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Chiral deformations of Yang-Mills and Gravity theories

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Abstract

In this thesis we present a modification of Yang-Mills and Gravity theories both written in terms of gauge connections. In particular gravity is presented in its pure connection formulation realizing a Diffeomorphism Invariant Gauge Theory. The modifications do not change the kinematical content of the theories, but they add an infinite set of new vertices retaining the symmetries of the original Lagrangians without introducing higher derivatives in the equations of motion. These new interactions are chiral, leaving the original Yang Mills and Gravity theories as the only parity-invariant theories in the set. In the Yang-Mills case we proved that at tree level the theory is still constructible via the so-called BCFW recursion relations. Both theories are claimed to be closed under the renormalization group. The Yang-Mills Deformations are indeed proved to be one-loop renormalizable and an explicit diagrammatic computation of the simplest deformation beta-function is given. For gravity a new formulation is introduced that could lead to a similar result at one-loop.

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1

INTRODUCTION

The quest for a description of the fundamental laws of nature have led the development of a huge number of theories and models. Not many have actually made it so far. Of course, this is mainly due to the usually unforgiving reality of experiments. Nonetheless the constrains imposed by the very mathematical nature of the theories and the principles that we used as main guidelines to build such theories, also helped reducing the number of possible candidates. In retrospect the fact that very specific gauge theories (Yang-Mills theories) are basically the only fully (also quantum mechanically) established theories which we can quite safely rely on is actually a combination of the great amount of experimental data that they explain and the incredibly strict mathematical framework they are based on, i.e. Quantum Field Theory (QFT). The main conditions for physical applicability of QFT, namely renormalizability and unitarity are often not met at the same time. The fact that in combination with the symmetry breaking mechanism all these conditions are incredibly met led to the prominence of YM theories. Even the other great theory of the 20th century, gravity although experimentally highly vindicated resists the quantization process and does not meet the requirement of renormalizability.

Yang-Mills (YM) and General Relativity (GR) are, anyway, quite unique theories. Both are very constrained by the symmetries, by their number of degrees of freedom, by the number of derivatives in the equations of motion and by the possible interaction with matter. Ultimately they are constrained

by the very choice of the fundamental variables we use to describe them. It is certainly not a new idea to push the boundaries of those two theories and see how they can be modified. Einstein's gravity had to wait only a few years before Eddington proposed variations both in the way we write the theory and its dynamical content. Modifying theories and gravity in particular is always a difficult task if one does not want to spoil some or all of the original properties. Doing so, quite often, either introduces new degrees of freedom or modifies the equations of motion in a way that unphysical instabilities arise. If one wants to at least keep the symmetries (gauge or diffeomorphism) one can then add invariant higher order operators build from the curvatures. In the case of YM this has to be done in such a way that the new interactions do not contain derivatives of the curvature build from the connection. In GR even simple modifications lead to new degrees of freedom. General Lagrangians can be written in terms of functions with specific symmetry properties. For example a quite large set of extensions of GR are the theories $f(R_{\mu\nu\rho\sigma})$ where the Lagrangian depends on general invariants build from the Riemann curvature. It can be shown that those theories have 6 more degrees of freedom with respect to normal GR [1] and are plagued by instabilities. The simplest generalizations of Einstein's theory are considered to be $f(R)$ theories (see [2] and references within for a recent general account) where one modifies the Einstein-Hilbert Lagrangian, proportional to the Ricci scalar R, with a general function of R. They have more degrees of freedom, but they are considered to be free of instabilities and have been extensively used to build models that aim to address open cosmological problems [3, 4].

Another point of view on modifications of gravity comes from radically change the basic variable used to describe it. In GR one starts with a metric with a compatible connection (on the tangent bundle of the underlying space-time manifold), in contrast to a principal G-bundle connection used in YM theories. It is known since the early years of GR that one can describe it in terms of the connection instead of the metric. Still the theory is quite

different from other gauge theories. Anyway, it took many more years to fully realize the close relationships entertained by GR and YM. After the early works on the relations of gravity and self-duality [5, 6], Plebanski took a bold step in proposing a completely new first order Lagrangian for gravity using a $\mathfrak{su}(2)$ connection as basic variable [7] alongside a triple of 2-forms and a constraint. Some years later Asthekar found that one can embed the phase space of gravity into the one of a $\mathfrak{su}(2)$ Yang-Mills theory [8], which is just the Hamiltonian version of Plebanski's theory. Capovilla et al. [9, 10, 11] then managed to rewrite Plebanski's version of GR integrating out the 2-forms. A pure connection formulation of gravity was then recently found by Krasnov in [12]. These "gauge" formulations of gravity allow to avoid the usual uniqueness proofs of GR. Indeed, it turns out that gravity is the only diffeomorphism invariant theory of metrics, with second order equations of motion and two propagating degrees of freedom. But changing the nature of variables to gauge connections opens the possibility of an infinite class of gravity theory with the same properties just stated. Of course the symmetries will now include gauge rotations. These modifications have been introduced in the 1990's by Bengtsson in [13] and rediscovered by Krasnov in [14] and put into the Pure Connection framework in [12] where they were also dubbed Diffeomorphism Invariant Gauge theories (DIGT).

This thesis will be mainly devoted to the introduction of certain modifications (or deformations) of YM theory along with some further analysis of the DIGT program. The former has been studied by the present author et al. in [15]. We aim at giving a self-contained presentation of both theories. For the Deformations of Yang-Mills (DYM) case we will present the main results of the paper [15] concerning their tree-level scattering amplitudes and the one-loop renormalizability. For gravity we will present the general idea and properties of the modifications and describe an attempt at a formal reconstruction of them based on just the symmetries of the final theory. Moreover, we will use some techniques employed for DYM to compute the one-loop beta function for gravity expanded on a constant curvature background.

The DYM theories are an infinite class of theories that add to the original YM Lagrangian a series of new interaction depending only on gauge invariant powers of the self-dual part of the connection's curvature. Up to a total derivative even YM itself can be written in terms of this self-dual part only, so not only it is a part of this class but constitutes the basic theory that gives the kinematical structure to the whole set. By the virtue of these key building blocks, these modification do not modify some basic features of the mother theory, YM. In particular, the theory keeps the same number of on-shell degrees of freedom and the equation of motion do not have higher than second derivatives leading to instabilities, but only non-linearities in the first and second derivative of the connection. This should be compared to the already mentioned case of $f(R)$ theories of gravity. The general Lagrangian can, also in this case, be written compactly as:

$$S_{\text{DYM}} = \int d^4x M^4 f\left(\frac{F_{sd}}{M^2}\right). \tag{1}$$

Notice that we have introduced a mass scale M (which the quadratic in the curvature YM term is independent of) in such a way that the new interaction terms have a dimensionless coupling multiplied by the correct power of M. This also allows for a direct easy inspection that, taken separately, none of these new interactions is power counting renormalizable, see later. A few comments on the action are already possible. First, the theory is defined in 4-dimensions where the concept of self-duality is well defined. Second, if we are working with a Lorentzian metric, the self-dual part of the curvature is complex which makes the Lagrangian itself complex. This immediately raises the question of unitarity. We will show that the theory has a discrete PT-symmetry that could help recovering the unitarity of such theories. We have not yet completely solved this important problem but we are confident that the interesting properties we are about to spell out make the effort of investigating these theories well worth despite a lack of a clear answer about unitarity.

The self-dual nature of the action (1) carries two more properties. All the theories share a sector of solutions, namely YM instantons. They are defined as purely anti-selfdual solution of the equations of motion, i.e. connection whose curvature satisfies $F_{sd} = 0$. Moreover, all new theories are chiral, YM being the only non-chiral element of the class.

These theories should behave much worse than they actually do. Indeed, the interactions vertices have as many derivatives as their valence, even if only one per leg. It looks like one should expect the amplitude to diverge much worse than YM. Anyway, there is the special property that when two derivatives are on the same internal line of an amplitude this line "collapses" and a higher order vertex appear. After this collapse the effective amplitudes contain only at most one derivative per internal propagator. We will see that this single property is responsible for many of the improved properties of DYM.

We have performed an analysis of the tree-level amplitude using modern spinor helicity techniques [16] to show that the DYM theories are constructible recursively in a way which is similar to YM. It is known that if the amplitudes have nice fall-off properties under complex shifts of the external momenta (BCFW shift) one can separate any tree-level into products of lower order on-shell amplitudes. In this way declaring the value of the basic 3-point amplitudes one can reconstruct any amplitude by recursion [17, 18]. In YM, in this way, one does not need any Lagrangian to compute the amplitudes. In DYM, almost all the amplitude can be reconstructed this way. Of course, these theories carry an infinite set of coupling constant. One needs then to add the basic amplitudes coming from evaluating the new vertices with maximal number of derivatives on-shell, at each new order in external particle number. Once this is done any amplitude can be constructed recursively. This result is surprising because in a general effective field theory the number of derivatives in the vertices prevents the amplitude to have the desired fall-off conditions that guarantee the BCFW recursion relations

to work. It is the collapsing property stated before that makes DYM to be constructible.

The main interest in these new Lagrangians lies though in their renormalization properties. It is now appreciated that any possible higher operator with the associated coupling constant could in principle be added to the Lagrangian to make it "renormalizable". The question than is how to keep the theory's power of making predictions with all its (possibly infinite) coupling constants. It is the goal of the Asymptotic Safety scenario of Weinberg (see [19] for an historical account) to make sense of effective field theories with an infinite tower of interactions whose renormalization flow is still well enough behaved so that the coupling constants do not blow-up while the theory remains predictive. In particular one has to show that there are some trajectories of the flow that end at a fixed point and the surface of attractive directions towards this fixed point is finite dimensional. In the general setting there is usually no constraint on the form of interactions and also vertices that produce unstable higher than second order equations of motion can get added. Our hope is that the DYM is actually close under the renormalization flow. This flow is then one in the space of theories presented here and DYM could be a good setting for testing claims regarding the asymptotic safety scenario. The same hope is still up for the DIGT theories and the DYM certainly present a benchmark for ideas and techniques to develop the gravity side of this story. In the following chapters we will discuss a very promising result in this respect. The DYM are one-loop renormalizable. No new vertices get added at one-loop to cancel the arising divergences. We also computed in detail the beta function for the first, simplest new vertex and found it to be positive. What is interesting about the calculation is that, again, the key collapsing property discussed above gives rise to improved renormalization property and allows for many cancellations in the diagrams.

The thesis will be divided into two main parts. In the first part we will describe the Deformations of Yang-Mills. In particular in chapter 2 we will

introduce the theory and its properties. After a discussion about its Hamiltonian formulation, we will give the mode decomposition and spell out the Feynman rules for two different gauge fixing choices. Here we will show the collapsing properties that lies at the heart of our study. We will then talk about the scattering amplitudes and recursion relations. We will also show some explicit calculations of simple 4-points amplitudes.

Chapter (3) will be devoted to the renormalization properties of DYM. We will start with a general power-counting argument for the improved behavior at loop order due to the collapsing property. We will then perform the explicit calculation of the β-function for the first new coupling constant. This is one of the main results of [15] by the author et al. We end the chapter by a review of the proof of one-loop renormalizability of DYM appeared in [20].

The second part will start in chapter (4) with an introduction to the DIGT program. We will start discussing the Plebanski formulation of GR, from which the extended class of theories can be derived. We will then introduce the pure connection formalism, its symmetries and compare the linearized part of the Lagrangian to YM. We conclude the chapter with a discussion of its interactions and the program to check if also these theories of gravity are renormalizable.

We dedicate the last chapters to some developments in the study of DIGT. In chapter (5) we show how to reconstruct the whole linear and cubic level of the DIGT using as constraint only the form of its symmetries and the fact that they will produce up to second order field equations.

In the last chapter (3) we will introduce an intermediate formulation for DIGT which makes use of the connection plus another field and a set of constraints to select a specific theory. We will perform a one-loop calculation of the beta-function for a generic theory in the class when expanded on a constant curvature background. We match a result previously found in [21]. The novelty here is that this first order formulation allows us to follow the exact same calculation that proved successful in determining the full

β-function of DYM. Indeed, also in that case a first order formulation was used, similar to the mixed formulation of gravity presented in this chapter. We hope that this result could be extended to a more general one in which one expands the theory around a general background.

We conclude with a few remarks on what can be learned from further investigations on DYM. Especially, we are interested in how results and techniques used there can serve as a benchmark for new ideas that might also prove fruitful for similar studies in gravity.

2

DEFORMATIONS OF YANG-MILLS THEORIES

In this chapter we will introduce a new class of theories, that we developed in [15], modeled on the usual YM theory. The class describes an infinite set of chiral interactions that are non-renormalizable by power counting added on top the usual YM cubic and quartic ones.

The new interactions are gauge invariant and they do not produce higher order derivatives in the field equations, keeping the theory free of instabilities. These deformations of the YM Lagrangian unexpectedly behave in a way that is much similar to the original, renormalizable YM theory. Even if all the interactions have more derivatives in the vertices compared to YM, at most only one instance of momentum will appear in each internal line of the Feynman graphs. This is the same situation that we encounter in YM and this single property accounts for many of the interesting results that we will show. This situation is due to a "collapsing" property that states that every time two derivatives lie on the same propagator it collapses, being substituted by a delta function, and gives rise to a higher order vertex.

After a brief introduction of the general structure of the theory and its Hamiltonian formulation, we will set up the gauge fixing and Feynman rules that we will use in the following analysis. Here we will spell out the key property of the diagrams. After, we will see how these theories appear to be constructible by recursion relations (according to the now well established BCFW construction). In this way one could build all the scattering amplitudes from lower order amplitudes.

2.1 THE LAGRANGIAN

The Lagrangian of DYM do not modify the YM structure, but just adds new interaction vertices constructed from powers of the curvature of the connection A^a_μ defined on a principal $SU(N)$-bundle. We do not need to specify which group we are going to take, but we want to work in 4 dimensions for reasons that will be clear in a moment. If one is bound to add only terms that do not add higher orders derivatives in the equations of motion a possible Lagrangian would be:

$$\mathcal{L} = -\frac{1}{4g^2}F^2 + \frac{\alpha}{M^2}F^3 + \frac{\beta}{M^4}F^4 + \dots, \tag{2}$$

where $F = dA$ is the curvature, d being the covariant exterior derivative for the connection A. Here the coefficients have been written in a form such that a dimensionless part is isolated $\alpha, \beta \dots$. Of course inverse powers of mass get introduced showing that these new vertices are power-counting non-renormalizable. A possible, more compact way to write these theories would be:

$$\mathcal{L} = M^4 f(F/M^2), \tag{3}$$

where f is a gauge invariant function of the sole field strength. These theories have been studied and do not show the nice properties that we would like them to have. Therefore, we will further restrict the space of theories, requiring that all the theories share an instanton sector. Instantons are special solutions of YM theories in which the connection's curvature (a 2-form) has vanishing self-dual components. As it is well known, the Hodge duality operation (defined by the spacetime metric $\eta_{\mu\nu}$) splits the 6 dimensional vector space of 2-forms Λ^2 into two orthogonal subspaces, Λ^2_+, Λ^2_-, called self-dual, invariant under the duality operation:

$$\begin{aligned} {}^* &: \Lambda^2 \longrightarrow \Lambda^2, \\ \Lambda^2 &= \Lambda^2_+ \oplus \Lambda^2_-. \end{aligned} \tag{4}$$

Choosing a basis for these two spaces, $\Sigma^a_{\mu\nu}, \bar{\Sigma}^a_{\mu\nu}$, where $a = 1, 2, 3$ is the Lie algebra index, the curvature F splits into its self- and anti-self-dual components:

$$F^a_{\mu\nu} = F^a \Sigma^a_{\mu\nu} + \bar{F}^a \bar{\Sigma}^a_{\mu\nu}. \tag{5}$$

The Lagrangian for the Deformations of Yang-Mills theories takes then the simple form:

$$\mathcal{L} = M^4 f(F_{\text{sd}}/M^2), \tag{6}$$

where we denoted the self dual part of the curvature with F_{sd}.

2.1.1 *Spinor conversions*

The best way to deal with self duality is to use spinor formalism (see [22] and the appendix (A)). We shall denote as $S_\pm$ the two inequivalent irreducible representation of the Lorentz group of dimension 2. We will denote objects in $S_\pm$ as unprimed and primed spinors, i.e. $\lambda^A, \lambda^{A'}$. There exist then an isomorphism between the tensor product $S_+ \otimes S_-$ and Minkowski space. The isomorphism is given by the so called soldering form $\theta^{AA'}_\mu$ which converts spinor indices into spacetime ones. Each spacetime index gets replaced by two spinor indices (unprimed and primed). In particular, in our convention, the metric is transformed into:

$$\eta_{\mu\nu} \longrightarrow -\varepsilon_{AB}\varepsilon_{A'B'}. \tag{7}$$

where ε_{MN}, $\varepsilon_{M'N'}$ are the antisymmetric identity 2-spinors that are used to lower and raise spinor indices. The conventions used for lowering and raising indices and normalization are:

$$\begin{aligned} \lambda^A \varepsilon_{AB} &= \lambda_B, \\ \varepsilon^{AB} \lambda_B &= \lambda^A, \\ \varepsilon_A{}^A &= 2, \end{aligned} \tag{8}$$

and similar equations hold for primed indices. Antisymmetric 2-tensors can be split into (anti)-self-dual parts which in spinor terms are proportional to the identity unprimed and primed spinors respectively. In particular, our curvature splits as

$$F^a_{MM'NN'} = \tfrac{1}{2} F^a_{MN} \varepsilon_{M'N'} + \tfrac{1}{2} \bar{F}^a_{M'N'} \varepsilon_{MN}. \tag{9}$$

2.2 DYM IN SPINOR FORM AND EQUATIONS OF MOTION

Using what we have learned we can put the DYM into its final spinorial form:

$$\mathcal{L} = M^4 f(F_{AB}/M^2). \tag{10}$$

YM is still one of the theories because, modulo a surface term, it can be rewritten as:

$$\mathcal{L}_{YM} = -\frac{1}{4g^2}(F^a_{MN})^2. \tag{11}$$

The equations of motion for (10) are:

$$d_{N'M}\left(\frac{\partial f}{\partial F^a_{MN}}\right) = 0. \tag{12}$$

As it is simple to check, instantons of the form $F_{MN} = 0$ are solutions to these equations for any f since the partial derivative in the last eq. becomes just a constant.

It is important to note that the Lagrangian in eq. (10) can be cast into a first order formulation:

$$\mathcal{L} = B^{aMN} F^a_{MN} - M^4 V(B/M^2), \tag{13}$$

where B_{MN} is the self-dual component of a 2-form. In this case YM is singled out as one of the possible gauge invariant potentials:

$$\mathcal{L} = B^{aMN} F^a_{MN} - \frac{g^2}{2} B^2_{MN}. \tag{14}$$

The field equations for the first order formulation read:

$$F_{MN} = M^2 V'(B)_{MN}, \tag{15}$$

$$d\,B_{MN} = 0. \tag{16}$$

As an example we show the simplest modification of YM along these lines:

$$\mathcal{L} = -\frac{1}{4g^2}(F^a_{MN})^2 + \frac{\alpha}{3!g^2M^2} f^{abc} F^{a\ B}_{A} F^{bC}_{B} F^{c\ A}_{C}. \tag{17}$$

We will study (17) in detail, especially we will compute the running of its coupling constant in the next chapter.

These theories make sense clearly in 4-dimensions where a notion of self-duality is well defined. Also, they will be chiral (i.e. parity violating) for all modifications but the mother theory, YM.

2.3 HAMILTONIAN ANALYSIS

A $3+1$ decomposition and Hamiltonian analysis can be performed in a similar way as in the usual YM case. The only thing that we need to take care of is the self-dual nature of some of the fields involved. It is best to perform these calculations starting with the first order Lagrangian (13). The decomposition is the following:

$$F^a_{MN} = (2\mathrm{i}F^a_{0i} - \epsilon^{ijk}F^a_{jk})\frac{1}{\sqrt{2}}T^i_{MN}, \tag{18}$$

$$T^{iAB}T^{j\ C}_{B} = -\frac{1}{2}\delta^{ij}\epsilon^{AC} + \frac{1}{\sqrt{2}}\epsilon^{ijk}T^{kAC}, \tag{19}$$

$$F^a_{0i} = \dot{A}^a_i - d_i A^a_0, \tag{20}$$

where T^i_{AB} are multiples of the Pauli matrices[1]. We introduce a spatial new field P^{ai}, such that:

$$B^a_{MN} = \frac{1}{\mathrm{i}\sqrt{2}} P^{ai} T^i_{MN}. \tag{21}$$

1 These matrices can also be used to convert $SO(3)$ indices into a symmetric pair of spinor ones, as explained in appendix (A)

Inserting all the previous formulae in the original Lagrangian one finds:

$$\mathcal{L} = P^{ai}\dot{A}^a_i + A^a_0 d_i P^{ai} - \mathcal{H}, \tag{22}$$

where

$$H^{ai} = \frac{1}{2}\epsilon_{ijk}F^a_{jk}, \tag{23}$$

is the magnetic field and the Hamiltonian is:

$$\mathcal{H} = -\mathrm{i}P^{ai}H^a_i + M^4 V(P/M^2). \tag{24}$$

Let us specialize this to the case of YM and the first deformation (17). In the YM case the Hamiltonian is:

$$\mathcal{H}_{\mathrm{YM}} = \frac{g^2}{2}\left(P^{ai} - \frac{\mathrm{i}}{g^2}H^{ai}\right)^2 + \frac{1}{2g^2}(H^{ai})^2. \tag{25}$$

As it is usual in this case, one performs a transformation:

$$E^{ai} := P^{ai} - \frac{\mathrm{i}}{g^2}H^{ai}. \tag{26}$$

This is indeed a canonical transformation, i.e. it does not modify the symplectic structure. The Hamiltonian then becomes:

$$\mathcal{H} = \frac{g^2}{2}\left(E^{ai}\right)^2 + \frac{1}{2g^2}(H^{ai})^2. \tag{27}$$

It is important to note here that due to the very nature of the self-dual formulation, in Lorentz signature the fields become complex and some reality conditions need to be imposed. In YM this is an easy task as we just need to impose that the spatial connection is real so that the magnetic field is real and (26) is real. This brings to a possible drawback of these theories at least until further analysis is done. If we take the first of the deformations and compute the Hamiltonian in the same way as before we obtain:

$$\begin{aligned}\mathcal{H} &= \frac{g^2}{2}\left(E^{ai}\right)^2 + \frac{1}{2g^2}(H^{ai})^2 \\ &- \frac{\mathrm{i}\alpha g^4}{3M^2}f^{abc}\epsilon^{ijk}\left(E^a_i + \frac{\mathrm{i}}{g^2}H^a_i\right)\left(E^b_j + \frac{\mathrm{i}}{g^2}H^b_j\right)\left(E^c_k + \frac{\mathrm{i}}{g^2}H^c_k\right).\end{aligned} \tag{28}$$

With the reality condition we saw before this new Hamiltonian is not Hermitian with respect to the usual scalar product. This problem is common in theories that are described by complex Lagrangians. This single problem could in principle lead one to discard these theories altogether due to the possible non-unitarity of the dynamics. The whole picture could be saved by some discrete symmetry of the Hamiltonian. Indeed, all these theories are invariant under an antilinear symmetry:

$$E^{ai} \to -E^{ai}, \qquad H^{ai} \to H^{ai}, \tag{29}$$

followed by complex conjugation.

Recently, some authors [23, 24] have entertained the possibility of making sense of quantum systems defined by complex Hamiltonians. The interest in this field started considering certain $\mathcal{PT}$-symmetric Hamiltonians such as:

$$H = p^2 + \mathrm{i}kx^3 \qquad \text{where} \qquad k \in \mathbb{R}, \tag{30}$$

which can show a real spectrum. This line of research has shown that one could extract a quantum description of systems defined originally with a non-Hermitian Hamiltonian in the presence of a more general anti-linear symmetry.

Let us briefly review the idea behind these works. In quantum mechanics the implementation of the standard axioms requires that any observable $\mathcal{O}$ defined by an operator acting on $\mathcal{H}$ has a complete set of eigenvectors and that for any $\mathcal{O}$ and state $\psi \in \mathcal{H}$ the expectation value $\frac{\langle\psi|\mathcal{O}\psi\rangle}{\langle\psi\psi\rangle}$ is real. Hermiticity of any observable is then a necessary and sufficient condition to meet these requirements. Moreover, the evolution of the state is driven by a unitary operator $U(t,t_0)$: $\psi(t) = U(t,t_0)\psi(t_0)$ in order to preserve the norm of the state. The Shrödinger equation then reads $i\hbar\frac{d}{dt}U(t,t_0) = HU(t,t_0)$ and demands the Hamiltonian to be Hermitian. Hermiticity is then a key property for the implementation of the axioms of quantum mechanics. However, if H is not Hermitian with respect to the standard inner product but

has a real spectrum and a complete set of eigenvectors one can modify the inner product of $\mathcal{H}$ defining a new Hilbert space $\mathcal{H}'$. In the new system, H is Hermitian with respect to the new inner product.

Specifically, a linear Hermitian automorphism $\eta : \mathcal{H} \to \mathcal{H}$ is called a pseudo-metric operator. If it is positive definite it is just called a metric operator. The linear operator H (Hamiltonian) is then called pseudo-Hermitian if there exist a pseudo-metric η such that:

$$H^\dagger = \eta H \eta^{-1}. \tag{31}$$

One can then define a new Hilbert space $\mathcal{H}'$ endowed with the inner product:

$$\langle \phi, \psi \rangle_\eta = \langle \phi, \eta\psi \rangle, \tag{32}$$

where on the RHS we used the original inner product of $\mathcal{H}$. It is clear that if η is positive definite, then H acts on $\mathcal{H}'$ as an Hermitian operator.

A necessary and sufficient condition for pseudo-Hermiticity of H is the existence of an anti-linear map that commutes with the Hamiltonian. In our case this operator exists and it is defined by the transformations (29). We could at least in principle hope to make sense of unitarity of our theories through the procedure just reviewed, although to actually achieve a physical interpretation a more detailed investigation of the issue is in order. For the rest of the thesis we will put aside this problem which needs more careful consideration. We will focus instead on a formal study of the properties of the DYM theories.

2.4 MODE DECOMPOSITION AND HELICITY OPERATORS

We can take the free part of the YM Hamiltonian (25) to perform a mode decomposition of these theories, given that the only thing that the deformations do is to add new interaction vertices:

$$\mathcal{H}_{\text{free}} = -\mathrm{i} P^{ai} \epsilon^{ijk} \partial_j A^a_k + \frac{g^2}{2} (P^{ai})^2. \tag{33}$$

From the last formula the (free) equations of motion are:

$$g^2 P^{ai} = \frac{\partial}{\partial t} A^{ai} + \mathrm{i}\epsilon^{ijk}\partial^j A^{ak},$$
$$\frac{\partial}{\partial t} P^{ai} - \mathrm{i}\epsilon^{ijk}\partial^j P^{ak} = 0, \tag{34}$$

combining these equations to put them into a second order equation for the connection, we find that the transverse part of A^{ai}, i.e. $\partial_i A^{ai} = 0$ satisfies the wave equation. This is the usual story in YM and, of course, it is preserved in our theories. The mode decomposition in positive and negative helicities of the transverse part of the connection is then:

$$A_i^a = g \int \frac{d^3k}{(2\pi)^3 2\omega_k} \Big(\epsilon_i^-(a_k^-)^a e^{\mathrm{i}kx} + \epsilon_i^+(a_k^+)^a e^{\mathrm{i}kx} + \epsilon_i^+(a_k^-)^{\dagger a} e^{-\mathrm{i}kx} + \epsilon_i^-(a_k^+)^{\dagger a} e^{-\mathrm{i}kx} \Big), \tag{35}$$

where $kx = k^i - \omega_k t$ and for on shell gluons $w_k = |k|$. The polarization vectors satisfy:

$$\mathrm{i}\epsilon_i{}^{mn} k_m \epsilon_n^- = \omega_k \epsilon_i^-,$$
$$\epsilon_i^+ = (\epsilon_i^-)^*,$$
$$\epsilon_i^+ \epsilon_j^- + \epsilon_i^- \epsilon_j^+ = \Pi_{ij}, \tag{36}$$

where Π_{ij} is the projector on transverse components. In particular the last equation is a normalization condition that, combined with (35) and the canonical equal-time commutation relations:

$$[A_i^a(x), P_j^b(y)] = \mathrm{i}\delta^{ab}\Pi_{ij}\delta^3(x-y), \tag{37}$$

guarantees the commutation relations between the modes:

$$[(a_k^\pm)^a, (a_p^\pm)^{\dagger b}] = (2\pi)^3 2\omega_k \delta^{ab}\delta^3(k-p). \tag{38}$$

We will use the spinor technology for the most part of the following, we then need a decomposition into modes of the field written directly using

spinor indices. To start we remind that external on-shell momenta can be separated into the outer product of two spinors[2]:

$$(k^\mu)^2 = 0 \longrightarrow k^{AA'} = k^A k^{A'}. \tag{39}$$

This decomposition is defined up to a scaling of the two spinors. For real momenta this rescaling factor is just a phase and the two spinors are complex conjugate to each other. Usually though, one continues the momenta to the complex plane to obtain a non-zero result for the basic 3-amplitudes which will be used as seeds in reconstructing all tree level amplitudes from recursion relations (see later). This allows for a general complex rescaling:

$$k^A \longrightarrow \lambda k^A \qquad k^{A'} \longrightarrow \lambda^{-1} k^{A'} \tag{40}$$

The helicity spinors are found to be:

$$\epsilon^-_{MM'} = \mathrm{i}\frac{q_M k_{M'}}{\langle q\, k\rangle}, \qquad \epsilon^+_{MM'} = \mathrm{i}\frac{k_M q_{M'}}{[q\, k]}. \tag{41}$$

As it is customary one introduces two reference spinors q_M and $q_{M'}$ to account for the gauge invariance of the theory. The brackets in (41) are defined as:

$$\begin{aligned} \langle \lambda\, \mu\rangle &:= \lambda^A \mu_A, \\ [\lambda\, \mu] &:= \lambda_{A'} \mu^{A'}. \end{aligned} \tag{42}$$

The helicity spinors in (40) scale as:

$$\epsilon^-_{MM'} \longrightarrow \lambda^{-2}\epsilon^-_{MM'} \qquad \epsilon^+_{MM'} \longrightarrow \lambda^2 \epsilon^+_{MM'}, \tag{43}$$

and they are homogeneous of degree zero in q. Moreover, they satisfy the normalization:

$$(\epsilon^-)^M{}_{M'} (\epsilon^+)_M{}^{M'} = 1. \tag{44}$$

The amplitudes can then be computed according to the usual LSZ formula where the helicity spinors are inserted into the amputated Feynman diagrams as external legs.

2 This is due to the fact that the square of a 4-vector is translated in spinor form into the determinant of a 2x2 matrix. It is a simple algebraic fact that a matrix with vanishing determinant can be written as the product of two vectors.

2.5 GAUGE FIXING IN THE NEVEU-GERVAIS AND FEYNMAN GAUGE

The deformations do not alter the structure of the YM sector, so the gauge fixing will be inherited directly from the mother theory. The only difference will be to adapt this gauge fixing to the self-dual formulation. We will employ two different gauge fixings. As it is customary for studies in amplitudes and recursion relations we will use a Gervais-Neveu gauge. For the renormalization part we will use the usual Feynman gauge.

The difference between the two gauge choice is that while the Feynman gauge alters the free part of the Lagrangian only, the Gervais-Neveu gauge also modifies the interactions allowing for a simpler form of the Feynman rules that displays the color and tensor structure of the amplitude separately.

To start we explain how to fix the invariance in the Feynman gauge. We first absorb the YM coupling constant into the connection. In this way the free Lagrangian, the only part that will be affected by this procedure, looks like:

$$\mathcal{L}^{(2)} = -\frac{1}{2}(\partial_{MM'} A^a_N{}^{M'} + \partial_{NM'} A^a_M{}^{M'})\partial^M{}_{N'} A^{aNN'}. \tag{45}$$

we use the so called Shouten identity in the form:

$$\varepsilon_{AB}\varepsilon_{CD} = \varepsilon_{AC}\varepsilon_{BD} - \varepsilon_{AD}\varepsilon_{BC}. \tag{46}$$

It is clear that the antisymmetry in the last formula is respected and one can check that taking the trace leads to an identity. Let us employ this identity on the second term in the brackets in eq. (45):

$$\begin{aligned} \partial_{NM'} A^a_M{}^{M'} &= \partial_{AM'} A^a_B{}^{M'} \varepsilon_N{}^A \varepsilon_M{}^B \\ &= \partial_{AM'} A^a_B{}^{M'} \left(\varepsilon^{AB}\varepsilon_{NM} - \varepsilon_N{}^B \varepsilon^A{}_M\right) \\ &= \partial_{MM'} A^a_N{}^{M'} + \epsilon_{MN} \partial^E{}_{M'} A^a_E{}^{M'}. \end{aligned} \tag{47}$$

The Lagrangian becomes:

$$\mathcal{L}^{(2)} = -\partial_{MM'} A^a_N{}^{M'} \partial^M{}_{N'} A^{aNN'} + \frac{1}{2}(\partial^M{}_{M'} A^a_M{}^{M'})^2. \tag{48}$$

Now, we can perform an integration by part in the first term and disregard a total derivative. Given that the partial derivatives commute, the following contraction is necessarily antisymmetric, i.e. it is proportional to the identity ε:

$$\partial_{MM'}\partial^{M}{}_{N'} = \tfrac{1}{2}\varepsilon_{M'N'}\partial_{ME'}\partial^{ME'} = \tfrac{1}{2}\varepsilon_{N'M'}\partial^{\mu}\partial_{\mu}. \tag{49}$$

After another integration by parts we arrive at the usual (free) YM kinetic term:

$$\mathcal{L}^{(2)} = -\frac{1}{2}(\partial_{\mu}A^{a}_{\nu})^2 + \frac{1}{2}(\partial^{\mu}A^{a}_{\mu})^2. \tag{50}$$

It is a common procedure than to just add a gauge fixing tern to get rid of the second term. Then, the kinetic part of the Lagrangian is invertible and can be used to find the propagator. We also need to add a ghost term, namely:

$$\mathcal{L}_{\text{ghost}} = \bar{c}^{a}\partial^{\mu}d_{\mu}c^{a}, \tag{51}$$

where as usual the c^a, $\bar{c}^a$ fields are the ghost and anti-ghost.

To achieve a gauge fixing that it is more suitable for amplitude calculations let us mimic the previous procedure in a slightly different setting. We explicitly use the generators of the internal Lie algebra, T^a such that:

$$\text{Tr}(T^{a}T^{b}) = \delta^{ab}, \qquad [T^{a}, T^{b}] = \mathrm{i}\sqrt{2}T^{c}, \tag{52}$$

and define the matrix: $A_{MM'} = T^{a}A^{a}_{MM'}$. The self-dual part of the matrix-valued curvature is then:

$$F_{MN} = 2\partial_{(MM'}A_{N)}{}^{M'} + \frac{g}{\mathrm{i}\sqrt{2}}(A_{MM'}A_{N}{}^{M'} + A_{NM'}A_{M}{}^{M'}), \tag{53}$$

and the chiral Yang-Mills Lagrangian becomes:

$$\mathcal{L}_{\text{YM}} = -\frac{1}{4}\text{Tr}(F^{MN}F_{MN}). \tag{54}$$

now we define the matrix:

$$H = \partial^{\mu}A_{\mu} + \frac{g}{\mathrm{i}\sqrt{2}}A^{\mu}A_{\mu}. \tag{55}$$

It is not hard to see using the Shouten identity (46) and a series of manipulations similar in spirit to (47), that one can put the curvature matrix (53) in the form:

$$F_{MN} = 2\left(\partial_{MM'}A_N{}^{M'} + \frac{g}{i\sqrt{2}}A_{MM'}A_N{}^{M'}\right) + \epsilon_{MN}H. \tag{56}$$

The Lagrangian is then the trace over the Lie algebra generators of the square of the last formula:

$$\mathcal{L}_{\rm YM} = -{\rm Tr}\left(\partial_{MM'}A_N{}^{M'} + \frac{g}{i\sqrt{2}}A_{MM'}A_N{}^{M'}\right)^2 + \frac{1}{2}{\rm Tr}(H)^2. \tag{57}$$

The Gervais-Neveu gauge fixing term is then added to remove the H^2 term. It is simply $\mathcal{L}_{\rm GN} = -\frac{1}{2}{\rm Tr}(H)^2$. The free part now looks much like in the Feynman gauge.

$$\mathcal{L}_{\rm YM} + \mathcal{L}_{\rm GN} = -\frac{1}{2}{\rm Tr}(\partial^\mu A^\nu \partial_\mu A_\nu) + \mathcal{L}^{(3)}_{\rm YM} + \mathcal{L}^{(4)}_{\rm YM}, \tag{58}$$

with

$$\begin{aligned}\mathcal{L}^{(3)}_{\rm YM} &= ig\sqrt{2}\,{\rm Tr}(\partial^M{}_{M'}A^{NM'}A_{MN'}A_N{}^{N'}),\\ \mathcal{L}^{(4)}_{\rm YM} &= \frac{g^2}{2}{\rm Tr}(A^M{}_{M'}A^{NM'}A_{MN'}A_N{}^{N'}).\end{aligned} \tag{59}$$

One has to realize that in this formulation the colour structure is carried by the gauge generators T^a.

Let us write also the vertex from the minimal deformation (17) as a trace over Lie algebra generators:

$$\mathcal{L}_{\rm new} = -i\frac{\alpha g}{3\sqrt{2}M^2}{\rm Tr}(F_A{}^B F_B{}^C F_C{}^A). \tag{60}$$

Notice that the general structure of the vertex at higher order follows a similar structure. The curvature can be seen a $2Nx2N$ matrix: $F_{iA}{}^{jB}$. At fourth order in the curvature then also double trace terms will appear alongside the usual single trace:

$$({\rm Tr}F^2)^2 \qquad ({\rm Tr}F^4). \tag{61}$$

2.6 FEYNMAN RULES

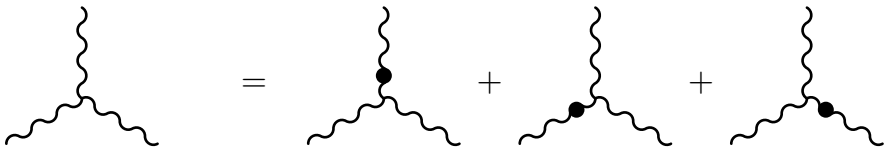
We are now ready to state the Feynman rules for the two previous gauge choices. Let us start with the Feynman gauge rules.

Propagator:

$$= \quad \frac{1}{\mathrm{i}k^2}(-\varepsilon_{MN}\varepsilon_{M'N'}). \tag{62}$$

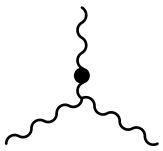
Cubic vertex:

We found it convenient, especially in view of a very important identity to be spelled out later, to split the full cubic vertex into three contributions:



(63)

where



$$= \quad gf^{abc}(kA(k))^{aAB}A^{b}_{AB'}(p)A^{c\,B'}_{B}(q), \tag{64}$$

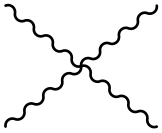
where

$$(kA(k))^{aAB} := 2k^{(A}_{A'}A^{aB)A'}(k). \tag{65}$$

All momenta are incoming. In the diagrams, a dot will correspond to an instance of derivative (and so a factor of the corresponding momentum) acting on a leg of the vertex following the rule (65). We need to clarify the meaning of eq. (64) a little further. The $A(k)$ act as placeholders. They will be replaced by helicity spinors following the rules of LSZ formula, or paired with another placeholder in a Wick contraction.

Quartic vertex:

This is the usual quartic vertex of YM:

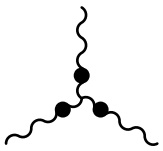


$$= -2ig^2 f^{abc} f^{cde} A^a_{(MM'}(k) A^b_{N)}{}^{M'}(l) A^{d(M}{}_{N'}(a) A^{eN)N'}(p)$$

$$+2 \text{ terms.} \tag{66}$$

New cubic and quartic vertices:

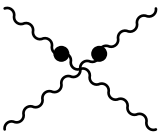
From (17) one can deduce the rules for the new vertices:



$$= \frac{\alpha g}{M^2} f^{abc} (kA(k))^a_A{}^B (pA(p))^b_B{}^C (qA(q))^c_C{}^A. \tag{67}$$

$$\tag{68}$$

There are instead 6 terms for the quartic new vertex. Two derivatives now act on 2 of the 4 legs. The quartic vertex rule then reads:



$$= -2i\frac{\alpha g^2}{M^2} f^{abc} f^{cde} (kA(k))^a_A{}^B (lA(l))^b_B{}^C A^d_{CM'}(q) A^{eAM'}(p)$$

$$+5 \text{ terms} \tag{69}$$

This are all the vertices that we need in this gauge. The term cubic in the curvature in eq. (17) produces higher order interaction but we will not need them.

We can now move on to the Gervais-Neveu gauge rules. These rules will be colour-ordered (or colour stripped), meaning that we fix the order of the labeling of the incoming lines to the vertex. The color-ordered amplitude will be stripped of the colour structure (the traces of matrices T^a). These amplitudes have the peculiar feature of being independently gauge invariant.

Cubic vertices:

In the following we will only need cubic vertices in this gauge to compute the seeds for the recursion relations. The higher order amplitudes will be found by lower order ones and not directly from the Feynman rules. The old colour ordered cubic vertex reads:

$$-\mathrm{i}g\sqrt{2}\,k^{M}_{M'}A^{NM'}(k)A_{MN'}(p)A_{N}{}^{N'}(q)+2 \text{ terms}, \tag{70}$$

the new one instead is given by:

$$-\mathrm{i}\frac{\alpha g}{\sqrt{2}M^2}(kA(k))_{A}{}^{B}(pA(p))_{B}{}^{C}(qA(q))_{C}{}^{A}, \tag{71}$$

where $2k^{M}{}_{M'}A^{NM'}(k) = (kA(k))^{MN}$. The propagator is the same as in the Feynman gauge case.

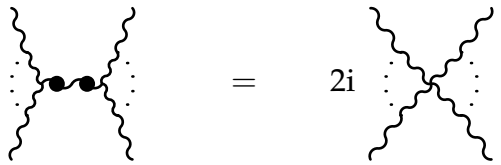
2.7 THE COLLAPSING DOTS IDENTITY

The dot notation seen in the previous section not only can be used as a simple computational tool. It is also of key importance because it clearly shows an important identity valid in these theories. Indeed, whenever two dots are on the same internal line, the entire line collapses into a (higher order) vertex. An internal line is formed when we connect two legs of two different vertices. This is a pictorial representation of a Wick contraction. In particular if the two legs that are Wick contracted both carry a momentum factor (which in turn comes from a derivative of the connection in the corresponding vertex operator) the following happens:

$$\begin{aligned}\left\langle\, (kA^{a})_{AB}\,(kA^{b})_{CD}\,\right\rangle &= \left\langle\, 2k_{(AA'}A^{a}_{B)}{}^{A'}\,2k_{(CC'}A^{b}_{D)}{}^{C'}\,\right\rangle \\ &= -4\delta^{ab}k_{(AA'}\frac{1}{\mathrm{i}k^2}\varepsilon^{A'C'}\varepsilon_{B)(D}k_{C)C'} \\ &= \frac{2}{\mathrm{i}}\delta^{ab}\varepsilon_{A(C}\varepsilon_{|B|D)}.\end{aligned} \tag{72}$$

We have substituted each Wick contraction (contraction of the placeholders in this notation) with a propagator (62). Then, contracting the primed indices and expanding the symmetrization (4 terms) one notices that for each

pair of momenta one has only the anti-symmetric part which is proportional to ε, the constant of proportionality being $-\frac{1}{2}k^2$. Ultimately this is the reason for the cancellation of the momentum in the denominator. What is left is an effective new higher order vertex. Schematically this looks like:



(73)

The overall plus sign comes from the fact that the two internal momenta factors are one incoming an the other outgoing. The other small dots indicate that the rest of the graph can be anything. But in the end all the amplitude will have at most one momentum for each internal line. It has to be stressed here that this identity do not hold for non-chiral theories, e.g. if we add in the vertices the missing anti-selfdual powers of the curvature. Another point of interest is that if we write the usual YM theory in its self-dual notation (this should not be confused with the so called self-dual YM which is a theory whose space of solution is composed of the sole instanton sector), this identity appears as well. It is then this identity that is crucial to show that the quartic vertex amplitudes cancel exactly with the cubic ones in which two dots lie on the internal lines. It is this very feature that tells us that YM can be actually recast as a theory containing only cubic vertices.

This identity, in particular in connection with the dot notation has been used throughout the paper [15] and has been proven to be of great help. Nevertheless, the identity has appeared in previous works by other authors ([25, 26]).

2.8 RECURSION RELATIONS AND AMPLITUDES

It is now well established that YM theories are constructible in the sense that starting with some basic amplitudes one can build all tree level amplitudes recursively. In principle there is no need for a specific Lagrangian to be defined. The seeds for the recursion relations will be the simple vertex

amplitudes, i.e. on-shell evaluation of the vertex operators. The special momenta kinematics tells us that these amplitude are actually zero, on-shell. To obtain non-zero results we analytically continue all the momenta. Having already computed all the vertices we only need a few relations for the contraction of the momenta with the positive and negative helicity spinors (41):

$$\begin{aligned}(k\epsilon^-)^{MN} &= 2k^{(M}{}_{M'}(\varepsilon^-)^{N)M'} = 0,\\ (k\epsilon^+)^{MN} &= 2k^{(M}{}_{M'}(\varepsilon^+)^{N)M'} = -2\mathrm{i}\,k^M k^N.\end{aligned} \tag{74}$$

The easiest vertex to compute is actually the new $3-$valent one, because it only has one term. There are three instances of momenta, one for each leg. It is then clear that if we insert any negative helicity , the amplitude will be zero. So, the only amplitude that has hope to be non-vanishing is the all-plus one:

$$\begin{aligned}\mathcal{A}^{+++}_{\text{new}} &= -\mathrm{i}\frac{\alpha g}{\sqrt{2}M^2}(1\varepsilon^+)_A{}^B(2\varepsilon^+)_B{}^C(3\varepsilon^+)_C{}^A\\ &= \frac{\alpha g}{\sqrt{2}M^2}1_A\,1^B\,2_B\,2^C\,3_C\,3^A\\ &= \frac{4\alpha g\sqrt{2}}{M^2}\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,1\rangle,\end{aligned} \tag{75}$$

where we labeled the three incoming momenta with numbers. This result is not new of course. The YM amplitudes are just a little more complicated to evaluate but the calculations are straightforward. There, the all-plus vertex amplitude is zero as well because all three parts of the amplitude will be proportional to $[q\,q] = 0$. The all-minus is zero because at least one negative helicity would be contracted with a momentum, which is zero by means of the first of eqs. (41).

Amplitudes for theories with higher powers of the curvature have already been studied. Here though, we are using only the self-dual part of those interactions and it is already apparent that these theories will be chiral, i.e.

the parity related amplitude $\mathcal{A}^{---}$ is not present. For completeness we compute the other YM amplitudes. The one plus, two minus amplitude is:

$$\mathcal{A}^{+--}_{\rm YM} = g\sqrt{2}\frac{\langle 1\,q\rangle^2}{\langle 2\,q\rangle\langle 3\,q\rangle}[2\,3] = g\sqrt{2}\frac{[2\,3]^3}{[1\,2][3\,1]}. \tag{76}$$

In the formula we used the momentum conservation $11' = 22' + 33'$ and contract with q, $2'$ and $3'$ repeatedly, to get the relations:

$$\begin{aligned}\frac{\langle 1\,q\rangle}{\langle 2\,q\rangle} &= -\frac{[2\,3]}{[3\,1]}\\ \frac{\langle 1\,q\rangle}{\langle 3\,q\rangle} &= -\frac{[2\,3]}{[1\,2]},\end{aligned} \tag{77}$$

to obtain the final result. The same can be done for the opposite helicity amplitude:

$$\begin{aligned}\mathcal{A}^{-++}_{\rm YM} &= -g\sqrt{2}\langle 2\,3\rangle\left(\frac{[1\,q]}{[2\,q]}\frac{\langle 3\,q\rangle}{\langle 1\,q\rangle} + \frac{[1\,q]}{[3\,q]}\frac{\langle 2\,q\rangle}{\langle 1\,q\rangle}\right)\\ &= -g\sqrt{2}\frac{\langle 2\,3\rangle[1\,q]}{\langle 1\,q\rangle[2\,q][3\,q]}(\langle 3\,q\rangle[3\,q] + \langle 2\,q\rangle[2\,q]).\end{aligned} \tag{78}$$

Again using momentum conservation one arrives at the final result:

$$\mathcal{A}^{-++}_{\rm YM} = g\sqrt{2}\frac{\langle 2\,3\rangle^3}{\langle 1\,2\rangle\langle 3\,1\rangle}, \tag{79}$$

which as expected is the complex conjugate of (76). We will use these amplitudes as seeds for the recursion relations.

2.8.1 *All negative and all-but-one negative amplitudes*

Before we consider the recursion relations we want to give a standard YM argument for the vanishing of the amplitudes which carry all negative and all-but-one negative amplitudes and see the identity (73) at work. If we take the all-negative amplitude with n external lines we can choose to set all the q spinors to be the same. There will be n instances of q in the numerator of the amplitude. Any time they contract with each other, the amplitude gives zero. So, the only chance we have to get a non-vanishing result is

to contract these spinors in the numerator with momenta coming from vertices with derivatives. Anyway, given the dot property (41) we can have at most one derivative per internal line. So let us take the case of trivalent amplitudes, i.e. diagrams containing only cubic vertices, including the new vertex with three derivatives. It is easy to check that for n external lines there are $n-2$ vertices and $n-3$ internal lines. No derivatives are on the external lines so the total number of momenta in the numerator is (due to the collapsing property) $n-3$ which is not sufficient to contract all the q's. These amplitudes are zero.

For the all-but one negative amplitudes instead we choose all the reference spinors to be the same as the only positive external momentum. The difference here is that we have $n-1$ unprimed reference spinors to contract with the numerator's momenta. In this case the number of derivatives after we used the collapsing property is at most $n-2$, given that one derivative can now be on the external positive line. Still it is not sufficient to contract all the reference spinors. Also these amplitudes are zero. For YM the collapsing property is not necessary of course to show the vanishing of the previous amplitudes. But it seems crucial in the case of chiral modifications.

2.8.2 *Recursion relations*

In the BCFW framework ([17], [18], one shifts a pair of momenta, introducing a complex parameter z. Under this shift one would like to find the large z behavior of the shifted amplitude. If certain conditions are met, all amplitudes can be reconstructed recursively.

The general idea behind the recursion relations is not complicated. The on-shell n-amplitude $\mathcal{A}_n$ is a function of the external on-shell momenta p_i with $i = 1 \dots n$, and $\Sigma_i p_i = 0$. One than takes complex shifts (for the

moment we can shift as many momenta as we want) of p_i by vectors r_i. We will see specific examples of these shifts later:

$$\hat{p}_i = p_i + z r_i \qquad z \in \mathbb{C}. \tag{80}$$

These new vectors follow a conservation law $\Sigma_i r_i = 0$, and they are orthogonal to each other and with p_i, for each i. These requirements are there to make the shifted momenta follow the same conservation law and on-shell condition of the unshifted ones. Moreover, if we take a sum over a subset I of momenta $P_I = \Sigma_{i\in I} p_i$ and shift it, then:

$$\hat{P}_I^2 = -\frac{P_I^2}{z_I}(z - z_I) \qquad z_I = \frac{P_I^2}{2P_i R_i}, \tag{81}$$

where we have defined $R_i = \Sigma_I r_i$. Now one considers the shifted amplitude $A(z)$ as a complex function. The original one is just $\mathcal{A}_n(z)$ evaluated at the origin. From the diagrammatic point of view a general shifted amplitude will have poles in z only at the propagators of the internal lines. At tree level these will be single poles of the form (81) and no pole will be sitting at the origin. The analytic structure of the amplitude is then encoded in the residues at these poles:

$$\mathcal{A}_n = -\sum_{z_i} \mathrm{Res}_{z\to z_I} \frac{\mathcal{A}_n(z)}{z} + B_n. \tag{82}$$

The boundary term B_n is the part of the amplitude that does not fall off for large z. We needs to prove that these terms are actually not present if we want to reconstruct the amplitudes from recursion relations. What the evaluation of the residues actually achieves is to put the internal line corresponding to the pole on-shell:

$$\mathcal{A}_n = \sum_{\text{diagrams}} \hat{A}_L(z_I)\frac{1}{P_I^2}\hat{A}_R(z_I). \tag{83}$$

The amplitude then factorizes into lower order on-shell amplitudes connected by an on-shell propagator. The sum has to be performed on all possible incoming particles states. It is clear that provided $B_n = 0$ for all n the tree level amplitude is recursively found by declaring the value of

some lower point amplitude (in YM, for example the 3-point amplitude that serves as seed for the recursion is completely determined by the little group scaling of the momenta spinors (40), so that no Lagrangian is actually needed).

We can now apply this framework to the theories at hand. Let us take a look first at general amplitudes constructed from the vertices that we have already encountered. In particular, we focus on amplitudes with at least one negative and one positive helicity gluon. We apply the following BCFW shift to those lines:

$$\begin{aligned} 1 &\to \hat{1} = 1 + z\, n, \\ n' &\to \hat{n}' = n' - z1'. \end{aligned} \tag{84}$$

From (41) we see that the helicity spinors behave as $\frac{1}{z}$ under this shift for large complex parameter z. As before the worst case scenario is the 3-valent amplitude. Each internal propagator that connects 1 with n contribute as $\frac{1}{z}$. There can be at most on derivative per each internal line due to the collapsing property each of which contributes as z. There can be no derivative on 1 because it is the negative helicity gluon and as we have seen the contraction $k\varepsilon_k^- = 0$. So we are left only with possible contraction on the helicity plus gluon with momentum n. All in all the large z behavior is, in the worst case, $\frac{1}{z}$. This guarantees the validity of the BCFW recursion relations for these amplitudes. There are some amplitudes missing, namely the all plus helicity ones. As we shall see the simple shift in (84) does not work in that case. A more involved shift is needed.

2.8.3 *MHV amplitudes*

The first non-vanishing amplitudes are the so called Maximally Helicity Violating. Historically, the name comes from the fact that, because of cross symmetry, the amplitude $\mathcal{A}(1^+2^+\dots n^+)$ with all outgoing particles corresponds to $1^-2^- \to 3^+\dots n^+$ with $1,2$ incoming. This is the maximum helic-

ity violation possible in a scattering process, but it is zero in YM. Also the next one with 2 negative helicity particles going into one negative and the rest positive is zero. So the first Maximally Helicity Violating process corresponds to the amplitude $\mathcal{A}(1^-2^-3^+\dots n^+)$. They have exactly two positive (or only two negative) helicity gluon. The BCFW relations become quite simple in this case:

$$\begin{aligned}&\mathcal{A}(1^-,2^-,3^-\dots,i^+,\dots,n-1^-,n^+)\\ =\ &\mathcal{A}(\hat{1}^-,2^-,\dots,i^+,\dots,\hat{P}^+)\frac{1}{P^2}\mathcal{A}(-\hat{P}^-,n-1^-,\hat{n}^+)\\ &+\mathcal{A}(1^-,2^-,\hat{P}^+)\frac{1}{P^2}\mathcal{A}(-\hat{P}^-,3^-,\dots,i^+,\dots,\hat{n}^+).\end{aligned} \tag{85}$$

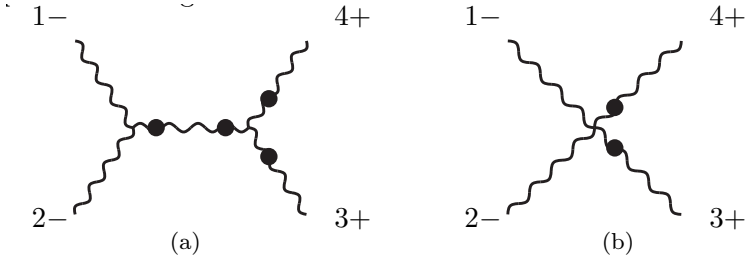
The first term vanishes on the pole (due to some special kinematics). The second term gives the usual YM result, also called Parke-Taylor formula [27]:

$$\mathcal{A}(1^-2^-3^+\dots n^+)=\frac{[1\,2]^4}{[1\,2][2\,3][3\,4][4\,5]\dots[n\,1]}. \tag{86}$$

It is instructive to compare two examples. The first is the usual 4 points MHV amplitude. The second is the $(--++)$ amplitude, the first modified by the new vertices.

$(--++)$ amplitude:

The following two new diagrams may contribute to this amplitude and so modify the MHV amplitude of YM:



but they precisely cancel each other. Of course the collapsing dots rule is at play. As already stated, the MHV amplitude is given by the usual YM Parke-Taylor formula. Let us do this simple exercise.

$$\mathcal{A}(1^-,2^-,3^+,4^+)=\mathcal{A}(\hat{1}^-,2^-,\hat{P}^+)\frac{-i}{P^2}\mathcal{A}(-\hat{P}^-,3^+,\hat{4}^+), \tag{87}$$

where we shifted $1, 4$. Notice that $P^2 = 2\langle 1\,2\rangle[1\,2]$. Now we just use the formulae for the vertices (76) and (79):

$$\mathcal{A}(1^-, 2^-, 3^+, 4^+) = -\mathrm{i}g^2 \frac{[1\,2]^3\langle 3\,4\rangle^3}{[\mathrm{P}\,1][2\,\mathrm{P}]\langle 1\,2\rangle[1\,2]\langle \mathrm{P}\,3\rangle\langle 4\,\mathrm{P}\rangle}. \tag{88}$$

with a little trick we can simplify this formula:

$$\begin{aligned}
[P\,1]\langle P\,3\rangle = -P_{A'}1^{A'}P_A 3^A = -1^{A'}3^A(3_A 3_{A'} + 4_A 4_{A'}) \\
= [1\,4]\langle 3\,4\rangle \\
[2\,P]\langle 4\,P\rangle = -2^{A'}4^A P_A P_{A'} = -2^{A'}4^A(1_A 1_{A'} + 2_A 2_{A'}) \\
= [1\,2]\langle 1\,4\rangle.
\end{aligned} \tag{89}$$

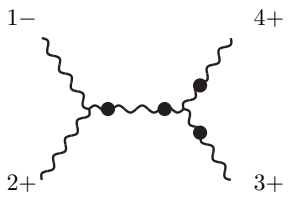
So the amplitude becomes:

$$\begin{aligned}
\mathcal{A}(1^-, 2^-, 3^+, 4^+) &= (-\mathrm{i})g^2 \frac{[1\,2]\langle 3\,4\rangle^2}{\langle 1\,2\rangle\langle 1\,4\rangle[1\,4]} \\
&= (-\mathrm{i})g^2 \frac{[1\,2]^4}{[1\,2][2\,3][3\,4][4\,1]},
\end{aligned} \tag{90}$$

which is the usual Parke-Taylor formula for this amplitude. In the last line we repeatedly used formulae extracted from the momentum conservation $11' + 22' = 33' + 44'$.

(-+++) amplitude:

As a second example we compute the first amplitude affected by the new vertex. It might look like the following diagram:



could contribute to the BCFW boundary. Indeed, a superficial count would lead to a $O(1)$ behavior for large z, due to the two dots on the internal line. However, this diagram actually behaves as $1/z$ because of the dots

collapsing rule. The amplitude can be constructed through BCFW shift and recursion and it is then given by:

$$\mathcal{A}(1^-,2^+,3^+,4^+) = \mathcal{A}(\hat{1}^-,2^+,\hat{P}^+)\frac{-i}{P^2}\mathcal{A}(-\hat{P}^-,3^+,\hat{4}^+) \\ +\mathcal{A}(\hat{1}^-,2^+,\hat{P}^-)\frac{-i}{P^2}\mathcal{A}(-\hat{P}^+,3^+,\hat{4}^+). \tag{91}$$

The first term is not contributing because $\mathcal{A}(\hat{1}^-,2^+,\hat{P}^+)$ vanishes at the pole, whereas the second term gives

$$\mathcal{A}(\hat{1}^-,2^+,\hat{P}^-)\frac{-i}{P^2}\mathcal{A}(-\hat{P}^+,3^+,\hat{4}^+) = \\ \sqrt{2}g\frac{\left[\hat{P}1\right]^3}{\left[12\right]\left[2\hat{P}\right]}\frac{-i}{P^2}2^2\sqrt{2}\frac{\alpha g}{M^2}\langle\hat{P}3\rangle\langle34\rangle\langle4\hat{P}\rangle. \tag{92}$$

We would not have this diagram in YM because there, the all plus 3-point sub-amplitude is vanishing. One can find the three useful relations:

$$\begin{aligned}[P\,1]\langle P\,3\rangle &= -[1\,2]\langle2\,3\rangle \\ [P\,1]\langle4\,P\rangle &= [1\,2]\langle2\,4\rangle \\ [2\,P]\langle4\,P\rangle &= [1\,2]\langle1\,4\rangle,\end{aligned} \tag{93}$$

to simplify the amplitude. Those formulae can be found using the trick in (89) or just evaluating $\hat{P}$ on shell. Indeed, we find the complex parameter z for which this happens, i.e. $z = \frac{\langle1\,2\rangle}{\langle2\,4\rangle}$. Then,

$$\hat{P} = \left(1_A + \frac{\langle1\,2\rangle}{\langle2\,4\rangle}4_A\right)1_{A'} + 2_A 2_{A'}. \tag{94}$$

The quantity in brackets is proportional to the unprimed spinor 2_A:

$$\left(1_A + \frac{\langle1\,2\rangle}{\langle2\,4\rangle}4_A\right) = \frac{\langle4\,1\rangle}{\langle4\,2\rangle}2_A. \tag{95}$$

The internal momentum, on-shell is then:

$$\hat{P} = 2_A\left(\frac{\langle4\,1\rangle}{\langle4\,2\rangle}1'_A + 2'_A\right). \tag{96}$$

One can now compute the brackets where $\hat{P}$ appears. The final result is then:

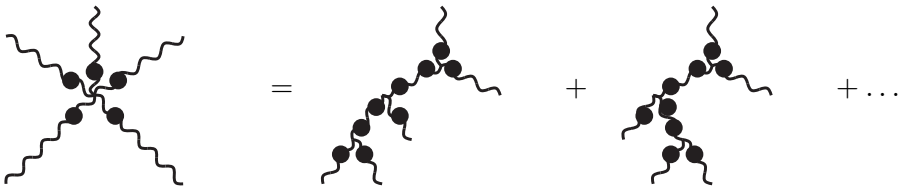
$$\mathcal{A}(1^-,2^+,3^+,4^+) = i\frac{4\alpha g^2}{M^2}\frac{\langle23\rangle\langle34\rangle}{\langle12\rangle\langle14\rangle}\langle24\rangle^2. \tag{97}$$

2.8.4 *The all plus amplitude*

The all-plus amplitudes would not fall off under the BCFW shift. Indeed, this shift (84), applied to a pair of positive helicity gluons would make the 3-point amplitude behave like $O(z)$ for large z. Another shift devised by Risager in [28] could then be used. It prescribes to shift all primed momentum spinor in the direction of the reference spinor, in such a way that the momentum conservation is preserved:

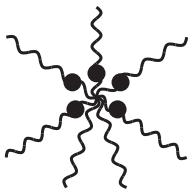
$$i' \to \hat{i}' = i' + z f_i q', \qquad \sum_i i f_i = 0. \tag{98}$$

Even in this case not all diagrams are reconstructible. There is a part of the amplitude, the BCFW boundary term, which does not have the required fall-off behavior. In general this would be a problem, but in this case we can isolate this part. If we start with just the new vertex with coupling constant α, the only diagrams that do not follow the $\frac{1}{z}$ fall-off are the ones made of new cubic vertices only. In view of the collapsing rule the sum of these amplitudes gives an amplitude proportional to a new vertex, which would come from one of the higher order operators in our class of theories:



Those terms should be included (with all their infinite new coupling constants) anyway as additional data. All in all the rest of the amplitude is completely determined by recursion. To see this, we use the collapsing rule until no internal line has more that one dot on it. The diagram would be composed of (possibly) effective higher order vertices with some lines emanating from them with no dots (an even number). We, of course, disregard the higher valent vertices that we described before as they will be added later. The dangerous situation arise in the presence of a line with a single

dot. The line has two ends one with a dot and the other without. The end with no dot is contracted with a higher valence vertex as in the following diagram:



There will be necessarily at least another line without a dot going into the vertex, otherwise we could not be able to contract the primed index of our incoming line. We focus on the most dangerous diagrams where all lines have exactly one dot. In this case we follow our original line into the other line that emanates from the vertex (their primed indices are contracted). The second one will have a dot on the other side as well. So the situation is: we have a pair of lines with one dot each on one side connected by the other side to a vertex. Here, the primed index of the propagators are contracted. The numerator from the first line dot is contracted to the analogous one of the second line. This gives $[q\,q] = 0$. This situation is exactly the dangerous one with a behavior of z^2. So the Risager's shift can be used to compute recursively all the amplitudes, apart from the ones in which all the internal lines collapsed. As pointed out, before these are of the same form of the amplitude coming from evaluating the vertices with maximum number of derivatives on-shell.

Let us concretely apply the Risager's shift to a simple amplitude.

Example:

We take the 4-points all-plus amplitude. Let us consider the following generalisation of the Risager shift on all four external legs:

$$\begin{aligned}
\hat{k}_1 &= k_1 k_1' \quad +0 \quad +z k_1\langle 34\rangle\eta' \quad +z\,k_1\langle 24\rangle\eta' \quad +z\,k_1\langle 23\rangle\eta' \\
\hat{k}_2 &= k_2 k_2' \quad +z k_2\langle 34\rangle\eta' \quad +0 \quad +z k_2\langle 41\rangle\eta' \quad +z k_2\langle 31\rangle\eta' \\
\hat{k}_3 &= k_3 k_3' \quad +z k_3\langle 42\rangle\eta' \quad +z k_3\langle 41\rangle\eta' \quad +0 \quad z k_3\langle 12\rangle\eta' \\
\hat{k}_4 &= k_4 k_4' \quad +z k_4\langle 23\rangle\eta' \quad +z k_4\langle 13\rangle\eta' \quad +z k_4\langle 12\rangle\eta' \quad +0
\end{aligned} \tag{99}$$

Let us also choose all reference spinors to be identical to the same η' introduced to define the shiftings. Since only the primed spinor are shifted, the polarisations are unchanged.

The diagrams that go to zero for large z are the one where there is at most one new cubic vertex. The s-channel residue is given by:

$$\begin{aligned}
& A(\hat{1}^+, 2^+, \hat{P}^-)\frac{-i}{P^2}A(-\hat{P}^+, 3^+, \hat{4}^+) \\
&= \sqrt{2}g\frac{\langle 12\rangle^3}{\langle 2\hat{P}\rangle\langle\hat{P}1\rangle}\,\frac{-i}{P^2}\,2^2\sqrt{2}\frac{\alpha g}{M^2}\langle\hat{P}3\rangle\langle 34\rangle\langle 4\hat{P}\rangle \\
& A(\hat{1}^+, 2^+, \hat{P}^+)\frac{-i}{P^2}A(-\hat{P}^-, 3^+, \hat{4}^+) \\
&= 2^2\sqrt{2}\frac{\alpha g}{M^2}\langle\hat{P}1\rangle\langle 12\rangle\langle 2\hat{P}\rangle\,\frac{-i}{P^2}\,\sqrt{2}g\frac{\langle 34\rangle^3}{\langle 4\hat{P}\rangle\langle\hat{P}3\rangle}.
\end{aligned} \tag{100}$$

For brevity we denote the shiftings (99) taken on legs 1 and 2 as

$$\begin{aligned}
\hat{k}_1' &= k_1' + z\,c_1\eta, \\
\hat{k}_2' &= k_2' + z\,c_2\eta'.
\end{aligned} \tag{101}$$

The on-shell condition at s-channel pole for $\hat{P}$ requires to compute the residue at $z = \frac{[1\,2]}{c_1[2\,\eta]-c_2[1\,\eta]}$. $\hat{P}$ is then found to be:

$$\hat{P} = \frac{1}{c_1\left[2\eta\right] - c_2\left[1\eta\right]}\left(k_1\left[1\eta\right] + k_2\left[2\eta\right]\right)\left(c_1 k_2' - c_2 k_1'\right), \tag{102}$$

where we have used:

$$\begin{aligned}
[2\,\eta]1_{A'} + [1\,2]\eta_{A'} &= [1\,\eta]1_{A'}, \\
[1\,\eta]2_{A'} - [1\,2]\eta_{A'} &= [2\,\eta]2_{A'}.
\end{aligned} \tag{103}$$

Finally, we extract the unprimed part of $\hat{P}$, i.e. $\hat{P}_A = 1_A[1\eta] + 2_A[2\eta] = -3_A[3\eta] - 4_A[4\eta]$ to get:

$$(-\mathrm{i})\frac{4\alpha g^2}{M^2}\frac{\langle 1\,2\rangle}{[3\,4][1\,\eta][2\,\eta][3\,\eta][4\,\eta]}\left(c_{12}^2 + c_{34}^2\right), \tag{104}$$

where we introduced the following anti-symmetric quantities

$$c_{ij} := [\eta\, i]\langle i\, j\rangle[j\,\eta]. \tag{105}$$

Repeating the same calculation for the t-channel, we find the total amplitude

$$A_{singular}(1^+,2^+,3^+,4^+) = \\ (-i)\frac{4\alpha g^2}{M^2}\frac{1}{[1\,\eta][2\,\eta][3\,\eta][4\,\eta]}\left(\frac{\langle 1\,2\rangle}{[3\,4]}\left(c_{12}^2 + c_{34}^2\right) + \frac{\langle 1\,4\rangle}{[3\,2]}\left(c_{14}^2 + c_{32}^2\right)\right). \tag{106}$$

Using the momentum conservation this becomes

$$A_{singular}(1^+,2^+,3^+,4^+) = \\ (-\mathrm{i})\frac{4\alpha g^2}{M^2}\frac{\langle 1\,2\rangle}{[3\,4][1\,\eta][2\,\eta][3\,\eta][4\,\eta]}\left(c_{12}^2 + c_{34}^2 - c_{14}^2 - c_{23}^2\right). \tag{107}$$

The momentum conservation implies that we have the following identities satisfied by the c's

$$\sum_j c_{ij} = 0. \tag{108}$$

Because of this we can rewrite the quantity in brackets in (107) as

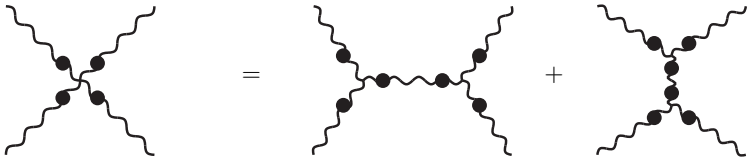
$$\begin{aligned}(c_{12} - c_{14})(c_{12} + c_{14}) + (c_{34} - c_{23})(c_{34} + c_{23}) \\ = (c_{12} - c_{14})(-c_{13}) + (-c_{31})(c_{34} + c_{23}) \\ = c_{13}(c_{34} + c_{23} - c_{12} + c_{14}) = 2c_{13}c_{42} = 2[1\eta]\,[2\eta]\,[3\eta]\,[4\eta]\,\langle 13\rangle\langle 42\rangle.\end{aligned} \tag{109}$$

Thus, the singular part of the 4-point all plus amplitude is finally given by

$$A_{singular}(1^+,2^+,3^+,4^+) = (-i)\frac{8\alpha g^2}{M^2}\frac{1}{[34]}\langle 13\rangle\langle 42\rangle\langle 12\rangle. \tag{110}$$

There are some diagrams that do not go to zero for large z. These are the BCFW boundary terms. They can be derived from a direct calculation of the

s and t-channel diagrams consisting of new cubic vertices only. In view of the dot collapsing rule their sum will contribute to an effective new vertex:



The boundary term is given by contracting these diagrams with the positive helicity polarisations. For the first one we get

$$\mathrm{i}\frac{16\alpha^2 g^2}{M^4}\langle 1\,2\rangle\langle 3\,4\rangle\left(\langle 1\,3\rangle\langle 2\,4\rangle+\langle 2\,3\rangle\langle 1\,4\rangle\right), \tag{111}$$

and the second one gives

$$\mathrm{i}\frac{16\alpha^2 g^2}{M^4}\langle 1\,4\rangle\langle 3\,2\rangle\left(\langle 1\,3\rangle\langle 4\,2\rangle+\langle 4\,3\rangle\langle 1\,2\rangle\right). \tag{112}$$

The sum is

$$A_{boundary}(1^+,2^+,3^+,4^+)=\mathrm{i}\,\frac{16\alpha^2 g^2}{M^4}\Big[\langle 1\,3\rangle^2\langle 2\,4\rangle^2-2\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,1\rangle\Big] \tag{113}$$

As we briefly discussed earlier in the context of the new vertices in the color ordered formalism, the F^4 operator will give rise to two contributions: the single trace and the double trace (see eq. (61)). The result (113) has two contributions that come exactly from the double and single trace, resp., evaluated on-shell. If we want to obtain all the DYM theories, we need to input these terms with a a new coupling constant at each order. These boundary terms then are just shifts in already present amplitudes proportional to arbitrary constants. We could also require the new interactions to have coupling constants such that the amplitudes they generate precisely cancel the boundary terms from lower order amplitudes. The theory becomes completely constructible and depends on just two coupling constant, namely g and α.

The constructible nature of the DYM theories was the first result of the paper [15]. In the next chapter, we will show a second important analysis concerning the one-loop renormalizability of these theories.

3

ONE-LOOP RENORMALIZATION OF DYM

We will continue here the discussion about the Deformations of YM theory presented in the last chapter. We will first perform a one-loop analysis of the easiest deformation (17). We will show a diagrammatic calculation of the arising divergences and counterterms needed to compute the beta function for the first coupling constant α. As in the case of the tree level recursion relations, the collapsing dot rule will play a fundamental role. Indeed, as it will be clear later, many diagrams written in the dotted formalism collapse into already present diagrams and cancel them. In the end the number of diagrams that we need to compute is just a small fraction of the total set. Later, we will also briefly show the idea behind a complete calculation of the one-loop beta function for the DYM performed in [20]. This calculation provides a proof of one-loop renormalizability for these theories giving hope to the claim that DYM are closed under renormalization. Also it allows to confirm the detailed calculation for the beta function of the coupling α. Finally it provides a general formalism that we are going to use for gravity in the last chapter of these thesis.

3.1 CHOICE OF THE RENORMALIZATION SCHEME

We are going to use the Feynman rules in the Feynman gauge obtained before (see eqs. (62) - (69)) Given that we want to compute only the beta function of the coupling α, we can perform this calculation having all the

external momenta on shell. The renormalization scheme that we are going to use is the usual Dimensional Regularization (Dim-Reg). This is the simplest scheme available and preserves gauge invariance. Nevertheless there is a little problem that we need to address.

In general, Dim-Reg works shifting the dimensionality of the manifold our fields are defined on. This is based on the fact that the loop integrals can be regularized in such a way that the arising divergences show themselves as poles in the dimension shift ε. As it is known this could create problems in theories with spinors if handled incorrectly. One cannot straightforwardly continue spinors to arbitrary dimension. In particular, the notion of self-duality our theories are based on cannot be extended to arbitrary dimension. We need to carefully check that when we do it the main relations between spinor quantities still hold. First, we will dimensionally continue only the loop momenta $k_{AA'}$. These are the ones we will integrate over. The momenta in four dimensions are related to the usual 4-momenta through:

$$k_{AA'} = -\theta^{\mu}_{AA'} k_{\mu} \tag{114}$$

where $\theta^{\mu}_{AA'}$ is the soldering form. Now, we could define k_{μ} in any dimension, but we need to continue the soldering forms as well. There is a known way to do it which is the same used in QED with fermions where one has to dimensionally continue the γ matrices (see the appendix in [16] for an account). There are some identities between the soldering forms that continue to hold in any dimension. In particular:

$$\theta^{(\mu A}{}_{E'} \theta^{\nu) BE'} = -\tfrac{1}{2} \eta^{\mu\nu} \varepsilon^{AB}. \tag{115}$$

In the last formula $\eta^{\mu\nu}$ is the d-dimensional metric but the Spinor indices continue to be 2-dimensional, in particular $\varepsilon_{AB}\varepsilon^{AB} = 2$. Now, if we contract the previous equation with $k_{\mu} k_{\nu}$ we arrive at the identity:

$$k^{A}{}_{E'} k^{BE'} = -\tfrac{1}{2} \varepsilon^{AB} k^2, \tag{116}$$

which is the crucial identity that we used in the proof of the dot identity (73). In what follows we will only use (116) and no other dimensionally

dependent identity. Moreover, any other object in the amplitude will be considered 4-dimensional, included the external helicity spinors. There is no need to continue these objects given that our Lagrangian is already defined in spinor form.

3.2 POWER COUNTING

Let us briefly discuss the power counting in the context of DYM. As it is well known in a general field theory, the loop diagrams in 4 dimensions have a superficial degree of divergence:

$$D = 4L - 2I + \sum_k k\, V_n^k, \tag{117}$$

where L is the number of loops, I is the number of internal lines and V_n^k is the number of n-valent vertices with k derivatives in them. If greater or equal to zero, D signals the presence of a potential divergence in the loop integrals. To simplify this formula one uses the topological structure of the diagrams to write:

$$L = I - \sum_k V_n^k + 1. \tag{118}$$

It is clear that combining the last two equations one arrives at Weinberg's formula:

$$D = 2L + 2 + \sum_k (k-2) V_n^k, \tag{119}$$

In YM this formula greatly simplifies due to the fact that this theory only carries two types of vertices, namely V_3^1 and V_4^0. With this knowledge one finds one more identity:

$$E + 2I = 3V_3^1 + 4V_4^0, \tag{120}$$

that leads to the simple formula for the superficial degree of divergence in YM theory:

$$D_{YM} = 4 - E. \tag{121}$$

This is the typical renormalizable theory pattern, in particular the degree of divergence do not depend on the loop order and it is actually positive or zero only up to the 4-point amplitude. We aim to modify this argument for the DYM case. A vertex of valency n, generally contains a number of derivatives that starts with n and decreases by 2. So one modifies the previous formulae accordingly:

$$\begin{aligned} D &= 4L - 2I + \sum_{n=3}^{\infty} \sum_{i=0}^{[n/2]} (n-2i) V_n^{n-2i}, \\ L &= I - \sum_{n=3}^{\infty} \sum_{i=0}^{[n/2]} V_n^{n-2i} + 1, \\ E + 2I &= \sum_{n=3}^{\infty} \sum_{i=0}^{[n/2]} n V_n^{n-2i}. \end{aligned} \tag{122}$$

Combining the first and last of these relations we obtain:

$$4L + E - D = \sum_{n=3}^{\infty} \sum_{i=0}^{[n/2]} (2i) V_n^{n-2i} \geq 0, \tag{123}$$

From the right hand side we see that the minimum in zero is reached for $i = 0$, i.e. for diagrams containing only vertices in which all the legs are saturated with derivatives. So we have:

$$D \leq 4L + E \tag{124}$$

This is actually an over-count because if all the legs contain a derivative then some of them will actually act on the external lines and will not contribute to the divergence. From (124) we see that any single theory of this type is badly divergent. They are even much more divergent than GR, for which $D \propto 2L$. Anyway the situation is improved slightly if one takes into account the collapsing dots rule. In this case there cannot be more that one derivative per internal line. If we denote by I_0 the lines with no derivatives, by I_1 the ones with one derivative and by E_1 the derivatives on the external lines, then:

$$D = 4L - I_1 - 2I_0 + E_1, \tag{125}$$

The largest possible degree of divergence at fixed loop order is given by diagrams in which $I_1 = I$ and $I_0 = 0$. From the second equation of (122) we see that one minimizes the number of internal lines by minimizing the number of vertices. The first non-trivial diagrams in Dim-Reg are the ones with at least 2 vertices:

$$I_{\min} = L + 1, \tag{126}$$

so that:

$$D_{\max} = 3L - 1 + E_1. \tag{127}$$

Some simple examples can easily show that:

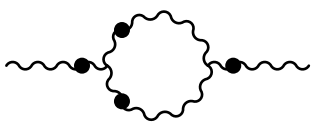
$$D_{\max} = \begin{cases} 3L - 1 + E, & L \text{ odd} \\ 3L - 2 + E, & L \text{ even} \end{cases} \tag{128}$$

The theories still behave worse than gravity, but a little bit better than in the previous analysis.

We can now proceed to the main calculation. We will add to the YM Lagrangian the first modification proportional to F^3_{sd}. There are two new vertices we are going to consider: one 3-valent with 3 derivatives and one 4-valent with 2 derivatives. We will not consider the diagrams built from YM vertices only. These diagrams will give the usual and well established results.

3.3 SELF-ENERGY DIAGRAMS

If we consider the renormalization of the 2-point function we need to disregard the diagrams with two dots on the internal lines. Those diagrams collapse to tadpoles and are vanishing in Dim-Reg. The only diagram left to analyze is then:



(129)

We will use this simple calculation to remind the general Dim-Reg steps and formulae in gauge theories but also to show a key property of the one-loop renormalization of DYM theories. Indeed, this diagram is quadratically divergent as it is apparent from the presence of two derivatives in the loop. This would be worrisome in general because it would require a counterterm of the form that is not present already in the Lagrangian. This is avoided by noting that this counterterm corresponds to a field redefinition (as we will see later).

For now we just apply the usual routine of Dim-Reg to the present diagram. We start by using the Feynman rules presented in the last chapter to build the skeleton of our amplitude.

$$\begin{aligned}\frac{\alpha g}{2M^2} f^{abc}(kA(k)))_A^{a\ B}(-lA(-l))_B^{b\ C}((l-k)A(l-k))_C^{c\ A} \\ g f^{def}(-kA(-k))^{dMN} A^e_{MM'}(-l+k)A_N^{f\ M'}(l).\end{aligned} \tag{130}$$

We remind the reader that we have used the placeholders A and that they will Wick-contract whenever they belong to the same line (i.e. loop momentum). This contraction will be replaced by an instance of propagator (62). We have also included a symmetry factor of $\frac{1}{2}$. After the contraction one gets:

$$\begin{aligned}\frac{\alpha g^2}{2M^2}(-C_2)(kA(k)))_A^{a\ B}(kA(-k))^{aMN} \\ 2l_{(BA'}\epsilon^{C)}{}_N \epsilon^{A'M'} 2(l-k)_{(CB'}\epsilon^{A)}{}_M \epsilon^{B'}{}_{M'} \frac{1}{\mathrm{i}l^2 \mathrm{i}(l-k)^2}\end{aligned} \tag{131}$$

Where C_2 is the quadratic Casimir of the gauge group that comes from the color contractions. It shoud be clear that we are for now understanding the integral over the loop momentum l. This integral can be dealt with passing to a Feynman parametrization:

$$\frac{1}{l^2(l-k)^2} = \int_0^1 dx \frac{1}{(l^2 - 2xlk + xk^2)^2}. \tag{132}$$

We then define the shift in the loop momentum:

$$\begin{aligned} l &= q + xk, \\ l-k &= q + (x-1)k. \end{aligned} \tag{133}$$

This shift is linear and does not affect the measure of the loop integration. The previous parametrization integral then becomes:

$$\frac{1}{l^2(l-k)^2} = \int_0^1 dx \frac{1}{(q^2 + x(1-x)k^2)^2}. \tag{134}$$

We will come back to this denominator later. Now we focus on the numerator of the diagram in (131).The primed indices of the loop momenta are contracted. We can then write the complex spinor contraction in the second line of (131) as:

$$\begin{aligned} I &= 2l_{(BA'}\epsilon^{C)}{}_N\epsilon^{A'M'}2(l-k)_{(CB'}\epsilon^{A)}{}_M\epsilon^{B'}{}_{M'} \\ &= -4l_{(BB'}\varepsilon^{C)}{}_N(l-k)_{(C}{}^{B'}\varepsilon^{A)}{}_M. \end{aligned} \tag{135}$$

As we have already discussed in section (3.1) we can make use of the formula (116) even if we are continuing the arbitrary dimension. The primed indices in the momenta of the last eq. are contracted and using (116) the unprimed ones will contract as well. One can then work out the spinor structure (products of ε) separately from the momenta. We then shift those momenta in the numerator as in (133). In Dim-Reg, only the pieces in the numerator quadratic in q and with no q at all will contribute. All in all, we get:

$$I = \frac{1}{2}(q^2 + x(x-1)k^2)(4\epsilon_M{}^A\epsilon_{BN} + \epsilon_B{}^A\epsilon_{MN}). \tag{136}$$

The second term contracts with a symmetric pair of indices in the external field $A(k)^{MN}$ and it cancels. We need to do a Wick rotation of the temporal part of the momenta $q^0 \to iq^0$ to compute the regularized part of the integrals. The overall diagram is then:

$$\int_0^1 dx \int \frac{d^dq}{(2\pi)^d} \frac{i\alpha g^2 C_2}{M^2} (kA(k)))^a_{AB} (kA(-k))^{aAB} \frac{(q^2-\Delta)}{(q^2+\Delta)^2}, \tag{137}$$

where $\Delta = x(1-x)k^2$. The usual formulae for Dim-Reg [29] apply:

$$\begin{aligned} \int \frac{d^dq}{(2\pi)^d} \frac{1}{(q^2+\Delta)^n} &= \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma\left(n-\frac{d}{2}\right)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{\left(n-\frac{d}{2}\right)} \\ \int \frac{d^dq}{(2\pi)^d} \frac{q^2}{(q^2+\Delta)^n} &= \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{d}{2} \frac{\Gamma\left(n-\frac{d}{2}-1\right)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{\left(n-\frac{d}{2}-1\right)}. \end{aligned} \tag{138}$$

We specialize the above equations to $n = 2$ and expand the gamma functions to get the behavior in the limit $d \to 4$, i.e. $\Gamma\left(2 - \frac{d}{2}\right) = \Gamma\left(\frac{\varepsilon}{2}\right) \to \frac{2}{\varepsilon} - \gamma + O(\varepsilon)$, where γ is the Euler-Mascheroni constant. The piece with q^2 in eq. (137) is (-2) times the one without q. We then obtain:

$$\int_0^1 dx \frac{2}{\varepsilon} \frac{i\alpha g^2 C_2}{(4\pi)^2 M^2} (kA(k)))^a_{AB} (kA(-k))^{aAB} (-3\Delta) \,. \tag{139}$$

the simple x-integration leads to the final result:

$$-\frac{1}{\epsilon} \frac{k^2}{M^2} \frac{i\alpha g^2 C_2}{(4\pi)^2} (kA(k)))^a_{AB} (kA(-k)))^{aAB}. \tag{140}$$

This divergence can be removed from the 2-point function by means of the field redefinition:

$$A \to A - \frac{1}{\epsilon} \frac{\alpha g^2 C_2}{(4\pi)^2} \frac{k^2}{M^2} A. \tag{141}$$

We know from (128) that at one loop, the maximal degree of divergence is $2 + E$ where E are the external legs. In the theories at hand all E-valent vertices with E derivatives in them are present. There is then hope to remove all the maximal degree divergences through the field redefinition (141). The remaining divergencies are logarithmic and they will contribute to the multiplicative renormalization of the coupling constants. In what follows we will find this multiplicative renormalization for the coupling α and its associated beta function.

3.4 $\beta-$FUNCTION FOR THE α COUPLING

In this section we will compute the diagrams that lead to the multiplicative renormalization of the α coupling. In general, our couplings have been separated into a mass term M which is just a bookkeeping device of the dimension of the corresponding operator and a dimensionless part. In Dim-Reg dimensionful couplings do not run. This is quite important because it tells us that the presence of a new operator (and a new coupling) will not affect the running of the YM coupling g. That is because the new vertex

carries a $\frac{1}{M^2}$ prefactor, so the divergence that the diagrams produce would require a counterterm of different mass dimension than the usual YM vertex. Given the previous result on self-energy diagrams, the new vertex do not change the multiplicative renormalization of the gauge field. We can than focus on the 3-valent diagrams only. We will evaluate the diagrams on-shell considering only all-plus polarizations.

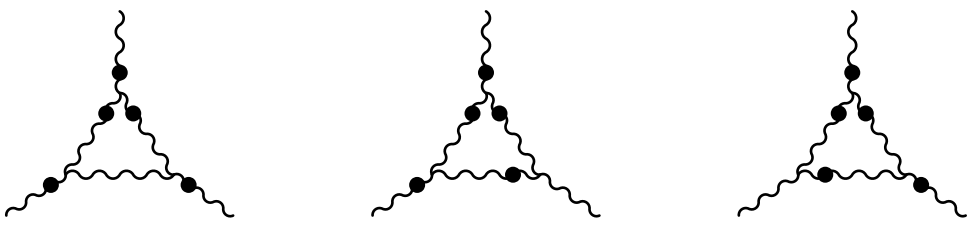
3.4.1 *Diagrammatics and cancellations*

We do not need to compute all the possible amplitudes divergencies. Indeed the collapsing dots rule and some superficial evaluation of the diagrams contractions allow us to select a much smaller set of remaining amplitudes. There will be only two types of diagrams: the ones in which we glue together three cubic vertices and the ones where we glue a cubic and a quartic vertex. In the following we will refer to the former as "triangles" and to the latter as "bubbles". We only need diagrams with at least one new vertex. First of all, if a 3-point amplitude has three new vertices it completely collapses and it is zero in Dim-Reg. If it has two new vertices and one old one, it collapses into one of these bubble diagrams:



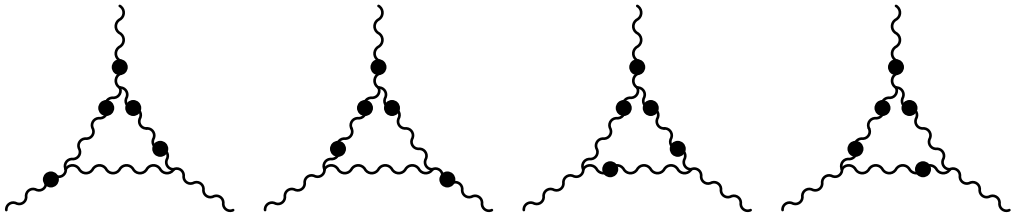
(142)

Anyway, they are proportional to k^2 and are thus zero on-shell. Therefore, as far as triangle diagrams are concerned, we only need the ones with one new and two old vertices. There are three diagrams that do not collapse



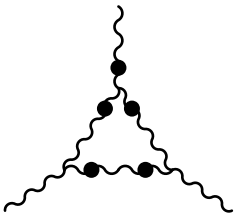
(143)

and five that collapse



(144)

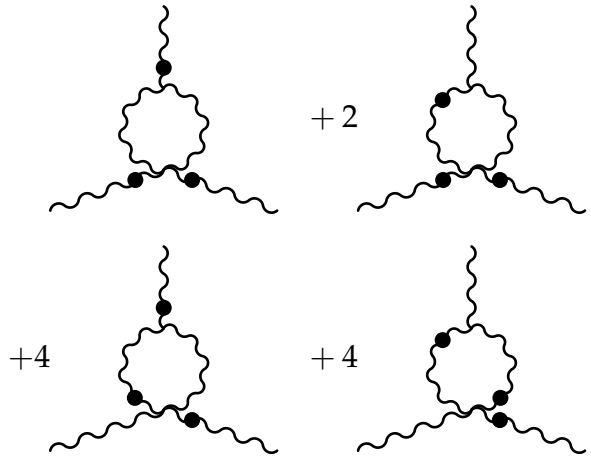
the last one of which



(145)

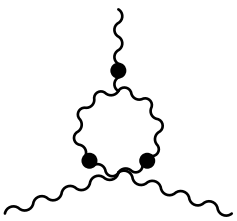
is proportional to k^2 and therefore vanishes on-shell.

For what concerns the bubble diagrams, we are left with the ones containing the old cubic and the new quartic vertex. In the cubic vertex there are 3 ways to attach the dotted line, six ways for the quartic. In total there are 18 diagrams, 6 of them are vanishing due to the collapsing rule. Each of the diagrams has a $\frac{1}{4}$ symmetry factor



(146)

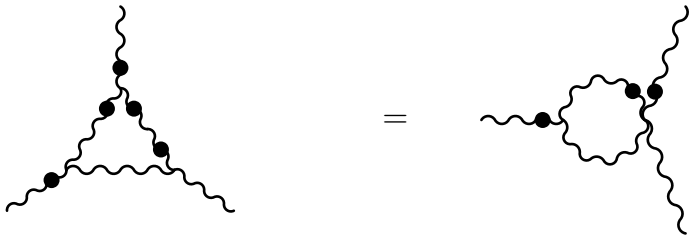
plus the last one which is zero on shell:



(147)

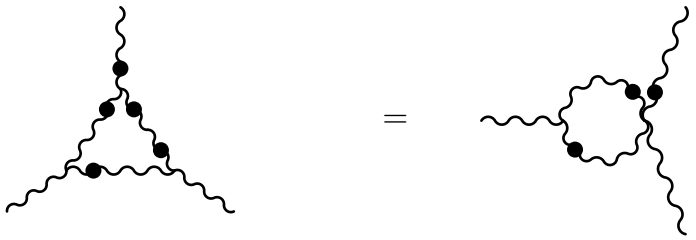
The numbers in (146) count the number of diagrams, in particular the last two diagrams are actually $2+2$ depending on where the dot goes in the bottom lines.

A careful analysis of the Feynman rules shows that the diagrams in (144) and the last two types of the bubble diagrams in (146) precisely cancel:



(148)

and

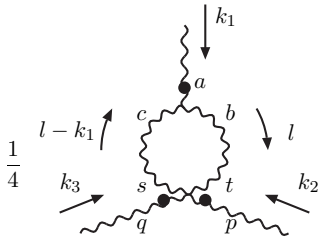


(149)

as well as similar relation for the other collapsing line. The diagrams precisely account for all the instances of the various diagrams and symmetry factors. All in all we are left with 4 diagrams to compute, which will be the task of the next subsection.

3.4.2 *Amplitude divergences*

We start with the easy bubble diagrams:



$$-\mathrm{i}\tfrac{\alpha g^3}{2M^2} f^{abc} f^{rpq} f^{rst} \left(k_1 A^a(k_1)\right)^{AB} A^b(-l)_{AB'} A^c(l-k_1)_B{}^{B'} \left(k_2 A^p(k_2)\right)_M{}^N \left(k_3 A^q(k_3)\right)_N{}^P A^s(-l+k_1)_{(PP'} A^t(l)^{M)P'}$$

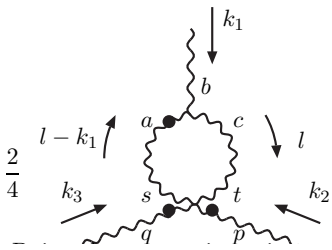
where 1/4 is the symmetry factor. As before the contractions of the placeholders for the internal gauge field lines are substituted by propagators. This leads to:

$$\frac{1}{4}(-2\mathrm{i})\frac{\alpha g^3}{M^2}(-C_2 f^{apq})\Big(k_1 A^a(k_1)\Big)^{AB}\Big(k_2 A^p(k_2)\Big)_M{}^N\Big(k_3 A^q(k_3)\Big)_N{}^P \\ \varepsilon_{B(P}\varepsilon^{B'}{}_{P'}\varepsilon_A{}^{M)}\varepsilon_{B'}{}^{P'}\frac{1}{\mathrm{i}l^2\mathrm{i}(l-k_1)^2}. \quad (150)$$

A straightforward calculation along the lines of the one we performed for the self-energy diagrams gives the divergent part of the loop diagram:

$$-\frac{\alpha g^3 C_2}{(4\pi)^2 M^2}\,\frac{2}{\varepsilon} f^{abc}\Big(k_1 A^a(k_1)\Big)_A{}^B\Big(k_2 A^b(k_2)\Big)_B{}^C\Big(k_3 A^c(k_3)\Big)_C{}^A. \quad (151)$$

The second bubble diagram is slightly harder:



$$(-\mathrm{i})\tfrac{\alpha g^3}{2M^2} f^{abc} f^{rpq} f^{rst} \\ \Big((l-k_1)A^a(l-k_1)\Big)^{AB} A^b(k_1)_{AB'} A^c(-l)_B{}^{B'} \\ \Big(k_2 A^p(k_2)\Big)_M{}^N\Big(k_3 A^q(k_3)\Big)_N{}^P A^s(-l+k_1)_{(PP'} A^t(l)^{M)P'}$$

The Wick contractions give:

$$(-\mathrm{i})\frac{\alpha g^3}{M^2}(-C_2) f^{bpq} A^b(k_1)_{AB'}\Big(k_2 A^p(k_2)\Big)_M{}^N\Big(k_3 A^q(k_3)\Big)_N{}^P \\ 2(l-k_1)^{(A}{}_{A'}\varepsilon_{(P}{}^{B)}\varepsilon_{P'}{}^{A'}\varepsilon_B{}^{M)}\varepsilon^{B'P'}\frac{1}{\mathrm{i}l^2\mathrm{i}(l-k_1)^2}. \quad (152)$$

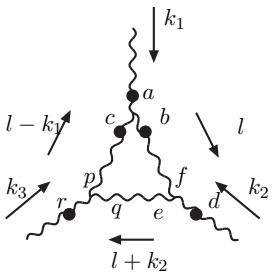
To evaluate this we use the fact when l is replaced with $l = q + xk_1$ only the k_1 term survives. Thus, after some manipulations of the spinor contractions, and using the on-shell property of A_1, we get

$$\frac{\alpha g^3}{M^2}\,C_2\, f^{bpq}\left(-\frac{1}{2}\Big(k_1 A^b(k_1)\Big)_P{}^M\Big(k_2 A^p(k_2)\Big)_M{}^N\Big(k_3 A^q(k_3)\Big)_N{}^P\right)\frac{\mathrm{i}(x-1)}{\mathrm{i}l^2\mathrm{i}(l-k_1)^2}.$$

Again, we use the same analysis of the loop integral divergence as before. This yields:

$$\frac{1}{4}\,\frac{\alpha g^3 C_2}{(4\pi)^2 M^2}\,\frac{2}{\varepsilon} f^{abc}\Big(k_1 A^a(k_1)\Big)_A{}^B\Big(k_2 A^b(k_2)\Big)_B{}^C\Big(k_3 A^c(k_3)\Big)_C{}^A. \quad (153)$$

The two triangle diagrams are a little more complicated than the last two.



$$
\begin{aligned}
&(\alpha g^3/2M^2) f^{abc} f^{def} f^{rpq} \\
&\Big(k_1 A^a(k_1)\Big)_A{}^B \Big(-l A^b(-l)\Big)_B{}^C \Big((l-k_1) A^c(l-k_1)\Big)_C{}^A \\
&\Big(k_2 A^d(k_2)\Big)^{MN} A^e(-l-k_2)_{MM'} A^f(l)_N{}^{M'} \\
&\Big(k_3 A^r(k_3)\Big)^{PQ} A^p(-l+k_1)_{PP'} A^q(l+k_2)_Q{}^{P'}
\end{aligned}
$$

Wick contracting gives

$$
\frac{\alpha g^3}{2M^2}\left(-\frac{C_2}{2}\right) f^{ade}\Big(k_1 A^a(k_1)\Big)_A{}^B \Big(k_2 A^d(k_2)\Big)^{MN} \Big(k_3 A^r(k_3)\Big)^{PQ}
$$

$$
2\, l_{(BB'}\epsilon^{C)}{}_N\, \epsilon^{B'M'}\, 2(l-k_1)_{(CC'}\epsilon^{A)}{}_P\, \epsilon^{C'}{}_{P'}\, \epsilon_{MQ}\, \epsilon_{M'}{}^{P'}\, \frac{1}{\mathrm{i}l^2 \mathrm{i}(l-k_1)^2 \mathrm{i}(l+k_2)^2}.
$$

We need a more general Feynmann parametrization for the denominator this time:

$$
\frac{1}{l^2(l-k_1)^2(l+k_2)^2} = 2\int_0^1 dx \int_0^x dy \frac{1}{\left(l^2 + x\left[(l-k_1)^2 - l^2\right] + y\left[(l+k_2)^2 - (l-k_1)^2\right]\right)^3}. \tag{154}
$$

After a shift

$$
l = q + xk_1 - y(k_2 + k_1), \tag{155}
$$

the previous denominator becomes $(q^2 + \Delta_{x,y})^3$, where $\Delta_{x,y}$ is a term independent of q. We are again in the cases covered by the Dim-Reg eqs. (138). From these formulae, specialized for the case $n = 3$, it is easy to check that the only numerator that produces a divergence is q^2. We then perform the shift (155) in the numerator also and keep only the quadratic terms in q. After some manipulation of the ε's contractions we get:

$$
2q^2 \epsilon_{NB} \epsilon_P{}^A \epsilon_{QM},
$$

plus a term proportional to $\varepsilon_B{}^A$ which vanishes when contracted to the rest of the expression. We remind that also an overall factor of i comes from

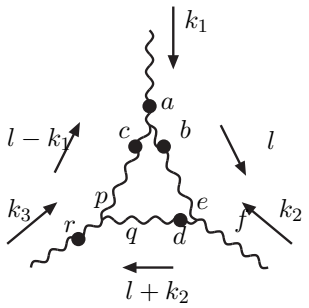
the Wick rotation. After we extracted the divergent part of the momenta's integrals the x, y integral is trivial:

$$2\int_0^1 dx \int_0^x dy = 1. \tag{156}$$

The divergent part of this diagram is thus

$$\frac{1}{2}\frac{\alpha g^3 C_2}{(4\pi)^2 M^2}\,\frac{2}{\epsilon} f^{abc}\Big(k_1 A^a(k_1)\Big)_A{}^B\Big(k_2 A^b(k_2)\Big)_B{}^C\Big(k_3 A^c(k_3)\Big)_C{}^A. \tag{157}$$

The other triangle diagram that we need to compute is



$$\begin{aligned}
&(\alpha g^3/M^2) f^{abc} f^{def} f^{rpq}\\
&\Big(k_1 A^a(k_1)\Big)_A{}^B\Big(-l A^b(-l)\Big)_B{}^C\Big((l-k_1)A^c(l-k_1)\Big)_C{}^A\\
&\big((-l-k_2)A^d(-l-k_2)\big)^{MN} A^e(l)_{MM'} A^f(k_2)_N{}^{N'}\\
&\Big(k_3 A^r(k_3)\Big)^{PQ} A^p(-l+k_1)_{PP'} A^q(l+k_2)_Q{}^{P'}
\end{aligned}$$

There are two diagrams where the dot is applied either to the k_3 or k_2 leg. These two diagrams give the same result and cancel the $1/2$ symmetry factor. The Wick contractions in this case are:

$$\begin{aligned}
&-\frac{\alpha g^3}{M^2}\left(-\frac{C_2}{2}\right) f^{afr}\Big(k_1 A^a(k_1)\Big)_A{}^B A^f(k_2)_N{}^{N'}\Big(k_3 A^r(k_3)\Big)^{PQ}\\
&2l_{(BB'}\varepsilon^{C)}{}_M\varepsilon^{B'}{}_{N'} 2(l-k_1)_{(CC'}\varepsilon^{A)}{}_P\varepsilon^{C'}{}_{P'} 2(l+k_2)^{(M}{}_{M'}\varepsilon^{N)}{}_Q\varepsilon^{M'P'}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\frac{1}{\mathrm{i}l^2\mathrm{i}(l-k_1)^2\mathrm{i}(l+k_2)^2}
\end{aligned} \tag{158}$$

where one extra overall minus comes from the fact that there are 3 Wick's contractions that produce 3 propagators, each carrying a minus sign. We perform then the usual algebraic manipulations and shift the loop momenta as in eq. (155). We then keep terms quadratic in q and replace $q_\mu q_\nu$ with $(1/4)q^2$. We replace the connections with the polarisation spinors using (74) which yields:

$$(-\mathrm{i})^3\left(4(1-3y)(12)(23)(31) + 2(1+3x)(12)(31)^2\frac{[p1]}{[p2]}\right),$$

where we denoted the reference spinor with p. The contractions involving this spinor can be replaced through relations found using momentum

conservation: $[p1]/[p2] = -(32)/(31)$. Thus, after the integration over the Feynman parameters we get

$$(-\mathrm{i})^3 4(12)(23)(31) = \frac{1}{2}\Big(k_1 A(k_1)\Big)_A{}^B \Big(k_2 A(k_2)\Big)_B{}^C \Big(k_3 A(k_3)\Big)_C{}^A.$$

Thus, finally the divergent part of this diagram is

$$-\frac{1}{4}\frac{\alpha g^3 C_2}{(4\pi)^2 M^2}\,\frac{2}{\epsilon} f^{abc}\Big(k_1 A^a(k_1)\Big)_A{}^B \Big(k_2 A^b(k_2)\Big)_B{}^C \Big(k_3 A^c(k_3)\Big)_C{}^A. \tag{159}$$

3.4.3 $\beta-$ *function result*

We now combine the 4 diagrams that we have computed (151), (153), (157) and (159) and we multiply by the 3! symmetry factor obtaining:

$$-6\,\frac{\alpha g^3 C_2}{(4\pi)^2 M^2}\,\frac{1}{\epsilon} f^{abc}\Big(k_1 A^a(k_1)\Big)_A{}^B \Big(k_2 A^b(k_2)\Big)_B{}^C \Big(k_3 A^c(k_3)\Big)_C{}^A, \tag{160}$$

The renormalizad Lagrangian is, schematically:

$$-\frac{1}{2} Z_A (\partial A)^2 + Z_g g \mu^{\epsilon/2} A^2 (\partial A) + Z_{\alpha g}\mu^{\epsilon/2}\frac{\alpha g}{M^2}(\partial A)^3 + \dots. \tag{161}$$

We will only need to write down the two counterterms Z_A and $Z_{\alpha g}$:

$$\begin{aligned} Z_{\alpha g} &= 1 + 6\frac{g^2 C_2}{(4\pi)^2}\,\frac{1}{\epsilon} \\ Z_A &= 1 + \frac{10}{3}\,\frac{g^2 C_2}{(4\pi)^2}\,\frac{1}{\epsilon}. \end{aligned} \tag{162}$$

From the two relations between the renormalized and bare couplings and fields:

$$Z_A A^2 = A_0^2, \qquad Z_{\alpha g}\alpha g \mu^{\epsilon/2} A^3 = (\alpha g)_0 A_0^3, \tag{163}$$

we get the relation:

$$(\alpha g)_0 = (Z_A)^{-\frac{3}{2}} Z_{\alpha g}\alpha g \mu^{\frac{\varepsilon}{2}}. \tag{164}$$

We take the logarithm of the last equation:

$$\begin{aligned} \ln(\alpha g)_0 &= \ln(\alpha g) + \frac{\varepsilon}{2}\ln\mu + \ln(Z_{\alpha g}) - \frac{3}{2}\ln(Z_A) \\ &= \ln(\alpha g)_0 = \ln(\alpha g) + \frac{\varepsilon}{2}\ln\mu + \frac{g^2 C_2}{(4\pi)^2 \varepsilon}. \end{aligned} \tag{165}$$

We now take the derivative with respect to $\ln \mu$, considering the bare quantities independent of the renormalization scale:

$$0 = \frac{1}{(\alpha g)_0}\frac{\partial (\alpha g)_0}{\partial \ln \mu} = \frac{1}{\alpha g}\frac{\partial \alpha g}{\partial \log(\mu)} + \frac{\epsilon}{2} + \frac{C_2}{(4\pi)^2 \varepsilon}\frac{\partial g^2}{\partial \ln \mu}, \tag{166}$$

The general argument along the same lines as the calculation presented in this section is used to compute the YM β-function and leads to $\frac{\partial g^2}{\partial \ln \mu} = -\varepsilon g^2 + \beta_{YM}$. Combining it with the last equation yields the final result for the α coefficient β-function:

$$\frac{\partial \alpha g}{\partial \log(\mu)} = \frac{\alpha g^3 C_2}{(4\pi)^2}. \tag{167}$$

The result is positive which means that this interaction grows stronger in the UV. If we compare the change in the YM coupling:

$$g^2(\mu) = \frac{g^2(\mu_0)}{1 + (11C_2/3(4\pi)^2) g^2(\mu_0) \log(\mu^2/\mu_0^2)} \tag{168}$$

with the change of the new coupling:

$$(\alpha g)(\mu) = (\alpha g)(\mu_0) \left(1 + (11C_2/3(4\pi)^2) g^2(\mu_0) \log(\mu^2/\mu_0^2)\right)^{3/11}, \tag{169}$$

it is clear that the new coupling running is very slow, i.e. logarithmic. Anyway, if we add the new interaction at some energy scale the departure from the YM will inevitably grow. Put it another way, the new interaction puts the theory out of the critical surface of the free UV fixed point of YM.

3.5 FULL ONE-LOOP β-FUNCTION

The result (167) has been confirmed and extended in [20] by Krasnov. There, a complete one-loop analysis of the DYM has been performed in the first order formalism. We want to briefly introduce the set up and final result on this calculation because it is quite instructive and a similar calculation will be performed in the last chapter of this thesis in the context of gravity.

In the last sections we expected that the parametric family of theories defined by (10) could be renormalized at one-loop order by a field redefinition

followed by a multiplicative renormalization of all the possible infinite coupling constants. In this way the family of theory as a whole, contains all the possible operators needed as counterterms. This expectation turns out to be true. In particular after some field redefinition the renormalization group flow is described by a β-function for the defining function f (or the relative potential V in the first order formalism):

$$\frac{\partial f(x)}{\partial \log \mu} = \beta_f(x), \tag{170}$$

let us briefly see how to get this result.

First we need to put the theory in the first order formalism. This is needed to directly and conveniently apply the Heat Kernel formalism for the computation of the functional determinants arising in the background field expansion. These determinants contain all the information about the one-loop divergences. It is much harder to do this in the second order formalism due to the fact that the arising operator is a non-standard Laplace operator.

We consider the action:

$$S[A, B] = \int B^{aMN} F^a_{MN} - M^4 V(B/M^2), \tag{171}$$

where F^a_{AB} is the self-dual part of the connection curvature, B^{aAB} is the self-dual part of 2-forms and V is a potential, a gauge invariant function of the B field only.

The linearization of (171) is an easy task:

$$\begin{aligned}\mathcal{L}^{(2)} = 2b^{aNM} d_{MM'} a^a_N{}^{M'} + f^{abc} B^{aMN} a^b_{MM'} a^c_N{}^{M'} \\ -\frac{1}{2}(V'')^{ab}_{MNPQ} b^{aMN} b^{bPQ},\end{aligned} \tag{172}$$

where $(V'')^{ab}_{MNPQ}$ is the matrix of second derivatives of the potential, $d_{MM'}$ is the covariant derivative with respect to the background field connection. We indicate with capital letters the background fields. The previous linearized Lagrangian shows that a gauge fixing procedure of the symmetries needs to be performed in order to have an invertible kinetic operator. Indeed, the first order differential operator in (172) sends the $4 \times \dim(\mathfrak{g})$ components of the connection into the $3 \times \dim(\mathfrak{g})$ ones of the B field.

One introduces then the YM gauge fixing fermion:

$$\Psi = 2\bar{c}^a d^M_{M'} a^a_M{}^{M'} - 2M^{ab}\bar{c}^a h^b_c, \tag{173}$$

Here $\bar{c}^a$ is an anti-ghost, h^a_c is an auxiliary field. The only difference with respect to the usual procedure is that here, we introduce some matrix M^{ab} instead of the gauge-fixing parameter.

The BRST variation of (173) is

$$s\Psi = 2\bar{c}^a d^M_{M'} d_M{}^{M'} c^a + 2h^a_c d^M_{M'} a^a_M{}^{M'} - 2M^{ab} h^a_c h^b_c. \tag{174}$$

As it is customary in the BRST treatment of gauge invariances the BRST-exact term (174) is added to the Lagrangian to remove the degeneracy of the kinetic operator. To see how this is achieved it is better to shift the b field and the potential V introducing the "missing" antisymmetric (i.e. trace) component in terms of the auxiliary field h_c and the matrix M:

$$\begin{aligned} \tilde{b}^{aNM} &= b^{aNM} + \epsilon^{NM} h^a_c. \\ (\tilde{V}'')^{ab}_{MNPQ} &= (V'')^{ab}_{MNPQ} - M^{ab}\epsilon_{MN}\epsilon_{PQ}. \end{aligned} \tag{175}$$

using the last two formulae in the linearized Lagrangian it is easy to see that the last two terms of (174) get canceled while the first is just the ghost sector Lagrangian. In terms of the new, shifted variables the gauge fixed linearization reads:

$$\begin{aligned} \mathcal{L}^{(2)} = 2\tilde{b}^{aNM} d_{MM'} a^a_N{}^{M'} + f^{abc} B^{aMN} a^b_{MM'} a^c_N{}^{M'} \\ -\frac{1}{2}(\tilde{V}'')^{ab}_{MNPQ} \tilde{b}^{aMN} \tilde{b}^{bPQ}, \end{aligned} \tag{176}$$

Notice that even if the form of the last Lagrangian is the same as before the $\tilde{b}$ field is not symmetric, now. For brevity, we can drop the tilde understanding the new symmetry property for this field.

We define the following Dirac spinor valued in the space $V = S_+ \otimes \mathfrak{g}$:

$$\begin{pmatrix} b \\ a \end{pmatrix}, \qquad b \in V \otimes S_+, \quad a \in V \otimes S_-, \tag{177}$$

The differential operator in (176) can be considered of Dirac-type and be written as:

$$D = \begin{pmatrix} A & \partial \\ \partial\!\!\!/^* & B \end{pmatrix}, \tag{178}$$

where the ∂ and $\partial\!\!\!/$ are the usual Dirac operators and A and B are the endomorphisms:

$$\begin{aligned} A_{AB} \equiv A^{ab}_{MANB} &= -\frac{1}{2}(\tilde{V}'')^{ab}_{MANB}. \\ B_{A'B'} \equiv B^{ab}_{MN}\epsilon_{A'B'} &= f^{aeb}B^{e}_{MN}\epsilon_{A'B'}. \end{aligned} \tag{179}$$

To use the Heat kernel method (see for example [30] for an exaustive discussion of use of this method in Quantum Field Theory) the arising Dirac operator (364) needs to be squared to obtain a Laplace type operator $\Delta = D^2$. In particular there is a unique way to put this Laplace operator in the standard form:

$$\Delta = -\mathcal{D}_\mu \mathcal{D}^\mu + E, \tag{180}$$

where $\mathcal{D}_\mu$ is some covariant derivative and E is an endomorphism. As it is well known in QFT the divergences at one loop are encoded in the logarithm of the determinant of such kinetic operators arising from the linearization of the action around a general background (i.e. the one loop effective action). Considering also the ghost piece the effective action is:

$$\Gamma_{\text{eff}} = -\frac{1}{2}\log\det(D) + \log(d_g^2), \tag{181}$$

As said, we need to square the Dirac operator to get a Laplace operator:

$$\Gamma_{\text{eff}} = -\frac{1}{4}\log\det(\Delta) + \log\det(d_g^2), \tag{182}$$

We use the formal relation $\log\det(\Delta) = \mathrm{Tr}\log(\Delta)$ and the integral representation of the logarithm:

$$\log\det(\Delta) = -\int_0^\infty \frac{dt}{t}\mathrm{Tr}\left(e^{-t\Delta}\right), \tag{183}$$

to rewrite the effective action (181). The so-called Heat-Kernel $\mathrm{Tr}\left(e^{-t\Delta}\right)$ has an expansion in powers of t:

$$\mathrm{Tr}\left(e^{-t\Delta}\right) = \int d^4x\sqrt{g}\frac{1}{(4\pi t)^2}\sum_{n=0}^{\infty} t^n a_n(E), \tag{184}$$

where a_n are the so called Seely-DeWitt coefficients. These coefficients can be written in terms of geometric invariants of the fiber bundle the operator is defined on. In particular, for QFT application, the fourth coefficient is related to the logarithmic divergences of the theory. This coefficient specialized for the operator Δ at hand reduces to (see e.g. [30])

$$a_4 = \frac{1}{(4\pi)^2}\frac{1}{12}\int \mathrm{Tr}\left(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} + 6E^2\right), \tag{185}$$

where $\mathcal{F}^{\mu\nu}$ is the curvature computed from the previously introduced covariant derivative $\mathcal{D}_\mu$. The fourth coefficient gives precisely the counterterms needed to reabsorb the divergences at one loop, from which it is easy to extract the β-function. We do not go into any detail of this quite technical computation here. We will show an explicit calculation along these lines in the last chapter in the context of gravity theories. We just want to show the final form of the beta function:

$$\frac{1}{(4\pi)^2}\frac{1}{6}\left(F_{AB}F^{AB} + 6A_{AB}{}^{AB}\tilde{B}^{MN}F_{MN} + 12A_{AM}{}^{BN}\tilde{B}_B{}^C A_{CN}{}^{DM}\tilde{B}_D{}^A\right), \tag{186}$$

or in terms of the second order defining function f:

$$\begin{aligned}\beta_f(x) = \frac{1}{(4\pi)^2}\frac{1}{6}\Big[&x^{ab}_{AB}x^{baAB} - 3((f'')^{-1})^{ab}_{AB}{}^{AB}(f')^{bc}_{MN}x^{caMN} \\ &+3((f'')^{-1})^{ab}_{AM}{}^{BN}(f')^{bcC}_{B}((f'')^{-1})^{cd}_{CN}{}^{DM}(f')^{daA}_{D}\Big].\end{aligned} \tag{187}$$

where $x^{ab}_{AB} = f^{acb}F^c_{AB}/M^2$ and $(f')^{ab}_{AB} := C^{aeb}(f')^e_{AB}$ and the primes indicate first and second derivatives with respect to x^a_{AB}. It is clear that the beta function does not contain derivatives of the curvature and it is already a gauge invariant function. Interestingly, those terms get canceled by a field redefinition of the same type as the one we have already encountered before. To check the formula (187), we can compute the β-function for the YM coupling g and the new coupling αg. We start from the Lagrangian:

$$\mathcal{L} = \frac{1}{4g^2}(F^a_{MN})^2 + \frac{\alpha}{3!g^2M^2}f^{abc}F^{a\,B}_{A}F^{bC}_{B}F^{c\,A}_{C}. \tag{188}$$

where we have used a different sign in front of the YM term appropriate for the euclidean setting in which these calculation are performed. The relevant quantities for the β-function are:

$$\begin{aligned}(f')^{a}_{AB} &= \frac{1}{2g^2}x^{a}_{AB} - \frac{\alpha f^{abc}}{2g^2}x^{b\ E}_{A}\ x^{c\ B}_{E}\\ ((f'')^{-1})^{ab}_{ABCD} &= 2g^2\delta^{ab}\varepsilon_{A(C}\varepsilon_{BD)} + 4g^2 f^{abc}\varepsilon_{C(A}\varepsilon_{EB)}x^{cE}{}_{D}.\end{aligned} \tag{189}$$

Using these formulae into (187) one gets the running of the coupling constants:

$$\begin{aligned}\frac{\partial}{\partial \log\mu}\left(\frac{1}{4g^2}\right) &= \frac{C_2}{(4\pi)^2}\frac{11}{6},\\ \frac{\partial}{\partial \log\mu}\left(\frac{\alpha}{3!g^2M^2}\right) &= \frac{C_2}{(4\pi)^2M^2}2\alpha.\end{aligned} \tag{190}$$

which after some easy manipulation of the derivatives on the left side become the running already computed in the last chapter plus the usual YM result:

$$\begin{aligned}\frac{\partial\, g}{\partial \log\mu} &= -\frac{11}{3}\frac{C_2\, g^3}{(4\pi)^2},\\ \frac{\partial\, \alpha g}{\partial \log\mu} &= \frac{\alpha g^3 C_2}{(4\pi)^2}.\end{aligned} \tag{191}$$

3.6 CONCLUSION AND REMARKS

It is a quite remarkable fact that DYM theories are sufficiently large to be closed under renormalization, at least at one loop, even if they are not the most general theories possible that are compatible with the symmetries. The DYM theories are also described in a very compact form and it is quite useful that one can extract the running of all the coupling constants from a single formula. Of course, one loop renormalizability is still far from being conclusive. Many theories pass this test but fail already at two loops (consider General Relativity, or the simple scalar field theory with a potential).

The collapsing dot property is the key ingredient for the good behavior of DYM, and we know that it is at work at all loops. This is automatically

satisfied in the first order formalism given that there are no derivatives in the vertices and the propagator is the inverse of a first-order operator (which can be thought as a derivative of a second-order operator). If this renormalizability result held at higher loop orders one could try to find the regimes at which one could trust the one-loop flow. More importantly one could be interested in finding the fixed points of this flow. In view of the asymptotic safety scenario one would be interested in the dimensionality of the critical surface of these fixed points (i.e. the points that are connected by the flow to the fixed points). Were it be finite dimensional the theory might be predictive even with its infinite set of coupling constants. Even if the hope of all-order renormalizability turns out to be too optimistic, it could be possible to find a reduced space of theories which are closed under renormalization (YM itself is a special case of this property).

Of course no physics can actually be extracted unless the crucial problem of unitarity of these theories is properly addressed. For sure much more work is needed to fully understand all the properties of the DYM theories.

In the next chapters we will show general ideas about some deformations of General Relativity which share a great deal of properties and features with DYM. The hope to find a similar renormalizability result for these theories of gravity actually lead to the investigation in DYM. We hope that many of the analysis here developed could be useful in the context of gravity.

4

DIFFEOMORPHISM INVARIANT GAUGE THEORIES

4.1 INTRODUCTION

In this chapter we will review some known facts about several formulation of General Relativity in terms of gauge fields instead of the metric. The first of formulation of the kind is due to Plebanski. It is a first order Lagrangian formulation that will serve as a model for any other attempts to write a gauge theory of gravity. This setting allows also to build an infinite class of theories that "deform" GR in a way that is close in spirit with DYM. We will start with a brief introduction to the Plebanski formulation, then we will see how to deform it to obtain new gravity theories. A pure connection formulation has been introduced by Krasnov in [31]. We will review it at the end of the chapter where we will also show that a form of renormalizability similar to the one claimed for DYM could hold.

4.2 PLEBANSKI FORMULATION OF GRAVITY

The idea behind Plebanski's first order action is to describe complex GR with a triple of 2-forms alongside a connection 1-form and a Lagrange multiplier. From the onset no metric is defined, and this concept only arise as a derived object after the equations of motion are solved. The role of the 2-forms field B^i is to provide the metric degrees of freedom, while the con-

nection field is later identified with the self-dual part of the spin connection. The Lagrange multiplier is then viewed as the self-dual part of the Weyl tensor. Let us show briefly how this is done.

The vector space of two forms can be split into two 3-dimensional orthogonal subspaces by the hodge duality operation, which needs a metric to be defined. On the other hand if we start with no metric and define a triple of 2-forms B^i to span such a 3-dimensional subspace this in turn defines the notion of Hodge duality. This operation maps these forms into their orthogonal ones, defining at the same time a notion of orthogonality for vectors. This provides a metric up to a conformal factor. However, a triple of 2-forms has 18 components to be fixed against the 10 needed for the metric. Plebanski introduced this constraint in his Lagrangian, such that the metric would come directly from the B field:

$$B^i \wedge B^j \propto \delta^{ij}, \tag{192}$$

which is called "metricity" or "simplicity" constraint. These are 5 equations leaving 13 components for B, which match the metric ones considering the 3 components of the $SO(3)$ rotations that act on the 2-forms.

An explicit form for the metric is due to Urbantke [6] and reads:

$$\sqrt{g} g_{\mu\nu} = \frac{1}{3} \varepsilon_{ijk} \varepsilon^{\alpha\beta\gamma\delta} B^i_{\mu\alpha} B^j_{\beta\gamma} B^k_{\delta\nu}. \tag{193}$$

In turn, of course, the 2-forms B^i are self-dual with respect to this metric.

To fix the signature of the metric and select a reality condition one has to supplement the B fields with additional constraints. For the split signature we require the B field to be real. For an Euclidean metric it is imposed that B is Hermitian with respect to an Hermitian product. The Lorentz condition needs special constraints, instead:

$$B^i \wedge \bar{B}^j = 0, \tag{194}$$

$$B^i \wedge B^i + \bar{B}^j \wedge \bar{B}^j = 0, \tag{195}$$

where the first of them is equivalent to the condition that the complex conjugate $\bar{B}^i$ is anti-self-dual. This means that two vectors are orthogonal if and

only if their conjugates are as well. This metric has to be conformal to a real metric. Moreover, the Hodge duality operator defined by this metric has eigenvalues $\pm i$ which implies that the determinant of g is negative. So the metric is Lorentzian. The second condition says that the volume form defined through B is real which excludes any purely imaginary conformal factor.

The action that Plebanski achieved is one in the B field plus a 1-form connection A. This connection is related to the 2-form through a compatibility condition:

$$DB^i = 0, \tag{196}$$

where D is the covariant exterior derivative built with the connection A. This is an algebraic condition in A which can be solved for it and implies that this connection is indeed the self-dual part of the spin connection compatible with the metric defined by B.

We have all the necessary objects to translate the Einstein equations into this formalism, now. To do so we have to remind an important way to rephrase the condition for a metric to be Einstein (i.e. Ricci flat). The Riemann tensor built from g can be viewed as a 2-form valued in the space of 2-forms. In this way one can achieve a complete self/anti-self-dual decomposition of this tensor. For completeness we state this decomposition here but we included a more detail account of this facts in the appendix (B). This decomposition is best seen using spinor indices:

$$\begin{aligned} R_{AA'BB'CC'DD'} &= X_{ABCD}\varepsilon_{A'B'}\varepsilon_{C'D'} + \Phi_{ABC'D'}\varepsilon_{A'B'}\varepsilon_{CD} \\ &+ \bar{\Phi}_{A'B'CD}\varepsilon_{AB}\varepsilon_{C'D'} + \bar{X}_{A'B'C'D'}\varepsilon_{AB}\varepsilon_{CD}, \end{aligned} \tag{197}$$

where the X field can be further decomposed into its totally symmetric Ψ and trace part:

$$X_{ABCD} = \Psi_{ABCD} + \Lambda(\varepsilon_{AC}\varepsilon_{BD} + \varepsilon_{AD}\varepsilon_{BC}), \tag{198}$$

and the two Φ fields are actually the same by virtue of the symmetries of the Riemann tensor:

Using this decomposition one can show that the Ricci tensor becomes:

$$R_{\mu\nu} \longrightarrow 6\Lambda\varepsilon_{AB}\varepsilon_{A'B'} - 2\Phi_{ABA'B'} \tag{199}$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} \longrightarrow R_{ABB'A'}$$

$$R_{ABB'A'} = -6\Lambda\varepsilon_{AB}\varepsilon_{A'B'} - 2\Phi_{ABA'B'}. \tag{200}$$

and the Einstein condition is:

$$R_{\mu\nu} = \lambda g_{\mu\nu} \qquad \Leftrightarrow \qquad \Phi_{ABA'B'} = 0, \quad \Lambda = \frac{1}{6}\lambda. \tag{201}$$

where λ is the cosmological constant. It is clear then, that the totally symmetric spinor Ψ and $\bar{\Psi}$ are nothing else than the part of the Riemann tensor not constrained by the Einstein equations, i.e. the self-dual and anti-self-dual parts of the Weyl tensor.

A closer look at eq. (197) shows that if we take the self dual part of the Riemann tensor (wrt the $CDC'D'$ indices) the metric is Einstein if and only if this part is self-dual (wrt the other indices). So Einstein equations are equivalent to the statement that the self-dual part of the Riemann tensor is self-dual as a two form. Let us write this statement using the $SO(3)$ indices notation also for future convenience:

$$F^i = F^{ij}\Sigma^j + \bar{F}^{ij}\bar{\Sigma}^j \longrightarrow \begin{array}{l} \bar{F}^{ij} = 0, \\ \mathrm{Tr}(F^{ij}) = 6\Lambda = \lambda \end{array} \tag{202}$$

where we denoted F^i as the self dual part of the curvature and with Σ^{ij} a basis of the vector space of 2-forms. The first of the equations to the right are the Einstein equations. The second states that the cosmological constant is the trace part of the component of the Riemann tensor not constrained by the Einstein equations.

Let us put everything together and state the Lagrangian for complex GR that produces the main equations of this section:

$$S_{\text{plebanski}} = \int B^i \wedge F^i - \frac{1}{2}\left(\Psi^{ij} - \frac{\Lambda}{3}\delta^{ij}\right) B^i \wedge B^j. \tag{203}$$

The B^i field is the usual triple of two forms, F^i is the curvature of the connection A^i. The symmetric traceless field Ψ^{ij} is a Lagrange multiplier and Λ

is a constant. The equations of motion are the three main equations spelled out in this section:

$$B^i \wedge B^j \propto \delta^{ij}, \tag{204}$$

$$DB^i = 0, \tag{205}$$

$$F^i = \left[\Psi^{ij} - \frac{\Lambda}{3}\delta^{ij}\right] B^j. \tag{206}$$

We finish this section summing up the content of these equations of motion. The Lagrangian (204) is defined without any metric, it is eq. (204) that provides us with one that also makes B self-dual with respect to it. The solution to (205) is the self dual part of the spin-connection. The last eq. (206) is now the statement that the self-dual part of the curvature is self-dual as a 2-form, i.e. Einstein equations. In turn this identifies the symmetric traceless field Ψ^{ij} with the self-dual part of the Weyl tensor and Λ with the cosmological constant.

The theory so built has had many developments. The one we are interested in is that eq. (204) can actually be dropped. The remaining relations and the symmetries acting on the fields are enough to close a system of differential and algebraic equations and define a class of new theories. Strikingly, these theories share more similarities with GR than one would expect.

4.3 CHIRAL MODIFICATIONS OF PLEBANSKI FORMULATION

The idea behind a class of modified gravity theory that we are about to review dates back to the work of Bengtsson [13], and has been rediscovered and further developed by Krasnov in the last few years.

The modifications rely on the fact that we can actually avoid the metricity constraint (192) and try to built a theory based on the same fields as in the Plebanski formulation. It can look strange that dropping the 5 equations of the metricity constraint one can still define a closed system of equations, but actually there are some identities that are sufficient to achieve this task.

Given that any (non-degenerate) triple of 2-forms defines a (conformal class of) metric these theories can still be regarded as gravitational.

The starting point is to simply keep the other two sets of equations (205) and (206). From the first one, the compatibility of the connection A with the 2-forms B one can find a unique solution for the former in terms of the latter. One can then find the curvature F of this connection, which in this case has no gravitational meaning. The 2-forms F^i can be decomposed into their self- and anti-self-dual part:

$$F^i = F^{ij}B^j + \bar{F}^{ij}\bar{B}^j, \tag{207}$$

where we have used B and $\bar{B}$ as a basis of 2-forms.

The other equation of motion:

$$F^i = \Phi^{ij}B^j, \tag{208}$$

where Φ^{ij} is a symmetric tensor, just states that the curvature of the connection A^i is self-dual. Of course, in the GR setting these system is clearly close being one of 18 equations for the 5 components of Φ (the trace is known and later identified with the cosmological constant) plus $18-5$ components of the B field constrained by the metricity condition.

What allows to close the systems are a kind of Bianchi identities. Let us state a Lagrangian that produces the previous equations of motion:

$$S = \int B^i \wedge F^i - \tfrac{1}{2}\Phi^{ij}B^i \wedge B^j, \tag{209}$$

and call $B^i \wedge B^j = h^{ij}$. The Bianchi identities for the connection together with the eq. (208) give:

$$(D\ \Phi^{ij}) \wedge B^j = 0. \tag{210}$$

Taking another derivative of the compatibility equation (205) and using the equations of motion gives:

$$\varepsilon^{ijk}\Phi^{jm}h^{mk} = 0 \tag{211}$$

these crucial equations close the system of eqs. of motion. Multipling the first of them, eq. (210) with the 1-form $\iota_\xi B^i$ for an arbitrary vector field ξ and then using the second, eq. (211), one obtains:

$$(d\Phi^{ij})h^{ij} = 0. \tag{212}$$

This equation implies that the second term in the action (209) is actually a function of the sole $B^i \wedge B^j$ combination. Indeed, defining $V = h^{ij}\Phi^{ij}$:

$$dV = \Phi^{ij} dh^{ij}. \tag{213}$$

The field Φ is not unknown as it can be expressed as the derivative of the "potential" V. The action (209) can be reexpressed as:

$$S = \int B^i \wedge F^i - \tfrac{1}{2} V(B^i \wedge B^j), \tag{214}$$

where V is a gauge invariant function. One can recast this Lagrangian in a way that is closer to the original Plebanski formulation. Indeed, Eq. (212) can be put into a new form splitting the two fields h and Φ into their trace and tracefree part:

$$H^{ij} d\Psi^{ij} = d\Lambda \tag{215}$$

$$h^{ij} = \delta^{ij} + H^{ij} \tag{216}$$

$$\Phi^{ij} = \Psi^{ij} + \frac{\Lambda}{3}\delta^{ij}. \tag{217}$$

Λ is a function, and it turns to be the Legendre transform of the function V. The action can then also be cast into the suggestive form:

$$S_{\text{plebanski}} = \int B^i \wedge F^i - \tfrac{1}{2}\left(\Psi^{ij} - \frac{\Lambda(\Psi)}{3}\delta^{ij}\right) B^i \wedge B^j. \tag{218}$$

Notice that Λ is a function of two complex variables, e.g $\mathrm{Tr}\Psi^2, \mathrm{Tr}\Psi^3$, two independent algebraic invariants of Ψ^{ij} seen as a $3x3$ symmetric traceless matrix on which we can act with $SO(3)$ rotations.

This infinite class of theories are still gravitational and they include GR as a special member. They can still be seen from a metric point of view as pointed out in [32]. A detailed Hamiltonian analysis (see [33]) shows

that these theories also have 2 on-shell degrees of freedom that come from a metric. In this sense they can be regarded as modifications of GR, in a similar way as we have seen in the deformation of YM theories. These theories, as in the DYM case, do not have higher derivatives in the equations of motion. This is a very important feature, given that usually, modifications of gravity do spoil this property leading to instabilities.

4.4 PURE CONNECTION FORMULATION

The class of theories described in the previous section was discovered by Bengtsson in [13] although the author of that paper used a formulation of the theory in which only the connection appeared. This pure connection formulation has been studied intensively in recent years as a new way to describe the gravitational perturbation theory. Ultimately, these theories were the reason behind the study of DYM presented at the beginning of this thesis. In the gravitational case as well, it is conjectured that the theory is closed under renormalization. We will present now the pure connection formulation and its properties and set the stage for some analysis of its interactions and one-loop renormalizability.

4.5 THE ACTION

The action in the PCF depends uniquely on a $SU(2) \sim SO(3)$ connection A^i on a manifold M. No metric is defined, and a function f with properties to be specified below, will select a specific theory out of an infinite class. Let us state the action, first:

$$S[A] = \frac{1}{\mathrm{i}} \int_M f(F^i \wedge F^j). \tag{219}$$

Where, from now on we will denote $X^{ij}\mathrm{vol} = F^i \wedge F^j$, with a choice of volume form. As should be clear, the previous action shares some similarities with the DYM one. The defining function f has the following properties:

- it is holomorphic
- it is gauge invariant, i.e. $f(X) = f(gXg^T)$ for $g \in SU(2)$
- it is of homogeneity degree 1, such that $f(F^i \wedge F^j) = f(X^{ij})(vol)$
- given X^{ij} is a symmetric matrix on which the group $SU(2)$ acts, the function f can be parametrized by 3 algebraic invariants of X.

It will be clear in a moment the relation between this story and the one described in the previous section, for now we can just state that although a metric is not present it is possible to construct one whenever the matrix X is non-degenerate. In this way one could rewrite these theories as metric ones but this is not the purpose of this section.

Let us examine the equations of motion. They are easily found to be:

$$D\frac{\partial f}{\partial X^{ij}}F^j = 0, \tag{220}$$

where D is the covariant exterior derivative wrt the connection A. First of all these are similar to the usual YM equations, i.e. $D * F^i = 0$. In this case we do not have any notion of Hodge duality, yet. This operation is replaced by the contraction with the derivative of the defining function. If we denote $B^i = \frac{\partial f}{\partial X^{ij}}F^j$ we can write (220) in a first order fashion:

$$\begin{aligned} B^i &= \frac{\partial f}{\partial X^{ij}}F^j, \\ DB^i &= 0. \end{aligned} \tag{221}$$

This first order formulation is nothing else that the theory already described in the last section, see eq. (214).

The functional for GR is quite non-linear:

$$= \frac{1}{16\pi G\Lambda}\int (\mathrm{Tr}(\sqrt{X}))^2. \tag{222}$$

As it is clear from this action, this formulation is not possible for $\Lambda = 0$. To briefly see that the last Lagrangian can actually describe GR we can write the equations of motion:

$$\begin{aligned} B^i &= \frac{\mathrm{Tr}(\sqrt{X})}{16\pi G\Lambda}((\sqrt{X})^{-1})^{ij}F^j, \\ DB^i &= 0. \end{aligned} \tag{223}$$

and compute the wedge product between the B^i fields just found:

$$B^i \wedge B^j \propto \delta^{ij}, \tag{224}$$

which is nothing else than the "metricity" condition of Plebanski.

To conclude this section we should stress a couple of important points. First, there are no dimensionful constants in the Lagrangian. The notion of Newton and cosmological constant can be introduced for the GR functional, e.g. deriving it directly from the first order Lagrangian. This definition would make EH and PCF Lagrangian agree when evaluated on a constant curvature background. The expansion around a constant curvature background brings in a dimensionful parameter, which is the curvature of the background. As we will see in a moment the interactions of any single theory are power-counting non-renormalizable.

4.6 LINEARIZATION

To better understand the degrees of freedom involved in this theory we perform a linearization around a simple background. We choose a constant curvature background (deSitter space). This background, being also a solution to Einstein equations can be described in terms of connections. The fact that the background has constant curvature can be stated as:

$$F^i = M_0^2\,\Sigma^i. \tag{225}$$

This background has the peculiar property:

$$F^i \wedge F^j \propto \delta^{ij} \tag{226}$$

This condition defines indeed the instantons (Einstein solution with zero self-dual part of the Weyl tensor, see [34] for a more mathematical oriented discussion). It is quite interesting that any theory of the form (219) shares the same instanton sector, which is another similarity of these theories and DYM. Indeed, all the defining functions collapse into one when evaluated on the identity (226). To see that these connections are actually solutions, just notice that the first derivative of the defining function with respect to the matrix X^{ij} evaluated at the identity is still a multiple of the identity. The equations of motion (220) then just become the Bianchi identities.

Let us compute the linearized Lagrangian around the instanton background. First, we find the second variation of the Lagrangian (219). It reads:

$$\begin{aligned}\delta S^2 = i \int 4\frac{\partial^2 f}{\partial X^{ij}\partial X^{kl}} \left(F^i \wedge Da^j\right) \left(F^k \wedge Da^l\right) \\ +2\frac{\partial f}{\partial X^{ij}} \left(Da^i \wedge Da^j + \varepsilon^{jkl} F^i a^k \wedge a^l\right),\end{aligned} \tag{227}$$

where we denoted a^i_μ the variation of the connection. We notice that with an integration by parts the second line cancel out, due to the fact that the first derivative of f is constant on the chosen background. To understand the second derivative of the defining function one can notice that the homogeneity condition on f can be stated as:

$$\frac{\partial f}{\partial X^{ij}} X^{ij} = 0. \tag{228}$$

Differentiating with respect to X leads to:

$$\frac{\partial^2 f}{\partial X^{ij}\partial X^{kl}} = 0. \tag{229}$$

On $X^{ij} \propto \delta^{ij}$ this implies that the second derivative is proportional to the projector onto the symmetric traceless tensors:

$$P^{ijkl} = \frac{1}{2}\left(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}\right) - \frac{1}{3}\delta^{ij}\delta^{kl}. \tag{230}$$

The second order action linearized on the constant curvature background then reads:

$$S^{(2)} = g_0 \int dx^4 \sqrt{-g} P^{ijkl} \left(\Sigma^{i\mu\nu} D_\mu a^j_\nu\right) \left(\Sigma^{k\sigma\rho} D_\sigma a^l_\rho\right). \tag{231}$$

where all the occurencies of M_0 cancel. To compute the scattering processes we would like to work on a flat background as it is usually done with the Einstein-Hilbert action. This is not possible in this formulation because in the connection formulation a flat background would correspond to $F^i = 0$. If we are interested in the high energy behavior, i.e. energies much higher than the one set by M_0, we can actually keep only the partial derivative part of the covariant ones in the last formula. Indeed, those terms become more important in the UV that the connection terms which are proportional to M_0. This procedure has to be done carefully when dealing with the interaction terms. For an account of this story see, [35].

Let us take a closer look at (231) and compare it with YM and usual GR. The action (219) is symmetric with respect to gauge transformations and diffeomorphism. We will write the infinitesimal transformations of the connection as:

$$\begin{aligned} \delta_\phi A^i_\mu &= D_\mu \phi^i, \\ \delta_\xi A^i_\mu &= \xi^\nu F^i_{\mu\nu}. \end{aligned} \tag{232}$$

The second equation describes an infinitesimal diffeomorphism transfomation corrected by a gauge transformation. Let us show that the linearized action is indeed invariant under the linearized version of (232). The full symmetry can also be easily checked directly. At the linearized level (232) takes the form:

$$\delta_\phi a^i_\mu = \partial_\mu \phi^i, \tag{233}$$

$$\delta_\xi a^i_\mu = \xi^\nu \Sigma^i_{\mu\nu}. \tag{234}$$

The variation of the linearized lagrangian wrt the diffeomorphisms is, leaving aside the unimportant parts:

$$P^{ijkl} \left(\Sigma^{i\mu\nu} \partial_\mu a^j_\nu\right) \left(\Sigma^{k\sigma\rho} \partial_\sigma \delta_\xi a^l_\rho\right) \tag{235}$$

$$= P^{ijkl} \left(\Sigma^{i\mu\nu} \partial_\mu a^j_\nu\right) \left(\Sigma^{k\sigma\rho} \Sigma^l_{\rho\lambda} \partial_\sigma \xi^\lambda\right). \tag{236}$$

With the help of the relation:

$$\Sigma^{i\mu\rho} \Sigma^j_{\rho\nu} = (-\delta^{ij} \eta^\mu{}_\nu + \varepsilon^{ijk} \Sigma^{k\mu}{}_\nu), \tag{237}$$

eq. (235) can be seen to be vanishing. Indeed, P^{ijkl} is the projector onto the symmetric traceless tensors, so both the contraction in the (kl) indices are zero.

The gauge invariance is even simpler to check. The variation of the action is:

$$P^{ijkl}\left(\Sigma^{i\mu\nu}\partial_\mu a^j_\nu\right)\left(\Sigma^{k\sigma\rho}\partial_\sigma\delta^\phi a^l_\rho\right) \tag{238}$$

$$= P^{ijkl}\left(\Sigma^{i\mu\nu}\partial_\mu a^j_\nu\right)\left(\Sigma^{k\sigma\rho}\partial_\sigma\partial_\rho\phi^l\right), \tag{239}$$

which is zero due to the antisymmetry of Σ.

To better appreciate the structure of the linearized action, it is best to use spinor indices for both the internal and spacetime indices. The Lagrangian than takes the very simple form:

$$\left(\partial^{(A}{}_{A'}a^{BCD)A'}\right)^2. \tag{240}$$

comparing it with the chiral version of YM:

$$\left(\partial^{(A}{}_{A'}a^{aB)A'}\right)^2, \tag{241}$$

we see that the two are quite similar. The connection in the YM case is a spinor in the $S_+ \otimes S_-$ representation. In the case of gravity a is a spinor in the $S_+^2 \otimes S_+ \otimes S_-$ representation which decomposes into:

$$S_+^2 \otimes S_+ \otimes S_- \sim S_+^3 \otimes S_- \oplus S_+ \otimes S_-. \tag{242}$$

It is clear that the Lagrangian (240) does not depend on the $S_+ \otimes S_-$ of the connection. Given that the linear diffeomorphisms (232) are just a shift of the connection exactly in this direction, fixing the diffeomorphism invariance amounts to set this component of the connection to zero.

The linearized Lagrangian for both theories are constructed in the same way by means of a Dirac operator. In the case of YM, this operator sends the spin one fields in $S_+ \otimes S_-$ to S_+^2. In the case of gravity, it sends the spin two fields in $S_+^3 \otimes S_-$ to S_+^4. We will see how this story works in more detail in the next chapter and how the knowledge of the symmetries actually fixes the

interactions. A detailed Hamiltonian analysis can be found in [31], which shows that among the 8 components on which the Lagrangian depends (independent components of $S^3_+ \otimes S_-$), there are 4 Lagrange multipliers for 4 constraints which leaves only 2 degrees of freedom.

4.7 INTERACTIONS AND RENORMALIZATION

We want to end this review chapter detailing some features of the interaction vertices of these theories. First it is better to normalize the action (231) such that there is no constant in front of it. This can be achieved by a rescaling of the connection's perturbation $a^i_\mu \to \sqrt{g_0}a^i_\mu$. In this way the interactions, at least at third order, take the schematic form:

$$\mathcal{L}^{(3)} = \frac{g^{(3)}}{(M_0)^2}(\partial a)^3 + \frac{1}{\sqrt{g_0}(M_0)^2}(\partial a)^3 + \frac{1}{\sqrt{g_0}}(\partial a)a^2. \tag{243}$$

A few comments are in order. First, we have understood the specific contractions inside these vertices, but we will write them explicitly in the following chapter. These vertices do not produce higher order derivatives in the equations of motion, only non-linearities in the derivative of the connection. This is a general feature of these theories as we have already pointed out. The first two terms are actually of the same type. The key difference lies in the coefficient $g^{(3)}$. It is actually a combination of an entirely new coefficient and the one already encountered g_0. This specific combination turns out to be vanishing for the functional corresponding to GR (222). This is a general pattern at all orders. It turns out that these vertices are the only parity-violating ones. As such, GR lies in the class of theories (219) as the only parity-preserving member[1]. The last term is the usual YM 3-vertex. When considering the scattering processes for these theories one has to be careful in taking the $M_0 \to 0$ limit to recover flat-background scattering of gravitons, for the vertices clearly blow up in this limit. Such a limit can be taken further in the calculation (i.e. after physical quantities have been computed).

1 This is another feature that DYM shares with these theories.

In so doing it has been shown in [36] that the YM vertex plays no role in these calculations and for the case of GR the other vertices reproduce the usual scattering processes as calculated from the Einstein-Hilbert action, expanded on a flat background. These theories have shown to be much more tractable than the usual EH action in this context. The structure and the number of vertices in the pure connection formalism are much simpler to deal with.

The general structure at higher order follows a similar pattern. The n-vertices contain at most n derivatives in the form $(\partial a)^n$. The subsequent terms change this by even numbers, e.g. $(\partial a)^{n-2k} a^{2k}$ with $k = 1, \ldots, \frac{n}{2}$.

To get an idea about how the renormalization could work for these theories we should make a couple of preliminary points. First, as it is clear from (243) the interactions contain inverse-mass coupling constant signaling that all the theories are as power-counting non-renormalizable as usual GR. As already pointed out many times the hope is that those theories (as in the case of DYM) contain enough vertices, as a class, that the counterterms that one has to introduce to renormalize the theory are already present in the class itself.

It is quite an involved process to translate the vertices contraction in the connection formulation to the metric one, but it can be done (at least on shell). For example the terms $(\partial a)^3$ are translated (schematically) into the form:

$$(\partial^2 h) hh + (\partial^2 h)^3, \tag{244}$$

where h is the metric perturbation. At each level, terms of the generic form $(Riemann)^n \sim (\partial^2 h)^n$ get produced. These terms are exactly of the type that actually arise in the quantum perturbative analysis of GR (at two loops, for example). This gives hope that the quantum counterterms needed are already present in the Lagrangian. Anyway, it is the general form of (219) that has to be preserved. For example, this Lagrangian does not contain interaction terms that would lead to higher than second order field equations,

even if the structure of the symmetries would allow that. In this regard, we shall see that the form of the vertices for those theories should be reconstructible exactly by the two key requirement of being invariant under gauge and diffeomorphisms and produce second order-field equations.

The renormalization process is then the one described by Weinberg's asymptotic safety scenario.

The beta-function of the entire class is of the same form as the Lagrangian itself, as we have seen e.g. at one loop in the DYM case. Now, we would have the flow:

$$\frac{\partial f(F \wedge F)}{\partial \ln \mu} = \beta(F \wedge F), \tag{245}$$

where $\beta(F \wedge F)$ should be then a gauge-invariant homogeneity degree one function of a $3x3$ symmetric matrix. As such the renormalization flow will be one in the space of complex functions of 3 complex variables.

5

RECONSTRUCTION OF THE PURE CONNECTION FORMALISM VERTICES

In this chapter we will take a more constructive approach and we will not suppose any specific Lagrangian for DIGT. Instead, we will see how imposing the symmetries one could in principle reconstruct the full theory. These procedures are based on a perturbative application of Nöther theorems. Starting from simple inputs like the basic fields and the order of the equations of motion (generally second) one fixes the quadratic part of the action invariant under some (local) transformations. One can then looks for a consistent way to couple the field with itself or other fields in a manner consistent with the symmetries. This requires also to find the right modification of the symmetry at higher orders in such a way that the algebra of those symmetries closes (see Wald's paper on this subject applied to spin 2 fields [37]). These procedures also can serve as a perturbative proof of uniqueness of the theories they are applied to. In the case of DIGT we know that, strictly no uniqueness can be achieved but what we learned is that just imposing the symmetries, one can reconstruct the full cubic vertex and no other terms, apart from the ones already contained in the DIGT formalism, appear without the need of the Lagrangian.

5.1 GENERAL PROCEDURE

We now detail the procedure that we are going to use for the rest of the chapter and give a few simple examples.

We want to write an action for a general field ψ_I, I being a placeholder for all its indices. We suppose that the action is symmetric under some local transformation. We expand the field symmetry transformations around some fixed configuration:

$$\delta\psi_I = \delta^{(0)}\psi_I + \delta^{(1)}\psi_I + \delta^{(2)}\psi_I \tag{246}$$

under both diffeomorphisms and gauge. In general the idea would be to consistently extend a theory starting from a quadratic action $S^{(2)}$ that is invariant under the zeroth order symmetries, i.e. $\delta^{(0)}S^{(2)} = 0$, to higher orders in the field. The consistency equations then read:

$$\begin{aligned} \delta^{(1)}S^{(2)} + \delta^{(0)}S^{(3)} &= 0, \\ \delta^{(2)}S^{(2)} + \delta^{(1)}S^{(3)} + \delta^{(0)}S^{(4)} &= 0, \end{aligned} \tag{247}$$

and so on, up to total derivatives. We then, at each order write an ansatz of the Lagrangian with all possible full contractions of the field and extract from (247) a system in the coefficients of the ansatz. The same procedure could be performed at the level of the equations of motion. Indeed, it is known for a local transformation of the fields:

$$\delta\psi_I = \mathcal{R}_I{}^J\,\lambda_J, \tag{248}$$

where λ_I is a local parameter, one can extract some off-shell relations between the equations of motion called Nöther identities:

$$0 = \delta_{\mathcal{R}}S = \int \mathcal{F}_I^{(1)}\delta\psi_I = \int \mathcal{F}_I^{(1)}\delta\mathcal{R}_I{}^J\lambda_J \quad \rightarrow \quad \mathcal{R}_I{}^J\mathcal{F}_I^{(1)} = 0 \tag{249}$$

where $\mathcal{F}_I^{(1)} = \frac{\partial\mathcal{L}}{\partial\psi_I} = 0$ are the equations of motion. The iterative procedure described above in eq. (247) can then be applied to the Nöther identities as well.

5.1.1 *Examples*

A very simple example of this procedure is given by YM. Indeed, one can form the most general second order contractions (up to total derivatives):

$$\alpha\partial_\mu A^a_\nu \partial^\mu A^{a\nu} + \beta\partial_\mu A^a_\nu \partial^\nu A^{a\mu} + \gamma A^{a\mu} A^a_\mu. \tag{250}$$

Imposing the zeroth-order gauge transformations $\delta A^a_\mu = \partial_\mu \phi^a$, one easily gets: $\alpha = -\beta, \gamma = 0$ which gives the usual linearized YM Lagrangian. The first order gauge transformations add the term $\delta^{(1)} A^a_\mu = f^{abc} A^b_\mu \phi^c$, which can be used to check that the constant appearing in the ansatz of the possible cubic term:

$$f^{abc} A^a_\mu A^b_\nu \partial^\mu A^{c\nu}, \tag{251}$$

is related to the overall constant α of the second order Lagrangian. Repeating the procedure one can find the quartic term in the same way, also one could add a set of invariants with a higher number of derivatives and check that they reorganize in the F^n operators already seen in the initial discussion of DYM.

The case of gauge and diffeomorphism invariant theories is more involved. As we will see in a moment.

5.2 LINEARIZED ACTION

In the gravity theories that we want to reconstruct we select a specific gauge group, namely $SO(3)$. Moreover, it is better to expand the connections transformations around a non-trivial instanton background, for which the curvature takes the form $F^i_{\mu\nu} \propto \Sigma^i_{\mu\nu}$ as we have already pointed out. This object can be now used, along with the background metric and the gauge group structure constants to make the contractions that will form the ansatz for the Lagrangian. This, although possible, leads to a proliferation of different

types of contractions related by several identities. It is better to have a formulation of the problem that spits the contractions into irreducible terms. This is achieved using the spinor formalism once again. As discussed before and in appendix (A) the connection can be seen as a spinor in $S^3_+ \otimes S_+ \otimes S_-$, namely $a^{AB}{}_{MM'}$. Indeed, one can convert the $SO(3)$ indices into a pair of symmetric spinor ones. In particular, in appendix (A) we show that the self dual forms $\Sigma^i_{\mu\nu}$ get transformed into the spinor:

$$\Sigma^{(AB)}_{MM'NN'} = \varepsilon^{(A}{}_M \varepsilon^{B)}{}_N \varepsilon_{M'N'}. \tag{252}$$

We want to obtain the most general quadratic action that is invariant under the zeroth-order gauge and diffeomorphism invariance:

$$\begin{aligned} \delta^{(0)}_\phi a^{AB}{}_{MM'} &= d_{MM'}\phi^{AB}, & (253)\\ \delta^{(0)}_\xi a^{AB}{}_{MM'} &= \Lambda\xi^{NN'}\varepsilon^{(A}{}_M\varepsilon^{B)}{}_N\varepsilon_{M'N'} \\ &= \Lambda\xi^{(A}{}_{M'}\,\varepsilon^{B)}{}_M & (254)\end{aligned}$$

where the covariant derivative is intended to be with respect to the background instanton connection. We will understand everywhere in the following that the connection is a symmetric spinor in the first two unprimed indices.

We start making a general ansatz for this action including all possible contractions of spinor indices of quadratic pieces. The only tensors that we can use to contract indices are the antisymmetric spinors ε^{AB}, $\varepsilon^{A'B'}$. The possible quadratic pieces are:

$$d_{MM'}a^{AB}{}_{NN'}d_{RR'}a^{CD}{}_{SS'}, \qquad a^{AB}{}_{MM'}a^{CD}{}_{NN'}. \tag{255}$$

5.2.1 *Primed indices contractions*

It is better to focus on primed indices, first. For the derivative piece in eq. (255), the possible primed contractions are three, but only 2 of them are linearly independent due to the Shouten identity:

$$\varepsilon^{M'R'}\varepsilon^{N'S'} = \varepsilon^{M'N'}\varepsilon^{R'S'} - \varepsilon^{M'S'}\varepsilon^{N'R'}. \tag{256}$$

We can always perform an integration by parts, and the two contractions will take the form:

$$a_{M'}d^{M'}d_{R'}a^{R'}, \qquad a_{M'}d_{R'}d^{M'}a^{R'}, \tag{257}$$

where for the moment we understand all unprimed indices contractions. These two pieces are related to each other by a commutator of the two derivatives. From now on we will denote all possible contractions of unprimed indices with $(\dots)$. The commutator can be expanded as in eq. (454), i.e.

$$a^{(\dots)M'}\left(\varepsilon_{R'}{}^{B'}\Box_{(\dots)}+\varepsilon_{(\dots)}\Box_{R'}{}^{M'}\right)a^{(\dots)R'}. \tag{258}$$

Let us consider the two pieces separately. The curvature spinors and their relations to the Einstein equations are briefly summarized in the appendix (B). The derivatives are with respect to the background instanton solution, so in our case $\Psi=0,\Phi=\bar{\Phi}=0$, (see Appendix (B)) and we make use of (452), (454) to prove that

$$\begin{aligned}\varepsilon_{R'}{}^{M'}\Box_{(\dots)}a^{R'} &\propto \Lambda\, a^{(\dots)M'}\\ \Box_{R'}{}^{M'}a^{R'} &\propto \Lambda\, a^{(\dots)M'},\end{aligned} \tag{259}$$

where in the second equation, we used the fact that the Weyl curvature spinor $\bar{\Psi}$ is totally symmetric. In the end, we can put this term in the form:

$$\Lambda\left(a^{(\dots)}{}_{B'}a_{(\dots)}{}^{B'}\right), \tag{260}$$

that we already included in the ansatz (see eq. (255)).

This allows us to reduce the ansatz for the primed contractions to:

$$d_{M'}a^{M'}d_{R'}a^{R'}, \qquad a_{B'}a^{B'} \tag{261}$$

where, as before, we understood all possible contractions of unprimed indices.

5.2.2 *Unprimed indices contractions*

Instead of just listing all possible contractions we try another route, namely we decompose the connection and its derivatives into irriducible (symmetric spinors) pieces. Remember that the primed contraction is fixed. We start with the decomposition:

$$\begin{aligned} d_{MM'} a^{AB}{}_{NN'} &\in S^2_+ \otimes S_+ \otimes S_+ \\ &\sim S^4_+ \oplus S^2_+ \oplus S^2_+ \oplus \text{trivial}, \qquad (262) \\ a^{ABC} &\in S^2_+ \otimes S_+ \\ &\sim S^3_+ \oplus S_+. \qquad (263) \end{aligned}$$

The six irriducible representations are:

$$\begin{aligned} a^{(ABC)} & \\ a^{AE}{}_E &\equiv \eta^A, \\ d^{(A}\, a^{BCD)} &\equiv (da)^{ABCD}, \\ d_E\, a^{(EAB)} &\equiv (da)^{AB}, \\ d^{(A}\, a^{B)E}{}_E &= d^{(A}\, \eta^{B)} \equiv (d\eta)^{AB}, \\ d_E\, a^{EF}{}_F &= d_E\, \eta^E \equiv (d\eta) \qquad (264) \end{aligned}$$

where we have introduced the spinor $\eta \in S_+ \otimes S_-$ and we have dropped the primed indices for brevity. In the following we will use this notation to understand that the contraction of the indices that do not appear explicitly is the standard one. The ones that do appear are taken to be totally symmetric.

Now we have all we need to write an ansatz for the linearized action:

$$\begin{aligned} \mathcal{L}_{ansatz} &= C_1\, d^{(A}\, a^{BCD)}\, d_A\, a_{BCD} + C_2 d_E\, a^{(EAB)}\, d^F\, a_{(FAB)} + C_3\, d^{(A}\eta^{B)}\, d_A\eta_B \\ &+ C_4\, d_E\, a^{(EAB)}\, d_A\eta_B + C_5\, d_E\, \eta^E\, d_F\, \eta^F + C_6\, a^{(ABC)}\, a_{ABC} \\ &+ C_7\, \eta_F\, \eta^F. \qquad (265) \end{aligned}$$

We will check the transformations of all the terms in the Lagrangian under zeroth order diffeos and gauge transformations and we will impose that the variation is zero as a condition on the coefficients of the ansatz.

We start with the diffeomorphisms. The variation of the connection comes from the η part only and it is just a shift, the variation of the fully symmetrized part vanishes, i.e. :

$$\begin{aligned} \delta_\xi \, a^{(ABC)} &= 0, \\ \delta_\xi \, \eta^A{}_{A'} &= -3\Lambda \, \xi^A{}_{A'}. \end{aligned} \tag{266}$$

The gauge transformations at this level also preserve the irreducible character of the spinors:

$$\begin{aligned} \delta^{(0)}_\phi a^{(AB}{}_{C)C'} &= d_{(CC'}\phi^{AB)}, \\ \delta^{(0)}_\phi a^{AC}{}_{CC'} = \delta^{(0)}_\phi \eta^A{}_{C'} &= d_{CC'}\phi^{CA}, \\ \delta^{(0)}_\phi (da)_{ABCD} &= d_{AD'} \, d_D{}^{D'} \phi_{BC} \\ &= \Box_{(AD}\phi_{BC)} \qquad = 0, \\ \delta^{(0)}_\phi (da)_{AB} &= \frac{2}{3} d^2 \phi_{AB} - \frac{8}{3}\Lambda \, \phi_{AB}, \\ \delta^{(0)}_\phi (d\eta)_{AB} &= -\frac{1}{2} d^2 \phi_{AB} - 4\Lambda \, \phi_{AB} \\ \delta^{(0)}_\phi (d\eta) &= 0. \end{aligned} \tag{267}$$

In the last formulae we always used the instanton background condition. From eq. (266), we know that C_1, C_2, C_6 are invariant under diffeos by themselves. The other variations are:

$$\delta_\xi \left(C_7 \, \eta_F \, \eta^F \right) = -6\Lambda \, C_7 \, \eta_F \, \xi^F, \tag{268}$$

$$\delta_\xi \left(C_5 \, d_E \, \eta^E \, d_F \, \eta^F \right) = -6\Lambda \, C_5 \, d_E \, \eta^E \, d_F \, \xi^F, \tag{269}$$

$$\delta_\xi \left(C_3 \, d_{(A}\eta_{B)} \, d^A \eta^B \right) = -6\Lambda \, C_3 \, d_{(A}\eta_{B)} \, d^A \xi^B, \tag{270}$$

$$\delta_\xi \left(C_4 \, d_E \, a^{(E}{}_{AB)} \, d^A \eta^B \right) = -3\Lambda \, C_4 \, d_E \, a^{(E}{}_{AB)} \, d^A \xi^B. \tag{271}$$

Due to the fact that we already used integration by parts to fix the primed contraction, these four terms are all different and independent and cannot be combined. This rules out the terms C_3, C_4, C_5, C_7.

The gauge transformations of the remaining terms can be simply evaluated using eqs. (267). We start with C_1:

$$\delta_\phi \left(C_1 \, d^{(A} \, a^{BCD)} \, d_A \, a_{BCD} \right) = 0, \tag{272}$$

The term C_1 then, is invariant by itself under both symmetries and can be included in the Lagrangian. Now we compute the variations of C_6, C_2:

$$\delta_\phi \left(C_6 \, a_{(ABC)} \, a^{ABC} \right) = 2C_6 \, a^{(ABC)}{}_{C'} \, d_{(C}{}^{C'} \phi_{AB}, \tag{273}$$

$$\delta_\phi \left(C_2 \, (da)_{AB} \, (da)^{AB} \right) = C_2 (da)_{AB} \left(\frac{4}{3} d_{FF'} d^{FF'} \phi^{AB} - \frac{16}{3} \Lambda \phi^{AB} \right) \tag{274}$$

We already excluded the possibility of integrating by parts, a move that could combine (273) with the second term in (274). Anyway, the D'Alambertian piece cannot be combined with anything left. Ultimately, we are left with just the fully symmetrized invariant:

$$\mathcal{L} = C_1 \, d_{(A} \, a_{BCD)} \, d^A \, a^{BCD} \, . \tag{275}$$

5.3 CUBIC INTERACTIONS

5.3.1 *General idea of the reduction of the ansatz*

Before digging into the Noether procedure to establish a unique cubic level, it would be better if we state the equations for the first order gauge and diffeo transformations applied to the connection. First of all the gauge transformations involves a product in the Lie algebra indices (through the structure constants). Our conventions for these product is the following:

$$X, Y \in S^2_+ \qquad \rightarrow \qquad \big[X, Y\big]^{AB} = 2 X^{(A}{}_E Y^{B)E}. \tag{276}$$

The gauge and diffeo transformation at first order in the connection are:

$$\begin{aligned} \delta^{(1)}_\phi a^{AB}{}_{CC'} &= \big[a_{CC'}, \phi\big]^{AB} \\ &= 2 \, a^{(A}{}_{ECC'} \phi^{EB)} \end{aligned} \tag{277}$$

$$\begin{aligned} \delta^{(1)}_\xi a^{AB}{}_{CC'} &= 2 \, d_{[CC'} a^{AB}{}_{DD']} \, \xi^{DD'} \\ &= \left(\varepsilon_{CD} \, d_{E(C'} \, a^{ABE}{}_{D')} + \varepsilon_{C'D'} \, d_{(CE'} \, a^{AB}{}_{D)}{}^{E'} \right) \xi^{DD'} \\ &= - \left(d_{E(C'} \, a^{ABE}{}_{D')} \, \xi_C{}^{D'} + d_{(CE'} \, a^{AB}{}_{D)}{}^{E'} \, \xi^D{}_{C'} \right). \end{aligned} \tag{278}$$

These transformations applied to the quadratic Lagrangian yield:

$$\delta^{(1)}_{\phi} S^{(2)} = 4\, C_1\, d_{AF'} \left(a_{BEC}{}^{F'} \phi^{E}{}_{D} \right) (da)^{ABCD}, \tag{279}$$

$$\delta^{(1)}_{\xi} S^{(2)} = -2\, C_1\, d_{AF'} \left(d_{E}{}^{(F'} a_{BC}{}^{E}{}_{D')}\, \xi_{D}{}^{D'} + d_{(DE'} a_{|BC|E)}{}^{E'}\, \xi^{EF'} \right) (da)^{ABCD} \tag{280}$$

Now, we will use the formulae for the irreducible decompositions of spinors:

$$a_{ABCC'} = -\frac{1}{3} \varepsilon_{BC} \eta_{AC'} - \frac{1}{3} \varepsilon_{AC} \eta_{BC'} + a_{(ABC)C'}, \tag{281}$$

$$\begin{aligned} \chi_{A(BCD)} = & -\frac{1}{4} \chi^{F}{}_{(CDF)} \varepsilon_{AB} - \frac{1}{4} \chi^{F}{}_{(BDF)} \varepsilon_{AC} \\ & -\frac{1}{4}\, \chi^{F}{}_{(BCF)} \varepsilon_{AD} + \chi_{(ABCD)}, \end{aligned} \tag{282}$$

to rewrite eqs. (279), (280) in terms of only irreducible parts of the connection:

$$\begin{aligned} \delta^{(1)}_{\phi} S^{(2)} = & - C_1\, (da)_{AB}\, \phi_{CD}\ (da)^{ABCD} \\ & - \frac{4}{3} C_1\ (d\eta)_{AB} \phi_{CD}\, (da)^{ABCD} \\ & - 4 C_1\, a_{(ABE)F'}\, d_{C}{}^{F'} \phi^{E}{}_{D}\, (da)^{ABCD} \\ & - \frac{4}{3} C_1\, a_{(ABC)F'}\, d_{F}{}^{F'} \phi^{F}{}_{D}\, (da)^{ABCD} \\ & + \frac{4}{3} C_1\, d_{AF'} \phi_{BC}\, \eta_{D}{}^{F'}\, (da)^{ABCD}. \end{aligned} \tag{283}$$

The part of (280) with all the derivatives evenly distributed can be written in terms of irreducibles:

$$\begin{aligned} \delta^{(1)}_{\xi} S^{(2)} = & - C_1\, (da)_{ABCD}\, (d\xi)\, (da)^{ABCD} \\ & - \frac{1}{2} C_1 (da)_{AB} (d\xi)_{CD}\, (da)^{ABCD} \\ & - \frac{2}{3} C_1 (d\eta)_{AB}\, (d\xi)_{CD}\, (da)^{ABCD} \\ & + 2 C_1 (da)_{ABE'F'} (d\xi)_{CD}{}^{E'F'}\, (da)^{ABCD} \\ & - \frac{4}{3} C_1 (d\eta)_{ABE'F'} (d\xi)_{CD}{}^{E'F'}\, (da)^{ABCD}. \end{aligned} \tag{284}$$

To prove these equations, one just needs to make repeated use of eqs. (281), (282).

The part of (280) with an uneven distribution of derivatives can be put, after some manipulation of the indices, into the form:

$$C_1 \Big(4\Lambda\, a_{ABCE'}\, \xi_{D}{}^{E'} (da)^{ABCD} \Big). \tag{285}$$

Schematically, the cubic terms will all have the form:

$$(da)^3 \qquad \text{or} \qquad (da\ a\ a), \tag{286}$$

and as in the previous analysis we will try to form all possible contractions in the primed and unprimed indices. The next step would be to match the zeroth order gauge and diffeo transformations (266), (267) of these contractions with similar terms coming from the transformation of the quadratic action under first order gauge and diffeo variations (277), (278).

$$\delta^{(1)}_{\phi} S^{(2)} \ \sim\ \delta^{(0)}_{\phi} (da)^3 \tag{287}$$

$$\delta^{(1)}_{\phi} S^{(2)} \ \sim\ \delta^{(0)}_{\phi} (da\ a\ a) \tag{288}$$

$$\delta^{(1)}_{\xi} S^{(2)} \ \sim\ \delta^{(0)}_{\xi} (da)^3 \tag{289}$$

$$\delta^{(1)}_{\xi} S^{(2)} \ \sim\ \delta^{(0)}_{\xi} (da\ a\ a) \tag{290}$$

5.3.2 $(da\ a^2)$ *interactions without* η *dependence*

The contractions of these interactions terms are in general of 6 types. There are three ways to contract the primed indices and 2 ways to contract the unprimed ones. This is easy to see because the (da)-type terms that have no dependence on η are just two, the S^4_a and S^2_a irreps. Not all of them are independent (or non-vanishing). A good representation of the independent ones is given by the following two terms:

$$Y_1 a_{(ABE)E'} a^{(E}{}_{CD)}{}^{E'}\ (da)^{ABCD}, \tag{291}$$

$$Y_2\ a_{(ABC)E'}\ a^{(BC}{}_{D)F'}\ d_E{}^{(E'} a^{(EAD)F')}. \tag{292}$$

Using integration by parts one can prove that the two last terms are related by a total derivative. So only the Y_1 term should be included in the ansatz.

The Y_1 interaction is invariant under zeroth order diffeos by itself, being independent from η. Let's work out the gauge transformations of this term, instead:

$$\begin{aligned} \delta^{(0)}_{\phi}\left(Y_1 a_{(ABE)E'} a^{(E}{}_{CD)}{}^{E'}\ (da)^{ABCD}\right) &= 2Y_1\ a_{(ABE)E'}\ \delta^{(0)}_{\phi}\left(a^{(E}{}_{CD)}{}^{E'}\right)\ (da)^{ABCD} \\ &= 2Y_1\ a_{(ABE)E'}\ d_{(C}{}^{E'} \phi_{D)}{}^{E}\ (da)^{ABCD} \end{aligned} \tag{293}$$

This combines with the third term in (283). The cancellation imposes:

$$C_1 = -\frac{1}{2}\gamma_2 \tag{294}$$

5.3.3 $(da\, a^2)$ *interactions with* η *dependence*

This terms have 4 primed indices. Following the strategy of the last section, there are only two independent contractions, the first one contains a contraction of these indices in (da) and the second which contains the symmetrization in (da). The possible unprimed contractions are 8. So we end up with 16 possible contractions. We list the unprimed contractions understanding that for each there are two primed ones, that we will denote with a label $(A)S$ for the (anti-)symmetric part:

$$\begin{aligned}
&P_1^{A/S}\ S_a^4 \otimes S_a^3 \otimes S_\eta \\
&P_2^{A/S}\ S_a^2 \otimes S_a^3 \otimes S_\eta \\
&P_3^{A/S}\ S_a^2 \otimes S_\eta \otimes S_\eta \\
&P_4^{A/S}\ S_a^2 \otimes S_a^3 \otimes S_a^3 \\
&P_5^{A/S}\ S_a^2 \otimes S_a^3 \otimes S_\eta \\
&P_6^{A/S}\ S_a^2 \otimes S_\eta \otimes S_\eta \\
&P_7^{A/S}\ S_a^0 \otimes S_a^3 \otimes S_a^3 \\
&P_8^{A/S}\ S_a^0 \otimes S_\eta \otimes S_\eta
\end{aligned}$$

It is not hard (but quite tedious) to show that some of them are related by integration by parts. In particular these integrations by parts fix the position of the derivative once and for all. The relations are:

$$\begin{aligned}
P_2^A &\sim P_1^A, P_1^S, P_2^S, P_4^S, P_7^A, \\
P_8^A &\sim P_6^S, \\
P_3^S &\sim P_5^A, P_5^S, \\
P_3^A &= P_4^A = P_7^S = P_8^S = 0.
\end{aligned} \tag{295}$$

The zeroth order gauge and diffeo transformations do not alter the irreps split. So it is not hard to see that the only term that can be combined with (283) is P_1^A.

Now, we compute the gauge and diffeo transformations of P_1^A and impose that they match some of the terms in (283), (284),

$$\begin{aligned}\delta_\phi^{(0)}\left(P_1^A\ a_{ABCF'}\ \eta_D{}^{F'}(da)^{ABCD}\right) &= P_1^A\ d_{CF'}\phi_{AB}\ \eta_D{}^{F'}(da)^{ABCD} \\ &+ P_1^A\ a_{ABCF'}\ d_F{}^{F'}\phi_D{}^F\ (da)^{ABCD}. \quad (296)\\ \delta_\xi^{(0)}\left(P_1^A\ a_{ABCF'}\ \eta_D{}^{F'}(da)^{ABCD}\right) &= -3\Lambda\ P_1^A\ a_{ABCF'}\ \xi_D{}^{F'}(da)^{ABCD} \quad (297)\end{aligned}$$

The gauge and diffeo transformation impose exactly the same relation between the coefficients of this vertex and the quadratic Lagrangian, indeed one can compare the gauge variations (296) with the last two terms in (283), that yields:

$$C_1 = \frac{3}{4}P_1^A, \tag{298}$$

the same relation is enforced by comapring the diffeo variation (297) with the last term in (285).

5.3.4 $(da)^3$ *interactions, unprimed indices contraction*

To restrict the number of possible independent primed contraction, we seek for a basis of them, modulo Shouten identities, as we have done in the quadratic Lagrangian case. There are 15 possible contractions, and 15 possible Shouten identities. Not all of them are independent. In the end the basis is composed by only 5 terms, a convenient choice is:

$$d_{A'}a^{A'}d_{B'}a^{B'}d_{C'}a^{C'} \tag{299}$$

$$d_{A'}a^{A'}d_{B'}a_{C'}d^{B'}a^{C'} \tag{300}$$

$$d_{A'}a_{C'}d_{B'}a^{B'}d^{A'}a^{C'} \tag{301}$$

$$d_{A'}a_{B'}d^{A'}a^{B'}d_{C'}a^{C'} \tag{302}$$

$$d_{A'}a_{B'}d_{C'}a^{A'}d^{B'}a^{C'} \tag{303}$$

All the linear combination of unprimed contractions of any of the three ones with one canonical primed contraction $d_{A'}a^{A'}$ would yield the same expression. So they are not really independent once one considers the unprimed contraction, too. Also the last one will not be present in our discussion because it cannot be combined in any possible way given that that any variation coming from the quadratic Lagrangian carries at least one $d_{A'}a^{A'}$ contraction.

5.3.5 $(da)^3$ *interactions,* η *- independent terms*

We will focus now on the first two types of primed contractions that we will denote "AS" and "S", because they correspond, resp., to the totally (anti-) symmetric part in all (da).

If we consider just the S^4_a and S^2_a irreps there are 8 possible unprimed contractions for each primed one. Of course not all of them are independent or non vanishing. The full list is:

$$\begin{aligned} &S^4_a \otimes S^4_a \otimes S^4_a, \quad S^4_a \otimes S^2_a \otimes S^2_a, \\ &S^4_a \otimes S^4_a \otimes S^2_a, \quad S^2_a \otimes S^4_a \otimes S^2_a \\ &S^4_a \otimes S^2_a \otimes S^4_a, \quad S^2_a \otimes S^2_a \otimes S^4_a \\ &S^2_a \otimes S^4_a \otimes S^4_a, \quad S^2_a \otimes S^2_a \otimes S^2_a. \end{aligned} \tag{304}$$

The independent and non-vanishing ones are:

$$\begin{aligned} \text{For AS} \quad &\begin{cases} K_1\ S^4_a \otimes S^4_a \otimes S^4_a, & K_1\ (da)_{ABCD}(da)^{AB}{}_{EF}(da)^{CDEF} \\ K_2\ S^4_a \otimes S^2_a \otimes S^2_a. & K_2\ (da)_{ABCD}(da)^{AB}(da)^{CD} \end{cases} \\ \text{For S} \quad &\begin{cases} K_3\ S^4_a \otimes S^4_a \otimes S^4_a, & K_3\ (da)_{ABCD}(da)^{AB}{}_{EFE'F'}(da)^{CDEFE'F'} \\ K_4\ S^4_a \otimes S^4_a \otimes S^2_a, & K_4\ (da)_{ABCD}(da)^{ABC}{}_{EE'F'}(da)^{DEE'F'} \\ K_5\ S^4_a \otimes S^2_a \otimes S^2_a, & K_5\ (da)_{ABCD}(da)^{AB}{}_{E'F'}(da)^{CDE'F'} \\ K_6\ S^2_a \otimes S^4_a \otimes S^2 & K_6\ (da)_{AB}(da)^{AB}{}_{CDE'F'}(da)^{CDE'F'}. \end{cases} \end{aligned} \tag{305}$$

5.3.6 $(da)^3$ *interactions,* η *- dependent terms*

Now we consider also the other two irreps of unprimed indices of (da), namely S^2_η and S^0_η. Of course also in this case we focus on the AS and S primed contractions. Following the same analysis of the previous section the non-vanishing and independent ones are:

$$\text{For AS} \quad \begin{cases} L_1\ S^4_a \otimes S^2_a \otimes S^2_\eta, & L_1\ (da)_{ABCD}(da)^{AB}(d\eta)^{CD} \\ L_2\ S^4_a \otimes S^2_\eta \otimes S^2_\eta, & L_2\ (da)_{ABCD}(d\eta)^{AB}(d\eta)^{CD}. \end{cases}$$

$$\text{For S} \quad \begin{cases} L_3\ S^4_a \otimes S^4_a \otimes S^2_\eta, & L_3\ (da)_{ABCD}(da)^{ABC}{}_{EE'F'}(d\eta)^{DEE'F'} \\ L_4\ S^4_a \otimes S^2_a \otimes S^2_\eta, & L_4\ (da)_{ABCD}(da)^{AB}{}_{E'F'}(d\eta)^{CDE'F'} \\ L_5\ S^4_a \otimes S^2_\eta \otimes S^2_\eta, & L_5\ (da)_{ABCD}(d\eta)^{AB}{}_{E'F'}(d\eta)^{CDE'F'} \\ L_6\ S^2_a \otimes S^4_a \otimes S^2_\eta, & L_6\ (da)_{AB}(da)^{AB}{}_{CDE'F'}(d\eta)^{CDE'F'} \\ L_7\ S^2_\eta \otimes S^4_a \otimes S^2_a, & L_7\ (d\eta)_{AB}(da)^{AB}{}_{CDE'F'}(da)^{CDE'F'} \\ L_8\ S^2_\eta \otimes S^4_a \otimes S^2_\eta, & L_8\ (d\eta)_{AB}(da)^{AB}{}_{CDE'F'}(d\eta)^{CDE'F'} \\ L_9\ S^2_a \otimes S^2_\eta \otimes S^2_\eta, & L_9\ (da)_{AB}(d\eta)^{A}{}_{CE'F'}(d\eta)^{BCE'F'} \\ L_{10}\ S^2_\eta \otimes S^2_\eta \otimes S^2_\eta. & L_{10}\ (d\eta)_{AB}(d\eta)^{AB}{}_{CDE'F'}(d\eta)^{CDE'F'} \end{cases} \tag{306}$$

The last list is for the terms including some occurrences of the trace part $(d\eta) = d_{AA'}\eta^{AA'}$:

$$\text{For AS} \quad \begin{cases} G_1\ S^4_a \otimes S^4_a \otimes S^0_\eta, & G_1\ (da)_{ABCD}(da)^{ABCD}(d\eta), \\ G_2\ S^2_a \otimes S^2_a \otimes S^0_\eta, & G_2\ (da)_{AB}(da)^{CD}(d\eta), \\ G_3\ S^2_a \otimes S^2_\eta \otimes S^0_\eta, & G_3\ (da)_{AB}(d\eta)^{CD}(d\eta), \\ G_4\ S^2_\eta \otimes S^2_\eta \otimes S^0_\eta, & G_4\ (d\eta)_{AB}(d\eta)^{CD}(d\eta), \\ G_5\ S^0_\eta \otimes S^0_\eta \otimes S^0_\eta, & G_3\ (d\eta)(d\eta)(d\eta). \end{cases} \tag{307}$$

(308)

5.3.7 *Gauge invariance check, $(da)^3$ terms*

K_1 is invariant by itself under all the symmetries. It has to be included as it is and it brings in a new coupling constant. A new (only one) coupling constant should arise at each order. GR arises as the only and unique theory that respect also parity invariance. Indeed the K_1 term spoils this discrete symmetry.

It is quite easy to see that K_6 has no chances to survive. Indeed it is diffeo invariant but its gauge transformation can be matched by terms $L_6 - L_{10}$. The diffeo transformations of these terms cannot to be combined either between themselves nor with diffeos of the quadratic action. Also, the remaining two, i.e. K_3, K_4 , have to be dropped, because as we pointed out before gauge and diffeos do not modify the structure of the irreps split. We focus now on the terms K_2 and K_5, the only ones that, at the end of the day, will survive.

$$\begin{aligned}
&\delta_\phi\ K_2\ (da)_{ABCD}(da)^{AB}(da)^{CD} \\
&= 2(da)_{ABCD}(da)^{AB}\delta_\phi\ (da)^{CD} \\
&= 2\ K_2\ (da)_{ABCD}(da)^{AB}\ d_{FC'}d^{(FC'}\phi^{CD)} \\
&= 2\ K_2\ (da)_{ABCD}(da)^{AB}\ \frac{1}{3}\big(d_{FC'}d^{FC'}\phi^{CD} + 2\ d_{FC'}d^{CC'}\phi^{FD}\big) \\
&= 2\ K_2\ (da)_{ABCD}(da)^{AB}\ \frac{2}{3}\big(d_{FC'}d^{FC'}\phi^{CD} + \Box_F{}^C\phi^{FD}\big) \\
&= 2\ K_2\ (da)_{ABCD}(da)^{AB}\ \frac{2}{3}\big(d_{FC'}d^{FC'}\phi^{CD} - 4\ \Lambda\phi^{CD}\big)
\end{aligned} \tag{309}$$

$$\begin{aligned}
&K_5\ \delta_\phi(da)_{ABCD}\ (da)^{AB}{}_{B'C'}\ (da)^{CDB'C'} \\
&= 2\ K_5\ (da)_{ABCD}\ (da)^{AB}{}_{B'C'}\ \delta_\phi\ (da)^{CDB'C'} \\
&= \frac{4}{3}\ K_5\ (da)_{ABCD}\ (da)^{AB}{}_{B'C'}\ d_{(F}{}^{(B'}d^{C)C')}\phi^{FD}
\end{aligned} \tag{310}$$

The second term in (309) can be combined with C_1, the other terms can be combined only after taking into account the L terms.

$$\begin{aligned} & L_1\ \delta_\phi\ (da)_{ABCD}(da)^{AB}(d\eta)^{CD} \\ &= L_1\ (da)_{ABCD}\ d_{EB'}d^{(EB'}\phi^{AB)}(d\eta)^{CD} \\ &+ L_1\ (da)_{ABCD}(da)^{AB}\ d^{C}{}_{F'}d_{G}{}^{F'}\phi^{GD} \\ &= \frac{1}{3}\ L_1\ (da)_{ABCD}\ (d\eta)^{CD}\big(2\ d_{FF'}d^{FF'}\phi^{AB} - 8\ \Lambda\phi^{AB}\big) \\ &+ \frac{1}{2}\ L_1\ (da)_{ABCD}(da)^{AB}\ \big(-d_{FF'}d^{FF'}\ \phi^{CD} - 8\ \Lambda\ \phi^{CD}\big). \end{aligned} \tag{311}$$

$$\begin{aligned} & L_2\ \delta_\phi\ (da)_{ABCD}(d\eta)^{AB}(d\eta)^{CD} \\ &= 2\ L_2\ (da)_{ABCD}\ \delta_\phi(d\eta)^{AB}(d\eta)^{CD} \\ &= L_2\ (da)_{ABCD}\ \big(-d_{FF'}d^{FF'}\ \phi^{AB} - 8\ \Lambda\ \phi^{AB}\big)(d\eta)^{CD}. \end{aligned} \tag{312}$$

The term K_5 is combined together with L_4 and L_5.

$$\begin{aligned} & L_4\ \delta_\phi\ (da)_{ABCD}\ (da)^{AB}{}_{B'C'}(d\eta)^{CDB'C'} \\ &= \frac{2}{3}L_4\ (da)_{ABCD}\ d_{(EB'}d^{A)}{}_{C'}\phi^{EB}\ (d\eta)^{CDB'C'} \\ &+ L_4\ (da)_{ABCD}(da)^{AB}{}_{B'C'}\ d^{(CB'}d_{F)}{}^{C'}\phi^{DF}. \end{aligned} \tag{313}$$

$$\begin{aligned} & L_5\ \delta_\phi\ (da)_{ABCD}\ (d\eta)^{AB}{}_{B'C'}(d\eta)^{CDB'C'} \\ &= 2\ L_5\ (da)_{ABCD}\ d_{(EB'}d^{A)}{}_{C'}\phi^{EB}\ (d\eta)^{CDB'C'}. \end{aligned} \tag{314}$$

the G_1 term is invariant by itself under the gauge transformation. All others G_k terms cannot be combined in any possible way. After the gauge transformation we can state a first system of equations for the coefficients of the ansatz. As it will be clear soon, this system does not close by itself but divides into two sub systems. They will be connected and consistent only after the diffeos are taken into account. The systems of coefficients after gauge transformations then reads:

$$\begin{cases} C_1 = -6\Lambda L_1, \\ K_2 = \frac{3}{8}L_1, \\ L_2 = \frac{2}{3}L_1. \end{cases} \qquad \begin{cases} \frac{4}{3}K_5 = -L_4, \\ \frac{1}{3}L_4 = -L_5, \end{cases} \tag{315}$$

As already stated, they are decoupled, but they get coupled by the system for the diffeos, that also provides some of the same equations of the previous system, as consistency requires. We have already computed the relevant diffeo transformations of the quadratic action in (284), now we apply the very simple linear transformation (266) on the cubic terms that survived under gauge transformations, namely K_2, K_5, L_1, L_2, L_4, L_5, G_1. In particular G_1 matches the first term in (284),

$$C_1 = -3\Lambda G_1. \tag{316}$$

Only one coefficient, C_1, remains, and it is an overall normalization constant.

$$\begin{cases} -\frac{1}{2}C_1 - 3\Lambda L_1 = 0, \\ -\frac{2}{3}C_1 - 6\Lambda L_2 = 0 \end{cases} \tag{317}$$

This enforces the same equations as in (315). The diffeos for L_4, L_5, yield:

$$\begin{cases} 2C_1 - 3\Lambda L_4 = 0, \\ -\frac{4}{3}C_1 - 6\Lambda L_5 = 0, \end{cases} \tag{318}$$

solving for L_4, L_5, gives the equations between these terms in (315), while solving for C_1 gives:

$$\begin{cases} C_1 = \frac{9}{2}\Lambda L_5, \\ C_1 = -\frac{3}{2}\Lambda L_4, \\ C_1 = 2\Lambda K_5. \end{cases} \qquad \begin{cases} C_1 = -16\Lambda K_2, \\ C_1 = -6\Lambda L_1, \\ C_1 = -9\Lambda L_2. \end{cases} \tag{319}$$

The final Lagrangian is the following:

$$\begin{aligned} \mathcal{L} = {} & C_1\, d_{(A}\, a_{BCD)}\, d^A\, a^{BCD} + C_1\frac{4}{3}\, a_{ABCF'}\, \eta_D{}^{F'} (da)^{ABCD} \\ & - C_1 2\, a_{(ABE)E'} a^{(E}{}_{CD)}{}^{E'}\, (da)^{ABCD} + K_1\, (da)_{ABCD} (da)^{AB}{}_{EF} (da)^{CDEF} \\ & + \frac{C_1}{\Lambda}\Big(-\frac{1}{16}\, (da)_{ABCD} (da)^{AB} (da)^{CD} + \frac{1}{2}\, (da)_{ABCD} (da)^{AB}{}_{E'F'} (da)^{CDE'F'} \\ & - \frac{1}{6}\, (da)_{ABCD} (da)^{AB} (d\eta)^{CD} - \frac{1}{9}\, (da)_{ABCD} (d\eta)^{AB} (d\eta)^{CD} \\ & + -\frac{2}{3}\, (da)_{ABCD} (da)^{AB}{}_{E'F'} (d\eta)^{CDE'F'} + \frac{2}{9}\, (da)_{ABCD} (d\eta)^{AB}{}_{E'F'} (d\eta)^{CDE'F'} \\ & - \frac{1}{3}\, (da)_{ABCD} (da)^{ABCD} (d\eta)\Big). \end{aligned} \tag{320}$$

This last formula is the full quadratic plus cubic action. It looks a complicated result, but it is on the contrary a quite simple one. A perturbative analysis of the scattering amplitude would require this Lagrangian to be gauge fixed. As we have already seen, the fixing of the diffeomorphisms is achieved by just setting the η components to zero. This is remarkable because the fixed cubic vertex would have then just four terms. Moreover, one of them is the only one that contributes to the chiral modifications (K_1). There are only two coupling constants, which is what we expected. The situation is not much different from what we have seen in the DYM theory, where at cubic level only one vertex (with maximal number of derivatives) is added on top of the non chiral theory. All other terms in eq. (320) correspond to GR. In comparison, we then clearly see that this formulation is much more economical than the usual metric formulation.

6

THE DIGT IN MIXED FORMULATION

6.1 INTRODUCTION

In this last chapter we will provide a description for the DIGT theories in terms of a slightly different set of variables. This version of the Lagrangian for the family of theories discussed in the previous chapters could be more suitable for the calculations we want to perform at the quantum level.

The following description has been pointed out for the first time in [11], but it has rarely been used as a starting point for actual computations. We want to start this chapter detailing the transition from the usual Plebanski-like formulation of the family of gravity theories to the new set of variables. The selection of a specific member from the class is provided by a choice of a constraint. After a few examples of this selection procedure, we will discuss the one-loop beta function calculation for the whole set of theories when expanded on an instanton background. The following formulation allows us, at least in the case of this expansion, to use the same techniques developed for the case of DYM theories. Further work and probably a more involved choice of the gauge fixing, is necessary to extend this results to a general background.

6.2 FROM PLEBANSKI TO THE MIXED FORMULATION

We saw in the previous chapters that from the Plebanski formulation of GR, one can construct a whole infinite family of new theories which "deform" the original action, retaining some of the properties of Einstein theory. We rewrite here the Lagrangian:

$$S = \frac{1}{8\pi G}\int B^i \wedge F^i - \frac{1}{2}\left(\Psi^{ij} + \frac{\Lambda(\Psi)}{3}\delta^{ij}\right) B^i \wedge B^j. \tag{321}$$

As we already stated before, B^i is an $su(2)$-valued two-form, F^i is the curvature of an $su(2)$ connection A, and Ψ^{ij} is valued in the symmetric-traceless power of the lie algebra $su(2)$ and acts as a Lagrange multiplier in the GR case (where the function Λ is actually a constant) enforcing the metricity condition $B^i \wedge B^j \propto \delta^{ij}$.

When the matrix $\Psi^{ij} + \frac{\Lambda(\Psi)}{3}\delta^{ij}$ is invertible, we can integrate out the B field using the equations of motion:

$$F^i = \left(\Psi^{ij} + \frac{\Lambda(\Psi)}{3}\delta^{ij}\right) B^i, \tag{322}$$

to get:

$$S = \int \left(\Psi^{ij} + \frac{\Lambda(\Psi)}{3}\delta^{ij}\right)^{-1} F^i \wedge F^i \tag{323}$$

where we also absorbed the constants in front of the integral in the definition of the fields.

To make one step further we call the matrices (depending on the function Λ) M^{ij} and consider them as a new variable, also symmetric in its indices. To select a specific member of this family of theories we introduce a Lagrange multiplier μ and a constraint on M:

$$S = \int M^{ij} X^{ij} - \mu g(M), \tag{324}$$

where in the last formula we used the already introduced matrix:

$$X^{ij} = \frac{1}{4}\varepsilon^{\mu\nu\rho\sigma} F^i_{\mu\nu} \wedge F^j_{\rho\sigma}. \tag{325}$$

Although this formulation is not very different from neither the pure connection or the Plebanski one, it provides an interesting setting to perform quantum computations as we will see in the following.

6.3 A STRING THEORY ANALOGY

As an example of this kind of formulations one can think of the relation between the Nambu-goto and Polyakov actions in String theories. Introducing the worldsheet metric:

$$\eta_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu}, \tag{326}$$

one can write the two lagrangians as follows:

$$S_{NG} = \int d\sigma d\tau \sqrt{\det(\partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu})}, \tag{327}$$

$$S_{NG} = \int d\sigma d\tau \sqrt{\det \eta}\, \eta^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu}, \tag{328}$$

where σ, τ are the worldsheet coordinates and g is the space-time metric. The last equation is analogous to eq. (323). Indeed if we call:

$$M^{\alpha\beta} = \sqrt{\det \eta}\, \eta^{\alpha\beta}, \tag{329}$$

we can write the Polyakov action in the following suggestive way:

$$\int d\sigma d\tau M^{\alpha\beta} \Psi_{\alpha\beta} - \mu g(M), \tag{330}$$

where we called $\Psi_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu}$ and we introduced the constraint $g(M) = \det(M) - 1$. The equation of motion that we need to go back to the Nambu-Goto action are simply:

$$\begin{aligned} \Psi_{\alpha\beta} &= \mu \det M \, M^{-1}_{\alpha\beta}, \\ \det M &= 1. \end{aligned} \tag{331}$$

taking the determinant of the first of eqs. (331) and using the second one then we arrive at:

$$\det \Psi_{\alpha\beta} = \mu^2. \tag{332}$$

if we put this equation back into the original action, it yields:

$$\int d\sigma d\tau\ \mu = \int d\sigma d\tau \sqrt{\det \Psi_{\alpha\beta}} = \int d\sigma d\tau \sqrt{\det(\partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu})}, \tag{333}$$

which is the Nambu-Goto action.

6.4 EXAMPLES

6.4.1 *GR action*

Before dwelling into the main calculation of this chapter, let us show a few examples of the constraints that can be chosen in this formulation. The first one is the usual GR. The constraint in this case is:

$$g(M) = \mathrm{Tr}(M^{-1}) - \alpha. \tag{334}$$

The equation of motion in this case reads:

$$X^{ij} = -\mu M^{ij}. \tag{335}$$

To convince ourselves that this is indeed GR, we can integrate out the M field, which would leave us with the pure connection formulation already encountered in the previous chapters. To get this result we simply take the inverse of the last equation and trace both sides:

$$\mathrm{Tr}\frac{\sqrt{X}}{-\mu} = \mathrm{Tr}(M^{-1}) = \alpha. \tag{336}$$

The Lagrangian of eq. (324) then becomes:

$$\begin{aligned} S &= \int \mathrm{Tr}(\sqrt{-\mu}\sqrt{X}^{-1}X) \\ &= \int \mathrm{Tr}(\frac{\sqrt{X}}{\Lambda})\mathrm{Tr}(\sqrt{X}^{-1}X) \\ &= \frac{1}{\alpha}\int (\mathrm{Tr}(\sqrt{X}))^2 \end{aligned} \tag{337}$$

which is the functional for GR that we have already found in chapter (4), eq. (222). One can fix the constant to make it agree with the standard pure connection formulation for GR, namely $\alpha = 16\pi G\Lambda$.

6.4.2 $\det M = 1$ *theory*

If we choose the constraint g to be:

$$g(M) = detM - 1, \tag{338}$$

we get a theory that closely resembles the string theory example of the last section. Indeed the equation of motion is:

$$X^{ij} = \mu(M^{ij})^{-1}. \tag{339}$$

taking the determinant of the last equation and using the unimodularity condition into the Lagrangian one arrives at:

$$S = \int (\mu)^{\frac{1}{3}} = \int (\det X)^{\frac{1}{3}}. \tag{340}$$

6.4.3 *Instanton Theory*

A third example of a theory that can be studied in this formulation is self-dual gravity. It has been called this way because it provides only gravitational instantons as solutions to its equations of motion. As we have already discussed these are Einstein metrics with vanishing (anti)-self-dual Weyl tensor. The constraint that produces this theory is:

$$g(M) = \mathrm{Tr}M - \Lambda. \tag{341}$$

The equations of motion are then simply:

$$F^i \wedge F^j \propto \delta^{ij}, \tag{342}$$

which, as we have already discussed, is an equation that leads to instanton metrics. This theory is a quantum finite one at all loops, despite having a negative mass coupling constant. For an account of this theory in this formulation see [38].

6.5 INSTANTON BACKGROUND β-FUNCTION FOR THE FAMILY OF GRAVITY THEORIES

6.5.1 *Expansion of the action on an instanton background*

We want to expand a general member of the family of theories provided by the action

$$S[A, M] = \int M^{ij} F^i \wedge F^j - \mu g(M), \tag{343}$$

around an instanton background, which is defined by the relations:

$$\begin{aligned} X_0^{ij} &= F_0^i \wedge F_0^j \propto \delta^{ij}, & (344)\\ M_0^{ij} &= \delta^{ij} & (345) \end{aligned}$$

The field strength and the auxiliary field expansion around the background, up the second order is:

$$\begin{aligned} F^i &= F_o^i + d_A a^i + \frac{1}{2}[a, a]^i, \\ M^{ij} &= \delta^{ij} + \psi^{ij}, \end{aligned} \tag{346}$$

where the variation for $\delta M = \psi$ is symmetric. To start we compute the first variation of the Lagrangian on the background, which of course will provide the equations of motion

$$\delta\mathcal{L} = \psi^{ij} F_0^i \wedge F_0^j + 2M_0^{ij} F_0^i \wedge d_A a^j - \delta\mu\, g(M_0) - \mu_0 \left(\frac{\partial g}{\partial M_0^{ij}} \psi^{ij} \right). \tag{347}$$

We compute the second variation and we divided by two to obtain the Lagrangian expanded to second order

$$\begin{aligned} \mathcal{L}^{(2)} &= \Bigg(2\psi_0^{ij} F_0^i \wedge da^j + M_0^{ij} da^i \wedge da^j + \tfrac{1}{2} M_0^{ij} F_0^i \wedge [a, a]^j \\ &\quad - \delta\mu \left(\tfrac{\partial g}{\partial M_0^{ij}} \psi^{ij} \right) - \tfrac{1}{2}\mu_0 \left(\tfrac{\partial^2 g}{\partial M_0^{ij} \partial M_0^{kl}} \psi^{ij} \psi^{kl} \right) \Bigg). \end{aligned} \tag{348}$$

We can now use the expansions written in eq. (346) to simplify the last formula. Indeed, the fact that M_0 is just the identity on this background,

leads to the mutual cancellation of the second and third term, after an integration by parts. Moreover, the equation of motion for the connection on the background reads:

$$\delta_{ij} = X_{ij}\Big|_{bg} = \mu_0 \, \frac{\partial g}{\partial M^{ij}}\Big|_{bg}. \tag{349}$$

Remembering that $X^{ij} = \frac{1}{4}\varepsilon_{\mu\nu\rho\sigma}F^{i\mu\nu}F^{j\rho\sigma}$, we can use the fact that F_0 is self-dual and write the Lagrangian as:

$$\mathcal{L}^{(2)} = \psi^{ij}F^i_{0\mu\nu}d^\mu a^{j\nu} - \delta\mu[\mathrm{Tr}(\psi)] - \frac{1}{2}\mu_0\left(\frac{\partial^2 g}{\partial M_0^{ij}\partial M_0^{kl}}\psi^{ij}\psi^{kl}\right), \tag{350}$$

given that now the constraint imposes the tracelessness of ψ, the second derivative of g appearing in the last formula is proportional to the projector onto the symmetric traceless part of 3x3 matrices:

$$\frac{\partial^2 g}{\partial M^{ij}\partial M^{kl}}\psi^{ij}\psi^{kl}\Big|_{bg} \propto P_{ij|kl} := \left(\delta_{i(k}\delta_{l)j} - \frac{1}{3}\delta_{ij}\delta_{kl}\right). \tag{351}$$

The calculation we are about to perform does not depend on the overall normalization of the Lagrangian so we can absorb all the constants into a single one. The final Lagrangian, expanded to second order in the variations is then:

$$\mathcal{L}^{(2)} = \psi^{ij}\Sigma^i_{\mu\nu}da^{j\mu\nu} - kP_{ij|kl}\,\psi^{ij}\psi^{kl}. \tag{352}$$

This Lagrangian is a first order version of the one obtained from the pure connection point of view eq. (231)

6.5.2 *Gauge fixing*

To compute the one loop beta function through the Heat kernel method we need to gauge fix the previous Lagrangian. The gauge freedom is manifest from the fact that the differential operator in (352) maps the 12 components of the connection into the 5 ones of the field ψ. The best setting to do this calculation is the spinor formalism. The field ψ is traceless and symmetric

which in spinor terms means that it is a completely symmetric four spinor, $\psi^{(ABCD)}$.

The fixing of the diffeomorphism proceeds as usual. The Lagrangian does not depend on the $S_+ \otimes S_-$ part of the connection, so we can project the connection onto the other irreducible part, $S^3_+ \otimes S_-$. This reduces the component of the connection to 8, fixing the 4 parameters of the diffeomorphism invariance. The gauge invariance can be dealt with in a similar way as the one employed in the DYM case. We introduce a YM like gauge fixing fermion and the corresponding ghosts. Instead of reducing the components in the connection we shift the ψ field so that it will acquire the missing 3 components in the kinetic term producing a invertible Dirac operator. Let us show how this is done in detail.

The projection of the connection $a^{AB}{}_{MM'}$ onto the $(\frac{3}{2}, \frac{1}{2})$ representation is easily achieved symmetrizing all the unprimed indices. This fixes the diffeomorphisms as already stated. To fix the remaining gauge we use a projected Lorentz condition:

$$(da)_{(3,1)}{}^i_\mu \equiv d^\mu (P^{(3,1)}{}_\mu{}^{i\nu j}\, a^j)_\nu = 0, \tag{353}$$

where the projector in the last formula reads:

$$P^{(3,1)}{}_{\mu i \nu j} = \frac{2}{3}\left(\delta_{ij} g_{\mu\nu} - \frac{1}{2}\varepsilon_{ijk}\Sigma^k_{\mu\nu}\right). \tag{354}$$

To achieve the gauge fixing we use the BRST method. The BRST transformations are the same as in the Yang-Mills case, including the auxiliary field b and the anti-ghost $\bar{c}$:

$$\begin{aligned} sA^i_\mu &= d_\mu c^i, \\ sc^i &= \frac{1}{2}[c,c]^i, \\ s\bar{c}^i &= b^i, \\ sb &= 0. \end{aligned} \tag{355}$$

We add to the Lagrangian (352) the gauge fixing term:

$$\begin{aligned} s\psi &= s\left(\alpha\, \bar{c}^i (da)^i_{(3,1)} - \beta\, \bar{c}^i b^i)\right), \\ &= \alpha\, b^i (da)^i_{(3,1)} - \beta\, b^i b^i - \alpha\, \bar{c}^i d(P_{(3,1)} dc^i), \end{aligned} \tag{356}$$

where we have suppressed the contracted Lorentz indices for brevity. We will take care of the ghost term at the end, so we can now focus on understanding how the gauge fixing works. First, we rewrite the Lagrangian plus the first two terms of the last formula in spinor indices:

$$\begin{aligned}\mathcal{L}^{(2)}_{gf} &= \sqrt{2}\,\psi^{(ABCD)}\,d_{AM'}a_{BCD}{}^{M'} - k\,\psi^{(ABCD)}\psi_{ABCD} \\ &+ \alpha\,b^{AB}\,d^{C}{}_{M'}a_{ABC}{}^{M'} - \beta\,b^{AB}b_{AB}. \end{aligned} \tag{357}$$

The auxiliary field b carries exactly the 3 d.o.f. we need to add to the field ψ in order to have an invertible kinetic operator. We define then:

$$\psi^{(ABC)D} = \psi^{(ABCD)} - b^{(AB}\varepsilon^{C)D}. \tag{358}$$

Using this shift we can combine the second and last term in eq. (357) in the following way:

$$\begin{aligned}&-k\,\psi^{(ABCD)}\psi_{ABCD} - \beta\,b^{AB}b_{AB} = \\ &-k\,\left(\psi^{(ABC)D}\psi_{ABCD} - 2\psi^{(ABC)}{}_{C}b_{AB} + b^{(AB}\varepsilon^{C)}{}_{C}b_{AB}\right) - \beta\,b^{AB}b_{AB}, \\ &-k\,\left(\psi^{(ABC)D}\psi_{ABCD} - b^{(AB}\varepsilon^{C)}{}_{C}b_{AB}\right) - \beta\,b^{AB}b_{AB}. \end{aligned} \tag{359}$$

Where in the last line we used the identity:

$$\psi^{(ABC)}{}_{C} = b^{(AB}\varepsilon^{C)}{}_{C}, \tag{360}$$

obtained tracing eq. (358). Expanding the symmetrization we can combine the second and third term:

$$-k\,\psi^{(ABC)D}\psi_{ABCD} - \left(\frac{4}{3}k + \beta\right)\,b^{AB}b_{AB}. \tag{361}$$

Clearly we greatly simplify the Lagrangian setting the last term to zero, choosing $\beta = -\frac{4}{3}k$.

The first and third terms in the Lagrangian (357) also simplify due to the shift of eq. (358).

$$\begin{aligned}&\sqrt{2}\,\psi^{(ABCD)}\,d_{AM'}a_{BCD}{}^{M'} + \alpha\,b^{AB}\,d^{C}{}_{M'}a_{ABC}{}^{M'} = \\ &\sqrt{2}\,\psi^{(ABC)D}\,d_{DM'}a_{ABC}{}^{M'} + \sqrt{2}\,b^{AB}\,d^{C}{}_{M'}\,a_{ABC}{}^{M'} + \alpha\,b^{AB}\,d^{C}{}_{M'}a_{ABC}{}^{M'}. \end{aligned} \tag{362}$$

The last two terms cancel choosing $\alpha = -\sqrt{2}$.

The final, gauge fixed Lagrangian, takes the simple form:

$$\mathcal{L}^{(2)} = 2\psi^{(ABC)D}\, d_{DM'} a_{ABC}{}^{M'} - k\,\psi^{(ABC)D}\psi_{ABCD}. \tag{363}$$

In the last formula we have normalized the Lagrangian, multiplying it by $\sqrt{2}$ and absorbed this factor into the constant k. This overall factor does not effect the calculation we are about to perform.

6.5.3 *Dirac operator*

The differential operator of the Lagrangian we obtained is invertible now. It can be written in matrix form:

$$D = \begin{pmatrix} A & \not{\partial} \\ \not{\partial}^* & 0 \end{pmatrix}, \tag{364}$$

where

$$\not{\partial} : V \otimes S_- \to V \otimes S_+, \quad \not{\partial}^* : V \otimes S_+ \to V \otimes S_-$$

are Dirac operators, $V = S^3_+$, and the endomorphism A is just $A = -k\mathbb{1}_{S_+}$. The operator (364) acts on V-valued Dirac spinors,

$$\begin{pmatrix} \psi \\ a \end{pmatrix}, \qquad \psi \in V \otimes S_+, \quad a \in V \otimes S_-,$$

and the chiral Dirac operators $\not{\partial}, \not{\partial}^*$ know how to act on V-valued sections. Now we need to extract information about the one-loop beta function of this theory from the differential operator at hand. To do so we notice that, being D a Dirac operator, its square is of Laplace type. This allows us to use the Heat Kernel technology. At the end of the day the information about the logarithmic divergencies and beta function will be encoded as usual in the fourth HK coefficient, but this time we will need to divide the result by 2 given that in the Dirac case the d.o.f. are doubled.

6.5.4 *The Laplace operator*

Naively taking the square of the Dirac operator D produces:

$$D^2 = \begin{pmatrix} \not{d}\not{d}^* + A^2 & A\not{d} \\ \not{d}^* A & \not{d}^*\not{d} \end{pmatrix}. \tag{365}$$

We will now show how this operator can be put (uniquely) into the more familiar form for Laplace operators:

$$-\mathcal{D}^\mu \mathcal{D}_\mu + E. \tag{366}$$

First of all, $\not{d}$ and $\not{d}*$ act on spinors as Dirac operators. Let $\psi \in S_+$ and $\psi' \in S_-$:

$$\begin{aligned} (\not{d}\psi')_A &= d_{AA'}\psi'^{A'}, \\ (\not{d}\psi)_{A'} &= d^A{}_{A'}\psi_A. \end{aligned} \tag{367}$$

the differential operators in the diagonal of (365) can be then rewritten as:

$$\begin{aligned} (\not{d}\not{d}^*\psi)_A = d_{AA'}d^{A'E}\psi_E = -\frac{1}{2}d^M{}_{M'}d_M{}^{M'}\psi_A + \frac{1}{2}\Box_A{}^E\psi_E, \\ (\not{d}^*\not{d}\psi')_{A'} = d_{A'}{}^E d_{EE'}\psi^{E'} = -\frac{1}{2}d^M{}_{M'}d_M{}^{M'}\psi_{A'} + \frac{1}{2}\Box_{A'E'}\psi^{E'}, \end{aligned} \tag{368}$$

where $\Box_{AB}$ and $\Box_{A'B'}$ are curvature terms defined by:

$$2d_{[\mu}d_{\nu]}t_{\rho\sigma} = \left(\varepsilon_{AB}\Box_{A'B'} + \varepsilon_{A'B'}\Box_{AB}\right) t^C{}_D{}^{E'}{}_{F'} \tag{369}$$

$$\begin{aligned} \Box_{A'B'}t^C{}_D{}^{E'}{}_{F'} &= \bar{X}_{A'B'Q'}{}^{E'}t^C{}_D{}^{Q'}{}_{F'} - \bar{X}_{A'B'F'}{}^{Q'}t^C{}_D{}^{E'}{}_{Q'} \\ &+ \bar{\Phi}_{A'B'Q}{}^C t^Q{}_D{}^{E'}{}_{F'} - \bar{\Phi}_{A'B'D}{}^Q t^C{}_Q{}^{E'}{}_{F'} \end{aligned} \tag{370}$$

$$\begin{aligned} \Box_{AB}t^C{}_D{}^{E'}{}_{F'} &= \bar{X}_{ABQ}{}^C t^Q{}_D{}^{E'}{}_{F'} - \bar{X}_{ABD}{}^Q t^C{}_Q{}^{E'}{}_{F'} \\ &+ \Phi_{ABQ'}{}^{E'}t^C{}_D{}^{Q'}{}_{F'} - \Phi_{ABF'}{}^{Q'}t^C{}_D{}^{E'}{}_{Q'} \end{aligned} \tag{371}$$

and $\bar{X}, X, \bar{\Phi}, \Phi$ are the 4 parts in which the Riemann tensor decomposes, see appendix (B) for a discussion of the curvature spinors. The square of D, normalized for future convenience, reads then:

$$2D^2 = \begin{pmatrix} -d^\mu d_\mu + 2\Box + 2A^2 & -2C^\mu d_\mu \\ -2C^{*\mu}d_\mu & -d^\mu d_\mu + 2\Box' \end{pmatrix}. \tag{372}$$

In the last formula we have also rewritten the off-diagonal terms using two operators:

$$C_\mu : V \otimes S_- \to V \otimes S_+, \qquad C^*_\mu : V \otimes S_+ \to V \otimes S_-$$

The components of C_μ, C^*_μ are given by

$$C^M{}_{M'}{}^A{}_{A'} = \epsilon_{A'M'} A^{AM}, \qquad C^{*M}{}_{M'}{}^A{}_{A'} = \epsilon_{M'A'} A^{MA}. \tag{373}$$

From the last formulae it is quite simple to guess that the first order differential operator that puts D^2 into (366) is:

$$\mathcal{D}_\mu = \begin{pmatrix} d_\mu & C_\mu \\ C^*_\mu & d_\mu \end{pmatrix}. \tag{374}$$

Squaring this operator one extracts the endomorphism E:

$$E = \begin{pmatrix} 2\Box + 2A^2 + C^\mu C^*_\mu & 0 \\ 0 & 2\Box' + C^*_\mu C^\mu \end{pmatrix}. \tag{375}$$

Actually the second and third term in the upper diagonal of E cancel each other, indeed:

$$\begin{aligned} C^\mu C^*_\mu &= (C^M{}_{M'})^{AA'} (C^*_M{}^{M'})_{A'}{}^B \\ &= \varepsilon^{A'}{}_{M'} A^{AM} \varepsilon^{M'}{}_{A'} A_M{}^B \\ &= -2A^{AM} A_M{}^B \\ &= -2A^2. \end{aligned} \tag{376}$$

Moreover the last term in the lower diagonal is just proportional to the trace of A^2:

$$\begin{aligned} C^*_\mu C^\mu &= (C^{*M}{}_{M'})^E{}_{A'} (C_M{}^{M'})_{EB'} \\ &= \varepsilon_{M'A'} A^{ME} \varepsilon_{B'}{}^{M'} A_{EM} \\ &= \varepsilon_{B'A'} A^{EM} A_{EM} \\ &= -\mathrm{tr}(A^2) \mathbb{1}_{S_-}. \end{aligned} \tag{377}$$

All in all the endomorphism E gets simplified in the following way:

$$E = \begin{pmatrix} 2\Box & 0 \\ 0 & 2\Box' - \mathrm{tr}(A^2)\mathbb{1}_{S_-} \end{pmatrix}. \tag{378}$$

one very important operator that we will need in the following is the curvature:

$$\mathcal{F}_{\mu\nu} = 2\mathcal{D}_{[\mu}\mathcal{D}_{\nu]} = \begin{pmatrix} \Omega_{\mu\nu} + C_\mu C_\nu^* - C_\nu C_\mu^* & 0 \\ 0 & \Omega_{\mu\nu} + C_\mu^* C_\nu - C_\nu^* C_\mu \end{pmatrix}. \quad (379)$$

where $\Omega_{\mu\nu} = d_\mu d_\nu - d_\nu d_\mu$

6.5.5 *The Heat Kernel calculation*

The beta function is known to be related to the fourth coefficient of the heat kernel expansion of the relevant operator of the quadratic expansion of the Lagrangian (around a background). In the case of the first order operator at hand we just need to compute the relevant coefficient for the associated second order operator (365) and divide the result by two. The heat kernel formula for a curved background is:

$$\begin{aligned} a_4 &= \frac{1}{(4\pi)^2}\int dx\ \mathrm{Tr}\Big(\frac{1}{12}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} + \frac{1}{2}E^2 + \frac{1}{6}ER \\ &+ \frac{1}{72}R^2 - \frac{1}{180}R_{\mu\nu}R^{\mu\nu} + \frac{1}{180}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\Big), \end{aligned} \quad (380)$$

where the trace is taken in the full space of spinors valued in $V = S^3_+$. In the last formula the terms that actually depends on the operator D^2 are the first three, so we compute them first.

The easiest term is the third, i.e. $\frac{1}{6}ER$:

$$\begin{aligned} \frac{1}{6}ER &= \frac{2}{3}\lambda\ \mathrm{Tr}E \\ &= \frac{2}{3}\lambda\ \mathrm{tr}_V\left(-2\lambda + \mathrm{tr}A^2\right) \\ &= -\frac{32}{3}\lambda^2 + \frac{16}{3}\lambda\ \mathrm{tr}A^2, \end{aligned} \quad (381)$$

where we used the fact that, on our background, $R = 4\lambda$, and

$$\begin{aligned} 2\Box_{AB}\psi^B &= -\lambda\ \psi_A \\ 2\Box_{A'B'}\psi^{B'} &= -\lambda\ \psi_{A'}. \end{aligned} \quad (382)$$

Moreover the trace produces a factor of $dim(S^3_+ \otimes S_+) = 8$

The next term in the Heat Kernel formula is $\frac{1}{2}E^2$:

$$\begin{aligned}\frac{1}{2}\mathrm{Tr}\ E^2 &= \frac{1}{2}\mathrm{tr}_V\left(\lambda^2 + \left(\mathrm{tr}A^2 - \lambda\right)^2\right)\\ &= 8\lambda^2 - 8\mathrm{tr}A^2\ \lambda + 4(\mathrm{tr}A^2)^2. \end{aligned} \tag{383}$$

The square of the curvature is the most difficult term to compute. Formally it is the sum of the squares of the upper and lower diagonal terms of the curvature $\mathcal{F}_{\mu\nu}$:

$$\begin{aligned}&\frac{1}{12}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} =\\ &\frac{1}{12}\Big\{\mathrm{tr}_+\left([d_\mu, d_\nu] + C_\mu C^*_\nu - C_\mu C^*_\nu\right)^2 + \mathrm{tr}_-\left([d_\mu, d_\nu] + C^*_\mu C_\nu - C^*_\mu C_\nu\right)^2\Big\} =\\ &\frac{1}{12}\Big\{\mathrm{tr}_+\left([d_\mu, d_\nu]^2 + 2([d_\mu, d_\nu]\,(C^\mu C^{*\nu} - C^\nu C^{*\mu}) + \left(C_\mu C^*_\nu - C_\nu C^*_\mu\right)^2\right) +\\ &\mathrm{tr}_-\left([d_\mu, d_\nu]^2 + 2([d_\mu, d_\nu]\,(C^{*\mu} C^\nu - C^{*\nu} C^\mu) + \left(C^*_\mu C_\nu - C^*_\nu C_\mu\right)^2\right)\Big\}\end{aligned} \tag{384}$$

where in the last formula we used the shorthand notation $\mathrm{tr}_\pm = \mathrm{tr}_{S^3_+ \otimes S_\pm}$. The last equation will give three contributions, so we split the calculation into these separate pieces. These contributions will match similar ones in the previous HK terms (381), (383).

Let us start with the last terms of the traces in (384). To compute these pieces we first express the antisymmetric product of C's in terms of the endomorphism A and its algebraic invariants, i.e. :

$$\begin{aligned}&\left(C^*_\mu C_\nu - C^*_\nu C_\mu\right)^{A'}{}_{B'}\ a^{B'} =\\ &\left((C^{*M}{}_{M'})^{AA'}(C^N{}_{N'})_{AB'} - (C^{*N}{}_{N'})^{AA'}(C^M{}_{M'})_{AB'}\right) a^{B'} =\\ &\left(A^{(MA}A^{N)}_A \varepsilon^{A'}{}_{B'}\varepsilon_{M'N'} + \mathrm{tr}A^2\ \varepsilon^{A'}{}_{(M'}\varepsilon_{N')B'}\varepsilon_{MN}\right) a^{B'} =\\ &\mathrm{tr}A^2\ \varepsilon^{A'}{}_{(M'}\varepsilon_{N')B'}\varepsilon_{MN}\ a^{B'},\end{aligned} \tag{385}$$

$$\begin{aligned}&\left(C_\mu C^*_\nu - C_\nu C^*_\mu\right)^{A'}{}_{B'}\ \psi_B =\\ &(C^M{}_{M'})^{AA'}(C^{*N}{}_{N'})_{AB'}\ \psi_B - (C^N{}_{N'})^{AA'}(C^{*M}{}_{M'})_{AB'}\ \psi_B =\\ &-2\ \varepsilon_{M'N}\ A^{(M}_A A^{N)B}\ \psi_B\end{aligned} \tag{386}$$

Now we can square these last terms and take the trace. For eq. (386) this yields:

$$\begin{aligned}\mathrm{Tr}\left(-2\,\varepsilon_{M'N}\,A_A^{(M}A^{N)B}\right)^2 &= -8A_{A(M}A_{N)}{}^BA_B{}^MA^{NA}\\ &= \mathrm{tr}_V\left(-4\,\mathrm{tr}A^4-4\,(\mathrm{tr}A^2)^2\right)\\ &= \mathrm{tr}_V\left(-6(\mathrm{tr}A^2)^2\right),\end{aligned}\tag{387}$$

where in the last line we used the fact that, given the antisymmetry of A:

$$\mathrm{tr}A^4=\frac{1}{2}(\mathrm{tr}A^2)^2.\tag{388}$$

For what concerns eq. (385) then, we have:

$$\left(\mathrm{tr}A^2\,\varepsilon^{A'}{}_{(M'}\varepsilon_{N')B'}\varepsilon_{MN}\right)^2=\mathrm{tr}_V\left(-6(\mathrm{tr}A^2)^2\right).\tag{389}$$

In total these terms contribute as:

$$\frac{1}{12}\mathrm{tr}_V\left(-6(\mathrm{tr}A^2)^2\right)=-4(\mathrm{tr}A^2)^2,\tag{390}$$

which cancels exactly the same quantity in eq. (383). A calculation along the same lines shows that also the terms proportional to $\lambda\,\mathrm{tr}(A^2)$ cancel out. This should not surprise, indeed the endomorphism A contains the constant k in the original Lagrangian. This constant cannot have any relevance to the HK coefficient. If we put our Lagrangian in a second order form, integrating out the auxiliary field ψ, k will appear as an overall normalization constant. The HK calculations are unaffected by such normalization. The last two terms, i.e. the trace of the square of the curvatures are the most cumbersome to compute. The task is purely algebraic and gives the same result for both terms. We will show only the lower diagonal contribution:

$$\begin{aligned}[d_\mu,d_\nu]^2\,P_{3,1}a^i_\mu &= \left(2\Box_{AB}\Box^{AB}+2\Box_{A'B'}\Box^{A'B'}\right)a^{C'DEF}\\ &= -2\lambda^2\,a^{C'DEF}.\end{aligned}\tag{391}$$

Of course this formula, on a general background, contains several terms proportional to the curvature operators Ψ and Φ introduced in eq. (454). The former is nothing else than the self-dual part of the Weyl tensor while

the latter is zero if and only if the Einstein equations are satisfied. All in all given the choice for our background this terms are zero leaving only the single contribution displayed in the last formula.

The last thing to do is to take the trace, which is now a very simple task, giving:

$$\mathrm{Tr}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} = -16\lambda^2. \tag{392}$$

Another way to do the same calculation would be to use Lorentz indices. One should be careful and use the projector $P_{(3,1)}$ introduced in eq. (354). This calculation gives:

$$\mathrm{Tr}\,\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} = -\frac{32}{3}\lambda^2 - 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}, \tag{393}$$

which is of course the same as before, given that on our background:

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{8\lambda^2}{3}. \tag{394}$$

The last three geometric invariants in the HK formula are easy to compute. For clarity we state all the relevant terms and we divide by two each term to account for the doubling of degrees of freedom for the Dirac operator:

$$\begin{aligned}
\frac{1}{2}\mathrm{Tr}\,E^2 &= 4\lambda^2 \\
\frac{1}{6}\mathrm{Tr}\,RE &= -\frac{16}{3}\lambda^2 \\
\frac{1}{12}\mathrm{Tr}\,\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} &= -\frac{8}{9}\lambda^2 - \frac{1}{6}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \\
\frac{1}{72}\mathrm{Tr}\,R^2 &= \frac{124}{72}\lambda^2 = \frac{16}{9}\lambda^2 \\
-\frac{1}{180}\mathrm{Tr}\,R_{\mu\nu}R^{\mu\nu} &= -\frac{8}{180}\lambda^2 g_{\mu\nu}g^{\mu\nu} = -\frac{8}{45}\lambda^2 \\
\frac{1}{180}\mathrm{Tr}\,R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} &= \frac{2}{45}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}
\end{aligned} \tag{395}$$

So the overall contribution to the HK formula for the operator D is:

$$a_4^{(D)} = -\frac{28}{45}\lambda^2 - \frac{11}{90}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \tag{396}$$

6.6 THE GHOST HEAT KERNEL COEFFICIENT

We left aside the part of the Lagrangian containing ghosts. Let us write it again here:

$$\begin{aligned}\mathcal{L}_{\text{ghost}} &= -\frac{3}{2}\,\bar{c}^i d(P_{(3,1)} dc^i) \\ &= \bar{c}^i \left(\delta_{ij} d_\nu - \frac{1}{2}\varepsilon_{ijk}\Sigma^k_{\mu\nu} d^\mu\right) d^\nu c^i. \qquad (397)\end{aligned}$$

(398)

The coefficient of $\frac{3}{2}$ is added so that the differential operator had the right normalization. The derivatives in the last term are automatically antisymmetrized by Σ. Therefore, the ghost Lagrangian can be simply written as:

$$\begin{aligned}\mathcal{L}_{\text{ghost}} &= \bar{c}^i \left(\delta_{ij} d^2 - \frac{1}{4}\varepsilon_{imn}\varepsilon_{mkj}\Sigma^n_{\mu\nu} F^{k\mu\nu}\right) c^i \\ &= \bar{c}^i \left(d^2 + \frac{2}{3}\lambda\right) c^i. \qquad (399)\end{aligned}$$

The relevant operator for the ghost sector is, then:

$$D_{\text{ghost}} = -d^2 - \frac{2}{3}\lambda. \qquad (400)$$

In this case the endomorphism is just $E_{gh} = \frac{2}{3}\lambda$. The curvature is easily found to be:

$$(\Omega_{\mu\nu})^i_j\, c^j = [D_{gh}, D_{gh}] = \varepsilon^{ijk} F^k_{\mu\nu} c^j, \qquad (401)$$

and the trace of the square of the last operator reads:

$$\operatorname{tr} \Omega_{\mu\nu}\Omega^{\mu\nu} = -\frac{8}{3}\lambda^2. \qquad (402)$$

It is important to remember that now the trace is done over the 3 dimensional space of ghosts fields. Using the same Heat Kernel formula eq. (380) as before, we arrive at the coefficient for the ghost sector:

$$a_4^{gh} = \frac{107}{45}\lambda^2 + \frac{1}{60} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \qquad (403)$$

6.7 FINAL RESULT AND REMARKS

Putting together all the results so far obtained we express the beta function for the diffeomorphism invariant gauge theories, expanded on an instanton background as:

$$\beta^{\text{1-loop}}_{\text{gauge}} = \frac{1}{(4\pi)^2}\int dx\ \sqrt{g}\left(-\frac{242}{45}\lambda^2 - \frac{7}{45}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right). \tag{404}$$

One can compare the same result for the metric GR case:

$$\beta^{\text{1-loop}}_{\text{gauge}} = \frac{1}{(4\pi)^2}\int dx\ \sqrt{g}\left(-\frac{522}{45}\lambda^2 + \frac{53}{45}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right). \tag{405}$$

These two results always differ by an integer as already pointed out in [21] (the difference being a topological quantity). Moreover, the first result (404) is only valid on a instanton background while eq. (405) is valid on any Einstein metric. The beta function in eq. (404) does not help differentiate the various theories defined by different constraints (or functions f in the pure connection formalism). This is because any theory in the DIGT class collapses to the same Lagrangian on an instanton background. On the contrary, the GR result is a statement on the running of the dimentionless quantity ΛG. Indeed, the Einstein-Hilbert action on the background becomes:

$$S_{\text{EH}} = -\frac{\lambda}{8\pi G}\int \delta^4 x\sqrt{g}. \tag{406}$$

So that the flow equation reads:

$$\frac{\partial}{\partial \ln \mu}\left(-\frac{\lambda}{8\pi G}\right) = -\frac{58\lambda^2}{5(4\pi)^2}, \tag{407}$$

or, considering a fixed λ, which is the constant curvature of the background:

$$\frac{\partial(\lambda G)}{\partial \ln \mu} = \frac{29(\lambda G)^2}{5\pi}, \tag{408}$$

To get a full result for the β-function of the DIGT that gives the running for the infinite coupling constant of these theories we should expand the Lagrangian on a general background. This expansion combined with the

first order formulation presented here should lead to a result close in spirit with what obtained in the DYM case. Nevertheless, expanding on a general background modifies the Dirac operator, given that the cancellation in (348) would not take place. Details of this calculation have not been worked out yet. We think that this formulation, or similar first order ones could be better suited for this kind of calculations as the DYM result shows.

7

CONCLUSIONS

In this thesis we have discussed how to modify Yang-Mills theories and General Relativity adding, on top of the usual interactions, an infinite-parameter class of new vertices.

The main results of the thesis concern the deformed YM theory introduced in [15] by the author et al. The key restriction for these modifications is that they do not increase the number of derivatives present in the equations of motion while retaining the gauge symmetry. The resulting new vertices are made up from invariant powers of the self-dual part of the field strength tensor.

Two where the main results found. At least at tree level, the recursion relations (BCFW) present in the original YM theory still work. These relations allow to reconstruct all possible tree level amplitudes from a basis of simple "seed" amplitudes. The details of this analysis are spelled out in Chapter 2. In the same chapter we introduced the key relation eq. 72 we believe being the main responsible for much of the results found for DYM theories. This property of the diagrams states that whenever two different vertices both connect through the legs upon which a derivative acts the resulting propagator with two derivatives gets substituted by a higher order vertex. Diagrammatically the propagator collapses into a vertex. A lower number of internal derivatives in the amplitudes improves the behavior of the theory in the BCFW framework and brings hope for a new renormalizability

claim of DYM. Given that this property works at loop level as well, it would be interesting to see how the recursion relations work beyond tree level.

The second result found was indeed a better behavior under renormalization than expected. We claimed that the DYM Lagrangian could be in fact closed under renormalization, meaning that the new class of vertices is big enough that no other counterterms are necessary to absorb the quantum divergencies. This would be a first step towards an asymptotic safety scenario for this kind of theories. In chapter 3 we first showed how the collapsing property improves the superficial degree of divergence count, then we explicitly computed the beta function of the simplest deformation coefficient. In particular the flow of coefficients associated to lower order operators is not affected by higher order ones. We have also showed that DYM is in fact fully renormalizable at one-loop. Even if this result is quite unexpected, it does not say much about the full renormalizability of the theory at all loops. Thus, there are two possible line of research that could be pursued. On one hand one can try to better understand the flow induced by the DYM beta-function of eq. 187 finding, for example, its fixed points. The behavior of the flow around those fixed points would be crucial in the asymptotic safety claim for these theories. On the other hand for the theory to be trusted, at least at perturbative level a direct two-loop beta function calculation is necessary. Some attempts have already been made but more work is needed.

In the second part of this thesis we have described a connection-based formulation of gravity which shares many important features with the DYM theory. This formulation allows to deform GR into an infinite-parameters class of gravity theory. We have shown how to cast GR into this new formulation starting from the Plebanski formulation and how dropping a key constraint allows for the introduction of new vertices. We have analyzed in particular two different sets of basic fields choices. A pure connection formulation in which only an $su(2)$ connection appears, which has been used to reconstruct the theory up to cubic level using only the knowledge of the final symmetry, i.e. gauge rotations and diffemorphisms. The result-

ing Lagrangian, once these symmetry are fixed, is quite simple (especially compared to the metric-based formulation of GR). In the last chapter of the thesis we tried to address the problem of renormalization of gravity in the same fashion as already described for DYM. We found that a particular reformulation of the theory in terms of gauge connection and an auxiliary field plus a constraint used to select a specific element in the class of theories could be best suited to compute the 1-loop renormalization flow. This formulation allowed us to employ the same Heat Kernel technique used to find the full 1-loop beta function in the case of DYM to compute the divergences of the 1-loop effective action in the gravity case when the theory is expanded around a instanton background. While the result found is not enough to discriminate between different theories in the class, given that the theory shares the same instanton sector of solutions, the similarities with the DYM case led us to believe that this particular formulation is the most promising one. We hope that a more in depth analysis could complete the 1-loop beta function calculation for these diffeomorphism invariant gauge theories giving new hope for their claimed renormalizability.

A

SPINOR TECHNIQUES AND CONVERSIONS

In this appendix we will give a self-contained review of the spinor formalism. This is quite useful when one deals with self-dual objects, and its use is also common nowadays in the literature about scattering amplitudes. In particular, we will list a series of formulae for the conversion of space-time and SU(2) indices into spinor ones. We will also set our conventions at least in the $(-,+,+,+)$ Lorentzian case.

A.1 SOLDERING FORM

We define the objects in the spaces of unprimed and primed spinors $S_\pm$ as the ones that transform under the two inequivalent irreducible representations of the Lorentz group. We have the isomorphism:

$$S_+ \otimes S_- \longrightarrow \mathcal{M}_4, \tag{409}$$

where $\mathcal{M}_4$ is the 4-dim Minkowski space, endowed with the usual metric $\eta_{\mu\nu}$ (we will use the $(-,+,+,+)$ signature). The isomorphism is defined by an object, the soldering form, which we use to convert Lorentz indices into a pair of prime/unprimed spinor ones. The soldering form is also defined to be Hermitian:

$$(\theta_\mu^{AA'})^* = \theta_\mu^{AA'}. \tag{410}$$

The metric can be then translated as follows:

$$\eta_{\mu\nu} = -\theta_\mu{}^{A}{}_{A'}\theta_\nu{}^{B}{}_{B'}\varepsilon_{AB}\varepsilon^{A'B'}, \tag{411}$$

where ε_{AB} and $\varepsilon_{A'B'}$ are the (antisymmetric) identities in S_+ and S_- respectively which can be used to raise and lower spinor indices.

$$\begin{aligned} \lambda^A\varepsilon_{AB} &= \lambda_B, \\ \varepsilon^{AB}\lambda_B &= \lambda^A, \\ \varepsilon_A{}^A &= 2. \end{aligned} \tag{412}$$

The minus sign in eq. (411) arises because we are working with an Hermitian metric and $(-,+,+,+)$ signature at the same time. This formula also sets a convention we will use often in the main text which we will call "natural" contraction:

$$\eta_{\mu\nu} = \theta_\mu{}^{A}{}_{A'}\theta_{\nu A}{}^{A'}, \tag{413}$$

where the unprimed indices get contracted form top left to bottom right and vice-versa for the primed indices. We will see these contractions in the spinor helicity formalism for scattering amplitude calculations, where contractions of two spinors are denoted by brackets:

$$\begin{aligned} \langle\psi\,\xi\rangle &= \psi^A\xi_A \\ [\psi'\,\xi'] &= \psi_{A'}\xi^{A'} \end{aligned} \tag{414}$$

A.2 SPINOR PROPERTIES

The following properties will be useful in translating tensorial identities into spinorial form. First we state that $SL(2,\mathbb{C})$ acts irreducibly on spaces of completely symmetric spinors S_+^n or S_-^n. Moreover, any spinor can be decomposed into a sum of lower order symmetric spinors (times appropriate ε). A simple example of this is the decomposition of a spinor in $S_+ \otimes S_+$:

$$\phi_{AB} = \phi_{(AB)} - \tfrac{1}{2}\phi^C{}_C\varepsilon_{AB}. \tag{415}$$

From the last relation we learn that $S_+ \otimes S_+ = S_+^0 \oplus S_+^2$. The general case is:

$$S_+^m \otimes S_+^n = \bigoplus_{k=0}^{min(m,n)} S_+^{m+n-2k}. \tag{416}$$

One can easily prove that the dimension of a symmetric spinor space is $dim(S_+^n) = n + 1$ which is in accordance with eq. (416).

Two important type of tensors that we need to analyse from the spinorial point of view are the symmetric and antisymmetric 2-tensors. On symmetric 2-tensors $t_{\mu\nu}$ one can define the operation of the trace reversal. Decomposing $t_{\mu\nu}$ into its symmetric traceless and trace part we obtain:

$$t_{\mu\nu} = s_{\mu\nu} + g_{\mu\nu}t, \qquad t = \frac{1}{4}t_\rho{}^\rho. \tag{417}$$

The trace reversal is defined by:

$$\hat{t}_{\mu\nu} = t_{\mu\nu} - 2g_{\mu\nu}t \tag{418}$$

It is an easy task to check that this operation in spinor form amounts to swap the order of two indices:

$$\hat{t}_{ABA'B'} = t_{ABB'A'}. \tag{419}$$

In the main text we will extensively use the concept of self-duality. The spinor formalism is the best suited to deal with this kind of situations. First we need to transform the completely antisymmetric tensor $\varepsilon_{\mu\nu\rho\sigma}$ into its spinorial counterpart:

$$\varepsilon_{\mu\nu\rho\sigma} \to i\varepsilon_{AC}\varepsilon_{BD}\varepsilon_{A'D'}\varepsilon_{B'C'} - i\varepsilon_{A'C'}\varepsilon_{B'D'}\varepsilon_{AD}\varepsilon_{BC}. \tag{420}$$

This object defines the duality operation that maps n-forms into $(d-n)$-forms. A 2-form $F_{\mu\nu} = -F_{\nu\mu}$ is translated into:

$$F_{\mu\nu} \to F_{ABA'B'} = \tfrac{1}{2}F_{AE'B}{}^{E'}\varepsilon_{A'B'} + \tfrac{1}{2}F_{A'EB'}{}^{E}\varepsilon_{AB}. \tag{421}$$

An easy calculation reveals that the duality operation:

$$F^*_{\mu\nu} = \tfrac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}, \tag{422}$$

can be simply rewritten in spinor language as:

$$F_{ABA'B'} = \mathrm{i}F_{ABB'A'}. \tag{423}$$

From the last formula one can see that the decomposition (421) is one in the so-called self- and anti-self-dual parts of the form F.

A.3 SELF-DUAL BASIS AND SO(3) INDICES

To perform the Hamiltonian decomposition of DYM in the main text we needed to translate the spatial indices into a pair of symmetric spinor ones. It is clear that the there should exist an isomorphism $T^i_{(AB)}$ to achieve this. It will not be hard to see that this map is realized by a multiple of the Pauli matrices. First we need to write into spinor form a basis of (anti)-self-dual two forms $\Sigma^i_{\mu\nu}$. Here $i = 1, 2, 3$ since they span a 3-dimensional subspace. The main relation they satisfy is the following:

$$\Sigma^i_\mu{}^\rho \Sigma^{j\rho}{}_\nu = -\delta^{ij} g_{\mu\nu} - \varepsilon^{ijk}\Sigma^k_{\mu\nu}. \tag{424}$$

Given that they are defined to be (anti)-self-dual it follows that:

$$\Sigma^i_{\mu\nu} \to \Sigma^i_{ABA'B'} = \Sigma^i_{AB}\varepsilon_{A'B'}, \tag{425}$$

where Σ^i_{AB} is the symmetric contraction seen in the last section. It is clear that Σ^i_{AB} and T^i_{AB} should be related. Indeed, we fix the spatial soldering form T^i_{AB} by requiring that it maps the spatial identity δ^{ij} into the identity of S^2_+ (which as we saw before is of course 3-dimensional):

$$\delta^{ij} T^{iAB} T^{iCD} = \varepsilon^{(AC}\varepsilon^{B)D}. \tag{426}$$

Using eq. (424) and (426) we fix the relation:

$$T^{i}{}_{AB} = \frac{1}{\sqrt{2}}\Sigma^i_{AB}. \tag{427}$$

such that the T^i satisfy the multiplication rule:

$$T^{iAE} T^{j\,B}_E = -\frac{1}{2}\delta^{ij} + \frac{1}{\sqrt{2}}\varepsilon^{ijk} T^{kAB}, \tag{428}$$

which shows that $T^i = -i\sigma^i$ where σ^i are the Pauli matrices.

These relations are extremely useful in translating spatial and SO(3) indices into spinor ones. As an example from (428) and (426) one can find a formula to translate the totally antisymmetric object ε^{ijk}:

$$\sqrt{2}\varepsilon^{(AB)}{}_{(CD)(EF)}\Psi^{(EF)} = -\varepsilon^{A(C}\Psi^{D)B} - \varepsilon^{B(C}\Psi^{D)A}, \tag{429}$$

or:

$$\sqrt{2}\varepsilon^{(AB)}{}_{(CD)(EF)}\Psi^{(CD)}\Phi^{EF} = \Psi^{AE}\Phi_E{}^B + \Psi^{BE}\Phi_E{}^A. \tag{430}$$

Another useful rule is the full conversion of the self-dual basis Σ. From (426) and (425) we find:

$$\Sigma^i_{\mu\nu} \to \Sigma^{AB}{}_{MM'NN'} = \sqrt{2}\varepsilon^{(A}{}_M\varepsilon^{B)}{}_N\varepsilon_{M'N'}. \tag{431}$$

B

CURVATURE SPINORS

B.1 RIEMANN TENSOR DECOMPOSITION

In this appendix we will show how to decompose the Riemann tensor in its irreducible spinor representations and how to characterize in this way Einstein and conformally flat solutions. We will follow mainly [22]. The symmetries of the Riemann tensor are sufficient to convert it into a pure spinorial simple form. The skew-symmetry in the first and last pair of indices allows us to write the decomposition:

$$\begin{aligned} R_{\mu\nu\rho\sigma} &= X_{ABCD}\varepsilon_{A'B'}\varepsilon_{C'D'} + \Phi_{ABC'D'}\varepsilon_{A'B'}\varepsilon_{CD} \\ &+ \bar{\Phi}_{A'B'CD}\varepsilon_{AB}\varepsilon_{C'D'} + \bar{X}_{A'B'C'D'}\varepsilon_{AB}\varepsilon_{CD} \end{aligned} \tag{432}$$

Where:

$$X_{ABCD} = \frac{1}{4}R_{AX'B}{}^{X'}{}_{CY'D}{}^{Y'} \tag{433}$$

$$\Phi_{ABC'D'} = \frac{1}{4}R_{AX'B}{}^{X'}{}_{YC'}{}^{Y}{}_{D'} \tag{434}$$

and the bar denotes the complex conjugate. Some properties directly follow from the skew symmetries of the Riemann tensor:

$$X_{ABCD} = X_{(AB)(CD)} \tag{435}$$

$$\Phi_{ABC'D'} = \Phi_{(AB)(C'D')} \tag{436}$$

$$X_{ABCD} = X_{CDAB} \tag{437}$$

$$\Phi_{ABC'D'} = \bar{\Phi}_{ABC'D'} \tag{438}$$

$$X_{A(BC)}{}^{A} = 0 \tag{439}$$

The $\Phi_{ABC'D'}$ spinor correspond then to a real tensor $\Phi_{\mu\nu}$, symmetric and traceless. The Riemann tensor can be seen as a (2,2)-form so we can apply the Hodge operator to the first (resp. second) pair of indices. Schematically the action that this operator produces in the decomposition is the following:

$$R_{\mu\nu\rho\sigma} = X + \Phi + \bar{\Phi} + \bar{X}, \tag{440}$$

$$R^{*}_{\mu\nu\rho\sigma} = -\mathrm{i}X + \mathrm{i}\Phi - \mathrm{i}\bar{\Phi} + \mathrm{i}\bar{X}, \tag{441}$$

$${}^{*}R_{\mu\nu\rho\sigma} = -\mathrm{i}X - \mathrm{i}\Phi + \mathrm{i}\bar{\Phi} + \mathrm{i}\bar{X}, \tag{442}$$

$${}^{*}R^{*}_{\mu\nu\rho\sigma} = +X - \Phi - \bar{\Phi} + \bar{X}. \tag{443}$$

The last symmetry of the Riemann tensor, namely $R_{\{\mu\nu\rho\}\sigma} = 0$ is equivalent to $R^{*}_{\mu\nu}{}^{\nu\rho} = 0$. Applying this last equation to (440) and using (436) one obtains:

$$X_{AB}{}^{B}{}_{C}\varepsilon_{A'C'} = \bar{X}_{A'B'}{}^{B'}{}_{C'}\varepsilon_{AC} \tag{444}$$

given the skew symmetry in AC of X. Contracting the remaining free indices one obtains an equivalent formulation of the sought-after remaining symmetry of the Riemann tensor:

$$\Lambda = \bar{\Lambda} \qquad \text{where} \qquad \Lambda \equiv \frac{1}{6} X_{AB}{}^{AB}, \tag{445}$$

Given the anti-symmetry in AC we can write:

$$X_{AB}{}^{B}{}_{C} = 3\Lambda\varepsilon_{AC}, \tag{446}$$

and a similar equation holds also for primed indices. To write the Einstein equations explicitly we need the Ricci tensor, written in spinor form:

$$R_{\mu\nu} = 6\Lambda\varepsilon_{AB}\varepsilon_{A'B'} - 2\Phi_{ABA'B'} \tag{447}$$

From here we can see that the trace-free part of the Ricci tensor and the Einstein tensor (i.e. trace-reversed) are:

$$\begin{aligned} R_{\mu\nu} - \frac{1}{4}g_{\mu\nu} &= -2\Phi_{\mu\nu}, & (448)\\ G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} = R_{ABB'A'} \\ &= -6\Lambda\varepsilon_{AB}\varepsilon_{A'B'} - 2\Phi_{\mu\nu}. & (449) \end{aligned}$$

To obtain the last line we used the fact that the operation of trace-reversal amounts in spinor terms only to an exchange in a pair of (un-)primed indices. The Einstein equations then can be re-written in spinor form as:

$$R_{\mu\nu} = \lambda g_{\mu\nu} \qquad \Leftrightarrow \qquad \Phi_{ABA'B'} = 0, \quad \Lambda = \frac{1}{6}\lambda. \tag{450}$$

It is clear from (440) and (443) that they can also be stated as follows:

$${}^{*}R^{*}_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}. \tag{451}$$

The last equation is equivalent to the statement that the self-dual part of the Riemann tensor is self-dual as a two-form. Moreover, X components are not constrained by the Einstein equations and only the symmetric traceless part is (as we already know). In particular the X part will be related to the Weyl tensors. Now we analyse the X component of the Riemann tensor. It can be split into a totally symmetric part (Ψ) plus other remaining parts, given the already stated symmetries:

$$X_{ABCD} = \Psi_{ABCD} + \Lambda(\varepsilon_{AC}\varepsilon_{BD} + \varepsilon_{AD}\varepsilon_{BC}) \tag{452}$$

Now we are in a position to restate the Riemann tensor decomposition we started from (432):

$$\begin{aligned} R_{\mu\nu\rho\sigma} &= \Psi_{ABCD}\varepsilon_{A'B'}\varepsilon_{A'B'C'D'} + \Phi_{ABC'D'}\varepsilon_{A'B'}\varepsilon_{CD} \\ &+ \bar{\Phi}_{A'B'CD}\varepsilon_{AB}\varepsilon_{C'D'} + \bar{\Psi}_{A'B'C'D'}\varepsilon_{AB}\varepsilon_{CD} \\ &+ 2\Lambda(\varepsilon_{AC}\varepsilon_{BD}\varepsilon_{A'C'}\varepsilon_{B'D'} - \varepsilon_{AD}\varepsilon_{BC}\varepsilon_{A'D'}\varepsilon_{B'C'}). \end{aligned} \tag{453}$$

The Weyl tensor is the combination of the two (Ψ) part, that in turn constitute its self-dual and anti-self-dual components.

B.2 SPINOR FORM OF COMMUTATORS OF COVARIANT DERIVATIVES

From the very definition of the curvature tensor the commutator of two derivatives applied to a general form gives various combinations of the Riemann tensor contracted with the form itself. Here we just state the formulae of the commutator on a general $(2,2)$-spinor:

$$2d_{[\mu}d_{\nu]}t_{\rho\sigma} = (\varepsilon_{AB}\Box_{A'B'} + \varepsilon_{A'B'}\Box_{AB})\, t^{C}{}_{D}{}^{E'}{}_{F'} \tag{454}$$

$$\begin{aligned}\Box_{A'B'}t^{C}{}_{D}{}^{E'}{}_{F'} &= \bar{X}_{A'B'Q'}{}^{E'}t^{C}{}_{D}{}^{Q'}{}_{F'} - \bar{X}_{A'B'F'}{}^{Q'}t^{C}{}_{D}{}^{E'}{}_{Q'} \\ &+ \bar{\Phi}_{A'B'Q}{}^{C}t^{Q}{}_{D}{}^{E'}{}_{F'} - \bar{\Phi}_{A'B'D}{}^{Q}t^{C}{}_{Q}{}^{E'}{}_{F'}\end{aligned} \tag{455}$$

$$\begin{aligned}\Box_{AB}t^{C}{}_{D}{}^{E'}{}_{F'} &= X_{ABQ}{}^{C}t^{Q}{}_{D}{}^{E'}{}_{F'} - X_{ABD}{}^{Q}t^{C}{}_{Q}{}^{E'}{}_{F'} \\ &+ \Phi_{ABQ'}{}^{E'}t^{C}{}_{D}{}^{Q'}{}_{F'} - \Phi_{ABF'}{}^{Q'}t^{C}{}_{D}{}^{E'}{}_{Q'}\end{aligned} \tag{456}$$

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