)_{n \in \mathbb{N}}$ be a convergent sequence in I with limit y . Suppose that $y_n \neq y$ for all $n \in \mathbb{N}$. Then there is a sequence $(f_n)_{n \in \mathbb{N}}$ in $C^m(I)$ satisfying both

$$\|f_n^{(m)}\|_\infty = f_n^{(m)}(y_n) = 1 = -f_n^{(m)}(y) \quad (n \in \mathbb{N}) \quad (5.4.1)$$

and $\lim_{n \rightarrow \infty} \max\{\|f_n^{(k)}\|_\infty : 0 \leq k < m\} = 0$.

Proof. Let w be the ‘wavelet’ defined by setting $w(x) = 0$ for $|x| > 1$ and

$$w(x) = 1 - |x| \quad \text{for } |x| \leq 1.$$

Setting $w_n(x) = w(3x/|y - y_n|)$ for $n \in \mathbb{N}$ and $x \in \mathbb{R}$, we obtain a sequence of continuous functions for which

$$\|w_n\|_\infty = w(0) = 1 \quad \text{and} \quad \lim_{n \rightarrow \infty} \|w_n\|_{L^1(\mathbb{R})} = 0.$$

Now set

$$f_n(x) = \frac{1}{(m-1)!} \int_{\frac{1}{2}(y_n+y)}^x (x-t)^{m-1} [w_n(t-y_n) - w_n(t-y)] dt$$

for $x \in I$ and $n \in \mathbb{N}$.

Fixing x and n , differentiating under the integral sign now gives

$$f_n^{(k)}(x) = \frac{1}{(m-1-k)!} \int_{\frac{1}{2}(y_n+y)}^x (x-t)^{m-1-k} [w_n(t-y_n) - w_n(t-y)] dt$$

for $k \leq m-1$, and $f_n^{(m)}(x) = w_n(x-y_n) - w_n(x-y)$. It is clear from the integral formula that $\|f_n^{(k)}\|_\infty = O(\|w_n\|_{L^1(\mathbb{R})})$ as $n \rightarrow \infty$. Property (5.4.1) follows from the fact that $\|w_n\|_\infty = 1$ (for all n) and the support of $x \mapsto w_n(x-y)$ is always disjoint from that of $x \mapsto w_n(x-y_n)$. $\square$

Lemma 5.4.2. Let T be a unital endomorphism of $C^m(\bar{\Omega})$, let ϕ be the associated self-map, and choose any $i \in \{1, \dots, d\}$. Suppose that there is a convergent sequence $(p_n)_{n \in \mathbb{N}}$ with limit $p \in \Omega$ such that $\phi_i(p_n) \neq \phi_i(p)$ for each $n \in \mathbb{N}$. Then there is a sequence $(u_n)_{n \in \mathbb{N}}$ in $C^m(\bar{\Omega})$ such that

$$\left\| (D^m T u_n)(p_n) - \frac{1}{m!} S^m(D\phi_i(p_n)) \right\| \rightarrow 0,$$

$$\left\| (D^m T u_n)(p) + \frac{1}{m!} S^m(D\phi_i(p)) \right\| \rightarrow 0,$$

$$\text{and } \sum_{k=0}^m \frac{1}{k!} \sup_{x \in \Omega} \|(D^k u_n)(x)\| \rightarrow \frac{1}{m!}$$

as $n \rightarrow \infty$.

Proof. Applying the previous lemma with

$$I = \chi_i(\bar{\Omega}) \quad , \quad y_n = \phi_i(p_n) \quad \text{and} \quad y = \phi_i(p),$$

we have access to a sequence $(f_n)_{n \in \mathbb{N}}$ in $C^m(I)$ satisfying $\lim_{n \rightarrow \infty} \|f_n^{(k)}\|_\infty = 0$ for $0 \leq k \leq m-1$ for which

$$(i) \quad f_n^{(m)}(\phi_i(p_n)) = 1;$$

$$(ii) \quad f_n^{(m)}(\phi_i(p)) = -1; \text{ and}$$

$$(iii) \quad \|f_n^{(m)}\|_\infty = 1$$

for every n .

Now let $(u_n)_{n \in \mathbb{N}}$ be the sequence of functions given by

$$u_n(x) = f_n(x_i)$$

for each $x \in \Omega$ and $n \in \mathbb{N}$. Then since $u_n \circ \phi = f_n \circ \phi_i$, we have

$$\begin{aligned} (D^m T u_n)(x) &= D^m (f_n \circ \phi_i)(x) \\ &= \sum_{\langle \alpha \rangle = m} \frac{1}{\alpha!} f_n^{(|\alpha|)}(\phi_i(x)) \phi_i^{[m]}(x)^\alpha \\ &= \frac{1}{m!} f_n^{(m)}(\phi_i(x)) S^m(D\phi_i(x)) + \dots \end{aligned}$$

for each $n \in \mathbb{N}$ and $x \in \Omega$, where the dots denote terms which contain a factor of $f_n^{(k)}(\phi_i(x))$ for some $k < m$. Having arranged it so that $\|f_n^{(k)}\|_\infty \rightarrow 0$ as $n \rightarrow \infty$ for $k < m$, we now see that the numbers

$$\gamma_n := \sup_{x \in \Omega} \left\| (D^m T u_n)(x) - \frac{1}{m!} f_n^{(m)}(\phi_i(x)) S^m(D\phi_i(x)) \right\|$$

tend to 0 as $n \rightarrow \infty$. Using properties (i) and (ii),

$$\left\| (D^m T u_n)(p_n) - \frac{1}{m!} S^m(D\phi_i(p_n)) \right\|$$

and

$$\left\| (D^m T u_n)(p) + \frac{1}{m!} S^m(D\phi_i(p)) \right\|$$

are just

$$\left\| (D^m T u_n)(x) - \frac{1}{m!} f_n^{(m)}(\phi_i(x)) S^m(D\phi_i(x)) \right\|$$

with $x = p_n$ and $x = p$ respectively; both quantities are clearly always dominated by the corresponding γ_n . It remains only to show that

$$\sum_{k=0}^m \frac{1}{k!} \sup_{x \in \Omega} \|(D^k u_n)(x)\| \rightarrow \frac{1}{m!}$$

as $n \rightarrow \infty$.

Since $\|S^k(\theta)\| = k!\|\theta\|^k$ for $\theta \in E^*$, we see that

$$\begin{aligned} \|D^k u_n(x)\| &= \left\| \frac{1}{k!} f_n^{(k)}(\chi_i(x)) S^k(D\chi_i(x)) \right\| \\ &= |f_n^{(k)}(\chi_i(x))| \|D\chi_i(x)\|^k \\ &= |f_n^{(k)}(\chi_i(x))|, \end{aligned}$$

for each $n \in \mathbb{N}$, $x \in \Omega$ and $k \leq m$. That

$$\sum_{k=0}^m \frac{1}{k!} \sup_{x \in \Omega} \|(D^k u_n)(x)\| = \sum_{k=0}^m \frac{1}{k!} \|f_n^{(k)}\|_\infty \rightarrow \frac{1}{m!}$$

is now clear from property (iii) and the fact that $\lim_{n \rightarrow \infty} \|f_n^{(k)}\|_\infty = 0$ for $0 \leq k \leq m-1$. $\square$

For what follows, all operator norms, essential norms etc. are computed relative to the norm on $C^m(\bar{\Omega})$ given by

$$\|u\|'_{m,\infty,\Omega} = \sum_{k=0}^m \frac{1}{k!} \sup_{x \in \Omega} \|D^k u(x)\|$$

for $u \in C^m(\bar{\Omega})$, as this reduces the number of $m!$ s which occur. To avoid the profusion of subscripts which will otherwise result, this norm will henceforth be denoted simply $\|\cdot\|$. With this understood, let us now make a note of a certain ‘convergence obstruction’ associated with the special sequences constructed above.

Lemma 5.4.3. Let T be a unital endomorphism of $C^m(\bar{\Omega})$ with self-map ϕ . Let $p \in \Omega$ and $i \in \{1, \dots, d\}$. Then there is a sequence $(u_n)_{n \in \mathbb{N}}$ in $C^m(\bar{\Omega})$ with

$$\lim_{n \rightarrow \infty} \|u_n\| = \frac{1}{m!} \tag{5.4.2}$$

and the following property: for any $f \in C^m(\bar{\Omega})$ and any subsequence $(u_{s(n)})_{n \in \mathbb{N}}$,

$$\frac{1}{m!} \|D\phi_i(p)\|^m \leq \limsup_{n \rightarrow \infty} \|u_{s(n)} \circ \phi - f\|. \tag{5.4.3}$$

Proof. We may assume that $D\phi_i(p) \neq 0$, as otherwise there is nothing to prove. By considering the initial value problem

$$\begin{cases} \gamma'(t) &= \nabla\phi_i(\gamma(t)) \quad (t > 0) \\ \gamma(0) &= p, \end{cases}$$

or otherwise, we can clearly obtain a sequence $(p_n)_{n \in \mathbb{N}}$ converging to p from outside $\{x \in \Omega : \phi_i(x) = \phi_i(p)\}$ (that is, with $\phi_i(p_n) \neq \phi_i(p)$ for every n). Invoking the previous lemma, we can therefore immediately arm ourselves with a sequence $(u_n)_{n \in \mathbb{N}}$ in $C^m(\bar{\Omega})$ for which

$$\left\| (D^m T u_n)(p_n) - \frac{1}{m!} S^m(D\phi_i(p_n)) \right\| \rightarrow 0,$$

$$\left\| (D^m T u_n)(p) + \frac{1}{m!} S^m(D\phi_i(p)) \right\| \rightarrow 0,$$

and $\|u_n\| \rightarrow \frac{1}{m!}$ as $n \rightarrow \infty$. Now fix any $f \in C^m(\bar{\Omega})$, and let $(u_{s(n)})_{n \in \mathbb{N}}$ be any subsequence of $(u_n)_{n \in \mathbb{N}}$. Then for each $n \in \mathbb{N}$,

$$\begin{aligned} D^m(T u_{s(n)} - f)(p_{s(n)}) &= \left(\frac{1}{m!} S^m(D\phi_i(p_{s(n)})) - D^m f(p_{s(n)}) \right) \\ &\quad + \left[(D^m T u_{s(n)})(p_{s(n)}) - \frac{1}{m!} S^m(D\phi_i(p_{s(n)})) \right], \end{aligned}$$

and

$$\begin{aligned} D^m(T u_{s(n)} - f)(p) &= - \left(\frac{1}{m!} S^m(D\phi_i(p)) + D^m f(p) \right) \\ &\quad + \left[(D^m T u_{s(n)})(p) + \frac{1}{m!} S^m(D\phi_i(p)) \right]. \end{aligned}$$

Since the familiar square bracketed terms here converge to 0 in norm as $n \rightarrow \infty$, we have firstly

$$\begin{aligned} \limsup_{n \rightarrow \infty} \|D^m(T u_{s(n)} - f)(p_{s(n)})\| &= \lim_{n \rightarrow \infty} \left\| \frac{1}{m!} S^m(D\phi_i(p_{s(n)})) - D^m f(p_{s(n)}) \right\| \\ &= \left\| \frac{1}{m!} S^m(D\phi_i(p)) - D^m f(p) \right\|, \end{aligned}$$

by continuity of f and ϕ_i , and also

$$\limsup_{n \rightarrow \infty} \|D^m(Tu_{s(n)} - f)(p)\| = \left\| \frac{1}{m!} S^m(D\phi_i(p)) + D^m f(p) \right\|.$$

Combining these two gives

$$\left\| \frac{1}{m!} S^m(D\phi_i(p)) \pm D^m f(p) \right\| \leq \limsup_{n \rightarrow \infty} \left(\sup_{x \in \Omega} \|D^m(Tu_{s(n)} - f)(x)\| \right),$$

whence

$$\frac{1}{m!} \|S^m(D\phi_i(p))\| \leq \limsup_{n \rightarrow \infty} \left(\sup_{x \in \Omega} \|D^m(Tu_{s(n)} - f)(x)\| \right).$$

Since the quantity on the left of this inequality is $\|D\phi_i(p)\|^m$, we have

$$\begin{aligned} \frac{1}{m!} \|D\phi_i(p)\|^m &\leq \limsup_{n \rightarrow \infty} \frac{1}{m!} \sup_{x \in \Omega} \|D^m(Tu_{s(n)} - f)(x)\| \\ &\leq \limsup_{n \rightarrow \infty} \|Tu_{s(n)} - f\|, \end{aligned}$$

and the proof is complete. $\square$

This latest lemma gives us an alternative proof of the fact, established by Kamowitz in the 1980s, that the only compact endomorphisms of $C^m(\bar{\Omega})$ are those with constant self-maps. To see this, let T be a compact endomorphism of $C^m(\bar{\Omega})$, let $i \in \{1, \dots, d\}$ and $p \in \Omega$. Then the lemma above gives us access to a bounded sequence $(u_{i,n})_{n \in \mathbb{N}}$ for which $(Tu_{i,n})_{n \in \mathbb{N}}$ *cannot have a convergent subsequence* unless $D\phi_i(p) = 0$; the fact that T is compact means it must, so $D\phi_i(p) = 0$. Applying this argument for each $i \in \{1, \dots, d\}$ and $p \in \Omega$, we see that ϕ is constant on Ω .

As the next result shows, the same idea can also be used to prove more quantitative results. The argument here is based on that used in Zheng's paper, [56].

Theorem 5.4.4. Let T be a unital endomorphism of $C^m(\bar{\Omega})$ with self-map ϕ . Then

$$\sup_{x \in \Omega} \|D\phi(x)\|^m \leq \|T\|_e.$$

Proof. Fix any $i \in \{1, \dots, d\}$ and $p \in \Omega$. By the previous lemma, we can find a sequence $(u_{i,n})_{n \in \mathbb{N}}$ in $C^m(\bar{\Omega})$ such that $\|u_{i,n}\| \rightarrow (m!)^{-1}$ possessing the following property: given any subsequence $(u_{i,s(n)})_{n \in \mathbb{N}}$ and $f \in C^m(\bar{\Omega})$,

$$\frac{1}{m!} \|D\phi_i(p)\|^m \leq \limsup_{n \rightarrow \infty} \|Tu_{i,s(n)} - f\|. \quad (5.4.4)$$

Let $K : C^m(\bar{\Omega}) \rightarrow C^m(\bar{\Omega})$ be any compact operator. Then, since $(u_{i,n})_{n \in \mathbb{N}}$ is a bounded sequence, there is a subsequence $(u_{i,s(n)})_{n \in \mathbb{N}}$ for which $(Ku_{i,s(n)})_{n \in \mathbb{N}}$ converges in $C^m(\bar{\Omega})$. Letting

$$f = \lim_{n \rightarrow \infty} Ku_{i,s(n)},$$

and noting that

$$\|Tu_{i,s(n)} - f\| = \|(T - K)u_{i,s(n)} + (Ku_{i,s(n)} - f)\|$$

for each n , we see immediately that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \|Tu_{i,s(n)} - f\| &\leq \|T - K\| \lim_{n \rightarrow \infty} \|u_{i,s(n)}\| \\ &\leq \frac{1}{m!} \|T - K\|. \end{aligned}$$

Using (5.4.4), we therefore see that

$$\|D\phi_i(p)\|^m \leq \|T - K\|.$$

With this conclusion available irrespective of our initial choices of K and i , we have

$$\left(\max_{i=1}^d \|D\phi_i(p)\| \right)^m \leq \|T\|_e.$$

Given that we have craftily chosen our norm on $E = \mathbb{C}^d$ to be the ‘infinity norm’ $X \mapsto \max_{i=1}^d |X_i|$, the quantity in the brackets is actually just the operator norm $\|D\phi(p)\|$. The result follows. $\square$

Corollary 5.4.5. Under the assumptions of Theorem 5.4.4, we also have the estimate $r(D\phi(p))^m \leq r_e(T)$ for each fixed point p of ϕ in $\bar{\Omega}$.

§5.5. Upper bounds for $r_e(T)$

Let Ω be a bounded domain in $\mathbb{R}^d$, let T be a unital endomorphism of $C^m(\bar{\Omega})$, and let ϕ be the associated self-map of $\bar{\Omega}$. By Corollary 5.4.5, we have

$$r_e(T) \geq r(D\phi(p))^m \text{ when } \phi^{\circ\infty}(\bar{\Omega}) = \{p\}$$

for some $p \in \bar{\Omega}$. In this section, we show that this inequality can be strengthened to an equality when $\bar{\Omega}$ is Whitney regular (thereby obtaining partial confirmation of Conjecture 5.3.9). The following two lemmas will be helpful.

Lemma 5.5.1 (Peetre-Tatar). Let X, Y and Z be Banach spaces, and let $(K, L) \in K(X, Y) \times L(X, Z)$. Suppose that there is a $C > 0$ for which

$$\|u\|_X \leq C(\|Ku\|_Y + \|Lu\|_Z) \quad (u \in X).$$

Then L is upper semi-Fredholm.

Lemma 5.5.2 (c.f. [17], Corollary 4.11). Let X be an infinite dimensional Banach space, let T be a bounded operator on X , and let

$$\sigma_{\text{usF}}(T) = \{\lambda \in \mathbb{C} : \lambda - T \text{ is not upper semi-Fredholm}\}.$$

Then $r_e(T) = \sup\{|\lambda| : \lambda \in \sigma_{\text{usF}}(T)\}$.

In our application of Lemma 5.5.1, L will be the operator D^m of m -fold differentiation, and K will be the inclusion map $C^m(\bar{\Omega}) \rightarrow C^{m-1}(\bar{\Omega})$; it is easy to see that the latter is compact when $\bar{\Omega}$ is Whitney regular. Our interest in Lemma 5.5.2 derives from the following immediate corollary; this does not appear to be in the literature.

Corollary 5.5.3. Let X and Y be infinite dimensional Banach spaces, and let $L : X \rightarrow Y$ be an upper semi-Fredholm operator. Let $T \in L(X)$, let $U \in L(Y)$, and suppose that $LT - UL$ is compact. Then $r_e(T) \leq r_e(U)$.

Proof. Let $\lambda \in \mathbb{C} \setminus \sigma_e(U)$. Then, as the composition of two upper semi-Fredholm operators, $(\lambda - U)L$ is upper semi-Fredholm.

As the sum of $(\lambda - U)L$ and the compact operator $UL - LT$, $L(\lambda - T)$ is upper semi-Fredholm, hence so is $\lambda - T$. Having shown that $\sigma_{\text{usF}}(T) \subseteq \sigma_e(U)$, the result follows from Lemma 5.5.2. $\square$

Remark 5.5.4. This result can, of course, also be applied to render an alternative proof of the fact that $r_e(T|F) \leq r_e(T)$ whenever T is a bounded operator on a Banach space and F is a closed T -invariant subspace.

The following lemma is a summary of our current position, and provides a clue as to where we go next.

Lemma 5.5.5. Let Ω be a bounded Whitney regular domain in $\mathbb{R}^d$, let T be a unital endomorphism of $C^m(\bar{\Omega})$, and let ϕ be the associated self-map of $\bar{\Omega}$. Let U be a bounded operator on $C(\bar{\Omega}, S^m E^*)$ for which $D^m T - U D^m$ is compact. Then $r_e(T) \leq r_e(U)$. In particular, $r_e(T) \leq r(U)$.

Unsurprisingly, our next move is to identify a suitable operator to play the role of U . Fortunately, we need look no further than the natural pullback operator defined (with ϕ as in the preceding lemma) by setting

$$(U_\phi f)(x) = \frac{1}{m!} f(\phi(x)) S^m(D\phi(x))$$

for $x \in \Omega$ and $f \in C(\bar{\Omega}, S^m E^*)$. This will be referred to as a ‘pullback operator’ simply because

$$\langle (U_\phi f)(x), (X_1, \dots, X_m) \rangle = \langle f(\phi(x)), (D\phi(x)X_1, \dots, D\phi(x)X_m) \rangle$$

whenever $X_1, \dots, X_m \in E$ (this matches the definition of the usual pullback operation defined for exterior differential forms). Given our choice of norm on $C(\bar{\Omega}, S^m E^*)$, this formula also makes clear that

$$\|U_\phi\| \leq \sup\{\|D\phi(x)\| : x \in \bar{\Omega}\}^m.$$

Given any $x \in \Omega$ and $n \in \mathbb{N}$, expanding using the chain rule reveals that

$$\|D(\phi^{\circ n})(x)\| \leq \prod_{i=1}^n \sup\{\|D\phi(y)\| : y \in \phi^{\circ(i-1)}(\bar{\Omega})\}.$$

Combining these last two inequalities (and assuming that ϕ is not constant), we therefore find that

$$\begin{aligned} \|U_\phi^n\|^{1/n} &\leq \sup\{\|D(\phi^{on})(x)\| : x \in \bar{\Omega}\}^{m/n} \\ &\leq \left(\prod_{i=1}^n \sup\{\|D\phi(y)\| : y \in \phi^{\circ(i-1)}(\bar{\Omega})\} \right)^{m/n}, \end{aligned}$$

for each $n \in \mathbb{N}$, so $r(U_\phi) \leq \sup\{\|D\phi(x)\| : x \in \phi^{\circ\infty}(\bar{\Omega})\}^m$ by the spectral radius formula. We are now ready to prove the following theorem.

Theorem 5.5.6. Let Ω be a bounded Whitney regular domain in $\mathbb{R}^d$, let T be a unital endomorphism of $C^m(\bar{\Omega})$, and let ϕ be the associated self-map of $\bar{\Omega}$. Then

$$r_e(T) \leq \sup\{\|D\phi(x)\| : x \in \phi^{\circ\infty}(\bar{\Omega})\}^m.$$

Moreover, $r_e(T) = r(D\phi(p))^m$ when $\phi^{\circ\infty}(\bar{\Omega}) = \{p\}$ for some $p \in \bar{\Omega}$.

Proof. Let $u \in C^m(\bar{\Omega})$ and $x \in \Omega$. Then, once more using the fact that $\langle \alpha \rangle = m = |\alpha|$ only when $\alpha_1 = m$ and $\alpha_i = 0$ for $i > 1$, we see that

$$(D^m T u - U_\phi D^m u)(x) = \sum_{\substack{\langle \alpha \rangle = m \\ |\alpha| \leq m-1}} \frac{1}{\alpha!} (D^{|\alpha|} u)(\phi(x)) \phi^{[m]}(x)^\alpha.$$

It is clear from this formula that there are bounded operators $(S_\alpha)_{\langle \alpha \rangle = m}$ for which

$$D^m T - U_\phi D^m = \sum_{\substack{\langle \alpha \rangle = m \\ |\alpha| \leq m-1}} S_\alpha D^{|\alpha|}.$$

As $D^{|\alpha|}$ is compact for all $|\alpha| \leq m-1$, the same is true of $D^m T - U_\phi D^m$. In light of Lemma 5.5.5 and our recent estimate for $r(U_\phi)$, this is enough for us to be able to conclude that

$$r_e(T) \leq \sup\{\|D\phi(x)\| : x \in \phi^{\circ\infty}(\bar{\Omega})\}^m.$$

In the case where $\phi^{\circ\infty}(\bar{\Omega})$ consists of a single point $p \in \bar{\Omega}$, either ϕ is constant or $r_e(T) = r_e(T^n)^{1/n} \leq \|D(\phi^{on})(p)\|^{m/n}$ for every n . The fact that p is a fixed point of ϕ in this situation ensures that $D(\phi^{on})(p) = D\phi(p)^n$ for every n , so $r_e(T) \leq r(D\phi(p))^m$; Corollary 5.4.5 does the rest. $\square$

The following result applies even to domains which are not Whitney regular. However, this comes at the price of having to assume slightly more about the behaviour of ϕ .

Theorem 5.5.7. Let Ω be a bounded domain in $\mathbb{R}^d$, let T be a unital endomorphism of $C^m(\bar{\Omega})$, and let ϕ be the associated self-map of $\bar{\Omega}$. Then the conclusions of Theorem 5.5.6 also apply if $\phi(\Omega_0) \subseteq \Omega_0$ for some convex relatively-open neighbourhood Ω_0 of $\phi^{\circ\infty}(\bar{\Omega})$ in $\bar{\Omega}$.

Proof. Since Ω_0 is convex, it satisfies the strong local Lipschitz condition. This ensures that the restriction map $C^m(\bar{\Omega}) \rightarrow C^m(\bar{\Omega}_0)$ is onto, hence that $C^m(\bar{\Omega}_0)$ is isomorphic to $C^m(\bar{\Omega})/M(\Omega_0)$. Applying Proposition 2.5.5, the nonzero spectrum of T is the same as that of the endomorphism T_0 induced by ϕ on $C^m(\bar{\Omega}_0)$. In particular, these operators have the same essential spectral radius. The result now follows from applying the previous result to T_0 . $\square$

We end this section with some remarks about the relationship between an endomorphism $T : u \mapsto u \circ \phi$ and the associated pullback operator U_ϕ . For the purposes of this discussion, Ω is an arbitrary bounded domain in $\mathbb{R}^d$, T is a unital endomorphism of $C^m(\bar{\Omega})$, ϕ is the associated self-map, and U_ϕ is the pullback operator it induces on $C(\bar{\Omega}, S^m E^*)$. For $1 \leq i, j \leq m$, define operators

$$T_{ij} : C(\bar{\Omega}, S^j E^*) \rightarrow C(\bar{\Omega}, S^i E^*)$$

by setting

$$(T_{ij}f)(x) = \sum_{\substack{\langle \alpha \rangle = i \\ |\alpha| = j}} \frac{1}{\alpha!} f(\phi(x)) \phi^{[m]}(x)^\alpha \quad (5.5.1)$$

for every $f \in C(\bar{\Omega}, S^j E^*)$ and $x \in \Omega$. Letting T_{00} be the composition operator ϕ induces on $C(\bar{\Omega})$, and writing D^0 for the inclusion map of $C^m(\bar{\Omega})$ into $C(\bar{\Omega})$, the Faa-di-Bruno formula gives

$$\begin{bmatrix} D^0 T u \\ D T u \\ D^2 T u \\ D^3 T u \\ \vdots \\ D^m T u \end{bmatrix} = \begin{bmatrix} T_{00} & 0 & 0 & 0 & \dots & 0 \\ 0 & T_{11} & 0 & 0 & \dots & 0 \\ 0 & T_{21} & T_{22} & 0 & \dots & 0 \\ 0 & T_{31} & T_{32} & T_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & T_{m1} & T_{m2} & T_{m3} & \dots & T_{mm} \end{bmatrix} \begin{bmatrix} D^0 u \\ D u \\ D^2 u \\ D^3 u \\ \vdots \\ D^m u \end{bmatrix} \quad (5.5.2)$$

for each $u \in C^m(\bar{\Omega})$.

To write this formula more compactly, let us introduce an operator

$$\mathbf{D} : C^m(\bar{\Omega}) \longrightarrow C(\bar{\Omega}) \times C(\bar{\Omega}, E^*) \times \cdots \times C(\bar{\Omega}, S^m E^*) \quad (5.5.3)$$

by insisting that $\mathbf{D}$ send any $u \in C^m(\bar{\Omega})$ to the column vector with entries $D^0 u, \dots, D^m u$. After setting

$$\hat{T} = \begin{bmatrix} T_0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & T_{11} & 0 & 0 & \cdots & 0 \\ 0 & T_{21} & T_{22} & 0 & \cdots & 0 \\ 0 & T_{31} & T_{32} & T_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & T_{m1} & T_{m2} & T_{m3} & \cdots & T_{mm} \end{bmatrix}$$

we can now write equation (5.5.2) simply as $\mathbf{D}T = \hat{T}\mathbf{D}$. Since $\mathbf{D}$ is manifestly bounded below, we see that T is (up to similarity) just a restriction of $\hat{T}$, a bounded operator on

$$C(\bar{\Omega}) \times C(\bar{\Omega}, E^*) \times \cdots \times C(\bar{\Omega}, S^m E^*).$$

A brief inspection of formula (5.5.1) reveals that the operators along the diagonal are all pullback operators, and that $T_{mm} = U_\phi$.

§5.6. $C^1(\bar{\Omega})$ as a Q -algebra

Having spent a great deal of this chapter implicitly considering $C^m(\bar{\Omega})$ as a closed *subspace* of a larger Banach space, we now examine a construction which allows us to consider $C^1(\bar{\Omega})$ explicitly as a *quotient* of a uniform algebra by a closed ideal. Sufficient conditions for endomorphisms of $C^1(\bar{\Omega})$ to ‘lift’ to this larger algebra are also discussed.

The algebra described below is the several variable case of a construction of Kaisjer from [32]; some of the details for the one dimensional case can be found on page 68 of Palmer’s book, [49]. It is clear that Kaisjer was aware of the manner in which his construction generalises to the situation where $\bar{\Omega}$ is replaced by a compact smooth manifold. However, I shall give the details for the benefit of the reader.

Let Ω be a bounded domain in $\mathbb{R}^d$, and let Δ be the closed unit polydisk in $\mathbb{C}^d$. Let us write

$$C(\bar{\Omega}) \boxtimes A(\Delta)$$

for the set of all $f \in C(\bar{\Omega} \times \Delta)$ for which all of the slices $f_x : \Delta \ni z \mapsto f(x, z)$ belong to $A(\Delta)$. The operator

$$Q : C(\bar{\Omega}) \boxtimes A(\Delta) \longrightarrow C(\bar{\Omega})$$

given by $(Qf)(x) = f(x, 0)$ for each $f \in C(\bar{\Omega}) \boxtimes A(\Delta)$ and $x \in \bar{\Omega}$ is clearly a unital algebra homomorphism.

Given $f \in C(\bar{\Omega}) \boxtimes A(\Delta)$, $i \in \{1, \dots, d\}$, and a test function $\varphi \in \mathcal{D}(\Omega)$, let

$$\theta_{i,\varphi}(f) = \int_{\Omega} [(Qf)(x)(D_i\varphi)(x) + \varphi(x)(\partial_i f_x)(0)] \, dx, \quad (5.6.1)$$

where $\partial_i = \partial/\partial z_i$ is the i -th complex partial derivative operator on $A(\Delta)$.

With access to the Cauchy integral formula

$$(\partial_i f_x)(0) = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{f(x, \zeta e_i)}{\zeta^2} \, d\zeta,$$

it is straightforward to show that $x \mapsto (\partial_i f_x)(0)$ is a continuous function on $\bar{\Omega}$ for each $f \in C(\bar{\Omega}) \boxtimes A(\Delta)$, and that the operator

$$D'_i : C(\bar{\Omega}) \boxtimes A(\Delta) \longrightarrow C(\bar{\Omega})$$

defined by $(D'_i f)(x) = (\partial_i f_x)(0)$ is continuous. This means, in particular, that each $\theta_{i,\varphi}$ is a bounded linear functional on $C(\bar{\Omega}) \boxtimes A(\Delta)$, and that

$$\mathfrak{A}^1(\Omega) = \bigcap \{ \text{Ker}(\theta_{i,\varphi}) : \varphi \in \mathcal{D}(\Omega), i = 1, \dots, d \}$$

is a closed subspace of $C(\bar{\Omega}) \boxtimes A(\Delta)$.

Our next task is to show that $\mathfrak{A}^1(\Omega)$ is actually a closed *subalgebra*, and that the restriction Q_0 of Q to $\mathfrak{A}^1(\Omega)$ maps $\mathfrak{A}^1(\Omega)$ onto $C^1(\bar{\Omega})$. The key to doing this is the observation that if $\theta_{i,\varphi}(f) = 0$, then $D_i Q_0 f = D'_i f$, as distributions on Ω .

Having noted already that $D_i Q_0 f = D'_i f$ and $D'_i f \in C(\bar{\Omega})$ for $f \in \mathfrak{A}^1(\Omega)$, this implies that each of the distributions $D_i Q_0 f$ may actually be represented by a continuous function on $\bar{\Omega}$, hence that Q_0 maps $\mathfrak{A}^1(\Omega)$ into $C^1(\bar{\Omega})$. As the intersection of the kernels of the derivations

$$D_i Q_0 - D'_i \quad (i = 1, \dots, d),$$

it is now also clear that $\mathfrak{A}^1(\Omega)$ is an algebra.

It now remains only to show that

$$Q_0 : \mathfrak{A}^1(\Omega) \longrightarrow C^1(\bar{\Omega})$$

is surjective (as a unital homomorphism from the semi-simple commutative Banach algebra $\mathfrak{A}^1(\Omega)$ into the Banach algebra $C^1(\bar{\Omega})$, Q_0 is automatically continuous). To see this, consider the operator $S : C^1(\bar{\Omega}) \longrightarrow C(\bar{\Omega}) \boxtimes A(\Delta)$ given by

$$(Su)(x, z) = u(x) + Du(x)z$$

for each $u \in C^1(\bar{\Omega})$ and $(x, z) \in \bar{\Omega} \times \Delta$. An easy calculation shows that $\text{Im}(S) \subseteq \mathfrak{A}^1(\Omega)$, and that $Q_0 S$ is the identity on $C^1(\bar{\Omega})$. The surjectivity of Q_0 is then immediate.

Having identified Q_0 as a surjective unital algebra homomorphism from $\mathfrak{A}^1(\Omega)$ onto $C^1(\bar{\Omega})$, we see that

$$\mathfrak{A}^1(\Omega)/\text{Ker}(Q_0) \cong C^1(\bar{\Omega})$$

as unital Banach algebras. Given the topic of this thesis, one question which immediately presents itself is the following one: given a unital endomorphism T of $C^1(\bar{\Omega})$, when is T similar to $\tilde{T}/\text{Ker}(Q_0)$ for some unital endomorphism $\tilde{T}$ of $\mathfrak{A}^1(\Omega)$? The following theorem contains a partial answer.

Theorem 5.6.1. Let T be a unital endomorphism of $C^1(\bar{\Omega})$ for which the associated self-map ϕ satisfies

$$\max_{j=1}^d \|D\phi_j(x)\| \leq 1 \quad (x \in \bar{\Omega}). \quad (5.6.2)$$

Then $T \cong \tilde{T}/\text{Ker}(Q_0)$ for a unital endomorphism $\tilde{T}$ of $\mathfrak{A}^1(\Omega)$.

Proof. Requirement (5.6.2) is really the requirement that $D\phi(x)$ maps Δ into itself, hence setting

$$(\tilde{T}f)(x, z) = f(\phi(x), D\phi(x)z)$$

gives us a unital endomorphism $\tilde{T}$ of $C(\bar{\Omega} \times \Delta)$. Our first task is to show that $\tilde{T}$ restricts to give an endomorphism of $\mathfrak{A}^1(\Omega)$. Fix $f \in \mathfrak{A}^1(\Omega)$ and $x \in \bar{\Omega}$. As $\mathfrak{A}^1(\Omega)$ is a subalgebra of $C(\bar{\Omega}) \boxtimes A(\Delta)$, the slice function

$$f_x : \Delta \longrightarrow \mathbb{C}$$

belongs to $A(\Delta)$. Noting that

$$(\tilde{T}f)_x(z) = f_{\phi(x)}(D\phi(x)z)$$

for $z \in \Delta$, we see that $(\tilde{T}f)_x$ is obtained by composing $f_{\phi(x)} \in A(\Delta)$ with a complex linear map; it is therefore certainly an element of $A(\Delta)$. We must now show that

$$(DQ\tilde{T}f)(x)e_i = \partial_i(\tilde{T}f)_x(0)$$

for $x \in \bar{\Omega}$ and $i \in \{1, \dots, d\}$. By the chain rule,

$$\begin{aligned} \partial_i(\tilde{T}f)_x(0) &= Df_{\phi(x)}(D\phi(x)0)D\phi(x)e_i \\ &= Df_{\phi(x)}(0)D\phi(x)e_i, \end{aligned} \tag{5.6.3}$$

where $Df_{\phi(x)}$ is the usual Fréchet derivative of $f_{\phi(x)}$. Since $f \in \mathfrak{A}^1(\Omega)$,

$$Df_y(0) = DQf(y)$$

for each $y \in \bar{\Omega}$. In particular,

$$Df_{\phi(x)}(0) = (DQf)(\phi(x)). \tag{5.6.4}$$

For $y \in \bar{\Omega}$, we also have

$$(Q\tilde{T}f)(y) = f(\phi(y), D\phi(y)0) = (Qf)(\phi(y)). \tag{5.6.5}$$

Combining this observation with (5.6.4),

$$(DQ\tilde{T}f)(x)e_i = (DQf)(\phi(x))D\phi(x)e_i = Df_{\phi(x)}(0)D\phi(x)e_i.$$

Comparing this with (5.6.3), we have

$$(DQ\tilde{T}f)(x)e_i = Df_{\phi(x)}(0)D\phi(x)e_i = \partial_i(\tilde{T}f)_x(0)$$

for each i , so $\tilde{T}f \in \mathfrak{A}^1(\Omega)$. That $\tilde{T}$ is an endomorphism of $\mathfrak{A}^1(\Omega)$ is now clear. To see that $\tilde{T}/\text{Ker}(Q_0)$ is similar to T , we have only to note that (5.6.5) is really just the statement: $Q_0\tilde{T} = TQ_0$. $\square$

Since endomorphisms of $\mathfrak{A}^1(\Omega)$ are power bounded, it is clear that the same is true of any endomorphism of $C^1(\bar{\Omega})$ which satisfies (5.6.2); it is not clear to me whether (by replacing Δ with a T -dependent polydisk Δ_T), every power bounded endomorphism of $C^1(\bar{\Omega})$ can be realised as a quotient of an endomorphism of an appropriate uniform algebra.

Remarks.

- In the paper, [54], mentioned at the beginning of this chapter, Tagaki, Miura and Takahasi also show that the weighted composition operator $fC_\phi : C(S) \rightarrow C(S)$ has closed range if and only if there is some $r > 0$ for which

$$\phi(\{x \in S : f(x) \neq 0\}) = \phi(\{x \in S : |f(x)| \geq r\}).$$

With this result in hand, an easy similarity argument shows that an endomorphism $T_\phi : f \mapsto f \circ \phi$ of $C^1([0, 1])$ has closed range if and only if

$$\phi(\{x \in [0, 1] : \phi'(x) \neq 0\}) = \phi(\{x \in [0, 1] : |\phi'(x)| \geq r\})$$

for some $r > 0$.

List of symbols

Some standard symbols, not explained in the text, appear below. Unless otherwise stated, compact Hausdorff spaces and domains in $\mathbb{R}^d$ are assumed to be non-empty, and all vector spaces and algebras are assumed to have $\mathbb{C}$ as their field of scalars.

$C(S)$	For a compact Hausdorff space S , this is the algebra of all continuous complex-valued functions on S , with the topology of uniform convergence. For $f \in C(S)$, $\ f\ _\infty = f _S = \sup_{x \in S} f(x) $.
$\text{Lip}(X, \rho)$	For a set X with a metric $\rho : X \times X \longrightarrow [0, +\infty)$, this is the algebra of all <u>bounded</u> complex valued functions on X for which there is an $M > 0$ satisfying $ f(x) - f(x') \leq M\rho(x, x') \quad (x, x' \in X)$.
$C^m([0, 1])$	For $m \in \mathbb{N}$, this is the algebra of all m -times continuously differentiable complex valued functions on $[0, 1]$, with the topology of uniform convergence of derivatives of all orders $\leq m$.
$C^m(\Omega)$	For an open subset Ω of $\mathbb{R}^d$, this is the set of all complex valued functions on Ω with m -times continuously Fréchet differentiable real and imaginary parts.
$C^m(\bar{\Omega})$	For a bounded domain $\Omega \subset \mathbb{R}^d$, this is the algebra of all $f \in C^m(\Omega)$ which, together with their partial derivatives of orders $\leq m$, have extensions in $C(\bar{\Omega})$. Although the domain Ω clearly determines $C^m(\bar{\Omega})$ uniquely, the same is not generally true of its closure $\bar{\Omega}$. The same is probably true for the Hölder algebras defined below.
$C^{m, \gamma}(\bar{\Omega})$	For $0 < \gamma < 1$ and a bounded domain $\Omega \subset \mathbb{R}^d$, this is the algebra of all $f \in C^m(\Omega)$ which, together with their partial derivatives of orders $\leq m$, belong to $\text{Lip}(\bar{\Omega}, \rho_e^\gamma)$, where ρ_e is the restriction to $\bar{\Omega}$ of the Euclidean metric.

$L(X, Y)$	For normed spaces X and Y , this is the space of all bounded linear maps from X into Y , equipped with the usual operator norm.
$X^*, L(X)$	These are shorthand for $L(X, \mathbb{C})$ and $L(X, X)$ respectively.
$L_F(X)$	For a closed subspace F of a Banach space X , this is the closed unital subalgebra of $L(X)$ consisting of all $T \in L(X)$ such that $T(F) \subseteq F$.
$F^\perp$	For a subset F of a normed space X , this is the space of all $\mu \in X^*$ for which $\mu(F) \subseteq \{0\}$.
$K(X, Y)$	For Banach spaces X and Y , this is the space of all compact operators mapping X into Y . The symbol $K(X)$ is used in place of $K(X, X)$.
$\mathfrak{F}(X, Y)$	For normed spaces X and Y , this is the space of all finite rank operators mapping X into Y . The symbol $\mathfrak{F}(X)$ is used in place of $\mathfrak{F}(X, X)$.
$\ T\ _e$	For a bounded operator $T : X \longrightarrow Y$ between Banach spaces X and Y , this is the quantity $\ T + K(X, Y)\ = \inf\{\ T + V\ : V \in K(X, Y)\}.$
T^*	For normed spaces X and Y , and $T \in L(X, Y)$ this is the element of $L(Y^*, X^*)$ given by $(T^*\mu)(x) = \mu(Tx) \quad (x \in X, \mu \in Y^*).$
$\mathcal{D}(\Omega)$	For an open subset Ω of $\mathbb{R}^d$, this is the space of all C^∞ functions on Ω with compact support.
$\mathcal{O}(\Omega)$	For an open subset Ω of $\mathbb{C}^n$, this is the algebra of all complex valued holomorphic functions on Ω , with the topology of uniform convergence on compact subsets.

$\mathcal{O}(\sigma)$	For a compact subset σ of $\mathbb{C}^n$, this is the algebra of germs of holomorphic functions over σ with the inductive limit topology (the inductive limit being taken in the category of locally convex spaces).
$\mathbb{C}[t]_{\leq k}$	This is the algebra of all polynomials of degree $\leq k$ in the indeterminate t , with operations carried out modulo t^{k+1} . It is assumed that $t^j \neq 0$ for $j \leq k$.
$\widehat{K}$	For a compact subset K of $\mathbb{C}$, this is the polynomially convex hull of K .
$\sigma(T)$	For a bounded operator T on a Banach space X , this is the set of all complex λ for which $\lambda - T$ has no inverse in $L(X)$. The symbol $\rho(T)$ is used for the complement of $\sigma(T)$ with respect to $\mathbb{C}$.
$r(T)$	Given X and T as above, this is the spectral radius of T relative to $L(X)$.
Φ_A	For a commutative unital Banach algebra A , this is the set $\text{Hom}(A, \mathbb{C})$ of all characters on A , equipped with the Gel'fand topology.
$\mathbb{C}^S$	For a set S , this is the algebra of <i>all</i> $\mathbb{C}$ -valued functions on S .
$\sum A$	For a finite subset $A = \{a_1, \dots, a_n\}$ of an Abelian group $(\Gamma, +)$, this symbol denotes the element of Γ given by the 'sum' $a_1 + a_2 + \dots + a_n$. This 'sum of a set' notation is used only in Section 2.4.
$D\phi$	This symbol is used in two ways. In Chapter 4, it is used to denote the usual Fréchet derivative ϕ' of a C^1 self-map $\phi = (\phi_1, \phi_2)$ of $\mathbb{R}^2$. Later, in Chapter 5, it denotes the <i>complexification</i> of the usual Fréchet derivative of ϕ . In both cases, $D\phi$ evaluates at $x \in \Omega$ to give a linear map whose (i, j) -th matrix entry with respect to the standard basis is $(D_j \phi_i)(x)$.

$\mathbb{Z}_+, \mathbb{N}$	The first of these symbols denotes the set of all non-negative integers; the second stands for $\mathbb{Z}_+ \setminus \{0\}$.
$\text{Pol}_k(\mu)$	For a symmetric k -form μ on a vector space E , this is the homogeneous polynomial of degree k given by $\text{Pol}_k(\mu)(X) = \mu(X, \dots, X)$ for $X \in E$.
$C(S, E)$	For a normed space E and a compact Hausdorff space S , this is the space of all continuous E -valued functions on S , equipped with the norm given by $\ f\ = \sup_{x \in S} \ f(x)\ _E$ for $f \in C(S, E)$.

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