



FINITE ELEMENT IMPLEMENTATION OF KOITER'S
INITIAL POST-BUCKLING ANALYSIS

by

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Abstract

Thin-wall structures as the main components in the aircrafts, as well as in many other types of structures, such as automotive, trains, ships, etc., are exposed to complex loading conditions. For some parts of these structures, e.g. the upper surface of wings of aircrafts, buckling is the most common mechanism of failure which designers have to ensure their safety with confidence. Prediction of buckling loads as an eigenvalue problem has become a standard provision from most of commercial FE codes as typical design tools nowadays. However, experienced designers and engineers are fully aware of the uncertainties in the structural collapsing load, as some structures could collapse at a load level significantly lower than the predicted buckling load. On the other hand, there are also structures which can sustain increasing loads after buckling to levels much higher than the buckling load. Such phenomenon has been well accommodated and explained by Koiter's initial post-buckling theory which was established in 1940s. After Koiter's thesis was translated into English, it quickly gained its popularity in 1960-70s and a lot of applications of it were made mostly by analytical means, whilst its implementation in FEM has been extremely disappointing, although attempts in this direction could be traced as far back as early 1970s shortly after Koiter's thesis was made known. Most successful applications of Koiter's theory were made to relatively simple structures due to the limitation of analytical means of analysis. In fact, there has been no lack of publications involving FEM of Koiter's theory in the past half a century. Nevertheless, the fact is that the theory has not been incorporated in any mainstream commercial FEM code yet.

As an alternative, some commercial FEM codes, e.g. Abaqus, Ansys and Nastran, offer facilities for users to by-pass the bifurcation point in order to obtain the post-buckling deformation after introducing a certain initial imperfection. This is inevitably a

highly nonlinear problem and hence computationally expensive. An even more worrying fact is that it might not provide what designers need whilst having potential to mislead designer. In fact, researchers had been there as early as 1940s to look for post-buckling equilibrium path but the endeavour wound up without any major breakthrough soon after Koiter's theory became known. Through Koiter's theory, it was understood that the actual buckling behaviour, i.e. whether collapsing prematurely or capable of sustaining higher loads, depends not only on the post-buckling equilibrium path but even more on the stability of the equilibrium state, which is not what a nonlinear analysis could provide. The efforts made to obtain the post-buckling deformation are believed to be on a wrong track. Successful implementation of Koiter's theory in FEM as a numerical approach would make them redundant, as it did once before when analytical approaches were employed.

In order to achieve this goal, the first task in the present project is to identify the obstacles which have prevented Koiter's theory from being implemented satisfactorily through FEM. A closer examination suggests that Koiter's theory has not been incorporated in mainstream commercial FEM codes not without genuine reasons. However, researchers seemed to shy away from revealing the problems directly and explicitly before the difficulties could be addressed appropriately. The objective of this project is to clear these obstacles and to demonstrate fully its feasibility. These key obstacles are found as follows.

- 1) A key output from Koiter's theory, initial post-buckling coefficient b has been found mesh sensitive. One would have to endure the mesh sensitivity if conventional finite elements were employed or to employ nonconventional finite elements specially formulated, e.g. having higher order of continuity, to avoid mesh sensitivity. None of them would be attractive to any commercial FE code developer.

2) To evaluate the same initial post-buckling coefficient as mentioned above, a set of special simultaneous linear equations as the governing equation for the secondary perturbation mode have to be solved, where a convincing solution lacks. To facilitate the analysis, an iterative approach was proposed and believed to be employed in all previous FE implementations. It is again unattractive to commercial FE code developers.

3) There is lack of verification for the FE implemented Koiter's theory. A conventional and effective way of verifying FE approaches has been proposed to compare with available analytical results. There has been no lack of analytical results, e.g. a simply support square plate under biaxial compression, after the developments in 1960-70s. Yet, no comparison has been made to analytical results as far as the applications to plates and shells are concerned. The FE implemented Koiter's theory has therefore been left in a state where even a basic level of verifications has not been fulfilled. No one has acknowledged the fact in the open literatures.

The first obstacle identified above is directly associated with the very basis of FE formulation. Koiter's theory was based on the total potential energy (TPE). A natural translation of it led to the N-notation formulation of FEM for the geometrically nonlinear problem. Whilst this was seemingly smooth and straightforward, the N-notation appeared to be a stumbling block, as the stiffness matrices obtained could not be easily expressed explicitly. Even if one managed to do so, they were not unique. As a consequence, the N-notation FE formulation of the geometrically nonlinear problem had been left as a 'symbolism', not until recently when this issue has been resolved by revealing the full commutativity of displacements in strain energy density as its intrinsic property and hence in the FE formulation based on it. Based on this, the stiffness matrices can be expressed not only explicitly but also uniquely. In absence of an operational N-notation presentation of Koiter's theory previously, an alternative was to follow the principle of virtual work

(PVW), which was equivalent mathematically, yet significantly different for FE implementation. The alternative expressions of the initial post-buckling coefficients have enjoyed great successes in the development of Koiter's theory in analytical approaches and they have been taken as the only viable means in almost all attempts of FE implementation. However, it did not come without a price, mesh sensitive being the penalty. The direct symptom was a kind of numerical locking phenomenon which is a purely numerical problem as it has never emerged in analytical solutions. A closer examination would reveal the direct use of stresses in the expression of b . Lower accuracy of stresses than displacements is a generic deficiency to FE which is well-known. The lack of accuracy in stresses has therefore been identified as the reason. Suffice it to say, locking has never emerged in analytical solutions and unconventional finite elements employed to avoid locking followed nothing but usual means of improving the accuracy of stress evaluation. The locking issue, once appropriately identified, has been successfully and completely resolved after re-establishing the N-notation FE formulation. When a and b were expressed and evaluated from the N-notation formulation, numerical locking was no longer present because of its reduced reliance on evaluated stresses. Numerical examples have been shown and the first obstacle is believed to have been overcome satisfactorily.

As for the second obstacle identified, a closed form solution has been found to the secondary perturbation displacement in this project. It is mathematically rigorous and computational efficient. The way of obtaining it has been proven to be probably more rational than the same in analytical approaches, e.g. in terms of the existence of the solution, which was mostly taken for granted previously. A rigorous proof has been provided. It is also applicable to problems involving multiple buckling modes. With the closed form solution, the second obstacle has been satisfactorily resolved.

The third obstacle was perhaps the fatal one for the adoption of Koiter's theory in commercial FE codes. Without convincing verification, it would be against the basic principles of commercial software. It has been shown that the absence of such comparison was not an oversight of previous researchers and the FE results differed from analytical results widely, if one wished to reveal the problem. After a careful examination, the cause for the difference has been identified. In fact, it was because of the different boundary conditions employed in both analyses. Once the boundary conditions in the FE analysis are prescribed as the same as in the analytical solution, excellent agreement with analytical results can be obtained as have been shown in the thesis.

In order to make comparisons with another highly relevant class of structures, shallow curved panels under axial compression, efforts have been made in the reproduction of an analytical solution. In fact, an analytical solution was available but it was derived under the Love-Kirchhoff (L-M) plate and shell theory. With all available finite elements formulated based on the Reissner-Mindlin (R-M) hypothesis, an update is required in order to make sure that comparisons can be made alike with alike. The analytical results obtained degenerates to L-K solution when transverse shear stiffness is set to infinity, which can be further specialised to that of plates when curvature vanishes, leading the results which have been employed in the verification mentioned above when biaxial loading is reduced to a uniaxial loading condition. By degenerating the obtained analytical results here directly to plates, one finds that the expression of b reproduces that as directly obtained for plates. Any variation of its value will be due to the variation in the buckling load. However, the comparisons obtained between the FE and analytical results can be more relevant when the analytical results are based on the R-M theory.

For curved panels as a slightly more sophisticated case than flat plates, the boundary conditions in various regimes implied in the analytical solution have been

unravelling and prescribed to the numerical reproduction. Different from flat plate, in-plane shears emerge in the buckling and post-buckling regimes. Moreover, the tangential displacements obtained from the analytical solution have been prescribed as essential boundary condition in order to precisely determine the buckling modes and the secondary perturbation modes. Excellent agreements of values of b have been achieved based on the accurate determination of secondary perturbation modes. While without appreciation of such boundary condition, *viz.* the normal stresses as the natural boundary condition and tangential displacements as the essential boundary condition, it is impossible to reproduce the analytical results via finite element method.

As a further application of the FE formulation of Koiter's theory, the mode interaction problem has been analysed. The buckling of a simply supported cylinder under axial compression has been known as a multiple-mode problem. The analytical expressions of Koiter has been resorted to for the detailed illustrations of buckling modes in terms of three displacements for both axisymmetric and non-axisymmetric buckling modes. The boundary conditions hence have been specified and employed accordingly. The numerical buckling loads show excellent agreement with the analytical ones along with the buckling modes. It can be found that the eigenvalues are clustered within a small range of the buckling load, rather than exactly repeated eigenvalues. By using the eigenvectors obtained for all modes within the cluster, the initial post-buckling analyses have been carried out in an approximate sense of mode interaction. Out of the analysis, the severity of the effects of mode interaction on the initial post-buckling behaviour has been demonstrated.

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Abbreviations and Notations

Abbreviations

FEM	Finite element method
IPBC	Initial post-buckling coefficient
L-K	Love-Kirchhoff plate theory
PVW	Principle of virtual work
R-M	Reissner-Mindlin plate theory
SED	Strain energy density
TPE	Total potential energy

Notations

A	Displacement interpolation matrix
B	Strain operator in finite element
C	Material stiffness
F	Discretised loading vector
K₀	Linear elastic stiffness matrix
K_t	Tangential stiffness matrix
N₁	Linear incremental stiffness matrices
N₂	Quadratic incremental stiffness matrices
q	Displacement vector
q⁽⁰⁾	Pre-buckling displacement
q⁽¹⁾	Buckling displacement
q⁽²⁾	Secondary perturbation mode
q^c	Complementary displacement
q^p	Particular displacement
S	Symmetric matrix operator in flip transformation
T	Auxiliary functional matrix in orthogonality condition
ε	Green strain
σ	Piola-Kirchhoff stress
ψ	Commutative and symmetric stiffness matrix operator

a	First initial post-buckling coefficient
b	Second initial post-buckling coefficient
B	Breadth of plate or shell
D	Bending rigidity, defined as $Et^3/12(1-\nu^2)$
E	Young's modulus of homogenous isotropic material
F	Airy stress
H	Amplitude of deflection
k	Defined as mB/L
L	Length of plate or shell
L_{FF}	Differential operator
m	Number of half waves in axial direction
n	Number of waves in circumferential direction
N_{xx}	In-plane stress resultant in x direction
N_{yy}	In-plane stress resultant in y direction
N_{xy}	In-plane shear stress resultant
M	Bending moment
Q	Commutative strain energy density
P	Defined as $m\pi R/L$
R	Radius of shell or cylinder
s	Coefficient of complementary solution
t	Thickness of plate or shell
U	Strain energy density
u	In-plane displacement in x direction
v	In-plane displacement in y direction
w	Later displacement
β	Defined as π/B
Γ	Commutative and symmetric strain density energy
δ_{ij}	Kronecker delta
η	Aspect ratio of plate, defined as L/B

λ	Loading parameter
λ^c	Buckling load
μ	Imperfection amplitude
ν	Poisson's ratio
ξ	Perturbation coefficient
Π	Total potential energy
φ	Rotation
ω_{ij}	Deflection coefficient of secondary perturbation mode
∇^2	Laplacian operator
∇_L	Strain operator, linear with derivatives of displacement
∇_N	Strain operator, quadratic with derivatives of displacement

Chapter 1 Introduction

1.1 Background

Thin-wall structures as the main components in the aircrafts, as well as in many other types of structures, such as automotive, trains, ships, etc., are exposed to complex loading conditions. For some parts of these structures, e.g. the up surface of wings of aircrafts, buckling is the most common mechanism of failure which designers have to ensure their safety with confidence. Prediction of buckling load as an eigenvalue problem has become a standard provision from most of commercial FE codes as typical design tools nowadays. However, experienced designers and engineers are fully aware of the uncertainties in the structural collapsing load, as some structures could collapse at a load level significantly lower than the predicted buckling load. On the other hand, there are also structures which can sustain increasing loads after buckling to levels much higher than the buckling.

The mathematical formulation of the buckling problem can be traced back to Euler [1]. The problem was systematically formulated in Timoshenko and Gere [2] as a classic textbook. The approach taken has been referred to as the classic buckling theory. As a mechanical problem, it attracted a great attention from the 1940s onwards when shell structures start to be employed in the aerospace industry, where pronounced discrepancies were observed between theoretical prediction based on the classic buckling theory and experimental data, in particular, for cylindrical shells under axial compression, as can be seen from Figure 1.1 [3]. In order to understand the discrepancies, great efforts were made by including the non-linear terms in the energy expression so that the post-buckling equilibrium paths could be obtained, giving birth to a variety of theories based on post-buckling analysis, primarily, conducted analytically, to name but a few [4-14]. Most of the efforts were brought to an unsettled end rather abruptly when Koiter's initial post-

buckling theory became known to the English world [15] in 1960s, although it was originally published in 1945 but in Dutch.

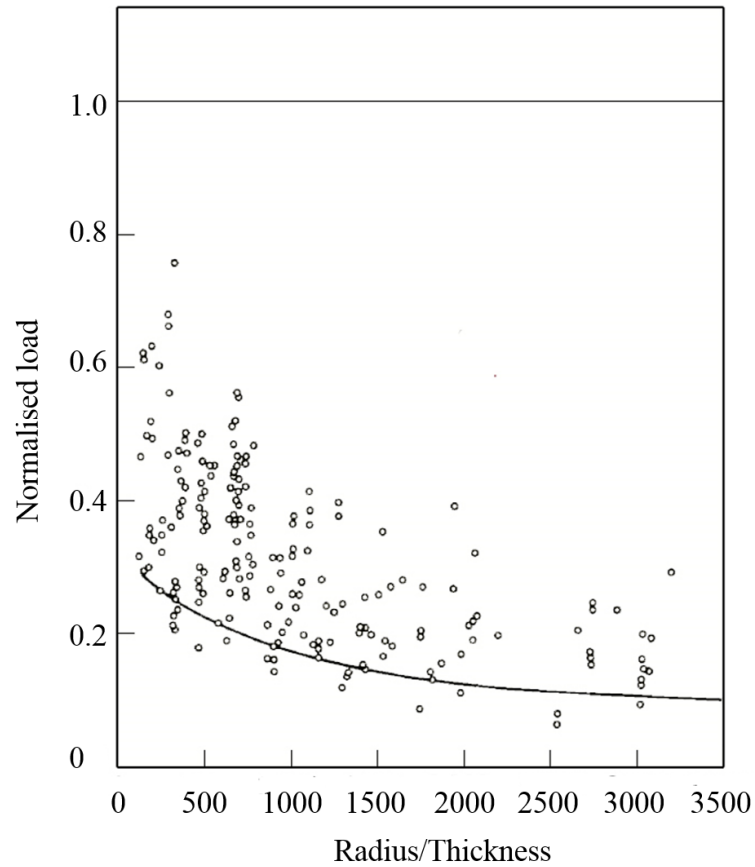


Figure 1.1 Comparisons of experimental data and classic solution of cylinders under compression

Another rather complementary development was pursued by Stein [16, 17] in 1950s. Instead of looking into the post-buckling regime, it investigated with great care the pre-buckling deformation leading to buckling. An inconsistency was identified in the boundary conditions between the idealistic ones as assumed in the classic buckling analysis and the realistic ones as could be practically achieved in experiments. By imposing the realistic boundary conditions in his analysis, Stein managed to show a noticeable reduction of the buckling load from that by the classic buckling analysis, which established a fact that the consistency in the boundary conditions for pre-buckling analysis and those for the buckling analysis mattered. He then validated his findings by

conducting extremely well designed experiments employing near-perfect specimens with the boundaries kept closely to the simply supported conditions as employed in his analytical solution. Convincing results were obtained confirming the fact that remaining discrepancies in addition to those introduced by the inconsistency in the boundary conditions were attributed to the presence of various imperfections, in particular, initial geometric deviation from the ideal shape of the structure. The high sensitivity of axially compressed cylinders to initial imperfections became established. However, Stein's consistent pre-buckling theory did not explain why the structure considered was imperfection sensitive and it did not specify what type of structures were imperfection sensitive as apparently not every structure behaved the same.

By focusing on the stability of the equilibrium in the neighbourhood of buckling state, the buckling problem was examined by Koiter using his initial post-buckling theory through a perturbation approach [15]. Without having to deal with the highly nonlinear problem for the advanced post-buckling deformation, useful information about the behaviour of the structure, such as the stability of the equilibrium state at the point of buckling and the sensitivity of the structure to initial imperfections, could be obtained. Whilst the post-buckling deformation could be constructed in asymptotic manner, the theory was not meant to be for this purpose. The true value of it was to achieve a much deeper understanding of the nature of buckling by examining the behaviour of the structure at the point of buckling from its initial post-buckling perspective. This theory gained its overwhelming popularity for over two decades after Koiter's thesis was translated in English following a period of hibernation for about two decades from its first publication. Given the timing, most accounts produced then were based on analytical means [18-39] as reviewed in [40]. Due to the limitations of analytical research, most of the results obtained were only applicable to relatively simple structures with simple

loading and boundary conditions. This restricted its value as a design tool to benefit designers and analysts of realistic structures which are normally of complexity of structural configurations, such as stiffened panels involving heterogeneous materials, e.g. laminated composites, subjected to general loading and boundary conditions.

The versatile applications of Koiter's theory relies on its successful implementation through the finite element method (FEM). However, the implementations of FEM to mechanical problems have not always been a smooth path. For instance, it took researchers all over the world about a full decade to overcome the difficulties associated with the C^1 continuity for the plate and shell applications of FEM before a formulation arising from the degenerated 3D brick element [41]. When it was initially proposed, it was meant to be for thick plates and shells only. The reduced integration technique brought it forward for general-purpose applications, regardless the thickness [42]. The implementation of FEM to Koiter's theory proves to be an even longer and more winding path. Given the fact that attempts of FEM implementation of Koiter's theory could be traced a far back as early 1970s [43], there has not been a single mainstream commercial FEM code, e.g. Abaqus [44], Ansys [45] or Nastran [46], that has incorporated it as a standard provision to benefit users. Those who understand Koiter's theory would never agree that the theory was not worthy implementing.

It should be noted that, whilst the research activities on Koiter's theory were at their height, the development of commercial FEM code was still being incubated [47-67]. Having said so, since Koiter's theory was formulated squarely based on the total potential energy, it should be a relatively simple step to translate it into an appropriate FEM formulation before being implemented systematically. However, it simply has not

happened in this way. There have not been many, if any, researchers challenged the state-of-the-art, at least not explicitly.

Since the buckling problem in engineering practices cannot be avoided, engineers still have to face them. Beyond the classic buckling analysis, the best the existing FEM buckling facility can offer directly is to allow consistent boundary conditions for buckling and pre-buckling analyses, or even allowing the nonlinearity within the pre-buckling phase to be incorporated, thus reproducing Steins analysis for general types of structures [16, 17]. It would not result in any assessment on the imperfection sensitivity of the structure. The catch here is the fact the some structures are insensitive to imperfections whilst others are deemed to be sensitive. One cannot apply a single measure blindly to all structures without either risking the safety of the structure or overdesigning the structure unduly, neither being acceptable, especially for aerospace applications. As an alternative, very much in desperation, the following approach has been offered by some commercial codes and it seems being adopted by researchers from time to time. In this approach, the buckling problem, whether its pre-buckling behaviour is deemed to be linear or nonlinear, the nonlinear algorithm for finite deformation option will be employed. An incremental iterative method, such as Newton-Raphson as is often the case, is therefore adopted which often falsifies a nonlinear analysis. However, this alone is still not sufficient. An iterative algorithm, in particular, the Newton-Raphson method, tends to fall apart at the point of buckling, given the existence of multiple solutions for equilibrium there. One has to artificially introduce a small imperfection to facilitate the analysis to by-pass the buckling point before entering the regime in a close neighbourhood of the post-buckling equilibrium paths. Even one manages to get there computationally, there is lack of guidance how the results should be interpreted before they demonstrate their engineering relevance. Without the knowledge of Koiter's theory, such obtained results could easily

be misleading, as will be demonstrated later. One is reminded that such efforts are not new at all. Before Koiter's theory was known to the English world, the hope of understanding the nature of the buckling problem was placed on the attainment of post-buckling equilibrium paths. The only difference was they were obtained analytically and now they can be obtained numerically. One way or another, the efforts did not resolve the problem then and will be unlikely to resolve the problem now. The previous campaign wrapped up as Koiter's theory was introduced whilst the means of solving the problem were analytically based. It can be expected that once Koiter's theory is successfully implemented through FEM, it will reproduce the same story numerically this time.

1.2 The Obstacles to the FEM implementation of Koiter's Theory

A closer examination suggests that the FEM implementation of Koiter's theory has been in this position not without genuine reasons. However, researchers seemed to shy away from revealing the problems directly and explicitly before the difficulties could be addressed appropriately. The grand aim of this thesis is to resurrect Koiter's theory by demonstrating the feasibility of its FE implementation although it will take a lot more than just this thesis to achieve. It is therefore good to identify key obstacles as follows, which will have to be overcome before developers of commercial FE codes would seriously contemplate the possibility of incorporating it into their codes.

1) A key output from Koiter's theory, initial post-buckling coefficient b , which is by large relatively more important piece of information as the outcome of the theory, has been found mesh sensitive. One would have to endure the mesh sensitivity if conventional finite elements were employed or to employ nonconventional finite elements specially formulated, e.g. having higher order of continuity, to avoid mesh sensitivity [47, 53, 60]. None of them would be attractive to any commercial FE code developer.

2) To evaluate the same initial post-buckling coefficient as mentioned above, a set of special simultaneous linear equations as the governing equation for the secondary perturbation displacements has to be solved, where a convincing solution lacks.

3) As far as plates and shells are concerned, the obtained results using FE so far in the literature have never been compared with available analytical results. The FE implemented Koiter's theory has therefore been left in a state where even basic level of verifications has not been fulfilled. Out of the author's intensive investigation on this particular issue, it can be confirmed that the absence of such comparison was not an oversight of previous researchers. It is associated with issues so profound that it will have to be addressed before a meaningful verification can be achieved.

1.3 Aim and Objectives

The aim of this research is to implement Koiter's initial post-buckling theory through the FEM by overcoming the obstacles as identified above. Without successful FEM implementation, the fate of Koiter's theory would be doomed in terms of practical engineering applications and the fact is that it has been in hibernation for a few decades already such that young generations of engineers have hardly heard of it, let alone making good use of it. Its full potential will be realised once it is incorporated in commercial FE codes and accessible to ordinary users and structural designers and analysts can then start to benefit from this extremely helpful theoretical framework in order to tackle the buckling problem with confidence. However, this would not happen until the feasibility has been fully demonstrated. This has been the aim of the present research project, which will promote the move for commercial FE code developers to start to contemplate Koiter's theory in their codes, given the outcomes of this thesis and other associated publications.

To achieve the aim of this research, the following objectives need to be accomplished.

- 1) To overcome the mesh sensitivity issue associated with the FEM evaluation of one of the initial post-buckling coefficients as a key output from the Koiter's theory;
- 2) To formulate Koiter's theory and to present it in the form ready for FE implementation, after reflecting necessary measures taken in order to achieve the objective as given above;
- 3) To implement the formulation through an in-house development FE code in order to demonstrate its feasibility;
- 4) To resolve any numerical issue associated with the implementation as stated above;
- 5) To verify the implementation against available analytical solutions;
- 6) To extend the applications of the FE implemented Koiter's theory as far as possible in order to demonstrate the feasibility of its FE implementation.

1.4 Thesis Layout

The thesis consists of nine chapters. Apart from the introduction as presented in this chapter, the layout of the remaining chapters is outlined as follows.

Chapter two is devoted to the literature review which covers major subjects relevant to the present research. The finite element formulations for the geometrically nonlinear problem have been briefly reviewed, giving particular attention to the developments around the so-called B-notation and N-notation [68], as well as their application to the FE implementation of Koiter's theory. It is clear that the B-notation formulation encountered a critical problem, mesh sensitivity or sometimes described as a

locking phenomenon. This has attracted a significant attention in the past decades and yet still presents as a major obstacle to the successful FE implementation of Koiter's theory. On the other hand, the N-notation has never been appropriately formulated. Before Koiter's theory was made known to the English world, there had been a lot of work aiming at solving nonlinear governing equations to obtain the post-buckling equilibrium paths. They were mostly dealt with analytically. As a result, the solvable problems were very limited, e.g. cylinders under axial compression. There was a significant amount of experimental work on the subject buckling. Koiter's theory came to light to address the significant discrepancies between the results as obtained from the classic buckling theory and experimental observations arising in 1940s. Most of its applications were made using analytical means in the 1970s, given the lack of commercial FE codes then. The existing work on the numerical implementation of Koiter's theory was also reviewed.

The concept of commutative strain energy density is introduced in Chapter three, which lays the firm basis for the N-notation for the finite element formulation for the first time. The observation and consequences of the lack of commutativity in strain energy density expression have been fully elaborated and resolved. Based on the commutative strain energy density expression, when finite elements are formulated following the N-notation, the symmetry of stiffness matrices has been achieved which are uniquely and explicitly expressed. The secant stiffness matrix and tangent stiffness matrix have been expressed in a consistent and unique way.

In Chapter four, Koiter's initial post-buckling theory has been presented taking advantage of the new and consistently obtained N-notation for serious finite element implementation, after an appropriate recapitulation of the basics of Koiter's theory. The

governing equation of buckling and secondary perturbation mode have been obtained along with the expressions of two IPBCs for both single and multiple mode systems.

A closed-form solution to the secondary perturbation mode has been presented in Chapter five. The governing equation of secondary perturbation mode has been identified as a set of simultaneous linear algebraic equations with singular coefficient matrix, subject to additional orthogonality condition(s). The closed-form solution is made possible by considering it as the sum of a complementary and a particular solution, which is a standard mathematical approach for linear equations, algebraic or differential [69]. The buckling mode provides a complementary solution naturally. After obtaining a particular solution, the complete solution can be constructed in a closed form without having to go through any iterative process. This proposed method is applicable to both single and multiple mode systems. Being mathematically rigorous, the solution has also been verified through other examples both analytically and numerically. Apart from this, some numerical examples employed in the open publications have been reproduced via the FE formulation for Koiter's theory using N-notation and closed-form solution for secondary perturbation form in the in-house code. The problem of mesh sensitivity encountered when calculating the second initial post-buckling coefficient has been completely resolved, a better feasibility and precision shown when compared to approaches in the open publications.

To validate the formulation and implementation, simply supported flat plates have been analysed using an in-house developed FE code. Numerical results obtained are compared with the analytical solutions in Chapter 6. The displacement and traction boundary conditions employed in the analytical solution have been revealed. By using the identical boundary conditions, the numerical results show an excellent agreement with

the analytical solutions in terms of two initial post-buckling coefficients and the two in-plane displacements of secondary perturbation mode.

To provide another case for the numerical results to compare with, an analytical investigation of shallow shells under axial compression based on Reissner-Mindlin (R-M) theory has been carried out in the seventh Chapter. The results, including the buckling load, coefficient of deflection of secondary perturbation mode and expression of second IPBC have been obtained and they reconciled precisely their counterparts based on Love-Kirchhoff (L-K) theory, whilst resulting in discrepancies for buckling loads and second IPBC for moderately thick shells. Similarly, both the natural and essential boundary condition in various regimes have been unravelled and precisely prescribed into the numerical models. It has been found out that in the thin panels, both L-K and R-M theories could deliver close solution in terms of buckling loads and two initial post-buckling coefficients which have been excellently reproduced via the numerical solutions. While for the moderately thick panels, the disparities between the L-K theory and R-M theory become evident. However, the numerical results are still in good agreement with the ones from R-M theory by using the consistent boundary conditions employed in the analytical solution.

In the eighth Chapter, a multiple mode system has been considered, viz. cylinder under axial compression. Basic analytical expressions for this problem from Koiter has been briefly recapitulated in order to identify the boundary conditions. The boundary conditions for both axisymmetric and non-axisymmetric buckling modes have been recovered and employed respectively. The numerically obtained family of lowest eigenvalues are actually not exactly repeated eigenvalues, but a clustered pack of them in the neighbourhood of the value of the analytical buckling load. Analyses of initial post-

buckling behaviours have been conducted. For one axisymmetric buckling mode considered as the outcomes of a single mode buckling problem, unstable initial post-buckling equilibrium and hence imperfection sensitivity are predicted. Approximating the cluster of eigenvalues as repeated ones with their corresponding eigenvectors are considered as multiple modes of buckling, it has been shown that the interactions of these modes tend to exacerbate the initial post-buckling instability and initial imperfection behaviour.

In the final Chapter, a summary of this thesis has been made with major achievements throughout the project and conclusions arrived through the research carried out along with a list of suggestions for the future work.

Chapter 2 Literature Review

2.1 Introduction

This chapter is devoted to the literature review regarding the endeavours in the problem of buckling last century and, in particular, Koiter's theory and its applications to the stability of thin-walled structures through both analytical and numerical means.

Since the first half of the last century, huge sums of efforts and resources have been invested in the structural problems associated with buckling one way or another. Latest developments includes POSICOSS [70] and COCOMAT [71], etc.

A massive amount of efforts was made on the buckling of cylinder under axial compression, to be specific, due to the erratic disparities between the analytical and experimental results. To understand reason behind, analytical work on the post-buckling equilibrium path was carried out firstly by Karman and Tsien [4]. The boundary condition as an inevitable factor was also taken into account to interpret the difference [16, 17, 72-75]. Besides, advanced manufacturing techniques was introduced to produce nearly perfect specimen to study the influence of initial geometrical imperfection [76-79]. All the relevant work will be summarised in the coming sections.

However, out of the endeavours, no clear conclusion was drawn to the phenomenon, *viz.* why some structures could sustain the increasing load after bypassing the buckling point while others could display a premature failure prior to the buckling load as predicted in analytical solution. Koiter's initial post-buckling theory gave rise to a definite answer by focusing at the equilibrium path in the neighbourhood of bifurcated point using perturbation method [15]. The stability of structure in the initial post-buckling regime could be easily predicted through the two terms, *viz.* 3rd order and 4th order energy variations which later were simplified as initial post-buckling coefficients by Budiansky and Hutchinson [80] through an asymptotic expression of sustaining load at

the bifurcated point. Extensive analytical investigations based on such theory were conducted in 1960s and 1970s, claiming a huge success [40]. While its implementation into finite element does not seem as successful as the analytical application. To explicitly seek for the obstacles which prevent its implementation, a thorough literature review is provided latter.

2.2 Geometrically Nonlinear Problem in Finite Element Formulations

To characterise the geometrically nonlinear structural behaviours, the principle of virtual work (PVW) based on the equilibrium state of structure which was referred to B-notation by Zienkiewicz [81] was firstly employed and revealed its prominence in numerical applications. An alternate method was proposed by Mallett and Marcal [82] as referred to N-notation, the principle of total potential energy. The most evident feature of N-notation is that the stiffness matrices maintain consistent throughout three equations, viz. the total potential energy, equilibrium equation and incremental equilibrium equation with different multipliers as follows.

$$\Pi = \mathbf{q}^T \left(\frac{1}{2} \mathbf{K}_0 + \frac{1}{6} \mathbf{N}_1 + \frac{1}{12} \mathbf{N}_2 \right) \mathbf{q} - \mathbf{q}^T \mathbf{F} \quad (2.1)$$

$$\left(\mathbf{K}_0 + \frac{1}{2} \mathbf{N}_1 + \frac{1}{3} \mathbf{N}_2 \right) \mathbf{q} - \mathbf{F} = \mathbf{0} \quad (2.2)$$

$$(\mathbf{K}_0 + \mathbf{N}_1 + \mathbf{N}_2) \Delta \mathbf{q} - \Delta \mathbf{F} = \mathbf{0} \quad (2.3)$$

where $\mathbf{K}_0$ is the linear elastic stiffness matrix. $\mathbf{N}_1$ and $\mathbf{N}_2$ are the linear and quadratic incremental stiffness matrices of displacement $\mathbf{q}$ respectively. $\mathbf{F}$ is discretised loading vector applied at the conservative system associated with $\mathbf{q}$. Π is denoted as the total potential energy. Some predicaments were encountered to generate the consistent stiffness matrices at the beginning of using N-notation as pointed out by Rajasekaran and Murray in [83], ‘the difficulty with the notations arises from the fact that the expressions used to evaluate $\mathbf{N}_1$ and $\mathbf{N}_2$ are not unique’ although one could conceive the stiffness

matrices that sufficed the equations (2.1-2.3). The lack of uniqueness of $\mathbf{N}_1$ and $\mathbf{N}_2$ not only made accuracy and reliability of N-notation doubtful but also prohibited the practicability into FEM. Massive computer memory space would be needed so that nowadays computational capacity even could not meet the requirements as concluded by Gallagher [84]. Some further studies on N-notation were conducted by Felippa [85], Wood and Schrefler [68] to correlate the different stiffness notations in B-notation and N-notation. However, their endeavour did not resolve the problem of lack of uniqueness. Based on the achievement of the second chapter of this thesis, an explicit reconciliation of these two notations has been made by Li *et al.* in [86].

2.3 Analytical Solutions on Buckling Problem

The first introduction of buckling problem was investigated by Euler through a simple illustration of column subject to compressive loading [1]. The critical load which causes the structure member to have a sudden deflection was characterised as the buckling load and it was perfectly reproduced through the experimental observation. For a more sophisticated structural configuration, the theoretical investigation of cylinder subject to axial compression was initiated by Lorenz, Timoshenko and Southwell [87]. The very first attempt was based on some simplified models where a linear eigenvalue problem was considered and solved without considering the nonlinearity in the displacement-strain relation. However, the theoretical prediction could not be realised any more in the experimental environments. The experimental data was erratic, being a fraction of the analytical solution. Moreover, the catastrophic failure afflicted the cylinder immediately after the buckling. Massive investigations into this problem to explain the discrepancy between them were conducted as reviewed as follows.

2.3.1 Investigation of Post-buckling Equilibrium Path

After the observation of the great discrepancy between experimental data and theoretical predictions, the reason was considered to be due to the sensitivity to imperfections. To consider such imperfection, namely the deviation from the cylindrical shape, the finite displacement theory also known as large displacement theory was proposed by Donnell [88] by including the nonlinear terms of strains in the energy expression. With sensible analytical expressions of the imperfection, the effect of such imperfection onto the failing stress was studied and this theory initiated a wave of substantial research on this subject in the later decades.

Instead of merely focusing at the buckling load, the equilibrium path after buckling point was studied by Karman and Tsien in [89] using the example of spherical shell subject to the external pressure. The load was found to decrease rapidly with increasing deformation after passing the buckling point before reaching at a low-bound value. Some researchers attempted to consider such low-bound value as a safe design value. As an extension of such work, the cylinder subject to axial compression was studied by Leggett [5] who found that the smallest load in the post-buckling regime was about one third of the classic value. By adopting different parameters to characterise the deflection in the post-buckling regime, the minimum post-buckling stress coefficient was determined as 0.195 by Michielsen [6], 0.182 by Kempner [7], 0.1496 by Sobey [12] , 0.0656 by Almroth [10] and 0.0518 by Sobey [90] respectively with the classic buckling stress coefficient being 0.605 of Et/R with E being Young's modulus, t thickness and R the radius of cylinder.

While such investigation of post-buckling equilibrium path came to end when Hoff *et al.* published their results in [13]. They concluded that the minimum post-buckling load approached zero when increasing wave number parameter and diminishing the

thickness of cylinder. In their research, the Yoshimura buckling pattern was also obtained [8].

By reviewing these publications, one can note that it is unsatisfactory to understand the buckling problem simply by exploring the post-buckling equilibrium path, at least for cylinders. For more information on such topic, one can refer to the review paper [91].

2.3.2 Analytical Development Considering Geometrical Imperfection

Since the initial geometrical imperfection which was unavoidable in the practical scenarios due to the manufacturing technology was unanimously acknowledged as the main reason why cylinders featured a premature failure prior to the theoretical buckling load, the imperfection was directly taken into account in the analytical investigations.

An independent research apart from his pioneering work as in [15] was conducted by Koiter [19] to analyse effect of the axisymmetric imperfection where the buckling mode was assumed to be non-axisymmetric. The result in this paper perfectly reproduced the same conclusion in [15] when the amplitude of imperfection was sufficiently small, e.g. the dramatic reduction of load could be overserved. A thorough investigation was conducted by him in [92] to compare the results from different methods as used in [15] and [19] respectively where the thickness variation was also taken into account. Similar research was undertaken by Babcock in [93, 94] where however the buckling modes were assumed to be axisymmetric.

The effects of amplitude and wavelength of axisymmetric imperfection were investigated by Tennyson and Muggeridge in [95]. Different from the previous discussion, the practical situation in the experiment was taken into account, namely the end constraints. It was found that the falling stress from this exact model was not noticeably

lower than that from perfect model. A great correlation with Koiter's work was achieved and same conclusion was made that a small amplitude of imperfection could lead to a dramatic reduction of sustaining load.

A dimple which existed at small portion of cylinder was considered in [96] to analyse its influence. A local axisymmetric imperfection was assumed to represent the small deviation. The nonlinear pre-buckling deformation along with the edge condition was incorporated into the analytical derivation to imitate a relatively realistic situation. The analytical solution was found to be in good agreement with the experimental results. A less severe impact of such local imperfection on the load-carrying capacity was observed than that of imperfection along the entire length of cylinder.

The effect of orientation of axisymmetric imperfection was studied by Arbocz and Sechler in [97]. Their conclusion was that axisymmetric imperfection pointing inward could only give rise to a considerably small limit load while the pointing outward one could lead to a comparable buckling load as the classic value. The non-symmetric imperfections were also included. The types of representations of the non-symmetric imperfection would make a significant difference that the half-wave cosine expression would yield a buckling load at the same level of classic value, while the half-wave sine expression would result into a catastrophic loading which was much smaller than the classic one.

Apart from the axisymmetric imperfection as an isolated and ideal scenario, arbitrary imperfection which was assumed to be more practical was adopted to analyse its effect on the final buckling load.

As an extension of work performed by Donnell [88], Sobey presented a theoretical analysis of cylinder under axial compression with arbitrary imperfection using double

Fourier series [90]. A sufficient choices of amplitudes of imperfection were adopted to work out its effect on the buckling load. However, in his work, the imperfection mode assumed was more of mathematical meaning rather than practical reality. Independently, the same assumption of imperfection mode was made by Thurston and Freeland [98]. Newton's method was adopted to solve the partial differential equations. With proper manipulations, their work could be simply reconciled with Koiter's [19] and Hutchison's [99] works where an axisymmetric and two-term imperfections were respectively assumed.

In Fersht's thesis [100], in order to achieve more realistic description of imperfection modes, they were expressed into an exponential cosine function and Gaussian distribution respectively. After the parametric study, the curve obtained there of load vs amplitude of imperfection was identical to Koiter's result [19] when axisymmetric imperfection was considered.

The three-dimensional plots of imperfections of cylinder specimens were obtained through data recording processing by Arbocz and Babcock in [101]. After analysing the distribution of imperfection, a combination of one axisymmetric and two asymmetric buckling modes was used to express a relatively general imperfection as assumed by Imbert in [102] as well which was incorporated into the theoretical development. A good agreement from analytical solutions and experimental results was obtained. Arbocz [103] used double Fourier series to represent the imperfection. The coefficients there were determined according to the three-dimensional plots obtained from the experimental specimens. By adopting such realistic imperfection, the buckling load of cylinder was generated which was favourably comparable to the experimental data.

In Shen and Chen's work [104], the initial perfection was simply assumed to be one type of non-axisymmetric buckling mode. Based on the boundary layer theory which included the edge effect, cylinder subject to the combined loading of external pressure and axial compression was investigated. They concluded that with the increasing the ratio of external pressure to axial compression, the post-buckling behaviour switched from an unstable state to stable state.

However, no matter what type of imperfection was assumed through whatever methods, a consistent conclusion was made that the cylinder subject to the axial compression is indeed sensitive to the initial imperfection. For more relevant information, one can refer to review papers from Stein [105] and Simites [91]. However, the structures studied there were mainly cylinder as a simplest structural configuration. For a more sophisticated applications, such as stiffened cylinders, the analytical solution may be too complicated to obtain. No one seems to be interested in venturing this tedious aspect of development. As a more convenient tool, the finite element method has been commonly employed to analyse the structural behaviour. However, to analyse the influence of imperfection with finite element method, each type of imperfection has to be incorporated individually to run a post-buckling analysis using incremental iterative approach. Considering the highly nonlinear structural behaviour in the advanced post-buckling regime, the computational cost would be significant, especially in the preliminary phase of design of structure.

2.3.3 Theoretical Approaches with Consistent Boundary Conditions

In parallel to the study of geometrical imperfection, an inconsistency between experimental applications and analytical assumptions was the boundary conditions. For instance, the pre-buckling regime in the analytical derivation was often assumed to be

pure membrane state which indicated that the cylinder ends were freely to expand to assure a uniform in-plane stresses and a constant radial deflection, while in practical or experimental scenarios, this assumption cannot be realised. The study of such discrepancy was conducted by Stein [16, 17] analytically through large displacement theory using consistent boundary condition for pre-buckling and buckling regimes. He disclosed that the inclusion of such edge restraint consistently would result into approximately 50% reduction of classical value.

Apart from the edge condition in the pre-buckling regime, the effects of the boundary conditions of buckling were also investigated by Hoff [73, 74] and Almroth [75, 106] whose results were summarised by Arbocz and Babcock in [101] as follows.

Table 2.1 Reduction factor due to the influence of boundary conditions on the buckling load of compressive cylinders

Boundary conditions	Hoff [73, 74]	Almroth [75, 106]
$w^{(1)} = w_{,xx}^{(1)} = N_{xx}^{(1)} = N_{xy}^{(1)} = 0$	0.5	0.502
$w^{(1)} = w_{,xx}^{(1)} = u^{(1)} = N_{xy}^{(1)} = 0$	0.5	0.503
$w^{(1)} = w_{,xx}^{(1)} = v^{(1)} = N_{xx}^{(1)} = 0$	1.0	0.844
$w^{(1)} = w_{,xx}^{(1)} = u^{(1)} = v^{(1)} = 0$	1.0	0.867
$w^{(1)} = w_{,x}^{(1)} = N_{xx}^{(1)} = N_{xy}^{(1)} = 0$	1.0	0.908
$w^{(1)} = w_{,x}^{(1)} = u^{(1)} = N_{xy}^{(1)} = 0$	1.0	0.926
$w^{(1)} = w_{,x}^{(1)} = v^{(1)} = N_{xx}^{(1)} = 0$	1.0	0.910
$w^{(1)} = w_{,x}^{(1)} = u^{(1)} = v^{(1)} = 0$	1.0	0.926

It can be observed that different boundary conditions applied in the buckling regime indeed influenced the buckling loads however less pronounced as compared with geometrical imperfection and in cases they had little effects. Other studies on this subject could be found in [107-111] and [112, 113] for the laminated structure while with the

similar conclusion drawn. Detailed reviews on this subject was made in [91, 101, 105] individually for anyone interested.

2.4 Experimental Researches of Buckling

Besides the analytical investigation, a great amount of experimental activities were accomplished in last century. For the very first attempts, no particular attention was paid to the quality of the specimens, in another word the cylinders were imperfection embedded without the forecast of problem of imperfection sensitivity. The first one who pointed out that the classic value cannot be reproduced in the experimental method was Robertson [114] in whose research discrepancy between them was disclosed. Lundquist conducted a series of experimental research, viz. cylinder subject to torsion [115], pure bending [116], compression [117], combined transverse shear and bending [118], to establish a database for the further investigation. One way or another, the collapsing load observed in the experiments cannot reach at the classic values. Some other experimental data could be found in [119-125]. They all revealed the fact that cylinder under axial compression was extremely sensitive to the initial imperfection.

To further demonstrate the conclusion made there, some nearly perfect specimens were fabricated using advanced manufacturing technologies. In Tennyson's work [76-79], the cylinder specimens were manufactured from a photoelastic plastic spun-cast. A careful attention was paid to the end support as well to produce a uniform loading condition in the pre-buckling regime. The results were such extraordinary that the classic value was reproduced there. The wave shape buckling modes as assumed in the theoretical solution were observed in the experiments followed by a diamond shape mode. Some carefully manufactured specimens were also tested by Babcock and Sechler [94], Almroth, Holmes and Brush [126] and Horton and Durham [127]. The excellent

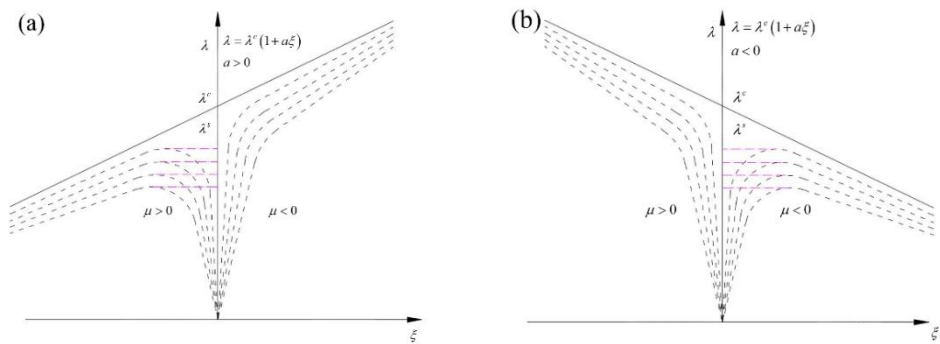
agreement of the experimental data with classic solution indicated that as long as the imperfection of cylinder could be minimised along with the excellent reproduction of boundary conditions employed in the analytical solution, the classic value could be reproduced through experimental method. In another word, the reason why the previous experimental data scattered, being only a fraction of classic value, is the existence of initial imperfection of specimens and mismatch of the boundary condition. A more detailed literature review on the experimental study of nearly-perfect cylinders can be found in [105].

2.5 Koiter's Initial Post-buckling Theory

At an early stage of the pursuit of minimum post-buckling load, Koiter already questioned its correctness used as a measure of load-carrying capacity for the cylinder. Instead of exploring the advanced post-buckling regime, he focused at equilibrium state at the neighbourhood of bifurcated point [15]. By using perturbation method to the initial post-buckling displacement field, loading parameter can be expressed in terms of an asymptotic series of a perturbation parameter, where the first perturbation mode is the buckling mode. In most problems, often the secondary perturbation mode (SPM) or even higher orders may be required. By substituting them into the total potential energy, the first derivative of that with respect to the perturbation parameter which is normally assumed to be the amplitude of the buckling mode gives rise to the equilibrium equation. The asymptotic expression of loading parameter was introduced by Budiansky and Hutchinson [21] in the presence of two initial post-buckling coefficients (IPBCs) a and b as follows.

$$\lambda = \lambda^c \left(1 + a\xi + b\xi^2 + \dots \right) \quad (2.4)$$

where the λ^c is the buckling load, ξ the perturbation parameter. The first derivative of equilibrium state with the aid of (2.4) leads to the determination of a . When a vanishes, the second derivative results in b . With the knowledge of the values of these two IPBCs, the stability of the structure at the initial post-buckling regime and its sensitivity to imperfection could be assessed as sketched in Figure 2.1 where μ is the amplitude of imperfection. It can be observed from (a) and (b) that as long as a is non-vanishing, the initial post-buckling behaviour could be unstable depending on the direction of bifurcation. The structure would be sensitive to the imperfection of one sense but not the other. Given the random nature of imperfection in the practical scenarios, imperfection sensitivity will have to be assumed and a premature collapse of structure could be envisaged. If a vanishes, the value of b has to be resorted to in order to facilitate a prediction. When b is negative, the initial post-buckling behaviour is unstable and the structure will be sensitive to the imperfection. A premature collapse of structure will be also envisaged. For an imperfection sensitive structure, any small amplitude of imperfection could lead to a substantial reduction of collapsing load as explained through Figure 2.1 (e). However if b is positive, structure is capable of sustaining addition load above the buckling load and structure is immune to the initial imperfection.



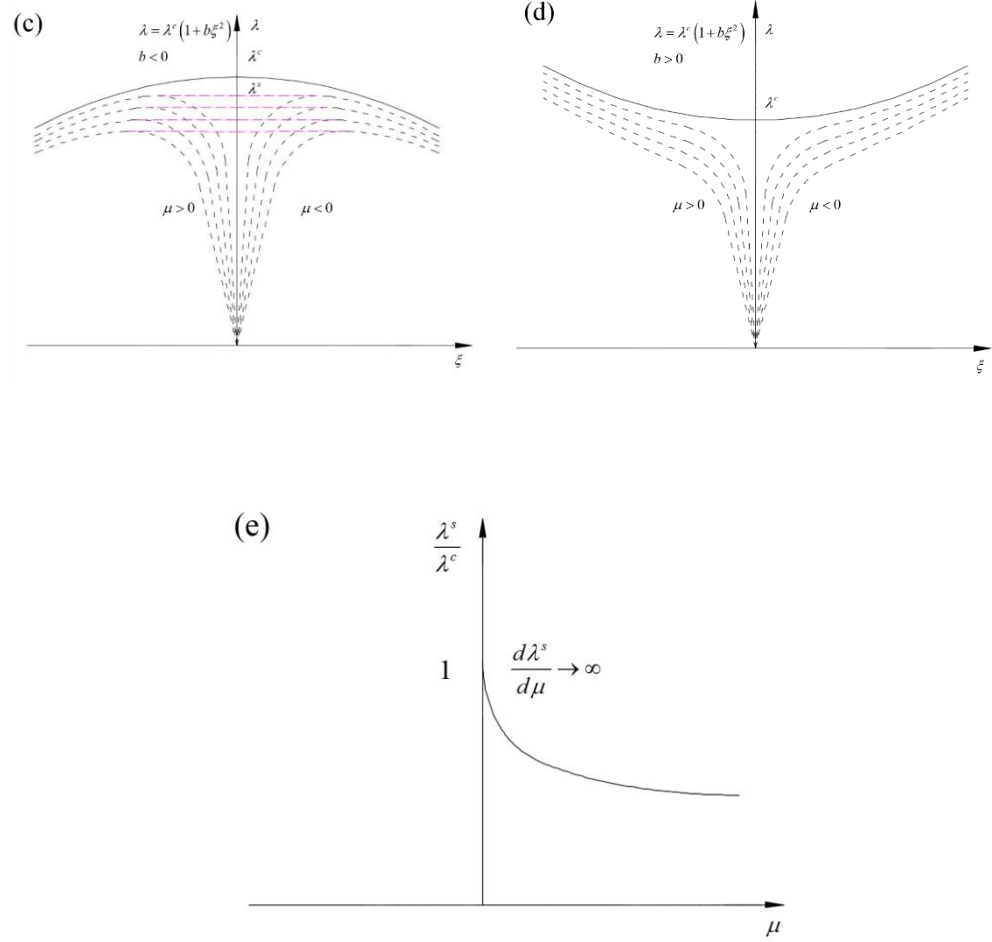


Figure 2.1 Initial post-buckling behaviour and imperfection sensitivity

Without having to deal with the mathematical difficulties in solution of highly nonlinear post-buckling equilibrium equations and constructing the full post-buckling equilibrium paths, the stability of initial post-buckling behaviour can be easily obtained through the values of two IPBCs. For value of a , it can be determined after the eigenvalue analysis based on the obtained buckling mode and buckling load. If a is non-vanishing, the initial post-buckling behaviour can be thus sufficiently described. If a vanishes, another step has to be taken to solve the SPM to determine b . It is worth noting that it is uncommon for b to be zero. However, if it happens, further perturbations will be required. Due to the simplicity of such theory, it is highly desirable.

2.5.1 Analytical Studies Using Koiter's Theory

The first application of initial post-buckling theory was of course by Koiter's himself in his thesis where the theory was proposed. In his pioneering work [15], a plate and a cylinder, both simply supported, were studied. A stable and an unstable initial post-buckling behaviours were obtained respectively. In addition, a curved panel under axial compression was also investigated and it was found that the initial post-buckling behaviour switched from being stable to unstable with the increase of the curvature [18].

A systematic formulation of Koiter's theory was presented by Budiansky and Hutchinson using the principle of virtual work [21] where the two IPBCs were introduced. As an extension of their work, by considering a nonlinear pre-buckling deformation and random imperfection, the basic derivation of two IPBCs and imperfection sensitivity were obtained by Cohen [26].

The buckling problem of cylinder was revisited using initial post-buckling theory by many researchers. For instance, cylinders with two types of stiffeners, axial stringers and ring stringers, subject to axial compression and hydrostatic pressure were quantitatively studied by Hutchinson and Amazigo [22]. It was found out that for the axially stiffened cylinder subject to axial compression, although both outside and inside stiffening strengthened capacity of buckling resistance, the outside-stiffened cylinder was more imperfection-sensitive than the one with inside stiffeners. When the loading was hydrostatic pressure, the disparity of outside and inside stiffening was less prominent. A suggestion for the optimum design was a combination of axial and ring stringers. The initial post-buckling behaviour of oval cylindrical shell, *i.e.* with elliptical cross section, under axial compression was determined by Hutchinson [28]. Similar to circular cylinder, such structure featured an unstable initial post-buckling behaviour and was imperfection

sensitive. Another loading condition, viz. combined axial compression and bending, with the same structural configuration was studied by Kempner and Chen [37]. Being more complicated, the doubly curved shells were studied by Hutchinson [24]. He found that the stability of structural behaviour after buckling totally depended on the two curvatures, not definitely being stable or unstable. Other similar researches could be found in [25, 128].

Apart from perfect structures, initial post-buckling theory was also applied to investigate the influence of imperfection. In Dym and Hoff's work [27], various imperfection modes were considered to study their influences on the buckling problem of axially compressed cylinder. They confirmed Koiter's conclusion that a small amplitude of imperfection could sharply reduce the maximum loading. Besides, the trigonometric imperfection could have similar detrimental effects as the axisymmetric imperfection. Muggeridge focused on the axisymmetric imperfection [96], including over length imperfection, local inward dimple and random axisymmetric imperfection. His results were respectively compared with the analytical work from Koiter [19], Hutchinson, Amazigo and Budiansky through private communications, and excellent agreements were achieved. Amazigo analysed the influence of the random imperfection [33]. The equations of imperfection sensitivity for modal imperfection, random imperfection and local imperfection were individually obtained.

Initial post-buckling theory found itself more instrumental when dealing with the problem of mode interaction. Ho employed this theory to explore the low bound after buckling load of a multiple-mode system [36]. He found out that the overall minimum value was attained when the imperfection corresponded to the steepest post-buckling path. An investigation was made to the axially stiffened cylindrical shells by Byskov [38, 39,

129]. Two types of imperfection were considered there, viz. overall mode and local mode which was present between adjacent stiffeners. The adverse influence of mode interaction to the buckling load was confirmed through this particular example.

The initial post-buckling behaviours of some composite structures were also investigated in [49, 130-138]. A comprehensive review on the analytical investigation through Koiter's theory was made in [40] and [35]. Although application of such theory through analytical means peaked in last century, the popularity seems to disappear in recent decades. One reason is that nowadays not many researchers still have the appetite to derive complicated equations. On the other hand, analytical solutions are often restrictive and only applicable to some specific structural configurations, boundary and loading conditions. For practical applications, structures are often much more complicated, such as those with stiffeners or cutouts for structural considerations. In the case of laminated composites, the stacking sequence could also make a difference. For these cases, the analytical solution would be virtually impossible. The only way to explore this theory for its versatile applications will have to be through its implementation using finite elements.

2.5.2 Numerical Implementation of Koiter's Theory

2.5.2.1 Mesh sensitivity Problem Using B-notation

To extend the applicability and feasibility of Koiter's theory, two computational modules, ANILISA and DISDECO, were developed by researchers at the Delft University of Technology where Koiter used to work [139-142]. The modules aimed to analyse the buckling problem of thin-walled structures and imperfection sensitivity. Various combinations of loading conditions were applicable but limited. The conjecture behind was to use numerical method to solve the Donnell-type governing equations of

buckling. However, as this approach did not have the versatility of the finite element method, its applications were bound to be restrictive and would not attract wide acceptance. A code FASOR of similar functionality was cited in [143].

The early stage of FEM implementation of Koiter's theory was made through to beams and frame structures [144]. The problem of mesh sensitivity of the second IPBC was identified. Since two terms in the numerator of the second IPBC shared close absolute values of opposite signs, in other word they could nearly cancel each other. If there was any evaluation not sufficiently accurately determined, it would leave a significant numerical error in the value of the second IPBC evaluated. The same observation can be made to the applications to plates and shells. However, it is worth noting that the locking phenomenon was not observed in analytical solutions, suggesting the numerical nature of the problem. To exactly hold the Bernoulli hypothesis, the continuity of the gradients of displacements should be guaranteed to deliver a precise initial post-buckling analysis.

The second IPBC in most FEM implementation of Koiter's theory was evaluated using the B-notation of the finite element formulation as was derived by Budiansky and Hutchinson's based on the principle of virtual work. Stresses were involved explicitly in such derived expression of second IPBC, while the such stresses normally cannot be precisely determined due to the different degrees of approximation of the in-plane and lateral displacements in the common finite element formulation as pointed out by Olesen and Byskov [145]. Their remedy to the problem was to use Lagrangian multipliers to introduce an auxiliary displacement field in each finite element.

In Poulsen and Damkilde's work [53], a method of direct determination of the secondary perturbation mode was provided. By making a local correction to every element concerned through additional degrees of freedom, the local bending moment

which was normally omitted in the common finite element formulation was taken back into consideration so that the initial post-buckling field could be precisely determined. Through a case study, it was demonstrated that the problem of mesh sensitivity was not present after such a treatment.

A high continuity (HC) finite element was formulated by Lanzo *et al.* [47]. The use of HC interpolation was to maintain the C^1 displacement continuity as stressed by Pignataro and Carlo in [144]. A series of numerical studies were conducted on plates under various boundary conditions and loading conditions and the mesh sensitivity of the second IPBC was greatly eased. Based on such achievement, some other research activities were conducted to demonstrate the advantage of Koiter's theory over incremental iterative method [48, 52] and study the influence of residual stresses on imperfection sensitivity [55].

Tiso *et al.* [58] dealt with the problem by linearising the shape function of the displacements in the quadratic terms of the kinematic model. As an extension of such work, Rahman and Jansen analysed the initial post-buckling behaviours of conical and cylindrical composite shells where the single mode and multiple mode systems were respectively considered based on FE code DIANA [59, 62]. The numerical results were compared with computational module ANILISA and the ones from nonlinear analysis. Good agreements were achieved.

Other endeavours to eliminate the problem of mesh sensitivity could be found in [50, 51, 54, 57, 146-148]. More recently, a corotational finite element formulation was employed to implement Koiter's theory by Garcea, Madeo *et al.* in [60, 149]. The essence of this type of element is to use a local corotational frame which moves along with the element to describe the deformation within each element. The advantage is that strains

could be obtained through a linear strain-displacement expression because the displacement in the local corotational frame is presumably small so that insignificant numerical errors are realised. With this element, the fourth-order derivatives of strain energy was obtained when implementing Koiter's theory in 3D beam assemblages. A fast convergence in the mesh density was achieved. For a more general application, a 4-noded shell element with 6 dofs (degrees of freedom) at each node MISS-4 based on Reissner-Mindlin theory was formulated using the corotational approach and applied to a folded structures [64] by Zagari, Madeo, *et al.* Excellent performance of such element has been demonstrated on avoiding the mesh sensitivity phenomenon.

A solid-shell element [150] has also been incorporated for the implementation of Koiter's analysis. Its advantage in the nonlinear analysis of thin-walled structures is that only translational degrees of freedom have been employed using the 3D continuum strain measures without involving rotations into the complicated computation. In fact, the so-called rotation in the shell element is nothing but a far-fetched terminology. The authentic description should be displacement gradient and the concept of rotations should not have been introduced in the first place, which was in fact the consequence of a logical fallacy. For detailed argument, one is referred to [151].

While it was not as easy as expected, when using such element the locking phenomenon were encountered to calculate the second IPBC. Thanks to contributions in [152, 153], the locking problem has been mitigated. Depending on such development, Magisano *et al.* implemented Koiter's theory to analyse the multi-layered composite structures [154]. Several numerical tests have been carried out and the results have been compared with the ones using Riks algorithm which is capable of delivering an accurate post-buckling equilibrium path. The excellent agreement between them proved the

validity of element proposed in the Koiter's initial post-buckling analysis. Its advantages in the geometrically nonlinear analysis have been elaborated in [155].

A post-buckling optimisation strategy has been proposed by Liguori, *et al.* using the same solid-shell element [156]. Koiter's theory makes such optimisation procedures feasible due to its computational efficiency. The objective of the optimisation is to seek for the optimal collapse load in the initial post-buckling regime rather than the buckling load and the imperfection has also been considered. Two numerical tests, laminated curved shells either with or without stiffeners, have been conducted to determine the optimal stacking sequence which maximises the collapse load. After the optimisation, the structural behaviour has been switched from unstable and imperfection sensitive to a stable initial post-buckling behaviour.

To improve the efficiency from design to analysis, the isogeometric analysis (IGA) which unifies the Computer Aided Design (CAD) and finite element analysis (FEA) into one straightforward procedure surfaced [157]. Due to its precise reproduction of complicated geometry, it becomes attractive in the buckling analysis. Leonetti, *et al.* made an attempt to amalgamate IGA and Koiter's theory for the analysis of composite shells [158]. A comparative study has been undertaken firstly on a square composite plate under compression to test the performance of various elements with higher order continuity to avoid the locking problem. The agreement of initial post-buckling equilibrium path of a curved panel with the ones using Riks algorithm has confirmed its feasibility and effectiveness of such combination of IGA and Koiter's theory. Some additional information on implementation of Koiter's theory could be found by Byskov in [159].

However, no matter which effort was made as reviewed above, using specially-formulated elements through particular integration schemes seems inevitable. These non-

conventional finite elements will never be attractive to any mainstream commercial FE code developer. Ideally, the type of elements which is applicable to the conventional eigenvalue analysis should be applicable to the initial post-buckling analysis as well. It is impractical for a commercial finite element code to incorporate any particular type of elements merely for implementation of Koiter's theory. Therefore, it seems that a successful implementation of Koiter's theory relies on a more effective solution to the locking problem which apparently has not been achieved yet.

2.5.2.2 Iterative Scheme of Solving Secondary Perturbation Mode

The problem of mesh sensitivity arose when calculating the second IPBC. While this has attracted a lot of attentions, the accurate and efficient determination of secondary perturbation mode (SPM) has not given much emphasis relatively. A way of obtaining the secondary perturbation mode was made in [160] using iterative scheme which was adopted in [47, 50]. The procedures have been briefly addressed herein. The governing equation of SPM and orthogonality condition between them are given as follows.

$$\mathbf{K}_t \mathbf{q}^{(2)} = \mathbf{f} \quad (2.5)$$

$$\left(\mathbf{q}^{(2)} \right)^T \mathbf{T} \mathbf{q}^{(1)} = 0 \quad (2.6)$$

where $\mathbf{K}_t$ is the tangential stiffness matrix at the bifurcated point. $\mathbf{q}^{(1)}$ and $\mathbf{q}^{(2)}$ are the buckling mode and SPM respectively. $\mathbf{T}$ is a positive definite matrix as an auxiliary means to introduce orthogonality condition between $\mathbf{q}^{(1)}$ and $\mathbf{q}^{(2)}$. The problem can be considered as the minimisation problem with respect to $\mathbf{q}^{(2)}$ subject to a condition (2.6). By introducing a Lagrangian multiplier α to eliminate the orthogonality condition, the objective function can be given as

$$G = \frac{1}{2} \left(\mathbf{q}^{(2)} \right)^T \mathbf{K}_t \mathbf{q}^{(2)} - \mathbf{f}^T \mathbf{q}^{(2)} + \alpha \left(\mathbf{q}^{(1)} \right)^T \mathbf{T} \mathbf{q}^{(2)} \quad (2.7)$$

The first derivative of G with respect to $\mathbf{q}^{(2)}$ leads to

$$\mathbf{K}_t \mathbf{q}^{(2)} = \mathbf{f} - \alpha \mathbf{T} \mathbf{q}^{(1)} = \bar{\mathbf{f}} \quad (2.8)$$

Multiplying the above with $\mathbf{q}^{(1)}$ leads to the determination of α as

$$\alpha = \frac{(\mathbf{q}^{(1)})^T \mathbf{f}}{(\mathbf{q}^{(1)})^T \mathbf{T} \mathbf{q}^{(1)}} \quad (2.9)$$

By utilising the following iteration procedure

$$\mathbf{r}_j = \bar{\mathbf{f}} - \mathbf{K}_t \mathbf{q}_j^{(2)} \quad (2.10a)$$

$$\mathbf{q}_{j+1}^{(2)} = \mathbf{q}_j^{(2)} - \mathbf{K}_0^{-1} \mathbf{r}_j \quad (j=0,1,2,\dots) \quad (2.10b)$$

and assuming the $\mathbf{q}_0^{(2)} = \mathbf{0}$. On convergence, $\mathbf{q}^{(2)}$ would be obtained. In (2.10), $\mathbf{K}_0$ is the initial stiffness matrix. With a close examination of equation (2.9), one could find out that α is always zero which can be readily demonstrated by left multiplying equation (2.5) with $\mathbf{q}^{(1)}$ bearing in mind that $\{\mathbf{q}^{(1)}\}^T \mathbf{K}_t = \mathbf{0}$.

While it was claimed that the iterative scheme converges quickly, there did not seem to be any rigorous proof which guarantees the convergence. It is not clear how the scheme should be applied in presence of multiple buckling modes. Computationally, one would always have concerns about the numerical efficiency and accuracy of an iterative scheme. While the iterative scheme did provide a solution in facilitating the FE implementation of Koiter's theory to some problems, it is far from ideal in order to impress a commercial FE code developer.

2.6 Post-buckling Behaviours of Thin-walled Structures

It is well known that the cylindrical or curved shells is acutely susceptible to the buckling problem as reviewed above. While reinforced with stiffeners, panels as a whole

might be capable of sustaining more loading even though the panel may have buckled locally in the early stage. Sometimes, the margin from the onset of buckling to the collapse of the structure can be substantial. Due to the allowance of buckling in the design of some of the typical structures, e.g. airframes, post-buckling behaviour of thin-walled structures has always been a hotspot of research in order to extend the loading limit in the preliminary phase of design so as to reduce the weight and maximise the effectiveness of the structure. In addition to optimisation of structural configuration, the emergence of composite materials also significantly promotes the mechanical performance due to their superb specific stiffness. Based on such objective, the POSICOSS project was launched by the European Commission [70]. To achieve such goal, some representative specimens were firstly manufactured and tested to identify the material properties, followed by stiffened panels to investigate the post-buckling response. Comprehensive experimental studies were carried out by Abramovich with his colleagues as follows. Nine curved blade stiffened panels of different heights and number of stiffeners were designed, manufactured and tested under axial compression till the ultimate failure [161]. Prior to the tests, the initial geometrical imperfection of each specimen was scanned. The strain distribution from gauges and shortening were recorded as well to validate the numerical simulation using Abaqus. Although there appeared the noticeable disparity between experimental observation and numerical predictions, no clear explanation was provided in that paper. Four torsion boxes made up of two identical stiffened cylindrical panels and two unstiffened aluminium panels were tested under torsion, axial compression and a combination of them [162]. Two types of stringers had been considered, i.e. blade-type stringers and J-stringers with either short or long flange. All the specimens appeared to possess a pronounced load-carrying capacity in the post-buckling regimes. An excellent agreement between numerical and experimental analyses was achieved for the pure axial

compression and torsion. While for the loading of combined compression and torsion, the experimental results were consistently lower than the numerical ones which was attributed to the repeated loadings from the first two tests that introduced the residual stress and unexpected degradation of material. As an extension of the experiments, four additional stiffened cylindrical panels with J-type stringers under axial compression were tested as well [163]. Based on these experimental observations, some meaningful design guidelines were derived. For instance, (i) the increase of stringer dimensions could not always guarantee the improvement of panel buckling load. The positive influence terminated as the stringers were stiff enough to represent a clamped boundary condition. (ii) the local buckling of the stringers before the overall buckling of panel should be avoided in order to maximise the mechanical performance in the post-buckling regime. (iii) the effective width method has been modified to be more realistic and practical by taking into the consideration of cross-section of stringers. Another experimental work of stiffened cylindrical shells under axial compression and torsion respectively was conducted by Bisagni and Cordisco [164]. To provide sufficient information for the validation of numerical models and design guidelines, the specimens with various thickness of skin and different number of stringers were designed and produced for the tests. The experiments indicated that the stiffened cylindrical shells exhibited a satisfactory post-buckling capacity which increased the confidence of alleviating the design limit and exploiting the post-buckling reserve of thin-walled structures. Similar work was also carried out in [165]. A conclusion made there was the influence of boundary conditions on the scattered data. It was believed that the boundary conditions in separate tests of the same kind could hardly be identical while the impact of such deviation of boundary conditions to the experimental results had not been addressed. For

imperfection sensitive structures, such impact could be substantial as shown in cylindrical shells under axial compression.

The work of POSICOSS project was mainly based on the exploitation of the post-buckling reserve capacity rather than blindly increasing the buckling load. However, the effective range of post-buckling regime was not clearly identified. There were two main factors that limited the range, i.e. the nonlinear geometrical deformation and the initiation of degradation of materials which triggered a successor project COCOMAT also funded by European Commission [71]. In order to reduce the weight of structures and making most use of the post-buckling regime reserve, the current design rules had to be modified to accomplish the goal. The most primary adjustment was to push the ultimate load to the collapse load as close as possible with the admission of degradation in the safe region. While under no circumstance the onset of degradation could emerge prior to the limit load. Hence, the precise identification of the initiation of degradation became the key technique.

After the observation of sufficient experiments of stiffened composite structures under various loading, it was found that the rapid and explosive failure took place immediately after the debonding between the stiffeners and the skin. Hence a comprehensive work was undertaken by Orifici and his colleagues on this topic to precisely predict the onset of failure and the process of degradation at the interface so as to simulate the mechanical response of the structure as a whole in the post-buckling regime [166-171]. 25 skin-stiffener sections were manufactured and tested subject to three sorts of loading conditions, i.e. antisymmetric, symmetric pull and symmetric push in [167]. A subroutine implementing the ‘degenerated Tsai’ criterion [172] has been incorporated to determine the initiation of delamination. A great effort was made to

specify the characteristic area of element to give rise to reasonable results because of the stress concentration and severe nonlinearity. Later on, such technique was applied to a global-local analysis to predict the post-buckling response of a fuselage-representative curved panel [169]. The local model simulated the skin-stiffener interface where the first failure was likely to take place with the boundary conditions which allowed the results readily transferred between the local and global models. As a result, such an approach was capable of predicting the experimental results. Besides the intact specimens, the pre-damaged ones with debonding between skin and stiffener were also manufactured and tested under compression [170]. The multi-point constraints were employed to capture the development of interlaminar failure, i.e. the delamination growth, via the virtual crack closure technique (VCCT). A consistent prediction of structural behaviour, damage evolution and collapse load was obtained through the numerical method as observed in experiment. Similar work could also be found in [171] with the same technique. In the cited papers above, the initial geometrical imperfections were not considered without a reason. As concluded in Koiter's theory, the presence of imperfection could substantially reduce the load-carrying capacity for the imperfection-sensitive structures. In addition to the amplitude of imperfection, the sign of imperfection could make a huge difference in the case of asymmetric initial post-buckling behaviour. Hence any post-buckling analysis without introduction of imperfection of various amplitudes and both signs seems unconvincing. Some other researches on the subject of buckling and post-buckling structures including experimental, analytical and numerical studies were compiled in the book [173].

More recently, Bohlooly *et al.* proposed an analytical approach to predict the buckling and post-buckling equilibrium path of stiffened composite double curved panels with nonlinear elastic supports [174]. The loading condition was comprised of in-plane

compression, lateral pressure, thermal environment and their combinations. Various types of simply supported condition were considered as well. Love-Kirchhoff theory was employed to derive the lateral equilibrium and in-plane compatibility equation. The explicit expressions of buckling load and post-buckling load-deflection were obtained through the Galerkin method. A Rayleigh-Ritz approach was proposed by Oliveri and Milazzo to study the post-buckling behaviour of variable angle tow composite stiffened panels [175]. The first order shear deformation theory has been employed with the consideration of geometrical nonlinearity via Von Karman assumption. A computational code was developed for the implementation of the approach and the results showed a great consistence with the ones from FEM. Trabelsi *et al.* focused on the post-buckling analysis of functionally graded material structure under thermal environment [176]. Three types of thermal distributions in the thickness direction have been accounted for, *viz.* uniform, linear and nonlinear. The analytical outcomes were verified through the comparisons with numerical results.

The post-buckling behaviour of thin-walled structures keeps motivating researchers to devote their time and efforts. The outcomes of such endeavour deserve a lot of credits. However, without sufficient confidence on the buckling problem, the move to post-buckling seems hasty. Existing commercial finite element codes can only provide eigenvalue and eigen-vectors as the outcomes of buckling analysis. Such information is too limited to draw a complete picture of structural behaviour at the neighbourhood of bifurcated point. Once Koiter's initial post-buckling theory is implemented through finite elements, it will inevitably enhance the understanding of buckling problem and raise the level of confidence on the predicted buckling load. This will guide designers of engineering structures to exploit their potential in the post-buckling regime.

2.7 Summary

Some significant aspects of buckling problem pertaining to current research have been reviewed in this chapter, including the development of analytical solution to equilibrium path in the post-buckling regime, techniques to manufacture nearly perfect cylinder for the experimental studies and analytical applications of Koter's theory. The significance of FE implementation of such theory has been emphasised. The obstacles have been identified which will be addressed in the subsequent chapters of this thesis. Besides, the work on the post-buckling behaviour of thin-walled structures has also been reviewed to provide a broad picture of the buckling problem.

Chapter 3 Fully Commutative Strain Energy Density

3.1 Introduction

The geometrically nonlinear problem has been perceived as well-established since analyses of this kind are routinely carried out by users of commercial FEM codes, e.g. [44-46], as a standard provision of these codes. However, there is one issue which has never been properly raised and investigated, namely, the lack of commutativity of displacements in the various stages of FEM formulation, all associated with the expression of strain energy density (SED) as will be explored in this chapter. The issue has always been sidestepped when encountered without being identified as a problem but can be revealed straightforwardly if one attempts to derive the secant stiffness matrix. Using conventional notations which are provided in Appendix A to avoid any confusion, the strain energy in a finite element can be given as follows.

$$\begin{aligned} U^e &= \frac{1}{2} \int_{\Omega^e} \left(\boldsymbol{\varepsilon}_L^T \mathbf{C} \boldsymbol{\varepsilon}_L + \frac{1}{2} \boldsymbol{\varepsilon}_L^T \mathbf{C} \boldsymbol{\varepsilon}_N + \frac{1}{2} \boldsymbol{\varepsilon}_N^T \mathbf{C} \boldsymbol{\varepsilon}_L + \frac{1}{4} \boldsymbol{\varepsilon}_N^T \mathbf{C} \boldsymbol{\varepsilon}_N \right) d\Omega \\ &= \frac{1}{2} \mathbf{q}^T \left[\int_{\Omega^e} \left(\mathbf{B}_L^T \mathbf{C} \mathbf{B}_L + \frac{1}{2} \mathbf{B}_L^T \mathbf{C} \mathbf{B}_N(\mathbf{q}) + \frac{1}{2} \mathbf{B}_N^T(\mathbf{q}) \mathbf{C} \mathbf{B}_L + \frac{1}{4} \mathbf{B}_N^T(\mathbf{q}) \mathbf{C} \mathbf{B}_N(\mathbf{q}) \right) d\Omega \right] \mathbf{q} \end{aligned} \quad (3.1)$$

Where Ω^e is the domain of the finite element concerned and $\mathbf{q}$ the nodal displacement vector discretised from an otherwise continuous displacement field $\mathbf{u}$. In order to establish the equilibrium equation, variation of the strain energy (3.1) is required as given below, where advantage has been taken of the symmetry of the $\mathbf{C}$ matrix, the obvious commutativity between $\mathbf{q}$ and $\delta \mathbf{q}$ in $\mathbf{B}_N(\delta \mathbf{q}) \mathbf{q}$ as explained in Appendix A and the fact that any term in the above expression, e.g. $\frac{1}{2} \mathbf{q}^T \int_{\Omega^e} \frac{1}{2} \mathbf{B}_N^T(\mathbf{q}) \mathbf{C} \mathbf{B}_L d\Omega \mathbf{q}$, is a scalar and its transpose is equal to itself.

$$\delta U^e = \delta \mathbf{q}^T \left[\int_{\Omega^e} \left(\mathbf{B}_L^T \mathbf{C} \mathbf{B}_L + \frac{1}{2} \mathbf{B}_L^T \mathbf{C} \mathbf{B}_N(\mathbf{q}) + \mathbf{B}_N^T(\mathbf{q}) \mathbf{C} \mathbf{B}_L + \frac{1}{2} \mathbf{B}_N^T(\mathbf{q}) \mathbf{C} \mathbf{B}_N(\mathbf{q}) \right) d\Omega \right] \mathbf{q} \quad (3.2)$$

The integral inside the square brackets in (3.2) should give the secant stiffness matrix of the element but it has never been accepted because it lacks symmetry, as stated in the second edition of [177]. Whilst the first and fourth terms inside the integral are symmetric, the remaining part, i.e. the sum of the second and third terms, does not look symmetric superficially because

$$\begin{aligned} \left(\frac{1}{2} \mathbf{B}_L^T \mathbf{C} \mathbf{B}_N(\mathbf{q}) + \mathbf{B}_N^T(\mathbf{q}) \mathbf{C} \mathbf{B}_L \right)^T &= \frac{1}{2} \mathbf{B}_N^T(\mathbf{q}) \mathbf{C} \mathbf{B}_L + \mathbf{B}_L^T \mathbf{C} \mathbf{B}_N(\mathbf{q}) \\ &\neq \frac{1}{2} \mathbf{B}_L^T \mathbf{C} \mathbf{B}_N(\mathbf{q}) + \mathbf{B}_N^T(\mathbf{q}) \mathbf{C} \mathbf{B}_L \end{aligned} \quad (3.3)$$

The above derivation followed the so-called N-notation (a terminology introduced in [68] as being one approach to the FEM formulation, in contrast to the other commonly used approach referred to in [68] as the B-notation) but has always been avoided in textbooks, e.g. [177-181], because it was seen as a dead end. If one could swap the position of some of the displacements involved, e.g. the $\delta \mathbf{q}$ and the $\mathbf{q}$ at the end of the

expression $\delta \mathbf{q}^T \left(\int_{\Omega^e} \mathbf{B}_N^T(\mathbf{q}) \mathbf{C} \mathbf{B}_L d\Omega \right) \mathbf{q}$, to give $\mathbf{q}^T \left(\int_{\Omega^e} \mathbf{B}_N^T(\mathbf{q}) \mathbf{C} \mathbf{B}_L d\Omega \right) \delta \mathbf{q}$ without

affecting the value of the expression, the inequality (3.3) would become an equality and the secant stiffness matrix could readily be made symmetric. Unfortunately, commutativity in matrix expressions is generally not applicable. Without a properly defined secant stiffness matrix, equilibrium conditions cannot be stated in explicit form. Common practice is to establish equilibrium indirectly by integrating stresses whilst jumping to the tangential stiffness matrix directly [178-181] in terms of the formulation of element stiffness matrices, employing the so-called B-notation [68]. There have been limited attempts [68, 83, 85] to unify expressions of secant and tangential stiffness matrices but the obtained expressions lacked uniqueness, as admitted in [83, 85]. It will be shown later in this chapter that the lack of symmetry and uniqueness is in fact a false

impression. A more fundamental and generic issue surfaces when one tries to implement Koiter's initial post-buckling theory [15] through FEM as it places more demands on commutativity of displacements in energy expressions, referred to as full commutativity, as will be detailed later in this chapter.

Existing accounts [182, 183] of the FEM formulation of Koiter's initial post-buckling theory tended to shy away from having such commutativity fully established. There have not been many attempts to adopt the N-notation for the FEM formulation in general since the publication of [182, 183], let alone its implementation for Koiter's theory. The difficulty has been circumvented by following the B-notation formulation of Koiter's theory [21]. The price paid for this convenience is, however, that the results become mesh sensitive as will be elaborated further in fourth chapter, noting that mesh sensitivity is undesirable in any FEM analysis.

The objectives of this chapter are twofold. On the fundamental theoretical development side, it will establish the full commutativity of the SED as one of its intrinsic properties, and provide an explicitly and uniquely presented FEM formulation in N-notation based on the full commutativity established. The failure to appreciate such full commutativity is responsible for the fact that the secant stiffness matrix has never been explicitly and uniquely obtained. This long-standing issue will be resolved completely in this chapter. On the practical side, and as a direct application of the obtained fully commutative SED, the FEM implementation of Koiter's theory using N-notation is herewith made possible thus freeing it from the problem of mesh sensitivity that occurs in available accounts. As will be introduced in more detail in fourth chapter of this thesis, Koiter's theory provides deep insight into the buckling problem of structures and hence has tremendous potential in practical engineering applications. However, due to the lack of an acceptable FEM implementation, this theory has been largely neglected, at least as

far as mainstream commercial FEM codes are concerned, wasting the opportunity to exploit it as a potentially valuable approach to address the uncertainties associated with the buckling problem. The outcome of this chapter will hopefully lead to the resurrection of Koiter's theory as a design analysis tool, providing structural designers and engineers with the means to evaluate the confidence on their obtained buckling strengths of structures. Lack of confidence in predicted results has been the main challenge to existing forms of buckling analysis, the reason being the fact that some structures are extremely sensitive to initial imperfections whilst others may be not. No assessment can currently be made of the sensitivity of the obtained buckling load to the initial imperfection from the conventional buckling analysis and Koiter's theory fills this gap precisely and neatly.

3.2 The Concept of Full Commutativity and Apparent Symmetry

The strain energy density U exists for any geometrically nonlinear problem and can be expressed as a sum of homogeneous terms from the 1st to the 4th orders as follows

$$U = U_1 + U_2 + U_3 + U_4 \quad (3.4)$$

where the exact expressions of each term are provided in Appendix A.

Continuous displacement fields $\mathbf{u}$, $\mathbf{u}'$, $\mathbf{u}''$ and $\mathbf{u}'''$ are introduced to facilitate the following discussion, and they are assumed to be mutually independent. They are discretised into $\mathbf{q}$, $\mathbf{q}'$, $\mathbf{q}''$ and $\mathbf{q}'''$ for FEM applications, as already employed in expressing (3.1). In the context of Koiter's theory, these displacement fields can be the fundamental equilibrium path $\mathbf{u}_0$, various orders of perturbation displacements, $\mathbf{u}_1$, $\mathbf{u}_2, \dots$, etc., respectively, as well as displacement variation $\delta\mathbf{u}$. In order to illustrate the concept of commutativity, consider the strain energy at a specific deformation state which is described as the sum of independent displacement fields. Take the third order term of the SED as given in (A-5) in Appendix A for example

$$\begin{aligned}
U_3(\mathbf{u} + \mathbf{u}' + \mathbf{u}'') &= \frac{1}{2}(\mathbf{u} + \mathbf{u}' + \mathbf{u}'')^T \nabla_L^T \mathbf{C} \nabla_N (\mathbf{u} + \mathbf{u}' + \mathbf{u}'')(\mathbf{u} + \mathbf{u}' + \mathbf{u}'') \\
&= U_{111}(\mathbf{u}, \mathbf{u}, \mathbf{u}) + U_{111}(\mathbf{u}, \mathbf{u}, \mathbf{u}') + U_{111}(\mathbf{u}, \mathbf{u}, \mathbf{u}'') + \dots \\
&\quad + U_{111}(\mathbf{u}, \mathbf{u}', \mathbf{u}) + U_{111}(\mathbf{u}, \mathbf{u}', \mathbf{u}') + U_{111}(\mathbf{u}, \mathbf{u}', \mathbf{u}'') + \dots \\
&\quad + U_{111}(\mathbf{u}', \mathbf{u}, \mathbf{u}) + U_{111}(\mathbf{u}', \mathbf{u}, \mathbf{u}') + U_{111}(\mathbf{u}', \mathbf{u}, \mathbf{u}'') + \dots \\
&\quad + U_{111}(\mathbf{u}', \mathbf{u}', \mathbf{u}) + U_{111}(\mathbf{u}', \mathbf{u}', \mathbf{u}') + U_{111}(\mathbf{u}', \mathbf{u}', \mathbf{u}'') + \dots \\
&\quad + U_{111}(\mathbf{u}'', \mathbf{u}, \mathbf{u}) + U_{111}(\mathbf{u}'', \mathbf{u}, \mathbf{u}') + U_{111}(\mathbf{u}'', \mathbf{u}, \mathbf{u}'') + \dots \\
&\quad + U_{111}(\mathbf{u}'', \mathbf{u}', \mathbf{u}) + U_{111}(\mathbf{u}'', \mathbf{u}', \mathbf{u}') + U_{111}(\mathbf{u}'', \mathbf{u}', \mathbf{u}'')
\end{aligned} \tag{3.5}$$

where

$$U_{111}(\mathbf{u}, \mathbf{u}', \mathbf{u}'') = \frac{1}{2} \mathbf{u}^T \nabla_L^T \mathbf{C} \nabla_N (\mathbf{u}') \mathbf{u}'' \tag{3.6a}$$

$$U_{111}(\mathbf{u}', \mathbf{u}, \mathbf{u}'') = \frac{1}{2} (\mathbf{u}')^T \nabla_L^T \mathbf{C} \nabla_N (\mathbf{u}) \mathbf{u}'' \tag{3.6b}$$

Relationship (3.5) introduces a trilinear form and its typical expression has been given in (3.6a). Some of the displacements in (3.6a) can swap their positions without affecting the value of the expression, e.g. $\mathbf{u}'$ and $\mathbf{u}''$. In other words, $U_{111}(\mathbf{u}, \mathbf{u}', \mathbf{u}'')$ is commutative as far as $\mathbf{u}'$ and $\mathbf{u}''$ are concerned. This commutativity is described as partial commutativity of the trilinear form $U_{111}(\mathbf{u}, \mathbf{u}', \mathbf{u}'')$ in the present chapter whilst $\mathbf{u}$ and $\mathbf{u}'$ cannot be swapped without affecting the value of $U_{111}(\mathbf{u}, \mathbf{u}', \mathbf{u}'')$, in general, because

$$U_{111}(\mathbf{u}, \mathbf{u}', \mathbf{u}'') = \frac{1}{2} \mathbf{u}^T \nabla_L^T \mathbf{C} \nabla_N (\mathbf{u}') \mathbf{u}'' \neq \frac{1}{2} (\mathbf{u}')^T \nabla_L^T \mathbf{C} \nabla_N (\mathbf{u}) \mathbf{u}'' = U_{111}(\mathbf{u}', \mathbf{u}, \mathbf{u}'') \tag{3.7}$$

In much the same way, a bilinear form $U_{11}(\mathbf{u}, \mathbf{u}')$ and a quad-linear form $U_{1111}(\mathbf{u}, \mathbf{u}', \mathbf{u}'', \mathbf{u}''')$ can also be introduced from U_2 and U_4 , respectively,

$$U_{11}(\mathbf{u}, \mathbf{u}') = \frac{1}{2} \mathbf{u}^T \nabla_N^T (\mathbf{u}') \boldsymbol{\sigma}_I + \frac{1}{2} \mathbf{u}^T \nabla_L^T \mathbf{C} \nabla_L \mathbf{u}' \tag{3.8}$$

$$U_{1111}(\mathbf{u}, \mathbf{u}', \mathbf{u}'', \mathbf{u}''') = \frac{1}{8} \mathbf{u}^T \nabla_N^T (\mathbf{u}') \mathbf{C} \nabla_N (\mathbf{u}'') \mathbf{u}''' \tag{3.9}$$

A property of *full commutativity* for bilinear, trilinear and quad-linear forms can be introduced as one of the key subjects of the present chapter if any pair of displacement

fields involved in the expression concerned can swap their positions in an expression without affecting the value of the expression.

Along with full commutativity, another closely related property can be introduced as follows. To formulate element stiffness matrices in FEM, it is desirable to express energy and its first and second variations in a form of $\mathbf{u}^T \mathbf{M} \mathbf{u}'$, with $\mathbf{M}$ being a matrix operator, as shown in (3.1) and (3.2), for instance. However, not every term is readily given in this form explicitly, e.g. $\mathbf{u}^T \nabla_N^T(\mathbf{u}') \boldsymbol{\sigma}_I$ as a part of $U_{11}(\mathbf{u}, \mathbf{u}')$ as given in (3.8). One of the tasks of the present chapter is to make such terms into $\mathbf{u}^T \mathbf{M} \mathbf{u}'$ form. Once a bilinear, trilinear or quad-linear form is so expressed, it will be referred to as apparently symmetric in the present chapter if $\mathbf{M}$ is given as a symmetric matrix operator. The lack of symmetry in the secant stiffness matrix as shown in (3.2) resulted from the lack of such apparent symmetry in the third order term of the SED.

Some of the difficulties in the FEM formulation of problems involving geometrical nonlinearity resulting from the lack of full commutativity and apparent symmetry are listed as follows.

- 1) Whilst the B-notation [68] delivers a symmetric tangential stiffness matrix [178-181], the secant stiffness matrix cannot be obtained explicitly.
- 2) Following the N-notation [68], one can obtain the tangential and secant stiffness matrices directly and make them symmetric [68, 83, 85]. However, without achieving the full commutativity of SED, the obtained expressions of stiffness matrices lack uniqueness [83, 85].

Just as the secant and tangential stiffness matrices can be considered as the outcomes of the first and second orders of variations of the strain energy, higher orders of variations are required in order to implement Koiter's theory [15]. The earliest versions of FEM

implementation of Koiter's theory tended to follow the N-notation [182, 183] for the expressions of the so-called initial post-buckling coefficients (IPBCs). However, the weaknesses of the N-notation as described above prevented it from being practically exploited. Instead, alternative expressions for these IPBCs were brought forward [21]. Unfortunately, they suffer from numerical 'locking' or mesh sensitivity [47, 53, 54, 60, 184]. The solution to the problem has so far been either to employ a much more refined mesh than that used for conventional buckling analysis in order to evaluate the IPBCs, or to employ specially formulated elements [53, 54, 60, 64, 154], to name but a few of the available unattractive options.

It should be pointed out that, in analytical applications of Koiter's theory, a displacement field, such as the buckling mode or any perturbation modes at high orders, is described by a single scalar giving the amplitude of an assumed pattern of the displacement field in the form of a scalar function. Scalars are always commutative and commutativity has therefore never been an issue there. In addition, the 'locking' problem is also absent in analytical approaches. However, the applicability of analytical approaches is of course restricted to simple structures under simple loading and boundary conditions, requiring FEM to be used for analysing structures of realistic complexity, giving rise to the difficulty of mesh sensitivity. The present chapter aims to demonstrate that this difficulty can be removed by employing the N-notation after the establishment of the fully commutative SED.

3.3 Fully Commutative SED

Due to the symmetries of stress, strain and stiffness tensors and the definition of the Green strain, the following relationships always hold.

$$\nabla_N(\mathbf{u})\mathbf{u}' = \nabla_N(\mathbf{u}')\mathbf{u} \quad (3.10a)$$

$$\mathbf{u}^T \nabla_L^T \mathbf{C} \nabla_L \mathbf{u}' = (\mathbf{u}')^T \nabla_L^T \mathbf{C} \nabla_L \mathbf{u} \quad (3.10b)$$

Given the above, it is clear that $U_{11}(\mathbf{u}, \mathbf{u}')$ as given in (3.8) is already fully commutative, i.e.

$$U_{11}(\mathbf{u}, \mathbf{u}') = U_{11}(\mathbf{u}', \mathbf{u}) \quad (3.11a)$$

However, relationships (3.10) deliver to $U_{111}(\mathbf{u}, \mathbf{u}', \mathbf{u}'')$ and $U_{1111}(\mathbf{u}, \mathbf{u}', \mathbf{u}'', \mathbf{u}''')$ only partial commutativity between some pairs of $\mathbf{u}$, $\mathbf{u}'$, $\mathbf{u}''$ and $\mathbf{u}'''$ as follows.

$$U_{111}(\mathbf{u}, \mathbf{u}', \mathbf{u}'') = U_{111}(\mathbf{u}, \mathbf{u}'', \mathbf{u}') \quad (3.11b)$$

$$\begin{aligned} U_{1111}(\mathbf{u}, \mathbf{u}', \mathbf{u}'', \mathbf{u}''') &= U_{1111}(\mathbf{u}', \mathbf{u}, \mathbf{u}'', \mathbf{u}''') = U_{1111}(\mathbf{u}, \mathbf{u}', \mathbf{u}''', \mathbf{u}'') = U_{1111}(\mathbf{u}', \mathbf{u}, \mathbf{u}''', \mathbf{u}'') \\ &= U_{1111}(\mathbf{u}''', \mathbf{u}'', \mathbf{u}', \mathbf{u}) = U_{1111}(\mathbf{u}'', \mathbf{u}''', \mathbf{u}', \mathbf{u}) = U_{1111}(\mathbf{u}''', \mathbf{u}'', \mathbf{u}, \mathbf{u}') = U_{1111}(\mathbf{u}'', \mathbf{u}''', \mathbf{u}, \mathbf{u}') \end{aligned} \quad (3.11c)$$

In general, however, commutativity cannot be blindly assumed between other pairs of $\mathbf{u}$, $\mathbf{u}'$, $\mathbf{u}''$ and $\mathbf{u}'''$, as may be observed from (3.7).

It is desirable to find alternative but equivalent expressions to U_{111} and U_{1111} , denoted as Q_{111} and Q_{1111} which are fully commutative, i.e.

$$\begin{aligned} Q_{111}(\mathbf{u}, \mathbf{u}', \mathbf{u}'') &= Q_{111}(\mathbf{u}, \mathbf{u}'', \mathbf{u}') = Q_{111}(\mathbf{u}', \mathbf{u}, \mathbf{u}'') \\ &= Q_{111}(\mathbf{u}', \mathbf{u}'', \mathbf{u}) = Q_{111}(\mathbf{u}'', \mathbf{u}, \mathbf{u}') = Q_{111}(\mathbf{u}'', \mathbf{u}', \mathbf{u}) \end{aligned} \quad (3.12a)$$

$$\begin{aligned} Q_{1111}(\mathbf{u}, \mathbf{u}', \mathbf{u}'', \mathbf{u}''') &= Q_{1111}(\mathbf{u}, \mathbf{u}', \mathbf{u}''', \mathbf{u}'') = \dots\dots\dots \\ &= \text{all together 24 permutations} = \dots\dots\dots = Q_{1111}(\mathbf{u}''', \mathbf{u}'', \mathbf{u}', \mathbf{u}). \end{aligned} \quad (3.12b)$$

As U_{11} is fully commutative, Q_{11} is identical to U_{11} .

The trilinear form is considered first. The expansion given in (3.5) offers a hint to the construction of Q_{111} from U_{111} . If all possible permutations ($P_3^3 = 3! = 6$) of $\mathbf{u}$, $\mathbf{u}'$ and $\mathbf{u}''$ in U_{111} as appearing in (3.5) have been included, one obtains

$$\begin{aligned} Q_{1111}(\mathbf{u}, \mathbf{u}', \mathbf{u}'', \mathbf{u}''') &= Q_{1111}(\mathbf{u}, \mathbf{u}', \mathbf{u}''', \mathbf{u}'') = \dots\dots\dots \\ &= \text{all together 24 permutations} = \dots\dots\dots = Q_{1111}(\mathbf{u}''', \mathbf{u}'', \mathbf{u}', \mathbf{u}). \end{aligned} \quad (3.13)$$

It is clear that swapping any pair of $\mathbf{u}$, $\mathbf{u}'$ and $\mathbf{u}''$ only results in a reshuffle of the six terms inside the brackets, e.g. between pair $\mathbf{u}$ and $\mathbf{u}'$

$$Q_{111}(\mathbf{u}', \mathbf{u}, \mathbf{u}'') = \left(U_{111}(\mathbf{u}', \mathbf{u}, \mathbf{u}'') + U_{111}(\mathbf{u}', \mathbf{u}'', \mathbf{u}) + U_{111}(\mathbf{u}, \mathbf{u}', \mathbf{u}'') \right. \\ \left. + U_{111}(\mathbf{u}, \mathbf{u}'', \mathbf{u}') + U_{111}(\mathbf{u}'', \mathbf{u}', \mathbf{u}) + U_{111}(\mathbf{u}'', \mathbf{u}, \mathbf{u}') \right) \equiv Q_{111}(\mathbf{u}, \mathbf{u}', \mathbf{u}'') \quad (3.14)$$

Given the partial commutativity of U_{111} as shown in (3.11b), $Q_{111}(\mathbf{u}, \mathbf{u}', \mathbf{u}'')$ given in (3.13) can be reduced to

$$Q_{111}(\mathbf{u}, \mathbf{u}', \mathbf{u}'') = 2U_{111}(\mathbf{u}, \mathbf{u}', \mathbf{u}'') + 2U_{111}(\mathbf{u}', \mathbf{u}, \mathbf{u}'') + 2U_{111}(\mathbf{u}'', \mathbf{u}', \mathbf{u}) \\ = \mathbf{u}^T \nabla_L^T \mathbf{C} \nabla_N (\mathbf{u}') \mathbf{u}'' + (\mathbf{u}')^T \nabla_L^T \mathbf{C} \nabla_N (\mathbf{u}) \mathbf{u}'' + (\mathbf{u}'')^T \nabla_L^T \mathbf{C} \nabla_N (\mathbf{u}') \mathbf{u} \quad (3.15a)$$

The trilinear form $Q_{111}(\mathbf{u}, \mathbf{u}', \mathbf{u}'')$ given above is thus fully commutative and it is straightforward to verify that it satisfies (3.12a).

Similarly, a fully commutative fourth order expression $Q_{1111}(\mathbf{u}, \mathbf{u}', \mathbf{u}'', \mathbf{u}''')$ can be constructed from all $(P_4^4 = 4! = 24)$ possible permutations of $\mathbf{u}, \mathbf{u}', \mathbf{u}''$ and $\mathbf{u}'''$ in U_{1111} .

After making use of the partial commutativity (3.11c), it can be expressed as

$$Q_{1111}(\mathbf{u}, \mathbf{u}', \mathbf{u}'', \mathbf{u}''') = 8U_{1111}(\mathbf{u}, \mathbf{u}', \mathbf{u}'', \mathbf{u}''') + 8U_{1111}(\mathbf{u}, \mathbf{u}'', \mathbf{u}', \mathbf{u}''') + 8U_{1111}(\mathbf{u}, \mathbf{u}''', \mathbf{u}', \mathbf{u}'') \\ = \mathbf{u}^T \nabla_N^T (\mathbf{u}') \mathbf{C} \nabla_N (\mathbf{u}'') \mathbf{u}''' + \mathbf{u}^T \nabla_N^T (\mathbf{u}'') \mathbf{C} \nabla_N (\mathbf{u}') \mathbf{u}''' + \mathbf{u}^T \nabla_N^T (\mathbf{u}''') \mathbf{C} \nabla_N (\mathbf{u}') \mathbf{u}'' \quad (3.15b)$$

One can introduce functions $Q_3(\mathbf{u})$ and $Q_4(\mathbf{u})$ from Q_{111} and Q_{1111} as

$$Q_3(\mathbf{u}) = \frac{1}{6} Q_{111}(\mathbf{u}, \mathbf{u}, \mathbf{u}) = U_3(\mathbf{u}) = U_{111}(\mathbf{u}, \mathbf{u}, \mathbf{u}) \quad (3.16a)$$

$$Q_4(\mathbf{u}) = \frac{1}{24} Q_{1111}(\mathbf{u}, \mathbf{u}, \mathbf{u}, \mathbf{u}) = U_4(\mathbf{u}) = U_{1111}(\mathbf{u}, \mathbf{u}, \mathbf{u}, \mathbf{u}) \quad (3.16b)$$

It is clear that Q_3 and Q_4 are identical to U_3 and U_4 in their roles as parts of the SED, respectively, but Q_3 and Q_4 are associated with the fully commutative trilinear and quad-linear forms. If Q_3 and Q_4 are employed instead of U_3 and U_4 in the SED, they will not alter the value of SED in any way. However, any subsequent manipulations on the displacement in the relevant energy term, such as variations, will be indifferent to the position of the displacement within the expression, as a straightforward and important application of the full commutativity, i.e.

$$\begin{aligned}\delta Q_3(\mathbf{u}) &= \frac{1}{6} Q_{111}(\delta \mathbf{u}, \mathbf{u}, \mathbf{u}) + \frac{1}{6} Q_{111}(\mathbf{u}, \delta \mathbf{u}, \mathbf{u}) + \frac{1}{6} Q_{111}(\mathbf{u}, \mathbf{u}, \delta \mathbf{u}) \\ &= \frac{1}{2} Q_{111}(\delta \mathbf{u}, \mathbf{u}, \mathbf{u}) = \frac{1}{2} Q_{111}(\mathbf{u}, \delta \mathbf{u}, \mathbf{u}) = \frac{1}{2} Q_{111}(\mathbf{u}, \mathbf{u}, \delta \mathbf{u}) = Q_{12}(\delta \mathbf{u}, \mathbf{u})\end{aligned}\quad (3.17a)$$

and similarly

$$\delta Q_4 = \frac{1}{6} Q_{1111}(\delta \mathbf{u}, \mathbf{u}, \mathbf{u}, \mathbf{u}) = Q_{13}(\delta \mathbf{u}, \mathbf{u}) \quad (3.17b)$$

$$\delta^2 Q_3 = Q_{111}(\delta \mathbf{u}, \mathbf{u}, \delta \mathbf{u}) = 2Q_{21}(\delta \mathbf{u}, \mathbf{u}) \quad (3.18a)$$

$$\delta^2 Q_4 = \frac{1}{2} Q_{1111}(\delta \mathbf{u}, \mathbf{u}, \mathbf{u}, \delta \mathbf{u}) = Q_{22}(\delta \mathbf{u}, \mathbf{u}) \quad (3.18b)$$

where the subscripts correspond to the order of the variable in the variable list in the same way as was originally introduced by Koiter in [15]. The SED expression obtained through Q_3 and Q_4 will be called the *fully commutative SED*. With this property, the variations of them resemble the derivatives of power functions. However, if δQ_3 and δQ_4 are directly derived from (3.16) with Q_3 and Q_4 with Q_{111} and Q_{1111} as expressed in (3.15), one might still find the lack of apparent symmetry at some points resembling the observation made to (3.2). It should be pointed out that the full commutativity was taken for granted in [15] by Koiter without proof or guidance how to obtain this property in general. Given the full commutativity as an intrinsic property of SED established mathematically above, it is now known with assurance that the matrix involved in (3.2) is symmetric, although this symmetry is not yet obvious. The lack of apparent symmetry in (3.2) is only due to the way in which the expression is presented. Apparent symmetry will emerge after various terms obtained above have been further streamlined by applying to them an identical transformation to be described in the next section.

3.4 The Flip Transformation

The second order term in the Green strain gives rise to a factor of $\nabla_N(\mathbf{u})\mathbf{u}$ appearing in various places as shown above. Whilst the displacements involved in it are

commutative, as shown in (3.10a), many energy terms involving this do not have apparently symmetric appearances, i.e. not in a form of $\mathbf{u}^T \mathbf{M} \mathbf{u}'$ with $\mathbf{M}$ being a symmetric matrix operator. For instance, $\mathbf{u}$ and $\mathbf{u}'$ in $\mathbf{u}^T \nabla_N^T (\mathbf{u}') \boldsymbol{\sigma}_I$ as a term in U_{11} are both located biasedly on the same side of $\boldsymbol{\sigma}_I$. They tend to cause difficulties in FEM formulation if not treated appropriately. In fact, these terms can be expressed differently in appearance but identically in value. One can easily prove the following relationship.

$$\mathbf{u}^T \nabla_N^T (\mathbf{u}') \boldsymbol{\sigma}_I \equiv (\mathbf{u}')^T \mathbf{S}_I \mathbf{u} \equiv \mathbf{u}^T \mathbf{S}_I \mathbf{u}' \quad (3.19a)$$

where

$$\mathbf{S}_I = \begin{bmatrix} \nabla_1 \\ \nabla_2 \\ \nabla_3 \end{bmatrix}^T \begin{bmatrix} \mathbf{s}_I & 0 & 0 \\ 0 & \mathbf{s}_I & 0 \\ 0 & 0 & \mathbf{s}_I \end{bmatrix} \begin{bmatrix} \nabla_1 \\ \nabla_2 \\ \nabla_3 \end{bmatrix} = \sum_{i=1}^3 \nabla_i^T \mathbf{s}_I \nabla_i \quad (3.19b)$$

$$\nabla_1 = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ \frac{\partial}{\partial y} & 0 & 0 \\ \frac{\partial}{\partial z} & 0 & 0 \end{bmatrix}, \quad \nabla_2 = \begin{bmatrix} 0 & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & \frac{\partial}{\partial z} & 0 \end{bmatrix} \quad \text{and} \quad \nabla_3 = \begin{bmatrix} 0 & 0 & \frac{\partial}{\partial x} \\ 0 & 0 & \frac{\partial}{\partial y} \\ 0 & 0 & \frac{\partial}{\partial z} \end{bmatrix}. \quad (3.19c)$$

$\boldsymbol{\sigma}_I$ and $\mathbf{s}_I$ represent the same stress but expressed in contracted and matrix forms, respectively, as introduced in Appendix A. (3.19a) is an identical transformation through which the asymmetric expression has been turned into an apparently symmetric one with $\mathbf{S}_I$ being a symmetric matrix operator. This transformation has often been employed, e.g. in [178, 179], but the author failed to trace it back to its source where it was originally introduced and it does not seem to have been named either. It is almost certain that it was first brought in when tackling the buckling problem using finite elements when the so-called geometric or initial stress stiffness matrix was introduced. For the sake of easy reference, the author suggests naming it the ‘flip transformation’ because it flips one of the two displacements on one side of the stress to other side, leading to the two

displacements being placed symmetrically on both sides of the symmetric matrix $\mathbf{S}_I$. The derivation can be found in [178, 179] but is provided in Appendix B in the context of independent displacements to be consistent with the present context.

Similarly, the flip transformation also applies to higher order terms of stress

$$\mathbf{u}^T \nabla_N^T(\mathbf{u}') \mathbf{C} \nabla_L \mathbf{u}'' \equiv \mathbf{u}^T \nabla_N^T(\mathbf{u}') \mathbf{s}_L(\mathbf{u}'') \equiv \mathbf{u}^T \mathbf{S}_L(\mathbf{u}'') \mathbf{u}' \quad (3.20a)$$

$$\mathbf{u}^T \nabla_N^T(\mathbf{u}') \mathbf{C} \nabla_N(\mathbf{u}'') \mathbf{u}''' \equiv \mathbf{u}^T \nabla_N^T(\mathbf{u}') \mathbf{s}_{11}(\mathbf{u}'', \mathbf{u}''') \equiv \mathbf{u}^T \mathbf{S}_{11}(\mathbf{u}'', \mathbf{u}''') \mathbf{u}' \quad (3.20b)$$

where $\mathbf{S}_L$ is similar to $\mathbf{S}_I$ but with $\mathbf{s}_I$ replaced by $\mathbf{s}_L$, whilst $\mathbf{S}_{11}(\mathbf{u}'', \mathbf{u}''')$ is obtained by replacing $\mathbf{s}_I$ with $\mathbf{s}_{11}(\mathbf{u}'', \mathbf{u}''')$ whose contracted form can be obtained as

$$\boldsymbol{\sigma}_{11}(\mathbf{u}'', \mathbf{u}''') = \mathbf{C} \nabla_N(\mathbf{u}'') \mathbf{u}''' \quad (3.20c)$$

One can obtain $\mathbf{S}_N(\mathbf{u}'')$ from $\mathbf{S}_{11}(\mathbf{u}'', \mathbf{u}''')$, as will be required later, if $\mathbf{u}''' = \mathbf{u}''$

$$\mathbf{S}_N(\mathbf{u}'') = \sum_{i=1}^3 \nabla_i^T \boldsymbol{\sigma}_N(\mathbf{u}'') \nabla_i = \frac{1}{2} \sum_{i=1}^3 \nabla_i^T \boldsymbol{\sigma}_{11}(\mathbf{u}'', \mathbf{u}'') \nabla_i \quad (3.21)$$

To reflect the terminology of ‘apparent symmetry’, it would be wrong to call expressions, such as $\mathbf{u}^T \nabla_N^T(\mathbf{u}') \boldsymbol{\sigma}_I$, asymmetric, given its equivalence to $\mathbf{u}^T \mathbf{S}_I \mathbf{u}'$. It only appears not to be symmetric, whilst the flip transformation makes the symmetry apparent without changing the content of the expression.

Although $Q_{11}(=U_{11})$, Q_{111} and Q_{1111} as given in (3.8) and (3.15a) and (3.15b) are fully commutative, they are not apparently symmetric. With the flip transformation they can be re-written into fully commutative and apparently symmetric forms as follows, denoted as ψ_{11} , ψ_{111} and ψ_{1111} , respectively.

$$\psi_{11}(\mathbf{u}, \mathbf{u}') = \mathbf{u}^T \boldsymbol{\Psi}_0 \mathbf{u}' \quad (3.22a)$$

$$\psi_{111}(\mathbf{u}, \mathbf{u}', \mathbf{u}'') = \mathbf{u}^T \boldsymbol{\Psi}_1(\mathbf{u}'') \mathbf{u}' \quad (3.22b)$$

$$\psi_{1111}(\mathbf{u}, \mathbf{u}', \mathbf{u}'', \mathbf{u}''') = \mathbf{u}^T \boldsymbol{\Psi}_{11}(\mathbf{u}'', \mathbf{u}''') \mathbf{u}' \quad (3.22c)$$

where

$$\Psi_0 = \nabla_L^T \mathbf{C} \nabla_L + \mathbf{S}_I \quad (3.23a)$$

$$\Psi_1(\mathbf{u}'') = \nabla_L^T \mathbf{C} \nabla_N(\mathbf{u}'') + \nabla_N^T(\mathbf{u}'') \mathbf{C} \nabla_L + \mathbf{S}_L(\mathbf{u}'') \quad (3.23b)$$

$$\Psi_{11}(\mathbf{u}'', \mathbf{u}''') = \nabla_N^T(\mathbf{u}'') \mathbf{C} \nabla_N(\mathbf{u}''') + \nabla_N^T(\mathbf{u}''') \mathbf{C} \nabla_N(\mathbf{u}'') + \mathbf{S}_{11}(\mathbf{u}'', \mathbf{u}''') \quad (3.23c)$$

The first term in the expression of Ψ_0 in (3.23a) is symmetric. The first two terms in the expressions of $\Psi_1(\mathbf{u}'')$ in (3.23b) and $\Psi_{11}(\mathbf{u}'', \mathbf{u}''')$ in (3.23c) forms a symmetric matrix operator. Since the remaining terms associated with $\mathbf{S}$ take a form similar to (3.19b), Ψ_0 , $\Psi_1(\mathbf{u}'')$ and $\Psi_{11}(\mathbf{u}'', \mathbf{u}''')$ are therefore all symmetric. $\psi_{11}(\mathbf{u}, \mathbf{u}')$, $\psi_{111}(\mathbf{u}, \mathbf{u}', \mathbf{u}'')$ and $\psi_{1111}(\mathbf{u}, \mathbf{u}', \mathbf{u}'', \mathbf{u}''')$ as given in (3.22) can therefore easily be shown to be fully commutative whilst being apparently symmetric. Taking $\psi_{111}(\mathbf{u}, \mathbf{u}', \mathbf{u}'')$ for example,

$$\begin{aligned} \psi_{111}(\mathbf{u}', \mathbf{u}, \mathbf{u}'') &= (\mathbf{u}')^T \left(\nabla_L^T \mathbf{C} \nabla_N(\mathbf{u}) + \nabla_N^T(\mathbf{u}) \mathbf{C} \nabla_L + \mathbf{S}_L(\mathbf{u}) \right) \mathbf{u}'' \\ &= \mathbf{u}^T \mathbf{S}_L(\mathbf{u}') \mathbf{u}'' + \mathbf{u}^T \nabla_N^T(\mathbf{u}') \mathbf{C} \nabla_L \mathbf{u}'' + \mathbf{u}^T \nabla_L^T \mathbf{C} \nabla_N(\mathbf{u}') \mathbf{u}'' = \psi_{111}(\mathbf{u}, \mathbf{u}', \mathbf{u}''). \end{aligned} \quad (3.24)$$

It can be seen that full commutativity may not be a property possessed by an individual term, e.g. $\mathbf{u}^T \nabla_N^T(\mathbf{u}') \mathbf{C} \nabla_L \mathbf{u}''$, but when all three terms in $\psi_{111}(\mathbf{u}, \mathbf{u}', \mathbf{u}'')$ are put together, collectively they deliver the full commutativity for $\psi_{111}(\mathbf{u}, \mathbf{u}', \mathbf{u}'')$ as a trilinear form which is closely associated with the third order term in the SED expression. The same applies to second and fourth order terms.

It should be pointed out that ψ_{11} , ψ_{111} and ψ_{1111} are identical to U_{11} , Q_{111} and Q_{1111} , respectively, since the flip transformation is meant to be an identical transformation. The difference between them is only the presentation. The second, third and fourth order terms of the SED can now be given in their fully commutative and apparently symmetric forms as

$$\Gamma_2 = \frac{1}{2} \psi_{11}(\mathbf{u}, \mathbf{u}) = \frac{1}{2} \mathbf{u}^T \Psi_0 \mathbf{u} \quad (3.25a)$$

$$\Gamma_3 = \frac{1}{6} \psi_{111}(\mathbf{u}, \mathbf{u}, \mathbf{u}) = \frac{1}{6} \mathbf{u}^T \Psi_1(\mathbf{u}) \mathbf{u} \quad (3.25b)$$

$$\Gamma_4 = \frac{1}{24} \psi_{1111}(\mathbf{u}, \mathbf{u}, \mathbf{u}, \mathbf{u}) = \frac{1}{12} \mathbf{u}^T \Psi_2(\mathbf{u}) \mathbf{u} \quad (3.25c)$$

where

$$\Psi_2(\mathbf{u}) = \frac{1}{2} \Psi_{11}(\mathbf{u}, \mathbf{u}) \quad (3.26)$$

with Ψ_0 being introduced in (3.23a) whilst $\Psi_1(\mathbf{u})$ in (3.23b) having $\mathbf{u}''$ there replaced by $\mathbf{u}$. The SED and its variations can be obtained as follows.

$$U = U_1 + U_2 + U_3 + U_4 = U_1 + \Gamma_2 + \Gamma_3 + \Gamma_4 = U_1 + \mathbf{u}^T \left(\frac{1}{2} \Psi_0 + \frac{1}{6} \Psi_1 + \frac{1}{12} \Psi_2 \right) \mathbf{u} \quad (3.27a)$$

$$\delta U = \delta U_1 + \delta \Gamma_2 + \delta \Gamma_3 + \delta \Gamma_4 = \delta U_1 + \delta \mathbf{u}^T \left(\Psi_0 + \frac{1}{2} \Psi_1 + \frac{1}{3} \Psi_2 \right) \mathbf{u} \quad (3.27b)$$

$$\delta^2 U = \delta^2 \Gamma_2 + \delta^2 \Gamma_3 + \delta^2 \Gamma_4 = \delta \mathbf{u}^T (\Psi_0 + \Psi_1 + \Psi_2) \delta \mathbf{u} \quad (3.27c)$$

where the matrices of partial differential operators $\left(\frac{1}{2} \Psi_0 + \frac{1}{6} \Psi_1 + \frac{1}{12} \Psi_2 \right)$,

$\left(\Psi_0 + \frac{1}{2} \Psi_1 + \frac{1}{3} \Psi_2 \right)$ and $(\Psi_0 + \Psi_1 + \Psi_2)$ are all symmetric and expressed explicitly.

The expressions of the coefficient arrays for the second, third and fourth order terms of the SED can be claimed to be unique under the conditions of full commutativity of ψ_{11} , ψ_{111} and ψ_{1111} . Their uniqueness can be argued in exactly the same manner as coefficient matrix for the quadratic form [185] where the criterion is the symmetry of the coefficient matrix. A quadratic form $\mathbf{x}^T \mathbf{M} \mathbf{x}$ can have an infinite number of different $\mathbf{M}$

matrices delivering the same function, e.g. $\mathbf{x}^T \mathbf{M} \mathbf{x} = x_1^2 + 2x_1x_2 + x_2^2 = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}^T \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$

$= \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}^T \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \dots$, but the commutativity of its corresponding bilinear form, i.e.

$\mathbf{x}^T \mathbf{M} \mathbf{y} = \mathbf{y}^T \mathbf{M} \mathbf{x}$ delivers a unique $\mathbf{M}$ which symmetric. The difference the third and fourth order terms is that the coefficient matrices are replaced by 3D and 4D arrays, respectively. Lack of appreciating the full commutativity prevented the reconciliation of different expressions for the same energy term, leading to the failure to deliver unique and symmetric stiffness matrices in the past.

The apparent symmetry as achieved through the use of flip transformation allows ψ_{11} , ψ_{111} and ψ_{1111} to be expressed explicitly as given in (3.22) and consequently the SED and its variations in the form of (3.27). They in turn deliver the secant and tangential stiffness matrices explicitly and systematically once applied to the FEM formulation as will be shown in the next section.

The full commutativity and apparent symmetry have thus been established as an intrinsic property of the SED which can be revealed only after the elaborations made above when it is presented uniquely and explicitly in the form of (3.27a).

3.5 Application in Finite Element Formulation of Geometrically Nonlinear

Problem

The stiffness matrices of a finite element for the geometrically nonlinear problem can be derived from the SED. Controversies in this field are rooted in the lack of full commutativity. To demonstrate the significance of the establishment of full commutativity, derivations are presented in this section, with the displacement field discretised as given in (A-6). With (3.25),

$$\Gamma_2 = \frac{1}{2} \mathbf{q}^T (\mathbf{A}^T \Psi_0 \mathbf{A}) \mathbf{q} = \frac{1}{2} \mathbf{q}^T \left(\mathbf{B}_L^T \mathbf{C} \mathbf{B}_L + \sum_{i=1}^3 \mathbf{b}^T \mathbf{s}_i \mathbf{b} \right) \mathbf{q} \quad (3.28a)$$

$$\Gamma_3 = \frac{1}{6} \mathbf{q}^T (\mathbf{A}^T \Psi_1 \mathbf{A}) \mathbf{q} = \frac{1}{6} \mathbf{q}^T \left(\mathbf{B}_L^T \mathbf{C} \mathbf{B}_N(\mathbf{q}) + \mathbf{B}_N^T(\mathbf{q}) \mathbf{C} \mathbf{B}_L + \sum_{i=1}^3 \mathbf{b}^T \mathbf{s}_i \mathbf{b} \right) \mathbf{q} \quad (3.28b)$$

$$\Gamma_4 = \frac{1}{12} \mathbf{q}^T (\mathbf{A}^T \Psi_2 \mathbf{A}) \mathbf{q} = \frac{1}{12} \mathbf{q}^T \left(\mathbf{B}_N^T(\mathbf{q}) \mathbf{C} \mathbf{B}_N(\mathbf{q}) + \sum_{i=1}^3 \mathbf{b}^T \mathbf{s}_i \mathbf{b} \right) \mathbf{q} \quad (3.28c)$$

The matrices involved in the above expressions can all be given explicitly, owing to the apparently symmetric presentation of the above energy terms, for a given type of element in which the displacement field is interpolated in the form of (A-6). In order to obtain the stiffness matrices for an element, one can substitute the fully commutative terms (3.28) into the SED and its variations (3.27) for the element.

$$U^e = \mathbf{q}^T \mathbf{h}^e + \mathbf{q}^T \mathbf{K}_e^e \mathbf{q} \quad (3.29a)$$

$$\delta U^e = \delta \mathbf{q}^T \mathbf{h}^e + (\delta \mathbf{q})^T \mathbf{K}_s^e \mathbf{q} \quad (3.29b)$$

$$\delta^2 U^e = \delta \mathbf{q}^T \mathbf{K}_t^e \delta \mathbf{q} \quad (3.29c)$$

where the superscript e indicates that the quantity is associated with the element concerned, and

$$\mathbf{h}^e = \int_{\Omega^e} \mathbf{B}_L^T \boldsymbol{\sigma}_I d\Omega \quad (3.30a)$$

$$\mathbf{K}_e^e = \frac{1}{2} \mathbf{K}_0^e + \frac{1}{6} \mathbf{N}_1^e + \frac{1}{12} \mathbf{N}_2^e \quad (3.30b)$$

$$\mathbf{K}_s^e = \mathbf{K}_0^e + \frac{1}{2} \mathbf{N}_1^e + \frac{1}{3} \mathbf{N}_2^e \quad (3.30c)$$

$$\mathbf{K}_t^e = \mathbf{K}_0^e + \mathbf{N}_1^e + \mathbf{N}_2^e \quad (3.30d)$$

whilst

$$\mathbf{K}_0^e = \int_{\Omega^e} \mathbf{A}^T \boldsymbol{\Psi}_0 \mathbf{A} d\Omega = \mathbf{K}^e + (\mathbf{K}_0^\sigma)^e \quad (3.31a)$$

$$\mathbf{N}_1^e = \int_{\Omega^e} \mathbf{A}^T \boldsymbol{\Psi}_1 \mathbf{A} d\Omega = (\mathbf{N}_1^q)^e + (\mathbf{K}_1^\sigma)^e \quad (3.31b)$$

$$\mathbf{N}_2^e = \int_{\Omega^e} \mathbf{A}^T \boldsymbol{\Psi}_2 \mathbf{A} d\Omega = (\mathbf{N}_2^q)^e + (\mathbf{K}_2^\sigma)^e \quad (3.31c)$$

$$\mathbf{K}^e = \int_{\Omega^e} \mathbf{B}_L^T \mathbf{C} \mathbf{B}_L d\Omega \quad (3.32a)$$

$$(\mathbf{N}_1^q)^e = \int_{\Omega^e} (\mathbf{B}_L^T \mathbf{C} \mathbf{B}_N(\mathbf{q}) + \mathbf{B}_N^T(\mathbf{q}) \mathbf{C} \mathbf{B}_L) d\Omega \quad (3.32b)$$

$$(\mathbf{N}_2^q)^e = \int_{\Omega^e} \mathbf{B}_N^T(\mathbf{q}) \mathbf{C} \mathbf{B}_N(\mathbf{q}) d\Omega \quad (3.32c)$$

$$(\mathbf{K}_0^\sigma)^e = \sum_{i=1}^3 \int_{\Omega^e} \mathbf{b}_i^T \mathbf{s}_I \mathbf{b}_i d\Omega \quad (3.32d)$$

$$(\mathbf{K}_1^\sigma)^e = \sum_{i=1}^3 \int_{\Omega^e} \mathbf{b}_i^T \mathbf{s}_L \mathbf{b}_i d\Omega \quad (3.32e)$$

$$(\mathbf{K}_2^\sigma)^e = \sum_{i=1}^3 \int_{\Omega^e} \mathbf{b}_i^T \mathbf{s}_N \mathbf{b}_i d\Omega \quad (3.32f)$$

$\mathbf{K}_s^e$ and $\mathbf{K}_t^e$ are conventionally known as the secant and tangential stiffness matrices. For the ease of association with $\mathbf{K}_s^e$ and $\mathbf{K}_t^e$, it is suggested that $\mathbf{K}_e^e$ be called ‘the energy stiffness matrix’ since it is directly associated with the expression of strain energy. They have all been expressed explicitly and symmetrically above. $\mathbf{K}_s^e$ is related to $\mathbf{K}_e^e$ in a similar way as $\mathbf{K}_t^e$ is related to $\mathbf{K}_s^e$. This reveals the generic relationships amongst $\mathbf{K}_e^e$, $\mathbf{K}_s^e$ and $\mathbf{K}_t^e$. A different presentation of these stiffness matrices has also been obtained in [68, 83, 181] but they admitted the lack of uniqueness and left the problem behind unresolved. Given the full commutativity as an intrinsic property of the SED, the uniqueness of SED expression and stiffness matrices derived from it is guaranteed. It is straightforward to reveal that the secant stiffness matrix as shown in (3.2) can be identically transformed to $\mathbf{K}_s^e$ after the flip transformation has been applied to some of the terms as provided in Appendix B to bring a resolution to a longstanding issue associated with the secant stiffness matrix.

In (3.31a), $\mathbf{K}_0^e$ is the stiffness matrix in linear problems, which incorporates the contribution from the initial stress through the so-called initial stress stiffness matrix $(\mathbf{K}_0^\sigma)^e$. $\mathbf{h}^e$ can be considered to be an extra load resulting from initial stresses. For problems without initial stresses, $(\mathbf{K}_0^\sigma)^e$ and $\mathbf{h}^e$ can simply be dropped without affecting the other terms. $\mathbf{N}_1^e$ and $\mathbf{N}_2^e$ are the stiffness matrices due to the nonlinearity, being linear and quadratic functions of $\mathbf{q}$, respectively.

3.6 Summary

In this chapter, the concept of full commutativity of displacements in the expressions for the strain energy density within the geometrically nonlinear problem has been introduced for the first time, and established firmly such that full commutativity can now be identified as an intrinsic property of the SED.

When this fully commutative SED is employed in the formulation of the FEM, as a direct application, all relevant stiffness matrices in the so-called N-notation can be obtained uniquely. In order for them to be expressed explicitly, the SED also needs to be put into an apparently symmetric form by applying the mathematically identical flip transformation to some of the terms of the SED, as has also been established in this chapter. The stiffness matrices thus obtained can then be used to construct the strain energy, equilibrium equation and the incremental equilibrium equation in a consistent manner. The N-notation FEM formulation has thus been fully and rigorously established. Prior to this, the N-notation formulation has remained as a symbolism, particularly within the FEM implementation of Koiter's theory.

Appendix A Basic Equations and Notations

Basic notations involved in this paper are defined as follows. For a linearly elastic continuum as dealt with in this paper, the strain energy density U exists and it can be expressed as

$$U(\mathbf{u}) = \boldsymbol{\varepsilon}^T \boldsymbol{\sigma}_I + \frac{1}{2} \boldsymbol{\varepsilon}^T \boldsymbol{\sigma} \quad (\text{A-1})$$

where $\boldsymbol{\varepsilon}$ is the Green strain, $\boldsymbol{\sigma}_I$ the initial Cauchy stress and $\boldsymbol{\sigma}$ the second Piola-Kirchhoff stress.

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_L + \boldsymbol{\varepsilon}_N, \quad \boldsymbol{\varepsilon}_L = \nabla_L \mathbf{u}, \quad \boldsymbol{\varepsilon}_N = \frac{1}{2} \nabla_N (\mathbf{u}) \mathbf{u} \quad (\text{A-2})$$

where $\mathbf{u} = \{u \quad v \quad w\}^T$ is the displacement field, and

$$\nabla_L = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix}, \quad \nabla_N(\mathbf{u}) = \begin{bmatrix} \frac{\partial u}{\partial x} \frac{\partial}{\partial x} & \frac{\partial v}{\partial x} \frac{\partial}{\partial x} & \frac{\partial w}{\partial x} \frac{\partial}{\partial x} \\ \frac{\partial u}{\partial y} \frac{\partial}{\partial y} & \frac{\partial v}{\partial y} \frac{\partial}{\partial y} & \frac{\partial w}{\partial y} \frac{\partial}{\partial y} \\ \frac{\partial u}{\partial z} \frac{\partial}{\partial z} & \frac{\partial v}{\partial z} \frac{\partial}{\partial z} & \frac{\partial w}{\partial z} \frac{\partial}{\partial z} \\ \frac{\partial u}{\partial y} \frac{\partial}{\partial z} + \frac{\partial u}{\partial z} \frac{\partial}{\partial y} & \frac{\partial v}{\partial y} \frac{\partial}{\partial z} + \frac{\partial v}{\partial z} \frac{\partial}{\partial y} & \frac{\partial w}{\partial y} \frac{\partial}{\partial z} + \frac{\partial w}{\partial z} \frac{\partial}{\partial y} \\ \frac{\partial u}{\partial z} \frac{\partial}{\partial x} + \frac{\partial u}{\partial x} \frac{\partial}{\partial z} & \frac{\partial v}{\partial z} \frac{\partial}{\partial x} + \frac{\partial v}{\partial x} \frac{\partial}{\partial z} & \frac{\partial w}{\partial z} \frac{\partial}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial}{\partial z} \\ \frac{\partial u}{\partial x} \frac{\partial}{\partial y} + \frac{\partial u}{\partial y} \frac{\partial}{\partial x} & \frac{\partial v}{\partial x} \frac{\partial}{\partial y} + \frac{\partial v}{\partial y} \frac{\partial}{\partial x} & \frac{\partial w}{\partial x} \frac{\partial}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial}{\partial x} \end{bmatrix} \quad (\text{A-3})$$

where operator ∇_L and $\nabla_N(\mathbf{u})$ are both defined according to the Green strain with the latter containing homogeneous linear expressions of the derivatives of $\mathbf{u}$. These operators apply to the displacement field on its right, e.g. $\nabla_L \mathbf{u}$ and $\nabla_N(\mathbf{u}) \mathbf{u}$. Transposing these expressions reverses the sequence of these operators, e.g. $\mathbf{u}^T \nabla_L^T$ and $\mathbf{u}^T \nabla_N^T(\mathbf{u})$. Given the expression of $\nabla_N(\mathbf{u})$, $\mathbf{u}$ and $\delta \mathbf{u}$ in $\nabla_N(\delta \mathbf{u}) \mathbf{u}$ are commutative, i.e. $\nabla_N(\delta \mathbf{u}) \mathbf{u} = \nabla_N(\mathbf{u}) \delta \mathbf{u}$.

Under the assumption of linear elasticity, the constitutive relationship can be given as

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_I + \mathbf{C} \boldsymbol{\varepsilon} = \boldsymbol{\sigma}_I + \mathbf{C} \boldsymbol{\varepsilon}_L + \mathbf{C} \boldsymbol{\varepsilon}_N = \boldsymbol{\sigma}_I + \boldsymbol{\sigma}_L + \boldsymbol{\sigma}_N \quad (\text{A-4})$$

where $\mathbf{C}$ is the material stiffness, $\boldsymbol{\sigma}_L$ and $\boldsymbol{\sigma}_N$ are linear and quadratic parts of the second Piola-Kirchhoff stress in their contracted form, with $\boldsymbol{\sigma}_N$ representing the effects of geometrical nonlinearity. In some parts of the derivation involved in the paper, the matrix form these stresses are required and they will be denoted as $\mathbf{S}$, $\mathbf{S}_I$ and $\mathbf{S}_N$, respectively.

Having introduced the notations above, (A-1) can be partitioned according to the orders of displacements as follows.

$$U = U_1 + U_2 + U_3 + U_4 \quad (\text{A-5a})$$

where

$$U_1 = \boldsymbol{\varepsilon}_L^T \boldsymbol{\sigma}_I = \mathbf{u}^T \nabla_L^T \boldsymbol{\sigma}_I \quad (\text{A-5b})$$

$$U_2 = \boldsymbol{\varepsilon}_N^T \boldsymbol{\sigma}_I + \frac{1}{2} \boldsymbol{\varepsilon}_L^T \mathbf{C} \boldsymbol{\varepsilon}_L = \frac{1}{2} \mathbf{u}^T \nabla_N^T (\mathbf{u}) \boldsymbol{\sigma}_I + \frac{1}{2} \mathbf{u}^T \nabla_L^T \mathbf{C} \nabla_L \mathbf{u} \quad (\text{A-5c})$$

$$U_3 = \frac{1}{2} (\boldsymbol{\varepsilon}_L^T \mathbf{C} \boldsymbol{\varepsilon}_N + \boldsymbol{\varepsilon}_N^T \mathbf{C} \boldsymbol{\varepsilon}_L) = \boldsymbol{\varepsilon}_L^T \mathbf{C} \boldsymbol{\varepsilon}_N = \frac{1}{2} \mathbf{u}^T \nabla_L^T \mathbf{C} \nabla_N (\mathbf{u}) \mathbf{u} \quad (\text{A-5d})$$

$$U_4 = \frac{1}{2} \boldsymbol{\varepsilon}_N^T \mathbf{C} \boldsymbol{\varepsilon}_N = \frac{1}{8} \mathbf{u}^T \nabla_N^T (\mathbf{u}) \mathbf{C} \nabla_N (\mathbf{u}) \mathbf{u} \quad (\text{A-5e})$$

The subscripts to various terms of U above indicate the orders of power in displacement. Each of U_1 to U_4 is a homogeneous function of $\mathbf{u}$ and hence can be expressed as a linear, quadratic, cubic or quartic form of $\mathbf{u}$ as in (A-5b) to (A-5e), respectively. Geometrical nonlinearity is brought into the problem through U_3 and U_4 which are often referred to as higher order terms whilst U_1 and U_2 pertain to the conventional linear problem.

To derive the stiffness matrices of a finite element for the geometrically nonlinear problem the displacement field in a finite element can be discretised as follows.

$$\mathbf{u} = \mathbf{A} \mathbf{q} \quad (\text{A-6})$$

where $\mathbf{q}$ is the nodal displacement vector for the element corresponding to the displacement field $\mathbf{u}$ and $\mathbf{A}$ is the displacement interpolation matrix which is formed from shape functions.

$$\boldsymbol{\varepsilon}_L = \mathbf{B}_L \mathbf{q}, \quad \boldsymbol{\varepsilon}_N = \frac{1}{2} \mathbf{B}_N (\mathbf{q}) \mathbf{q} \quad (\text{A-7})$$

where

$$\mathbf{B}_L = \nabla_L \mathbf{A} \quad (\text{A-8a})$$

$$\mathbf{B}_N (\mathbf{q}) = \nabla_N (\mathbf{u}) \mathbf{A} = \nabla_N (\mathbf{q}) \mathbf{A} \quad \text{with } \nabla_N (\mathbf{A} \mathbf{q}) \text{ denoted later as } \nabla_N (\mathbf{q}) \quad (\text{A-8b})$$

$\mathbf{B}_L$ is independent of $\mathbf{q}$, whilst all the elements of $\mathbf{B}_N$ are homogeneous linear expressions of $\mathbf{q}$, which represents the effects of geometrical nonlinearity in the kinematic equation. Matrices $\mathbf{B}_L$ and $\mathbf{B}_N$ may vary with the type of element, including truss, membrane and brick continuum elements and beam, plate and shell structural elements. Following an argument that was made earlier, $\mathbf{q}$ and $\delta\mathbf{q}$ involved in $\mathbf{B}_N(\delta\mathbf{q})\mathbf{q}$ are commutative, i.e. $\mathbf{B}_N(\delta\mathbf{q})\mathbf{q} = \mathbf{B}_N(\mathbf{q})\delta\mathbf{q}$.

Discretising the term in ψ_{11} associated with $\mathbf{S}_I$ as in (3.22a) leads to

$$\mathbf{u}^T \mathbf{S}_I \mathbf{u}' = \mathbf{q} \left(\sum_{i=1}^3 \mathbf{b}_i^T \mathbf{s}_I \mathbf{b}_i \right) \mathbf{q}' \quad (\text{A-9a})$$

where

$$\mathbf{b}_i = \nabla_i \mathbf{A} \quad (\text{A-9b})$$

Similarly, those in ψ_{111} and ψ_{1111} associated with $\mathbf{S}_L$ and $\mathbf{S}_{11}$ as in (3.22b-c) are discretised to

$$\mathbf{u}^T \mathbf{S}_L(\mathbf{q}'') \mathbf{u}' = \mathbf{q} \left(\sum_{i=1}^3 \mathbf{b}_i^T \mathbf{s}_L(\mathbf{u}'') \mathbf{b}_i \right) \mathbf{q}' \quad \text{with } \mathbf{s}_L(\mathbf{q}'') = \mathbf{s}_L(\mathbf{A}\mathbf{q}'') = \mathbf{s}_L(\mathbf{u}'') \quad (\text{A-10})$$

$$\mathbf{u}^T \mathbf{S}_{11}(\mathbf{u}'', \mathbf{u}''') \mathbf{u}' = \mathbf{q} \left(\sum_{i=1}^3 \mathbf{b}_i^T \mathbf{s}_{11}(\mathbf{u}'', \mathbf{u}''') \mathbf{b}_i \right) \mathbf{q}' \quad \text{with } \mathbf{s}_{11}(\mathbf{q}'', \mathbf{q}''') = \mathbf{s}_{11}(\mathbf{u}'', \mathbf{u}''') \quad (\text{A-11})$$

whilst

$$\mathbf{u}^T \mathbf{S}_N(\mathbf{u}) \mathbf{u} = \frac{1}{2} \mathbf{q} \left(\sum_{i=1}^3 \mathbf{b}_i^T \mathbf{s}_N(\mathbf{u}, \mathbf{u}) \mathbf{b}_i \right) \mathbf{q} \quad \text{with } \mathbf{s}_N(\mathbf{q}) = \mathbf{s}_N(\mathbf{u}) \quad (\text{A-12})$$

as a part of (3.25c) after discretisation.

Appendix B Derivation of the Flip Transformation and an Example of Its Application

$$\begin{aligned}
\mathbf{u}^T \nabla_N^T(\mathbf{u}') \boldsymbol{\sigma}_I &= \left(\frac{\partial u}{\partial x} \frac{\partial u'}{\partial x} + \dots \right) \sigma_{Ix} + \left(\frac{\partial u}{\partial y} \frac{\partial u'}{\partial y} + \dots \right) \sigma_{Iy} + \left(\frac{\partial u}{\partial z} \frac{\partial u'}{\partial z} + \dots \right) \sigma_{Iz} \\
&+ \left(\frac{\partial u}{\partial y} \frac{\partial u'}{\partial z} + \frac{\partial u'}{\partial y} \frac{\partial u}{\partial z} + \dots \right) \tau_{Iyz} + \left(\frac{\partial u}{\partial z} \frac{\partial u'}{\partial x} + \frac{\partial u'}{\partial z} \frac{\partial u}{\partial x} + \dots \right) \tau_{Izx} \\
&+ \left(\frac{\partial u}{\partial x} \frac{\partial u'}{\partial y} + \frac{\partial u'}{\partial x} \frac{\partial u}{\partial y} + \dots \right) \tau_{Ixy} \\
&= \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial z} \end{Bmatrix}^T \begin{bmatrix} \sigma_{Ix} & \tau_{Ixy} & \tau_{Ixz} \\ \tau_{Ixy} & \sigma_{Iy} & \tau_{Iyz} \\ \tau_{Ixz} & \tau_{Iyz} & \sigma_{Iz} \end{bmatrix} \begin{Bmatrix} \frac{\partial u'}{\partial x} \\ \frac{\partial u'}{\partial y} \\ \frac{\partial u'}{\partial z} \end{Bmatrix} + \begin{Bmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial z} \end{Bmatrix}^T \begin{bmatrix} \sigma_{Ix} & \tau_{Ixy} & \tau_{Ixz} \\ \tau_{Ixy} & \sigma_{Iy} & \tau_{Iyz} \\ \tau_{Ixz} & \tau_{Iyz} & \sigma_{Iz} \end{bmatrix} \begin{Bmatrix} \frac{\partial v'}{\partial x} \\ \frac{\partial v'}{\partial y} \\ \frac{\partial v'}{\partial z} \end{Bmatrix} \\
&+ \begin{Bmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \\ \frac{\partial w}{\partial z} \end{Bmatrix}^T \begin{bmatrix} \sigma_{Ix} & \tau_{Ixy} & \tau_{Ixz} \\ \tau_{Ixy} & \sigma_{Iy} & \tau_{Iyz} \\ \tau_{Ixz} & \tau_{Iyz} & \sigma_{Iz} \end{bmatrix} \begin{Bmatrix} \frac{\partial w'}{\partial x} \\ \frac{\partial w'}{\partial y} \\ \frac{\partial w'}{\partial z} \end{Bmatrix} \\
&= u \nabla^T \mathbf{s}_I \nabla u' + v \nabla^T \mathbf{s}_I \nabla v' + w \nabla^T \mathbf{s}_I \nabla w' \\
&= \mathbf{u}^T \nabla_1^T \mathbf{s}_I \nabla_1 \mathbf{u}' + \mathbf{u}^T \nabla_2^T \mathbf{s}_I \nabla_2 \mathbf{u}' + \mathbf{u}^T \nabla_3^T \mathbf{s}_I \nabla_3 \mathbf{u}' \\
&= \mathbf{u}^T \mathbf{S}_I \mathbf{u}'
\end{aligned} \tag{B-1}$$

where $\mathbf{s}_I$ and ∇_i have been given in (3.19b) and (3.19c). Applying the flip transformation to discretised energy terms, e.g. those involved in (3.2), one can readily recover the symmetry of the secant stiffness matrix as in (3.2).

$$\begin{aligned}
\delta U^e &= \delta \mathbf{q}^T \left[\int_{\Omega^e} \left(\mathbf{B}_L^T \mathbf{C} \mathbf{B}_L + \frac{1}{2} \mathbf{B}_L^T \mathbf{C} \mathbf{B}_N(\mathbf{q}) + \mathbf{B}_N^T(\mathbf{q}) \mathbf{C} \mathbf{B}_L + \frac{1}{2} \mathbf{B}_N^T(\mathbf{q}) \mathbf{C} \mathbf{B}_N(\mathbf{q}) \right) d\Omega \right] \mathbf{q} \\
&= \delta \mathbf{q}^T \left[\mathbf{K}_0^e + \frac{1}{2} (\mathbf{N}_1^q)^e + \frac{1}{3} (\mathbf{N}_2^q)^e + \int_{\Omega^e} \left(\frac{1}{2} \mathbf{B}_N^T(\mathbf{q}) \mathbf{C} \mathbf{B}_L + \frac{1}{6} \mathbf{B}_N^T(\mathbf{q}) \mathbf{C} \mathbf{B}_N(\mathbf{q}) \right) d\Omega \right] \mathbf{q} \\
&= \delta \mathbf{q}^T \left[\mathbf{K}_0^e + \frac{1}{2} (\mathbf{N}_1^q)^e + \frac{1}{3} (\mathbf{N}_2^q)^e + \frac{1}{2} \sum_{i=1}^3 \int_{\Omega^e} \mathbf{b}_i^T \mathbf{s}_L \mathbf{b}_i d\Omega + \frac{1}{3} \sum_{i=1}^3 \int_{\Omega^e} \mathbf{b}_i^T \mathbf{s}_N \mathbf{b}_i d\Omega \right] \mathbf{q} \\
&= \delta \mathbf{q}^T \left[\mathbf{K}_0^e + \frac{1}{2} (\mathbf{N}_1^q)^e + \frac{1}{3} (\mathbf{N}_2^q)^e + \frac{1}{2} (\mathbf{K}_1^\sigma)^e + \frac{1}{3} (\mathbf{K}_2^\sigma)^e \right] \mathbf{q} \\
&= \delta \mathbf{q}^T \left[\mathbf{K}_0^e + \frac{1}{2} \mathbf{N}_1^e + \frac{1}{3} \mathbf{N}_2^e \right] \mathbf{q} = \delta \mathbf{q}^T \mathbf{K}_s^e \mathbf{q}
\end{aligned} \tag{B-2}$$

where various stiffness matrices have been given in (3.30) to (3.32). Without making the relevant expression apparently symmetric through the flip transformation, the secant stiffness matrix cannot be obtained explicitly.

Chapter 4 Initial Post-buckling Theory Based on the N-notation of the FEM Formulation

4.1 Introduction

The initial post-buckling theory was established by Koiter in the 1940s [15] although it did not become known to the rest of the world outside the Netherlands until some 20 years later. It was the first ever attempt at offering an insight into various buckling phenomena through the concept of stability of equilibrium states, in particular, those in the initial post-buckling regime. Through this, the concept of imperfection sensitivity of a structure was quantitatively established and directly associated with the stability of the initial post-buckling equilibrium state. The stability of the initial post-buckling equilibrium state and the sensitivity of the buckling load to the initial imperfection could all be described through two non-dimensional parameters, a and b , often termed as IPBCs [40]. Research activities on the theory peaked in the 1970s [19-30] when the development of commercial FEM codes was still in its infancy. However, modern mainstream commercial FEM codes have never managed to incorporate Koiter's theory.

Buckling analysis is one of the basic modules in modern commercial FEM codes [44-46]. However, structural engineers and designers are very aware of the limitations of the predicted buckling load. Structures could collapse at a much lower load level than the predicted buckling load due to their imperfection sensitivity. As the buckling analysis alone is insufficient to provide any information in this regard, investigations have been taken to the post-buckling regime which could be traced as far back as to the 1940s [4] with intensified activities in the 1960s, to name but a few [10-13]. The outcomes of such efforts, whilst having enhanced the understanding of the post-buckling deformation,

proved to be a rather inefficient approach to understanding buckling behaviour. A modern equivalent of such efforts is to analyse structures with artificially introduced imperfections, often as an initial deflection taking the pattern of the buckling mode, taking advantage of the computing power nowadays and the options available in mainstream FEM codes. Such analyses are computationally expensive and time-consuming. A proper assessment will have to involve separate runs having imperfections of either sense one at a time and they will have to be run over a range of magnitudes of such artificially introduced imperfection in order to be representative. This is often missing from such attempts. The computational cost could be overwhelming, bearing in mind that each of such an analysis must be conducted as a nonlinear problem. In most of such endeavours, the ultimate objective is in fact some assurance on the predicted buckling load. Much of the useful part of the outcomes of the above computationally expensive and time-consuming exercises could have been condensed into the two IPBCs, a and b , if Koiter's theory had been adopted. Their evaluations involve little more than a conventional buckling analysis, typically the extraction of the lowest order of eigenvalue, in terms of computational efforts. To be specific, the evaluation of a requires no more information than has been made available after the buckling analysis. That for b needs slightly more effort since one has to solve another specially formulated stiffness equation to obtain the 2nd order of displacement perturbation. This is a linear problem. Computationally, it is equivalent to a single iteration in a conventional incremental-iterative nonlinear analysis. Koiter's theory is called the initial post-buckling theory and it does produce a prediction of the post-buckling deformation. However, it is meant to be applicable to the initial phase of the post-buckling regime in an asymptotic sense. It is usually insufficient to offer meaningful understanding of post-buckling deformation in a broad sense, if such information is genuinely required. The true value of Koiter's theory is to provide an

efficient tool for designers dealing with buckling as they will have much improved level of confidence on the buckling load predicted if it is accompanied by IPBCs a and b . In this regard, through the development of the theory in the 1970-80s, mostly by analytical means, a reasonable understanding has been achieved in terms of their imperfection sensitivity of simple structures, such as plates and cylindrical shells. However, modern structures tend to have intricate details, e.g. panels with hat-shaped stiffeners, often involving the use of new materials, such as composites, and their imperfection sensitivity is no longer obvious before an appropriate evaluation has been carried out.

While, when attempting to implement Koiter's theory into finite element environment using B-notation for the formulation, the severe problem of mesh sensitivity of second IPBC b raised [47, 53, 54, 60, 184] because of the involvement of stresses. The remedy to the problem is to employ some specially-formulated elements which was not attractive to any finite element code developer. In this chapter, Koiter's theory will be re-derived based on the N-notation with the property of commutative displacements as achieved in previous chapter so that it is expected to solve mesh sensitivity problem with only using conventional elements.

4.2 Brief Summary of Koiter's Theory

In order to present chapter in self-contained form, the main procedure of Koiter's theory is provided briefly in its FEM presentation. With the FEM discretisation, after assembling all elements involved in the structure, the total potential energy for a conservative elastic system can be given as

$$\Pi(\mathbf{q}) = \Pi_1(\mathbf{q}) + \Pi_2(\mathbf{q}) + \Pi_3(\mathbf{q}) + \Pi_4(\mathbf{q}) \quad (4.1)$$

where the same $\mathbf{q}$ has been used for the nodal displacement vector for the whole structure without being confused with its counterpart for an element. Having employed the fully

commutative and apparently symmetric SED to formulate element stiffness matrices as shown in the previous section, the full commutativity of $\mathbf{q}$ is present in Π_2 , Π_3 and Π_4 for the structure. Equilibrium is defined by the first variation of Π

$$\delta\Pi(\mathbf{q}) = \Pi_1(\delta\mathbf{q}) + \Pi_{11}(\mathbf{q}, \delta\mathbf{q}) + \Pi_{21}(\mathbf{q}, \delta\mathbf{q}) + \Pi_{31}(\mathbf{q}, \delta\mathbf{q}) = 0 \quad (4.2)$$

where the subscript rules for Π remain the same as for Q in previous chapter in line with Koiter's original notation [15].

As conventionally assumed, there exists a fundamental equilibrium path given as

$$\mathbf{q}^{(0)} = \mathbf{q}^{(0)}(\lambda) \quad \text{satisfying} \quad \mathbf{q}^{(0)}(0) = \mathbf{0}. \quad (4.3)$$

Where λ the load parameter and λ^c the critical load. The critical condition for buckling is met as λ increases from 0 to λ^c , when the second variation as follows ceases to be positive definite, i.e.

$$\delta^2\Pi(\mathbf{q}) = \Pi_{11}(\delta\mathbf{q}, \delta\mathbf{q}) + \Pi_{111}(\mathbf{q}^{(0)}, \delta\mathbf{q}, \delta\mathbf{q}) + \Pi_{211}(\mathbf{q}^{(0)}, \delta\mathbf{q}, \delta\mathbf{q}) \geq 0 \quad (4.4)$$

whilst there exists a nontrivial displacement variation $\delta\mathbf{q} = \mathbf{q}^{(1)}$ that results in a zero value for the second variation above, i.e.

$$\Pi_{11}(\mathbf{q}^{(1)}, \delta\mathbf{q}) + \Pi_{111}(\mathbf{q}^{(0)}, \mathbf{q}^{(1)}, \delta\mathbf{q}) + \Pi_{211}(\mathbf{q}^{(0)}, \mathbf{q}^{(1)}, \delta\mathbf{q}) = 0 \quad (4.5)$$

which gives the governing equation for the eigenvalue problem where the lowest eigenvalue determines the magnitude of $\mathbf{q}^{(0)}$ at buckling, i.e. λ^c , and eigenvector $\mathbf{q}^{(1)}$ provides the buckling mode. The superscripts in the round brackets signifies the order of perturbation, viz. 0 for the pre-buckling regime and 1 for the first order perturbation namely buckling and 2 for the second order perturbation, etc.

The classical buckling problem finishes at this point. However, structures tend to exhibit a range of dramatically different behaviours. Some can support further loads after buckling whilst others collapse long before the classical buckling load is reached. Koiter

established [15, 20] that the different responses were directly related to the state of stability of the post-buckling equilibrium path at the bifurcation point. If it was stable, the structure would be insensitive to the initial imperfection and it could sustain higher levels of load without catastrophic collapse. Otherwise, the structure would be sensitive to the initial imperfection and it would tend to collapse at a load significantly lower than the predicted buckling load, depending on the magnitude of the initial imperfection. In order to obtain the initial post-buckling path and to assess the stability of the equilibrium state on the initial post-buckling path, an asymptotic expression of it can be assumed as an implicit function of ξ in its parametric form with parameter $\mathbf{q}=\mathbf{q}(\lambda)$ as follows.

$$\mathbf{q} = \mathbf{q}^{(0)} + \xi \mathbf{q}^{(1)} + \xi^2 \mathbf{q}^{(2)} + \dots \text{ for single-mode system} \quad (4.6a)$$

$$\mathbf{q} = \mathbf{q}^{(0)} + \xi_i \mathbf{q}_i^{(1)} + \xi_i \xi_j \mathbf{q}_{ij}^{(2)} + \dots \text{ for multiple-mode system} \quad (4.6b)$$

$$\lambda = \lambda^c (1 + a\xi + b\xi^2 + c\xi^3 + \dots) \text{ for single-mode system} \quad (4.7a)$$

$$\xi_i (\lambda - \lambda^c) = \lambda^c (\xi_j \xi_k a_{ijk} + \xi_j \xi_k \xi_l b_{ijkl} + \dots) \text{ for multiple-mode system} \quad (4.7b)$$

where ξ is also called the perturbation parameter in the context of asymptotic analysis which signifies the amplitude of the buckling mode as a measure of the magnitude of the initial post-buckling deformation, $a, b, \dots$ are the so-called IPBCs, $\mathbf{q}^{(1)}$ is the buckling mode assumed to have already being obtained from (4.5), $\mathbf{q}^{(2)}$ is the secondary perturbation mode for the determination of b . For initial post-buckling analysis, the fundamental path $\mathbf{q}^{(0)}$ is assumed to be known and it can be expanded as a Taylor series around the bifurcation point as

$$\mathbf{q}^{(0)} = \mathbf{q}_c^{(0)} + (\lambda - \lambda^c) \dot{\mathbf{q}}_c^{(0)} + \dots \quad (4.8)$$

For cases where Koiter's theory applies, buckling is characterised by bifurcation from the fundamental equilibrium path. In most of such cases, the pre-buckling phase of the fundamental path involves small deformation and can be determined from a linear analysis as a reasonable approximation which is usually incorporated as a part of the

buckling analysis. Koiter's theory does not apply to the other type of buckling at a maximum load where snap-through takes place, in which pre-buckling deformation is typically highly nonlinear.

When the asymptotic series (4.6-4.7) are substituted into the total potential energy expression, its first derivative with respect to ξ gives the equilibrium condition for the post-buckling path and can be considered as an implicit function $\lambda=\lambda(\xi)$. If a and b (4.7) are considered as the 1st and 2nd order of derivatives of λ with respect to ξ at $\xi=0$ and hence $\lambda=\lambda^c$, their expressions can be obtained as follows.

$$a = -\frac{3}{2\lambda^c} \frac{\Pi_3(\mathbf{q}^{(1)}) + \Pi_{13}(\mathbf{q}_c^{(0)}, \mathbf{q}^{(1)})}{\Pi_{12}(\dot{\mathbf{q}}_c^{(0)}, \mathbf{q}^{(1)}) + \Pi_{112}(\mathbf{q}_c^{(0)}, \dot{\mathbf{q}}_c^{(0)}, \mathbf{q}^{(1)})} \quad (4.9a)$$

$$a_{ijk} = -\frac{1}{4\lambda^c} \frac{\Pi_{111}(\mathbf{q}_i^{(1)}, \mathbf{q}_j^{(1)}, \mathbf{q}_k^{(1)}) + \Pi_{1111}(\mathbf{q}_c^{(0)}, \mathbf{q}_i^{(1)}, \mathbf{q}_j^{(1)}, \mathbf{q}_k^{(1)})}{\Pi_{12}(\dot{\mathbf{q}}_c^{(0)}, \mathbf{q}_i^{(1)}) + \Pi_{112}(\mathbf{q}_c^{(0)}, \dot{\mathbf{q}}_c^{(0)}, \mathbf{q}_i^{(1)})} \quad (4.9b)$$

$$b = -\frac{2}{\lambda^c} \frac{\Pi_4(\mathbf{q}^{(1)}) - (\Pi_2(\mathbf{q}^{(2)}) + \Pi_{12}(\mathbf{q}_c^{(0)}, \mathbf{q}^{(2)}) + \Pi_{22}(\mathbf{q}_c^{(0)}, \mathbf{q}^{(2)}))}{\Pi_{12}(\dot{\mathbf{q}}_c^{(0)}, \mathbf{q}^{(1)}) + \Pi_{112}(\mathbf{q}_c^{(0)}, \dot{\mathbf{q}}_c^{(0)}, \mathbf{q}^{(1)})} \quad (a=0) \quad (4.10a)$$

$$b_{ijkl} = -\frac{1}{12\lambda^c} \frac{\Pi_{1111}(\mathbf{q}_i^{(1)}, \mathbf{q}_j^{(1)}, \mathbf{q}_k^{(1)}, \mathbf{q}_l^{(1)}) - 6 \left(\Pi_{11}(\mathbf{q}_{ij}^{(2)}, \mathbf{q}_{kl}^{(2)}) + \Pi_{111}(\mathbf{q}_c^{(0)}, \mathbf{q}_{ij}^{(2)}, \mathbf{q}_{kl}^{(2)}) + \Pi_{22}(\mathbf{q}_0^c, \mathbf{q}_{ij}^{(2)}, \mathbf{q}_{kl}^{(2)}) \right)}{\Pi_{12}(\dot{\mathbf{q}}_0^c, \mathbf{q}_i^{(1)}) + \Pi_{112}(\mathbf{q}_0^c, \dot{\mathbf{q}}_0^c, \mathbf{q}_i^{(1)})} \quad (\forall a_{ijk} = 0) \quad (4.10b)$$

In most practical analyses, determination of a and b is sufficient. As established in Koiter's theory, the most important characteristics of the initial post-buckling behaviour can be determined using only these constants, provided that they do not vanish simultaneously.

If $a \neq 0$, the initial post-buckling path given (4.7) can be truncated at the first order of the asymptotic series as an approximation and one does not even need to evaluate b as it contributes only an asymptotically higher order term. In this case, an asymmetric

bifurcation is expected which predicts an unstable initial post-buckling behaviour in general and leads to imperfection sensitivity. Given the asymmetric nature of the bifurcation, if the structure is sensitive to an imperfection of one sense, e.g. positive, it is insensitive to the same imperfection of the other sense, i.e. negative. This is why if one adopts the nonlinear post-buckling analysis approach, imperfections of both senses must be assessed separately to form a complete view.

When $a=0$, one has to proceed to the second order perturbation. If $b \neq 0$, (4.7) can be truncated at the second order term for initial post-buckling behaviour leading to a symmetric bifurcation. In order to evaluate b , the second order displacement perturbation $\mathbf{q}^{(2)}$ is required. The governing equation can be obtained from the second order terms of ξ in the asymptotically expanded total potential energy as [15, 20]

$$\begin{aligned} & \Pi_{11}(\delta\mathbf{q}, \mathbf{q}^{(2)}) + \Pi_{111}(\delta\mathbf{q}, \mathbf{q}^{(0)}, \mathbf{q}^{(2)}) + \Pi_{121}(\delta\mathbf{q}, \mathbf{q}^{(0)}, \mathbf{q}^{(2)}) \\ & = -\Pi_{12}(\delta\mathbf{q}, \mathbf{q}^{(1)}) - \Pi_{112}(\delta\mathbf{q}, \mathbf{q}^{(0)}, \mathbf{q}^{(1)}) \end{aligned} \quad \text{for single-mode system} \quad (4.11a)$$

$$\begin{aligned} & \Pi_{11}(\delta\mathbf{q}, \mathbf{q}_{ij}^{(2)}) + \Pi_{111}(\delta\mathbf{q}, \mathbf{q}^{(0)}, \mathbf{q}_{ij}^{(2)}) + \Pi_{121}(\delta\mathbf{q}, \mathbf{q}^{(0)}, \mathbf{q}_{ij}^{(2)}) \\ & = -\frac{1}{2}\Pi_{111}(\delta\mathbf{q}, \mathbf{q}_i^{(1)}, \mathbf{q}_j^{(1)}) - \frac{1}{2}\Pi_{1111}(\delta\mathbf{q}, \mathbf{q}^{(0)}, \mathbf{q}_i^{(1)}, \mathbf{q}_j^{(1)}) \end{aligned} \quad \text{for multiple-mode system} \quad (4.11b)$$

subject to an orthogonality condition between $\mathbf{q}^{(1)}$ and $\mathbf{q}^{(2)}$ as

$$\Pi_{11}(\dot{\mathbf{q}}_c^{(0)}, \mathbf{q}^{(1)}, \mathbf{q}^{(2)}) + \Pi_{1111}(\mathbf{q}_c^{(0)}, \dot{\mathbf{q}}_c^{(0)}, \mathbf{q}^{(1)}, \mathbf{q}^{(2)}) = 0 \quad \text{for sing-mode system} \quad (4.12a)$$

$$\Pi_{11}(\dot{\mathbf{q}}_c^{(0)}, \mathbf{q}_i^{(1)}, \mathbf{q}_{jk}^{(2)}) + \Pi_{1111}(\mathbf{q}_c^{(0)}, \dot{\mathbf{q}}_c^{(0)}, \mathbf{q}_i^{(1)}, \mathbf{q}_{jk}^{(2)}) = 0 \quad \text{for multiple-mode system} \quad (4.12b)$$

Higher orders of perturbation could be made in theory in the case that b vanishes but this situation rarely arise in practice.

4.3 Koiter's Theory Presented in the N-notation of FEM

Koiter's theory has been summarised as above. However, it has not been implemented in any mainstream commercial FEM code. In fact, there has been no lack of desire to achieve this, early attempts could be traced back to the 1970s [182, 183] and

numerous more recent efforts could be found including those in [47, 48, 50-53, 56-59, 61, 62, 184]. The development of the FEM implementation of Koiter's theory has suffered from a number of key obstacles. One of them was referred to as the 'locking' phenomenon as originally defined in [184]. It will be resolved as one of the main outcomes of the present chapter. The remaining obstacles have also been resolved to present them soon in subsequent chapter.

Making use of the SED and FEM formulation as presented in previous section, key governing equations as presented above can be expressed in the FEM context as follow.

$$\Pi = \mathbf{q}^T \left(\frac{1}{2} \mathbf{K}_0 + \frac{1}{6} \mathbf{N}_1 + \frac{1}{12} \mathbf{N}_2 \right) \mathbf{q} - \mathbf{q}^T \mathbf{F} \quad \text{c.f.(4.1)} \quad (4.13)$$

$$\left(\mathbf{K}_0 + \frac{1}{2} \mathbf{N}_1 + \frac{1}{3} \mathbf{N}_2 \right) \mathbf{q} - \mathbf{F} = 0 \quad \text{c.f.(4.2)} \quad (4.14)$$

$$(\mathbf{K}_0 + \mathbf{N}_1 + \mathbf{N}_2) \mathbf{q}^{(1)} = \mathbf{0} \quad \text{c.f.(4.5)} \quad (4.15)$$

$$a = -\frac{1}{2\lambda^c} \frac{(\mathbf{q}^{(1)})^T \mathbf{N}_1 (\mathbf{q}^{(1)}) \mathbf{q}^{(1)} + 2(\mathbf{q}^{(1)})^T \mathbf{N}_2 (\mathbf{q}^{(1)}) \mathbf{q}_c^{(0)}}{(\mathbf{q}^{(1)})^T \mathbf{N}_1 (\mathbf{q}^{(1)}) \dot{\mathbf{q}}_c^{(0)} + 2(\mathbf{q}_c^{(0)})^T \mathbf{N}_2 (\mathbf{q}^{(1)}) \dot{\mathbf{q}}_c^{(0)}} \quad \text{c.f.(4.9a)} \quad (4.16a)$$

for sing-mode system

$$a_{ijk} = -\frac{1}{2\lambda^c} \frac{(\mathbf{q}_i^{(1)})^T \mathbf{N}_1 (\mathbf{q}_j^{(1)}) \mathbf{q}_k^{(1)} + (\mathbf{q}_i^{(1)})^T \mathbf{N}_{11} (\mathbf{q}_j^{(1)}, \mathbf{q}_k^{(1)}) \mathbf{q}_c^{(0)}}{(\mathbf{q}_i^{(1)})^T \mathbf{N}_1 (\mathbf{q}_i^{(1)}) \dot{\mathbf{q}}_c^{(0)} + 2(\mathbf{q}_c^{(0)})^T \mathbf{N}_2 (\mathbf{q}_i^{(1)}) \dot{\mathbf{q}}_c^{(0)}} \quad \text{c.f.(4.9b)} \quad (4.16b)$$

for multiple-mode system

$$b = -\frac{1}{3\lambda^c} \frac{(\mathbf{q}^{(1)})^T \mathbf{N}_2 (\mathbf{q}^{(1)}) \mathbf{q}^{(1)} - 6(\mathbf{q}^{(2)})^T (\mathbf{K}_0 + \mathbf{N}_1 (\mathbf{q}_c^{(0)}) + \mathbf{N}_2 (\mathbf{q}_c^{(0)})) \mathbf{q}^{(2)}}{(\mathbf{q}^{(1)})^T \mathbf{N}_1 (\mathbf{q}^{(1)}) \dot{\mathbf{q}}_c^{(0)} + 2(\mathbf{q}_c^{(0)})^T \mathbf{N}_2 (\mathbf{q}^{(1)}) \dot{\mathbf{q}}_c^{(0)}} \quad \text{c.f.(4.10a)} \quad (4.17a)$$

for sing-mode system

$$b_{ijkl} = -\frac{1}{6\lambda^c} \frac{(\mathbf{q}_i^{(1)})^T \mathbf{N}_{11} (\mathbf{q}_k^{(1)}, \mathbf{q}_l^{(1)}) \mathbf{q}_j^{(1)} - 6(\mathbf{q}_{ij}^{(2)})^T (\mathbf{K}_0 + \mathbf{N}_1 (\mathbf{q}_c^{(0)}) + \mathbf{N}_2 (\mathbf{q}_c^{(0)})) \mathbf{q}_{kl}^{(2)}}{(\mathbf{q}_i^{(1)})^T \mathbf{N}_1 (\mathbf{q}_i^{(1)}) \dot{\mathbf{q}}_c^{(0)} + 2(\mathbf{q}_c^{(0)})^T \mathbf{N}_2 (\mathbf{q}_i^{(1)}) \dot{\mathbf{q}}_c^{(0)}} \quad \text{c.f.(4.10b)} \quad (4.17b)$$

for multiple-mode system

$$(\mathbf{K}_0 + \mathbf{N}_1 + \mathbf{N}_2)\mathbf{q}^{(2)} = -\frac{1}{2}\mathbf{N}_1(\mathbf{q}^{(1)})\mathbf{q}^{(1)} - \mathbf{N}_2(\mathbf{q}^{(1)})\mathbf{q}_c^{(0)} \quad c.f.(4.11a) \quad (4.18a)$$

for sing-mode system

$$(\mathbf{K}_0 + \mathbf{N}_1 + \mathbf{N}_2)\mathbf{q}_{ij}^{(2)} = -\frac{1}{2}\mathbf{N}_1(\mathbf{q}_i^{(1)})\mathbf{q}_j^{(1)} - \frac{1}{2}\mathbf{N}_{11}(\mathbf{q}_i^{(1)}, \mathbf{q}_j^{(1)})\mathbf{q}_c^{(0)} \quad c.f.(4.11b) \quad (4.18b)$$

for multiple-mode system

$$(\mathbf{q}^{(1)})^T \left(-\mathbf{N}_1(\dot{\mathbf{q}}_c^{(0)}) - \mathbf{N}_{11}(\dot{\mathbf{q}}_c^{(0)}, \mathbf{q}_c^{(0)}) \right) \mathbf{q}^{(2)} = 0 \quad c.f.(4.12a) \quad (4.19a)$$

$$(\mathbf{q}_i^{(1)})^T \left(-\mathbf{N}_1(\dot{\mathbf{q}}_c^{(0)}) - \mathbf{N}_{11}(\dot{\mathbf{q}}_c^{(0)}, \mathbf{q}_c^{(0)}) \right) \mathbf{q}_{jk}^{(2)} = 0 \quad c.f.(4.12b) \quad (4.19b)$$

where $\mathbf{F}$ is the external loading vector whilst various symbols associated with the FEM formulation are inherited from the previous section in a straightforward manner except that they have been assembled from all elements involved in the structure. It is found that these equations could be consistently corresponding to the ones in 4.2, especially the singular coefficient matrix is shared by governing equations of both buckling and SPM and in fact also by all higher orders of perturbation, because these governing equations are all derived from the equilibrium equation as different orders of perturbation. All orders of perturbation are with reference to the equilibrium at the bifurcation point. The governing equations are incremental forms of the equilibrium equation and the coefficient matrix is the tangential stiffness matrix. At the bifurcation point, the tangential stiffness matrix is singular. The singularity of such coefficient has presented a numerical difficulty when solving for the SPM. The iterative method has been the only available means of solving the problem in the literature. Due to its disadvantages, a closed-form solution will be delivered in next chapter in order to provide an alternative using rigorous mathematical derivations.

The expressions of a and b above were obtained as early as in 1971 [182]. However, they have remained unimplemented because the nonlinear stiffness matrices $\mathbf{N}_1$ and $\mathbf{N}_2$ have been a ‘symbolism’ as referred to in the first edition of [177]. In obtaining them in [182], the full commutativity of various displacements within each of the terms

appearing in the expressions had been taken for granted, which did not stand scrutiny if one wished to adopt it in a commercial code.

4.4 Summary

Based on the full commutativity and the apparent symmetry as established earlier in previous chapter, the Koiter's initial post-buckling theory has been re-derived based on it. The two IPBCs have been successfully expressed with the commutative N-notation along with the determination of secondary perturbation mode and the orthogonality condition between buckling mode and secondary perturbation mode for both single-mode system and multiple-mode system. The practicability of N-notation when used in Koiter's theory will be tested in the coming chapter.

Chapter 5 A Closed-form Solution for the Secondary Perturbation

Displacement and Benefits of N-notation on Mesh Sensitivity

5.1 Introduction

The provision of buckling strength prediction from mainstream commercial FE codes is generally unsatisfactory for practising structural designers and analysts, despite the ever-improving capability of nonlinear analysis. Solvers available nowadays are in fact sufficient for the class of the so-called snap-through buckling problem, which features a maximum load point on the equilibrium path where snap-through takes place. The equilibrium path before snap-through is usually highly nonlinear due to finite deformation. There is another, practically much large, class of buckling problem characterised by a bifurcation proceeded by usually fairly linear pre-buckling equilibrium path. The prediction of such bifurcation point is straightforward as an eigenvalue problem which has never been any problem practically. However, designers are well aware of the fact that it is simply not reliable to treat the predicted the buckling load defined by the bifurcation as the strength of the structure. Some structures can sustain much higher loads above the load at the bifurcation point with increasing deformation, whilst others could collapse at load levels much lower than the predicted buckling load. This phenomenon started troubling engineers in 1940s [4] which led to a great effort trying to obtain the post-buckling equilibrium path, e.g. [10, 13], in order to understand the phenomenon. The campaign ended rather abruptly with the emergence of Koiter's initial post-buckling theory in late 1960s [15] although it was established in 1945 before it appeared in the English world in 1963 [20]. In essence, Koiter's theory revealed the significance of the post-buckling behaviour on the buckling load. It was in fact not the post-buckling equilibrium path but the stability of the equilibrium state on that path, in particular, that

at the bifurcation point on the post-buckling path. In other words, being able to find the post-buckling equilibrium path is not enough, no matter how far one could follow the path beyond the bifurcation point. What one really needs is the stability state of the equilibrium on the post-buckling path, which none of the existing mainstream commercial code is capable of providing. If one could assess the stability state, there would not be much need to go very far beyond the bifurcation point. That in a close neighbourhood of the bifurcation point or that right at the bifurcation point but on the post-buckling equilibrium path will be sufficient. Koiter proved as a part of his initial post-buckling theory that this state of stability dictated the sensitivity of the actual buckling load to the initial imperfection the presence of which is inevitable in reality due to manufacturing or testing practicality. Structures of stable initial post-buckling are able to sustain higher loads whilst those tend to collapse prematurely are due to their unstable initial post-buckling equilibrium path. Koiter did not only prove the qualitative relationship as stated above, he also managed to associate the actual collapsing load quantitatively and simply with the magnitude of the initial imperfection in an asymptotic manner.

Koiter's theory liberated researchers in the 1960-70s from their fruitless pursuit for the post-buckling path. Unfortunately, like in other areas, history repeats itself periodically. Having been disappointed by the FE predicted buckling strength, there have been a lot of efforts nowadays devoted again to obtaining the post-buckling equilibrium path using the facility available in existing commercial FE codes, e.g. Abaqus [44], taking their advantage of the nonlinear analysis capability. Such analyses are computationally expensive and yet the outcomes would only become meaningful if the user has good grasp of the principles of Koiter's theory, which does not usually seem the case in reality. One of the attractive features of Koiter's theory is that in order to reach an informative assessment of the buckling strength, the extra computational cost is virtually negligible,

although the theoretical development supporting the conclusions is highly mathematical and sophisticated. The outcome is amazingly simple and the interpretation of the results, most straightforward, all rest on the magnitude of two so-called initial post-buckling coefficients in most cases, conventionally denoted as a and b . Guided by these two values, structural designers would have much improved confidence on the relevance of the predicted buckling load to the actual buckling strength of the structure concerned. 1970s were also the era when modern commercial FE codes started being incubated. Half a century on, the development of FE capabilities of these codes has been sensational. However, given the practical significance of Koiter's theory and given the attempts of FE implementation of it since early 1970s [182], it is still unavailable from any mainstream commercial FE code, for which disappointment is an absolute understatement. A bleak reality is that Koiter and his theory are being forgotten by structural designers and analysts, instead of being a useful tool at their fingertips to benefit them.

A closer examination suggests that it has been in this position not without genuine reasons. However, researchers seemed to shy away from revealing the problems directly and explicitly before the difficulties could be addressed appropriately. The grand aim of this chapter is to resurrect Koiter's theory by demonstrating the feasibility of its FE implementation although it will take a lot more than just this chapter to achieve. It is therefore good to identify key obstacles which will have to be overcome before developers of commercial FE codes would seriously contemplate the possibility of incorporating it into their codes. One of the obstacles is that to evaluate the second initial post-buckling coefficient b , a set of special simultaneous linear equations as the governing equation for the secondary perturbation displacements has to be solved, where a convincing solution lacks. A more detailed description will be provided later as it is the subject of the present chapter.

In order to evaluate b , one has to solve the secondary perturbation equation for the bifurcated equilibrium path. The solution gives the secondary perturbation displacements which constitutes an essential part in the expression of b . This involves solving a set of simultaneous linear algebraic equations. The difficulty associated with this set of equations is that the coefficient matrix is singular, although the existence of the solution can be guaranteed. The existence of the solution also comes with the fact that there exists infinite number of solutions, in general. The uniqueness will be reinstated after imposing one or more orthogonality condition(s), depending on the degree of singularity. If it involves a small number of equations, as it was often the case in analytical solutions, one or more dependent equation(s) which can be linearly expressed by other equations could be relatively easily identified and filtered out before topping up the series of equations with the orthogonality condition(s) to remove the singularity. The solution can then proceed in a straightforward manner. However, in an FE application, one often faces a large number of simultaneous equations. Out of such a large number of them, which should be filtered out does not seem to be a straightforward decision to make without implementing a significant amount of manipulations. This presented as an obstacle to the FE implementation of Koiter's theory, at least if one wishes to bring it to a level commercial FE code developers would feel at ease with.

Given this practical problem, there has not been much effort in order to resolve the problem in the literature, except those to be described in the next paragraph, although every user of the theory has to face the problem. In fact, most accounts in the literature tended to avoid mentioning it as a problem. Mathematically, stating the fact of the existence and uniqueness of the solution, as it is indeed the case for this particular problem, would be sufficient, although no proof seems to have been offered in the literature. On the practical side, in order to solve the problem and deliver a solution for engineers to

facilitate their structural design, the solution had to be actually found. Not many seemed to care to deal with the problem.

There have been a number of publications [47, 50, 52, 57, 146, 160] in the literature where a complete description of the solution procedure could be found. In these accounts, some iterative schemes were proposed to solve this set of simultaneous equations. Iterations turned a linear problem, requiring solving a set of simultaneous equations once, into a much more demanding one like a nonlinear problem where a solution required solving simultaneous equations many times. In addition to the increased computational cost and loss of numerical accuracy, there was no rigorous proof of guaranteed convergence of such an iterative scheme, although the convergence was claimed to be rapid for this particular problem. Good iterative schemes did seem to deliver a converged solution but the iterative approach in principle was far from being ideal.

To the best knowledge of author, there has been only one account in the literature which described a non-iterative approach intuitively as ‘the solution can therefore be found simply by adding a support that eliminates the singularity and afterwards the orthogonality condition can be applied’ [53]. However, no instruction was given where the support should be added and why. Whilst the intuition stood some truth, it was far from rigor and it did not seem to have attracted much attention afterwards. In the present chapter, mathematical rigor will be given to the approach presented which leads to a closed-form solution. The governing equation will be expressed in this chapter using the N-notation finite element formulation as established in Chapter 3. To express the orthogonality condition, a positive definite matrix is required [20]. A matrix corresponding to the load derivative of the secondary perturbation of the total potential energy will be chosen to deliver the orthogonality condition in line with other attempts. As it is a closed-form solution, no iteration will be required. Mathematically it is an exact

solution if numerical errors involved in the solution of linear simultaneous equations are excluded. It will be shown that the solution involves solving a set of linear simultaneous equations whilst the remaining manipulations, though mathematically essential, are trivial in terms of computational cost.

5.2 Positive Definite Matrix Employed in Orthogonality Condition

To uniquely determine the secondary perturbation displacement, it is required to be orthogonal to the buckling mode as an auxiliary condition as

$$\left(\mathbf{q}^{(1)}\right)^T \mathbf{T} \mathbf{q}^{(2)} = 0 \quad (5.1)$$

According to [20], the orthogonal condition can be introduced through an arbitrary positive definite matrix. The one as given in (5.1) has been employed widely [47, 50, 52, 53, 57, 146, 160]. However, its positive definiteness does not seem obvious to every reader. A brief elaboration is therefore provided here for the sake of completeness. Given the fundamental equilibrium path, it is customary to assume that the equilibrium is stable before reaching the buckling point, i.e. $\lambda < \lambda^c$. The 2nd order variation of the total potential energy Π therefore positive definite, i.e.

$$\delta^2 \Pi = \delta \mathbf{q}^T \left(\mathbf{K}_0 + \mathbf{N}_1 \left(\mathbf{q}^{(0)} \right) + \mathbf{N}_2 \left(\mathbf{q}^{(0)} \right) \right) \delta \mathbf{q} > 0 \text{ for } \delta \mathbf{q} \neq \mathbf{0} \quad (5.2)$$

As $\mathbf{q}^{(0)}$ is a function of load parameter, it can be Taylor expanded at $\lambda = \lambda^c$. Accordingly, the above expression can also be expanded at $\lambda = \lambda^c$ and truncated at the linear term as

$$\begin{aligned} \delta^2 \Pi \approx & \delta \mathbf{q}^T \left(\mathbf{K}_0 + \mathbf{N}_1 \left(\mathbf{q}_c^{(0)} \right) + \mathbf{N}_2 \left(\mathbf{q}_c^{(0)} \right) \right) \delta \mathbf{q} \\ & + \left(\lambda - \lambda^c \right) \delta \mathbf{q}^T \left(\mathbf{N}_1 \left(\dot{\mathbf{q}}_c^{(0)} \right) + \mathbf{N}_{11} \left(\mathbf{q}_c^{(0)}, \dot{\mathbf{q}}_c^{(0)} \right) \right) \delta \mathbf{q} > 0 \end{aligned} \quad (5.3)$$

Given the criterion for bucking, the first term on the right hand side of the above is semi-positive definite. As (5.2) applies for any $\lambda < \lambda^c$, the 2nd term on the right hand side of the above must be positive definite, i.e.

$$(\lambda - \lambda^c) \delta \mathbf{q}^T \left(\mathbf{N}_1(\dot{\mathbf{q}}_c^{(0)}) + \mathbf{N}_{11}(\mathbf{q}_c^{(0)}, \dot{\mathbf{q}}_c^{(0)}) \right) \delta \mathbf{q} > 0 \quad (5.4)$$

Given, matrix $-\left(\mathbf{N}_1(\dot{\mathbf{q}}_c^{(0)}) + \mathbf{N}_{11}(\mathbf{q}_c^{(0)}, \dot{\mathbf{q}}_c^{(0)})\right)$ must be positive definite. Hence the auxiliary functional matrix $\mathbf{T}$ is assigned as

$$\mathbf{T} = -\left(\mathbf{N}_1(\dot{\mathbf{q}}_c^{(0)}) + \mathbf{N}_{11}(\mathbf{q}_c^{(0)}, \dot{\mathbf{q}}_c^{(0)})\right) \quad (5.5)$$

5.3 The Closed-form Solution

5.3.1 The Construction of the Solution

For the derivation in this section, the buckling will be assumed to be of a single mode. The problem of multiple mode buckling will be addressed in the next section. For the sake of convenience in demonstrating the mathematical manipulations, governing equation of secondary perturbation mode (4.18a) is denoted as follows as an abbreviation.

$$\mathbf{K}_t \mathbf{q} = \mathbf{f} \quad (5.6a)$$

$$\begin{bmatrix} K_{1,1} & \cdots & K_{1,i} & \cdots & K_{1,n} \\ \vdots & \ddots & \vdots & & \vdots \\ K_{i,1} & \cdots & K_{i,i} & \cdots & K_{i,n} \\ \vdots & & \vdots & \ddots & \vdots \\ K_{n,1} & \cdots & K_{n,i} & \cdots & K_{n,n} \end{bmatrix} \begin{Bmatrix} q_1 \\ \vdots \\ q_i \\ \vdots \\ q_n \end{Bmatrix} = \begin{Bmatrix} f_1 \\ \vdots \\ f_i \\ \vdots \\ f_n \end{Bmatrix} \quad (5.6b)$$

This is a set of simultaneous linear nonhomogeneous equations. In general, the solution to a linear problem can always be constructed from a complementary solution $\mathbf{q}^c$ and a particular solution $\mathbf{q}^p$. The complementary solution $\mathbf{q}^c$ satisfies the homogeneous counterpart of (5.6)

$$\mathbf{K}_t \mathbf{q}^c = \mathbf{0} \quad (5.7)$$

whilst the particular solution $\mathbf{q}^p$ can be anyone satisfying (5.6). Apparently, none of the complementary solution $\mathbf{q}^c$ and the particular solution $\mathbf{q}^p$ is unique. For any constant s , one can easily verify that $s\mathbf{q}^c$ also gives a complementary solution whilst $\mathbf{q}^c + \mathbf{q}^p$ is also a particular solution. According to linear algebra [69], if the buckling is of a single mode,

the rank of $\mathbf{K}_t$ is $(n-1)$, i.e. the solution to (5.7) is determined to a single arbitrary constant s . Thus, if $\mathbf{q}^c$ is a complementary solution, any complement solution can be given as $s\mathbf{q}^c$. The complete solution to (5.6), if it exists, can be expressed as

$$\mathbf{q} = s\mathbf{q}^c + \mathbf{q}^p \quad (5.8)$$

With the problem mathematically formulated as above, a closed-form solution will be sought for in the following subsections with appropriate mathematical considerations duly made.

5.3.2 The Complementary Solution

Given the buckling equation (4.15), it is clear that the buckling mode $\mathbf{q}^{(1)}$ is a complementary solution, as

$$\mathbf{K}_t \mathbf{q}^{(1)} = \mathbf{0} \quad (5.9)$$

Apparently, (5.9) is identical to (4.15) whilst adopting the abbreviated notation. Thus

$$\mathbf{q}^c = \mathbf{q}^{(1)} = \begin{bmatrix} q_1^{(1)} & \cdots & q_{i-1}^{(1)} & q_i^{(1)} & q_{i+1}^{(1)} & \cdots & q_n^{(1)} \end{bmatrix}^T \quad (5.10)$$

Substituting $\mathbf{q}^{(1)}$ into (5.7) or (5.9), one has

$$q_1^{(1)} \begin{Bmatrix} K_{1,1} \\ \vdots \\ K_{1,i} \\ \vdots \\ K_{1,n} \end{Bmatrix} + \cdots + q_{i-1}^{(1)} \begin{Bmatrix} K_{i-1,1} \\ \vdots \\ K_{i-1,i} \\ \vdots \\ K_{i-1,n} \end{Bmatrix} + q_i^{(1)} \begin{Bmatrix} K_{i,1} \\ \vdots \\ K_{i,i} \\ \vdots \\ K_{i,n} \end{Bmatrix} + q_{i+1}^{(1)} \begin{Bmatrix} K_{i+1,1} \\ \vdots \\ K_{i+1,i} \\ \vdots \\ K_{i+1,n} \end{Bmatrix} + \cdots + q_n^{(1)} \begin{Bmatrix} K_{n,1} \\ \vdots \\ K_{n,i} \\ \vdots \\ K_{n,n} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{Bmatrix} \quad (5.11)$$

Assuming that the i^{th} component of $\mathbf{q}^{(1)}$ is of the highest absolute magnitude, one is sure that

$$q_i^{(1)} \neq 0 \quad (5.12)$$

Otherwise, the buckling mode would be a null vector which is not allowed as the buckling mode is supposed to be a non-trivial solution. With condition (5.12), one can re-write (5.11) into

$$\begin{Bmatrix} K_{i,1} \\ \vdots \\ K_{i,i} \\ \vdots \\ K_{i,n} \end{Bmatrix} = -\frac{q_1^{(1)}}{q_i^{(1)}} \begin{Bmatrix} K_{1,1} \\ \vdots \\ K_{1,i} \\ \vdots \\ K_{1,n} \end{Bmatrix} - \dots - \frac{q_{i-1}^{(1)}}{q_i^{(1)}} \begin{Bmatrix} K_{i-1,1} \\ \vdots \\ K_{i-1,i} \\ \vdots \\ K_{i-1,n} \end{Bmatrix} - \frac{q_{i+1}^{(1)}}{q_i^{(1)}} \begin{Bmatrix} K_{i+1,1} \\ \vdots \\ K_{i+1,i} \\ \vdots \\ K_{i+1,n} \end{Bmatrix} - \dots - \frac{q_n^{(1)}}{q_i^{(1)}} \begin{Bmatrix} K_{n,1} \\ \vdots \\ K_{n,i} \\ \vdots \\ K_{n,n} \end{Bmatrix} \quad (5.13)$$

The above equation states that the i^{th} column of the $\mathbf{K}_t$ matrix can be linearly expressed in terms of other columns of the same matrix, with i corresponds to the row number of the buckling mode of the highest absolute magnitude. As $\mathbf{K}_t$ is a symmetric matrix, the above statement can be made equally for the i^{th} row of matrix $\mathbf{K}_t$. The implication is that the i^{th} equation in (5.7) is not independent.

5.3.3 Existence of the Solution for the Nonhomogeneous Equations

In Koiter's theory, there has never been specific proof of the existence of the solution for the secondary perturbation equation. In analytical approaches, usually the proof came with the fact that a solution had been actually found. This is not an ideal position for numerical approaches. For instance, when facing non-converging iterations, one would not be sure if it was because of the inexistence of the solution or improper iteration algorithm. A clear statement on the existence of the solution is helpful. For (5.6) to have a solution, according to linear algebra strictly, one should have

$$\text{rank} \left(\begin{bmatrix} K_{1,1} & \dots & K_{1,i} & \dots & K_{1,n} \\ \vdots & \ddots & \vdots & & \vdots \\ K_{i,1} & \dots & K_{i,i} & \dots & K_{i,n} \\ \vdots & & \vdots & \ddots & \vdots \\ K_{n,1} & \dots & K_{n,i} & \dots & K_{n,n} \end{bmatrix} \right) = \text{rank} \left(\begin{bmatrix} K_{1,1} & \dots & K_{1,i} & \dots & K_{1,n} & f_1 \\ \vdots & \ddots & \vdots & & \vdots & \vdots \\ K_{i,1} & \dots & K_{i,i} & \dots & K_{i,n} & f_i \\ \vdots & & \vdots & \ddots & \vdots & \vdots \\ K_{n,1} & \dots & K_{n,i} & \dots & K_{n,n} & f_n \end{bmatrix} \right) \quad (5.14)$$

This can also be interpreted as f_i can be expressed in terms of the rest of the components of $\mathbf{f}$ in the same way as the i^{th} row of $\mathbf{K}_t$ is expressed in terms of other rows of $\mathbf{K}_t$, as in (5.13), i.e.

$$f_i = -\frac{q_1^{(1)}}{q_i^{(1)}} f_1 - \dots - \frac{q_{i-1}^{(1)}}{q_i^{(1)}} f_{i-1} - \frac{q_{i+1}^{(1)}}{q_i^{(1)}} f_{i+1} - \dots - \frac{q_n^{(1)}}{q_i^{(1)}} f_n \quad (5.15)$$

If (5.15) is re-written into $\mathbf{f}^T \mathbf{q}^{(1)} = 0$, its satisfaction can be readily verified after (5.6) is transposed before being multiplied by $\mathbf{q}^{(1)}$, given condition (4.15). The existence of solutions to (5.6) is thus guaranteed.

5.3.4 The Particular Solution

Equations (5.13) and (5.15) imply that the i^{th} equation is dependent and can be removed from (5.6) without affecting anything, assuming that the i^{th} component of $\mathbf{q}^{(1)}$ is the one with the highest absolute magnitude. After removing the i^{th} equation, the governing equation for the particular solution can be re-written into

$$\begin{bmatrix} K_{1,1} & \cdots & K_{1,i-1} & K_{1,i+1} & \cdots & K_{1,n} \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ K_{i-1,1} & \cdots & K_{i-1,i-1} & K_{i-1,i+1} & \cdots & K_{i-1,n} \\ K_{i+1,1} & \cdots & K_{i+1,i-1} & K_{i+1,i+1} & \cdots & K_{i+1,n} \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ K_{n,1} & \cdots & K_{n,i-1} & K_{n,i+1} & \cdots & K_{n,n} \end{bmatrix} \begin{bmatrix} q_1^p \\ \vdots \\ q_{i-1}^p \\ q_{i+1}^p \\ \vdots \\ q_n^p \end{bmatrix} = \begin{bmatrix} f_1 \\ \vdots \\ f_{i-1} \\ f_{i+1} \\ \vdots \\ f_n \end{bmatrix} - \begin{bmatrix} K_{1,i} \\ \vdots \\ K_{i-1,i} \\ K_{i+1,i} \\ \vdots \\ K_{n,i} \end{bmatrix} q_i^p \quad (5.16)$$

where the coefficient matrix on the left hand side is now positive definite, assuming that the buckling is of a single mode. As stated previously, the particular solution is not unique. Anything satisfying (5.16) gives a particular solution. An examination of (5.16) reveals that q_i^p is a free factor that can be set to any fixed value to produce a particular solution. Since only one particular solution is required which can be anyone, the arbitrariness of q_i^p offers a great convenience. If one chooses $q_i^p = 0$, it will be of exactly the same

effects as constraining the degree of freedom q_i^p from (5.6) [69]. The solution then will deliver a particular solution.

It should be pointed out that although a particular solution can be found by constraining a dof, it cannot be an arbitrary dof. If the dof had not been properly chosen, for instance, if the component of the buckling mode chosen was associated with the dof which happened to have a zero value, this would be equivalent to eliminating an independent equation rather than a dependent one. Even if the component of the buckling mode associated with the dof was not exactly zero, the problem could become ill-conditioned if it was small relatively to the amplitude of the buckling mode. Choosing the i^{th} component of $\mathbf{q}^{(1)}$ of the highest absolute magnitude is to ensure that a dependent equation gets eliminated on one hand and that the problem is numerically well-posed on the other hand.

Having had the dof corresponding to q_i^p constrained, the singularity of $\mathbf{K}_t$ has been eliminated. This is equivalent to solving the following equation

$$\begin{bmatrix} K_{1,1} & \cdots & K_{1,i} & \cdots & K_{1,n} \\ \vdots & \ddots & \vdots & & \vdots \\ K_{i,1} & \cdots & \infty & \cdots & K_{i,n} \\ \vdots & & \vdots & \ddots & \vdots \\ K_{n,1} & \cdots & K_{n,i} & \cdots & K_{n,n} \end{bmatrix} \begin{Bmatrix} q_1^p \\ \vdots \\ q_i^p \\ \vdots \\ q_n^p \end{Bmatrix} = \begin{Bmatrix} f_1 \\ \vdots \\ f_i \\ \vdots \\ f_n \end{Bmatrix} \quad (5.17)$$

where ∞ is infinite but can be represented numerically by the highest positive value the computer or compiler allows, as the higher it is, the closer will q_i^p be to zero and hence more accurately approximating the particular solution.

Solving (5.17) results in a particular solution $\mathbf{q}^p$. It should be pointed out that due to the non-unique nature of the particular solution, constraining q_i^p is not the only way of

obtaining a particular solution. In fact, q_i^p can be set to any value in (5.16) in order to deliver a particular solution. Setting it to zero is undoubtedly the simplest option which is practically effective and mathematically rigorous.

5.3.5 Orthogonality Condition and the Uniqueness of the Solution

Having obtained the complementary solution and a particular solution as presented in the previous subsections, the complete solution as shown in (5.8) can be constructed, which satisfies the governing equation (5.6) identically. However, it does not necessarily meet the requirement of the orthogonality condition (5.1). To impose it, one has

$$\left(\mathbf{q}^{(1)}\right)^T \mathbf{T} \left(s\mathbf{q}^c + \mathbf{q}^p\right) = 0 \quad (5.18)$$

There is only one special value of s that will satisfy the orthogonality condition and it can be determined as follows, given the normalisation condition

$$\left(\mathbf{q}^{(1)}\right)^T \mathbf{T} \mathbf{q}^{(1)} = 1 \quad (5.19)$$

And $\mathbf{q}^c = \mathbf{q}^{(1)}$ so that

$$s = -\left(\mathbf{q}^{(1)}\right)^T \mathbf{T} \mathbf{q}^p \quad (5.20)$$

The secondary perturbation displacement can thus be obtained as

$$\mathbf{q}^{(2)} = -\left(\left(\mathbf{q}^{(1)}\right)^T \mathbf{T} \mathbf{q}^p\right) \mathbf{q}^{(1)} + \mathbf{q}^p \quad (5.21)$$

The uniqueness of s eliminates any arbitrariness from the side of the complementary solution. One might argue that a different value of s will be obtained if a different particular solution has been employed to construct (5.8). It is in fact true. However, as will be proven in Appendix C that the solution as expressed in (5.8) or (5.21) is unique and will not be compromised by the choice of a different particular solution. It

is straightforward to verify that (5.21) satisfies both the governing equation and the orthogonal condition and therefore it gives the secondary perturbation displacement $\mathbf{q}^{(2)}$. It has been obtained in a closed-form which is exact if numerical errors in obtaining $\mathbf{q}^{(1)}$ and $\mathbf{q}^p$ are excluded.

In terms of computational demands, $\mathbf{q}^{(1)}$ has been made available already through the conventional buckling analysis. One has to solve only one more set of simultaneous linear equations as given in (5.17). This is a well-presented problem as the coefficient matrix is symmetric, positive definite and, more importantly, banded in an identical form to the stiffness matrix for the structure concerned. It should also be readily available from the preceding buckling analysis without having to generate it again. Having solved (5.17) for $\mathbf{q}^p$, the construction of $\mathbf{q}^{(2)}$ according to (5.21) is computationally effortless.

Reflecting on the statement made in [53] as quoted in the Introduction, the present chapter has turned it into a practical procedure users can follow. More importantly, the mathematical rigor of the procedure has been fully established in this chapter, which can be extended to embrace broader generality as will be addressed in the next section.

5.4 Buckling of Multiple Modes

When the buckling is of multiple modes, the closed-form solution can be constructed in a similar way whilst taking due account of the existence of these modes. Assume that, corresponding to the buckling load, the eigenvalue is repeated h times, there are h independent eigenvectors, denoted as $\mathbf{q}_{(n)}^{(1)}$ ($n=1,2,\dots,h$), and these eigenvectors as well as the buckling load have all been obtained already. For each of eigenvectors, the row number of the component of the highest absolute magnitude can be identified and denoted as $i_{(n)}$. All eigenvectors can be normalised as

$$\left(\mathbf{q}_{(j)}^{(1)}\right)^T \mathbf{T} \mathbf{q}_{(n)}^{(1)} = \delta_{jn} \quad (j, n = 1, 2, \dots, h) \quad (5.22)$$

where δ_{jn} is the Kronecker delta. The complementary solution can then be constructed from these eigenvectors. There are h dependent equations in (5.6) which can be eliminated one by one by constraining the $i_{(n)}$ -th degree of freedom in the structure. This will recover the positive definiteness of $\mathbf{K}_t$ before (5.6) can be solved for a particular solution $\mathbf{q}^p$. Then the complete solution to the governing equation can be constructed as

$$\mathbf{q}^{(2)} = \sum_{n=1}^h s_n \mathbf{q}_{(n)}^{(1)} + \mathbf{q}^p \quad (5.23)$$

Given (5.22), using orthogonality condition (5.1) for each of the multiple eigenvectors, constants s_n can be determined as

$$s_n = -\left(\mathbf{q}_{(n)}^{(1)}\right)^T \mathbf{T} \mathbf{q}^p \quad (n = 1, 2, \dots, h) \quad (5.24)$$

Substituting the above into (5.23), one obtains a closed-form solution again

$$\mathbf{q}^{(2)} = -\sum_{n=1}^h \left(\left(\mathbf{q}_{(n)}^{(1)}\right)^T \mathbf{T} \mathbf{q}^p \right) \mathbf{q}_{(n)}^{(1)} + \mathbf{q}^p \quad (5.25)$$

The mathematical rigor is as tight as the single mode buckling scenario. Apparently, the single buckling mode case can be considered as a special case of the present multiple mode formulation when $h=1$. The existence and uniqueness can be argued in the same way as that for the single mode buckling case as provided in the previous section.

With an investigating mind, it is not difficult to facilitate another approach for a closed-form solution. If one put (5.6a) and (5.1) together as a set of over-determined simultaneous equations, i.e. that having more equations than unknowns, denoted as

$$\mathbf{A} \mathbf{q}^{(2)} = \mathbf{b} \quad (5.26)$$

where $\mathbf{A}$ is an $(n+m) \times n$ matrix and $\mathbf{b}$ is an $(n+m)$ -dimensional vector. The least square technique can be employed to convert (5.26) into

$$(\mathbf{A}^T \mathbf{A}) \mathbf{q}^{(2)} = \mathbf{A}^T \mathbf{b} \quad (5.27)$$

where $\mathbf{A}^T \mathbf{A}$ is an $n \times n$ matrix and $\mathbf{A}^T \mathbf{b}$ is an n -dimensional vector. (5.27) can be solved to obtain the same results as the approach presented above. However, the disadvantage of this approach is that the coefficient matrix $\mathbf{A}^T \mathbf{A}$ is no longer banded like the stiffness matrices. It will be computationally more demanding and hence will not be pursued any further.

5.5 Numerical Examples

5.5.1 A Verification through a Trivial Manually Solvable

A simple example can be put up as follows with

$$\mathbf{K}_0 = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}, \quad \mathbf{K}_0^\sigma = -\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \mathbf{f} = \begin{Bmatrix} 1 \\ -1 \\ 1 \end{Bmatrix}$$

of which a single eigenvalue and its corresponding normalised eigenvector can be found as

$$\lambda^c = 1 \quad \text{and} \quad \mathbf{q}^{(1)} = \frac{1}{\sqrt{6}} \begin{Bmatrix} 1 \\ 2 \\ 1 \end{Bmatrix}$$

Given the magnitudes of the components of $\mathbf{q}^{(1)}$, one can easily identify that $i=2$.

Apparently, $\mathbf{f}$ satisfies (5.14). The governing equation for $\mathbf{q}^{(2)}$ becomes

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{Bmatrix} q_1^{(2)} \\ q_2^{(2)} \\ q_3^{(2)} \end{Bmatrix} = \begin{Bmatrix} 1 \\ -1 \\ 1 \end{Bmatrix}$$

Solving

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & \infty & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{Bmatrix} q_1^p \\ q_2^p \\ q_3^p \end{Bmatrix} = \begin{Bmatrix} 1 \\ -1 \\ 1 \end{Bmatrix}$$

One can obtain that

$$\mathbf{q}^p = \begin{Bmatrix} 0.5 \\ 0 \\ 0.5 \end{Bmatrix}$$

After determining $\mathbf{s}$ according to (5.20), the complete solution can be obtained from (5.21) as

$$\mathbf{q}^{(2)} = \frac{1}{3} \begin{Bmatrix} 1 \\ -1 \\ 1 \end{Bmatrix}$$

If the obtained result is substituted back into both the governing equation and the orthogonality condition, they are satisfied exactly. The solution has therefore been verified.

In order to draw a comparison with Lanzo's iterative approach [47], results are also obtained from such scheme and those after 6 and 12 iterations are shown as follows, respectively, as opposed to the initial value in order to offer an indication of the rate of convergence.

$$\mathbf{q}_{0\text{-iteration}}^{(2)} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}, \quad \mathbf{q}_{6\text{-iterations}}^{(2)} = \begin{Bmatrix} 0.33325 \\ -0.33325 \\ 0.33325 \end{Bmatrix} \quad \text{and} \quad \mathbf{q}_{12\text{-iterations}}^{(2)} = \begin{Bmatrix} 0.3333331 \\ -0.3333331 \\ 0.3333331 \end{Bmatrix}.$$

Each of the two could offer as an approximation depending on the requirement of accuracy. It is clear that the approximation, no matter how accurate it is, is unnecessary if a closed-form solution can be found, especially if the closed-form solution can be found with less computational efforts and is of wide applicability.

5.5.2 A Manually Solvable Case Involving Multiple Eigenvectors

If one replaces the $\mathbf{K}_0$ matrix as in the previous subsection by the one below

$$\mathbf{K}_0 = \begin{bmatrix} 1 & -2 & 2 \\ -2 & -2 & 4 \\ 2 & 4 & -2 \end{bmatrix}$$

whilst everything else remains the same, one finds a repeated eigenvalue at

$$\lambda_{(1)}^c = \lambda_{(2)}^c = 2$$

The eigenvectors associated with the repeated eigenvalue are

$$\mathbf{q}_{(1)}^{(1)} = \frac{1}{\sqrt{5}} \begin{Bmatrix} 2 \\ -1 \\ 0 \end{Bmatrix} \text{ and } \mathbf{q}_{(2)}^{(1)} = \frac{1}{3\sqrt{5}} \begin{Bmatrix} 2 \\ 4 \\ 5 \end{Bmatrix} \text{ satisfying } \left(\mathbf{q}_{(1)}^{(1)} \right)^T \mathbf{T} \mathbf{q}_{(2)}^{(1)} = 0.$$

Given the magnitudes of the components of the eigenvectors, one can easily identify that the first and the third equations can be eliminated from the governing equation for $\mathbf{q}^{(2)}$

$$\begin{bmatrix} -1 & -2 & 2 \\ -2 & -4 & 4 \\ 2 & 4 & -4 \end{bmatrix} \begin{Bmatrix} q_1^{(2)} \\ q_2^{(2)} \\ q_3^{(2)} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2 \\ -2 \end{Bmatrix} \quad \text{to give} \quad \begin{bmatrix} \infty & -2 & 2 \\ -2 & -4 & 4 \\ 2 & 4 & \infty \end{bmatrix} \begin{Bmatrix} q_1^p \\ q_2^p \\ q_3^p \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2 \\ -2 \end{Bmatrix}$$

leading to

$$\mathbf{q}^p = -\frac{1}{2} \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}.$$

From (5.24), one has $s_1 = -\frac{1}{2\sqrt{5}}$ and $s_2 = \frac{2}{3\sqrt{5}}$. Substituting them in (5.23), one

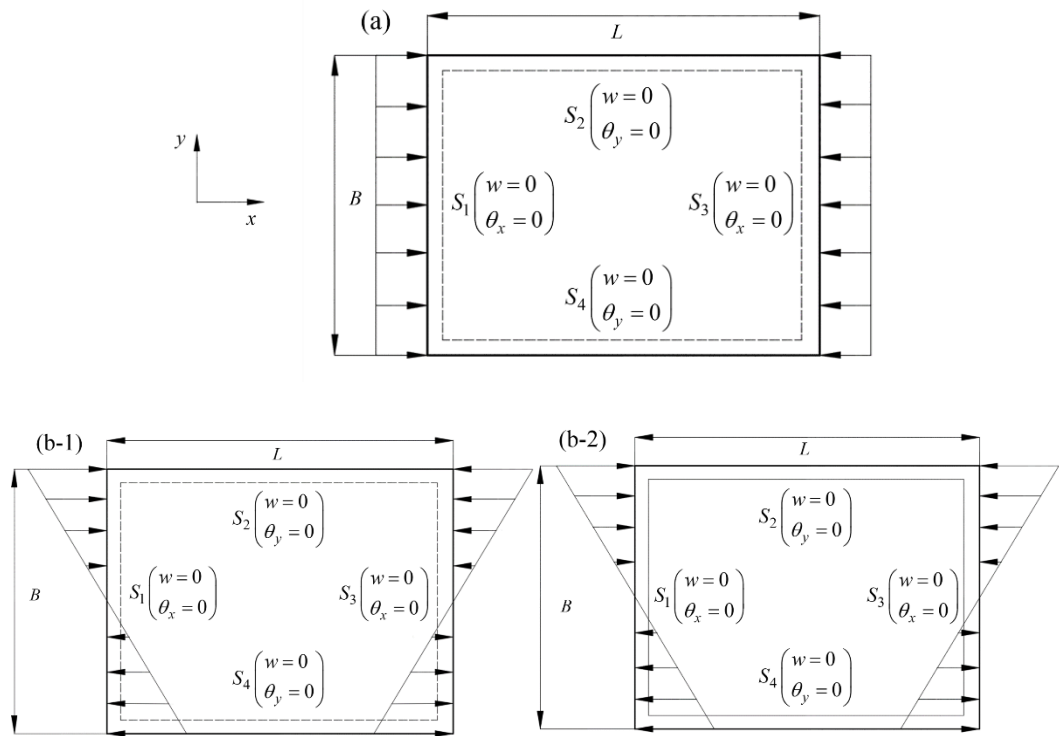
obtains

$$\mathbf{q}^{(2)} = \frac{1}{9} \begin{Bmatrix} -1 \\ -2 \\ 2 \end{Bmatrix}$$

which satisfies both the governing equation and the orthogonality conditions and therefore is a solution and the only solution.

5.5.3 Examples of Structural Applications

From the simple examples shown in the previous subsections, the approach established in this chapter has been nicely verified. To further demonstrate the practicality and reliability of this closed-form approach in structural analysis using FEM, the same set of plates has been resorted to under loading conditions as illustrated in Figure 5.1 as in [47]. The plates are considered to take a range of aspect ratios and are assumed to be simply supported (a), (b-1) and (c-1) or clamped (b-2) and (c-2) with in-plane constraints as specified in Figure 5.1. It is subjected to a number of loading conditions, viz. (a) uniaxial compression, (b) pure in-plane bending and (c) pure shear, as illustrated in Figure 5.1. The Young's modulus of the material of the plate is $E=210GPa$ and its Poisson's ratio $\nu=0.3$. The thickness of the plate is $t=1mm$.



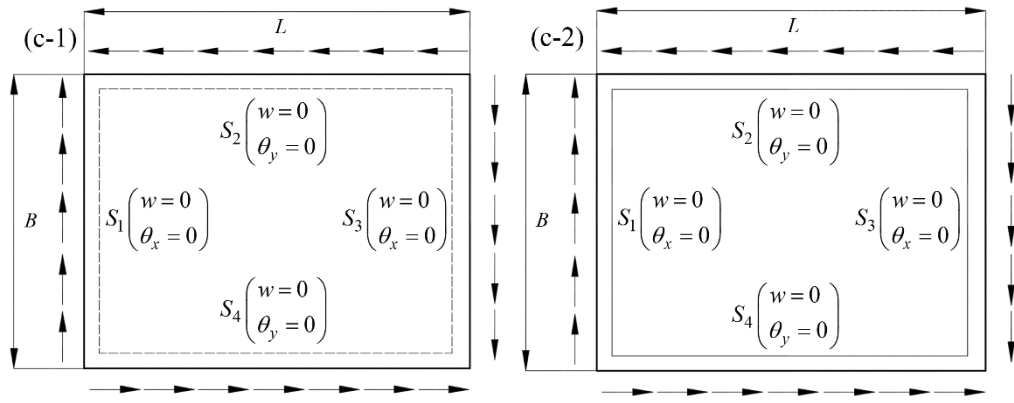


Figure 5.1 Rectangular plate under different loading and boundary conditions

Table 5.1 Numerical results of secondary perturbation mode

Case	Mesh	$\ \mathbf{K}\bar{\mathbf{q}}^{(2)} - \mathbf{f}\ _2 / \ \mathbf{f}\ _2$	$(\mathbf{q}^{(1)})^T \mathbf{T}\bar{\mathbf{q}}^{(2)}$
(a) L/B=3	21×7	9.406×10^{-15}	3.041×10^{-29}
	33×11	1.790×10^{-14}	1.911×10^{-28}
	49×15	3.860×10^{-14}	2.709×10^{-27}
(b-1) L/B=1	9×9	2.564×10^{-15}	-5.103×10^{-28}
	15×15	6.733×10^{-15}	-1.500×10^{-28}
	25×25	1.897×10^{-14}	2.617×10^{-27}
	33×33	3.289×10^{-14}	3.536×10^{-28}
(b-2) L/B=1	9×9	2.903×10^{-15}	1.031×10^{-28}
	15×15	7.433×10^{-15}	6.563×10^{-29}
	25×25	2.304×10^{-14}	1.159×10^{-27}
	33×33	4.373×10^{-14}	1.838×10^{-30}
(c-1) L/B=2	15×7	5.464×10^{-15}	-1.313×10^{-29}
	25×11	1.443×10^{-14}	1.002×10^{-27}
	33×17	2.558×10^{-14}	3.074×10^{-29}
(c-2) L/B=2	15×7	6.786×10^{-15}	-5.251×10^{-29}
	25×11	1.689×10^{-14}	1.633×10^{-28}
	33×17	3.451×10^{-14}	1.933×10^{-28}

To support the solution derived in the present thesis, Table 5.1 shows the numerical errors in the evaluated secondary perturbation mode $\mathbf{q}^{(2)}$. The satisfaction of the governing equation (4.15) and the orthogonality condition (5.1) put the correctness of the approach firmly beyond doubt. Again, mesh sensitivity does not feature this part of the analysis either, as it should not.

5.6 The Problem of Mesh Sensitivity

In the literature, most attempts of evaluating the IPBCs a and b through FEM have employed the expressions as obtained in [21].

$$a = -\frac{3}{2\lambda_c} \frac{\int_{\Omega} \boldsymbol{\sigma}_1^T \nabla_N (\mathbf{q}^{(1)}) \mathbf{q}^{(1)} d\Omega}{\int_{\Omega} (\boldsymbol{\sigma}_0^c)^T \nabla_N (\mathbf{q}^{(1)}) \mathbf{q}^{(1)} d\Omega} \quad (5.28)$$

$$b = -\frac{1}{\lambda^c} \frac{\int_{\Omega} (\boldsymbol{\sigma}_2^T \nabla_N (\mathbf{q}^{(1)}) \mathbf{q}^{(1)} + 2\boldsymbol{\sigma}_1^T \nabla_N (\mathbf{q}^{(1)}) \mathbf{q}^{(2)}) d\Omega}{\int_{\Omega} (\boldsymbol{\sigma}_0^c)^T \nabla_N (\mathbf{q}^{(1)}) \mathbf{q}^{(1)} d\Omega} \quad (5.29)$$

where $\nabla_N (\mathbf{q}^{(1)})$ is defined in the same way as (A-8b) but with $\mathbf{q}$ replaced by $\mathbf{q}^{(1)}$, $\boldsymbol{\sigma}_0^c$ the stress state on the fundamental equilibrium path at the point of buckling, and $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ are the first and second order perturbation stresses defined as follows (to be distinguished from $\boldsymbol{\sigma}_L$ and $\boldsymbol{\sigma}_N$).

$$\boldsymbol{\sigma}_0^c = \mathbf{C} \left(\nabla_L + \frac{1}{2} \nabla_N (\mathbf{q}_c^{(0)}) \right) \mathbf{q}_c^{(0)} \quad (5.30)$$

$$\boldsymbol{\sigma}_1 = \mathbf{C} \left(\nabla_L + \nabla_N (\mathbf{q}_c^{(0)}) \right) \mathbf{q}^{(1)} \quad (5.31)$$

$$\boldsymbol{\sigma}_2 = \mathbf{C} \left(\left(\nabla_L + \nabla_N (\mathbf{q}_c^{(0)}) \right) \mathbf{q}^{(2)} + \frac{1}{2} \nabla_N (\mathbf{q}^{(1)}) \mathbf{q}^{(1)} \right) \quad (5.32)$$

Equations (5.28) and (5.29) are mathematically equivalent to (4.16a) and (4.17a), respectively. However, the equivalence for b was greatly compromised numerically when implemented through FEM, as the two terms in the numerator of (5.29) are of opposite senses whilst having similar magnitude. The sum is often one or more orders of magnitude

smaller, resulting in the so-called ‘locking’ phenomenon [184]. The fact that it has never appeared in analytical solutions [21] suggests that it is a numerical problem associated with accuracy. The underlying reason was in fact the generic shortcoming of FEM in which stresses are obtained inherently at a lower order of accuracy than displacements. However, in the B-notation, stresses are involved in the expressions for a and b . It is inevitable to undermine the numerical accuracy, causing the locking phenomenon as observed and documented in the literature, e.g. [47, 53, 54, 60, 184]. The consequence is that a much more refined mesh would be required for initial post-buckling analysis than that which would usually be sufficient to deliver satisfactory buckling analysis. This would undermine the practical usefulness of an FEM code. To improve the accuracy in order to avoid the mesh sensitivity, specially formulated finite elements have been adopted using unconventional shape functions, e.g. those having higher orders of continuity [47]. Neither enduring the mesh sensitivity nor employing specially formulated elements is an attractive property to FEM code developers. It was undoubtedly one of the major reasons why the theory has never been incorporated in any mainstream FEM code.

Since (4.17a) no longer relies on stresses, one can expect it to offer improved performance. This will be demonstrated in the next section through example applications. It will be shown that it has in fact removed the mesh sensitivity completely without having to use a specially formulated element.

To demonstrate the advantages of the formulation as presented in Chapter 4, it is applied here to the plates presented in [47] as used in previous section. Given the absence of Koiter’s theory from any mainstream commercial FEM code, the present analyses were conducted using an in-house developed FEM code employing the 3D degenerated 8-noded shell element with reduced integration [177, 178]. Prior to the applications shown in this chapter, the code had been extensively verified against various sources, including

analytical solutions wherever available and the results obtained from Abaqus as shown in Figure 5.2. For the present application, the obtained results have been listed in and compared with available results in [47]. As an illustration of the reliability of the in-house code, the results for the buckling loads obtained using Abaqus have also been included, where agreement is seen to be excellent.

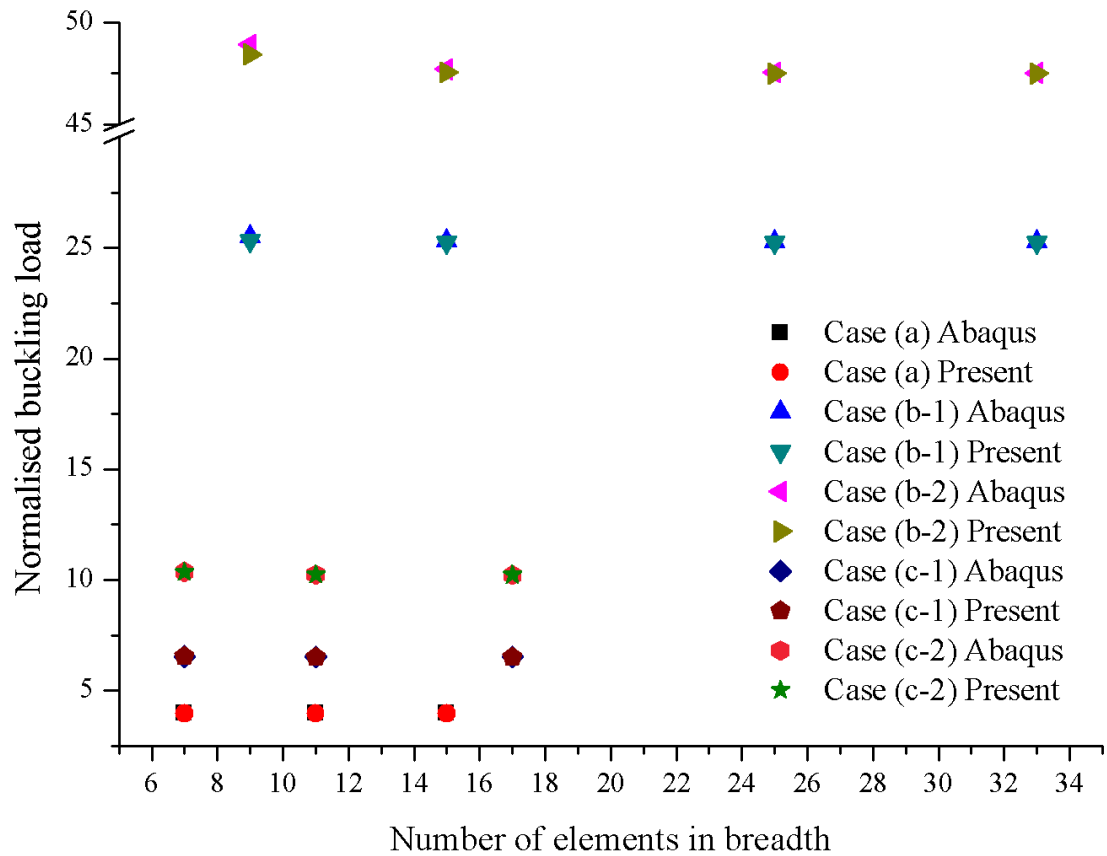


Figure 5.2 Comparisons between Abaqus and in-house code for rectangular plates

Table 5.2 Initial post-buckling coefficients of simply supported rectangular plate

Case	Mesh	Normalised buckling load			IPBC b		Partitions of b such that $b=b_1-b_2$	
		$12(1-\nu^2)B^2\lambda^c/\pi^2Et^3$						
		Ref.[47]	Abaqus	Present	Ref.[47]	Present	b_1	b_2
(a) $L/B=3$	21×7	4.033	3.969	3.968	0.2203	0.2108	1.7716	1.5608
	33×11	4.014	3.974	3.974	0.2213	0.2190	1.9215	1.7025
	49×15	4.007	3.974	3.974	0.2217	0.2149	1.8838	1.6689
(b-1) $L/B=1$	9×9	26.18	25.49	25.31	0.2236	0.2160	2.0664	1.8504
	15×15	25.76	25.32	25.25	0.2252	0.2156	2.0441	1.8285
	25×25	25.61	25.27	25.25	0.2194	0.2097	2.0129	1.8032
	33×33	25.58	25.26	25.25	0.2193	0.2104	2.0257	1.8153
(b-2) $L/B=1$	9×9	53.49	48.90	48.44	0.3400	0.2838	1.3404	1.0566
	15×15	49.74	47.71	47.55	0.2978	0.2926	1.3913	1.0987
	25×25	48.46	47.55	47.50	0.2978	0.2806	1.3352	1.0546
	33×33	48.16	47.53	47.50	0.2924	0.2854	1.3569	1.0715
(c-1) $L/B=2$	15×7	6.777	6.537	6.537	0.07297	0.07305	1.2558	1.1828
	25×11	6.639	6.522	6.523	0.07174	0.07273	1.2635	1.1908
	33×17	6.587	6.521	6.521	0.07164	0.07226	1.2758	1.2036
(c-2) $L/B=2$	15×7	11.41	10.37	10.37	0.1566	0.1357	1.3057	1.1700
	25×11	10.67	10.23	10.23	0.1383	0.1330	1.2982	1.1652
	33×17	10.42	10.22	10.22	0.1340	0.1326	1.2924	1.1598

The IPBCs as given in (4.16a) and (4.17a) have been evaluated. As the problem is for a flat plate, IPBC a should vanish identically. This was also verified in the numerical results obtained through the in-house code, which offered further assurance of the accuracy of the in-house code from another perspective in the post-buckling regime, although there was no need to show this trivial result in Table 5.2.

The predicted IPBC b using the in-house FEM code has been shown in Table 5.2.

As Abaqus does not incorporate Koiter's theory, no results can be obtained from it.

Results published in [47] have been resorted to in order to verify the present predictions where excellent agreement is obvious. Over the range of mesh densities considered, the mesh convergence occurs in a consistent way between the buckling load and IPBC b , i.e. a mesh good enough for buckling analysis is also good enough for initial post-buckling analysis.

The agreements with [47] and Abaqus verify the correctness of the formulation and the implementation. A much more important aspect is that the present analysis employs a conventional type of finite element, viz. 3D degenerated 8-noded shell with reduced integration, available commonly in commercial FEM codes, whilst that in [47] was based on specially formulated finite elements with a higher order of continuity which is unavailable in any mainstream commercial code.

It should be pointed out that an analytical solution for b is available in [35] for a problem corresponding to Figure 5.1 Case (a), given as

$$b = \frac{3}{8}(1 - \nu^2) = 0.34125 \quad (5.33)$$

which is significantly different from what was obtained in [47] as well as the present prediction. It is most disturbing to enquire why the disparity has never been flagged up and explained appropriately. There appeared to be reluctant in the literature, e.g.[53], to refer to Figure 5.1 Case (a). It would be extremely negative for any serious developer of FEM code considering adaption of Koiter's theory, if such the discrepancy was left unaddressed. The author has identified the source of the problem and resolved it. Since it is not simply a numerical error and takes a considerable elaboration on both numerical and analytical sides in order to clarify the position, it will have to be presented in a coming chapter. For one's assurance, the result as presented in Table 5.1 is correct for the problem

as defined in Figure 5.1 (a), which is however subtly different from that solved analytically in [35].

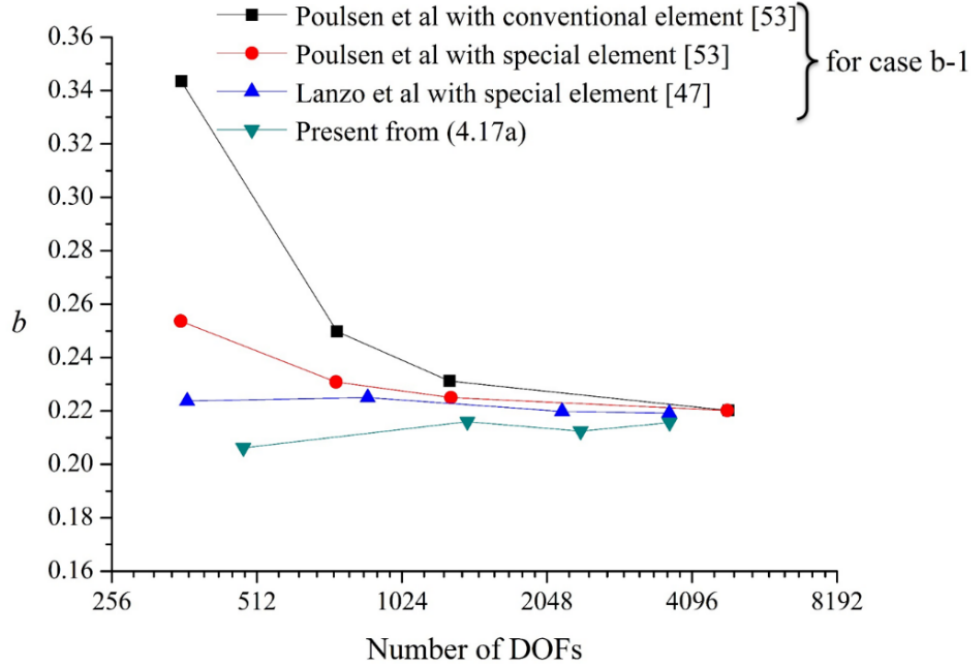


Figure 5.3 Mesh convergence for Case (b-1) as sketched in Figure 5.1

As mesh sensitivity has been one of the key issues regarding the FEM implementation of Koiter's theory, it has been investigated by plotting the predicted b for Case (b-1) shown in Figure 5.1 against the degrees of freedom (dofs) involved in the mesh with the template found in [53], as shown in Figure 5.3. Over the range of the number of dofs as shown, no mesh sensitivity can be observed using (4.17a). It should be noted that the full commutativity and apparent symmetry come without any extra computational effort or cost. If anything, it has made the FEM formulation clearer and more streamlined. In contrast, the curve as was found in [53] with conventional elements using the popular expression (5.29) showed strong mesh sensitivity over the same range of number of dofs. Other results shown in Figure 5.3 including those from [47] show improvements in mesh sensitivity but at the price of employing specially formulated finite elements. In fact, the

present formulation demonstrated better convergence in general with a number of more pronounced cases shown bold in Table 5.2.

The reason for the improvement in the numerical accuracy when using (4.17a) instead of (5.29) is believed to have resulted from the much reduced direct involvement of stresses in the expression of b and therefore any undue numerical errors are avoided. Also included in Table 5.1 are the values of the two terms of b if expression (4.17a) split according to its numerator into two separate terms, b_1 and b_2 , so that $b=b_1-b_2$. It can be observed that b is indeed an order of magnitude smaller than either b_1 or b_2 and hence ‘locking’ is likely [47, 53, 54, 60, 184], especially if the terms involved could not be evaluated sufficiently accurately. However, having reduced the direct involvement of stresses in these expressions, numerical errors due to FEM should have been reduced in comparison with that evaluated using (5.29). The freedom from mesh sensitivity in the present results is therefore by no accident.

Use of (4.17a) brings forward a satisfactory solution and locking does not appear whilst using a conventional type of C^0 element. This has been achieved based on the N-notation which was not possible practically previously. With the fully commutative and apparently symmetric SED, the N-notation is now fully established. This should open the gate for Koiter’s theory to be incorporated in mainstream commercial FEM codes.

5.7 Summary

In this chapter, the major obstacles in the FE implementation of Koiter’s initial post-buckling theory have been revealed explicitly for the first time with a grand aim of convincing commercial FE code developers for them to incorporate the theory in their codes to benefit structural designers and analysts. To tackle one of the obstacles, a closed-form solution has been obtained for the secondary perturbation displacement. The

difficulty associated with the problem is the singularity of the coefficient matrix of the governing equation although it is supplemented with some orthogonality condition(s). Most publications on FE implementation of Koiter's theory avoided recognising this as a problem. Those cared to reveal their way of solving the problem followed some iterative schemes which might not sound very appealing to commercial FE code developers. The present solution is based firmly on mathematical basis and constructed rigorously from a complementary solution and a particular solution as an approach generally applicable to all linear equations. The buckling mode has been identified as a complementary solution. A particular solution can be obtained from the governing equation directly after constraining the degree of freedom corresponding to the component of the maximum absolute value in the buckling mode. The solution can then be uniquely determined after imposing the orthogonality condition. The mathematical procedure has been extended to problems of multiple mode buckling. The solution is not only analytically neat but computationally efficient. The procedure has been demonstrated through simple examples where exact solutions can be obtained manually as a basic verification. To further demonstrate its applicability in the FE implementation of the initial post-buckling theory, it has been applied to a number of structures under various loading and boundary conditions. The accuracy has been demonstrated.

Without using unconventional types of elements, Koiter's theory has been incorporated in the in-house code through the newly-established N-notation. To demonstrate its advancement for solving the problem of mesh sensitivity when calculating the second IPBC, study cases which were employed in open publications have been revisited. Through the comparison of results, the mesh sensitivity has never been encountered anymore. Thus, Koiter's theory is expected to have a more promising future

since the obstacles have been removed completely when implemented in finite element environment.

Appendix C Uniqueness of the Secondary Perturbation Displacement

Denoting (5.16) as

$$\mathbf{K}'\mathbf{q}^p = \mathbf{f} + q_i^p \mathbf{b} \quad (\text{C-1})$$

For an arbitrary value of q_i^p , the solution can be expressed as

$$\mathbf{q}^p = \mathbf{q}_0^p + q_i^p \mathbf{q}_A^p \quad (\text{C-2})$$

where $\mathbf{q}_0^p$ is obtained from (C-1) with $\mathbf{f}$ alone as the right hand side, which is therefore identical to $\mathbf{q}^p$ in (5.18-5.20), and $\mathbf{q}_A^p$ with $\mathbf{b}$ alone as the right hand side. According to (5.8), one has

$$\mathbf{q}^2 = s' \mathbf{q}^c + \mathbf{q}_0^p + q_i^p \mathbf{q}_A^p \quad (\text{C-3})$$

Orthogonality condition (5.1) leads to

$$s' = -(\mathbf{q}^1)^T \mathbf{T}(\mathbf{q}_0^p + q_i^p \mathbf{q}_A^p) = s - q_i^p (\mathbf{q}^1)^T \mathbf{T} \mathbf{q}_A^p \quad (\text{C-4})$$

where s has been given in (5.20). Substitute (C-4) into (C-3), bearing in mind of (5.10), one has

$$\mathbf{q}^2 = s \mathbf{q}^1 - q_i^p \left((\mathbf{q}^1)^T \mathbf{T} \mathbf{q}_A^p \right) \mathbf{q}^1 + \mathbf{q}_0^p + q_i^p \mathbf{q}_A^p \quad (\text{C-5})$$

With the commutativity as established in Chapter three, one has

$$\mathbf{q}^2 = s \mathbf{q}^1 - q_i^p \left((\mathbf{q}^1)^T \mathbf{T} \mathbf{q}^1 \right) \mathbf{q}_A^p + \mathbf{q}_0^p + q_i^p \mathbf{q}_A^p = s \mathbf{q}^1 - q_i^p \mathbf{q}_A^p + \mathbf{q}_0^p + q_i^p \mathbf{q}_A^p = s \mathbf{q}^1 + \mathbf{q}_0^p \quad (\text{C-6})$$

Thus, (5.8) remains the only solution. The above arguments can be extended to multiple modes buckling problem.

Chapter 6 Verifications for the FEM Implemented Koiter's Theory through a Simply Supported Plate under Biaxial Compression

6.1 Introduction

Initial post-buckling theory was established by Koiter in 1940s and it provides a unique insight into the buckling problem by investigating the structural stability in the neighbourhood of the bifurcation point using a perturbation method [15]. However, Koiter's theory has not been incorporated into any mainstream commercial finite element (FE) code, which casts a mysterious veil on the theory. Attempts of its FE implementation can be traced back to as early as 1970s when the pioneers of commercial FE codes were being incubated [43, 84, 183]. Some difficulties in the FE implementation of Koiter's theory were reported associated with the mesh sensitivity due to a locking phenomenon [47, 53, 54, 60, 184]. However, there has been no lack of mesh sensitive problems in engineering, e.g. fracture mechanics. It did not stop users applying FE and the penalty was to bear with the demand on mesh refinements. In fact, as far as the mesh sensitivity in the FE implementation of Koiter's theory was concerned, there had been some approaches which did ease the problem, e.g. by using specially formulated elements [47, 53, 54], although the solutions were far from being ideal. A new initiative has been taken in the previous chapter. The problem underlying the mesh sensitivity has been identified as one of a pure numerical nature since the locking phenomenon has never emerged in analytical solutions. A typical reason for such locking problems in general is the lack of numerical accuracy, such as shear locking [178]. With this in mind when the expression in question was re-examined, it was identified that it involved stresses explicitly. It was well-known that FEM loses a degree of its accuracy when evaluating stresses from displacements. It prompted the author seeking for an alternative expression. The N-

notation of FE formulation was been reactivated and established fully after the introduction of the concept of fully commutative strain energy. When the refreshed N-notation was employed for the FE implementation of Koiter's theory, the mesh sensitivity disappeared completely.

Whilst there has been lack of explicit explanation about the absence of Koiter's theory in commercial FE codes, there is a subtle observation which seemed to have been overlooked, at least in the open literature. There existed a large number of analytical results as a result of efforts of applications of Koiter's theory in the 1970-80s, one of them being a simply supported square plate subjected to biaxial compression [35]. Instead verifying the FE prediction against available analytical solutions, other cases were dealt in [47, 53, 54] where no analytical solutions were available. No one seemed keen to use an analytical solution to verify his/her FE predictions. There was in fact one case as analysed in [47, 53], i.e. a square plate under uniaxial compression, for which the analytical solution in [35] was readily applicable by reducing the result from biaxial to uniaxial compression. The numerical results were simply presented, turning a blind eye to the existing analytical solution. There has been absolutely no record whatsoever in the literatures of a successful comparison with an analytical solution as far as the applications of FE implemented Koiter's theory to plates and shells are concerned. In the meantime, applications of the theory have been made to structures involving a higher level of sophistications, e.g. channel sections [67, 186] and cylindrical shell under axial compression [62], employing composite materials one way or another, with the latter addressing a more complicated aspect of Koiter's theory, the mode interaction.

If one cared to venture a comparison with available analytical result, he/she would soon find him/herself in an embarrassing position. The FE results simply do not agree with the analytical ones, no matter how refined the meshes are. A likely conclusion might

have been drawn in the mind of many that the FE implemented Koiter's theory could not be verified by analytical results. Without appropriate verification, incorporation of the theory into a commercial FE code is simply unthinkable. The author believes that it is the key reason why it has been absent from mainstream commercial FE codes, although it has never been openly acknowledged.

The objective of this chapter is to offer such a verification as a first step in the right and meaningful direction before commercial FE code developers could start contemplating the possibility of incorporating the theory into their codes. The first task is to identify the reason for the disparity which does not seem to be a question ever asked properly by researchers. After a close examination of the problem, it is not difficult to realise that, whilst addressing the seemingly identical problem, the boundary conditions employed show significant differences.

Different boundary conditions employed in the buckling problem between pre-buckling and buckling analyses have been the norm, in particular, in analytical approaches for shell structures. Typically, in order to obtain a membrane stress state for the pre-buckling as has always been assumed in the classical buckling theory, the boundary constraints are appropriately relaxed to allow a membrane state for the pre-buckling deformation. Realistic boundary conditions, e.g. simple support or clamped, are imposed for the phase of buckling analysis. By imposing consistent the boundary conditions for both phases of the analysis, Stein [16, 17] managed to demonstrate noticeable differences which were also experimentally validated. In his analysis, the nonlinearity for the pre-buckling deformation was also taken into account. It was clear that different boundary conditions could make a difference.

In the subsequent sections, differences in boundary conditions will be revealed between second order perturbation and analysis phases preceding it, viz. pre-buckling and

buckling. The effects of inconsistent boundary conditions are assessed. One will find that the mismatch in the boundary conditions can result in disagreement between the FE results and the analytical predictions so significantly that they are no longer representative for each other. However, in order to identify the problem and correct from it, it is essential to have necessary understanding of the approach taken for the analytical solution and the underlying details of FEM in terms of the treatments for essential and natural boundary conditions [181]. The inability of reproducing analytical results from FE implemented Koiter's theory will be shown to be due to the lack of such understanding, which rendered a crucial fact hidden from researchers' intuition hitherto.

6.2 Governing Equations

The research and applications of Koiter's theory took a substantial share of activities in structural mechanics in 1970-80s where analytical approaches dominated the solution techniques. One of the simplest cases analysed then was a simply supported square plate under uniform biaxial compression [35]. For plates and shells, typically, analytical formulations were based on a mixed formulation involving the deflection for the out-of-plane deformation and an Airy's stress function for the in-plane stresses, presented as the out-of-plane equilibrium and in-plane strain compatibility conditions as the governing equations. An asymptotic analysis was then applied following Koiter's original approach. In this chapter, a simply supported rectangular plate under uniformly distributed biaxial compression as sketched in Figure 6.1 will be considered as a minor extension from [35].

The governing equations for the initial post-buckling analysis are all based on von Karman equation for large deflection

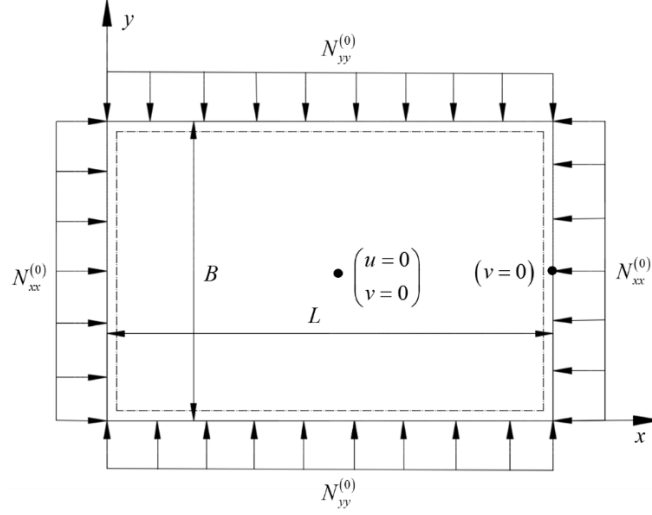


Figure 6.1 Simply supported shell under biaxial compression

$$\nabla^4 F + Et(w_{,xx}w_{,yy} - w_{,xy}w_{,xy}) = 0 \quad (6.1a)$$

$$D\nabla^4 w - (F_{,yy}w_{,xx} + F_{,xx}w_{,yy} - 2F_{,xy}w_{,xy}) = 0 \quad (6.1b)$$

where $D=Et^3/12(1-\nu^2)$ gives the bending rigidity of the plate, E and ν are the Young's modulus and Poisson's ratio of the material, respectively and t is the thickness of plate.

$\nabla^4 = (\nabla^2)^2$ with ∇^2 being the Laplacian operator. w and F are the out-of-plane deflection and Airy stress respectively which can be asymptotically expressed for the initial post-buckling analysis expressed as

$$F = F^{(0)} + \xi F^{(1)} + \xi^2 F^{(2)} + \dots \quad (6.1c)$$

$$w = w^{(0)} + \xi w^{(1)} + \xi^2 w^{(2)} + \dots \quad (6.1d)$$

The substitution of them into (6.1a) and (6.1b) gives rise to the governing equations for various orders of perturbations as follows.

Pre-buckling:

$$\nabla^4 F^{(0)} + Et(w_{,xx}^{(0)}w_{,yy}^{(0)} - w_{,xy}^{(0)}w_{,xy}^{(0)}) = 0 \quad (6.2a)$$

$$D\nabla^4 w^{(0)} - (F_{,yy}^{(0)}w_{,xx}^{(0)} + F_{,xx}^{(0)}w_{,yy}^{(0)} - 2F_{,xy}^{(0)}w_{,xy}^{(0)}) = 0 \quad (6.2b)$$

Buckling:

$$\nabla^4 F^{(1)} + Et \left(w_{,xx}^{(0)} w_{,yy}^{(1)} + w_{,xx}^{(1)} w_{,yy}^{(0)} - 2w_{,xy}^{(0)} w_{,xy}^{(1)} \right) = 0 \quad (6.3a)$$

$$D\nabla^4 w^{(1)} - \left(F_{,yy}^{(0)} w_{,xx}^{(1)} + F_{,xx}^{(0)} w_{,yy}^{(1)} - 2F_{,xy}^{(0)} w_{,xy}^{(1)} \right) - \left(F_{,yy}^{(1)} w_{,xx}^{(0)} + F_{,xx}^{(1)} w_{,yy}^{(0)} - 2F_{,xy}^{(1)} w_{,xy}^{(0)} \right) = 0 \quad (6.3b)$$

Second order perturbation:

$$\nabla^4 F^{(2)} + Et \left(w_{,xx}^{(1)} w_{,yy}^{(1)} - w_{,xy}^{(1)} w_{,xy}^{(1)} \right) + Et \left(w_{,xx}^{(0)} w_{,yy}^{(2)} + w_{,xx}^{(2)} w_{,yy}^{(0)} - 2w_{,xy}^{(0)} w_{,xy}^{(2)} \right) = 0 \quad (6.4a)$$

$$D\nabla^4 w^{(2)} - \left(F_{,yy}^{(0)} w_{,xx}^{(2)} + F_{,xx}^{(0)} w_{,yy}^{(2)} - 2F_{,xy}^{(0)} w_{,xy}^{(2)} \right) - \left(F_{,yy}^{(2)} w_{,xx}^{(0)} + F_{,xx}^{(2)} w_{,yy}^{(0)} - 2F_{,xy}^{(2)} w_{,xy}^{(0)} \right) - \left(F_{,yy}^{(1)} w_{,xx}^{(1)} + F_{,xx}^{(1)} w_{,yy}^{(1)} - 2F_{,xy}^{(1)} w_{,xy}^{(1)} \right) = 0 \quad (6.4b)$$

Analytical solutions are obtained by finding Airy's stress function and deflection for each order of the above governing equations as will be provided in the next section.

6.3 Analytical Solution

In the subsections to follow, the analytical solutions for pre-buckling, buckling and the second order of perturbation will be presented to facilitate the subsequent development.

6.3.1 Pre-buckling

In the analytical solution [35], the pre-buckling deformation is assumed to be a pure membrane state under uniform biaxial stress.

$$N_{xx}^{(0)} = -\lambda k_x, \quad N_{yy}^{(0)} = -\lambda k_y \quad \text{and} \quad k_x N_{xy}^{(0)} = 0 \quad (6.5)$$

$$w^{(0)} = 0 \quad (6.6)$$

where N_{xx} , N_{yy} and N_{xy} are the membrane forces, λ is the load parameter, k_x and k_y the normalised membrane forces with respect to λ , and w the deflection. Special cases of uniaxial compression in x direction can be obtained by setting $k_x=1$ and $k_y=0$. Similarly, other special cases can be generated by taking different combinations of these two parameters.

6.3.2 Buckling

Buckling represents the first order perturbation. The solution is presented below, satisfying the associated governing equations. The Airy stress can be readily given as [35]

$$F^{(1)} = 0 \quad (6.7)$$

and hence the corresponding membrane forces can be obtained accordingly as

$$N_{xx}^{(1)} = F_{,yy}^{(1)} = 0 \quad (6.8a)$$

$$N_{yy}^{(1)} = F_{,xx}^{(1)} = 0 \quad (6.8b)$$

$$N_{xy}^{(1)} = -F_{,xy}^{(1)} = 0 \quad (6.8c)$$

For the buckling mode satisfying the simply supported boundary conditions for any rectangular plate, the first order perturbation of the deflection can be given below [2] as a more general form than that of a square plate [35],

$$w^{(1)} = H \sin \frac{m\pi x}{\eta B} \sin \frac{n\pi y}{B} \quad (6.9)$$

where $\eta=L/B$ is the aspect ratio, t , L and B are the maximum deflection of buckling mode, the length and breadth of the plate, respectively, and m and n the number of half-waves along the x and y directions, respectively. It was proven [187] that between m and n , at least one of them must be 1. As a further step forward, it can be determined [188] which should be 1, given the load ratio, aspect ratio and the stiffness ratio in the case of orthotropic plate. The buckling load can be determined by substituting (6.7) and (6.9) into (6.3b) as

$$\lambda^c = \frac{\pi^2 D}{\eta^2 B^2} \frac{m^4 + 2\eta^2 m^2 n^2 + \eta^4 n^4}{m^2 k_x + n^2 \eta^2 k_y} \quad (6.10a)$$

which reduces to that as given in [35] as follows for a square plate

$$\lambda^c = \frac{4\pi^2 D}{B^2 (k_x + k_y)} \quad (6.10b)$$

6.3.3 The Second Order Perturbation

In an initial post-buckling analysis according to Koiter's theory [15], the second order perturbation is often required. Given (6.5), (6.6) and (6.8), the following satisfies (6.4b)

$$w^{(2)} = 0 \quad (6.11)$$

Given (6.9), the following satisfies (6.4a)

$$F^{(2)} = \frac{EH^2t}{32} \left(\left(\frac{\eta n}{m} \right)^2 \cos \frac{2m\pi x}{\eta B} + \left(\frac{m}{\eta n} \right)^2 \cos \frac{2n\pi y}{B} \right) \quad (6.12)$$

The normalised membrane forces of the second order perturbation can be obtained as

$$N_{xx}^{(2)} = F_{,yy}^{(2)} = -\frac{EH^2t}{8} \left(\frac{m\pi}{\eta B} \right)^2 \cos \frac{2n\pi y}{B} \quad (6.13a)$$

$$N_{yy}^{(2)} = F_{,xx}^{(2)} = -\frac{EH^2t}{8} \left(\frac{n\pi}{B} \right)^2 \cos \frac{2m\pi x}{\eta B} \quad (6.13b)$$

$$N_{xy}^{(2)} = -F_{,xy}^{(2)} = 0 \quad (6.13c)$$

It should be noted that the membrane forces obtained above apply to the plate as the domain where the problem is defined, as well as to the boundary. The distributions of $N_{xx}^{(2)}$ and $N_{yy}^{(2)}$ along the boundary are illustrated in Figure 6.2, which are the conditions the solution obtained above satisfying at the boundary.

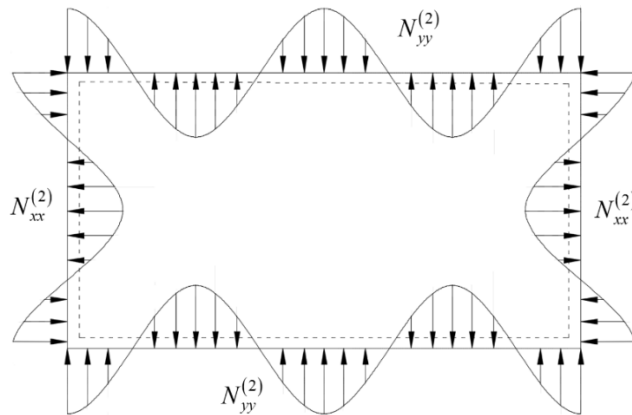


Figure 6.2 Boundary conditions of plates subject to biaxial compression in initial post-buckling regime

6.3.4 The Initial Post-buckling Coefficients

Given the solutions obtained above, the initial post-buckling coefficients (IPBC) a and b can be evaluated from their expressions established in [35] as

$$a = -\frac{3}{2\lambda^c} \frac{\int_A N_{\alpha\beta}^{(1)} w_{,\alpha}^{(1)} w_{,\beta}^{(1)} dA}{\int_A N_{\alpha\beta}^{(0)} w_{,\alpha}^{(1)} w_{,\beta}^{(1)} dA} \quad (\alpha, \beta = x, y) \quad (6.14a)$$

$$b = -\frac{1}{\lambda^c} \frac{\int_A \left(2N_{\alpha\beta}^{(1)} w_{,\alpha}^{(1)} w_{,\beta}^{(2)} + N_{\alpha\beta}^{(2)} w_{,\alpha}^{(1)} w_{,\beta}^{(1)} \right) dA}{\int_A N_{\alpha\beta}^{(0)} w_{,\alpha}^{(1)} w_{,\beta}^{(1)} dA} \quad (\alpha, \beta = x, y) \quad (6.14b)$$

By substituting (6.8) and (6.10a) into (6.14a), and (6.9) and (6.11) into (6.14b), one can obtain the IPBCs explicitly as

$$a = 0 \quad (6.14c)$$

$$b = \frac{3(1-\nu^2)H^2}{4t^2} \frac{m^4 + \eta^4 n^4}{m^4 + \eta^4 n^4 + 2\eta^2 m^2 n^2} \quad (6.14d)$$

which again can reduce to the case in [35] for the square plate by setting $H=t$ that

$$b = \frac{3(1-\nu^2)}{8} \quad (6.14e)$$

For plates, a vanishes identically whilst the value of b is always positive regardless the dimensions of the plate for all realistic materials. According to Koiter's theory [15], the sense of b as obtained signifies that plates under biaxial compression is stable in its initial post-buckling regime and therefore capable of sustaining a higher load than the predicted buckling whilst the equilibrium follows the bifurcated path. The structure is insensitive to the initial imperfections.

6.4 Boundary Condition Employed in Analytical Solution

In obtaining the analytical solutions, such as the ones as presented in the previous section, F and w were usually assumed satisfying apparent boundary conditions, e.g.

simple support conditions, whilst the solution is mostly to determine the coefficients through the governing equations. However, there were other boundary conditions which were implied and did not have to take any particular value to deliver the solution. Such boundary conditions were largely unattended in analytical approaches whilst leaving the solutions to suit themselves. The fact that they were unattended did not mean the absence of them. If the solution is to be reproduced by another means, e.g. FE, these boundary conditions will have to be duly represented. Otherwise, theoretically equivalent solutions cannot be expected. All boundary conditions, apparent and implied, will be presented in this section. This will help to identify an oversight when verifying the FE implemented Koiter's theory against analytical solutions.

In terms of FE applications, one needs to distinguish between essential and natural boundary conditions in the terminology of variational principles on which FEM is based. In FE analysis, a vanishing natural boundary condition can be ignored whilst being strictly correct, since its satisfaction will be met approximately as a result of energy minimisation. Non-vanishing natural boundary conditions are dealt with as loads prescribed at the boundary and their satisfaction will be approximated also by the process of energy minimisation. Essential and natural boundary conditions are treated differently in FE analyses. The former are constraints. They should be imposed at relevant degrees of freedom (dofs) and hence will be satisfied precisely. The latter are dealt with as loads prescribed at the relevant dofs. The satisfaction of equilibrium conditions at these dofs will be approximated on the basis of energy minimisation. It should be pointed out that imposing natural boundary conditions as if they were essential boundary conditions, i.e. requiring the strict satisfaction of relationships amongst the gradients of displacements (or rotations as generalised displacements, in the case of bending) would compromise the minimisation process and hence give a worse off solution in general, as illustrated in [189].

For bending, the simply supported condition is relatively straightforward to interpret. In the analytical solution, a sinusoidal deflection function fulfils all necessary considerations. For FE modelling, a vanishing deflection along the boundary suffices the simple support condition, whilst the zero bending moments are also a part of the condition but they are natural boundary conditions and can be legitimately ignored. Readers are reminded that, since the analytical solution was based on the Love-Kirchhoff hypothesis, the bending aspect allows only two conditions (deflection and bending moment) at any point on the boundary whilst, for the FE analysis which is based on the R-M hypothesis, it requires three instead (deflection, bending moment and twisting moment).

In the analytical solution, the in-plane boundary conditions are not prescribed before a definitive solution had been obtained. They are therefore implied. A blind spot can be easily generated here because, once the solution has been obtained, usually one cannot be bothered about the boundary conditions associated with the solution. As far as the analytical solution is concerned, there does not seem anything wrong with the mentality. A simple observation is that the in-plane boundary conditions implied by the analytical solution have never been made explicit. However, for FEM applications, these boundary conditions have to be correctly reflected before an equivalent solution can be sought for.

For this reason, boundary conditions implied (in-plane) and prescribed (bending) in the analytical solution have been summarised in Table 6.1, which should be prescribed in the FE analyses of the pre-buckling, buckling and post-buckling deformations if a solution equivalent to the analytical one is expected.

Table 6.1 Boundary conditions either prescribed to or implied by the analytical solution for flat plat under biaxial compression

		In-plane				Out-of-plane			
		$x = 0 \text{ \& } L$		$y = 0 \text{ \& } B$		$x = 0 \text{ \& } L$		$y = 0 \text{ \& } B$	
Pre-buckling	Boundary conditions	$N_{xx}^{(0)} = -k_x, N_{xy}^{(0)} = 0$		$N_{yy}^{(0)} = -k_y, N_{xy}^{(0)} = 0$			$M_{xx}^{(0)} = 0$		$M_{yy}^{(0)} = 0$
	Nature	Natural		Natural		$w^{(0)} = 0$	$M_{xy}^{(0)} = 0$	$w^{(0)} = 0$	$M_{xy}^{(0)} = 0$
	Alternative	$u^{(0)} _{x=0 \text{ or } L} = \pm \frac{L}{2Et} (k_x - \nu k_y)$	$N_{xy}^{(0)} = 0$	$v^{(0)} _{y=0 \text{ or } B} = \pm \frac{B}{2Et} (k_y - \nu k_x)$	$N_{xy}^{(0)} = 0$				
	Nature	Essential	Natural	Essential	Natural	Essential	Natural	Essential	Natural
Buckling	Boundary conditions	$N_{xx}^{(1)} = 0, N_{xy}^{(1)} = 0$		$N_{yy}^{(1)} = 0, N_{xy}^{(1)} = 0$			$M_{xx}^{(1)} = 0$		$M_{yy}^{(1)} = 0$
	Nature	Natural		Natural		$w^{(1)} = 0$	$\varphi_x^{(1)} = 0$	$w^{(1)} = 0$	$\varphi_y^{(1)} = 0$
	Alternative	$u^{(1)} = 0$	$N_{xy}^{(1)} = 0$	$v^{(1)} = 0$	$N_{xy}^{(1)} = 0$				
	Nature	Essential	Natural	Essential	Natural	Essential	Natural	Essential	Natural
Post-buckling	Boundary conditions	$N_{xx}^{(2)} = -\frac{EH^2t}{8} \left(\frac{m\pi}{L} \right)^2 \cos \frac{2m\pi y}{B}$		$N_{yy}^{(2)} = -\frac{EH^2t}{8} \left(\frac{n\pi}{B} \right)^2 \cos \frac{2m\pi x}{L}$			$M_{xx}^{(2)} = 0$		$M_{yy}^{(2)} = 0$
		$N_{xy}^{(2)} = 0$		$N_{xy}^{(2)} = 0$		$w^{(2)} = 0$	$\varphi_x^{(2)} = 0$	$w^{(2)} = 0$	$\varphi_y^{(2)} = 0$
	Nature	Natural		Natural					
	Alternative	$u^{(2)} _{x=0 \text{ or } L} = \pm \frac{m^2\pi^2 H^2}{16L}$	$N_{xy}^{(2)} = 0$	$v^{(2)} _{y=0 \text{ or } B} = \pm \frac{n^2\pi^2 H^2}{16B}$	$N_{xy}^{(2)} = 0$				
	Nature	Essential	Natural	Essential	Natural	Essential	Natural	Essential	Natural

In FE analysis, homogeneous natural boundary conditions, e.g. $N_{xy}^{(0)}$, $N_{xx}^{(1)}$, $N_{xy}^{(1)}$ and $N_{xy}^{(2)}$ on $x=0$ and L , should be left unattended. This is achieved by leaving the corresponding displacements free from any constraints. Any non-vanishing right hand side in a natural boundary condition, such as the in-plane forces $N_{xx}^{(2)}$ on $x=0$ and L and $N_{yy}^{(2)}$ on $y=0$ and B for the post-buckling phase, should be prescribed as load applied to the structure along the relevant parts of the boundary. Failing to represent them in FE analysis results in discrepancies between the FE results and analytical solution such that they are no longer equivalent to each other and comparisons between them are not meaningful as will be shown later through examples. If one chooses to employ boundary conditions in the form of in-plane forces in the case of an FE analysis, some additional constraints will be required to eliminate the in-plane rigid body motions.

However, one will find the following alternative, but exactly equivalent, more convenient to apply. This involves turning natural boundary conditions into essential ones. This is not to impose natural boundary conditions as if they were essential boundary conditions, as this is wrong. The procedure suggested here is equivalent to that as follows. (a) Solving the problem first under the natural boundary conditions concerned. Having solved the problem, displacements can be found explicitly. (b) Imposing the found displacements as essential boundary conditions directly instead of the original natural boundary conditions, identical results should be expected.

The actual implementation of this procedure is to keep the part (a) within the regime of analytical solution to have the boundary conditions implied by the analytical solution expressed analytically, before being applied in the FE analysis as part (b).

To elaborate the above stated procedure through the present problem concerned, consider the second order perturbation first. If the in-plane strains are expressed in terms of the membrane forces, before substituting them into the following second order perturbation of the kinematic equation

$$\begin{Bmatrix} \varepsilon_{xx}^{(2)} \\ \varepsilon_{yy}^{(2)} \\ \tau_{xy}^{(2)} \end{Bmatrix} = \begin{Bmatrix} u_{,x}^{(2)} \\ v_{,y}^{(2)} \\ v_{,x}^{(2)} + u_{,y}^{(2)} \end{Bmatrix} + \frac{1}{2} \begin{Bmatrix} w_{,x}^{(1)} w_{,x}^{(1)} \\ w_{,y}^{(1)} w_{,y}^{(1)} \\ 2w_{,x}^{(1)} w_{,y}^{(1)} \end{Bmatrix} + \begin{Bmatrix} w_{,x}^{(0)} w_{,x}^{(2)} \\ w_{,y}^{(0)} w_{,y}^{(2)} \\ w_{,x}^{(0)} w_{,y}^{(2)} + w_{,y}^{(0)} w_{,x}^{(2)} \end{Bmatrix} \quad (6.15)$$

one can obtain the second order perturbation in-plane displacements by integrating the displacement gradients $u_{,x}^{(2)}$ and $v_{,y}^{(2)}$ whilst employing the third equation to determine arbitrary integration functions, as a routine exercise in elasticity [2].

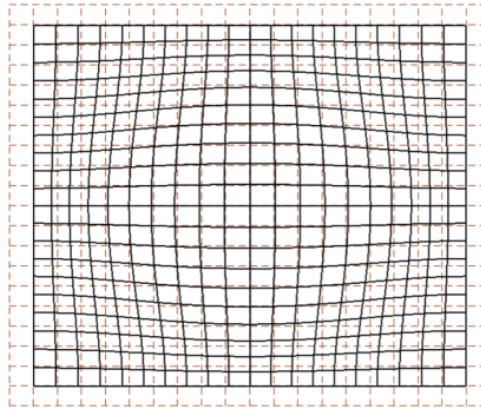
$$u^{(2)} = -\frac{m\pi H^2}{16L} \left(\frac{2m\pi}{L} \left(x - \frac{L}{2} \right) + \frac{(m^2 - \nu\eta^2 n^2)}{m^2} \sin \frac{2m\pi x}{L} - \sin \frac{2m\pi x}{L} \cos \frac{2n\pi y}{B} \right) \quad (6.16a)$$

$$v^{(2)} = -\frac{\pi n H^2}{16B} \left(\frac{2\pi n}{B} \left(y - \frac{B}{2} \right) + \frac{\eta^2 n^2 - \nu m^2}{\eta^2 n^2} \sin \frac{2n\pi y}{B} - \cos \frac{2m\pi x}{L} \sin \frac{2n\pi y}{B} \right) \quad (6.16b)$$

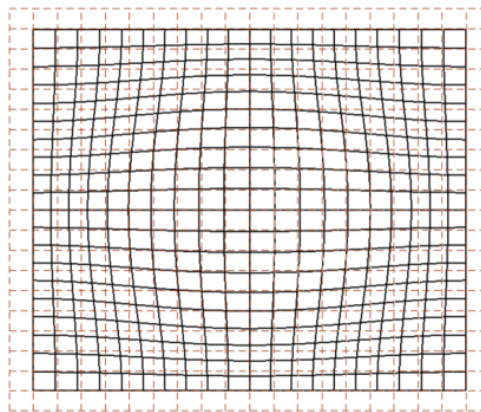
where the rigid body motions have been eliminated by constraining the in-plane translations and rotation at the centre of the plate. As a displacement field for in-plane deformation, it is plotted in Figure 6.3 (a). The displacement normal to the boundary is obtained as

$$u^{(2)} \Big|_{x=0 \text{ or } L} = \pm \frac{m^2 \pi^2 H^2}{16L} \text{ or } v^{(2)} \Big|_{y=0 \text{ or } B} = \pm \frac{n^2 \pi^2 H^2}{16B} \quad (6.17)$$

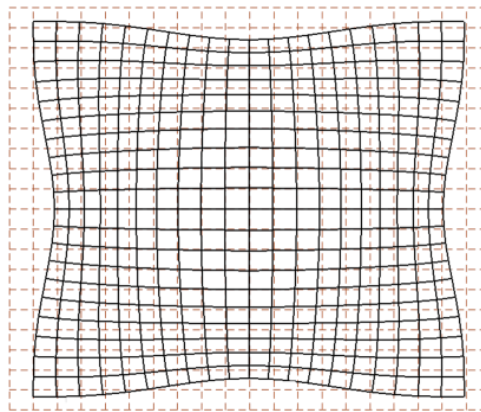
Apparently, it is constant along each side as is also observed in Figure 6.3 (a). In other words, the sinusoidal membrane force boundary conditions as given in Table 6.1 can be equivalently replaced by the displacement boundary conditions (6.17) in FE analysis.



(a)



(b)



(c)

Figure 6.3 Deformed (solid black) versus undeformed (dashed red) meshes for the 2nd order perturbation in-plane displacement field (amplified by 800 times) (a) analytical, (b) FE with consistent boundary conditions and (c) FE with free in-plane boundary conditions

Following the same argument, the membrane force boundary conditions for pre-buckling and buckling as also given in Table 6.1 can also be replaced accordingly by the displacement boundary conditions as

$$u^{(0)}\Big|_{x=0 \text{ or } L} = \pm \frac{L}{2Et} (k_x - \nu k_y) \text{ and } v^{(0)}\Big|_{y=0 \text{ or } B} = \pm \frac{B}{2Et} (k_y - \nu k_x) \quad (6.18)$$

$$u^{(1)}\Big|_{x=0 \text{ or } L} = 0 \text{ and } v^{(1)}\Big|_{y=0 \text{ or } B} = 0 \quad (6.19)$$

The right hand sides involved in (6.17)-(6.19) are all different. The commonality is that the sides of the plate will keep straight after pre-buckling, buckling and second order perturbation deformation, respectively. The elaboration above is to prove that the rows entitled of ‘Boundary conditions’ and the rows ‘Alternative’ in Table 6.1 are equivalent and one can employ either to produce identical results up to the numerical accuracy allowed by the computer.

It should be pointed out that the boundary conditions given in in (6.17)-(6.19), i.e. prescribing the displacement normal to the boundary, are sufficient for FE analyses as far as the in-plane aspect of the problem is concerned. However, for the completeness as a mathematical problem, all of them should be topped up with another boundary condition in the tangential direction which can be conveniently given as the vanishing shear force condition as given in Table 6.1. As it is homogeneous, it can be left unattended in the FE analysis. However, it should not be replaced by vanishing tangential displacement as it would lead to a rather different problem altogether.

For correct implementation of Koiter’s theory, it is crucial to keep the consistency in the displacement boundary conditions between buckling and initial post-buckling analysis, i.e. the dofs constrained must be the same, since the governing equations for these two phases of analysis share the same coefficient matrix (stiffness matrix).

Otherwise, the coefficient matrices would be different after the constraints have been imposed, compromising the requirement generic to Koiter's theory.

It should be noted that the boundary conditions as implied in the analytical solutions are by no means the only possible form of boundary conditions and they are not necessarily the most realistic ones, either. They take their form simply to allow an analytical solution to be obtained. A slight deviation from them, even if they were meant to be more realistic, could prevent the possibility of achieving an analytical solution.

6.5 Numerical Results and Discussions

To facilitate the comparison, the plate subject to uniaxial compression has been given a breadth of 100 mm with a range of lengths from 100 mm to 300 mm so that the buckling modes will have different numbers of half waves in the longitudinal direction. The thickness of plate is assigned to be 1 mm to ensure that the plate is sufficiently thin in order to be in line with the analytical solution. The material is assumed to be homogenous, isotropic and linearly elastic with the Young's modulus of 210 *GPa* and Poisson's ratio 0.3. As demonstrated in Chapter five, no mesh sensitivity is present in the problem and an element size of 5×5mm, i.e. 20 elements in the direction of the breadth, is sufficient to deliver converged results for all cases considered.

The boundary conditions in various regimes aforementioned have been prescribed in consistence with these implied in the analytical analysis in order to reproduce the analytical results. The in-plane boundary conditions and their alternative as provided in Table 6.1 produced almost identical results with errors well below 1%. The numerical results including buckling loads and values of a and b have been provided in Table 6.2 and compared with analytical results in the case of uniaxial compression as a special case

of biaxial compression. For the buckling load, Abaqus [44] predictions are also included as a verification case. However, Abaqus cannot produce any results beyond buckling.

The agreements in the buckling loads can be observed in Table 6.2 amongst all predictions. In particular, the agreement between the results from Abaqus and the in-house code offers a degree of assurance of the correct implementation of FEM in the in-house code as also shown in Figure 6.4. The marginal but consistent underestimation of the analytical results by FEM can be attributed, in presence of unavoidable numerical errors, to the fact that the analytical analysis was based on the Love-Kirchhoff theory whilst FEM was on the R-M theory which is meant to be more compliant than the former due to the allowance for transverse shear deformation.

The values of a have been obtained as numerical zero from FE as opposed to exact zero from the analytical results. The most important outcome is the present predictions of b . Although a close comparison is expected as presented in Table 6.2, such comparison has never been achieved previously. No one has ever attempted such comparison in the open literature. This is therefore the first ever serious verification as far as the FE implementation of Koiter's theory for plates and shells is concerned. With verification like this, commercial FE code developers can at least start to contemplate the possibility of incorporating Koiter's theory into their codes to offer structural designers and analysts a much needed design tool.

As a further justification of the results obtained, the in-plane displacement field in the case of square plate is shown in Figure 6.3 (b) in the same fashion as Figure 6.3 (a) from the analytical solution and also in perfect agreement.

However, if one leaves the boundary free from any in-plane constraints, as was the case in producing the results in the column labelled as 'Free BC' in Table 6.2, the

deformed boundary will not keep straight anymore as shown in Figure 6.3 (c). This reveals the disparity between this treatment of the boundary conditions and those implied by the analytical solution. Apparently, the level of agreement with analytical results is unacceptable as any kind of valid verification. It should be pointed out that, since the mesh has converged, the agreement cannot be improved by mesh refinement. Results as shown in Chapter five were all generated in this way in order to compare with results in [47].

At this stage, a statement needs to be made retrospectively on the results as shown Chapter five which were in good agreement with those published in [47]. However, none of them would serve as valid verification. To avoid any confusion, the second initial post-buckling coefficient in [47] was twice of the value of b as that as defined in the analytical solution and therefore should be halved to give 0.1824 as shown in Table 6.2 in order to make a valid comparison. The fact that the present analysis has reconciled the FE results with the analytical one should offer a further endorsement to the results presented [47], although the results there were hopeless to be considered as a convincing verification case.

To reiterate the point made earlier, the boundary conditions implied by the analytical solution are by no means correct or realistic, they were mere those which allowed an analytical solution to be obtained. Numerical reproduction of the analytical results serves the purpose of verification of the FE formulation and the corresponding coding. Without such verification, it would not be possible to accept the code as a practical design tool. Convincing verification can only be achieved by close comparison. In order to obtain a close comparison, one has to ensure that the problem is represented identically for the analytical solution as well as for the numerical solution. Any mismatch there compromised the comparison and then the verification, without suggesting any of these solutions being necessarily wrong.

It is well-known that the buckling behaviour is sensitive to boundary conditions. It has been shown now that the initial post-buckling behaviour is also so, if not even more so. Just like the necessity of consistent boundary conditions for verification, it is equally essential in practical applications, the prescription of boundary conditions will have to reflect the reality truthfully for the FE predictions to be relevant. Structure designers and analysts should be fully aware of the uncertainties involved in the boundary conditions in the structures they are dealing with in order to produce a valid prediction.

Extending the aspect ratio from 1 for square plate to 2 and 3, different results were obtained in [47] as provided in Table 6.2 whereas those as obtained analytically according to (6.14d) keep constant as a further disparity. With the boundary conditions as implied by the analytical solution employed in the present analysis, the analytical results can be reproduced consistently, which verifies the present FE implementation of Koiter's theory, at least for the test cases as represented by the analytical solution.

As a further demonstration, the results for biaxial compression have been presented in Table 6.3 over a range of values of aspect ratios and load ratios and comparisons of buckling loads against with Abaqus have also presented in Figure 6.5. The number of half-waves were determined analytically according to [188] and also reproduced in the FE results. Agreements in all aspects are obvious. Readers are pointed to a subtle detail. The case of $\eta=0.5$ and $k_x/k_y=0.2$ is the same as that of $\eta=2$ and $k_x/k_y=5$ if one swaps the length L and breadth B and the x and y coordinates. Identical results have been obtained from two separate runs, as a 'sanity check'. The successful FE implementation of Koiter's theory has been convincingly demonstrated through the case of a simply supported plate subjected to biaxial compression.

Table 6.2 Results for uniaxial compression

Aspect ratio	m	Buckling load(N/mm)			<i>a</i>		<i>b</i>			
		Analytical	Abaqus	Present	Analytical	Present	Analytical	Present	Free BC	Reference [47]
1.0	1	75.92	75.58	75.28		1.707×10^{-16}	0.3413	0.3440	0.1779	0.1824
1.2	1	78.47	78.05	77.88		8.526×10^{-17}	0.3523	0.3547	0.1635	N/A
1.4	1	84.84	84.45	84.29		6.619×10^{-16}	0.3771	0.3789	0.1345	N/A
1.6	2	79.76	79.36	79.22		3.505×10^{-16}	0.3577	0.3599	0.2227	N/A
1.8	2	76.77	76.39	76.25		2.560×10^{-18}	0.3450	0.3472	0.2176	N/A
2.0	2	75.92	75.56	75.43	0	2.422×10^{-16}	0.3413	0.3434	0.2058	0.2118
2.2	2	76.61	76.27	76.13		1.001×10^{-15}	0.3443	0.3464	0.1900	N/A
2.4	2	78.47	78.14	78.02		9.339×10^{-16}	0.3523	0.3545	0.1718	N/A
2.6	3	77.49	77.14	77.02		1.822×10^{-16}	0.3481	0.3485	0.2389	N/A
2.8	3	76.28	75.95	75.83		5.281×10^{-16}	0.3429	0.3436	0.2276	N/A
3.0	3	75.92	75.60	75.48		2.407×10^{-16}	0.3413	0.3417	0.2148 ¹	0.2217

¹ This was produced with a 60×20 mesh as opposed to 0.2149 from a 49×15 mesh.

Table 6.3 Results for biaxial compression

Aspect ratio η	k_x/k_y	m	n	Buckling load (N/mm)			a	b		
				Analytical	Abaqus	Present	Analytical	Present	Analytical	Present
0.5	0.2	1	2	12.65	12.59	12.57	0	1.639×10 ⁻¹⁵	0.3413	0.3434
	1	1	1	23.73	23.65	23.63		4.815×10 ⁻¹⁶		0.4652
	5	1	1	5.649	5.632	5.625		6.990×10 ⁻¹⁶	0.4641	0.4654
	10	1	1	2.893	2.885	2.881		1.658×10 ⁻¹⁵		0.4654
1.0	0.2	1	1	12.65	12.60	12.55		6.985×10 ⁻¹⁶		0.3440
	1	1	1	37.96	37.79	37.64		6.257×10 ⁻¹⁶	0.3413	0.3440
	5	1	1	12.65	12.60	12.55		3.401×10 ⁻¹⁶		0.3440
	10	1	1	6.902	6.871	6.843		2.374×10 ⁻¹⁶		0.3440
2.0	0.2	1	1	5.649	5.632	5.625		1.178×10 ⁻¹⁶	0.4641	0.4654
	1	1	1	23.73	23.65	23.62		2.518×10 ⁻¹⁵		0.4652
	5	2	1	12.65	12.59	12.57		1.452×10 ⁻¹⁶	0.3413	0.3434
	10	2	1	6.902	6.869	6.857		8.853×10 ⁻¹⁷		0.3434
3.0	0.2	1	1	4.585	4.577	4.574		2.370×10 ⁻¹⁵	0.5597	0.5600
	1	1	1	21.09	21.05	21.04		1.499×10 ⁻¹⁵		0.5596
	5	2	1	12.29	12.25	12.23		3.446×10 ⁻¹⁶	0.3917	0.3932
	10	3	1	6.902	6.872	6.861		4.938×10 ⁻¹⁶	0.3413	0.3394

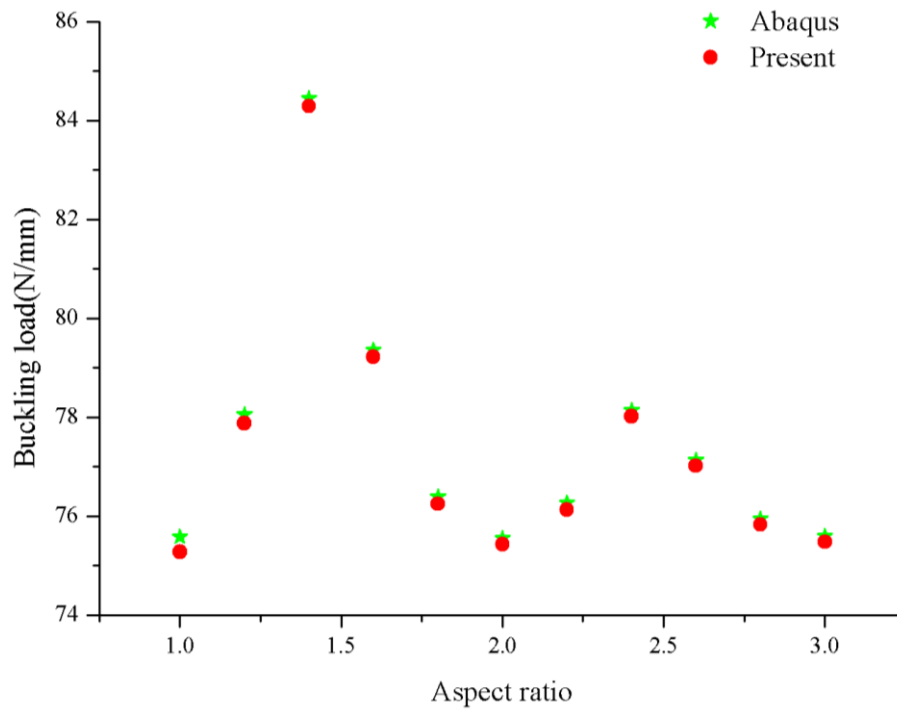


Figure 6.4 Comparisons of buckling loads against Abaqus for uniaxial compression

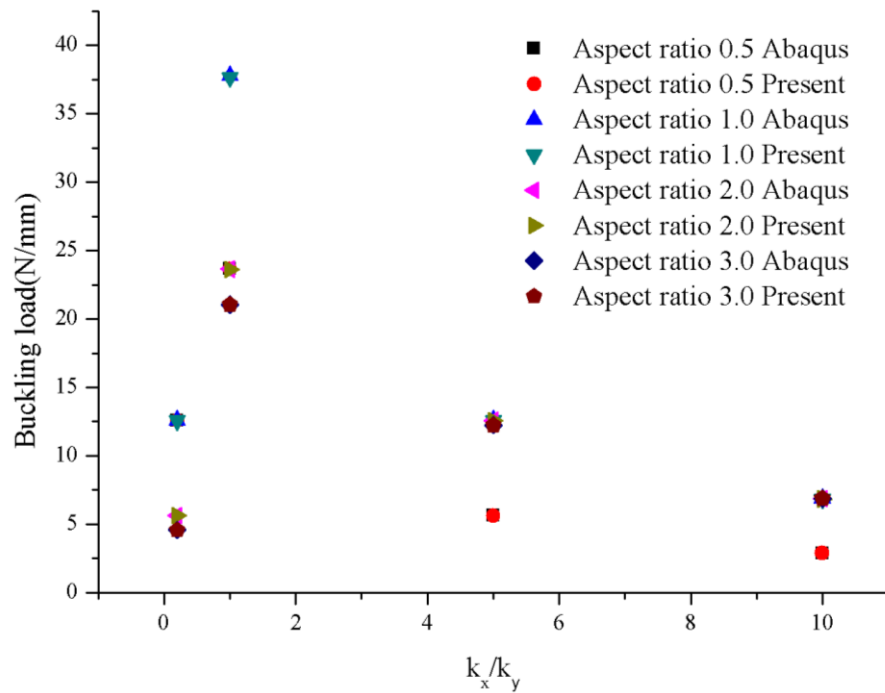


Figure 6.5 Comparisons of buckling loads against Abaqus for biaxial compression

As a final point, the excellent comparisons as shown in Table 6.2 and Table 6.3 can be compromised to different extents when there are other eigenvalues in the

neighbourhood of the lowest one. In these cases, the accuracy of the obtained buckling load and buckling mode can be compromised depending on how close those eigenvalues are. When they become too close to each other, one ought to address the issue as a problem of mode interaction, which will be a subject of a subsequent publication in terms of FE implementation whilst the concept is well-known.

6.6 Summary

The first ever serious verification for the FE implemented Koiter's theory has been satisfactorily made in this chapter, as far as its application to plates are concerned, after nearly half a century long struggle. The failure of reproduction of analytical solutions in the past was due to the lack of equivalent interpretation of the boundary conditions implied in the analytical solutions. It has been shown that, once the correct interpretation of boundary conditions implied by the analytical solution have been employed, FE results in perfect agreement with the analytical ones can be obtained. Any mismatch in such boundary conditions can result in errors significantly enough to distort the FE predictions beyond recognition. This also reveals the sensitivity of post-buckling analysis to the boundary conditions, especially that for the second order perturbation. Other than these, there is nothing wrong with the theory and its potential for industrial applications.

For commercial FE developers, the lack of verification was undoubtedly the most critical obstacle to the acceptance of the theory in its FE implementation. With the outcomes from this chapter, commercial FE code developers should be encouraged to review their position in the incorporation of the theory into a part of the buckling analysis module for the benefit of structure designers and analysts. Assisted with the initial post-buckling analysis, users would have much improved confidence on the buckling load as

predicted by the FE code. More extensive verifications will be pursued as the next phase of development.

Chapter 7 Analytical Solution of Initial Post-buckling Problem of Orthotropic Panels Subject to Axial Compression Based on Reissner-Mindlin Theory as Another Verification for FEM Implementation

7.1 Introduction

The first order transversely shear deformable plate and shell theory was independently introduced by Reissner [190, 191] to account for transverse shear deformation and Mindlin to incorporate rotational inertia [192] into plates, giving rise to what has been widely known as the Reissner-Mindlin (R-M) theory, although claims can be found to distinguish one another [193-195]. It was meant to improve the accuracy over the conventional Love-Kirchhoff (L-M) theory but subsequently find itself as an indispensable gateway for the development of finite elements for plates and shells by relaxing the continuity requirements from C^1 to C^0 [41]. It also allowed plates and shells to be unified seamlessly as far as FEM applications were concerned. Although the applications of such plate and shell element had encountered a major challenge in terms of shear locking when applied to thin plates and shells, the reduced integration technique was found effective to suppress the locking phenomenon [42]. By now, the R-M based shell elements have dominated all commercial FEM codes for plates and shells [44-46]. Whilst this has been a remarkable achievement in terms of the development of FEM, it causes a subtle problem when new developments in FEM seek for verifications by comparing with analytical results. Existing analytical results are mostly derived from the L-K theory. When any disparity is observed, one would often be left wondering whether it was because of the underlying plate and shell theory, i.e. comparing the R-M based

FEM results with L-K based analytical results, or something more substantial. The present chapter was driven by a need based on such a scenario.

A substantial effort has been made to implement Koiter's initial post-buckling theory [15] through FEM in previous chapters. There have been no lack of such attempts, given the fact that Koiter's theory reached its maturity and wide appreciation in the 1970s [18, 23, 29, 35]. However, the achievements so far have been rather bleak, if one is honest, in terms of its implementation through FEM. The obstacles regarding the accurate determination of SPM, mesh sensitivity when calculating b and verification of implemented Koiter's theory through a thin rectangular plate under biaxial compression have been resolved in the previous chapters. Having accomplished these, the author was intrigued to seek for more analytical results for the FEM analysis to compare with in order to establish the FEM implementation of Koiter's theory firmly. This was the motivation behind the present chapter, i.e. to generate an analytical solution for curved panels under uniform uniaxial compression. In fact, a solution had been available in [196] but it was published over 30 years ago in Chinese and hence it is difficult for readers nowadays to source. There was an unsatisfactory aspect of that solution to today's needs as it was based on the L-K theory. It is the objective of this chapter to update the solution onto the R-M framework so that it can be compared with FEM results on a like-to-like basis.

It is the objective of the present project to revive Koiter's theory and make it implementable to FEM which remains as a realm not achieved so far, given publications in this field from as early as 1970s [47, 55, 64, 184]. Given the verification achieved in Chapter 6 for plates, the present chapter is to offer a useful verification for shells. Those who understand Koiter's theory would tell that there was a characteristic difference between plates and shells in terms of initial post-buckling behaviours. Verification against

analytical solutions will be one of the cornerstones which has been unfortunately missing so far and consequently the bleak state-of-the-art of the development. Having conducted a case for plates in Chapter 6 successfully, it will be extended to shells in this chapter to build up the level of confidence on the FEM implementation of Koiter's theory.

7.2 Analytical Solution

A shallow cylindrical shell is sketched in Figure 7.1, with a radius of R , length of L and breadth of B as annotated. It is simply supported and under uniform axial compression. Given the loading condition, a membrane pre-buckling state can be assumed as $N_{xx}^{(0)} = -\lambda^c$ and $N_{yy}^{(0)} = N_{xy}^{(0)} = 0$.

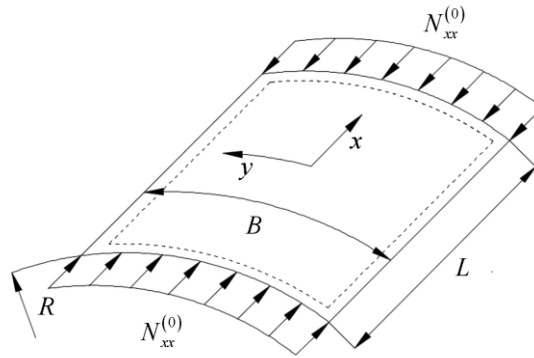


Figure 7.1 Shallow shell under axial compression

7.2.1 Governing Equations

In the framework of L-K theory, the Karman-Donnell equations [88] have been conventionally employed as the governing equations for buckling for cylindrical shells, which consists of two differential equations, one representing the equilibrium condition in the radial direction and the other the compatibility condition for the in-plane strains. An attempt was made in [197] to incorporate the effects of transverse shear whilst maintaining the Karman-Donnell framework. However, the level of sophistication escalated overwhelmingly. For orthotropic shells with cross sections of mid-plane

symmetry, a much simplified version can be obtained which deviated from the L-K counterpart only marginally.

The in-plane strain compatibility condition has been conveniently expressed in terms of the Airy stress F as given in [29].

$$L_{FF} \langle F \rangle + w_{,xx} w_{,yy} - w_{,xy}^2 - \frac{w_{,xx}}{R} = 0 \quad (7.1)$$

where the differential operator L_{FF} is defined by

$$L_{FF} \langle \rangle = a_{22} \langle \rangle_{,xxxx} + (2a_{12} + a_{66}) \langle \rangle_{,xxyy} + a_{11} \langle \rangle_{,yyyy} \quad (7.2)$$

in which a_{ij} are the components of $[a]$ which is the inverse matrix of $[A]$ as illustrated in Appendix D. The out-of-plane equilibrium equations are

$$N_{xx} w_{,xx} + 2N_{xy} w_{,xy} + N_{yy} w_{,yy} + A_{55} (w_{,xx} + \varphi_{x,x}) + A_{44} (w_{,yy} + \varphi_{y,y}) - \frac{N_{yy}}{R} = 0 \quad (7.3a)$$

$$D_{11} \varphi_{x,xx} + (D_{12} + D_{66}) \varphi_{y,xy} + D_{66} \varphi_{x,yy} - A_{55} (w_{,x} + \varphi_x) = 0 \quad (7.3b)$$

$$D_{66} \varphi_{y,xx} + (D_{12} + D_{66}) \varphi_{x,xy} + D_{22} \varphi_{y,yy} - A_{44} (w_{,y} + \varphi_y) = 0 \quad (7.3c)$$

where D_{ij} are the components of $[D]$, the bending stiffness matrix, and A_{44} and A_{55} are the transverse stiffnesses, with symmetric orthotropic laminate being assumed in this chapter. Unlike L-K theory, the out-of-plane equilibrium in the R-M theory is described by three equations which correspond the equilibrium conditions in lateral force and bending moments. To apply analysis according to Koiter's theory, the Airy stress (hence the stress resultants), the deflection and the rotations can be expanded in asymptotic series as

$$N_{\alpha\beta} = N_{\alpha\beta}^{(0)} + \xi N_{\alpha\beta}^{(1)} + \xi^2 N_{\alpha\beta}^{(2)} + \dots \quad (\alpha, \beta = x, y) \quad (7.4a)$$

$$w = w^{(0)} + \xi w^{(1)} + \xi^2 w^{(2)} + \dots \quad (7.4b)$$

$$\varphi_\alpha = \varphi_\alpha^{(0)} + \xi \varphi_\alpha^{(1)} + \xi^2 \varphi_\alpha^{(2)} + \dots \quad (7.4c)$$

where ξ is the perturbation parameter and can be assumed as the magnitude of buckling mode.

For pre-buckling, as in the classic buckling theory, membrane state is assumed.

This gives the following as the solution for the pre-buckling analysis.

$$N_{xx}^{(0)} = -1 \quad \text{and} \quad N_{xy}^{(0)} = N_{yy}^{(0)} = 0 \quad (7.5a)$$

$$w^{(0)} = \text{const.} \quad \text{and} \quad \varphi_x^{(0)} = \varphi_y^{(0)} = 0. \quad (7.5b)$$

7.2.2 First Order Perturbation

By substituting (6.1c) and (6.1d) into (7.1), the first order perturbation of the compatibility equations is obtained as

$$L_{FF} \langle F^{(1)} \rangle - \frac{w_{,xx}^{(1)}}{R} = 0 \quad (7.6)$$

Similarly, substitution of equation (7.4) into (7.3), the first order perturbation of out-of-plane equilibrium equations are obtained as follows.

$$A_{55} \left(w_{,xx}^{(1)} + \varphi_{x,x}^{(1)} \right) - \lambda^c w_{,xx}^{(1)} + A_{44} \left(w_{,yy}^{(1)} + \varphi_{y,y}^{(1)} \right) - \frac{N_{yy}^{(1)}}{R} = 0 \quad (7.7a)$$

$$D_{11} \varphi_{x,xx}^{(1)} + (D_{12} + D_{66}) \varphi_{y,xy}^{(1)} + D_{66} \varphi_{x,yy}^{(1)} - A_{55} \left(w_{,x}^{(1)} + \varphi_x^{(1)} \right) = 0 \quad (7.7b)$$

$$D_{66} \varphi_{y,xx}^{(1)} + (D_{12} + D_{66}) \varphi_{x,xy}^{(1)} + D_{22} \varphi_{y,yy}^{(1)} - A_{44} \left(w_{,y}^{(1)} + \varphi_y^{(1)} \right) = 0 \quad (7.7c)$$

The deflection of buckling mode can be readily assumed as

$$w^{(1)} = H \sin k\beta x \sin \beta y \quad (7.8)$$

where H is the amplitude of the deflection and $\beta = \pi/B$ by assuming that there is only one half wave in the circumferential direction without loss of generality. k is equal to mB/L and m is the number of half waves along the axial direction. Substituting (7.8) into (7.6)

results in an equation for the Airy stress function which can be satisfied by the following solution.

$$F^{(1)} = -\frac{k^2 H}{R\bar{\beta}^2 \bar{A}} \sin k\beta x \sin \beta y \quad (7.9)$$

where

$$\bar{A} = a_{22}k^4 + (2a_{12} + a_{66})k^2 + a_{11} \quad (7.10)$$

Thus, the first order perturbation of the stress resultants can be obtained as

$$N_{xx}^{(1)} = F_{,yy}^{(1)} = \frac{k^2 H}{R\bar{A}} \sin k\beta x \sin \beta y \quad (7.11a)$$

$$N_{yy}^{(1)} = F_{,xx}^{(1)} = \frac{k^4 H}{R\bar{A}} \sin k\beta x \sin \beta y \quad (7.11b)$$

$$N_{xy}^{(1)} = -F_{,xy}^{(1)} = \frac{k^3 H}{R\bar{A}} \cos k\beta x \cos \beta y \quad (7.11c)$$

The two rotations can be obtained from (7.7) and (7.8) in a similar manner as follows which satisfy the simply supported boundary conditions.

$$\varphi_x^{(1)} = H_x^{(1)} \cos k\beta x \sin \beta y \quad (7.12a)$$

$$\varphi_y^{(1)} = H_y^{(1)} \sin k\beta x \cos \beta y \quad (7.12b)$$

where $H_x^{(1)}$ and $H_y^{(1)}$ are the corresponding amplitudes of the rotations.

Substitution of equations (7.8), (7.11b) and (7.12) into (7.7) leads to the following eigenvalue problem

$$\begin{bmatrix} \beta^2 (A_{44} + k^2 A_{55} - \lambda^c k^2) + \frac{k^4}{R^2 \bar{A}} & k\beta A_{55} & \beta A_{44} \\ k\beta A_{55} & \beta^2 (k^2 D_{11} + D_{66}) + A_{55} & k\beta^2 (D_{12} + D_{66}) \\ \beta A_{44} & k\beta^2 (D_{12} + D_{66}) & \beta^2 D_{22} + (k\beta)^2 D_{66} + A_{44} \end{bmatrix} \begin{Bmatrix} H \\ H_x^{(1)} \\ H_y^{(1)} \end{Bmatrix} = \{0\} \quad (7.13)$$

If there exists a non-trivial solution for $\{H \ H_x^{(1)} \ H_y^{(1)}\}^T$, the determinant of coefficient matrix has to be equal to zero so that the buckling load can be obtained as

$$\lambda^c = \frac{(2kA_{44}A_{55}\bar{D}_{12} - k^2A_{55}^2\bar{G}_2 - A_{44}^2\bar{G}_1)}{k^2(\bar{G}_1\bar{G}_2 - \bar{D}_{12}^2)} + \left(A_{55} + \frac{A_{44}}{k^2} + \frac{k^2}{\beta^2 R^2 \bar{A}} \right) \quad (7.14)$$

where

$$\bar{G}_1 = (k\beta)^2 D_{11} + \beta^2 D_{66} + A_{55} \quad (7.15a)$$

$$\bar{G}_2 = \beta^2 D_{22} + (k\beta)^2 D_{66} + A_{44} \quad (7.15b)$$

$$\bar{D}_{12} = k\beta^2 (D_{12} + D_{66}) \quad (7.15c)$$

It is easy to observe the contribution from the transverse shear effects to the buckling load through the involvements of stiffness parameters A_{44} and A_{55} which do not appear in the L-K theory. The L-K theory can be reproduced from the R-M theory by setting transverse stiffnesses infinity.

Using the expressions for the IPBCs as provided in Appendix D, after substituting (7.8) and (7.11) into (D-6a), the first IPBC a can be determined as

$$a = \frac{8}{\lambda^c} \frac{1 - (-1)^m}{m\pi^2} \frac{k^2 t}{RA} \quad (7.16)$$

It can be found that when half waves in the buckling mode in the axial direction is an odd number, a is nonzero which implies an asymmetric bifurcation. The structure is unstable at the vicinity of bifurcation point and sensitive to imperfection. If the half wave number is even, or the radius of the shell becomes infinity, giving a flat panel physically, a vanishes and the initial post-buckling behaviour will depend on IPBC b as is the subject of the next section.

7.2.3 Second Order Perturbation

Secondary perturbation mode is essential for the determination of second IPBC b . The governing equations for the second order perturbation of the Airy stress function, deflection and rotations as follows.

$$L_{FF} \left\langle F^{(2)} \right\rangle + w_{,xx}^{(1)} w_{,yy}^{(1)} - w_{,xy}^{(1)} w_{,xy}^{(1)} - \frac{w_{,xx}^{(2)}}{R} = 0 \quad (7.17)$$

$$N_{xx}^{(1)} w_{,xx}^{(1)} + 2N_{xy}^{(1)} w_{,xy}^{(1)} + N_{yy}^{(1)} w_{,yy}^{(1)} - \lambda^c w_{,xx}^{(2)} - \frac{N_{yy}^{(2)}}{R} + A_{55} \left(w_{,xx}^{(2)} + \phi_{x,x}^{(2)} \right) + A_{44} \left(w_{,yy}^{(2)} + \phi_{y,y}^{(2)} \right) = 0 \quad (7.18a)$$

$$D_{11} \phi_{x,xx}^{(2)} + (D_{12} + D_{66}) \phi_{y,xy}^{(2)} + D_{66} \phi_{x,yy}^{(2)} - A_{55} \left(w_{,x}^{(2)} + \phi_x^{(2)} \right) = 0 \quad (7.18b)$$

$$D_{66} \phi_{y,xx}^{(2)} + (D_{12} + D_{66}) \phi_{x,xy}^{(2)} + D_{22} \phi_{y,yy}^{(2)} - A_{44} \left(w_{,y}^{(2)} + \phi_y^{(2)} \right) = 0 \quad (7.18c)$$

Without losing generality, the second order perturbation of deflection can be assumed as a double trigonometric series following Navier's approach, satisfying the simple support boundary conditions.

$$w^{(2)} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} H \omega_{ij} \sin ik_0 \beta x \sin j \beta y \quad (7.19)$$

Substituting (7.8) and (7.9) into (7.17), the $F^{(2)}$ can be determined from (7.17) as

$$F^{(2)} = \frac{H^2}{32} \left(\frac{1}{a_{22} k^2} \cos 2k \beta x + \frac{k^2}{a_{11}} \cos 2\beta y \right) - \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{H \omega_{ij}}{R j^2 \beta^2 \bar{A}_{ij}} \sin ik_0 \beta x \sin j \beta y \quad (7.20)$$

where

$$\bar{A}_{ij} = a_{22} \left(\frac{ik_0}{j} \right)^2 + (2a_{12} + a_{66}) + a_{11} \left(\frac{j}{ik_0} \right)^2 \quad (7.21)$$

Hence the stress resultants can be resolved correspondingly as

$$N_{xx}^{(2)} = F_{,yy}^{(2)} = -\frac{H^2 k^2 \beta^2}{8a_{11}} \cos 2\beta y + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{H \omega_{ij}}{R \bar{A}_{ij}} \sin ik_0 \beta x \sin j \beta y \quad (7.22a)$$

$$N_{yy}^{(2)} = F_{,xx}^{(2)} = -\frac{H^2 \beta^2}{8a_{22}} \cos 2k \beta x + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{H \omega_{ij}}{R \bar{A}_{ij}} \left(\frac{ik_0}{j} \right)^2 \sin ik_0 \beta x \sin j \beta y \quad (7.22b)$$

$$12N_{xy}^{(2)} = -F_{,xy}^{(2)} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{H \omega_{ij}}{R \bar{A}_{ij}} \frac{ik_0}{j} \cos ik_0 \beta x \cos j \beta y \quad (7.22c)$$

Similarly, the second order perturbation rotations can be assumed as double trigonometric series as well

$$\varphi_x^{(2)} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} H \theta_{xij}^{(2)} \cos ik_0 \beta x \sin j \beta y \quad (7.23a)$$

$$\varphi_y^{(2)} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} H \theta_{yij}^{(2)} \sin ik_0 \beta x \cos j \beta y \quad (7.23b)$$

where $\theta_{xij}^{(2)}$ and $\theta_{yij}^{(2)}$ are the rotation coefficients.

Substituting assumed expressions (7.11) and (7.22) along with (7.8) and (7.23) into (7.18), with the constant term expanded into a cosine Fourier series, the coefficient ω_{ij} can be determined as follows

$$\omega_{ij} = \frac{\frac{16H\beta^2}{\pi^2 R} \left(\left(\frac{k^4}{\bar{A}} + \frac{1}{8a_{22}} \right) \frac{i}{j(i^2 - 4m^2)} + \frac{k^4}{\bar{A}} \frac{j}{i(j^2 - 4)} \right)}{\bar{G}_{ij} - \lambda^c (ik_0 \beta)^2 + \frac{(ik_0)^2}{R^2 j^2 \bar{A}_{ij}}} \quad (i, j = 1, 3, 5, \dots) \quad (7.24)$$

in which

$$\bar{G}_{ij} = \frac{2ijk_0 \beta^2 A_{44} A_{55} \bar{D}_{12ij} - (ik_0 \beta A_{55})^2 \bar{G}_{2ij} - (j\beta A_{44})^2 \bar{G}_{1ij}}{\bar{G}_{1ij} \bar{G}_{2ij} - \bar{D}_{12ij}^2} + (ik_0 \beta)^2 A_{55} + (j\beta)^2 A_{44} \quad (7.25a)$$

$$\bar{G}_{1ij} = D_{11} (ik_0 \beta)^2 + D_{66} (j\beta)^2 + A_{55} \quad (7.25b)$$

$$\bar{G}_{2ij} = D_{22} (j\beta)^2 + D_{66} (ik_0 \beta)^2 + A_{44} \quad (7.25c)$$

$$\bar{D}_{12ij} = (D_{12} + D_{66}) (ijk_0 \beta^2) \quad (7.25d)$$

Having obtained the second order perturbation solution as above, the second IPBC b can be expressed as follows according to (D-6b).

$$b = \frac{H}{\lambda^c} \left(\frac{H\beta^2}{16} \left(\frac{k^2}{a_{11}} + \frac{1}{k^2 a_{22}} \right) + \frac{64}{R\pi^2} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left(\frac{k^2 \omega_{ij}}{\bar{A}} + \frac{\omega_{ij}}{2\bar{A}_{ij} j^2} \right) \frac{2i^2 - i^2 j^2 + 2m^2 j^2}{i(i^2 - 4m^2)} \frac{1}{j(j^2 - 4)} \right) \quad (i, j = 1, 3, 5, \dots) \quad (7.26)$$

When $a=0$, the stability of the initial post-buckling equilibrium path can be predicted by the sense of b . When it is positive, structure is stable at the vicinity of bifurcation point and able to sustain extra load after structure buckles. On the other hand,

if it becomes negative, structure will lose its load-carrying capacity before reaching the bifurcation point and the structure is sensitive to the initial imperfection.

7.3 Special Cases and Discussion

7.3.1 Special Case of Love-Kirchhoff Counterpart

In the previous section, the buckling load λ^c and coefficients ω_{ij} have been derived based on R-M theory. Given the relationship between the R-M theory and the L-K theory, the present outcomes can be specialised from the R-M based to the L-K based by setting the transverse shear stiffnesses to infinity. Thereby, buckling load (7.14) can be reduced as

$$\lambda^c = \frac{\beta^2 \bar{D}}{k^2} + \frac{k^2}{\beta^2 R^2 \bar{A}} \quad (7.27)$$

where

$$\bar{D} = D_{11}k^4 + 2(D_{12} + 2D_{66})k^2 + D_{22} \quad (7.28)$$

Equation (7.27) is exactly the same expression of buckling load derived from L-K theory as given in [196]. Similarly, the deflection of SPM (7.24) is simplified as

$$\omega_{ij} = \frac{\frac{16H\beta^2}{\pi^2 R} \left(\left(\frac{k^4}{\bar{A}} + \frac{1}{8a_{22}} \right) \frac{i}{j(i^2 - 4m^2)} + \frac{k^4}{\bar{A}} \frac{j}{i(j^2 - 4)} \right)}{H\beta^4 \bar{D}_{ij} - \lambda^c (ik_0\beta)^2 + \frac{(ik_0)^2}{R^2 j^2 \bar{A}_{ij}}} \quad (i, j = 1, 3, 5, \dots) \quad (7.29)$$

where

$$\bar{D}_{ij} = D_{11}(ik_0)^4 + 2(D_{12} + 2D_{66})(ijk_0)^2 + D_{22}j^4 \quad (7.30)$$

Again, equation (7.30) is identical to the one obtained in [196] based on L-K theory. The reconciliations of buckling load and deflection coefficient of SPM based on R-M theory have been successfully achieved to the ones based on L-K theory as expected.

By assuming an infinite transverse shear stiffness, the results deduced from R-M theory is able to be perfectly transformed to the ones from L-K theory. In this way, the results obtained herein have been justified through the ones from L-K theory.

7.3.2 Special Form for Flat Plate

The analytical investigations of initial post-buckling behaviours of shell under compression were undertaken in a limited number of papers. However, not only consistently derived from L-K theory, the boundary conditions were not simply supported, for instance the study objects in [18] were the curved panel between two adjacent stiffeners where the panel was not simply supported along the two axial edges. Although dealing with the same problem, the explicit expressions in terms of deflection coefficient ω_{ij} and Airy stress $F^{(2)}$ were not provided in [22]. However, as a specialised example of shell, the IPBCs of square flat panel were unanimously determined and addressed in [35]. As a special case, when radius R becomes infinity, curved panel degenerates to a flat plate.

$$b = \frac{t}{\lambda^c} \frac{t\beta^2}{16} \left(\frac{k^2}{a_{11}} + \frac{1}{k^2 a_{22}} \right) \quad (7.31)$$

This reproduces the result of the simply supported flat plate under uniform uniaxial compression. When the material is homogeneous and isotropic, one has

$$[a] = \frac{1}{Et} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} \text{ and } [D] = \frac{Et^3}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \quad (7.32)$$

where E is Young's modulus and ν Poisson's ratio. The substitutions of equation (7.32) into (7.27), (7.16), (7.31) with $B=L$, $m=1$ and $R \rightarrow \infty$ lead to respectively

$$\lambda^c = \frac{Et^3}{3(1-\nu^2)} \left(\frac{\pi}{L} \right)^2 \quad (7.33a)$$

$$a = 0 \quad (7.33b)$$

$$b = \frac{3}{8}(1 - \nu^2) \quad (7.33c)$$

All the equations in (7.33) are consistent with the counterparts in [35]. Consequently, it has been demonstrated that the outcomes of shell derived in section 7.2 is totally applicable to the flat panel as a specific case.

7.4 Numerical Reproductions of Shallow Panels under Axial Compression

The initial post-buckling analysis of shallow panels under axial compression based on R-M theory has been successfully derived in previous sections. Herein, the numerical reproduction of the analytical solution will be delivered. Similar to the flat panels, the boundary conditions in the analytical solution have to be excavated firstly for a precise correspondence. It is worth noting that a homogenous isotropic material has been considered here although the analytical derivation is applicable to orthotropic materials in general.

7.4.1 Boundary Conditions Employed or Implied in the Analytical Solution

As demonstrated in the previous chapters, the boundary conditions play a significant role in the initial post-buckling behaviour. In order to reproduce an analytical result, it is essential that the boundary conditions in the FE analysis are theoretically identical to those employed or implied in the analytical solution. For the shallow shells under uniaxial compression as a more sophisticated case, it will be more demanding to interpret the boundary conditions as will be elaborated later. For the ease of presentation, a cylindrical coordinate system is adopted to describe the deformation of shell in pre-buckling, buckling and post-buckling regimes with displacements denoted as u in the axial direction, v in the circumferential direction and w in the radial direction. Similar to the flat plate, a state of pure membrane should be assured in the pre-buckling stage. Hence the displacement in the radial direction should be free from constraint to allow free

expansion while no movement is allowed along the circumferential direction at the curved ends as shown in Table 7.1.

Unlike the flat plate, the in-plane shear stress resultant does not vanish anymore in the buckling and post-buckling regime of the curved panel. In the buckling analysis, such natural boundary condition has been implemented in an alternative way, viz. the constraints of tangential displacement on the edges as an essential boundary condition, which is expected to be equivalent to the natural boundary condition, as derived in Appendix (D-8). This has also been verified by the numerical results as presented in Table 7.2.

While in the post-buckling regime, the tangential displacements obtained through kinematic equation in (D-12) have been prescribed with the normal stress resultants as natural boundary conditions. This is apparently equivalent to prescribing the shear stress resultant as a natural boundary condition. This has also been verified by the numerical results as shown in Table 7.5.

Without the appreciation of the presence of tangential displacements along the edges in the post-buckling regime, the precise reproduction of analytical results is impossible. To demonstrate the impact of tangential displacements, one additional set of boundary conditions has been employed as a comparative study. The tangential displacements have been constrained identical to the way as in buckling stage, as denoted as ‘Intuitive BC’ in Table 7.5 without any imposition of forces. No results comparable with the analytical ones could be obtained as opposed to the correct results as also included in Table 7.5.

Table 7.1 Boundary conditions either prescribed to or implied by the analytical solution for curved shell under uniaxial compression

		In-plane				Out-of-plane			
		$x=0\&L$		$y=0\&B$		$x=0\&L$		$y=0\&B$	
Pre-buckling	Boundary conditions	$N_{xx}^{(0)} = -1, N_{xy}^{(0)} = 0$		$N_{yy}^{(0)} = 0, N_{xy}^{(0)} = 0$					
	Nature	Natural		Natural		$M_{xx}^{(0)} = 0$ $M_{xy}^{(0)} = 0$		$M_{yy}^{(0)} = 0$ $M_{xy}^{(0)} = 0$	
	Alternative	$u^{(0)}\Big _{x=0\text{ or }L} = \pm \frac{L}{2Et} k_x, v^{(0)}\Big _{x=0\text{ or }L} = 0$	$N_{xy}^{(0)} = 0$	$v^{(0)}\Big _{y=0\text{ or }B} = 0$	$N_{xy}^{(0)} = 0$				
	Nature	Essential	Natural	Essential	Natural	Natural		Natural	
Buckling	Boundary conditions	$N_{xx}^{(1)} = 0, N_{xy}^{(1)} = \frac{EHtk^3}{R(k^2+1)^2} \cos \beta y$		$N_{yy}^{(1)} = 0, N_{xy}^{(1)} = \frac{EHtk^3}{R(k^2+1)^2} \cos k \beta x$					
	Nature	Natural		Natural		$w^{(1)} = 0$	$M_{xx}^{(1)} = 0$ $\varphi_x^{(1)} = 0$	$w^{(1)} = 0$	$M_{yy}^{(1)} = 0$ $\varphi_y^{(1)} = 0$
	Alternative	$v^{(1)} = 0$	$N_{xx}^{(1)} = 0$	$u^{(1)} = 0$	$N_{yy}^{(1)} = 0$				
	Nature	Essential	Natural	Essential	Natural	Essential	Natural	Essential	Natural

		$N_{xx}^{(2)} = -\frac{EH^2tk^2\beta^2}{8}\cos 2\beta y$ $N_{xy}^{(2)} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{H\omega_{ij}}{R\bar{A}_{ij}}\frac{ik_0}{j}\cos j\beta y$	$N_{yy}^{(2)} = -\frac{EH^2t\beta^2}{8}\cos 2k\beta x$ $N_{xy}^{(2)} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{H\omega_{ij}}{R\bar{A}_{ij}}\frac{ik_0}{j}\cos ik_0\beta x$				
Boundary conditions							
Post-buckling	Nature	Natural	Natural	$w^{(2)} = 0$	$M_{xx}^{(2)} = 0$	$w^{(2)} = 0$	$M_{yy}^{(2)} = 0$
	Alternative (tangential)	$v^{(2)} = -\frac{(H\beta)^2}{8}\left(y - \frac{k^2\nu}{2\beta}\sin 2\beta y - \frac{B}{2}\right)$	$u^{(2)} = -\frac{H^2\beta^2k^2}{8}\left(x - \frac{\nu}{2k^3\beta}\sin 2k\beta x - \frac{L}{2}\right)$		$\phi_x^{(2)} = 0$		$\phi_y^{(2)} = 0$
	Nature	Essential	Essential				
	Alternative (normal)	$N_{xx}^{(2)} = -\frac{EH^2tk^2\beta^2}{8}\cos 2\beta y$	$N_{yy}^{(2)} = -\frac{EH^2t\beta^2}{8}\cos 2k\beta x$				
Nature		Natural	Natural	Essential	Natural	Essential	Natural

7.4.2 Numerical Results and Discussions

7.4.2.1 Buckling and the First IPBCs

Simply supported curved shells under uniaxial compression have been modelled using FEM. The thickness of curved shells is assigned to be 1 mm. The same homogeneous isotropic material as employed in flat plate is considered. Same type of element has been applied with approximately identical mesh density as used previously. The boundary conditions in the pre-buckling and buckling regimes have been prescribed as shown in Table 7.1.

Table 7.2 First IPBCs of shallow shells under axial compression

R/ mm	L/ mm	m	Buckling load (N/mm)					a		
			L-K	R-M	Abaqus	Present	error	L-K	R-M	Present
2000	30	1	844.8	838.5	822.1	816.9	-0.63%	0.05038	0.05075	0.05219
1000			848.3	842.0	825.6	820.4	-0.63%	0.1003	0.1011	0.1039
500			862.7	856.4	839.6	834.5	-0.61%	0.1973	0.1988	0.2043
2000	60	2	844.8	839.5	826.3	823.6	-0.33%			2.766×10^{-13}
1000			848.3	843.1	829.8	827.1	-0.33%	0.000	0.000	1.004×10^{-12}
500			862.7	857.5	843.8	841.3	-0.30%			6.310×10^{-13}
2000	90	3	844.8	838.5	827.7	823.9	-0.46%	0.01679	0.01842	0.01842
1000			848.3	842.0	831.2	827.4	-0.46%	0.03344	0.03611	0.03670
500			862.7	856.4	845.2	841.6	-0.43%	0.06577	0.07098	0.07219

The buckling analysis has been conducted using the in-house FE code and Abaqus separately. The obtained buckling loads are shown in Table 7.2 and Figure 7.2 and compared with the analytical results from both L-K and R-M theories. The buckling mode has been normalised by setting the maximum radial displacement equal to the thickness of curved shell. The dimensions of structures are listed in Table 7.2 along with the

analytical and numerical results. Shallow shells with three different axial lengths have been simulated. The lengths L are chosen for the purpose of producing one to three half waves to include scenarios of a being zero and nonzero. Arc lengths B of curved shells is kept constant for all cases. The analytical results, both buckling load and a , from R-M and L-K theories have been provided for comparison. Identical FE models as used in the in-house code have been created in Abaqus [44] as well for the verification of the buckling loads. Apparently, Abaqus is incapable of providing any initial post-buckling analysis.

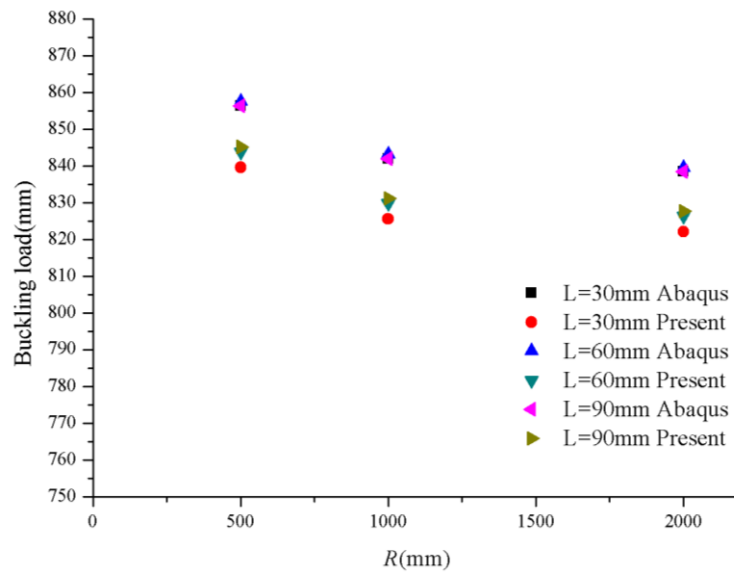


Figure 7.2 Comparisons of buckling loads for homogenous isotropic curved shells

Excellent agreement can be observed with insignificant errors complying with the expected trends. For instance, R-M is expected to show a relatively low stiffness than L-K and hence gives lower buckling loads. The marginal differences in the buckling loads between Abaqus and the in-house code are likely due to the different eigenvalue extraction algorithms and accuracy control. The lower values of the present results in contrast to analytical outcomes especially the ones from R-M plate theory may be attributed to the utilisation of reduced integration algorithm which could lead to compliant mechanical behaviour as opposed to the idealisation as assumed in the analytical solution.

More importantly, the satisfactory agreement of a with the analytical outcomes has been achieved for both non-vanishing and vanishing scenarios. The noticeable difference for the case with $L=90\text{mm}$ and $R=500\text{mm}$ is due to the fact that the 2nd lowest eigenvalue in this case is much closer to the lowest one. In this case, the buckling modes tend to interact so that the precision of buckling modes could be compromised to some extent. Hence, disparity between numerical and analytical results is a bit more pronounced than others.

In addition to the cases of the homogenous isotropic material, a composite case has also been considered to widen the scope of verification. Due to the limitation of analytical derivation in 7.2, only effectively orthotropic materials are applicable to the expressions of the two IPBCs as in (7.16) and (7.26). The material properties have been provided in Table 7.3. The stacking sequence is $[90/0/90/0]_s$ with thickness of each layer being 0.125mm. Same shell element has been employed to discretise the numerical model. Four combinations of dimensions have been attempted to form a study which contains the scenarios of a being vanishing and non-vanishing. The numerical results alongside structural dimension have been presented in Table 7.4.

Table 7.3 Material properties of lamina

$E_{11}(\text{GPa})$	$E_{22}(\text{GPa})$	$G_{12} = G_{13}(\text{GPa})$	$G_{23}(\text{GPa})$	ν_{12}
138	11.0	5.50	3.93	0.278

Table 7.4 Numerical results of composite curved shells

R (mm)	B (mm)	L (mm)	m	Buckling load (N/mm)		a	
				Analytical	Present	Analytical	Present
200	34.9	20	1	159.46	154.62	0.2251	0.2314
200	34.9	50	2	134.81	131.71	0.0000	0.1012×10^{-11}
200	34.9	80	3	131.56	128.73	0.0971	0.1009
200	34.9	110	4	130.57	127.85	0.0000	0.1275×10^{-9}

With the variations of length of curved panels, one can observe different numbers of half wave in the longitudinal direction to generate sufficient and distinctive findings for comparison. It is easy to find out that the buckling loads calculated through self-coded program are satisfactorily consistent with analytical ones. In addition, the values of initial post-buckling coefficients demonstrate satisfactory accuracy. Especially, for the cases with even numbers of half wave, a is negligibly small suggesting a symmetric bifurcation of initial post-buckling behaviour of curved panels.

7.4.2.2 Second IPBCs

In this subsection, attention has been paid to the second IPBCs of shells in the presence of vanishing values of a . Same homogenous isotropic material as used previously has been considered here. The arch length B and axial length L are 30mm and 60mm respectively so that the number of half waves along the axial direction remains as 2 and hence a always vanishes. Various radii have been chosen intentionally to represent shell from being extremely shallow to relatively deeply curved. In such arrangement, values of b could vary from being positive to negative as analytically investigated by Koiter in [18]. Same type of shell element has been chosen to discretise the numerical models. The boundary conditions as illustrated in Table 7.1 have been prescribed. It is worth noting that the boundary conditions in the alternative rows have been opted. For the pre-buckling analysis, both boundary conditions could produce the same results and there is no practical difficulties. However, if the prescribed tangential stress resultant was prescribed, the constraints imposed for the buckling and post-buckling analyses would not be the same anymore. Bearing in mind that the governing equations for buckling and the 2nd order perturbation share the same coefficient matrix on the left hand side. This condition would be violated if essential boundary conditions, i.e. constraints, were prescribed differently for these analyses. In the buckling analysis, prescribing natural

boundary condition is impractical because the actual load level was yet to be determined as the eigenvalue. The practical alternative will have to turn the natural boundary condition for the 2nd order perturbation to an essential one which has been presented in Table 7.1 as the alternative boundary condition for the post-buckling analysis. The set of boundary conditions, ‘Intuitive-BC’, has also been employed herein.

All the numerical results along with analytical outcomes have been collected and tabulated in Table 7.5. It is not surprising that excellent agreement of buckling loads has been realised when compared with the analytical results. Meanwhile, the values of a obtained are consistently negligibly small as a close representation of its vanishing value according to the analytical solution. Most significantly, by prescribing the boundary conditions exactly as implied in the analytical solution, the analytical results for b have been perfectly reproduced. The disparity between L-K and R-M seems most pronounced when the value of b is small, as shown in the case of $R=500\text{mm}$. For this particular case, the present FEM result, being based on the R-M formulation, agrees better with the R-M solution. For other cases, the differences between the two formulations are negligible and FEM predictions agree well with both generally. In addition, the switch of value b from positive to negative with the increase of curvature as predicted in the analytical solution by Koiter in [18] has been reproduced as well in such numerical study. This raises a point that, although higher buckling loads can be achieved at some specific aspect ratios, the increased buckling loads come with decreased stability in the initial post-buckling regime as indicated by the negative value of b and hence the sensitivity of the buckling loads to initial imperfections.

The misinterpretation of the boundary condition employed in the analytical solution, e.g. ‘Intuitive-BC’, could not deliver any reasonable results as shown in Table

7.5. Although such set of boundary conditions produce comparative results when $R=250\text{mm}$, it is more like a coincident than anything consistent.

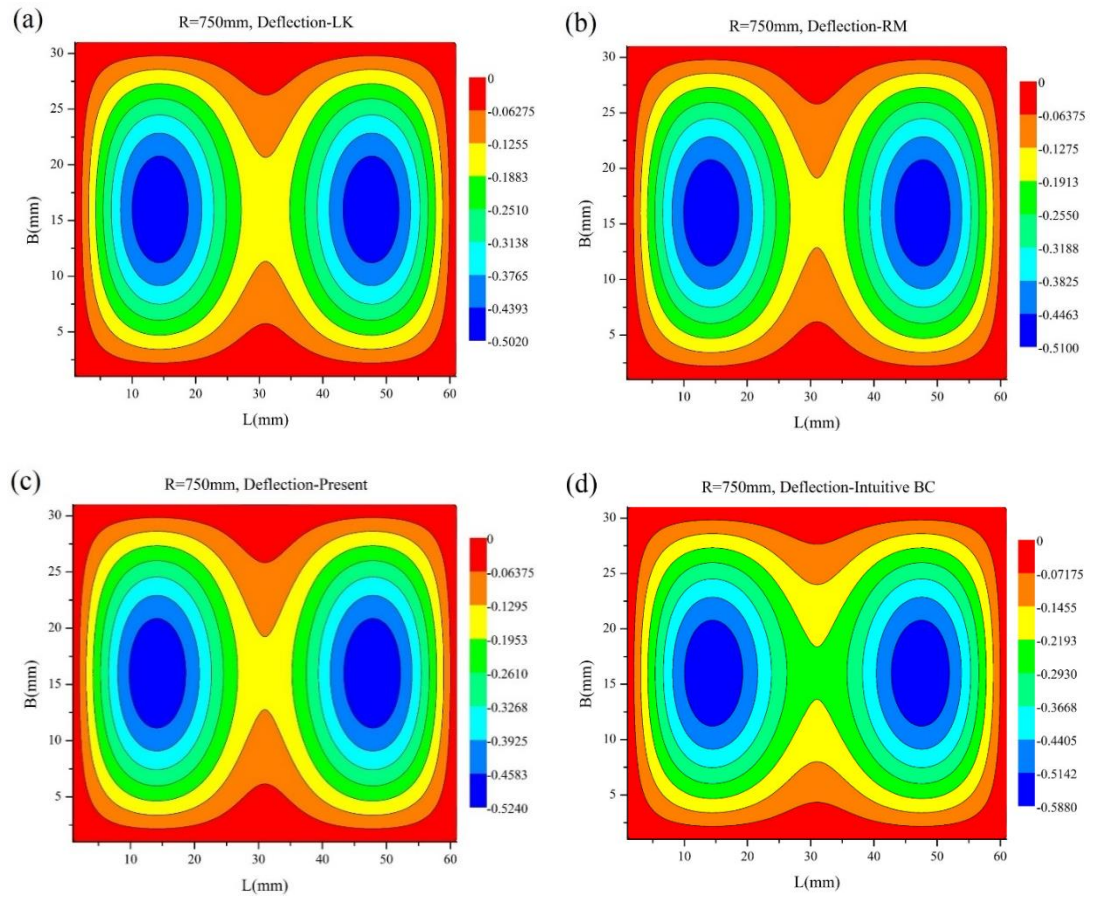


Figure 7.3 Deflection contours of SPM when $R=750\text{mm}$ and thickness= 1mm where (a) and (b) from analytical solution and (c-d) from numerical results

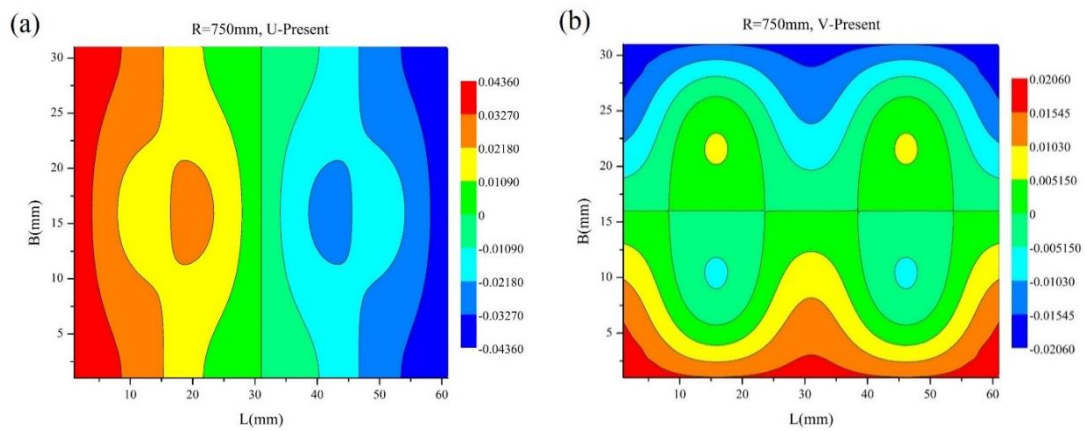


Table 7.5 Curved shells under axial compression with thickness 1mm

R/mm	Buckling load (N/mm)			a		b			
	L-K	R-M	Present	Analytical	Present	L-K	R-M	Present	Intuitive-BC
2000	844.8	839.5	823.6	0	2.766×10^{-13}	0.3245	0.3262	0.3379	0.4923
1000	848.3	843.1	827.1		1.004×10^{-12}	0.2743	0.2749	0.2837	0.4316
750	852.1	846.8	830.8		4.641×10^{-14}	0.2223	0.2218	0.2279	0.3692
500	862.7	857.5	841.3		6.310×10^{-13}	0.1047	0.0706	0.0688	0.1915
250	920.2	914.9	897.8		4.130×10^{-13}	-0.7111	-0.7337	-0.7763	-0.7523
100	1322	1317	1300		2.811×10^{-5}	-6.985	-7.269	-7.543	-8.189

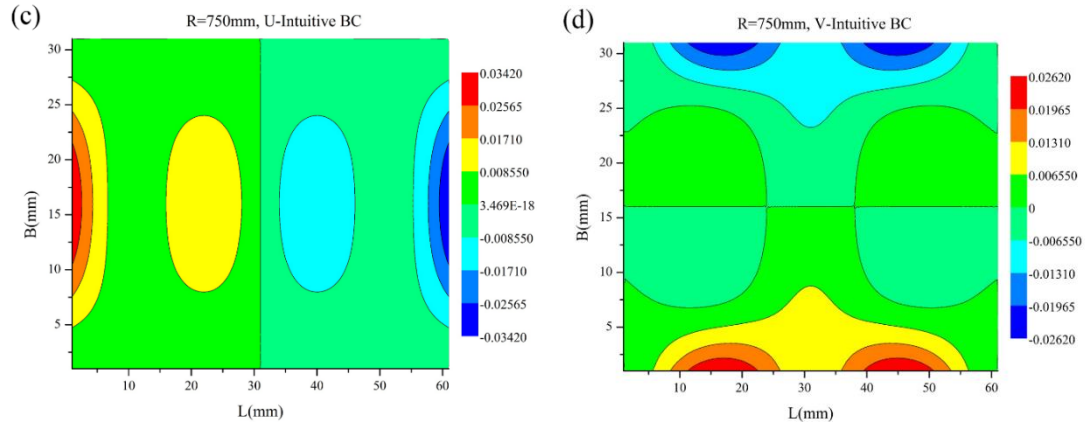


Figure 7.4 In-plane displacement contours of SPM when $R=750\text{mm}$ and thickness= 1mm

The precise reproduction of b in fact relies on the determination of SPM. In another word, the disparity of values of b from boundary conditions ‘Intuitive-BC’ stems from the incorrect SPMs as opposed to the analytical ones. The deflections of SPMs including both analytical ones and numerical results have been depicted in Figure 7.3. It can be found out that although they all feature the similar pattern, the amplitudes of SPMs are different. The L-K theory produces the lowest value -0.5020mm while -0.5100mm for R-M which complies with the common sense that plate of L-K often performs more stiffly than that of R-M. By imposing the consistent boundary condition in present analysis, the maximum amplitude of SPM produced here shows the perfect agreement with the analytical ones in contrast to the one from ‘Intuitive BC’.

In addition to the deflection, the contours of two in-plane displacements of SPMs have also been incorporated for comparison as shown in Figure 7.4. The absence of analytical counterparts is due to the fact that in Donnell’s formulation, the governing equation in the in-plane aspect is the compatibility condition of in-plane strains expressed in terms of Airy’s stress function. For an approximate solution of stresses as obtainable from the Airy’s stress function, one can obtain strains, but not the displacement by integration the strains, in general, because the approximate strains may not be integrable,

as demonstrated in Appendix (D-9). Through the numerical results, one can find out that more pronounced differences could be readily observed as a result of different boundary conditions prescribed. Similar to the flat plates, the edges keep straight in the plane for the SPM from the correct as shown in Figure 7.4 (a) and (b). However, this is not observed by the ‘Intuitive-BC’ cases as shown in Figure 7.4 (c) and (d) as indicated by different colours in these contour plots. Once again, the differences in the boundary conditions employed have been demonstrated graphically.

Table 7.6 Curved shells under axial compression with thickness 3mm

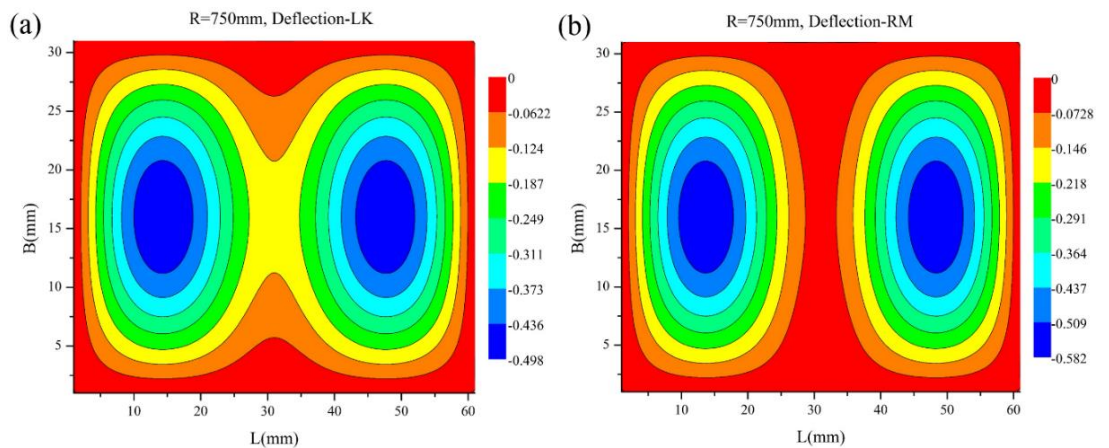
<i>R</i> /mm	Buckling load (N/mm)			<i>a</i>	<i>b</i>			
	L-K	R-M	Present	Analytical	Present	L-K	R-M	Present
2000	22780	21564	20545	0	1.102×10 ⁻¹⁴	0.3394	0.3582	0.3772
1000	22790	21574	20554		2.424×10 ⁻¹⁵	0.3338	0.3513	0.3692
750	22802	21586	20564		2.223×10 ⁻¹⁴	0.3280	0.3441	0.3615
500	22833	21618	20591		3.258×10 ⁻¹⁴	0.3114	0.3237	0.3391
250	23006	21790	20742		4.761×10 ⁻¹⁴	0.2223	0.2133	0.2193
100	24212	22996	21792		1.431×10 ⁻⁵	-0.3931	-0.5548	-0.6003

To further demonstrate the advancement of R-M theory for relatively thick plates and shells in contrast to the L-K theory as discussed in the previous chapter, the thickness has been set to be 3 mm with other dimensions of curved shells remaining unchanged. The consistent boundary conditions from analytical solution once again have been

prescribed in the numerical simulation. It is worth noting that the amplitude H of buckling modes have been assigned to be thickness of curved shells 3 mm.

Similar to the previous case, the L-K theory delivers a higher buckling loads than the R-M theory, and the FE results are closer to the ones from the R-M theory as expected. The values of a consistently remain negligible in line with the analytical solution. Although the tendency of values of b which switch from positive to negative has been maintained as in the cases of thickness=1mm, the increased thickness tend to delay the switch in comparison with Table 7.5. One can also observe that the FE results are in better consistence with the analytical ones based on the R-M theory than the L-K theory because the plate and shell elements in FEM are all formulated based on the R-M theory.

The deflection contours of SPM from L-K theory, R-M theory and present numerical solution have been listed in Figure 7.5 for the SPMs obtained with $R=750\text{mm}$. A noticeable difference between the deflections can be observed viz. the loci of L-K theory are connected to each other while for R-M theory they are completely separated which has been satisfactorily reproduced through the FE analysis as in Figure 7.5 (c). In addition, a better agreement of amplitude has been achieved between FE and R-M results, -0.602mm and -0.582 mm respectively as opposed to -0.498mm for L-K theory.



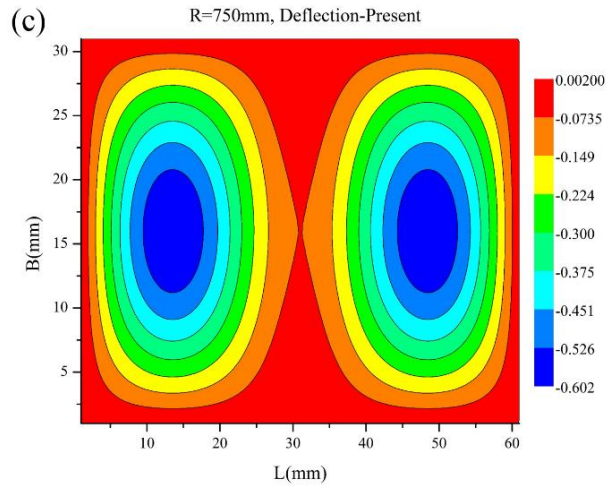


Figure 7.5 Deflection contours of SPM when $R=750\text{mm}$ and thickness= 3mm

7.5 Summary

In this chapter, initial post-buckling behaviours of simply supported curved panel has been completely re-derived based on Reissner-Mindlin theory. The governing equations of buckling and SPM including the effects of transverse shear have been generated along with the complete determinations of two IPBCs a and b , the expressions of which coincide identically with those as based on the Love-Kirchhoff theory though their numerical values may vary.

The derived analytical solution has been applied to a simply supported shallow shell under axial compression as a case study to obtain its initial post-buckling behaviour. With an explicit expression of buckling mode, the buckling load have been obtained analytically. Beyond that, the SPM including the deflection and two rotations has been obtained. The results have been justified against the ones from L-K theory as a special case when transverse shear stiffness becomes infinite.

The boundary conditions of various regimes in the analytical solution have been excavated to facilitate the FE analysis. For precise representation of the problem as analysed analytically, essential boundary conditions of in-plane displacements have been

obtained from the analytical solution where they were prescribed in the form of natural boundary conditions. The results from the analytical solutions of thin and moderately thick panels have been separately reproduced using FEM after adopting the consistent boundary conditions. It can be concluded that for thin panels, the L-K and R-M theories produces close results while for moderately thick panels, R-M theory will be more appropriate. The analytical solution based on R-M exhibits its prominence to verify the FE implemented initial post-buckling theory. Having verified the FE implemented Koiter's theory on flat plates in Chapter 6 and also on an example of shells here, it is time for mainstream commercial FE code developers to contemplate the incorporation of the theory in their codes to benefit structural designers.

Appendix D Basic Notations on R-M theory and Expressions of Two In-plane displacements

The total potential energy Π of shell structure subjected to conservative load $\mathbf{f}$ is

$$\Pi = \frac{1}{2} \int_{\Omega} \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon} d\Omega - \int_{\Omega} \mathbf{f} \cdot \mathbf{u} d\Omega \quad (\text{D-1})$$

where Ω is the integration domain over the whole structure and $\mathbf{u}$ a kinematically admissible displacement field. The strain-displacement relationship of R-M theory is defined as

$$\boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} = \begin{Bmatrix} u_{,x} \\ v_{,y} + w/R \\ v_{,x} + u_{,y} \\ w_{,x} + \varphi_x \\ w_{,y} + \varphi_y \end{Bmatrix} + z \begin{Bmatrix} \varphi_{x,x} \\ \varphi_{y,y} \\ \varphi_{y,x} + \varphi_{x,y} \\ 0 \\ 0 \end{Bmatrix} + \frac{1}{2} \begin{Bmatrix} w_{,x} w_{,x} \\ w_{,y} w_{,y} \\ w_{,x} w_{,y} + w_{,y} w_{,x} \\ 0 \\ 0 \end{Bmatrix} \quad (\text{D-2})$$

where u and v are the in-plane displacements, w the deflection and φ_x and φ_y angles of rotation of the normal to the reference surface towards the x and y -axes, respectively. They are also called generalised displacement for plates and shells. The stress-strain relationship is

$$\boldsymbol{\sigma} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{12} & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & Q_{66} & 0 & 0 \\ 0 & 0 & 0 & Q_{55} & 0 \\ 0 & 0 & 0 & 0 & Q_{44} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} \quad (\text{D-3})$$

where $[Q]$ is the stiffness matrix of the material of the shell, allowing material orthotropy.

Perturbation is applied to the generalised displacement in the same way as (7.4).

$$\mathbf{u} = \begin{Bmatrix} u \\ v \\ w \\ \varphi_x \\ \varphi_y \end{Bmatrix} = \begin{Bmatrix} u^{(0)} \\ v^{(0)} \\ w^{(0)} \\ \varphi_x^{(0)} \\ \varphi_y^{(0)} \end{Bmatrix} + \xi \begin{Bmatrix} u^{(1)} \\ v^{(1)} \\ w^{(1)} \\ \varphi_x^{(1)} \\ \varphi_y^{(1)} \end{Bmatrix} + \xi^2 \begin{Bmatrix} u^{(2)} \\ v^{(2)} \\ w^{(2)} \\ \varphi_x^{(2)} \\ \varphi_y^{(2)} \end{Bmatrix} + \dots \quad (\text{D-4})$$

After the substitution of (D-4) into (D-2) and (D-3) and integration of stresses within each layer along the thickness direction, the stress resultants in different orders of perturbation can be obtained as

$$\begin{Bmatrix} N_{xx}^{(r)} \\ N_{yy}^{(r)} \\ N_{xy}^{(r)} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(r)} \\ \varepsilon_{yy}^{(r)} \\ \gamma_{xy}^{(r)} \end{Bmatrix} \quad (\text{D-5a})$$

$$\begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} = \begin{Bmatrix} u_{,x}^{(0)} \\ v_{,y}^{(0)} + w^{(0)}/R \\ v_{,x}^{(0)} + u_{,y}^{(0)} \end{Bmatrix} + \frac{1}{2} \begin{Bmatrix} w_{,x}^{(0)} w_{,x}^{(0)} \\ w_{,y}^{(0)} w_{,y}^{(0)} \\ w_{,x}^{(0)} w_{,y}^{(0)} + w_{,y}^{(0)} w_{,x}^{(0)} \end{Bmatrix} \quad (\text{D-5b})$$

$$\begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix} = \begin{Bmatrix} u_{,x}^{(1)} \\ v_{,y}^{(1)} + w^{(1)}/R \\ v_{,x}^{(1)} + u_{,y}^{(1)} \end{Bmatrix} + \begin{Bmatrix} w_{,x}^{(0)} w_{,x}^{(1)} \\ w_{,y}^{(0)} w_{,y}^{(1)} \\ w_{,x}^{(0)} w_{,y}^{(1)} + w_{,y}^{(0)} w_{,x}^{(1)} \end{Bmatrix} \quad (\text{D-5c})$$

$$\begin{Bmatrix} \varepsilon_{xx}^{(2)} \\ \varepsilon_{yy}^{(2)} \\ \gamma_{xy}^{(2)} \end{Bmatrix} = \begin{Bmatrix} u_{,x}^{(2)} \\ v_{,y}^{(2)} + w^{(2)}/R \\ v_{,x}^{(2)} + u_{,y}^{(2)} \end{Bmatrix} + \frac{1}{2} \begin{Bmatrix} w_{,x}^{(1)} w_{,x}^{(1)} \\ w_{,y}^{(1)} w_{,y}^{(1)} \\ w_{,x}^{(1)} w_{,y}^{(1)} + w_{,y}^{(1)} w_{,x}^{(1)} \end{Bmatrix} + \begin{Bmatrix} w_{,x}^{(0)} w_{,x}^{(2)} \\ w_{,y}^{(0)} w_{,y}^{(2)} \\ w_{,x}^{(0)} w_{,y}^{(2)} + w_{,y}^{(0)} w_{,x}^{(2)} \end{Bmatrix} \quad (\text{D-5d})$$

The two IPBCs can be given as

$$a = -\frac{3}{2\lambda^c} \frac{\int_S N_{\alpha\beta}^{(1)} w_{,\alpha}^{(1)} w_{,\beta}^{(1)} dS}{\int_S N_{\alpha\beta}^{(0)} w_{,\alpha}^{(1)} w_{,\beta}^{(1)} dS} \quad (\alpha, \beta = x, y) \quad (\text{D-6a})$$

$$b = -\frac{1}{\lambda^c} \frac{\int_S \left(2N_{\alpha\beta}^{(1)} w_{,\alpha}^{(1)} w_{,\beta}^{(2)} + N_{\alpha\beta}^{(2)} w_{,\alpha}^{(1)} w_{,\beta}^{(1)} \right) dS}{\int_S N_{\alpha\beta}^{(0)} w_{,\alpha}^{(1)} w_{,\beta}^{(1)} dS} \quad (\alpha, \beta = x, y) \quad (D-6b)$$

where repeated subscripts implies summation and S is the surface area of shell. The expressions of two IPBCs here obtained from analysis based on the R-M theory look identical to the ones on the L-K theory [128]. However, the buckling load, buckling mode and secondary perturbation mode can be different from their L-K counterparts and, hence, the two IPBCs based on R-M theory are expected to be of different values from their L-K counterparts.

Through the substitutions of deflections in various regimes, *viz.* (7.5b), (7.8), into (D-5c) with the aid of (D-5a) where the stress resultants have been provided in (7.11), the two in-plane displacement $u^{(1)}$ and $v^{(1)}$ can be obtained as

$$u^{(1)} = \frac{Hk(\nu k^2 - 1)}{\beta R(k^2 + 1)^2} \cos k\beta x \sin \beta y \quad (D-7a)$$

$$v^{(1)} = \frac{1}{\beta} \frac{H}{R} \left(\frac{(\nu + 2)k^2 + 1}{(k^2 + 1)^2} \right) \sin k\beta x \cos \beta y \quad (D-7b)$$

It can be seen that in the buckling regime, the in-plane displacements along the edges are vanishing in the tangential direction. Namely,

$$u^{(1)} \Big|_{y=0 \& B} = 0 \quad (D-8a)$$

$$v^{(1)} \Big|_{x=0 \& L} = 0 \quad (D-8b)$$

By following the same procedure, the two in-plane displacements $u^{(2)}$ and $v^{(2)}$ can be obtained as

$$u^{(2)} = -\frac{H}{R} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{ik_0\beta} \left[\frac{1}{\left(\frac{ik_0}{j} + \frac{j}{ik_0}\right)^2} - \frac{\nu}{\left(\frac{j}{ik_0}\right)^2 \left(\frac{ik_0}{j} + \frac{j}{ik_0}\right)^2} \right] \varpi_{ij} \cos ik_0\beta x \sin j\beta y$$
(D-9a)

$$-\frac{1}{8}(H\beta)^2 \left(k^2 x + \frac{(k^2 - \nu)}{2k\beta} \sin 2k\beta x - \frac{k}{2\beta} \sin 2k\beta x \cos 2\beta y \right) + f(y) + \text{const}$$

$$v^{(2)} = -\frac{H}{R} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{j\beta} \left[\frac{1}{\left(\frac{j}{ik_0}\right)^2 \left(\frac{ik_0}{j} + \frac{j}{ik_0}\right)^2} - 1 - \frac{\nu}{\left(\frac{ik_0}{j} + \frac{j}{ik_0}\right)^2} \right] \varpi_{ij} \sin ik_0\beta x \cos j\beta y$$
(D-9b)

$$-\frac{(H\beta)^2}{8} \left(y - \frac{(k^2\nu - 1)}{2\beta} \sin 2\beta y - \frac{1}{2\beta} \cos 2k\beta x \sin 2\beta y \right) + g(x) + \text{const}$$

where $f(y)$ and $g(x)$ are two undetermined functions and they satisfy that

$$\frac{\partial g(x)}{\partial x} + \frac{\partial f(y)}{\partial y} = -\frac{(1+\nu)t}{R} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{\varpi_{ij} \cos ik_0\beta x \cos j\beta y}{\left(\frac{j}{ik_0}\right) \left(\frac{ik_0}{j} + \frac{j}{ik_0}\right)^2}$$
(D-10)

Hence, the tangential displacements along the corresponding edges are

$$u^{(2)} \Big|_{y=0 \& B} = -\frac{1}{8}(H\beta)^2 \left(k^2 x - \frac{\nu}{2k\beta} \sin 2k\beta x \right) + \text{const}$$
(D-11a)

$$v^{(2)} \Big|_{x=0 \& L} = -\frac{(H\beta)^2}{8} \left(y - \frac{k^2\nu}{2\beta} \sin 2\beta y \right) + \text{const}$$
(D-11b)

The constraint of rigid body movement could help determine the constants involved so that $u^{(2)}$ and $v^{(2)}$ can be eventually specified as

$$u^{(2)} \Big|_{y=0 \& B} = -\frac{H^2\beta^2 k^2}{8} \left(x - \frac{\nu}{2k^3\beta} \sin 2k\beta x - \frac{L}{2} \right)$$
(D-12a)

$$v^{(2)} \Big|_{x=0 \& L} = -\frac{(H\beta)^2}{8} \left(y - \frac{k^2\nu}{2\beta} \sin 2\beta y - \frac{B}{2} \right)$$
(D-12b)

It should be pointed out that these tangential displacements if imposed into the numerical models are of same effect as the shears as demonstrated in Table 7.2, Table 7.5 and Table 7.6.

Chapter 8 Initial Post-buckling of Mode-interacted Cylinder under Axial Compression

8.1 Introduction

The work in previous chapter is based on single-mode buckling where Koiter's theory has demonstrated its advancement in the prediction of initial post-buckling behaviour compared to the iterative incremental method. Most significantly, the implemented theory has been successfully validated through the perfect reproduction of simply supported plate under biaxial compression and curved shell under axial compression. While in fact, Koiter's theory should draw more attraction when dealing with the problem involving mode interaction. Bearing in mind the computational convergence problem when using iterative incremental method to estimate post-buckling behaviour of a single-mode system, the computational cost would be vast as far as the mode interaction takes part. While in Koiter's theory, the several simultaneous modes can be easily coped with at once and hence the initial post-buckling behaviour in the presence of these buckling modes can be readily predicted.

In this chapter, the problem with multiple modes buckling, such as simply supported cylinder subject to axial compression, will be revisited. Firstly, the reproduction of buckling of that has been conducted. With the aid of the attained buckling modes, the initial post-buckling of cylinder has been studies afterword.

8.2 Multi-mode Buckling of Cylinders under Axial Compression

The analytical solution of simply supported cylinder under axial compress as shown in Figure 8.1 was firstly performed by Lorenz, Timoshenko and Southwell [87] whose works were based on Love's general theory of thin shell, viz. small displacement

theory by neglecting the all the quadratic terms and the ones higher in the energy expression. However, the predicted buckling load could not be obtained in the experimental observations after a large number of cylinders were tested. The pronounced discrepancy between the theoretical and experimental results was explained by Donnell which was due to the initial geometrical imperfection, namely the deviation from the shape of cylinder. To imitate such imperfection in analytical solution, the large displacement theory was proposed with consideration of the nonlinear terms in the energy expression. This theory was predominately employed in the study of cylindrical shells. In Koiter's thesis, this theory was adopted as well to formulate the total potential energy of cylinders subject to axial compression.

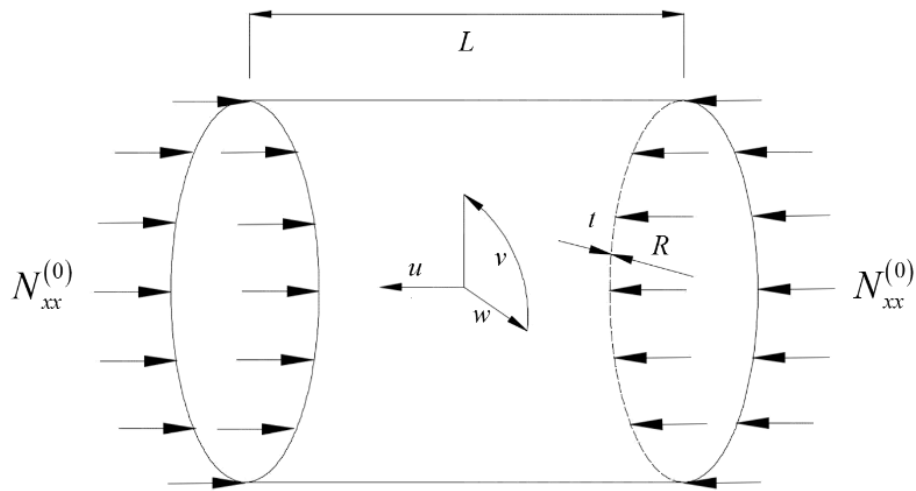


Figure 8.1 A cylinder under axial compression

8.2.1 Analytical Expressions from Koiter

The pre-buckling deformation there was assumed to be a pure membrane state. Thus, the displacement expressions in fundamental path can be assumed to be

$$\begin{aligned} u^{(0)} &= \text{const.} \\ v^{(0)} &= 0 \\ w^{(0)} &= \text{const.} \end{aligned} \tag{8.1}$$

The governing equation of buckling was expressed as

$$\left\{ \begin{array}{l} 2u_{,xx}^{(1)} + (1-\nu)u_{,yy}^{(1)} + (1+\nu)v_{,xy}^{(1)} + 2\nu w_{,x}^{(1)} - (1-\nu^2)\lambda u_{,yy}^{(1)} = 0 \\ 2(1+\nu)u_{,xy}^{(1)} + 2(1-\nu)v_{,xx}^{(1)} + 4v_{,yy}^{(1)} + 4w_{,y}^{(1)} \\ + J \left[2(1-\nu)v_{,xx}^{(1)} + v_{,yy}^{(1)} - (2-\nu)w_{,xxy}^{(1)} - w_{,yyy}^{(1)} \right] - 2(1-\nu^2)\lambda v_{,xx}^{(1)} = 0 \\ 4\nu u_{,x}^{(1)} + 4v_{,y}^{(1)} + 4w + J \left[-(2-\nu)v_{,xxy}^{(1)} - v_{,yyy}^{(1)} + w_{,xxxx}^{(1)} + 2w_{,xxyy}^{(1)} + w_{,yyyy}^{(1)} \right] + 4(1-\nu^2)\lambda w_{,xx}^{(1)} = 0 \end{array} \right. \quad (8.2)$$

where u and v are the two in-plane displacements in longitudinal and circumferential direction, respectively. w is the radial displacement. ν is the Poisson's ratio of homogenous isotropic material. J and the loading parameter λ are defined as

$$J = \frac{h^2}{3R^2} \quad (8.3)$$

$$\lambda = \frac{N_{xx}^{(0)}}{EH} \quad (8.4)$$

where H is the thickness of cylinder, R the radius of cylinder. E is the Young's modulus of material. $N_{xx}^{(0)}$ is the axial loading of per unit length of the circumference imposed at the two ends. The simply supported boundary conditions at the two ends were

$$\left. \begin{array}{l} u_{,x}^{(1)} \\ v^{(1)} \\ w^{(1)} \end{array} \right|_{x=0 \& l/R} = 0 \quad (8.5)$$

Two types of buckling modes were assumed while satisfying the simply supported boundary conditions, viz. non-axisymmetric and axisymmetric, as follows

$$\begin{aligned} u_{non}^{(1)} &= U_{non}^{(1)} \cos px \sin ny \\ v_{non}^{(1)} &= V_{non}^{(1)} \sin px \cos ny \end{aligned} \quad (8.6)$$

$$\begin{aligned} w_{non}^{(1)} &= W_{non}^{(1)} \sin px \sin ny \\ u_{axis}^{(1)} &= U_{axis}^{(1)} \cos px \end{aligned} \quad (8.7)$$

$$v_{axis}^{(1)} = V_{axis}^{(1)} \sin px$$

$$w_{axis}^{(1)} = W_{axis}^{(1)} \sin px$$

where $x \in [0, L/R]$, $y \in [0, 2\pi]$. U , V and W indicate the maximum amplitudes of corresponding displacements. p is equal to $m\pi R/L$. m is the number of half waves in the longitudinal direction. n is the number of full waves in circumferential direction. The substitutions of (8.5) and (8.6) into (8.1) lead to corresponding buckling loads as

$$\lambda_{non}^c = \frac{4(1-\nu^2)p^4 + J[2(1-\nu)p^2 + n^2][2(1+\nu)p^2 + n^2] - 2Jn^2[(2-\nu)p^2 + n^2][(2+\nu)p^2 + n^2] + J(p^2 + n^2)^4}{4(1-\nu^2)[(1+\nu)p^4 + p^2n^2 + p^2(p^2 + n^2)^2]} \quad (8.8)$$

$$\lambda_{axis}^c = \frac{1}{p^2} + \frac{Jp^2}{4(1-\nu^2)} \quad (8.9)$$

If the maximum deflection of buckling mode is normalised to be one, the buckling modes can be updated as

$$u_{non}^{(1)} = \frac{p(\nu p^2 - n^2)}{(p^2 + n^2)^2} \cos px \sin ny$$

$$v_{non}^{(1)} = -\frac{n((2+\nu)p^2 + n^2)}{(p^2 + n^2)^2} \sin px \cos ny \quad (8.10)$$

$$w_{non}^{(1)} = \sin px \sin ny$$

$$u_{axis}^{(1)} = \frac{\nu}{p} \cos px$$

$$v_{axis}^{(1)} = 0$$

$$w_{axis}^{(1)} = \sin px$$

(8.11)

It can be observed that the determined buckling loads of two types of buckling modes are different from each other. The buckling loads are not invariant of length due to the presence of p . However, in the classic buckling solution, the buckling load is a function of only radius and thickness in terms of structural dimensions but not the length.

Predictions using the finite element method tend to agree with the above analytical results better than the classic buckling load. Koiter did not make efforts to keep consistency in boundary conditions between pre-buckling and buckling. For pre-buckling, membrane state was employed to allow analytical solution to be obtained. If such analytical solutions are used as verification cases for the FE implemented Koiter's theory, the boundary conditions in the FE analysis should be kept consistent with those employed in the analytical solution, as illustrate in the previous chapter for plates. On the other hand, if one is to apply the FE implemented Koiter's theory to any practical problem, keeping the consistency in the boundary conditions will be more accurate. Relatively, as far as FE analysis is concerned, using consistent boundary conditions is simpler to achieve, while inconsistent boundary conditions require additional treatments as will be described below.

8.2.2 FEM Prediction of Buckling

8.2.2.1 Boundary Conditions

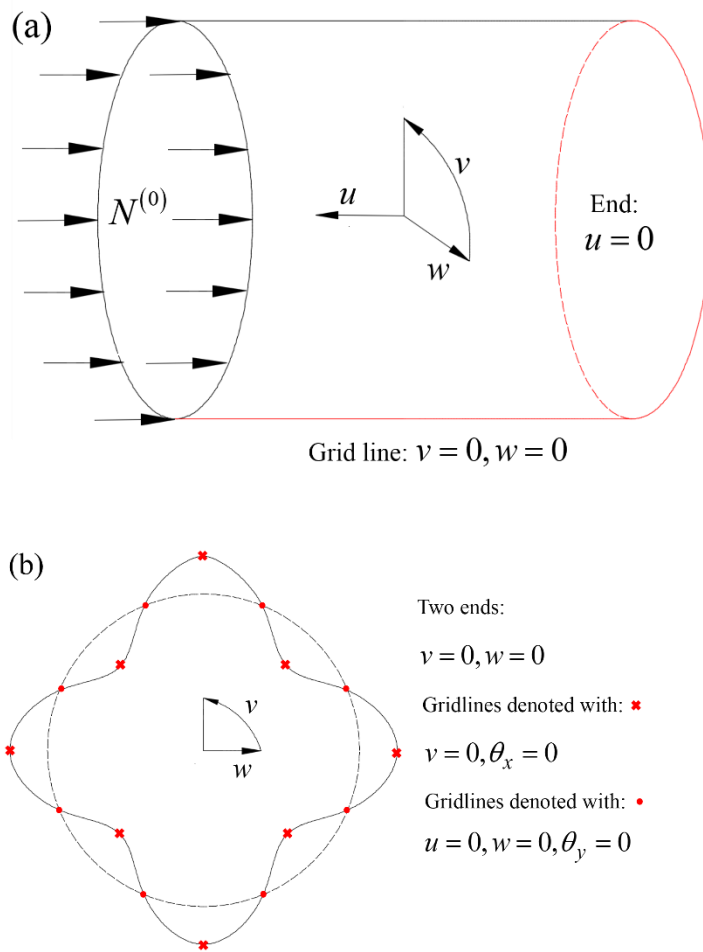
The problem could be analysed straightaway with simply supported boundary conditions at the both ends of the cylinder. However, FEM would result in a large number of eigenvalues clustered around the lowest one, giving rise to a variety of modes more than what have been predicted by the analytical solution as obtained in the previous subsection. Direct use of these modes in the subsequent initial-post-buckling analysis, while producing a valid solution to the practical problem concerned, would not be meaningful as one is seeking for verifications against available analytical solutions. To facilitate such a verification, one will have to filter out those modes as predicted by the analytical solution. By constraining appropriately along a set of grid lines according to the deformation as defined in the analytical displacement field, one will narrow the candidates to those modes as specified by the analytical solution.

The displacement field on the fundamental path in the pre-buckling regime is given as (8.1). To reproduce buckling modes as given in (8.6) and (8.7), the gridlines are chosen as the generators in the longitudinal direction and rings in the circumferential direction where deflection is maximum, zero and minimum in its distribution in these two directions, with one of each shown in Figure 8.2 (a). The vanishing displacements or rotations at these grid lines according to the analytical solution will be specified as boundary conditions to facilitate the FEM buckling analysis, although the gridlines are not all located at the boundary. As grid lines vary from mode to mode, a specific set of boundary conditions should be imposed for each individual buckling mode. As explained above, the objective of doing so is merely to reproduce the buckling modes as obtained in the analytical solution.

Consider the non-axisymmetric buckling modes as depicted in Figure 8.2 (b) as an example, where the half-wave number of the buckling mode is 4 in the circumferential direction, the displacement boundary conditions have been imposed according to the displacement expressions in (8.10). It can be observed that at the troughs and crests of the buckling modes denoted with crosses, the displacement in the circumferential direction vanishes along with rotation about the generator of the cylinder, coincide with a symmetry condition. At the intersection points with the undeformed profile denoted with dots, the axial displacement and radial displacement vanish along with the rotation about the circumferential direction, corresponding to an antisymmetric condition.

For axisymmetric buckling mode, it is much more straightforward to realise the conditions as implied by (8.11). A constraint of circumferential displacement has been imposed to the entire structure as illustrated in Figure 8.2 (c). The axial displacement at the both ends are of the same value and they can therefore be constrained without violating this condition. The radial displacement vanished at the both ends and it is

constrained there accordingly. It is worth noting that the analytical solution based on the Love-Kirchhoff hypothesis can be fully described by the three displacements, while the finite elements for plates and shells requires rotational degrees of dofs which are independent of the displacements in the R-M plate and shell theory. Thus the constraints on two rotations have to be specified as well for an accurate prescription of boundary conditions as shown in Figure 8.2.



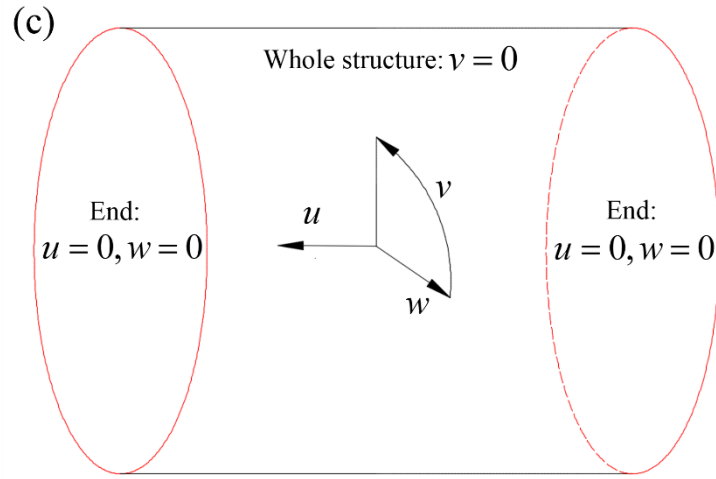


Figure 8.2 Boundary conditions in pre-buckling regime (a) and buckling regimes for non-axisymmetric buckling modes ($n=4$) (b) and axisymmetric buckling modes (c)

8.2.2.2 Numerical Results

A cylinder with 50mm radius, 200mm length and 1mm thickness has been taken as an example for the initial post-buckling analysis to demonstrate the effects of mode-interaction. The material is considered to be homogenous isotropic with Young's modulus 210 GPa and Poisson's ratio 0.3. The commonly adopted commercial finite element code Abaqus [44] has been employed for the eigenvalues analysis. To be consistent with the element incorporated in the in-house code, S8R5 has been used to discretise the numerical models. The circumference has been discretised into 120 equal sized elements so that buckling modes with 1-6 waves along the circumference of cylinder would have nodes located at the grid lines. Cylinder has been discretised with 76 equal sized elements in the axial direction.

The default arrangement for boundary conditions in Abaqus for pre-buckling and buckling analyses are consistent, as they would be in reality. However, this has been artificially overridden in order to compare directly with the analytical solution.

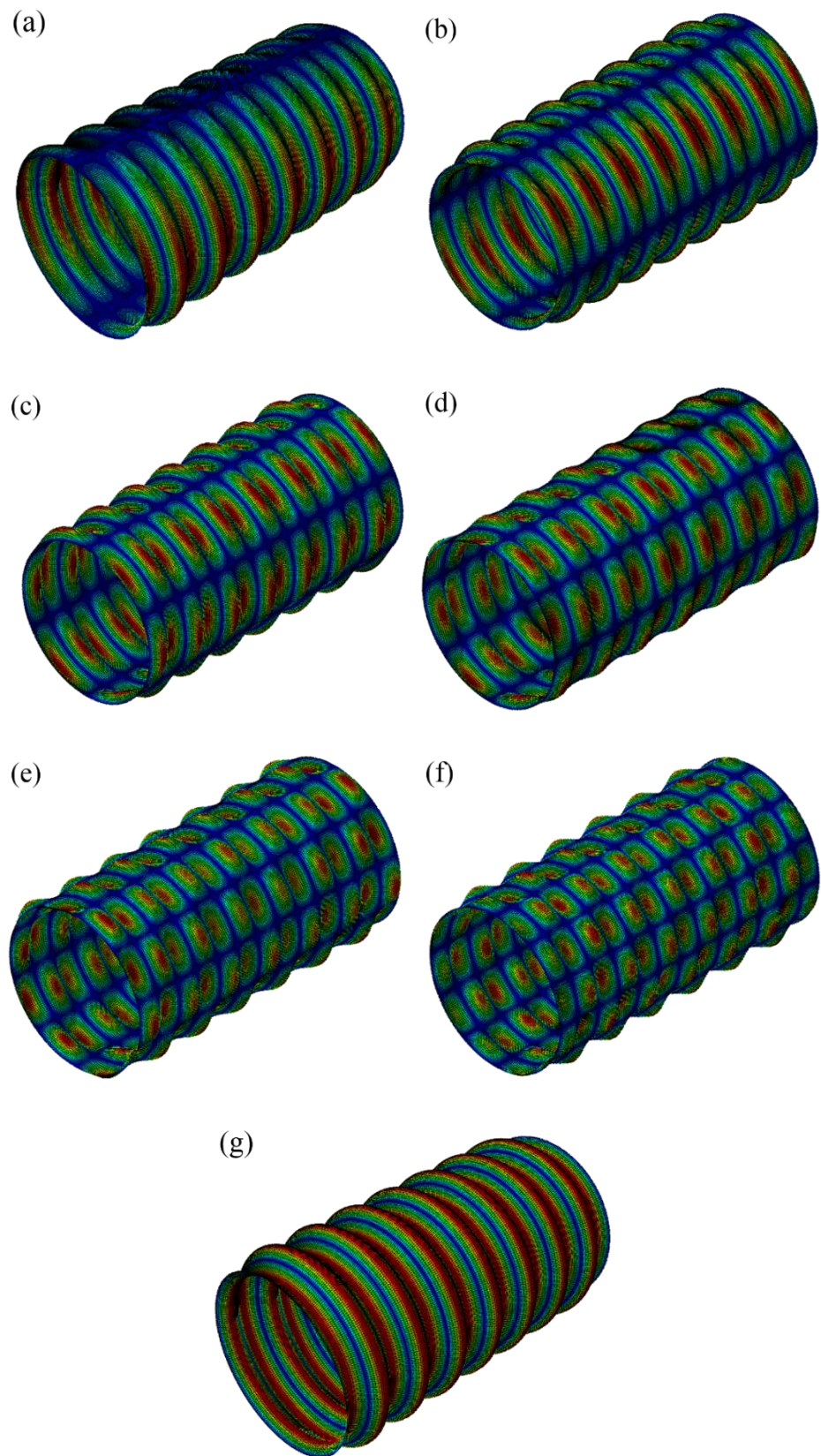


Figure 8.3 Non-axisymmetric buckling modes with (a) 1, (b) 2, (c) 3, (d) 4, (e) 5, (f) 6 waves in circumferential direction and axisymmetric buckling mode (g)

The representatives of buckling modes for different numbers of waves in circumferential direction have been provided in Figure 8.3. From the contours of buckling modes, one can clearly observe that by prescribing the constraints in the FE analysis consistent with the buckling modes as obtained in analytical solution as expressed in (8.10) and (8.11), these mode can be precisely reproduced through FEM for both non-axisymmetric and axisymmetric ones.

Table 8.1 Comparisons of buckling loads of cylinder under axial compression

n	m	Buckling load (N/mm)		Error	n	m	Buckling load (N/mm)		Error
		Analytical	FEM				Analytical	FEM	
0	15	2580.7	2562.9	-0.69%	1	15	2550.6	2578.5	1.09%
0	16	2544.6	2521.6	-0.90%	1	16	2522.5	2537.3	0.59%
0	17	2549.3	2521.0	-1.11%	1	17	2533.1	2536.3	0.13%
0	18	2588.1	2552.7	-1.37%	1	18	2576.3	2568.0	-0.32%
2	15	2535.1	2536.0	0.04%	3	1	2092.2	2070.1	-1.06%
2	16	2520.5	2514.3	-0.25%	3	15	2519.6	2515.4	-0.17%
2	17	2541.2	2527.3	-0.55%	3	16	2524.9	2514.1	-0.43%
2	18	2591.9	2569.7	-0.86%	3	17	2560.4	2542.2	-0.71%
4	2	2327.4	2297.7	-1.28%	5	3	2366.1	2341.5	-1.04%
4	14	2517.0	2511.6	-0.22%	5	4	2511.5	2482.4	-1.16%
4	15	2517.7	2506.9	-0.43%	5	13	2507.5	2497.8	-0.39%
4	16	2545.7	2528.8	-0.66%	5	14	2515.3	2501.7	-0.54%
6	4	2485.8	2459.2	-1.07%					
6	5	2432.6	2407.6	-1.03%					
6	6	2456.4	2433.3	-0.94%					
6	7	2482.8	2462.5	-0.82%					

The numerical results of buckling loads along with its analytical counterparts have been collected in Table 8.1. There are seven categories of buckling loads in terms of the number of waves in the circumferential direction. For each category, the four lowest

buckling loads have been chosen for comparison. Such arrangement is due to that the chosen buckling loads are approximately located at the Koiter's circle [15] however others are well away from it hence they are not considered in the current study. It can be found that all the numerical buckling loads show excellent agreement with the analytical ones with relative errors within $\pm 1.5\%$. It can be found that the eigenvalues corresponding to each mode do not fall to the exactly same value but cluster within a small range. It should be emphasised again that the excellent agreement achieved here attributes to the precise employment of the boundary conditions consistent between the FEM solution and the analytical solution.

8.3 Initial Post-buckling Behaviours and Mode Interaction

The erratic experimental results of collapse load of cylinder under compression prompted tremendous efforts on this subject. Eventually, the reason for the discrepancy was identified which lied in the fact that cylinder was extremely sensitive to initial imperfections, the presence of which was unavoidable. The analytical investigations by directly taking account into the imperfection were conducted in [14, 88, 90-104, 110, 128, 198-201], in some of which, axisymmetric imperfections were individually considered while others used double Fourier series to simulate a more general and realistic imperfections. With Koiter's initial post-buckling analysis, the study of imperfection is more straightforward. In addition to the individual imperfection mode, the interaction of different imperfections could also be readily attained.

For the class of problems characterised by asymmetric bifurcation, the buckling mode as obtained from a conventional eigenvalue analysis would be sufficient to facilitate the initial post-buckling analysis without having to pursue further perturbations. Cylinders under axial compression fall into this class. Unlike the single distinct mode

buckling problems, there is a cluster of modes corresponding to eigenvalues in the small neighbourhood of the lowest one. To investigate how these modes interact each other while corresponding to slightly different eigenvalues, the numerator of (4.16b) is resorted as an alternative as follows.

$$A_{ijk} = \left(\mathbf{q}_i^{(1)} \right)^T \mathbf{N}_1 \left(\mathbf{q}_j^{(1)} \right) \mathbf{q}_k^{(1)} + \left(\mathbf{q}_i^{(1)} \right)^T \mathbf{N}_{11} \left(\mathbf{q}_j^{(1)}, \mathbf{q}_k^{(1)} \right) \mathbf{q}_c^{(0)} \quad (8.12)$$

If this theory was implemented in any commercial FE code, the initial post-buckling analysis would be carried out much more conveniently and its outcomes be of great value to structural designers. Unfortunately, Koiter theory is not available in any commercial FEM code. The following approach has been taken to facilitate the evaluation of the effects of mode interaction. Use has been made of the in-house FE code to deliver the results.

The two lowest eigenvalues of each category, namely from $n=0-6$, have been extracted from the Abaqus .fil, giving 14 eigenmodes in total for the purpose of demonstration of versatility and simplification of Koiter's theory when dealing with mode interactions. They are fed into the in-house FE code for the initial post-buckling analysis by evaluating the expression (8.12). The result are shown in Table 8.2.

Table 8.2 The first IPBCs obtained if each eigenmode was perceived as a single distinct mode

No.	n	m	A_{iii}	No.	n	m	A_{iii}
1	0	16	8.665×10^{-1}	8	3	16	1.111×10^{-6}
2	0	17	-2.931×10^5	9t	4	2	-2.566×10^{-2}
3	1	16	-7.617×10^{-4}	10	4	15	-4.033×10^{-3}
4	1	17	6.297×10^{-4}	11	5	3	2.291×10^{-6}
5	2	16	-2.803×10^{-6}	12	5	4	2.885×10^{-8}
6	2	17	-1.773×10^{-6}	13	6	5	3.342×10^{-6}
7	3	1	-6.802×10^{-2}	14	6	6	-6.150×10^{-6}

It can be observed in Table 8.2 that the maximum absolute value of A_{iii} is at 2.931×10^5 obtained for the one corresponding to the axisymmetric mode which indicates that initial post-buckling behaviour is asymmetric and extremely unstable. Any imperfection in this mode embodied in the structure would lead to a catastrophic premature failure. Interestingly enough, other buckling modes produce relatively less pronounced values of A_{iii} as compared with A_{222} . As presented in an unfinished analytical work in Appendix E, such numerical results have been partly verified. The vanishing values of A_{iii} have been satisfactorily reproduced for asymmetric buckling modes and axisymmetric ones with even number of waves in circumferential direction. While axisymmetric buckling modes with odd numbers of waves in the circumferential direction would give rise to a non-vanishing value of A_{iii} as also shown in Table 8.2 and in Appendix (E-4b) which leads to A_{iii} equal to -7.906×10^3 with aid of (E-1), although discrepancies are present between the numerical and analytical results, which needs to be further explored in the future.

The 3rd variation of total potential energy as denoted P_3 for cylinder was initially derived in Koiter's thesis [15] in a general form. In another work of his [92], a simplified form was presented by eliminating the quadratic terms of derivatives of in-plane displacements which were considered to have a negligible effect on the value of P_3 . However, there does not seem to be any solid ground for such simplification. Practically, such simplified form could not deliver any non-vanishing value of P_3 as presented in Appendix (E-5) which therefore contradicted the general form as well as the numerical solution.

To demonstrate how the mode interaction exacerbates cylinder's initial post-buckling instability, some severest cases of them have been chosen without including all values as obtained and shown in Table 8.3.

Table 8.3 First IPBCs of interactive modes

No.	No.	No.	A_{ijk}	No.	No.	No.	A_{ijk}
1	7	8	1.937×10^6	1	13	14	-7.677×10^4
1	9	10	1.256×10^6	2	8	8	-7.870×10^4
4	6	7	1.066×10^6	1	1	2	-8.320×10^4
4	10	11	5.988×10^6	2	3	3	-8.372×10^4
4	7	10	2.789×10^5	2	10	10	-1.067×10^5
2	14	14	8.374×10^4	2	4	4	-1.458×10^5
3	8	10	4.926×10^4	3	10	12	-2.855×10^5
2	13	13	4.567×10^4	4	8	9	-6.978×10^5
4	5	8	3.366×10^4	3	5	7	-9.664×10^5

It can be noted that many more significant values of A_{ijk} have been obtained as a result of the interaction between different modes, some are substantially more pronounced than A_{222} . Each non-vanishing value indicates the unstable initial post-buckling behaviour associated with the modes involved. It should be noted that the outcomes listed in Table 8.3 is only small portion of the entire results, there are plenty of combinations of modes which could give rise to non-vanishing values of A_{ijk} . Hence, it is therefore evident that the structure will be sensitive a wide range of modes of initial imperfections, and the sensitivity exacerbates when the actual initial imperfection is present as a combination of some of the modes. Therefore, unless a specimen was made free from any initial imperfection, the structure will always collapse at a much reduced load level. Owing to the unpredictable nature of the modes and magnitudes of imperfections, one should never expect one experiment to be reproduced by another. For such problems, the so-called ‘experimental validation’ in its conventional sense would lose its value in engineering practices completely. In fact, it could be even worse as it would be badly misleading if one assumed the authoritarian role of a particular experimental result as the correct answer to a problem of this nature.

It is obvious that Koiter's theory is of great advantages in assessing the mode interaction problem as demonstrated through the example of simply supported cylinder under axial compression. From the computational point of view, the cost of conduct such an analysis after the conventional buckling analysis is negligible compared to the iterative incremental method, even in presence of mode interaction. With the outcomes of eigenvalue analysis, the first IPBCs could be straightforwardly generated. Only if all the first IPBCs turn out to be zero, a further perturbation will be required. While as demonstrated in chapter 5, the solution of SPMs is also effortless using closed-form scheme as opposed to the iterative one. Once SPMs is obtained, the calculation of the second IPBCs amount to solving a set of linear simultaneous equations equivalent to a single iteration and a nonlinear analysis.

It should be pointed out that in presence of interacting buckling modes, the iterative incremental method of analysing the buckling problem would not be practical anymore, given the inexhaustible available of combinations of modes and magnitudes of imperfections. The Koiter's theory is the remedy which is particularly competent in dealing with problems with mode interactions, bearing in mind that modern optimally designed structures tend to buckling in multiple modes typically, e.g. local and global buckling in stiffened panels. It is a pity that not designers nowadays are aware of the theory, let alone its merits. Its implementation in commercial FE codes must be pursued in order to make a difference, and ultimately, to benefit practising structural designers.

8.4 Summary

The analytical solution for the buckling of a simply supported cylindrical shell as obtained by from Koiter has been revisited in order to clarify the boundary conditions in pre-buckling regime and buckling regime. Using the ideal boundary conditions as implied

by the analytical solution, FEM results have been obtained with perfect agreement with the analytical solution, giving both axisymmetric and non-axisymmetric buckling modes. The obtained eigenmodes in fact correspond to cluster of eigenvalues falling within a small range of lowest.

Assuming these eigenmodes as interacting buckling modes, the initial post-buckling analysis has been conducted. When each eigenmode was perceived as an individual buckling mode, the most severe imperfection sensitivity had been identified as the axisymmetric buckling mode. However, the presence of mode interaction makes the situation much worse and more severe. Due to the unpredictable nature of initial imperfections and the extreme sensitivity of the structure to such imperfections, it is difficult to precisely determine the actual collapsing load for this type of structures. The initial post-buckling theory offers sufficient understanding into the nature and the behaviour of such structures which will be of invaluable confidence on dealing with such structures which is completely missing as the state-of-the-art.

Appendix E Third Order Variation of Potential Energy of Cylinder under Axial Compression

In Koiter's original work [15] , the 3rd order variation of potential energy P_3 was derived and exploited to predict the initial post-buckling of cylinder under axial compression. To relate to the expression in current study, its equivalence to A_{iii} has been made as

$$P_3 = \frac{A_{iii}}{6} \quad (E-1)$$

Two expressions of it were proposed in a generalised and simplified form as

$$P_3|_G = \frac{EH}{2R(1-\nu^2)} \int_0^{2\pi} \int_0^{L/R} \left\{ \begin{aligned} & (u_{,x} + \nu v_{,y} + \nu w) \left[(u_{,x})^2 + (v_{,x})^2 + (w_{,x})^2 \right] \\ & + (\nu u_{,x} + v_{,y} + w) \left[(u_{,y})^2 + (v_{,y} + w)^2 + (w_{,y} - \nu)^2 \right] \\ & + (1-\nu)(u_{,y} + v_{,x}) \left[u_{,x} u_{,y} + v_{,x} (v_{,y} + w) + w_{,x} (w_{,y} - \nu) \right] \end{aligned} \right\} dx dy \quad (E-2)$$

$$P_3|_S = \frac{EH}{2R(1-\nu^2)} \int_0^{2\pi} \int_0^{L/R} \left\{ \begin{aligned} & [u_{,x} + \nu(v_{,y} + w)] (w_{,x})^2 \\ & + (\nu u_{,x} + v_{,y} + w) (w_{,y})^2 \\ & + (1-\nu)(u_{,y} + v_{,x}) w_{,x} w_{,y} \end{aligned} \right\} dx dy \quad (E-3)$$

It is worth to note that the displacements involved are all of buckling modes. For the sake of brevity, the superscripts have been eliminated.

The substitutions of asymmetric buckling modes (8.6) and axisymmetric buckling modes (8.7) into the generalised form lead to

$$P_3|_{G-non} = 0 \quad (E-4a)$$

$$P_3|_{G-axis} = \frac{EHL}{3mR^2} \left((\cos(m\pi))^3 - 3\cos(m\pi) + 2 \right) \quad (E-4b)$$

It can be observed that the non-axisymmetric buckling modes would consistently deliver a vanishing values of P_3 . While for axisymmetric buckling modes, P_3 would be non-vanishing if the odd number of m surfaces which has been reproduced by the current numerical study as presented in Table 8.2 when $n=0$, $m=17$. While such numerical values does not agree with analytical one which will be examined in the future work.

Likewise, the substitutions of buckling modes into the simplified form would result in

$$P_3|_{S-non} = P_3|_{S-axis} = 0 \quad (E-5)$$

Interestingly enough, the simplified form of P_3 would be consistently vanishing no matter what buckling modes get involved which is in stark contrast to (E-4b) and the results provided in Table 8.2. Hence, the simplified form would no way generate reliable results to validate the numerical outcomes. The elimination of the quadratic forms of in-plane displacements therefore seems to make no sense. It is believed that if one attempts to

validate the numerical results via the analytical solution, the generalised form should be resorted to.

Chapter 9 Conclusions and Future Work

9.1 Conclusions

The aim of the research is to successfully implement Koiter's initial post-buckling theory into the finite element framework. Having succeeded in achieving the goal, the major outcomes of this thesis can be summarised as follows.

- 1) By revisiting the N-notation, the lack of uniqueness in the expressions for stiffness matrices has been resolved completely after introducing the concept of fully commutative strain energy. Symmetric stiffness matrices have been obtained in explicit expressions in which displacements involved in these expressions are fully commutative. A natural consequence is that the strain energy, the secant stiffness matrix and the tangential stiffness matrix can be simply and meaningfully related while previously such relationships were perceived as symbolisms only.
- 2) To implement Koiter's initial post-buckling theory into finite element environment, the theory has been re-derived based on the commutative N-notation. The governing equations of buckling and SPM have been obtained along with the orthogonality condition between buckling mode and SPM. Finally the expressions of two IPBCs have been expressed using the stiffness matrices in N-notation explicitly for both single and multiple mode systems.
- 3) As an important part in the determination of the second IPBC, a closed-form solution to the governing equations for SPM has been obtained. The solution is expressed as a combination of particular solution and complementary solution following a strict mathematical approach. The complementary solution could be readily proven to be the buckling mode. By constraining the dof in the coefficient matrix corresponding to the maximum displacement of the buckling mode, the particular solution can be obtained by solving a set of linear simultaneous equations. Through the orthogonality condition, the

coefficient of complementary solution can be easily determined so that the full solution can be presented in a closed-form. The solution obtained is applicable to both single and multiple mode systems. Two simple examples have been provided to verify its correctness.

4) Based on the commutative N-notation and closed-form solution to secondary perturbation mode, the Koiter's theory has been implemented into an in-house code. A number of examples which were employed previously in the literature have been analysed using the code. The mesh sensitivity problem for the second IPBC no longer emerges as a result of the use of the commutative N-notation in the FE implementation of Koiter's theory.

5) Given the availability of analytical solutions for the problem, no attempt has been made in the literature to use them to verify any of the existing FE implementation of Koiter's theory as far as plates and shells are concerned. The reason for the absence of such attempts has been revealed in this thesis to be the lack of agreement in results due to incorrect representation of the boundary conditions. Having prescribed the boundary conditions correctly, simply supported plates subject to biaxial compression have been conducted and perfect agreement has been achieved with the analytical results as the first ever serious verification for the FE implementation of Koiter's theory for its applications to plates and shells.

6) To provide another analytical solution to a problem of practical significance as a further verification of the FE implemented Koiter's theory, a simply supported curved panel subject to axial compression has been derived based on the R-M theory in order to bring forward a more meaningful comparison with FEM since plates and shells in modern FE codes are all based on the R-M theory. The buckling load along with the deflection coefficient of secondary perturbation mode and expressions of two IPBCs have been

obtained. It reconciles well with the results based on the Love-Kirchhoff. In order to simulate the case as represented by the analytical solution, once again, the boundary conditions as implied in the analytical solution have been prescribed and perfect agreements have been obtained with the analytical results.

7) Simply supported cylinder under axial compression has been studied as a multiple mode system. Similarly, the boundary conditions in pre-buckling and buckling regimes have been explored by carefully examining analytical expressions. Abaqus has been resorted to in order to provide accurate buckling modes while the in-house code has been dropped due to the subspace algorithm adopted which becomes inefficient when dealing with mode interaction problem. By using the consistent boundary conditions, excellent agreements of buckling loads for each buckling mode have been achieved. It is observed that such obtained buckling loads scatter within a narrow range rather than repeated eigenvalue. Taking them as a unique value as an approximation, the first IPBCs have been obtained. Considering each buckling mode individually, the axisymmetric buckling mode would show the most unstable behaviour in the initial post-buckling regime. However, after investigating the interactions between different modes, much more severe instability can be found.

9.2 Future Work

Having achieved what have been presented in the thesis, a number of areas have been identified where more efforts will be required as a future development further to the achievements of this thesis.

1) In the present work, the initial post-buckling behaviour of cylindrical shells has been focused on the first IPBCs. A thorough analytical study on this subject was undertaken by Koiter in [19, 202] where some analytical results were provided. It will be

important to verify the FE implemented Koiter's theory against that. In addition, analytical solutions available in the open publications to the best knowledge of author are confined to the single-mode scenarios. The one pertaining to the interactive modes demands to be derived in the future work.

2) The analytical solution obtained based on the R-M theory is only applicable to the single mode system. Extension would be of value to incorporate the possible interaction between two different modes for plates and curved panels.

3) A further verification can be made against the above case as another example of mode interaction.

4) In Koiter's theory, the effects of initial imperfection on the structure's collapsing load can be evaluated, which involves the evaluation of one parameter associated with each of the IPBCs. They can also be expressed in the context of FEM but unfortunately not attended in the present project. Given the N-notation obtained, they should be pursued as a future development.

5) In the current work, the main aim is to resurrect the Koiter's theory by resolving the obstacles which prohibited the successful implementation and consequently verifying the numerical results. Although the structural configurations employed in this thesis are rather simply, viz. plates, curved panels and cylinders, Koiter's theory is meant to be generally applicable to all structures exhibiting bifurcation buckling, in particular, assisted with FEM. If successful, initial post-buckling theory could bring forward a valuable approach to deal with buckling problem. It is not suitable for nonlinear post-buckling problems involving snap-through. In fact, the snap-through type of problems poses little challenge to modern FE code with advanced procedures of nonlinear analyses, for instance, by using the Riks method [203]. It is the bifurcation type of buckling that modern FE codes struggle to cross the bifurcation point. An available approach, e.g. as in

Abaqus, is to introduce some imperfections to avoid a discrete bifurcation before the Riks method can be applied at a much increased computational cost. This is not dealt with in this thesis as the facility has been established already in commercial FE codes.

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