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Derived Geometry and Higher Quantum Groups

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Cameron James Deverall Kemp

20407478

Supervised by

Robert Laugwitz
Alexander Schenkel

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Date _____ / _____ / _____

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Memento mori

"Whether you are man or woman, rich or poor, dependent or free, happy or unhappy; whether you bore in your elevation the splendour of the crown or in humble obscurity only the toil and heat of the day; whether your name will be remembered for as long as the world lasts, and so will have been remembered as long as it lasted, or you are without a name and run namelessly with the numberless multitude; whether the glory that surrounded you surpassed all human description, or the severest and most ignominious human judgment was passed on you - eternity asks you and every one of these millions of millions, just one thing: whether you have lived in despair or not, whether so in despair that you did not know that you were in despair, or in such a way that you bore this sickness concealed deep inside you as your gnawing secret, under your heart like the fruit of a sinful love, or in such a way that, a terror to others, you raged in despair. If then, if you have lived in despair, then whatever else you won or lost, for you everything is lost, eternity does not acknowledge you, it never knew you, or, still more dreadful, it knows you as you are known, it manacles you to yourself in despair!"

— Søren Aabye Kierkegaard, The Sickness Unto Death.

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If there is *any* profundity or penetrating depth to this thesis *whatsoever*, if its insights should resolve or clarify concepts which the zeitgeist is currently explicating, if it *could* help synthesise another bridge between seemingly distant fields of Mathematics, if it should form a stepping stone or guiding light for any mathematician of posterity who wishes to plunge into the labyrinth that is ∞-categorical deformation quantisation...then *all* such merits would be duly owed to my senseis, Robert Laugwitz and Alexander Schenkel. Whether it be their steadfast wisdom, their expansive creativity or their enduring patience, they have provided me with two examples of a supervisor which, should I ever become one then I will count myself blessed if I managed to emulate half the virtues in half the measure half of the time. When this project was first proposed, a primary goal was to use a computational approach to Derived Algebraic Geometry to show exactly how quasi-Lie bialgebra structures and symmetric invariant 2-tensors arise as shifted Poisson structures for ordinary Lie algebras. The scope of the project was then to explore such characterisation for algebraically richer objects such as differential graded Lie algebras. It's fair to say that we've accomplished much more than this; in fact, the graphical calculus of Chapter 3 came as something of a surprise during our investigation. What's more is that its details were pretty much done and dusted only 14 months after this PhD started. Again, that swift progression is credited to Alex and Robert; they are the ones who taught me the mainstay, showed me how to work out examples and demonstrated flexibility of understanding as the nature of the project was becoming clearer. It is thanks to their efforts that we then had that much more time to study the application of shifted Poisson structures to the realm of Higher Category Theory. Part of the beauty of that application was finding connection with the work of Lucio Simone Cirio and João Faria Martins on infinitesimal 2-braidings as pseudonatural transformations. Truth be told, there was a time when I thought that it was obvious that (and how) 2-shifted Poisson structures induce infinitesimal dg-braidings and that the latter would have nothing at all to do with pseudonatural transformations and other such higher categorical notions. Such unwarranted self-evidentiality came about because I rushed to conclusions, that door would have certainly remained forever unopened had it not been for Alex and Robert's exercise of due diligence. I thank them as well for the moments of bellyaching laughter, their excellent taste in cuisine, the conversations surrounding movies, music and philosophy.

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Abstract

This thesis presents a novel path to the explicit construction of models of 'higher quantum groups' by studying positively-shifted Poisson structures on a generic semi-free commutative differential graded algebra and investigating exactly how they induce infinitesimal deformations of the symmetric monoidal structure on the dg-category of semi-free dg-modules over such a commutative differential graded algebra.

First we develop a graphical calculus to characterise the shifted Poisson structures on finitely generated semi-free commutative differential graded algebras in terms of families of homotopy-coherent data adjoined to the corresponding finite-dimensional higher Lie algebra. Such a characterisation allows us to show that we recover the notion of higher Lie quasibialgebra due to Bai, Sheng and Zhu. When applied to an ordinary Lie algebra, we recover Safronov's result that the 1 and 2-shifted Poisson structure in this case are given by quasi-Lie bialgebra structures and invariant symmetric 2-tensors, respectively. We generalise these results to the case of a Lie 2-algebra and obtain 1, 2, 3 and 4-shifted Poisson structures which we interpret as semiclassical data of higher quantum groups. In particular, if we assume a 2-term differential graded Lie algebra and trivialise part of the data of a 2-shifted Poisson structure then we recover a special case of the notion of a symmetric quasi-invariant tensor in a differential crossed module due to Cirio and Faria Martins.

It is then shown that every 2-shifted Poisson structure on a finitely generated semi-free commutative differential graded algebra A defines a concrete infinitesimal 2-braiding on the homotopy 2-category of the symmetric monoidal dg-category of finitely generated semi-free dg-modules over A. This provides a concrete realisation, to first order in the deformation parameter, of the abstract deformation quantisation results in derived algebraic geometry due to Calaque, Pantev, Toën, Vaquié and Vezzosi. Of particular interest is the case when A is the Chevalley-Eilenberg algebra of a higher Lie algebra, where the braided monoidal deformations developed in this thesis may be interpreted as candidates for representation categories of higher quantum groups.

Upon choosing the usual ansatz of the braiding and associator as, respectively, the exponential and Drinfeld's Knizhnik-Zamolodchikov series of the infinitesimal 2-braiding, it is demonstrated that the hexagon axioms are obstructed at second order by modifications. These modifications are shown to satisfy the requisite axioms of a braided monoidal 2-category (i.e., the modifications are shown to be 'infinitesimal hexagonators') provided that the infinitesimal 2-braiding is totally γ-equivariant and coherent in Cirio and Faria Martins' sense. We show that those infinitesimal 2-braidings induced by 2-shifted Poisson structures are indeed totally γ-equivariant and we conjecture that coherency also holds by relating the condition to the third-weight component of the Maurer-Cartan equation that a 2-shifted Poisson structure definitively satisfies. We then demonstrate that the pentagonator is nontrivial by determining its third-order term.

Finally, we study the problem of sylleptic deformations and show that a 3-shifted Poisson structure induces "infinitesimal syllepses" which can be trivially integrated to all orders. We also show that a "coboundary" 2-shifted Poisson structure induces "infinitesimal coboundary syllepses" and these admit of a Cartier integration alongside that of infinitesimal 2-braidings and we carry this out to second and third order.

Conventions and notations

- All vector spaces, algebras, etc., in this thesis will be over a generic ground field $\mathbb{K}$ of characteristic 0. In Section 5.3 and Subsection 6.2.3, we specialise this to $\mathbb{K} = \mathbb{C}$.
- Every non-Lie algebra will be assumed to be unital associative though we may still state this every now and then merely for sake of clarity.
- Terms such as 'commutative', 'Jacobi identity' or 'Leibniz law', etc., will always refer to their graded analogues hence we suppress this adjective.
- For a mapping/equation, input elements will be assumed to be of homogeneous degree.
- A diagram in a category will be assumed to commute unless stated otherwise, e.g., if there is a filler 2-morphism.
- We use juxtaposition to denote: the composition of morphisms in a category, the composition of functors, the monoidal product of objects and the vertical composition of pseudonatural transformations and modifications.
- From Chapter 4 onwards, our base algebra R is a generic commutative associative unital $\mathbb{K}$-algebra. Think of this as $R = \mathbb{K}[\hbar]/(\hbar^n)$ for $n \in \mathbb{N}$ where $\hbar$ is the deformation parameter.
- Boldface is used to denote a category or highlight the term being defined whereas italics are used for emphasis. For example, a **cochain 2-category** is a $\mathbf{Ch}_R^{[-1,0]}$-category and infinitesimal 2-braidings are defined to be *pseudo*natural rather than 2-natural.
- The Einstein summation convention is assumed unless the context dictates otherwise.
- fg = finitely generated, fd = finite-dimensional, sf = semi-free, (n)SPS = $(n$-$)$shifted Poisson structure, CDGA = commutative differential graded algebra, DGLA = differential graded Lie algebra.
- CM = Cirio and Faria Martins, KZ = Knizhnik-Zamolodchikov.
- Every monoidal category of structured sets is assumed to be strict under Schauenburg's refined strictification theorem [Sch01, Theorem 4.3]. The only exception to this is second (and higher) order Cartier integration of infinitesimal braidings, i.e., Chapter 5 and Subsection 6.2.2 onwards.

Chapter 1

Introduction

Symmetry permeates the foundation of mathematical physics through the formalism of groups G and their representation theory $\rho : G \to \mathrm{Aut}(V)$, of particular interest is the notion of a smooth symmetry and this is encoded by the theory of Lie groups. It was Emmy Noether who rigorously proved the one-to-one correspondence between smooth symmetries and conservation laws thus providing a powerful method for integrating the governing equations of motion by way of reference to the associated invariants. Another way that Lie groups have entered our description of Nature's laws is through the idea of gauge field theory; indeed The Standard Model of Particle Physics has the gauge group $G = \mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)$ and this controls the way in which the strong, weak and electromagnetic, respectively, forces couple to matter.

Associated to every Lie group G is its Lie algebra $\mathfrak{g}$ which can be thought of as the 'infinitesimal' symmetries. The representation theory of a Lie algebra $\rho : \mathfrak{g} \to \mathrm{End}(V)$ linearises that of the Lie group and is much easier to study. Actually, one can construct a unital associative algebra known as the universal enveloping algebra $U\mathfrak{g}$ and a representation of $\mathfrak{g}$ is simply a module over $U\mathfrak{g}$. In quantum field theory, multiparticle states are modelled via the tensor product of vector spaces $V \otimes W$. If each of V and W are modules over $U\mathfrak{g}$ then the tensor product also is with the following action,

$$x.(v \otimes w) := (x.v) \otimes w + v \otimes (x.w) \quad , \tag{1.0.1}$$

where $x \in \mathfrak{g}$. The action (1.0.1) is rooted in the canonical coproduct $\Delta : U\mathfrak{g} \to U\mathfrak{g} \otimes U\mathfrak{g}$,

$$\Delta(x) := x \otimes 1 + 1 \otimes x \quad . \tag{1.0.2}$$

Likewise, the ground field $\mathbb{K}$ (which models the vacuum state) admits the trivial action

$$x.1 = 0 \tag{1.0.3}$$

and this corresponds to the trivial counit $\varepsilon : U\mathfrak{g} \to \mathbb{K}$,

$$\varepsilon(x) = 0 \quad . \tag{1.0.4}$$

The fact that (1.0.1)/(1.0.3) is well-defined is reflected in the fact that (1.0.2)/(1.0.4) extends to an algebra homomorphism. Furthermore, the associativity/unitality/commutativity of the tensor product is reflected in the coassociativity/counitality/cocommutativity of the coproduct. This example echoes the philosophy of Tannaka-Krein duality; that the symmetric monoidal linear structure on the category ${}_{U\mathfrak{g}}\mathbf{Mod}$ corresponds to the fact that $U\mathfrak{g}$ is a cocommutative bialgebra. On top of this, ${}_{U\mathfrak{g}}\mathbf{Mod}$ admits a closed structure and this is reflected in the fact that $U\mathfrak{g}$ is a Hopf algebra, i.e., there is an antipode $S : U\mathfrak{g} \to U\mathfrak{g}$ simply given by $S(x) = -x$.

A broad class of Lie algebras whose representation theory admit a nice decomposition property

are the semisimple Lie algebras (which includes the Standard Model gauge); their classification was initiated by Wilhelm Killing and rigorously finished by Élie Cartan. The subsequent representation theory was then worked out by Élie Cartan and Hermann Weyl. Another nice property of semisimple Lie algebras $\mathfrak{g}$ is their well-known 'rigidity', i.e., they admit only trivial deformations and thus can be thought of as already maximally general. In contrast to this, $U\mathfrak{g}$ *does* admit nontrivial Hopf algebra deformations meaning that the hitherto representation theory of semisimple Lie algebras on multiparticle/vacuum states (1.0.1)/(1.0.3) has been a special case.

1.1 A brief historical overview of Drinfeld-Jimbo quantum groups

The Yang-Baxter equation (YBE) first arose in the context of the (1+1)-dimensional many-body problem [Yan67] and exactly solvable lattice models [Bax72]. Specifically, given a vector space V, an automorphism $R \in \mathrm{Aut}(V \otimes V)$ is said to be an R-**matrix** if it solves the equation

$$R_{12}R_{23}R_{12} := (R \otimes \mathrm{id})(\mathrm{id} \otimes R)(R \otimes \mathrm{id}) = (\mathrm{id} \otimes R)(R \otimes \mathrm{id})(\mathrm{id} \otimes R) =: R_{23}R_{12}R_{23} \qquad (1.1.1)$$

in $\mathrm{Aut}(V \otimes V \otimes V)$. The equation (1.1.1) expresses a coherency condition for the scattering matrix R to factorise to that of the two-body problem, see [Jim89]. The YBE played a central rôle in the Leningrad school's development of the quantum inverse scattering method [FST79, KS79], a technique for reconstructing the potential from given scattering data of a (1+1)-dimensional quantum integrable system. The fact that the algebraic machinery of the quantum inverse scattering method looked like a deformed symmetry algebra motivated Kulish and Reshetikhin [KR81] to construct an explicit R-matrix acting on higher spin representations of $\mathfrak{sl}(2, \mathbb{C})$.

In parallel, the Knizhnik-Zamolodchikov (KZ) system of complex linear first-order partial differential equations were derived in [KZ84] from the SU(2) Wess–Zumino–Witten model in two-dimensional conformal field theory. In particular, the equations govern the correlation functions of the theory and it was demonstrated that the equations of the system are compatible with one another, i.e., the system is consistent.

Around this time, Drinfeld [Dri85] and Jimbo [Jim85] independently generalised the Kulish-Reshetikhin result and discovered the existence of nontrivial R-matrices on non-cocommutative Hopf algebra deformations $U_q\mathfrak{g}$ of the universal enveloping algebra of a semisimple/simple, respectively, complex Lie algebra $\mathfrak{g}$. In [Dri86, Section 10], Drinfeld abstracted the notion of a **quasitriangular** Hopf algebra $(H, \mu, \eta, \Delta, \varepsilon, S, R)$ as one equipped with an invertible element $R \in H \otimes H$ which controls the potential lack of cocommutativity,

$$\Delta^{\mathrm{op}}(x) = R\Delta(x)R^{-1} \quad , \qquad (1.1.2)$$

and satisfies the **hexagon** conditions:

$$[\Delta \otimes \mathrm{id}_H](R) = R_{13}R_{23} \qquad , \qquad [\mathrm{id}_H \otimes \Delta](R) = R_{13}R_{12} \quad . \qquad (1.1.3)$$

He then notes that the quasitriangular structure R satisfies the YBE (1.1.1) and provides, for H-modules $V, W \in {}_H\mathbf{Mod}$, an H-linear isomorphism from $V \otimes_H W$ to $W \otimes_H V$. For these reasons, it is sometimes called a *universal* R-matrix.

Motivated by the form of the KZ equations (and the consistency of the system), Kohno [Koh87, Subsection 1.1] defined, from the canonical symmetric invariant 2-tensor $t \in (\mathrm{Sym}^2\mathfrak{g})^{\mathfrak{g}}$ of a simple complex Lie algebra $\mathfrak{g}$ and a generic finite-dimensional representation $\rho : \mathfrak{g} \to \mathrm{End}(V)$, a flat connection on a trivial principal G-bundle over the **configuration space of** $n \in \mathbb{N}^*$ **distinguishable**

particles on the complex line $Y_n := \{(z_1, \ldots, z_n) \in \mathbb{C}^n | \text{ if } i \neq j \text{ then } z_i \neq z_j\}$. Such a flat connection is manifestly invariant under pullback by the natural action of S_n on Y_n thus descends to a flat connection over the configuration space of n indistinguishable particles on the complex line $X_n := Y_n/S_n$. This, in turn, provides a monodromy representation $\rho_n^{\mathrm{KZ}} : \pi_1(X_n) \to \mathrm{Aut}(V^{\otimes n})$ which is important because the fundamental group $\pi_1(X_n)$ was already well-known [FN62] to be **Artin's braid group on** n **strands** [Art25], i.e., $\{\sigma_i\}_{1 \leq i \leq n-1}$ such that $\sigma_i \sigma_j = \sigma_j \sigma_i$ if $|i-j| \geq 2$ and $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$. Kohno then showed in [Koh87, Subsection 1.2] that this monodromy representation is equivalent to the one induced by the R-matrices of the aforementioned Drinfeld-Jimbo quantum groups $U_q\mathfrak{g}$ which allows for a concrete description [Koh87, Theorem 1.2.8] via the explicit form in, say, [Jim86].

Given a bialgebra $(A, \mu, \eta, \Delta, \epsilon)$, the category of modules ${}_A\mathbf{Mod}$ is equipped with a canonical monoidal structure where the monoidal product comes from the coproduct Δ and its coassociativity allows one to use the usual associativity constraint from the category of vector spaces. With a view to generalising the associativity constraint, Drinfeld introduced [Dri89, (1.1)-(1.4)] the notion of a **quasibialgebra** $(A, \mu, \eta, \Delta, \varepsilon, \Phi)$ where: (A, μ, η) is a unital associative algebra, $\Delta : A \to A \otimes A$ is an algebra homomorphism, $\varepsilon : A \to \mathbb{K}$ is an algebra homomorphism which acts as a counit for the coproduct Δ and the **Drinfeld associator** $\Phi \in A \otimes A \otimes A$ is an invertible element such that:

$$[\mathrm{id} \otimes \Delta]\big(\Delta(a)\big) = \Phi[\Delta \otimes \mathrm{id}]\big(\Delta(a)\big)\Phi^{-1} \quad , \tag{1.1.4a}$$

$$[\mathrm{id} \otimes \mathrm{id} \otimes \Delta](\Phi)[\Delta \otimes \mathrm{id} \otimes \mathrm{id}](\Phi) = \Phi_{234}[\mathrm{id} \otimes \Delta \otimes \mathrm{id}](\Phi)\Phi_{123} \quad , \tag{1.1.4b}$$

$$[\mathrm{id} \otimes \varepsilon \otimes \mathrm{id}](\Phi) = 1 \quad . \tag{1.1.4c}$$

Even though the notion of a braided monoidal category had not been defined yet, he anticipated the need to define a quasitriangular structure R on a quasibialgebra by weakening the hexagon conditions (1.1.3) as in [Dri89, (3.9)]:

$$[\Delta \otimes \mathrm{id}_H](R) = \Phi_{312} R_{13} \Phi_{132}^{-1} R_{23} \Phi \quad , \quad [\mathrm{id}_H \otimes \Delta](R) = \Phi_{231}^{-1} R_{13} \Phi_{213} R_{12} \Phi^{-1} \quad . \tag{1.1.5}$$

Actually, the relaxed hexagon conditions (1.1.5) allow for the following simple concrete ansatz for a nontrivial R-matrix. Suppose we are given a generic complex Lie algebra $\mathfrak{g}$ and a symmetric invariant 2-tensor $t \in (\mathrm{Sym}^2\mathfrak{g})^{\mathfrak{g}}$, extend the cocommutative Hopf algebra structure of $U\mathfrak{g}$ to $(U\mathfrak{g})[[h]]$ in the trivial way; the ansatz $R = e^{\frac{h}{2}t}$ is clearly invertible in $(U\mathfrak{g})[[h]] \otimes (U\mathfrak{g})[[h]]$ and satisfies (1.1.2). Furthermore, in general, $e^{\frac{h}{2}(t_{13}+t_{23})} \neq e^{\frac{h}{2}t_{13}} e^{\frac{h}{2}t_{23}}$ and $e^{\frac{h}{2}(t_{13}+t_{12})} \neq e^{\frac{h}{2}t_{13}} e^{\frac{h}{2}t_{12}}$. Inspired by Kohno's results, Drinfeld [Dri90, Section 2] constructed an invertible formal power series $\Phi_{\mathrm{KZ}}(\mathfrak{a}, \mathfrak{b})$ via analysis of the KZ differential equations. He then showed that $\Phi := \Phi_{\mathrm{KZ}}(t_{12}, t_{23})$ is indeed a Drinfeld associator, i.e., satisfies (1.1.4) and (1.1.5).

Part of the motivation for Joyal and Street to introduce the concept of braided (strict) monoidal category ([JS91, Section 1 of Chapter 3])[JS93, Definition 2.1] was to formalise the intuition that the modules over a quasitriangular Hopf algebra should form such a category with the braiding being given by $\sigma := \mathrm{flip} \circ R$.

1.2 Derived Algebraic Geometry and shifted Poisson structures

Given the historical context in which quantum groups arose, it is justified and natural to search for a generalization of such structures to higher category theory. Such "higher quantum groups" are then expected to have applications in higher-dimensional: quantum integrable systems, conformal

field theory and topological phases of matter. There have been various attempts to develop a theory of higher quantum groups based on higher Lie algebras such as Lie 2-algebras, reaching from algebraic approaches [Maj12, BSZ13, BV15, CG23, LST23] to more categorical ones [CFM12, CFM15, CFM17]. These attempts are distinct from the approach to higher quantum groups via "categorification" in the sense of [CF94, CR08, Rou08, KL09] where higher quantum groups are constructed based on ordinary Lie algebras.

In this thesis, we initiate a new approach to higher quantum groups which is rooted in recent developments in Derived Algebraic Geometry, in particular the theory of shifted symplectic and shifted Poisson structures (SPS) on derived stacks [PTVV13, CPTVV17, Pri17]. Let us first quickly recall the most popular context for which ordinary Poisson structures (on manifolds or algebraic varieties) arise. Classical physical systems, when described within the Hamiltonian formalism, come equipped with a canonical Poisson bracket $\{\,\cdot\,,\,\cdot\,\} : A \otimes A \to A$ on the algebra $A := C^\infty(M)$ of functions on their phase space[1] M. This Poisson structure plays multiple important rôles: From the point of view of dynamics, the derivation $\{\,\cdot\,, H\} : C^\infty(M) \to C^\infty(M)$ defined by inserting the Hamiltonian $H \in C^\infty(M)$ of the system into the Poisson bracket generates the time evolution of observables $O \in C^\infty(M)$ through Hamilton's equations $\frac{d}{dt}O(t) = \{O(t), H\}$. From the point of view of quantization, the Poisson structure provides an 'initial datum' for the deformation quantization problem which consists of deforming the commutative algebra $C^\infty(M)$ of classical observables into a noncommutative algebra $C^\infty_\hbar(M)$ of quantum observables. We refer the reader to [LGPV13] for a thorough introduction to Poisson structures and to [Bor08] and [Esp15] for introductions to the field of deformation quantization in broad.

The theory of Poisson structures becomes even richer in (differential) graded geometric contexts in which the algebra $A = \bigoplus_{i\in\mathbb{Z}} A^i$ is graded. This is due to the fact that in this case the Poisson bracket $\{\,\cdot\,,\,\cdot\,\} : A \otimes A \to A$ can carry a non-trivial degree, which alters both its geometric and deformation theoretic aspects. Adopting the shifting conventions from the derived algebraic geometry literature [PTVV13, CPTVV17] and [Pri17, Pri18], an nSPS has by definition an underlying Poisson bracket $\{\,\cdot\,,\,\cdot\,\}$ of the opposite degree $-n$, where $n \in \mathbb{Z}$ is any integer. One of the prime examples is given by the antibracket from the Batalin-Vilkovisky (BV) formalism [BV81, Sch93], which according to these conventions is a (-1)SPS. While the quantization of ordinary (i.e. 0-shifted) Poisson structures yields noncommutative algebras, the BV quantization of (-1)SPS yields cochain complexes without any multiplication operation but a deformed differential.

The works [CPTVV17, Pri17] develop a precise definition for the concept of an nSPS on a large class of spaces, called derived Artin stacks. Furthermore, there are general existence theorems for deformation quantizations in the case of positive shifts $n \geq 1$ [CPTVV17] and also in some specific cases of non-positive shifts $n \in \{-2, -1, 0\}$ [Pri18a, Pri18c, Pri19]. Using a simplified language, the general picture which emerges from these works is as follows: An nSPS on a commutative differential graded algebra (CDGA) A is the initial datum for the quantization of A into an $\mathbb{E}_{n+1}$-algebra, or alternatively for the quantization of the symmetric monoidal dg-category ${}_A\mathbf{dgMod}$ of A-dg-modules into an $\mathbb{E}_n$-monoidal dg-category. Here $\mathbb{E}_n$ denotes the little n-disks operad, which encodes the behavior of moving around n-dimensional disks in $\mathbb{R}^n$ without colliding them. Since $\mathbb{E}_n$-operads become increasingly more commutative when increasing n, which loosely speaking results from the additional flexibility in higher-dimensional $\mathbb{R}^n$ to avoid collisions of disks by moving through the new dimensions, this implies that quantizations of nSPS become increasingly more commutative objects for larger n.

The first characterisation results came from Safronov [Saf21] who showed that the semiclassical data associated with ordinary quantum groups, namely quasi-Lie bialgebra structures and invariant symmetric 2-tensors, arise naturally as the 1- and 2SPS on the classifying stack $\mathrm{B}G = [\mathrm{pt}/G]$ of

[1] Recall that $M := T^*Q$ where Q is the configuration space of the physical system.

an algebraic group G, or its infinitesimal analogue given by the formal classifying stack $\mathrm{B}\mathfrak{g} = [\mathrm{pt}/\mathfrak{g}]$ of the Lie algebra $\mathfrak{g}$ of G. Our concrete proposal is to start from an L_∞-algebra $\mathfrak{g}$, which is a higher-categorical generalization of an ordinary Lie algebra, and to interpret the positively $(n \geq 1)$SPS on its formal classifying stack $\mathrm{B}\mathfrak{g} = [\mathrm{pt}/\mathfrak{g}]$ as the semi-classical data of higher quantum groups. The corresponding higher quantum groups can then be defined by the deformation quantization results from [CPTVV17] and they take the form of $\mathbb{E}_n$-monoidal deformations of the symmetric monoidal dg-category $\mathbf{wRep}(\mathfrak{g})$ of weak representations of $\mathfrak{g}$ on cochain complexes.

It is worthwhile to emphasize that, even though such quantizations always exist, they are not easy to work out explicitly since the results in [CPTVV17] are based on formality theorems of the $\mathbb{E}_{n+1}$-operads (for $n \geq 1$), see also [FW20]. Hence, it is a priori not evident how these abstract methods relate to the more explicit deformation techniques for ordinary Lie algebras which are based on the simple but powerful concept of an infinitesimal braiding, controlling the first-order deformation in $\hbar$, and its lift to a full deformation via a choice of Drinfeld associator series $\Phi(\mathfrak{a}, \mathfrak{b})$. In this vein, it was Cartier [Car93] who abstracted from Drinfeld's algebraic deformation quantisation results [Dri90, Section 2] the problem of integrating a symmetric infinitesimal braiding on a symmetric strict monoidal $\mathbb{K}$-linear category to construct a braided monoidal $\mathbb{K}[[h]]$-category with braiding $\sigma := \mathrm{flip} \circ e^{\frac{h}{2}t}$ and associator $\alpha := \Phi(t_{12}, t_{23})$, see [Kas95, Theorem XX.6.1]. Briefly, naturality of the infinitesimal braiding $t : \otimes \Rightarrow \otimes$ guarantees the **four-term relations**:

$$[t_{ij}, t_{ik} + t_{jk}] = 0 = [t_{jk}, t_{ij} + t_{ik}] \qquad , \qquad |\{i, j, k\}| = 3 \quad , \tag{1.2.1}$$

see [Kas95, Lemma XX.4.3]. Such relations (1.2.1) imply flatness of the **KZ connection**,

$$\nabla_{\mathrm{KZ}} := \hbar \sum_{1 \leq i < j \leq n} \left(\frac{\mathrm{d}z_i - \mathrm{d}z_j}{z_i - z_j} \right) t_{ij} \quad , \tag{1.2.2}$$

over the configuration space of n distinguishable particles on the complex line Y_n, see [Kas95, Proposition XIX.2.1]. The Drinfeld KZ series $\Phi_{\mathrm{KZ}}(t_{12}, t_{23})$ is then constructed as the non-singular factor of a parallel transport with respect to ∇_{KZ} along a straightforward path in Y_3, see [BRW23, Theorem 20], and flatness is essential to proving that the pentagon and hexagon axioms are indeed satisfied.

The main goal of this thesis is to display an explicit relationship between the abstract deformation results of [CPTVV17] and (higher) infinitesimal braidings. This is done by adopting a more concrete and computational approach to derived algebraic geometry which is rooted in Pridham's works [Pri17, Pri18]. In particular, we aim to show how 2SPS on semi-free (sf) CDGAs induce such higher infinitesimal braidings.

1.3 Cirio-Martins' infinitesimal 2-braidings as our path to higher quantum groups

The axiomatisation of braided monoidal 2-categories began with the work of Kapranov and Voevodsky [KV94a, KV94b]; the history of the development is an incredibly involved topic and we refer the interested reader to [CSP14, Section 2.1]. For our purposes, we briefly mention that the braiding/associator are *pseudo*natural and the pentagon/hexagon axioms are replaced by pentagonator/hexagonator modifications which satisfy higher coherence conditions.

With the goal of finding the appropriate 2-categorical analogue of the KZ connection, Cirio and Faria Martins (CM) [CFM12, CFM15, CFM17] considered the higher gauge theoretic concept of a 2-connection, see [BH11] for a lucid introduction. Specifically, [CFM12, Theorem 10] finds neces-

sary and sufficient conditions for their, by construction, fake flat KZ 2-connection to be 2-flat and [CFM12, Theorem 13] gives sufficient conditions for the 2-connection to descend to $X_n := Y_n/\mathrm{S}_n$. They then abstracted from these conditions the notion of a "Free infinitesimal 2-Yang-Baxter operator in a differential crossed module" [CFM17, Definition 26]. They reversed the order of this investigation in their latter work [CFM15], i.e., they *began* with the definition of an "infinitesimal 2-braiding" on a symmetric monoidal linear 2-category [CFM15, Definition 14]. Importantly, defining infinitesimal 2-braidings as *pseudo*natural endomorphisms of the monoidal product gave rise to obstructions to the four-term relations and these obstructions were determined in [CFM15, (21)]. They then provided the notions of a "coherent" infinitesimal 2-braiding [CFM15, Definition 17] and "totally symmetric" infinitesimal 2-braiding [CFM15, Definition 18]. The KZ 2-connection they describe in [CFM15, (58)] is, by construction, fake flat. Moreover, [CFM15, Theorem 23] proves that it is 2-flat if the infinitesimal 2-braiding is coherent and [CFM15, Theorem 24] proves that it descends to $X_n := Y_n/\mathrm{S}_n$ if the infinitesimal 2-braiding is totally symmetric.

A few points of comparison are in order:

1. CM centred their study on differential crossed modules (equivalently, 2-term DGLAs) rather than the more general notion of L_∞-algebras.

2. They consider the 2-category of "categorical representations" naturally assigned to every differential crossed module rather than the more extensive dg-category of weak representations.

3. Their "quasi-invariant symmetric tensors" [CFM15, Definition 30] are adapted to such conditions, i.e., they abstract such a definition so that they can say that quasi-invariant symmetric tensors on differential crossed modules induce coherent totally symmetric infinitesimal 2-braidings on the 2-category of categorical representations. In our case, 2SPSs are *not* defined with such pressing desideratum in mind.

Let us focus on the last point. It is already well-known exactly *how* a symmetric invariant 2-tensor (again, a 2SPS on the Chevalley-Eilenberg algebra of an ordinary Lie algebra) equips the symmetric monoidal linear category of representations $\mathbf{Rep}_{\mathbb{K}}\mathfrak{g}$ with an infinitesimal braiding. It is not known *that* a 2SPS on the Chevalley-Eilenberg algebra of an L_∞-algebra equips the symmetric monoidal dg-category of weak representations $\mathbf{wRep}_{\mathbb{K}}\mathfrak{g}$ with an infinitesimal dg-braiding[2], and it is certainly not known how. Actually, we can truncate such dg-categories to 2-categories and the latter point still applies. Furthermore, there is no a priori guarantee that such infinitesimal 2-braidings would possess the same properties as outlined by CM, i.e., there is nothing prohibiting the possibility that they would: fail to satisfy the infinitesimal hexagon relations on the nose, fail to be coherent/totally symmetric, only be 2-natural, etc. Fortune would have it that such abstract threats remain merely that, i.e., our infinitesimal 2-braidings will indeed turn out to be pseudonatural transformations which satisfy the infinitesimal hexagon relations, are coherent and totally symmetric.

As mentioned above, pseudonaturality of the infinitesimal 2-braiding leads to obstructions to the four-term relations and such four-term relations are essential for the pentagon and hexagon axioms to be upheld if we take as ansatz braiding $\sigma := \gamma\, e^{\frac{h}{2}t}$ and ansatz associator $\alpha := \Phi_{\mathrm{KZ}}(t_{12}, t_{23})$. Thus we expect nontrivial hexagonator and pentagonator modifications to appear as we attempt to Cartier integrate the infinitesimal 2-braiding.

[2]Let alone how such infinitesimal dg-braidings would be explicitly defined in terms of data and relations.

1.4 Summary of contents of this thesis

This thesis is based on the joint publication [KLS25], the joint preprint [KLS24] and the solo preprint [Kemp25].

Chapter 2 provides the necessary foundation and basic motivation for the latter chapters which can be summarised as follows: "2SPSs π on the Chevalley-Eilenberg algebra $\mathrm{CE}_{\mathfrak{g}}$ of a finite-dimensional (fd) Lie algebra $\mathfrak{g}$ are characterised as symmetric invariant 2-tensors $(\mathrm{Sym}^2\mathfrak{g})^{\mathfrak{g}}$ and it can be concretely shown exactly how they induce infinitesimal braidings t^{π} on the symmetric monoidal linear category of representations $\mathbf{Rep}_{\mathbb{K}}\mathfrak{g}$". Subsection 2.1.1 recalls the essential notions of braided monoidal categories, braided monoidal functors and monoidal natural transformations. A symmetric braiding is one which is involutive and a symmetric monoidal category can be used as the base for enrichment, Subsection 2.1.2 studies such notions and leads up to the concept of an infinitesimal braiding on a symmetric monoidal linear category, i.e., a Cartier category. Subsection 2.1.3 discusses the fundamental example of $\big(\mathbf{Rep}_{\mathbb{K}}\mathfrak{g}, \pi \in (\mathrm{Sym}^2\mathfrak{g})^{\mathfrak{g}}\big)$. Subsection 2.2.1 provides all the details surrounding the closed symmetric monoidal category of cochain complexes $\mathbf{Ch}$. These details are then made use of to define commutative algebra objects in $\mathbf{Ch}$ and modules over such commutative algebra objects, i.e., CDGAs and dg-modules, respectively. Subsection 2.2.3 recalls the closed symmetric monoidal structure on the category ${}_A\mathbf{dgMod}_0^{\mathrm{fgsf}}$ of sf dg-modules fg in degree 0 over a CDGA A. Our first example of a CDGA is provided by $\mathrm{CE}_{\mathfrak{g}}$ in Example 2.2.6 whereas our first example of a dg-module is provided by the Chevalley-Eilenberg module $\mathrm{CE}_{\mathfrak{g}}^V$ of a representation (V, ρ_V) in Example 2.2.10. Subsection 2.2.4 constructs the equivalence $\mathbf{Rep}_{\mathbb{K}}\mathfrak{g} \simeq {}_A\mathbf{dgMod}_0^{\mathrm{sf}}$ between symmetric monoidal linear categories. Section 2.3 provides the requisite concepts from Derived Algebraic Geometry. In particular, Subsection 2.3.1 defines the essential dg-modules of derivations and Kähler differential forms of a CDGA. The latter are used in Subsection 2.3.2 to define shifted symplectic structures and we arrive at our first characterisation result in Theorem 2.3.24 where it is shown that the only shifted symplectic structures on $\mathrm{CE}_{\mathfrak{g}}$ are the 2-shifted symplectic structures and these are characterised as non-degenerate invariant pairings on $\mathfrak{g}$. The former are used in Subsection 2.3.3 to define SPSs and we reproduce, in Theorem 2.3.40, Safronov's result [Saf21] that the 1 and 2SPSs on $\mathrm{CE}_{\mathfrak{g}}$ are given by quasi-Lie bialgebra structures and invariant symmetric 2-tensors, respectively.

Chapter 3 follows the joint preprint [KLS24] in developing a graphical calculus to characterise, in Corollary 3.1.6, the nSPSs on a generic fgsf CDGA (augmented over $\mathbb{K}$) in terms of families of homotopy-coherent data adjoined to the underlying fd L_∞-algebra $\mathfrak{g}$. Such a graphical calculus also allows us to reproduce the well-known result that fgsf CDGAs are in one-to-one correspondence with fd curved L_∞-algebras $\mathfrak{g}$ and we even provide the explicit Chevalley-Eilenberg differential in Construction 3.2.4. Subsection 3.2.1 applies this graphical calculus to an ordinary Lie algebra $\mathfrak{g}$ and we recover the above mentioned characterisation of Safronov [Saf21]. We then generalise these results in Subsection 3.2.2 to case of a Lie 2-algebra and characterise, for $n \in \{1, 2, 3, 4\}$, the nSPSs. The chapter finishes by making a further specification in Section 3.3 and considering concrete classes of examples of Lie 2-algebras (e.g., shifted cotangent Lie 2-algebras and string Lie 2-algebras) so as to *classify* the 2, 3 and 4SPSs.

Chapter 4 follows the joint publication [KLS25] and demonstrates in Theorem 4.2.12 that every 2SPS π on a fgsfCDGA A induces a γ-equivariant hexagonal infinitesimal 2-braiding t^{π} on the homotopy 2-category of the symmetric monoidal dg-category of fgsf A-dg-modules ${}_A\mathbf{C} := \big(\mathsf{Ho}_2\big({}_A\mathbf{dgMod}^{\mathrm{fgsf}}\big), \otimes_A, A, \gamma\big)$. To contextualise this result, Subsection 4.1.1 recalls our closed symmetric monoidal tricategory of cochain 2-categories, cochain 2-functors, pseudonatural transformations and modifications. Subsection 4.2.1 recalls the fact that we may form such a cochain 2-category of any dg-category via the homotopy 2-truncation 3-functor Ho_2, i.e., truncating the hom complexes to degrees $\{-1, 0\}$ by taking homotopies modulo 2-homotopies. Subsection 4.2.2

starts from taking such a truncation of ${}_A\mathbf{dgMod}^{\mathrm{fgsf}}$. The concrete infinitesimal 2-braiding that we construct, t^π, is composed of two natural transformations and a pseudonatural transformation. To get a more explicit handle, we evaluate the components of t^π on module generators in Remark 4.2.13. Section 4.3 makes use of the characterisations of Propositions 3.2.11 and Remark 3.3.10, i.e., the 2SPS on the Chevalley-Eilenberg algebra of an ordinary Lie algebra and a string Lie 2-algebra, respectively. We then investigate the conditions under which we can Cartier integrate (to all orders) as usual by choosing the braiding as $\sigma := \gamma\, e^{\frac{h}{2}t^\pi}$ and the associator as Drinfeld's KZ series $\alpha = \Phi_{\mathrm{KZ}}(t^\pi_{12}, t^\pi_{23})$. To be clear, because infinitesimal 2-braidings $t : \otimes \Rightarrow \otimes$ are defined as *pseudo*natural transformations, the 4-term relations $[t_{12}, t_{(12)3}] = 0 = [t_{23}, t_{1(23)}]$ are not necessarily satisfied but are obstructed by the homotopies $t_{t_{UV}\boxtimes 1_W}$ of Example 4.2.14 so we look for a full sub monoidal 2-category of ${}_A\mathbf{C}$ upon which such homotopies vanish.

The material in the solo preprint [Kemp25] has been split into Chapters 5 and 6. The former shows in Proposition 5.2.33 that a hexagonal infinitesimal 2-braiding t can be Cartier integrated to second-order if it is coherent and totally γ-equivariant in Cirio and Martins' sense [CFM15, Definitions 17 and 18, respectively]. The relevant notion we require is that of braided strictly-unital monoidal cochain 2-category as in Definition 5.1.3. Construction 5.2.11 builds infinitesimal hexagonators from the 4-term relationator modifications, $t_{t_{12}} : [t_{12}, t_{(12)3}] \Rrightarrow 0$ and $t_{t_{23}} : [t_{23}, t_{1(23)}] \Rrightarrow 0$. We note in Example 5.2.21 that t^π is indeed totally γ-equivariant and Remark 5.2.32 relates coherency to the third-weight component of the Maurer-Cartan equation that π must satisfy. The chapter ends in Section 5.3 by determining the third-order terms of the pentagonator and hexagonators.

Chapter 6 shows in Theorem 6.1.5 that 3SPSs induce γ-equivariant syllepses on ${}_A\mathbf{C}$ whereas Theorem 6.2.3 shows "coboundary" 2SPSs π induce "σ-equivariant infinitesimal coboundary syllepses" T on the first-order deformation of ${}_A\mathbf{C}$ with respect to t^π. Definition 6.1.1 states what we mean by a σ-equivariant syllepsis on a braided strictly-unital monoidal cochain 2-category and Definition 6.1.3 introduces our notion of an infinitesimal syllepsis on a symmetric strict monoidal cochain 2-category. Remark 6.1.4 discusses the curious fact that infinitesimal syllepses can be trivially integrated to a full syllepsis on the same symmetric strict monoidal cochain 2-category. Example 6.1.6 provides specific cases of such syllepses for the shifted cotangent Lie 2-algebra and centrally extended/Heisenberg/Abelian string Lie 2-algebra. Remark 6.2.1 and Definition 6.2.2 introduces our notion of an infinitesimal coboundary syllepsis and coboundary 2SPS, respectively. Example 6.2.4 demonstrates that the 2SPS on the Chevalley-Eilenberg algebra of the string Lie 2-algebra of a simple Lie algebra from Example 3.3.13 is a coboundary 2SPS. Infinitesimal coboundary syllepses admit of a Cartier integration alongside that of infinitesimal 2-braidings and we carry this out to second and third order in Subsections 6.2.2 and 6.2.3, respectively.

Appendix A supplies the necessary well-definition checks for the compositions of pseudonatural transformations and modifications between cochain 2-functors, i.e., that the composites indeed satisfy the requisite relations. The horizontal composition (4.1.30) is purported to be a pseudofunctor hence Subsection A.2.5 demonstrates that a generic component of the pseudo composition constraint, the exchanger, is indeed a modification between the relevant composite pseudonatural transformations and Subsection A.2.1 proves that the family of such components is natural. Subsection A.2.6 shows that the vertical/horizontal composition is indeed functorial/locally functorial, i.e., that the lateral and vertical/horizontal composition of modifications satisfies the exchange law.

1.4.1 Roadmap of results

This subsection presents an explanation of the results of the thesis, the order they will be obtained and the relations between them.

- Subsection 2.1.3 recalls how a quadratic Casimir $t \in (\mathrm{Sym}^2\mathfrak{g})^{\mathfrak{g}}$ induces an infinitesimal braiding on the category $\mathbf{Rep}_{\mathbb{K}}\mathfrak{g}$.

- Subsection 2.2.2 shows Lie algebras $\mathfrak{g}$ correspond to semi-free CDGAs generated in degree 1, i.e. Chevalley-Eilenberg algebras $\mathrm{CE}_{\mathfrak{g}} = \big(\mathrm{Sym}(\mathfrak{g}^*[-1]), \delta\big)$. Subsection 2.2.4 goes further and constructs the equivalence $\mathbf{Rep}_{\mathbb{K}}\mathfrak{g} \simeq {}_{\mathrm{CE}_{\mathfrak{g}}}\mathbf{dgMod}_0^{\mathrm{sf}}$ between symmetric monoidal linear categories.

- Subsection 2.3.3 offers a computational approach to Safronov's result [Saf21] that 1- and 2-shifted Poisson structures on $\mathrm{CE}_{\mathfrak{g}}$ are in one-to-one correspondence with, respectively, quasi-Lie bialgebra structures and quadratic Casimirs whereas n-shifted Poisson structures are trivial for $n \geq 3$.

- Section 3.1 begins from the premise that if $\mathfrak{g}$ is a finite-dimensional cochain complex then $\mathrm{Sym}(\mathfrak{g}^*[-1])$ is a finitely generated free CDGA. We recall from Remark 2.3.35 that a Maurer-Cartan polyvector whose terms all have weight 1 is the exact same thing as a Maurer-Cartan derivation, the latter deforms the differential of a free CDGA A so that it becomes semi-free. We specify this remark to the case of $\mathrm{Sym}(\mathfrak{g}^*[-1])$ and demonstrate that finitely generated semi-free CDGAs are in bijective correspondence with finite-dimensional curved L_∞-algebras. Note that this result allows us to write down our results on semiclassical limits of higher quantum groups in terms of semi-free CDGAs, where all complexity is in the differential and not on the infinite family of maps constituting the L_∞-algebra structure.

- An n-shifted Poisson structure is simply a Maurer-Cartan polyvector whose terms have weight at least 2 hence Corollary 3.1.6 says that a Maurer-Cartan polyvector whose terms have weight at least 1 is nothing more than an n-shifted Poisson structure on a generic finitely generated semi-free CDGA $A \cong \mathrm{CE}_{\mathfrak{g}}$ and we spell out exactly what the n-shifted Poisson structure looks like in terms of adjoined data and relations on the underlying curved L_∞-algebra $\mathfrak{g}$. This generalises Safronov's result because it characterises the shifted Poisson structures on a generic (finitely generated) semi-free CDGA in terms of adjoined structure to the underlying (finite-dimensional) curved L_∞-algebra thus providing semiclassical data of higher quantum groups, with such data arising naturally from geometric considerations.

- From Chapter 4 onward, we model the representations of a curved L_∞-algebra via the symmetric monoidal dg-category ${}_A\mathbf{dgMod}^{\mathrm{sf}}$, where A is a generic semi-free CDGA. The prospect of braided monoidal dg-categories is well beyond the scope of this thesis (in fact, braided monoidal 3-categories would be as well) thus we restrict to a 2-category of representations $\mathsf{Ho}_2({}_A\mathbf{dgMod}^{\mathrm{sf}})$ by truncating the hom complexes so they are concentrated in degrees $\{-1, 0\}$, i.e. considering semi-free dg-modules over A, A-linear cochain maps and A-linear homotopies up to 2-homotopies.

- Working with 2-categories means that one can consider braidings which are pseudonatural (rather than just 2-natural) thus we allow for the possibility that our infinitesimal 2-braiding will be such as well. The two different compositions of pseudonatural transformations does not satisfy the interchange law thus we are forced to consider, in Subsection 4.1.1, a tricategorical universe $\mathbf{dgCat}^{[-1,0],\mathrm{ps}}$ of $\mathbf{Ch}^{[-1,0]}$-categories, $\mathbf{Ch}^{[-1,0]}$-functors, pseudonatural transformations and modifications.

- The objective of Subsection 4.2.2 is to generalise the result of Subsection 2.1.3 and show how a 2-shifted Poisson structure on a semi-free CDGA A induces an infinitesimal 2-braiding on $\mathsf{Ho}_2({}_A\mathbf{dgMod}^{\mathrm{sf}})$.

- As in Theorem 4.2.12, more specifically (4.2.31b), such a concrete infinitesimal 2-braiding is nontrivially pseudonatural and this complicates the usual integration story which Cartier formalised. To be specific, naturality of an infinitesimal braiding implies that the four-term relations are satisfied whereas pseudonaturality of an infinitesimal 2-braiding implies they are not satisfied on the nose but only up to 'four-term relationator' modifications as in Example 4.2.14. For instance, let us flesh out Remark 4.3.7: given a finite-dimensional simple Lie algebra $\mathfrak{h}$ with chosen basis $\{x_a \in \mathfrak{h}\}$, we denote the structure constants as $[x_a, x_b] = f^c_{ab} x_c$. The Chevalley-Eilenberg algebra of the string Lie 2-algebra admits generators $\{s\theta^a\}$ and $s\nu$ of degree 1 and 2, respectively. Every 2-shifted Poisson structure takes the form $\pi = \tilde{\pi}^a_b s\theta^b \, s^3\frac{\partial}{\partial(s\theta^a)} s^3\frac{\partial}{\partial(s\nu)}$ where $\tilde{\pi}^a_b \in \mathbb{K}$ and $f^d_{bc}\,\tilde{\pi}^a_d - f^a_{bd}\,\tilde{\pi}^d_c + f^a_{cd}\,\tilde{\pi}^d_b = 0$. Considering semi-free dg-modules which model strict representations on finite-dimensional graded vector spaces, the differential on the module generators takes the form $\mathrm{d}_V(v_i) = V^j_{ai} s\theta^a \, v_j + V^j_i s\nu \, v_j$ where $V^j_{ai} \in \mathbb{K}$ is only nonzero for $|v_j| = |v_i|$ whereas $V^j_i \in \mathbb{K}$ is only nonzero for $|v_j| = |v_i| - 1$. The cochain components of the infinitesimal 2-braiding on module generators looks like

$$t_{UV}(u_\alpha \otimes v_i) \sim \tilde{\pi}^a_b s\theta^b (U^\beta_{a\alpha} V^j_i + U^\beta_\alpha V^j_{ai})\, u_\beta \otimes v_j \tag{1.4.1}$$

  and the 'four-term relationator' modifications look like

$$t_{t_{UV}\boxtimes \mathrm{id}_W}(u_\alpha \otimes v_i \otimes w_m) \sim \tilde{\pi}^a_b \tilde{\pi}^b_c s\theta^c (U^\beta_{a\alpha} V^j_i + U^\beta_\alpha V^j_{ai}) W^n_m \, u_\beta \otimes v_j \otimes w_n \quad . \tag{1.4.2}$$

- Anticipating that such 'four-term relationator' modifications will obstruct the hexagon and pentagon axioms from holding at higher orders in the deformation parameter $\hbar$, Section 5.1 specifies the definition of a braided monoidal bicategory to our context of $\mathbf{dgCat}^{[-1,0],\mathrm{ps}}$.

- Section 5.2 shows the pentagon axiom is satisfied at second order in the deformation parameter $\hbar$ whereas the hexagon axioms are obstructed by combinations of the 'four-term relationator' modifications and such obstructions satisfy the requisite axioms if the infinitesimal 2-braiding is totally symmetric and coherent [CFM15].

- Chapter 6 is all about syllepses and how to construct them from 2- and 3-shifted Poisson structures. First we specify the definition of a sylleptic monoidal category to our context of $\mathbf{dgCat}^{[-1,0],\mathrm{ps}}$, i.e. we introduce the notion of a sylleptic braided monoidal $\mathbf{Ch}^{[-1,0]}$-category at the start of Section 6.1.

- We then analyse the details of this definition in the special case where the syllepsis is on a symmetric strict monoidal $\mathbf{Ch}^{[-1,0]}$-category and introduce the idea of an infinitesimal syllepsis (on a symmetric strict monoidal $\mathbf{Ch}^{[-1,0]}$-category).

- Whereas the integration of infinitesimal braidings is nontrivial, that of infinitesimal syllepses is trivial and Subsection 6.1.1 shows how a 3-shifted Poisson structure on a semi-free CDGA A induces a syllepsis on the symmetric strict monoidal $\mathbf{Ch}^{[-1,0]}$-category $\mathsf{Ho}_2({}_A\mathbf{dgMod}^{\mathrm{sf}})$.

- Section 6.2 explores syllepses on the first- and second-order braided monoidal deformations with respect to an infinitesimal 2-braiding (i.e. Sections 4.2 and 5.2). Subsection 6.2.1 starts by introducing the definition of an infinitesimal coboundary syllepsis and shows how coboundary 2-shifted Poisson structures induce them on $\mathsf{Ho}_2({}_A\mathbf{dgMod}^{\mathrm{sf}})$. Subsection 6.2.2 shows that integrating the infinitesimal coboundary syllepsis to second-order requires making use of the fact that the underlying infinitesimal 2-braiding is coherent and totally symmetric.

Chapter 2

Preliminaries

2.1 Cartier categories

2.1.1 Braided monoidal categories

In this subsection we recall some important definitions and lemmas appearing in [Kas95, Chapter XI] and [Kas95, Chapter XIII] regarding braided monoidal[1] categories and their properties. In particular, the definition below is the same as that of [Kas95, Definition XI.2.1].

Definition 2.1.1. A **monoidal category** $\mathcal{C} = (\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ consists of six pieces of data:

(i) A category $\mathcal{C}$.

(ii) **Monoidal product** functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$. Functoriality means that: for $U \xrightarrow{f} V \xrightarrow{g} W$ and $U' \xrightarrow{f'} V' \xrightarrow{g'} W'$, the **interchange law** is upheld, i.e.,

$$(g \otimes g')(f \otimes f') = (gf) \otimes (g'f') : U\,U' \to W\,W' \tag{2.1.1a}$$

and, for every pair of objects and their identities, $U \overset{1_U}{=} U$ and $U' \overset{1_{U'}}{=} U'$, we have

$$1_U \otimes 1_{U'} = 1_{U\,U'} \quad . \tag{2.1.1b}$$

(iii) A functor $u : \mathbf{1} \to \mathcal{C}$ which maps the single object of $\mathbf{1}$ to the **monoidal unit** $I \in \mathcal{C}$.

(iv) A natural isomorphism $\alpha : \otimes\,(\otimes \times \mathrm{id}_{\mathcal{C}}) \xRightarrow{\sim} \otimes\,(\mathrm{id}_{\mathcal{C}} \times \otimes) : \mathcal{C} \times \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ called the **associator**. Explicitly, for every triple $U, V, W \in \mathcal{C}$, we have an isomorphism $(UV)W \xrightarrow{\alpha_{UVW}} U(VW)$ in $\mathcal{C}$ such that, for every triple of arrows $U \xrightarrow{f} U'$, $V \xrightarrow{g} V'$ and $W \xrightarrow{h} W'$,

$$\begin{array}{ccc} (UV)W & \xrightarrow{\alpha_{UVW}} & U(VW) \\ {\scriptstyle (f\otimes g)\otimes h}\downarrow & & \downarrow{\scriptstyle f\otimes(g\otimes h)} \\ (U'V')W' & \xrightarrow[\alpha_{U'V'W'}]{} & U'(V'W') \end{array} \tag{2.1.2}$$

(v) A natural isomorphism $\lambda : \otimes\,(u \times \mathrm{id}_{\mathcal{C}}) \xRightarrow{\sim} \mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ called the **left unitor**. Explicitly,

[1] Note that Kassel uses the adjective "tensor" where we use the adjective "monoidal".

for every $U \in \mathcal{C}$, we have an isomorphism $IU \xrightarrow{\lambda_U} U$ such that, for every arrow $U \xrightarrow{f} U'$,

$$\begin{array}{ccc} IU & \xrightarrow{\lambda_U} & U \\ {\scriptstyle 1_I \otimes f}\downarrow & & \downarrow{\scriptstyle f} \\ IU' & \xrightarrow[\lambda_{U'}]{} & U' \end{array} \tag{2.1.3}$$

(vi) A natural isomorphism $\rho : \otimes\,(\mathrm{id}_{\mathcal{C}} \times u) \xRightarrow{\sim} \mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ called the **right unitor**. Explicitly, for every $U \in \mathcal{C}$ we have an isomorphism $UI \xrightarrow{\rho_U} U$ such that, for every arrow $U \xrightarrow{f} U'$,

$$\begin{array}{ccc} UI & \xrightarrow{\rho_U} & U \\ {\scriptstyle f \otimes 1_I}\downarrow & & \downarrow{\scriptstyle f} \\ U'I & \xrightarrow[\rho_{U'}]{} & U' \end{array} \tag{2.1.4}$$

This data is required to satisfy the following two axioms:

1. **Pentagon**, i.e., for every quadruple of objects $U, V, W, X \in \mathcal{C}$,

$$\begin{array}{ccccc} & & \big((UV)W\big)X & & \\ & {\scriptstyle \alpha_{UVW}\otimes 1_X}\swarrow & & \searrow{\scriptstyle \alpha_{(UV)WX}} & \\ \big(U(VW)\big)X & & & & (UV)(WX) \\ {\scriptstyle \alpha_{U(VW)X}}\downarrow & & & & \downarrow{\scriptstyle \alpha_{UV(WX)}} \\ U\big((VW)X\big) & & \xrightarrow[1_U \otimes \alpha_{VWX}]{} & & U\big(V(WX)\big) \end{array} \tag{2.1.5}$$

2. **Triangle**, i.e., for every pair of objects $U, V \in \mathcal{C}$,

$$\begin{array}{ccccc} (UI)V & & \xrightarrow{\alpha_{UIV}} & & U(IV) \\ & {\scriptstyle \rho_U \otimes 1_V}\searrow & & \swarrow{\scriptstyle 1_U \otimes \lambda_V} & \\ & & UV & & \end{array} \tag{2.1.6}$$

If the associator and unitors are identities then we say $(\mathcal{C}, \otimes, I)$ is a **strict monoidal category**.

As always in Category Theory, objects are best understood by appreciating the structure-preserving maps between them hence, in order to understand monoidal categories, we recall the definition of lax monoidal functors between them as in [Kas95, Definition XI.4.1].

Definition 2.1.2. Let $\mathcal{C} = (\mathcal{C}, \otimes_{\mathcal{C}}, I_{\mathcal{C}}, \alpha^{\mathcal{C}}, \lambda^{\mathcal{C}}, \rho^{\mathcal{C}})$ and $\mathcal{D} = (\mathcal{D}, \otimes_{\mathcal{D}}, I_{\mathcal{D}}, \alpha^{\mathcal{D}}, \lambda^{\mathcal{D}}, \rho^{\mathcal{D}})$ be monoidal categories, a **lax monoidal functor** $F = (F, F^2, F^0) : \mathcal{C} \to \mathcal{D}$ consists of three pieces of data:

(i) A functor $F : \mathcal{C} \to \mathcal{D}$

(ii) A natural transformation called the **lax product constraint**,

$$\begin{array}{ccc} \mathcal{C}\times\mathcal{C} & \xrightarrow{\otimes_{\mathcal{C}}} & \mathcal{C} \\ {\scriptstyle F\times F}\downarrow & \overset{F^2}{\Rightarrow} & \downarrow{\scriptstyle F} \\ \mathcal{D}\times\mathcal{D} & \xrightarrow[\otimes_{\mathcal{D}}]{} & \mathcal{D} \end{array} \tag{2.1.7a}$$

Explicitly, for every pair of objects $U, V \in \mathcal{C}$, we have a morphism in $\mathcal{D}$,

$$F^2_{UV} : F(U)F(V) \to F(UV) \quad , \tag{2.1.7b}$$

such that, for every pair of arrows $U \xrightarrow{f} U'$ and $V \xrightarrow{g} V'$,

$$\begin{array}{ccc} F(U)F(V) & \xrightarrow{F^2_{UV}} & F(UV) \\ {\scriptstyle F(f)\otimes_{\mathcal{D}}F(g)}\downarrow & & \downarrow{\scriptstyle F(f\otimes_{\mathcal{C}}g)} \\ F(U')F(V') & \xrightarrow[F^2_{U'V'}]{} & F(U'V') \end{array} \tag{2.1.7c}$$

(iii) A natural transformation $F^0 : u_{\mathcal{D}} \to F\, u_{\mathcal{C}} : \mathbf{1} \to \mathcal{D}$, called the **lax unity constraint**, i.e.,

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ & {\scriptstyle u_{\mathcal{C}}}\nwarrow \quad \overset{F^0}{\Rightarrow} \quad \nearrow{\scriptstyle u_{\mathcal{D}}} & \\ & \mathbf{1} & \end{array} \tag{2.1.8}$$

Explicitly, a morphism $F^0 : I_{\mathcal{D}} \to F(I_{\mathcal{C}})$ in $\mathcal{D}$.

This data is required to satisfy the following two axioms:

1. **Lax associativity**, i.e., for every triple of objects $U, V, W \in \mathcal{C}$,

$$\begin{array}{ccc} \big(F(U)F(V)\big)F(W) & \xrightarrow{\alpha^{\mathcal{D}}_{F(U)F(V)F(W)}} & F(U)\big(F(V)F(W)\big) \\ {\scriptstyle F^2_{UV}\otimes_{\mathcal{D}}1_{F(W)}}\downarrow & & \downarrow{\scriptstyle 1_{F(U)}\otimes_{\mathcal{D}}F^2_{VW}} \\ F(UV)F(W) & & F(U)F(VW) \\ {\scriptstyle F^2_{(UV)W}}\downarrow & & \downarrow{\scriptstyle F^2_{U(VW)}} \\ F\big((UV)W\big) & \xrightarrow[F(\alpha^{\mathcal{C}}_{UVW})]{} & F\big(U(VW)\big) \end{array} \tag{2.1.9}$$

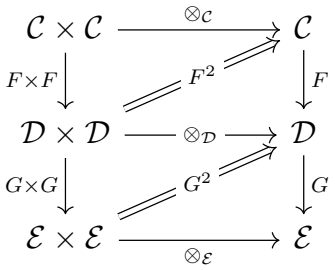
2. **Lax left and right unitality**, i.e., for every object $U \in \mathcal{C}$,

$$\begin{array}{ccc} I_{\mathcal{D}}F(U) & \xrightarrow{\lambda_{F(U)}} & F(U) \\ {\scriptstyle F^0\otimes_{\mathcal{D}}1_{F(U)}}\downarrow & & \uparrow{\scriptstyle F(\lambda_U)} \\ F(I_{\mathcal{C}})F(U) & \xrightarrow[F^2_{I_{\mathcal{C}}U}]{} & F(I_{\mathcal{C}}U) \end{array} \qquad \begin{array}{ccc} F(U)I_{\mathcal{D}} & \xrightarrow{\rho_{F(U)}} & F(U) \\ {\scriptstyle 1_{F(U)}\otimes_{\mathcal{D}}F^0}\downarrow & & \uparrow{\scriptstyle F(\rho_U)} \\ F(U)F(I_{\mathcal{C}}) & \xrightarrow[F^2_{UI_{\mathcal{C}}}]{} & F(UI_{\mathcal{C}}) \end{array} \tag{2.1.10}$$

An **oplax monoidal functor** instead has natural transformations $F^2 : F \otimes_{\mathcal{C}} \to \otimes_{\mathcal{D}} (F \times F)$, called **oplax product constraint**, and $F^0 : F\, u_{\mathcal{C}} \to u_{\mathcal{D}}$, called **oplax unity constraint**, which satisfy the above diagrams but with the direction of the arrow components of F^2 and F^0 reversed (whence we call such diagrams the **oplax associativity** and **oplax unitality**, respectively). A lax monoidal functor, (F, F^2, F^0), is simply called a **monoidal functor** if the lax product constraint, F^2, and the lax unity constraint, F^0, are both natural isomorphisms in which case we call them the **strong product constraint** and the **strong unity constraint**, respectively. A **strict monoidal functor** F is a lax monoidal functor such that the lax product constraint is an identity (i.e., $F^2 = \mathrm{Id}_{F \otimes_{\mathcal{C}}}$) and the lax unity constraint is an identity (i.e., $F^0 = \mathrm{Id}_{u_{\mathcal{D}}}$).

The process of understanding monoidal categories by appreciating monoidal functors can only take place by prescribing a well-defined composition to such structure-preserving maps.

Lemma 2.1.3. *Given the monoidal functors $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}$, we define the **composite lax product constraint** via the following pasting diagram*



(2.1.11a)

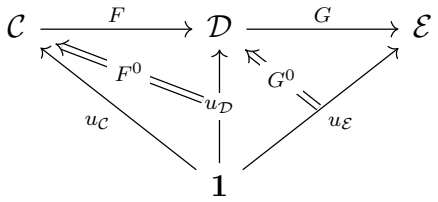
Explicitly, the natural transformation

$$(G\,F)^2 := (\mathrm{Id}_G * F^2) \circ (G^2 * \mathrm{Id}_{F \times F}) : \otimes_{\mathcal{E}} \big((G\,F) \times (G\,F)\big) \to (G\,F) \otimes_{\mathcal{C}} : \mathcal{C} \times \mathcal{C} \to \mathcal{E} \tag{2.1.11b}$$

is defined on every pair of objects $U, V \in \mathcal{C}$ by the following morphism in $\mathcal{E}$,

$$(G\,F)^2_{UV} := G\big(F^2_{UV}\big) G^2_{F(U)F(V)} : G\big(F(U)\big) \otimes_{\mathcal{E}} G\big(F(V)\big) \to G\big(F(UV)\big) \quad . \tag{2.1.11c}$$

*The **composite lax unity constraint** is defined via the following pasting diagram*



(2.1.12a)

Explicitly, the natural transformation

$$(G\,F)^0 := (\mathrm{Id}_G * F^0) \circ G^0 : u_{\mathcal{E}} \to (G\,F)\, u_{\mathcal{C}} : \mathbf{1} \to \mathcal{E} \tag{2.1.12b}$$

is defined by the following morphism in $\mathcal{E}$,

$$(G\,F)^0 := G(F^0) * G^0 : I_{\mathcal{E}} \to G\big(F(I_{\mathcal{C}})\big) \quad . \tag{2.1.12c}$$

*The **composite lax monoidal functor** $G\,F := \big(G\,F, (G\,F)^2, (G\,F)^0\big) : \mathcal{C} \to \mathcal{E}$ is indeed a lax monoidal functor from $\mathcal{C}$ to $\mathcal{E}$. Furthermore, if G and F are both (strict) monoidal functors then so too is $G\,F$. Likewise, two oplax monoidal functors can be composed.*

Proof. See [JY21, Lemma 4.1.29]. □

Clearly, the triple $\mathrm{id}_{\mathcal{C}} := (\mathrm{id}_{\mathcal{C}}, \mathrm{Id}_{\otimes_{\mathcal{C}}}, \mathrm{Id}_{u_{\mathcal{C}}})$ is a strict monoidal functor and serves as the identity on $\mathcal{C}$ for the above composition law. Furthermore, the composition law is also associative as demonstrated in [JY21, Theorem 4.1.30]. In sum, there exists a category **MonCat** of monoidal categories and lax monoidal functors between them with composition as above. Actually, there exists a 2-categorical structure on **MonCat** and the relevant 2-morphisms are described below.

Definition 2.1.4. Let $F = (F, F^2, F^0) : \mathcal{C} \to \mathcal{D}$ and $G = (G, G^2, G^0) : \mathcal{C} \to \mathcal{D}$ be lax monoidal functors. A **monoidal natural transformation** $\eta : F \to G : \mathcal{C} \to \mathcal{D}$ is a natural transformation which further satisfies the following two properties:

1. For every pair of objects $U, V \in \mathcal{C}$,

$$\begin{array}{ccc} F(U)F(V) & \xrightarrow{F^2_{UV}} & F(UV) \\ {\scriptstyle \eta_U \otimes_{\mathcal{D}} \eta_V}\downarrow & & \downarrow{\scriptstyle \eta_{UV}} \\ G(U)G(V) & \xrightarrow[G^2_{UV}]{} & G(UV) \end{array} \tag{2.1.13}$$

2. The following diagram in $\mathcal{D}$ commutes,

$$\begin{array}{ccc} & & F(I_{\mathcal{C}}) \\ & \overset{F^0}{\nearrow} & \downarrow{\scriptstyle \eta_{I_{\mathcal{C}}}} \\ I_{\mathcal{D}} & & \\ & \underset{G^0}{\searrow} & \\ & & G(I_{\mathcal{C}}) \end{array} \tag{2.1.14}$$

If either F or G (or both) were oplax then the notion of a monoidal natural transformation $\eta : F \to G$ would be the same as above but with appropriate reversal of arrows. A **monoidal natural isomorphism** $\eta : F \xRightarrow{\sim} G : \mathcal{C} \to \mathcal{D}$ is a monoidal natural transformation which is a natural isomorphism.

The fact that a monoidal natural transformation requires no further data and only requires that the natural transformation satisfies the above two properties means that the horizontal $*$ and vertical $\circ$ composition of these 2-morphisms is the same as in the 2-category **Cat** of categories, functors and natural transformations. The vertical composition is clearly well-defined as can be seen by stacking the above commutative diagrams whereas the well-definition of the horizontal composition requires a bit more work.

Definition 2.1.5. A monoidal functor $(F, F^2, F^0) : \mathcal{C} \to \mathcal{D}$ is said to be a **monoidal equivalence** if there exists another monoidal functor $(G, G^2, G^0) : \mathcal{D} \to \mathcal{C}$ together with monoidal natural isomorphisms $\eta : \mathrm{id}_{\mathcal{C}} \xRightarrow{\sim} G\,F$ and $\varepsilon : F\,G \xRightarrow{\sim} \mathrm{id}_{\mathcal{D}}$.

We denote such a monoidal equivalence as $(F, F^2, F^0) : \mathcal{C} \xrightarrow{\sim} \mathcal{D}$.

Remark 2.1.6. It is clear that a monoidal equivalence $(F, F^2, F^0) : \mathcal{C} \xrightarrow{\sim} \mathcal{D}$ provides an equivalence between the underlying categories $F : \mathcal{C} \xrightarrow{\sim} \mathcal{D}$ but what is surprising is that the converse holds. In other words, equipping an equivalence of categories $F : \mathcal{C} \xrightarrow{\sim} \mathcal{D}$ with a monoidal structure (F^2, F^0) induces a monoidal structure on the weak inverse $G : \mathcal{D} \to \mathcal{C}$ and the unit η and counit ε of the equivalence happen to respect this induced monoidal structure on the composites

$F\,G$ and $G\,F$, i.e., η and ε are accidentally monoidal natural isomorphisms. Specifically, for every pair of objects $X, Y \in \mathcal{D}$, the components of the strong[2] product constraint is defined by the following commutative diagram of isomorphisms in $\mathcal{C}$

$$\begin{array}{ccc} G\big(F[G(X)]F[G(Y)]\big) & \xleftarrow{G(F^2_{G(X)G(Y)})^{-1}} & G\big(F[G(X)G(Y)]\big) \\ {\scriptstyle G(\varepsilon_X \otimes_{\mathcal{D}} \varepsilon_Y)}\downarrow & & \uparrow{\scriptstyle \eta_{G(X)G(Y)}} \\ G(XY) & \xleftarrow[G^2_{XY}]{} & G(X)G(Y) \end{array} \tag{2.1.15}$$

The strong unity constraint is defined as

$$\begin{array}{ccc} I_{\mathcal{C}} & \xrightarrow{\eta_{\mathcal{I}_{\mathcal{C}}}} & G(F(I_{\mathcal{C}})) \\ & {\scriptstyle G^0}\searrow \quad \swarrow{\scriptstyle G(F^0)^{-1}} & \\ & G(I_{\mathcal{D}}) & \end{array} \tag{2.1.16}$$

A straightforward though tedious exercise in diagram chasing shows this data to satisfy the associativity/unitality axioms and that the unit $\eta : \mathrm{id}_{\mathcal{C}} \xRightarrow{\sim} G\,F$ and counit $\varepsilon : F\,G \xRightarrow{\sim} \mathrm{id}_{\mathcal{D}}$ are monoidal natural isomorphisms. △

In addition, recall the well-known fact that a functor is an equivalence of categories if and only if it is an essentially surjective fully faithful functor, see [Kas95, Proposition XI.1.5]. Putting both these remarks together, we get the following characterisation of a monoidal equivalence.

Proposition 2.1.7. *Given the monoidal categories $\mathcal{C}$ and $\mathcal{D}$, a monoidal equivalence is the same thing as an essentially surjective fully faithful monoidal functor $(F, F^2, F^0) : \mathcal{C} \xrightarrow{\sim} \mathcal{D}$.*

Remark 2.1.8. The pentagon axiom and triangle axiom of Definition 2.1.1 are sometimes referred to as **coherence conditions** and the reason is as follows. Ab extra those axioms, there are clearly multiple inequivalent paths[3] between objects and the pentagon/triangle diagrams are the two simplest examples. However, Mac Lane's coherence theorem [Mac98, Chapter VII.2] shows that if the pentagon and triangle axioms hold then, if two objects can be joined by a variety of paths of the type previously mentioned then they are all equal. An alternative way to understand this result is through Mac Lane's strictification theorem, see [Mac98, Chapter XI.3] and [Kas95, Chapter XI.5]; this tells us that given a monoidal category $\mathcal{C}$ we can define from it a strict monoidal category $\mathcal{C}^{\mathrm{str}}$ and construct a monoidal equivalence between them. The strict monoidal category $\mathcal{C}^{\mathrm{str}}$ has different objects from that of $\mathcal{C}$, Schauenburg's refined strictification theorem regarding monoidal categories of structured sets [Sch01, Theorem 4.3] avoids this problem. As in [Sch01, Definition 4.1], we say $\mathcal{C}$ is a **category of structured sets** if there exists a faithful functor $\mathcal{U} : \mathcal{C} \to \mathbf{Set}$ such that:

(i) Given $V \in \mathcal{C}$, $S \in \mathbf{Set}$ and $f : \mathcal{U}(V) \cong S$, there exists $W \in \mathcal{C}$ with $\mathcal{U}(W) = S$ and $g : V \cong W$ with $\mathcal{U}(g) = f$.

(ii) Given $g : V \cong W$ then $\mathcal{U}(g) = \mathrm{id}_{\mathcal{U}(V)}$ implies $g = \mathrm{id}_V$.

[2]Note that if F was equipped with a merely *lax* product/unity constraint then we would only be able to put an *oplax* product/unity constraint on G but this would not necessarily satisfy the oplax associativity/unitality axioms. It is this subtle fact that motivates restricting the definition of a monoidal equivalence (as in Definition 2.1.5) to the scope of monoidal functors. Furthermore, the composition of a lax monoidal functor with an oplax monoidal functor, or vice versa, is not well-definable.

[3]Those paths being made of the associator α, unitors λ/ρ and monoidal product $\otimes$ of the components of those natural isomorphisms with themselves and/or identity arrows.

Schauenburg's result [Sch01, Theorem 4.3] tells us that any monoidal structure on such a category of structured sets $\mathcal{C}$ is monoidally equivalent to a strict monoidal structure with the same monoidal unit. All the specific categories (e.g., **Vec**, $\mathbf{Rep}_{\mathbb{K}}\mathfrak{g}$, ${}_A\mathbf{dgMod}$ and $\mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}$) that we will be interested in this thesis will be categories of structured sets hence we assume their monoidal structure is strict though we relax this assumption in Chapter 5 and Subsection 6.2.2 onward. △

Definition 2.1.9. A **braided monoidal category** $\mathcal{C} = (\mathcal{C}, \otimes, I, \alpha, \lambda, \rho, \sigma)$ consists of the following two pieces of data:

(i) A monoidal category $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$.

(ii) A natural isomorphism $\sigma : \otimes \xRightarrow{\sim} \otimes\tau : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ called the **braiding**. Explicitly, for $U, V \in \mathcal{C}$, we have an isomorphism $UV \xrightarrow{\sigma_{UV}} VU$ such that, for $U \xrightarrow{f} U'$ and $V \xrightarrow{g} V'$,

$$\begin{array}{ccc} UV & \xrightarrow{\sigma_{UV}} & VU \\ {\scriptstyle f\otimes g}\downarrow & & \downarrow{\scriptstyle g\otimes f} \\ U'V' & \xrightarrow[\sigma_{U'V'}]{} & V'U' \end{array} \tag{2.1.17}$$

These two pieces of data are required to satisfy the following two axioms:

1. **Left hexagon**, i.e., for every triple of objects $U, V, W \in \mathcal{C}$,

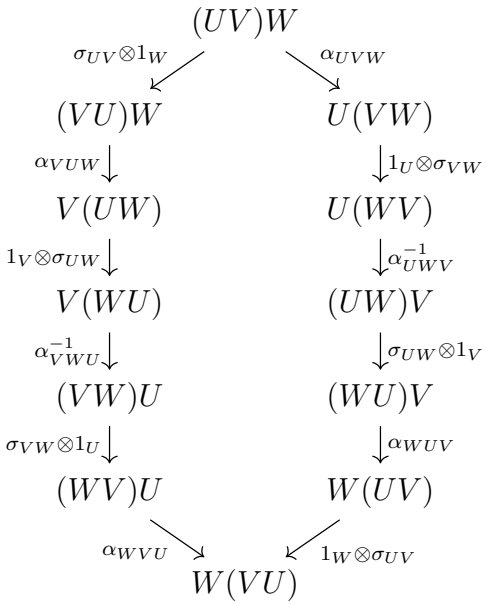
$$\begin{array}{ccccc} & & U(VW) & \xrightarrow{\sigma_{U(VW)}} & (VW)U & & \\ & \nearrow{\scriptstyle \alpha_{UVW}} & & & & \searrow{\scriptstyle \alpha_{VWU}} & \\ (UV)W & & & & & & V(WU) \\ & \searrow{\scriptstyle \sigma_{UV}\otimes 1_W} & & & & \nearrow{\scriptstyle 1_V\otimes\sigma_{UW}} & \\ & & (VU)W & \xrightarrow[\alpha_{VUW}]{} & V(UW) & & \end{array} \tag{2.1.18a}$$

2. **Right hexagon**, i.e., for every triple of objects $U, V, W \in \mathcal{C}$,

$$\begin{array}{ccccc} & & (UV)W & \xrightarrow{\sigma_{(UV)W}} & W(UV) & & \\ & \nearrow{\scriptstyle \alpha^{-1}_{UVW}} & & & & \searrow{\scriptstyle \alpha^{-1}_{WUV}} & \\ U(VW) & & & & & & (WU)V \\ & \searrow{\scriptstyle 1_U\otimes\sigma_{VW}} & & & & \nearrow{\scriptstyle \sigma_{UW}\otimes 1_V} & \\ & & U(WV) & \xrightarrow[\alpha^{-1}_{UWV}]{} & (UW)V & & \end{array} \tag{2.1.18b}$$

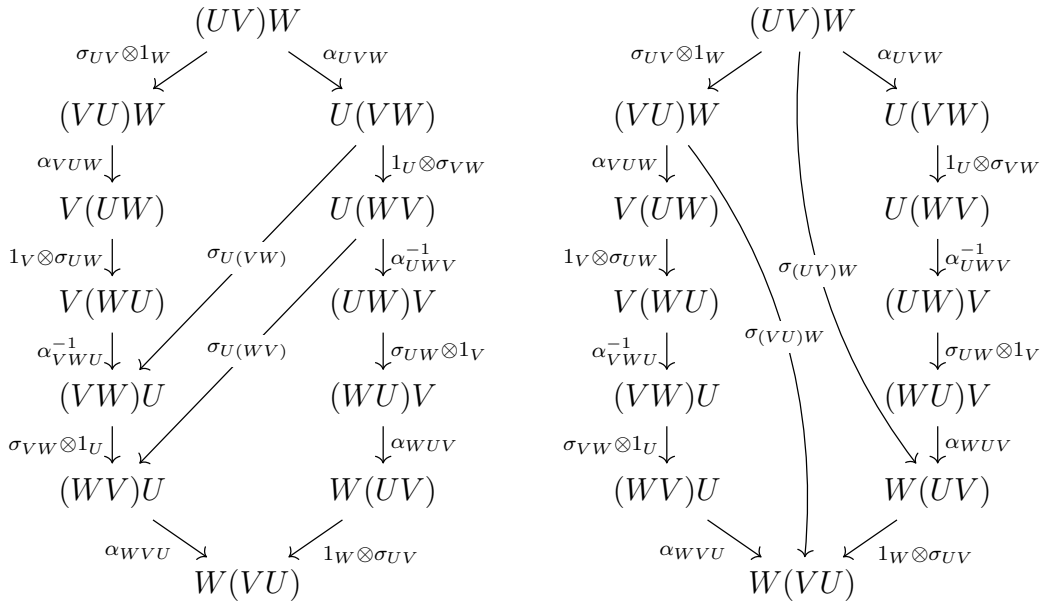
Braided monoidal categories satisfy what is called the **categorical Yang-Baxter equation** as exemplified by the following Lemma.

Lemma 2.1.10. *Given a triple of objects of a braided monoidal category $U, V, W \in \mathcal{C}$,*



(2.1.19)

Proof. Let us give *both* methods of proving this diagram to commute, the first method exploits the left hexagon axiom whereas the second method makes use of the right hexagon axiom. Each method divides the above dodecagon into two (left/right) hexagons and a naturality square, respectively, as follows



It should be obvious that the naturality squares are instances of (2.1.17); the left naturality square has $f = 1_U$ and $g = \sigma_{VW}$ whereas the right naturality square has $f = \sigma_{UV}$ and $g = 1_W$. □

Corollary 2.1.11. *Every object $V \in \mathcal{C}$ of a braided strict monoidal category $\mathcal{C} = (\mathcal{C}, \otimes, I, \sigma)$ is such that the automorphism $\sigma_{VV} \in \mathrm{Aut}(VV)$ satisfies the Yang-Baxter equation,*

$$(\sigma_{VV} \otimes 1_V)(1_V \otimes \sigma_{VV})(\sigma_{VV} \otimes 1_V) = (1_V \otimes \sigma_{VV})(\sigma_{VV} \otimes 1_V)(1_V \otimes \sigma_{VV}) \quad . \tag{2.1.20}$$

Remark 2.1.12. The reason why we have illustrated both methods of proof for Lemma 2.1.10 is because it will be relevant when discussing braided monoidal 2-categories as in Definition 5.1.3. In short, the braiding and associator of a braided monoidal 2-category are pseudonatural transformations and are no longer required to satisfy the pentagon/hexagon axioms but rather a braided

monoidal structure also incorporates the *data* of pentagonator/hexagonator modifications which fill such pentagon/hexagon diagrams. The fact that the braiding is a *pseudo*natural transformation means that the naturality squares (2.1.17) get similarly replaced by filler data. Taken together, one must then fill the above commutative subdiagrams of the above two methods of proof with the relevant 2-morphism data, it is then an *axiom* of a braided monoidal 2-category that the above two fillings are the same. Such an axiom is called the Breen polytope as in [CSP14, (SM5.a)=(SM5.b), Appendix C.2] or (5.1.13). △

Definition 2.1.13. Let $\mathcal{C} = (\mathcal{C}, \otimes_{\mathcal{C}}, I_{\mathcal{C}}, \alpha^{\mathcal{C}}, \lambda^{\mathcal{C}}, \rho^{\mathcal{C}}, \sigma^{\mathcal{C}})$ and $\mathcal{D} = (\mathcal{D}, \otimes_{\mathcal{D}}, I_{\mathcal{D}}, \alpha^{\mathcal{D}}, \lambda^{\mathcal{D}}, \rho^{\mathcal{D}}, \sigma^{\mathcal{D}})$ be braided monoidal categories, a **braided lax monoidal functor** $F = (F, F^2, F^0) : \mathcal{C} \to \mathcal{D}$ is a lax monoidal functor F which **respects the braiding**, i.e., for every pair of objects $U, V \in \mathcal{C}$,

$$\begin{array}{ccc} F(U)F(V) & \xrightarrow{F^2_{UV}} & F(UV) \\ {\scriptstyle \sigma^{\mathcal{D}}_{F(U)F(V)}}\downarrow & & \downarrow{\scriptstyle F(\sigma^{\mathcal{C}}_{UV})} \\ F(V)F(U) & \xrightarrow[F^2_{VU}]{} & F(VU) \end{array} \tag{2.1.21}$$

Because a braided lax monoidal functor is merely a lax monoidal functor which satisfies further properties, and requires no additional data, their composition is the same as that of lax monoidal functors and it is easy to check that such composition is still well-defined. Furthermore, the strict monoidal functor $\mathrm{id}_{\mathcal{C}} := (\mathrm{id}_{\mathcal{C}}, \mathrm{Id}_{\otimes_{\mathcal{C}}}, \mathrm{Id}_{u_{\mathcal{C}}})$ clearly respects the braiding. Better yet, a braided monoidal natural transformation is merely a monoidal natural transformation between braided monoidal categories hence we have another 2-category **BraMonCat** of braided monoidal categories, braided lax monoidal functors and monoidal natural transformations.

Definition 2.1.14. For $n \in \mathbb{N}^*$, the **braid group on** n **strands** B_n is the group with the $n-1$ generators $\{\sigma_i \in \mathrm{B}_n\}_{1 \leq i \leq n-1}$ subject to the following relations:

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad , \qquad 1 \leq i \leq n-1 \ , \tag{2.1.22}$$

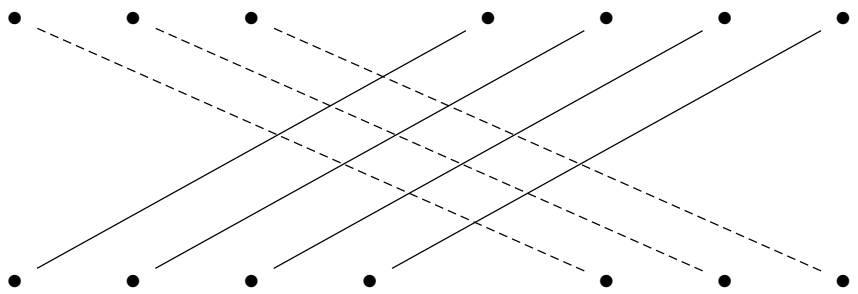
$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad , \qquad |i-j| \neq 1 \ . \tag{2.1.23}$$

We set $\mathrm{B}_1 = \{e\}$. The **braid category** $\mathcal{B}$ has objects given by the natural numbers $n, m \in \mathbb{N}$ with the hom-sets given by:

$$\mathcal{B}(n,m) := \begin{cases} \mathrm{id} \ , & n = m = 0 \\ \mathrm{B}_n \ , & n = m \in \mathbb{N}^* \\ \varnothing \ , & n \neq m \end{cases} \tag{2.1.24}$$

and the composition given by the group product of B_n. This category admits a monoidal structure given by **juxtaposition of braids**, i.e., $n \square m = n + m$ and, e.g., if $1 \leq i \leq n$ and $1 \leq j \leq m$ then $\sigma_i \square \sigma_j = \sigma_i \sigma_{j+n}$. This assignment extends to a well-defined functor and it is clear that this monoidal product is strictly associative and strictly unital with respect to the monoidal unit 0. Furthermore, there exists a well-defined braiding c on the strict monoidal category $\mathcal{B}$ given by

the crossing of strands $c_{n,m} : n + m \to m + n$ as in [Kas95, Chapter XIII.2], e.g.,



In sum, $(\mathcal{B}, \square, 0, c) \in \mathbf{BraMonCat}$.

Remark 2.1.15. Contrary to Remark 2.1.8, if two objects in a braided monoidal category admit a morphism made of the basic ingredients $\otimes, \alpha, \lambda, \rho, \sigma$ then there is no guarantee that there is only *one* such path. For example: 1_{VW}, $\sigma_{WV}\sigma_{VW}$, $\sigma_{WV}\sigma_{VW}\sigma_{WV}\sigma_{VW}$, etc. are all clearly inequivalent paths. Nevertheless these paths are all elements of the braid group on 2 strands $\mathrm{B}_2 \cong \mathbb{Z}$. In other words, there exists a weaker notion of "braided coherence" as in [Mac98, Chapter XI.5] or [Kas95, Theorem XIII.3.7]. Specifically, for every braided monoidal category $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho, \sigma)$, the functor

$$\mathrm{ev}_1 : \mathbf{BraMonCat}\left[(\mathcal{B}, \square, 0, c), (\mathcal{C}, \otimes, I, \alpha, \lambda, \rho, \sigma)\right] \to \mathcal{C} \qquad , \qquad F \mapsto F(1) \ , \tag{2.1.25}$$

is an equivalence of categories. △

Definition 2.1.16. A **symmetric monoidal category** $\mathcal{C} = (\mathcal{C}, \otimes, I, \alpha, \lambda, \rho, \gamma)$ is a braided monoidal category such that the braiding γ is **involutive**, i.e., for every pair of objects $V, W \in \mathcal{C}$,

$$\gamma_{VW} = \gamma_{WV}^{-1} \quad . \tag{2.1.26}$$

Note that if the natural transformation $\gamma : \otimes \Rightarrow \otimes\tau : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ satisfies (2.1.26) then the left hexagon axiom (2.1.18a) is equivalent to the right hexagon axiom (2.1.18b) hence we could instead define a symmetric braiding on a monoidal category as a natural involution which satisfies the left hexagon axiom (2.1.18a).

Remark 2.1.17. In contrast to Remark 2.1.15 and in conformity with Remark 2.1.8, a symmetric monoidal category is such that there is only *one* morphism between objects if it can be made with the basic ingredients. △

Definition 2.1.18. A **closed symmetric monoidal category** is a symmetric monoidal category $\mathcal{C} = (\mathcal{C}, \otimes, I, \alpha, \lambda, \rho, \gamma)$ such that, for every object $V \in \mathcal{C}$, the monoidal product functor $(\cdot)V : \mathcal{C} \to \mathcal{C}$ admits a right adjoint internal hom functor $(\cdot)^V : \mathcal{C} \to \mathcal{C}$. In other words, the following isomorphism

$$\mathcal{C}(UV, W) \cong \mathcal{C}(U, W^V) \tag{2.1.27}$$

is natural in $U, W \in \mathcal{C}$.

Example 2.1.19. The category of vector spaces and linear maps between them **Vec** is closed symmetric monoidal with: the monoidal product given by the tensor product $\otimes_{\mathbb{K}}$, the monoidal unit given by the ground field $\mathbb{K}$, the symmetric braiding given by the swap and the inner homs given by the standard vector space structure on the set of linear maps. ▽

2.1.2 Elements of Enriched Category Theory

In this subsection, we recall the basic notions of enriched categories/functors/natural transformations as found in, say, [Rie14, Chapter 3]. To this effect, we fix a base for enrichment given by the symmetric monoidal category $\mathcal{V} = (\mathcal{V}, \boxtimes, \mathbb{1}, a, l, r, c)$.

Definition 2.1.20. A category enriched over $\mathcal{V}$ or, in short, a **$\mathcal{V}$-category** $\underline{\mathcal{C}} = (\mathcal{C}, \circ, \mathrm{id})$ is given by the following four pieces of data:

(i) A class of objects $U, V, W \in \mathcal{C}$.

(ii) For each pair of objects $U, V \in \mathcal{C}$, a **$\mathcal{V}$-hom**, i.e., $\underline{\mathcal{C}}[U, V] \in \mathcal{V}$.

(iii) For each triple of objects $U, V, W \in \mathcal{C}$, a **composition $\mathcal{V}$-morphism**

$$\circ : \underline{\mathcal{C}}[V, W]\,\underline{\mathcal{C}}[U, V] \to \underline{\mathcal{C}}[U, W] \quad . \tag{2.1.28}$$

(iv) For each object $U \in \mathcal{C}$, an **identity $\mathcal{V}$-morphism**

$$\mathrm{id}_U : \mathbb{1} \to \underline{\mathcal{C}}[U, U] \quad . \tag{2.1.29}$$

These four pieces of data are subject to the following two axioms:

1. **Associativity of** $\circ$, i.e., for every quadruple of objects $U, V, W, X \in \mathcal{C}$,

$$\begin{array}{ccc} \left(\underline{\mathcal{C}}[W, X]\,\underline{\mathcal{C}}[V, W]\right)\underline{\mathcal{C}}[U, V] & \xrightarrow{a_{\underline{\mathcal{C}}[W,X]\underline{\mathcal{C}}[V,W]\underline{\mathcal{C}}[U,V]}} & \underline{\mathcal{C}}[W, X]\left(\underline{\mathcal{C}}[V, W]\,\underline{\mathcal{C}}[U, V]\right) \\ \downarrow{\scriptstyle \circ \boxtimes 1_{\underline{\mathcal{C}}[U,V]}} & & \downarrow{\scriptstyle 1_{\underline{\mathcal{C}}[W,X]} \boxtimes \circ} \\ \underline{\mathcal{C}}[V, X]\,\underline{\mathcal{C}}[U, V] & & \underline{\mathcal{C}}[W, X]\,\underline{\mathcal{C}}[U, W] \\ & {\scriptstyle \circ}\searrow \quad \swarrow{\scriptstyle \circ} & \\ & \underline{\mathcal{C}}[U, X] & \end{array} \tag{2.1.30}$$

2. **Unitality of identities**, i.e., for every pair of objects $U, V \in \mathcal{C}$,

$$\begin{array}{ccc} \mathbb{1}\,\underline{\mathcal{C}}[U, V] & \xrightarrow{c_{\mathbb{1}\underline{\mathcal{C}}[U,V]}} & \underline{\mathcal{C}}[U, V]\,\mathbb{1} \\ \downarrow{\scriptstyle \mathrm{id}_V \boxtimes 1_{\underline{\mathcal{C}}[U,V]}} & \searrow^{l_{\underline{\mathcal{C}}[U,V]}} \quad {}^{r_{\underline{\mathcal{C}}[U,V]}}\swarrow & \downarrow{\scriptstyle 1_{\underline{\mathcal{C}}[U,V]} \boxtimes \mathrm{id}_U} \\ \underline{\mathcal{C}}[V, V]\,\underline{\mathcal{C}}[U, V] & & \underline{\mathcal{C}}[U, V]\,\underline{\mathcal{C}}[U, U] \\ & {\scriptstyle \circ}\searrow \quad \swarrow{\scriptstyle \circ} & \\ & \underline{\mathcal{C}}[U, V] & \end{array} \tag{2.1.31}$$

   where the central triangle already commutes as in [Kas95, Proposition XIII.1.2], i.e., for every object $x \in \mathcal{V}$ we already have $r_x c_{\mathbb{1}x} = l_x$.

Example 2.1.21. A **linear category** is a **Vec**-category. In a linear category, we can linearly add morphisms and the composition is bilinear. ▽

Construction 2.1.22. Borrowing the notation of Definition 2.1.18, every closed symmetric monoidal category is canonically enriched over itself. For every object $U \in \mathcal{C}$, let us denote the unit and counit of the adjunction $(\cdot)U \dashv (\cdot)^U$ by $\eta^U : \mathrm{id}_{\mathcal{C}} \to \big((\cdot)U\big)^U$ and $\epsilon^U : (\cdot)^U U \to \mathrm{id}_{\mathcal{C}}$, respectively, then the structure of enrichment is provided as follows:

(i) The objects are those of $\mathcal{C}$ itself.

(ii) The $\mathcal{C}$-homs are simply given by the internal homs $\underline{\mathcal{C}}[V,W] := W^V \in \mathcal{C}$.

(iii) The composition $\mathcal{C}$-morphisms are given by the adjunct under $(\cdot)U \dashv (\cdot)^U$ to the composite

$$\underline{\mathcal{C}}[V,W](\underline{\mathcal{C}}[U,V]\,U) \xrightarrow{1_{\underline{\mathcal{C}}[V,W]} \otimes \epsilon_V^U} \underline{\mathcal{C}}[V,W]\,V \xrightarrow{\epsilon_W^V} W \quad . \tag{2.1.32}$$

Specifically, the composition is given by

$$\circ := \left(\epsilon_W^V(1_{\underline{\mathcal{C}}[V,W]} \otimes \epsilon_V^U)\right)^U \eta^U_{\underline{\mathcal{C}}[V,W]\underline{\mathcal{C}}[U,V]} : \underline{\mathcal{C}}[V,W]\,\underline{\mathcal{C}}[U,V] \to \underline{\mathcal{C}}[U,W] \quad , \tag{2.1.33}$$

and it can be checked that this composition is associative (2.1.30).

(iv) The identity $\mathcal{C}$-morphisms are given by the adjunct under $(\cdot)U \dashv (\cdot)^U$ to the left unitor component

$$\lambda_U : IU \cong U \quad , \tag{2.1.34}$$

or, specifically,

$$\mathrm{id}_U := \lambda_U^U \eta_I^U : I \to \underline{\mathcal{C}}[U,U] \quad , \tag{2.1.35}$$

and, again, it can be checked that this satisfies the unitality axiom (2.1.31).

▷

Definition 2.1.23. Let $\underline{\mathcal{C}}$ and $\underline{\mathcal{D}}$ be two $\mathcal{V}$-categories, a **$\mathcal{V}$-functor** $F : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ is given by the following two pieces of data:

(i) A map on objects; for every object $U \in \mathcal{C}$, an object $F(U) \in \mathcal{D}$.

(ii) For every pair of objects $U, V \in \mathcal{C}$, a $\mathcal{V}$-morphism between $\mathcal{V}$-homs,

$$F_{U,V} : \underline{\mathcal{C}}[U,V] \to \underline{\mathcal{D}}[F(U),F(V)] \quad . \tag{2.1.36}$$

These two pieces of data are subject to the following two axioms:

1. **Preservation of** $\circ$, i.e., for every triple of objects $U, V, W \in \mathcal{C}$,

$$\begin{array}{ccc} \underline{\mathcal{C}}[V,W]\,\underline{\mathcal{C}}[U,V] & \xrightarrow{\quad\circ\quad} & \underline{\mathcal{C}}[U,W] \\ \downarrow{\scriptstyle F_{V,W}\boxtimes F_{U,V}} & & \downarrow{\scriptstyle F_{U,W}} \\ \underline{\mathcal{D}}[F(V),F(W)]\,\underline{\mathcal{D}}[F(U),F(V)] & \xrightarrow{\quad\circ\quad} & \underline{\mathcal{D}}[F(U),F(W)] \end{array} \tag{2.1.37}$$

2. **Preservation of identities**, i.e., for every object $U \in \mathcal{C}$,

$$\begin{array}{ccc} \mathbb{1} & \xrightarrow{\mathrm{id}_U} & \underline{\mathcal{C}}[U,U] \\ & {\scriptstyle \mathrm{id}_{F(U)}}\searrow & \downarrow{\scriptstyle F_{U,U}} \\ & & \underline{\mathcal{D}}[F(U),F(U)] \end{array} \tag{2.1.38}$$

We say F is an **$\mathcal{V}$-equivalence** if it is essentially surjective and, for every pair of objects $U, V \in \mathcal{C}$, (2.1.36) is a $\mathcal{V}$-isomorphism.

Example 2.1.24. A **linear functor** is a **Vec**-functor, i.e., it is linear between hom spaces. ▽

Remark 2.1.25. The composition of $\mathcal{V}$-functors is the same as that of ordinary functors; the local $\mathcal{V}$-morphism between homs is simply given by $(G\,F)_{U,V} := G_{F(U),F(V)}F_{U,V}$. This is well-defined as can be seen by making use of the interchange law that $\boxtimes$ satisfies, i.e.,

$$\begin{aligned}(G_{F(V),F(W)} \boxtimes G_{F(U),F(V)})(F_{V,W} \boxtimes F_{U,V}) &= (G_{F(V),F(W)}F_{V,W}) \boxtimes (G_{F(U),F(V)}F_{U,V}) \\ &= (G\,F)_{V,W} \boxtimes (G\,F)_{U,V} \qquad .\end{aligned}$$

Composition of $\mathcal{V}$-functors is thus associative. The identity $\mathcal{V}$-functors $\mathrm{id}_{\underline{\mathcal{C}}}$ are simply given by identity on objects and identity $\mathcal{V}$-morphisms between homs which is also well-defined by virtue of the fact that $1_{\underline{\mathcal{C}}[V,W]} \boxtimes 1_{\underline{\mathcal{C}}[U,V]} = 1_{\underline{\mathcal{C}}[V,W]\,\underline{\mathcal{C}}[U,V]}$. These identity $\mathcal{V}$-functors are obviously unital with respect to the aforementioned composition. △

Definition 2.1.26. Given a pair of $\mathcal{V}$-functors $F, G : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$, a **$\mathcal{V}$-natural transformation** $\xi : F \Rightarrow G : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ is given by the following datum:

(i) For every object $U \in \mathcal{C}$, a $\mathcal{V}$-morphism

$$\xi_U : \mathbb{1} \to \underline{\mathcal{D}}[F(U), G(U)] \quad . \tag{2.1.40}$$

This datum is subject to the single axiom of **$\mathcal{V}$-naturality**, i.e., for every pair of objects $U, V \in \mathcal{C}$,

$$\begin{array}{ccccc}
\underline{\mathcal{D}}[F(U),F(V)] & \xleftarrow{F_{U,V}} & \underline{\mathcal{C}}[U,V] & \xrightarrow{G_{U,V}} & \underline{\mathcal{D}}[G(U),G(V)] \\
{\scriptstyle l^{-1}_{\underline{\mathcal{D}}[F(U),F(V)]}}\downarrow & & & & \downarrow{\scriptstyle r^{-1}_{\underline{\mathcal{D}}[G(U),G(V)]}} \\
\mathbb{1}\,\underline{\mathcal{D}}[F(U),F(V)] & & & & \underline{\mathcal{D}}[G(U),G(V)]\,\mathbb{1} \\
{\scriptstyle \xi_V \boxtimes 1_{\underline{\mathcal{D}}[F(U),F(V)]}}\downarrow & & & & \downarrow{\scriptstyle 1_{\underline{\mathcal{D}}[G(U),G(V)]} \boxtimes \xi_U} \\
\underline{\mathcal{D}}[F(V),G(V)]\,\underline{\mathcal{D}}[F(U),F(V)] & \xrightarrow{\circ} & \underline{\mathcal{D}}[F(U),G(V)] & \xleftarrow{\circ} & \underline{\mathcal{D}}[G(U),G(V)]\,\underline{\mathcal{D}}[F(U),G(U)]
\end{array} \tag{2.1.41}$$

Given a $\mathcal{V}$-functor $F : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$, the family of particular identity $\mathcal{V}$-morphisms $\{\mathrm{id}_{F(U)}\}_{U\in\mathcal{C}}$ obviously satisfy the $\mathcal{V}$-naturality condition (2.1.41) simply by using the unitality of identities (2.1.31) thus we denote such a $\mathcal{V}$-natural transformation as $\mathrm{Id}_F : F \Rightarrow F : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$.

Example 2.1.27. A **Vec**-natural transformation is simply a natural transformation between linear functors. ▽

Remark 2.1.28. The vertical composition of $\mathcal{V}$-natural transformations $F \overset{\xi}{\Rightarrow} G \overset{\eta}{\Rightarrow} H : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$, i.e., $\eta\xi : F \Rightarrow H$ is defined as follows

$$\begin{array}{ccc}
\mathbb{1} & \xrightarrow{(\eta\xi)_U} & \underline{\mathcal{D}}[F(U),H(U)] \\
{\scriptstyle l^{-1}_{\mathbb{1}} = r^{-1}_{\mathbb{1}}}\downarrow & & \uparrow{\scriptstyle \circ} \\
\mathbb{1}\,\mathbb{1} & \xrightarrow[\eta_U \boxtimes \xi_U]{} & \underline{\mathcal{D}}[G(U),H(U)]\,\underline{\mathcal{D}}[F(U),G(U)]
\end{array} \tag{2.1.42}$$

where the equality $l^{-1}_{\mathbb{1}} = r^{-1}_{\mathbb{1}}$ follows from Mac Lane's coherence theorem but is specifically shown, for example, in [Kas95, Lemma XI.2.3]. Likewise, the horizontal composition of $F \overset{\xi}{\Rightarrow} G : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ with $F' \overset{\upsilon}{\Rightarrow} G' : \underline{\mathcal{D}} \to \underline{\mathcal{E}}$, i.e., $F'F \overset{\upsilon * \xi}{\Longrightarrow} G'G : \underline{\mathcal{C}} \to \underline{\mathcal{E}}$ is defined as in (2.1.42) but now with $\upsilon_{G(U)} \boxtimes F'_{F(U),G(U)}(\xi_U)$ replacing the bottom arrow. An exercise in diagram chasing shows that such vertical/horizontal composition is: well-defined, associative, unital with respect to $\mathrm{Id}_F/\mathrm{Id}_{\mathrm{id}_{\underline{\mathcal{C}}}}$, satisfies the exchange law and preserves identities. △

Definition 2.1.29. We say $\xi : F \Rightarrow G$ is a **$\mathcal{V}$-natural isomorphism** if there exists a $\mathcal{V}$-natural transformation $\eta : G \Rightarrow F$ such that $\eta\xi = \mathrm{Id}_F$ and $\xi\eta = \mathrm{Id}_G$.

Putting Remarks 2.1.25 and 2.1.28 together, we obtain that $\mathcal{V}\mathbf{Cat}$ is a 2-category. In fact, it has a symmetric monoidal structure as exemplified by the following construction.

Construction 2.1.30. For $\mathcal{V}$-categories $\underline{\mathcal{C}}$ and $\underline{\mathcal{D}}$, the product $\underline{\mathcal{C}} \boxtimes \underline{\mathcal{D}}$ is the $\mathcal{V}$-category whose:

(i) Objects are denoted by juxtapositions UV where $U \in \underline{\mathcal{C}}$ and $V \in \underline{\mathcal{D}}$.

(ii) $\mathcal{V}$-homs are given by the obvious ones, i.e., $\big(\underline{\mathcal{C}} \boxtimes \underline{\mathcal{D}}\big)[UV, U'V'] := \underline{\mathcal{C}}[U, U']\,\underline{\mathcal{D}}[V, V']$.

(iii) Composition $\mathcal{V}$-morphisms are defined by the following commutative diagram in $\mathcal{V}$,

$$\begin{array}{ccc} (\underline{\mathcal{C}} \boxtimes \underline{\mathcal{D}})[U'V', U''V''](\underline{\mathcal{C}} \boxtimes \underline{\mathcal{D}})[UV, U'V'] & \xrightarrow{\circ_{\underline{\mathcal{C}}\boxtimes\underline{\mathcal{D}}}} & \underline{\mathcal{C}}[U, U'']\,\underline{\mathcal{D}}[V, V''] \\ \| & & \uparrow{\scriptstyle \circ_{\underline{\mathcal{C}}}\boxtimes\circ_{\underline{\mathcal{D}}}} \\ (\underline{\mathcal{C}}[U', U'']\underline{\mathcal{D}}[V', V''])(\underline{\mathcal{C}}[U, U']\underline{\mathcal{D}}[V, V']) & \xrightarrow[\cong]{} & (\underline{\mathcal{C}}[U', U'']\underline{\mathcal{C}}[U, U'])(\underline{\mathcal{D}}[V', V'']\underline{\mathcal{D}}[V, V']) \end{array} \tag{2.1.43}$$

where the bottom isomorphism is a combination of the associator a and braiding c of $\mathcal{V}$.

(iv) Identity $\mathcal{V}$-morphisms defined by the following commutative diagram in $\mathcal{V}$,

$$\begin{array}{ccc} \mathbb{1} & \xrightarrow{\mathrm{id}_{UV}} & (\underline{\mathcal{C}} \boxtimes \underline{\mathcal{D}})[UV, UV] \\ {\scriptstyle l_{\mathbb{1}}^{-1}=r_{\mathbb{1}}^{-1}}\downarrow & & \| \\ \mathbb{1}\mathbb{1} & \xrightarrow[\mathrm{id}_U\boxtimes\mathrm{id}_V]{} & \underline{\mathcal{C}}[U, U]\underline{\mathcal{D}}[V, V] \end{array} \tag{2.1.44}$$

For $\mathcal{V}$-functors $F : \underline{\mathcal{C}} \to \underline{\mathcal{C}}'$ and $G : \underline{\mathcal{D}} \to \underline{\mathcal{D}}'$, we define $F \boxtimes G$ as:

(i) Object map simply given by $UV \mapsto F(U)G(V)$.

(ii) Local $\mathcal{V}$-morphism between $\mathcal{V}$-homs given by

$$\begin{array}{ccc} (\underline{\mathcal{C}} \boxtimes \underline{\mathcal{D}})[UV, U'V'] & \xrightarrow{(F\boxtimes G)_{UV,U'V'}} & (\underline{\mathcal{C}}' \boxtimes \underline{\mathcal{D}}')[F(U)G(V), F(U')G(V')] \\ \| & & \| \\ \underline{\mathcal{C}}[U, U']\underline{\mathcal{D}}[V, V'] & \xrightarrow[F_{U,U'}\boxtimes G_{V,V'}]{} & \underline{\mathcal{C}}'[F(U), F(U')]\underline{\mathcal{D}}'[G(V), G(V')] \end{array} \tag{2.1.45}$$

For $\mathcal{V}$-natural transformations $\xi : F \Rightarrow F'$ and $\psi : G \Rightarrow G'$, we define $\xi \boxtimes \psi$ simply as

$$\begin{array}{ccc} \mathbb{1} & \xrightarrow{(\xi\boxtimes\psi)_{UV}} & (\underline{\mathcal{C}}' \boxtimes \underline{\mathcal{D}}')[F(U)G(V), F'(U)G'(V)] \\ {\scriptstyle l_{\mathbb{1}}^{-1}=r_{\mathbb{1}}^{-1}}\downarrow & & \| \\ \mathbb{1}\mathbb{1} & \xrightarrow[\xi_U\boxtimes\psi_V]{} & \underline{\mathcal{C}}'[F(U), F'(U)]\underline{\mathcal{D}}'[G(V), G'(V)] \end{array} \tag{2.1.46}$$

We define the monoidal unit $\underline{\mathbb{1}}$ as the $\mathcal{V}$-category with a single object and $\mathbb{1}$ as its enriched hom, the composition is simply given by $l_{\mathbb{1}} = r_{\mathbb{1}}$ and the identity is the equality $\mathbb{1} = \mathbb{1}$. The associator, unitors and symmetric braiding are simply borrowed from the symmetric monoidal structure of $\mathcal{V}$, using the latter the above data can be shown to be well-defined. ▷

This symmetric monoidal structure on $\mathcal{V}\mathbf{Cat}$ allows us to define a **braided monoidal $\mathcal{V}$-category** $\underline{\mathcal{C}} = (\underline{\mathcal{C}}, \otimes, I, \alpha, \lambda, \rho, \sigma)$ to consist of the same data and axioms of a braided monoidal category as in Definition 2.1.9 but now in the context of Enriched Category Theory, e.g., the monoidal product $\otimes : \underline{\mathcal{C}} \boxtimes \underline{\mathcal{C}} \to \underline{\mathcal{C}}$ and monoidal unit[4] $u : \underline{\mathbb{1}} \to \underline{\mathcal{C}}$ are $\mathcal{V}$-functors, the associator is a $\mathcal{V}$-natural isomorphism $\alpha : \otimes(\otimes \boxtimes \mathrm{id}_{\underline{\mathcal{C}}}) \Rightarrow \otimes(\mathrm{id}_{\underline{\mathcal{C}}} \boxtimes \otimes) a_{\underline{\mathcal{C}},\underline{\mathcal{C}},\underline{\mathcal{C}}} : (\underline{\mathcal{C}} \boxtimes \underline{\mathcal{C}}) \boxtimes \underline{\mathcal{C}} \to \underline{\mathcal{C}}$ and the composition in the pentagon axiom (2.1.5) makes use of the workaround (2.1.42). We say the braiding $\sigma : \otimes \Rightarrow \otimes c_{\underline{\mathcal{C}},\underline{\mathcal{C}}}$ is **symmetric** if it satisfies $(\sigma * \mathrm{Id}_{c_{\underline{\mathcal{C}},\underline{\mathcal{C}}}})\sigma = \mathrm{Id}_{\otimes}$.

Example 2.1.31. Example 2.1.27 told us that a linear natural transformation is nothing more nor less than a natural transformation between linear functors hence their horizontal and vertical composition does not require the workaround (2.1.42). Because of this, a **symmetric monoidal linear category** is simply a symmetric monoidal category as in Definition 2.1.16 where the composition and monoidal product are linear with respect to the vector space structure on the homs. ▽

Construction 2.1.32. A symmetric lax monoidal functor $\mathcal{F} : \mathcal{V} \to \mathcal{W}$ induces a **change of base 2-functor** $\mathcal{F}_* : \mathcal{V}\mathbf{Cat} \to \mathcal{W}\mathbf{Cat}$ as follows:

(i) Given $\underline{\mathcal{C}} \in \mathcal{V}\mathbf{Cat}$, define $\mathcal{F}_*(\underline{\mathcal{C}})$ as having the same objects but with the $\mathcal{W}$-homs given by $\mathcal{F}_*(\underline{\mathcal{C}})[U,V] = \mathcal{F}\big(\underline{\mathcal{C}}[U,V]\big) \in \mathcal{W}$ and the composition $\mathcal{W}$-morphisms defined via the lax product constraint $\mathcal{F}^2$ as follows

$$\begin{array}{ccc} \mathcal{F}\big(\underline{\mathcal{C}}[V,W]\big) \boxtimes^{\mathcal{W}} \mathcal{F}\big(\underline{\mathcal{C}}[U,V]\big) & \overset{\circ}{\dashrightarrow} & \mathcal{F}_*(\underline{\mathcal{C}})[U,W] \\ {\scriptstyle \mathcal{F}^2_{\underline{\mathcal{C}}[V,W]\underline{\mathcal{C}}[U,V]}}\downarrow & & \vdots \\ \mathcal{F}\big(\underline{\mathcal{C}}[V,W] \boxtimes^{\mathcal{V}} \underline{\mathcal{C}}[U,V]\big) & \xrightarrow[\mathcal{F}(\circ)]{} & \mathcal{F}\big(\underline{\mathcal{C}}[U,W]\big) \end{array} . \tag{2.1.47}$$

It can be shown, using the naturality (2.1.7) and associativity (2.1.9) of the lax product constraint $\mathcal{F}^2$ together with the associativity of the composition $\mathcal{V}$-morphisms (2.1.30), that the above defined composition $\mathcal{W}$-morphisms are also associative.

The identity $\mathcal{W}$-morphisms are defined via the lax unity constraint $\mathcal{F}^0$ as follows

$$\begin{array}{ccc} \mathbb{1}^{\mathcal{W}} & \overset{\mathrm{id}_U}{\dashrightarrow} & \mathcal{F}_*(\underline{\mathcal{C}})[U,U] \\ {\scriptstyle \mathcal{F}^0}\downarrow & & \vdots \\ \mathcal{F}\big(\mathbb{1}^{\mathcal{V}}\big) & \xrightarrow[\mathcal{F}(\mathrm{id}_U)]{} & \mathcal{F}\big(\underline{\mathcal{C}}[U,U]\big) \end{array} . \tag{2.1.48}$$

Again it can be shown, using the unitality (2.1.10) of the lax unity constraint $\mathcal{F}^0$ together with the unitality of the identity $\mathcal{V}$-morphisms (2.1.31), that the above defined identity $\mathcal{W}$-morphisms are also unital.

(ii) Given the $\mathcal{V}$-functor $F : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$, we define the $\mathcal{W}$-functor $\mathcal{F}_*(F) : \mathcal{F}_*(\underline{\mathcal{C}}) \to \mathcal{F}_*(\underline{\mathcal{D}})$ as having the same object map and the $\mathcal{W}$-morphism between $\mathcal{W}$-homs simply as

$$\begin{array}{ccc} \mathcal{F}_*(\underline{\mathcal{C}})[U,V] & \overset{\left(\mathcal{F}_*(F)\right)_{U,V}}{\dashrightarrow} & \mathcal{F}_*(\underline{\mathcal{D}})[F(U),F(V)] \\ \vdots & & \vdots \\ \mathcal{F}\big(\underline{\mathcal{C}}[U,V]\big) & \xrightarrow[\mathcal{F}(F_{U,V})]{} & \mathcal{F}\big(\underline{\mathcal{D}}[F(U),F(V)]\big) \end{array} . \tag{2.1.49}$$

[4]Which maps the single object of $\underline{\mathbb{1}}$ to $I \in \underline{\mathcal{C}}$.

This can be shown to preserve the newly defined $\circ$ by using its definition (2.1.47) together with the fact that the lax product constraint $\mathcal{F}^2$ is natural (2.1.7) and the assumption that F already satisfies (2.1.37). Likewise, showing that the newly defined identity $\mathcal{W}$-morphisms (2.1.48) are preserved only requires that we make use of the fact that F already satisfies (2.1.38).

(iii) Given a $\mathcal{V}$-natural transformation $\xi : F \Rightarrow G : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$, we define the $\mathcal{W}$-natural transformation $\mathcal{F}_*(\xi) : \mathcal{F}_*(F) \Rightarrow \mathcal{F}_*(G) : \mathcal{F}_*(\underline{\mathcal{C}}) \to \mathcal{F}_*(\underline{\mathcal{D}})$ as follows

$$\begin{array}{ccc} \mathbb{1}^{\mathcal{W}} & \xrightarrow{\left(\mathcal{F}_*(\xi)\right)_U} & \mathcal{F}_*(\underline{\mathcal{D}})[F(U), G(U)] \\ {\scriptstyle \mathcal{F}^0}\downarrow & & \vdots \\ \mathcal{F}\big(\mathbb{1}^{\mathcal{V}}\big) & \xrightarrow[\mathcal{F}(\xi_U)]{} & \mathcal{F}\big(\underline{\mathcal{D}}[F(U), G(U)]\big) \end{array} \quad . \tag{2.1.50}$$

That this satisfies $\mathcal{W}$-naturality requires making use of the fact that ξ is already $\mathcal{V}$-natural together with the fact that $\mathcal{F}$ is lax unital (2.1.10).

These assignments can be shown to be 2-functorial thus they preserve commutative/pasting diagrams in $\mathcal{V}\mathbf{Cat}$, e.g., such a change of base 2-functor takes a braided monoidal $\mathcal{V}$-category $\underline{\mathcal{C}}$ to a braided monoidal $\mathcal{W}$-category $\mathcal{F}_*(\underline{\mathcal{C}})$. ▷

Remark 2.1.33. Following on from Construction 2.1.22, every closed symmetric monoidal $\mathcal{C}$ is canonically a symmetric monoidal $\mathcal{C}$-category. If we have a symmetric lax monoidal 'forgetful' functor $\mathcal{U} : \mathcal{C} \to \mathbf{Vec}$ then Construction 2.1.32 allows us to consider $\mathcal{C}$ as a symmetric monoidal linear category. △

Definition 2.1.34. Given a symmetric strict monoidal linear category $\underline{\mathcal{C}} = (\underline{\mathcal{C}}, \otimes, I, \gamma)$, an **infinitesimal braiding** is a natural endomorphism of the monoidal product $t : \otimes \Rightarrow \otimes : \underline{\mathcal{C}} \boxtimes \underline{\mathcal{C}} \to \underline{\mathcal{C}}$ which satisfies the **infinitesimal hexagon relations**: for all $U, V, W \in \mathcal{C}$,

$$t_{U(VW)} = t_{UV} \otimes 1_W + (\gamma_{VU} \otimes 1_W)(1_V \otimes t_{UW})(\gamma_{UV} \otimes 1_W) \quad , \tag{2.1.51a}$$
$$t_{(UV)W} = 1_U \otimes t_{VW} + (1_U \otimes \gamma_{WV})(t_{UW} \otimes 1_V)(1_U \otimes \gamma_{VW}) \quad . \tag{2.1.51b}$$

We say it is γ**-equivariant** if it intertwines with the symmetric braiding, i.e.,

$$t_{UV} = \gamma_{VU} t_{VU} \gamma_{UV} \tag{2.1.52}$$

in which case we say $(\underline{\mathcal{C}}, \otimes, I, \gamma, t)$ is a **Cartier category**.

2.1.3 Infinitesimal braidings from symmetric invariant 2-tensors

Given a Lie algebra $\mathfrak{g} = (\mathfrak{g}, [\cdot, \cdot])$, a **representation** $U = (U, \rho_U)$ is a vector space U together with a **left $\mathfrak{g}$-action** given by the Lie algebra homomorphism $\rho_U : \mathfrak{g} \to \mathrm{End}(U)$. We denote the left $\mathfrak{g}$-action by

$$[\rho_U(x)](u) = x.u \tag{2.1.53}$$

in which case the fact that this left $\mathfrak{g}$-action is a Lie algebra homomorphism reads as

$$[x, y].u = x.(y.u) - y.(x.u) \quad . \tag{2.1.54}$$

Given another representation $V = (V, \rho_V)$, an **equivariant map** $f : U \to V$ is a linear map which preserves the $\mathfrak{g}$-action,

$$x.f(u) = f(x.u) \quad . \tag{2.1.55}$$

Equivariant maps can clearly be added and scaled and the composition (which is that of the underlying linear maps) is obviously bilinear, associative and unital with respect to the identity linear map hence $\mathbf{Rep}_{\mathbb{K}}\mathfrak{g}$ is a linear category. It admits a compatible symmetric monoidal structure with monoidal product $UV := (U \otimes V, \rho_{UV})$ where

$$x.(u \otimes v) = (x.u) \otimes v + u \otimes (x.v) \quad . \tag{2.1.56}$$

The linear monoidal product of equivariant maps $f : U \to U'$ and $g : V \to V'$ is simply their tensor product. The monoidal unit is $\mathbb{K}$ equipped with the trivial action,

$$x.1 = 0 \quad . \tag{2.1.57}$$

The symmetric braiding $\gamma_{UV} : UV \to VU$ is simply

$$\gamma_{UV}(u \otimes v) := v \otimes u \quad . \tag{2.1.58}$$

The Lie algebra $\mathfrak{g}$ forms a representation of itself known as the **adjoint representation** using which we have the following lemmas.

Lemma 2.1.35. *Given a representation $U = (U, \rho_U)$, the action:*

$$\rho_U : \mathfrak{g}U \to U \qquad , \qquad y \otimes u \mapsto y.u \quad , \tag{2.1.59}$$

is an equivariant map such that

$$\rho_{UV} = \rho_U \otimes 1_V + (1_U \otimes \rho_V)(\gamma_{\mathfrak{g}U} \otimes 1_V) \tag{2.1.60}$$

and the family of such equivariant maps is natural, i.e., given an equivariant map $f : U \to V$ we have

$$f\rho_U = \rho_V(1_{\mathfrak{g}} \otimes f) : \mathfrak{g}U \to V \quad . \tag{2.1.61}$$

Proof. To prove (2.1.59) is equivariant we have to show that

$$x.(y.u) = [x, y].u + y.(x.u) \tag{2.1.62}$$

but this obviously follows from (2.1.54). The relation (2.1.60) is simply the functional form of (2.1.56). Naturality (2.1.61) is immediate;

$$f(y.u) = y.f(u) \quad . \tag{2.1.63}$$

□

Lemma 2.1.36. *Given a symmetric invariant 2-tensor $\pi \in (\mathrm{Sym}^2\mathfrak{g})^{\mathfrak{g}}$ the following is an equivariant map,*

$$\pi : \mathbb{K} \to \mathrm{Sym}^2\mathfrak{g} \quad . \tag{2.1.64}$$

By $L_U : \mathbf{Rep}_{\mathbb{K}}\mathfrak{g} \to \mathbf{Rep}_{\mathbb{K}}\mathfrak{g}$ we mean the linear endofunctor:

$$V \mapsto UV \qquad , \qquad g \mapsto 1_U \otimes g \quad , \tag{2.1.65}$$

thus Lemma 2.1.35 can be rewritten as $\rho : L_{\mathfrak{g}} \Rightarrow \mathrm{id}$ and Lemma 2.1.36 gives a natural transfor-

mation $\pi : \mathrm{id} \Rightarrow L_{\mathrm{Sym}^2\mathfrak{g}}$. Putting this together, we construct an infinitesimal braiding as

$$\begin{array}{ccc} \otimes & \xrightarrow{\quad t \quad} & \otimes \\ {\scriptstyle \pi * \mathrm{Id}_\otimes}\Downarrow & & \Uparrow{\scriptstyle \mathrm{Id}_\otimes * (\rho \boxtimes \rho)} \\ L_{\mathrm{Sym}^2\mathfrak{g}} \otimes & \xrightarrow[\cong]{} & \otimes (L_\mathfrak{g} \boxtimes L_\mathfrak{g}) \end{array} \tag{2.1.66}$$

whose components are thus defined as:

$$t_{UV} := [\rho_U \otimes \rho_V][1_\mathfrak{g} \otimes \gamma_{\mathfrak{g}U} \otimes 1_V][\pi \otimes 1_{UV}] : UV \to UV \quad , \quad u \otimes v \mapsto \pi_{(1)}.u \otimes \pi_{(2)}.v \ . \tag{2.1.67}$$

That (2.1.67) is a natural endomorphism of the monoidal product $\otimes$ follows from the fact that we are horizontally/vertically composing natural transformations from Lemmas 2.1.35 and 2.1.36. The infinitesimal hexagon relations (2.1.51) being satisfied follows from the "primitivity" of the action maps (2.1.60) whereas the γ-equivariance (2.1.52) follows from π being a symmetric 2-tensor. Altogether, we have that $(\mathbf{Rep}_\mathbb{K}\mathfrak{g}, \otimes, \mathbb{K}, \gamma, \pi)$ is a Cartier category.

2.2 Basic notions of homological algebra

We recall some elementary concepts appearing in [CW94, Chapter 1], i.e., the definitions of cochain complex/map, tensor product/internal hom of complexes, cochain homotopies and the shift of a complex.

2.2.1 Cochain complexes

Definition 2.2.1. A **cochain complex** $V = (V^\sharp, \mathrm{d})$ consists of a **$\mathbb{Z}$-graded vector space**, i.e., a family

$$V^\sharp = \{V^i\}_{i\in\mathbb{Z}} \tag{2.2.1a}$$

of vector spaces labelled by integers $i \in \mathbb{Z}$ (called **degrees**) and a family

$$\mathrm{d} = \{\mathrm{d} : V^i \to V^{i+1}\}_{i\in\mathbb{Z}} \tag{2.2.1b}$$

of degree-increasing linear maps (called **differential**) which square to zero, i.e., $\mathrm{d}^2 = 0$. We say V is finite-dimensional (fd) if $\sum_{i\in\mathbb{Z}} \dim(V^i) < \infty$.

We denote the subspace of **i-cocycles** by

$$\mathsf{Z}^i(V) := \ker\{\mathrm{d} : V^i \to V^{i+1}\} \subseteq V^i \quad , \tag{2.2.2a}$$

and we denote the subspace of **i-coboundaries** as

$$\mathsf{B}^i(V) := \mathrm{Im}\{\mathrm{d} : V^{i-1} \to V^i\} \subseteq V^i \quad . \tag{2.2.2b}$$

From these two subspaces we construct the **i^{th}-cohomology** as the subquotient

$$\mathsf{H}^i(V) := \mathsf{Z}^i(V)/\mathsf{B}^i(V) \quad . \tag{2.2.2c}$$

A **cochain map** $f : V \to W$ is a family $f = \{f : V^i \to W^i\}_{i\in\mathbb{Z}}$ of degree-preserving linear maps which commutes with the differentials, i.e., $f\,\mathrm{d}_V = \mathrm{d}_W\,f$. A cochain map sends coboundaries to coboundaries and cocycles to cocycles thus induces a map $\mathsf{H}^i(V) \to \mathsf{H}^i(V)$.

Definition 2.2.2. A **quasi-isomorphism** is a cochain map $f : V \to W$ such that, for all degrees $i \in \mathbb{Z}$, the induced map between i^{th}-cohomologies $f : \mathsf{H}^i(V) \to \mathsf{H}^i(W)$ is an isomorphism.

We denote by **Ch** the category of cochain complexes and cochain maps. The category **Ch** carries the following standard closed symmetric monoidal structure. The **tensor product** $V \otimes W \in \mathbf{Ch}$ of two cochain complexes $V, W \in \mathbf{Ch}$ is given by

$$(V \otimes W)^i \;:=\; \bigoplus_{j\in\mathbb{Z}} \big(V^j \otimes W^{i-j}\big) \quad , \tag{2.2.3a}$$

for all $i \in \mathbb{Z}$, and the differential

$$\mathrm{d}(v \otimes w) \;:=\; (\mathrm{d}v) \otimes w + (-1)^{|v|}\, v \otimes (\mathrm{d}w) \quad , \tag{2.2.3b}$$

where $|\cdot|$ indicates the degree. The monoidal unit is given by the ground field $\mathbb{K} \in \mathbf{Ch}$, regarded as a cochain complex concentrated in degree 0 with trivial differential. The symmetric braiding is defined component-wise by the **Koszul sign rule**

$$\gamma_{V,W} \,:\, V \otimes W \;\to\; W \otimes V \quad , \quad v \otimes w \;\mapsto\; \gamma_{V,W}(v \otimes w) \;:=\; (-1)^{|v||w|}\, w \otimes v \,. \tag{2.2.4}$$

The **internal hom** $\underline{\hom}(V, W) \in \mathbf{Ch}$ between two cochain complexes $V, W \in \mathbf{Ch}$ is given by

$$\underline{\hom}(V, W)^i \;:=\; \prod_{j\in\mathbb{Z}} \mathrm{Hom}(V^j, W^{j+i}) \quad , \tag{2.2.5a}$$

for all $i \in \mathbb{Z}$, where Hom denotes the vector space of linear maps, and the differential

$$\partial L \;:=\; \mathrm{d}_W\, L - (-1)^{|L|}\, L\, \mathrm{d}_V \quad , \tag{2.2.5b}$$

This is indeed a *closed* monoidal structure given that we have the following natural isomorphism,

$$\mathbf{Ch}\,(U, \underline{\hom}(V, W)) \cong \mathbf{Ch}\,(U \otimes V, W) \quad . \tag{2.2.6}$$

Note that cochain maps $f : V \to W$ are precisely the 0-cocycles in the inner hom complex $\underline{\hom}(V, W)$, i.e., $f \in Z^0\,(\underline{\hom}(V, W))$ such that $\partial f = 0$. A **cochain homotopy** between two cochain maps

$$h : f \to g : V \to W \tag{2.2.7}$$

is a degree -1 element $h \in \underline{\hom}(V, W)^{-1}$ such that $\partial h = f - g$. The degree $i < -1$ elements of $\underline{\hom}(V, W)$ admit an interpretation in terms of higher cochain homotopies.

Definition 2.2.3. A **dg-category** is a **Ch**-category, i.e., a category with hom complexes and cochain composition/units.

Associated to any integer $n \in \mathbb{Z}$ is the **n-shift endofunctor** $[n] : \mathbf{Ch} \to \mathbf{Ch}$. To a cochain complex $V \in \mathbf{Ch}$ it assigns the **n-shifted cochain complex** $V[n] \in \mathbf{Ch}$ given by $V[n]^i := V^{i+n}$, for all $i \in \mathbb{Z}$, and $\mathrm{d}_{V[n]} := \mathrm{d}_V$. To a cochain map $f : V \to W$ it assigns the **n-shifted cochain map** $f[n] : V[n] \to W[n]$ given by $\{f[n]^i := f^{i+n} : V^{i+n} \to W^{i+n}\}_{i\in\mathbb{Z}}$. Note that $[0] = \mathrm{id}_{\mathbf{Ch}}$ is the identity and

$$[n]\,[m] = [n + m] \tag{2.2.8}$$

under composition. Recalling the tensor product (2.2.3), one obtains a natural isomorphism $[n] \cong \mathbb{K}[n] \otimes (\,\cdot\,) : \mathbf{Ch} \to \mathbf{Ch}$. To keep track of shifts in element-wise expressions, we will denote elements in $V[n]$ by $s^{-n}\, v \in V[n]$, where $s^{-n} \in \mathbb{K}[n]$ is the degree $-n$ element.

The interplay between shifts and the closed symmetric monoidal structure on **Ch** is as follows. For any $n, m \in \mathbb{Z}$ and $V, W \in \mathbf{Ch}$, we have a natural cochain isomorphism

$$V[n] \otimes W[m] \xrightarrow{\cong} (V \otimes W)[n+m] \quad , \quad s^{-n}\, v \otimes s^{-m} w \mapsto (-1)^{|v|m} s^{-n-m}(v \otimes w) \ , \tag{2.2.9}$$

which moves the shifts to the left. Note that this cochain isomorphism does not preserve the symmetric braiding on **Ch**, but rather one has that

$$\begin{array}{ccc} V[n] \otimes W[m] & \xrightarrow{\quad\gamma\quad} & W[m] \otimes V[n] \\ \cong\downarrow & & \downarrow\cong \\ (V \otimes W)[n+m] & \xrightarrow[(-1)^{nm}\,\gamma[n+m]]{} & (W \otimes V)[n+m] \end{array} \quad . \tag{2.2.10}$$

For the internal hom between $V, W \in \mathbf{Ch}$, we have a natural cochain isomorphism

$$\underline{\hom}(V,W) \ \xrightarrow{\cong}\ \underline{\hom}\big(V[n], W[n]\big) \quad , \quad L \ \mapsto\ L[n] \ :=\ \mathrm{id}_{\mathbb{K}[n]} \otimes L \quad . \tag{2.2.11}$$

The tensor product in this expression denotes the one of internal homs, which when evaluated on elements $s^{-n}\, v \in V[n]$ yields the Koszul sign $L[n](s^{-n}\, v) := (-1)^{|L|n}\, s^{-n}\, L(v)$. Furthermore, we have a natural cochain isomorphism

$$\underline{\hom}(V,W)[n] \ \xrightarrow{\cong}\ \underline{\hom}\big(V, W[n]\big) \tag{2.2.12}$$

that comes without Koszul signs and is given by regarding $s^{-n}\, L \in \underline{\hom}(V,W)[n]$ as the element $\widetilde{L} \in \underline{\hom}\big(V, W[n]\big)$ which is defined by the evaluation $\widetilde{L}(v) := s^{-n} L(v) \in W[n]$, for all $v \in V$.

2.2.2 Commutative differential graded algebras (CDGAs)

A **differential graded algebra** (in short, **DGA**) is a monoid object in the symmetric monoidal category **Ch**. More explicitly, a DGA is a triple $A = (A, \mu, \eta)$ consisting of a cochain complex $A = (A^\sharp, \delta) \in \mathbf{Ch}$ and two cochain maps:

$$\mu : A \otimes A \to A \quad , \qquad a \otimes b \mapsto ab \tag{2.2.13a}$$

(called **multiplication**) and

$$\eta : \mathbb{K} \to A \quad , \qquad 1 \mapsto \mathbb{1} \tag{2.2.13b}$$

(called **unit**) which satisfy the usual axioms of associativity,

$$a(bc) = (ab)c \quad , \tag{2.2.13c}$$

and unitality,

$$\mathbb{1}a = a = a\mathbb{1} \quad . \tag{2.2.13d}$$

The multiplication and unit being cochain maps means that we have:

$$\delta(ab) = \delta(a)b + (-1)^{|a|} a\delta(b) \quad , \tag{2.2.14}$$

$$\delta(\mathbb{1}) = 0 \quad , \tag{2.2.15}$$

respectively. A **commutative differential graded algebra** (in short, **CDGA**) is a DGA whose multiplication is commutative with respect to the symmetric braiding (2.2.4) on **Ch**, i.e.,

$$\mu = \mu \circ \gamma_{A,A} \qquad \Longrightarrow \qquad ab = (-1)^{|a||b|} ba \quad . \tag{2.2.16}$$

We denote by $\mathbf{CDGA} := \mathbf{CAlg}(\mathbf{Ch})$ the category of CDGAs whose morphisms are cochain maps $f : A \to B$ which preserve the multiplication and unit:

$$f(ab) = f(a)f(b) \qquad , \qquad f(\mathbb{1}) = \mathbb{1} \quad . \tag{2.2.17}$$

One class of CDGAs which is relevant for this thesis is that of **free CDGAs**, which are also known as symmetric algebras. Recall that the **symmetric algebra** associated with a cochain complex $V \in \mathbf{Ch}$ is given by the cochain complex

$$\mathrm{Sym}\, V \;:=\; \bigoplus_{n\geq 0} \mathrm{Sym}^n V \;:=\; \bigoplus_{n\geq 0} \big(V^{\otimes n}\big)_{S_n} \;\in\; \mathbf{Ch} \quad , \tag{2.2.18a}$$

where $V^{\otimes n} := V \otimes V \otimes \cdots \otimes V \in \mathbf{Ch}$ denotes the n-fold tensor power and $\big(V^{\otimes n}\big)_{S_n} \in \mathbf{Ch}$ are the coinvariants (i.e., quotient) of the permutation group action defined by the symmetric braiding (2.2.4). The multiplication is defined by

$$\mu\big([v_1 \otimes \cdots \otimes v_n] \otimes [v'_1 \otimes \cdots \otimes v'_m]\big) = [v_1 \otimes \cdots \otimes v_n \otimes v'_1 \otimes \cdots \otimes v'_m] \tag{2.2.18b}$$

and the unit is given by $\eta(1) = 1 \in \mathbb{K} = \mathrm{Sym}^0 V \subseteq \mathrm{Sym}\, V$. To ease notation, we shall denote elements of the symmetric algebra simply by $v_1\, v_2 \cdots v_n \in \mathrm{Sym}\, V$. We denote the full subcategory of such objects as $\mathbf{CDGA}^{\mathrm{f}}$.

Relaxing the constraint on the differential structure but maintaining the underlying graded commutative algebra structure results in the following more general concept.

Definition 2.2.4. A CDGA $A = (A^\sharp, \delta, \mu, \eta)$ is called **semi-free** (sf) if its underlying $\mathbb{Z}$-graded commutative algebra $A^\sharp = (A^\sharp, \mu, \eta)$ is free, i.e.,

$$A^\sharp \;\cong\; \mathrm{Sym}(V^\sharp) \tag{2.2.19}$$

in the category of graded-commutative algebras, for some $\mathbb{Z}$-graded vector space $V^\sharp$. An sfCDGA is called **finitely generated** (fg) if $V^\sharp$ is fd as in Definition 2.2.1. An sfCDGA is called **positively generated** if $V^i = 0$ for $i \leq 0$.

Remark 2.2.5. The Leibniz law (2.2.14) means the differential δ on a sfCDGA $A^\sharp \cong \mathrm{Sym}(V^\sharp)$ is completely determined by its restriction $\delta(v)$ to the generators. Hence, one can prescribe such differential by its values $\delta(v)$ on generators such that $|\delta(v)| = |v| + 1$ and $\delta^2(v) = 0$. △

Example 2.2.6. Suppose we are given a fd Lie algebra $\mathfrak{g} = (\mathfrak{g}, [\cdot,\cdot])$. We denote the basis elements by $\{x_i \in \mathfrak{g}\}_{1\leq i\leq \dim\mathfrak{g}}$ and the **structure constants** by $\mathfrak{g}^i_{jk} \in \mathbb{K}$ where

$$[x_j, x_k] = \mathfrak{g}^i_{jk} x_i \quad . \tag{2.2.20a}$$

The skew-commutative property of the Lie bracket $[\cdot,\cdot] : \bigwedge^2_{\mathbb{K}} \mathfrak{g} \to \mathfrak{g}$ translates into the following anti-symmetry of the lower indices,

$$\mathfrak{g}^i_{jk} = -\mathfrak{g}^i_{kj} \quad . \tag{2.2.20b}$$

The Jacobi identity

$$\big[[x_i, x_j], x_k\big] + \big[[x_j, x_k], x_i\big] + \big[[x_k, x_i], x_j\big] = 0 \tag{2.2.20c}$$

translates into the following relation between the structure constants,

$$\mathfrak{g}^l_{ij}\mathfrak{g}^m_{lk} + \mathfrak{g}^l_{jk}\mathfrak{g}^m_{li} + \mathfrak{g}^l_{ki}\mathfrak{g}^m_{lj} = 0 \quad . \tag{2.2.20d}$$

Because $\mathfrak{g}$ is fd, the dual vector space $\mathfrak{g}^* := \mathrm{Hom}(\mathfrak{g}, \mathbb{K})$ exists and is also fd. We denote the dual basis by $\{\theta^i \in \mathfrak{g}^*\}_{1\leq i\leq \dim\mathfrak{g}}$ hence the generators of the *free* graded-commutative algebra

$\mathrm{Sym}(\mathfrak{g}^*[-1])$ are denoted by $\{\vartheta^i \in \mathfrak{g}^*[-1]\}_{1\leq i\leq \dim\mathfrak{g}}$ where

$$\vartheta^j\vartheta^k = -\vartheta^k\vartheta^j \quad , \tag{2.2.21a}$$

thus

$$\mathrm{Sym}(\mathfrak{g}^*[-1]) = \bigoplus_{i=0}^{\dim\mathfrak{g}} \mathrm{Sym}^i(\mathfrak{g}^*[-1]) \quad . \tag{2.2.21b}$$

As mentioned in Remark 2.2.5, if we wish to construct a sfCDGA structure on $\mathrm{Sym}(\mathfrak{g}^*[-1])$ then we must first prescribe the differential on generators such that it increases the degree by 1. We define the differential by the dualised Lie bracket (e.g., [BPS22, Formula (2.11) of Example 2.7]),

$$\delta(\vartheta^i) = -\frac{1}{2}\mathfrak{g}^i_{jk}\vartheta^j\vartheta^k \quad . \tag{2.2.21c}$$

Now we force the Leibniz law to hold as in (2.2.14) and we must check that this differential indeed squares to zero,

$$\delta^2(\vartheta^i) = -\tfrac{1}{2}\mathfrak{g}^i_{jk}\delta(\vartheta^j\vartheta^k) = \tfrac{1}{4}\mathfrak{g}^i_{jk}\mathfrak{g}^j_{lm}\vartheta^l\vartheta^m\vartheta^k - \tfrac{1}{4}\mathfrak{g}^i_{jk}\mathfrak{g}^k_{lm}\vartheta^j\vartheta^l\vartheta^m = \tfrac{1}{2}\mathfrak{g}^i_{jk}\mathfrak{g}^j_{lm}\vartheta^l\vartheta^m\vartheta^k \,, \tag{2.2.22a}$$

where the second equality uses the derivation property and the last equality uses the reindexing $(j \leftrightarrow k)$ and the anti-symmetry of the lower indices (2.2.20b) on the second term. Setting this to zero and then stripping away the ϑ terms means that we require the following relation between the structure constants,

$$\mathfrak{g}^i_{jk}\mathfrak{g}^j_{lm} + \mathfrak{g}^i_{jl}\mathfrak{g}^j_{mk} + \mathfrak{g}^i_{jm}\mathfrak{g}^j_{kl} = 0 \quad . \tag{2.2.22b}$$

This is simply the relation between the structure constants (2.3.59b) which follows from the Jacobi identity. This example has proven the one-to-one correspondence between fd Lie algebras $(\mathfrak{g}, [\cdot,\cdot])$ and sfCDGAs $\big(\mathrm{Sym}(\mathfrak{g}^*[-1]), \delta\big)$ finitely generated in degree 1. We call the latter the **Chevalley-Eilenberg algebra**,

$$\mathrm{CE}_{\mathfrak{g}} := \big(\mathrm{Sym}(\mathfrak{g}^*[-1]), \delta\big) \quad , \tag{2.2.23}$$

of the fd Lie algebra $\mathfrak{g}$. In fact, CE is actually contravariantly functorial in the sense that there exists a well-defined contravariant equivalence,

$$\mathrm{CE} : \big(\mathbf{LieAlg}^{\mathrm{fd}}\big)^{\mathrm{op}} \simeq \mathbf{CDGA}_1^{\mathrm{fgsf}} \quad . \tag{2.2.24}$$

Given a Lie algebra homomorphism $f : \mathfrak{g} \to \mathfrak{h}$, we must define a CDGA morphism

$$\mathrm{CE}_f : \mathrm{CE}_{\mathfrak{h}} \to \mathrm{CE}_{\mathfrak{g}} \quad . \tag{2.2.25}$$

Semi-freeness means that it suffices to only prescribe a well-defined result on the generators of $\mathrm{CE}_{\mathfrak{h}}$. To that end, choose the basis $\{y_a \in \mathfrak{h}\}_{1\leq a\leq\dim\mathfrak{h}}$ and the shifted dual basis $\{\varphi^a \in \mathfrak{h}^*[-1]\}_{1\leq a\leq\dim\mathfrak{h}}$ so that if $f(x_i) = f^a_i y_a$ then

$$\mathrm{CE}_f(\varphi^a) := f^a_i\vartheta^i \quad . \tag{2.2.26}$$

The fact that f is a Lie algebra homomorphism translates to the index relation

$$f^a_i\mathfrak{g}^i_{jk} = \mathfrak{h}^a_{bc}f^b_j f^c_k \tag{2.2.27}$$

but this obviously implies that (2.2.26) lifts to a cochain map. ▽

2.2.3 dg-Modules over a CDGA

Associated with any CDGA A is its dg-category ${}_A\mathbf{dgMod}$ of **left A-dg-modules**. More explicitly, an object $M = (M^\sharp, \mathrm{d}, \triangleright)$ in this category is a cochain complex $M = (M^\sharp, \mathrm{d}) \in \mathbf{Ch}$ endowed with a cochain map

$$\triangleright : A \otimes M \to M\,, \qquad a \otimes m \mapsto a \triangleright m = am \tag{2.2.28a}$$

(called **left action**) which satisfies the usual left module axioms of associativity

$$(a\,a')m = a(a'm) \tag{2.2.28b}$$

and unitality

$$\mathbb{1}m = m \quad . \tag{2.2.28c}$$

The left action being a cochain map means that we have

$$\mathrm{d}(am) = (\delta a)m + (-1)^{|a|}a(\mathrm{d}m) \quad . \tag{2.2.28d}$$

The cochain complex of morphisms between two objects $M, N \in {}_A\mathbf{dgMod}$ is defined as the subcomplex

$$\underline{\hom}_A(M,N) \subseteq \underline{\hom}(M,N) \tag{2.2.29a}$$

of the internal hom in $\mathbf{Ch}$ containing the **A-linear** maps $f \in \underline{\hom}(M,N)$, i.e.,

$$f(am) = (-1)^{|f||a|}af(m) \quad . \tag{2.2.29b}$$

The property of A-linearity is clearly preserved under addition and taking the inner hom differential. The dg-category ${}_A\mathbf{dgMod}$ has a compatible closed symmetric monoidal structure with:

(i) Monoidal product given by the relative tensor product,

$$M \otimes_A N := \mathrm{colim}\Bigg(M \otimes N \;\underset{\mathrm{id}_M \otimes \triangleright}{\overset{(\triangleright \otimes \mathrm{id}_N)\,(\gamma_{M,A} \otimes \mathrm{id}_N)}{\leftleftarrows}}\; M \otimes A \otimes N \Bigg) \in {}_A\mathbf{dgMod}\,. \tag{2.2.30}$$

(ii) Monoidal unit given by the base CDGA $A = (A^\sharp, \delta, \mu) \in {}_A\mathbf{dgMod}$.

(iii) Symmetric braiding given by the Koszul sign rule,

$$\gamma_{MN} : M \otimes_A N \to N \otimes_A M \qquad , \qquad m \otimes_A n \mapsto (-1)^{|m||n|}n \otimes_A m \quad . \tag{2.2.31}$$

(iv) Internal hom $\underline{\hom}_A(M,N) \in {}_A\mathbf{dgMod}$ given by the cochain complex (2.2.29) with the left action

$$(af)(m) := af(m) \tag{2.2.32}$$

and, as before, we have natural isomorphisms

$${}_A\mathbf{dgMod}\,(L, \underline{\hom}_A(M,N)) \cong {}_A\mathbf{dgMod}\,(L \otimes_A M, N) \quad . \tag{2.2.33}$$

Definition 2.2.7. An A-dg-module $M = (M^\sharp, \mathrm{d}, \triangleright)$ is **semi-free** if its underlying $A^\sharp$-module $M^\sharp = (M^\sharp, \triangleright)$ is free, i.e.,

$$M^\sharp \;\cong\; A^\sharp \otimes V^\sharp \tag{2.2.34}$$

in ${}_{A^\sharp}\mathbf{gMod}$ for a $\mathbb{Z}$-graded vector space $V^\sharp$ where the left action is simply given by

$$a(a' \otimes v) = (a\, a') \otimes v =: a\, a' v \quad . \tag{2.2.35}$$

An sf A-dg-module is **finitely generated** if the total dimension

$$\dim(V^\sharp) := \sum_{i\in\mathbb{Z}} \dim(V^i) < \infty \tag{2.2.36}$$

of the $\mathbb{Z}$-graded vector space $V^\sharp$ is finite. Furthermore, we say that an sf A-dg-module is **generated in degree 0** if $V^\sharp = V^0$. We denote by

$${}_A\mathbf{dgMod}^{\mathrm{fgsf}} \subseteq {}_A\mathbf{dgMod} \tag{2.2.37}$$

the full dg-subcategory whose objects are all fgsf A-dg-modules. We further denote by

$${}_A\mathbf{dgMod}_0^{\mathrm{fgsf}} \subseteq {}_A\mathbf{dgMod}^{\mathrm{fgsf}} \tag{2.2.38}$$

the full dg-subcategory whose objects are all sf A-dg-modules finitely generated in degree 0.

Remark 2.2.8. The closed symmetric monoidal structure on ${}_A\mathbf{dgMod}$ restricts to a closed symmetric monoidal structure on the full dg-subcategory ${}_A\mathbf{dgMod}^{\mathrm{fgsf}} \subseteq {}_A\mathbf{dgMod}$. Indeed, the monoidal unit $A \in {}_A\mathbf{dgMod}$ is *singly* generated in degree 0 and *free* (for the simple reason that $A \cong A \otimes_A \mathbb{K}$ in ${}_A\mathbf{dgMod}$) and, given any two fgsf A-dg-modules $M, N \in {}_A\mathbf{dgMod}^{\mathrm{fgsf}}$ with $M^\sharp \cong A^\sharp \otimes V^\sharp$ and $N^\sharp \cong A^\sharp \otimes W^\sharp$, we have isomorphisms between graded $A^\sharp$-modules,

$$\big(M \otimes_A N\big)^\sharp \;\cong\; A^\sharp \otimes \big(V^\sharp \otimes W^\sharp\big) \tag{2.2.39a}$$

and

$$\underline{\hom}_A(M,N)^\sharp \;\cong\; A^\sharp \otimes \big(W^\sharp \otimes (V^\sharp)^*\big) \quad , \tag{2.2.39b}$$

which imply that $M \otimes_A N \in {}_A\mathbf{dgMod}^{\mathrm{fgsf}}$ and $\underline{\hom}_A(M,N) \in {}_A\mathbf{dgMod}^{\mathrm{fgsf}}$ are fgsf. Here $(V^\sharp)^*$ denotes the dual of the $\mathbb{Z}$-graded vector space $V^\sharp$, which exists since $V^\sharp$ is, by hypothesis, of finite total dimension. Analogous arguments show that the closed symmetric monoidal structure on ${}_A\mathbf{dgMod}^{\mathrm{fgsf}}$ restricts to a closed symmetric monoidal structure on ${}_A\mathbf{dgMod}_0^{\mathrm{fgsf}}$. △

Remark 2.2.9. The Leibniz law (2.2.28d) implies that the differential d on a sf A-dg-module $M^\sharp \cong A^\sharp \otimes V^\sharp$ is determined by its restriction $\mathrm{d}(w)$ to the generators. Hence, one can prescribe such differential by its values $\mathrm{d}(w)$ on generators such that $|\mathrm{d}(w)| = |w| + 1$ and $\mathrm{d}^2 w = 0$. △

2.2.4 Lie algebra representations as Chevalley-Eilenberg modules

Example 2.2.10. Given a fd Lie algebra $\mathfrak{g}$ and a representation (V, ρ_V), we follow Remark 2.2.9 and define d on the generators of the free $\mathrm{CE}^\sharp_\mathfrak{g}$-graded module $(\mathrm{CE}^\sharp_\mathfrak{g} \otimes V, \triangleright)$ by shifted resolution of the identity as follows,

$$\mathrm{d}v := \vartheta^i(x_i.v) \quad . \tag{2.2.40}$$

Now we force the Leibniz law as in (2.2.28d) and we check that d squares to zero on the generators,

$$\begin{aligned}
\mathrm{d}^2 v =& \delta\big(\vartheta^i\big)(x_i.v) - \vartheta^i \mathrm{d}(x_i.v) \\
=& -\frac{1}{2}\mathfrak{g}^i_{jk}\vartheta^j\vartheta^k(x_i.v) - \vartheta^i\vartheta^j\big(x_j.(x_i.v)\big) \\
=& -\frac{1}{2}\vartheta^j\vartheta^k([x_j,x_k].v) - \vartheta^j\vartheta^k\big(x_k.(x_j.v)\big) \\
=& -\frac{1}{2}\vartheta^j\vartheta^k\big(x_j.(x_k.v) - x_k.(x_j.v)\big) - \vartheta^j\vartheta^k\big(x_k.(x_j.v)\big)
\end{aligned} \tag{2.2.41}$$

where the third equality uses (2.2.20a) on the first term and the last equality uses (2.1.54). This last equality is obviously zero by using (2.2.21a) and a reindexing on the first term. We call

$$\mathrm{CE}^V_{\mathfrak{g}} := \big(\mathrm{CE}^\sharp_{\mathfrak{g}} \otimes V, \mathrm{d}, \triangleright\big) \tag{2.2.42}$$

the **Chevalley-Eilenberg module** of the Lie algebra representation (V, ρ_V). In fact, given an equivariant map $f : U \to V$ we define a $\mathrm{CE}_{\mathfrak{g}}$-linear cochain map:

$$\mathrm{CE}^f_{\mathfrak{g}} : \mathrm{CE}^U_{\mathfrak{g}} \to \mathrm{CE}^V_{\mathfrak{g}} \qquad , \qquad u \mapsto f(u) \quad . \tag{2.2.43}$$

The cochain property follows from the equivariance property,

$$\mathrm{CE}^f_{\mathfrak{g}}(\mathrm{d}u) = \mathrm{CE}^f_{\mathfrak{g}}(\vartheta^i x_i.u) = \vartheta^i f(x_i.u) = \vartheta^i x_i.f(u) = \mathrm{d}f(u) = \mathrm{dCE}^f_{\mathfrak{g}}(u) \quad . \tag{2.2.44}$$

$\triangledown$

Proposition 2.2.11. *Example 2.2.10 defines a linear equivalence* $\mathrm{CE}_{\mathfrak{g}} : \mathbf{Rep}_{\mathbb{K}}\mathfrak{g} \simeq {}_{\mathrm{CE}_{\mathfrak{g}}}\mathbf{dgMod}^{\mathrm{sf}}_0$.

Proof. Noting that

$$\mathrm{CE}^f_{\mathfrak{g}} := 1_{\mathrm{CE}_{\mathfrak{g}}} \otimes f \quad , \tag{2.2.45}$$

linear functoriality simply follows from that of the tensor product:

$$\mathrm{CE}^{f'+f}_{\mathfrak{g}} = \mathrm{CE}^{f'}_{\mathfrak{g}} + \mathrm{CE}^f_{\mathfrak{g}} \qquad , \qquad \mathrm{CE}^{1_U}_{\mathfrak{g}} = 1_{\mathrm{CE}^U_{\mathfrak{g}}} \qquad , \qquad \mathrm{CE}^{gf}_{\mathfrak{g}} = \mathrm{CE}^g_{\mathfrak{g}}\mathrm{CE}^f_{\mathfrak{g}} \quad . \tag{2.2.46}$$

To show essential surjectivity, consider a generic object $M \in {}_{\mathrm{CE}_{\mathfrak{g}}}\mathbf{dgMod}^{\mathrm{sf}}_0$. By Definition 2.2.7, there exists some vector space V such that

$$M^\sharp \cong \mathrm{CE}^\sharp_{\mathfrak{g}} \otimes V \tag{2.2.47}$$

as graded modules over $\mathrm{CE}^\sharp_{\mathfrak{g}}$. The differential on M transfers under this isomorphism to define a differential d on $\mathrm{CE}^\sharp_{\mathfrak{g}} \otimes V$ thence promoting the isomorphism as one between $\mathrm{CE}_{\mathfrak{g}}$-dg-modules. Evaluating the differential d on a generator in V, we have

$$\mathrm{d}v = \vartheta^i \rho_i(v) \tag{2.2.48}$$

where $\rho_i \in \mathrm{End}(V)$. The square-zero condition now implies relations on the ρ_i,

$$\mathrm{d}^2 v = \sum_{i>j} \vartheta^i\vartheta^j \left([\rho_i, \rho_j] - \mathfrak{g}^k_{ij}\rho_k\right)(v) = 0 \quad . \tag{2.2.49}$$

Explicitly, (2.2.49) states that the ρ_i satisfy the Lie algebra relations of $\mathfrak{g}$ and can be identified as $\rho_i := \rho(x_i)$ with $\rho : \mathfrak{g} \to \mathrm{End(V)}$ a Lie algebra homomorphism.

Faithfulness is obvious,

$$\mathrm{CE}_{\mathfrak{g}}^{f} = \mathrm{CE}_{\mathfrak{g}}^{f'} \quad \Longrightarrow \quad 1_{\mathrm{CE}_{\mathfrak{g}}} \otimes f = 1_{\mathrm{CE}_{\mathfrak{g}}} \otimes f' \quad \Longrightarrow \quad f = f' \,. \tag{2.2.50}$$

To prove fullness, note that a $\mathrm{CE}_{\mathfrak{g}}$-linear cochain map $\phi : \mathrm{CE}_{\mathfrak{g}}^{U} \to \mathrm{CE}_{\mathfrak{g}}^{V}$ is determined by its restriction to module generators, $\phi(u) \in V$, and the cochain condition $\phi(\mathrm{d}u) = \mathrm{d}\phi(u)$ reads

$$\phi(\vartheta^i x_i.u) = \vartheta^i x_i.\phi(u) \quad \Longrightarrow \quad \vartheta^i[\phi(x_i.u) - x_i.\phi(u)] = 0 \quad , \tag{2.2.51}$$

hence $\phi(x_i.u) = x_i.\phi(u)$ and $\phi : U \to V$ is an equivariant map. □

Proposition 2.2.12. $\mathrm{CE}_{\mathfrak{g}} : \mathbf{Rep}_{\mathbb{K}}\mathfrak{g} \simeq {}_{\mathrm{CE}_{\mathfrak{g}}}\mathbf{dgMod}_0^{\mathrm{sf}}$ *is a symmetric monoidal equivalence with strict unity constraint and strong product constraint*

$$(\mathrm{CE}_{\mathfrak{g}})^2_{UV} : \mathrm{CE}_{\mathfrak{g}}^{U} \otimes_{\mathrm{CE}_{\mathfrak{g}}} \mathrm{CE}_{\mathfrak{g}}^{V} \cong \mathrm{CE}_{\mathfrak{g}}^{UV} \qquad , \qquad u \otimes_{\mathrm{CE}_{\mathfrak{g}}} v \mapsto u \otimes v \quad . \tag{2.2.52}$$

Proof. Clearly the monoidal unit is strictly preserved,

$$\mathrm{CE}_{\mathfrak{g}}^{\mathbb{K}} = \mathrm{CE}_{\mathfrak{g}} \quad . \tag{2.2.53}$$

Let us show that (2.2.52) is cochain; we have

$$\mathrm{d}(u \otimes_{\mathrm{CE}_{\mathfrak{g}}} v) = \vartheta^i x_i.u \otimes_{\mathrm{CE}_{\mathfrak{g}}} v + u \otimes_{\mathrm{CE}_{\mathfrak{g}}} \vartheta^i x_i.v = \vartheta^i(x_i.u \otimes_{\mathrm{CE}_{\mathfrak{g}}} v + u \otimes_{\mathrm{CE}_{\mathfrak{g}}} x_i.v) \tag{2.2.54}$$

and this gets mapped to

$$\vartheta^i(x_i.u \otimes v + u \otimes x_i.v) = \vartheta^i x_i.(u \otimes v) = \mathrm{d}(u \otimes v) \quad . \tag{2.2.55}$$

It is obvious that (2.2.52) is a natural isomorphism which satisfies the lax associativity axiom (2.1.9) and preserves the symmetric braiding (2.1.21). □

2.3 Derived algebraic geometry of CDGAs

In order to describe the geometric concept of n-shifted Poisson structures on a CDGA, we first need to define what we mean by polyvectors in the context where the "function" algebra is a CDGA. For this reason, we first define derivations and then construct the complex of n-shifted polyvectors from such derivations. An n-shifted Poisson structure will later be defined as an n-shifted polyvector of a specific cohomological degree which satisfies the Maurer-Cartan equation in this context.

2.3.1 Derivations and Kähler differential forms

The cochain complex of **derivations** of a CDGA A is defined as the subcomplex

$$\mathsf{T}_A \subseteq \underline{\hom}(A, A) \in \mathbf{Ch} \tag{2.3.1a}$$

whose homogeneous elements $D \in \mathsf{T}_A$ satisfy the Leibniz rule, i.e.

$$D(a\,a') = (Da)\,a' + (-1)^{|D||a|}\,a\,(Da') \quad , \tag{2.3.1b}$$

for all homogeneous $a, a' \in A$. The cochain complex T_A is canonically an A-dg-module with the same left action as that of the internal homs of ${}_A\mathbf{dgMod}$, i.e., given by post-multiplication as in

(2.2.32). It further carries the structure of a Lie algebra object in **Ch**, i.e., it is a **dg-Lie algebra** (or **DGLA**), with cochain Lie bracket given by the commutator of composition

$$[\,\cdot\,,\,\cdot\,] : \mathsf{T}_A \otimes \mathsf{T}_A \;\to\; \mathsf{T}_A \quad , \quad D \otimes D' \;\mapsto\; [D, D'] := D \circ D' - (-1)^{|D||D'|}\, D' \circ D \quad , \tag{2.3.2}$$

for all homogeneous $D, D' \in \mathsf{T}_A$. For later reference let us spell out, in the following Lemma, the relevant properties which the commutator bracket satisfies. The graded-bilinearity is obvious hence we state only the remaining nontrivial properties that we have to check.

Lemma 2.3.1. *The commutator bracket* (2.3.2) *equips* T_A *with a DGLA structure which is a derivation of the A-dg-module structure:*

(i) Well-definedness: For all homogeneous $D, D' \in \mathsf{T}_A$*,*

$$[D, D'](ab) = [D, D'](a)b + (-1)^{|a|(|D|+|D'|)} a[D, D'](b) \quad . \tag{2.3.3a}$$

(ii) Cochain property: For all homogeneous $D, D' \in \mathsf{T}_A$*,*

$$\partial[D, D'] = [\partial D, D'] + (-1)^{|D|}[D, \partial D'] \quad . \tag{2.3.3b}$$

(iii) Antisymmetry: For all homogeneous $D, D' \in \mathsf{T}_A$*,*

$$[D, D'] \;=\; -(-1)^{|D||D'|}\,[D', D] \quad . \tag{2.3.3c}$$

(iv) Jacobi identity: For all homogeneous $D, D', D'' \in \mathsf{T}_A$*,*

$$\big[D, [D', D'']\big] \;=\; \big[[D, D'], D''\big] + (-1)^{|D||D'|}\,\big[D', [D, D'']\big] \quad . \tag{2.3.3d}$$

(v) Derivation property: For all homogeneous $a \in A$ *and* $D, D' \in \mathsf{T}_A$*,*

$$[D, a\,D'] \;=\; D(a)\,D' + (-1)^{|D||a|}\,a\,[D, D'] \quad . \tag{2.3.3e}$$

Proof. These are all elementary checks which we will spell out for concreteness below:

$$\begin{aligned}
[D, D'](ab) =& D(D'(ab)) - (-1)^{|D||D'|} D'(D(ab)) \\
=& D(D'(a)b + (-1)^{|a||D'|} aD'(b)) - (-1)^{|D||D'|} D'(D(a)b + (-1)^{|a||D|} aD(b)) \\
=& D(D'(a))b + (-1)^{|D|(|D'|+|a|))} D'(a)D(b) + (-1)^{|a||D'|} D(a)D'(b) + (-1)^{|a|(|D|+|D'|)} aD(D'(b)) \\
& - (-1)^{|D||D'|} D'(D(a))b - (-1)^{|a||D'|} D(a)D'(b) - (-1)^{|D|(|D'|+|a|)} D'(a)D(b) \\
& - (-1)^{|D||D'|+|a||D|+|a||D'|} aD'(D(b)) \\
=& [D, D'](a)b + (-1)^{|a|(|D|+|D'|)} a[D, D'](b) \quad .
\end{aligned}$$

$$\begin{aligned}
\partial[D, D'] =& \partial(D \circ D' - (-1)^{|D||D'|} D' \circ D) \\
=& \delta \circ D \circ D' - (-1)^{|D|+|D'|} D \circ D' \circ \delta - (-1)^{|D||D'|} \delta \circ D' \circ D + (-1)^{|D||D'|+|D|+|D'|} D' \circ D \circ \delta \\
=& \delta \circ D \circ D' - (-1)^{|D||D'|+|D'|} D' \circ \delta \circ D - (-1)^{|D|} D \circ \delta \circ D' + (-1)^{|D||D'|+|D|+|D'|} D' \circ D \circ \delta \\
& + (-1)^{|D|} D \circ \delta \circ D' - (-1)^{|D||D'|} \delta \circ D' \circ D - (-1)^{|D|+|D'|} D \circ D' \circ \delta + (-1)^{|D||D'|+|D'|} D' \circ \delta \circ D \\
=& [\delta \circ D, D'] - (-1)^{|D|}[D \circ \delta, D'] + (-1)^{|D|}[D, \delta \circ D'] - (-1)^{|D|+|D'|}[D, D' \circ \delta] \\
=& [\partial D, D'] + (-1)^{|D|}[D, \partial D'] \quad .
\end{aligned}$$

$$[D, D'] = D\circ D' - (-1)^{|D||D'|} D'\circ D = -(-1)^{|D||D'|}\Big(D'\circ D - (-1)^{|D'||D|} D\circ D'\Big) = -(-1)^{|D||D'|}[D', D] \quad .$$

$$\begin{aligned}
[D, [D', D'']] =& D \circ [D', D''] - (-1)^{|D|(|D'|+|D''|)}[D', D''] \circ D \\
=& D \circ D' \circ D'' - (-1)^{|D'||D''|} D \circ D'' \circ D' - (-1)^{|D|(|D'|+|D''|)} D' \circ D'' \circ D \\
&+ (-1)^{|D||D'|+|D'||D''|+|D''||D|} D'' \circ D' \circ D \\
=& D \circ D' \circ D'' - (-1)^{|D||D'|} D' \circ D \circ D'' - (-1)^{|D''|(|D|+|D'|)} D'' \circ D \circ D' \\
&+ (-1)^{|D||D'|+|D'||D''|+|D''||D|} D'' \circ D' \circ D \\
&+ (-1)^{|D||D'|} D' \circ D \circ D'' - (-1)^{|D||D'|+|D''||D|} D' \circ D'' \circ D - (-1)^{|D'||D''|} D \circ D'' \circ D' \\
&+ (-1)^{|D''|(|D|+|D'|)} D'' \circ D \circ D' \\
=& [[D, D'], D''] + (-1)^{|D||D'|}[D', [D, D'']] \quad .
\end{aligned}$$

$$\begin{aligned}
[D, a\, D'] =& D \circ (aD') - (-1)^{|D||aD'|}(aD') \circ D \\
=& D(a) D' + (-1)^{|D||a|} a\,(D \circ D') - (-1)^{|D||a|} a \Big((-1)^{|D||D'|} D' \circ D\Big) \\
=& D(a)\, D' + (-1)^{|D||a|}\, a\, [D, D'] \quad .
\end{aligned}$$

□

Remark 2.3.2. Note that we can use the canonical DGLA structure on T_A to write the (adjoint) differential of T_A, given by (2.2.5), as $\partial = [\delta, \cdot\,]$. △

Remark 2.3.3. In the case of a free CDGA $A = \operatorname{Sym} V$, each derivation is completely determined by its restriction to the generators of $V \subseteq \operatorname{Sym} V$, i.e., we have a cochain isomorphism

$$\mathsf{T}_{\operatorname{Sym} V} \xrightarrow{\cong} \underline{\hom}\big(V, \operatorname{Sym} V\big) \quad . \tag{2.3.4a}$$

The inverse is given explicitly by extending each homogeneous $L \in \underline{\hom}(V, \operatorname{Sym} V)$ to a derivation D_L on $\operatorname{Sym} V$ via the Leibniz rule

$$D_L(v_1 \cdots v_n) := \sum_{i=1}^{n} (-1)^{|L| \sum_{j=1}^{i-1} |v_j|}\, v_1 \cdots v_{i-1}\, L(v_i)\, v_{i+1} \cdots v_n \tag{2.3.4b}$$

for all homogeneous $v_1, \ldots, v_n \in V$. Furthermore, the action of A on T_A is by post-multiplication hence we have the following isomorphism

$$\mathsf{T}_A \cong A \otimes \underline{\hom}(V, \mathbb{K}) \tag{2.3.4c}$$

as A-dg-modules. In other words, if the ground CDGA A is free (i.e., $A = \operatorname{Sym} V$) then the derivations T_A form a *free* A-dg-module.

Actually, we can generalise this to the case of a sfCDGA A. A derivation is still completely determined by its restriction to the generators and all we are to do to lift the derivation to the whole complex is to force the Leibniz law as in (2.3.1b) and then semi-freeness of A guarantees consistency of this lift, i.e., there is no enforced relation between the generators of the underlying graded commutative algebra $A^\sharp$ which we could possibly spoil. Likewise, the action is still by post-multiplication hence we have the following isomorphism

$$\mathsf{T}_A^\sharp \cong A^\sharp \otimes \underline{\hom}(V, \mathbb{K})^\sharp \tag{2.3.4d}$$

as $A^\sharp$-graded-modules. In other words, the derivations T_A form a sf A-dg-module[5] with differential $\partial = [\delta, \cdot]$. Furthermore, if we denote the generators of V by $\{v_i \in V\}_{i\in I}$ then the generators of the derivations (as a sf A-dg-module) are given by $\left\{\frac{\partial}{\partial v_i} \in \mathsf{T}_A\right\}_{i\in I}$ and they are such that

$$\left[\frac{\partial}{\partial v_i}, \frac{\partial}{\partial v_j}\right](v_k) = \left(\frac{\partial}{\partial v_i} \circ \frac{\partial}{\partial v_j} - (-1)^{|v_i||v_j|}\frac{\partial}{\partial v_j} \circ \frac{\partial}{\partial v_i}\right)(v_k) = \frac{\partial}{\partial v_i}(\delta^j_k \mathbb{1}) - (-1)^{|v_i||v_j|}\frac{\partial}{\partial v_j}(\delta^i_k \mathbb{1}) = 0 \tag{2.3.4e}$$

where δ^j_k is the Kronecker delta and we have used the well-known fact that the derivation of an algebra unit is always zero[6]. Again, a derivation of a sfCDGA is completely determined by its specified value on the generators of the CDGA hence we have

$$\left[\frac{\partial}{\partial v_i}, \frac{\partial}{\partial v_j}\right] = 0 \quad . \tag{2.3.4g}$$

△

Example 2.3.4. Given a fd Lie algebra $\mathfrak{g}$, construct the corresponding sfCDGA fg in degree 1 as the Chevalley-Eilenberg algebra $\mathrm{CE}_\mathfrak{g} := \big(\mathrm{Sym}(\mathfrak{g}^*[-1]), \delta\big)$ of Example 2.2.6. Remark 2.3.3 tells us that the derivations are given by the following sf $\mathrm{CE}_\mathfrak{g}$-dg-module fg in degree -1,

$$\mathsf{T}_{\mathrm{CE}_\mathfrak{g}} \cong \big(\mathrm{CE}^\sharp_\mathfrak{g} \otimes (\mathfrak{g}^*[-1])^*, \partial, \triangleright\big) \quad . \tag{2.3.5a}$$

We can be more explicit and specifically determine the result of the differential ∂ on the generators $\frac{\partial}{\partial(\vartheta^i)} \in (\mathfrak{g}^*[-1])^*$ in terms of something familiar from Example 2.2.10,

$$\begin{aligned}
\left(\partial\frac{\partial}{\partial(\vartheta^j)}\right)(\vartheta^i) &= \left[\delta, \frac{\partial}{\partial(\vartheta^j)}\right](\vartheta^i) \\
&= \left(\delta \circ \frac{\partial}{\partial(\vartheta^j)} + \frac{\partial}{\partial(\vartheta^j)} \circ \delta\right)(\vartheta^i) \\
&= \delta(\delta^i_j \mathbb{1}) - \frac{1}{2}\mathfrak{g}^i_{kl}\frac{\partial}{\partial(\vartheta^j)}\big(\vartheta^k\vartheta^l\big) \\
&= -\frac{1}{2}\mathfrak{g}^i_{kl}\delta^k_j\vartheta^l + \frac{1}{2}\mathfrak{g}^i_{kl}\vartheta^k\delta^l_j \\
&= \mathfrak{g}^i_{kj}\vartheta^k
\end{aligned}$$

where the last equality comes from a reindexing on the first term. In other words,

$$\partial\frac{\partial}{\partial(\vartheta^j)} = \mathfrak{g}^i_{kj}\vartheta^k\frac{\partial}{\partial(\vartheta^i)} \quad . \tag{2.3.5b}$$

Comparing (2.3.5b) with (2.2.40) gives us that, as graded vector spaces,

$$(\mathfrak{g}^*[-1])^* \cong \mathfrak{g}[1] \quad , \qquad \frac{\partial}{\partial(\vartheta^i)} \mapsto s^{-1}x_i \quad , \tag{2.3.5c}$$

we rewrite (2.3.5b) as

$$\partial(s^{-1}x_j) = \mathfrak{g}^i_{kj}\vartheta^k s^{-1}x_i = \vartheta^k s^{-1}[x_k, x_j] = \vartheta^k\left(x_k.\big(s^{-1}x_j\big)\right) \quad . \tag{2.3.5d}$$

[5]Generated in the dual degrees of those of the generators of A.

[6]Simply because the unit $\eta : \mathbb{K} \to A$ is a cochain map hence $|\mathbb{1}| = 0$,

$$D(\mathbb{1}) = D(\mathbb{1}\mathbb{1}) = D(\mathbb{1})\mathbb{1} + \mathbb{1}D(\mathbb{1}) = D(\mathbb{1}) + D(\mathbb{1}) = 2D(\mathbb{1}) \quad \Longrightarrow \quad D(\mathbb{1}) = 0 \quad . \tag{2.3.4f}$$

In sum, the derivations of the Chevalley-Eilenberg algebra of a fd Lie algebra are isomorphic (as $\mathrm{CE}_{\mathfrak{g}}$-dg-modules) to the Chevalley-Eilenberg module of the 1-shifted adjoint representation of the Lie algebra,

$$\mathsf{T}_{\mathrm{CE}_{\mathfrak{g}}} \cong \mathrm{CE}_{\mathfrak{g}}(\mathfrak{g}_{\mathrm{ad}}[1]) \quad . \tag{2.3.5e}$$

The semi-freeness of $\mathrm{CE}_{\mathfrak{g}}$ means that (2.3.4g) still holds hence

$$[s^{-1}x_j, s^{-1}x_k] = 0 \quad . \tag{2.3.5f}$$

$\triangledown$

Let us recall a standard deformation construction for CDGAs which will be relevant in the next section.

Definition 2.3.5. A degree 1 derivation $\alpha \in \mathsf{T}^1_A$ on a CDGA $A = (A, \mu, \eta)$ is called a **Maurer-Cartan derivation** (in the DGLA T_A) if it satisfies the Maurer-Cartan equation

$$\partial\alpha + \tfrac{1}{2}\,[\alpha, \alpha] \;=\; 0 \quad . \tag{2.3.6}$$

Given any Maurer-Cartan derivation $\alpha \in \mathsf{T}^1_A$, one may deform the differential $\delta \in \mathsf{T}^1_A$ on A to a new differential

$$\delta_\alpha \;:=\; \delta + \alpha \;\in\; \mathsf{T}^1_A \tag{2.3.7}$$

which, as a consequence of the Maurer-Cartan equation, squares to zero $\delta_\alpha^2 = 0$. One checks that endowing the resulting cochain complex $A_\alpha := (A, \delta_\alpha) \in \mathbf{Ch}$ with the given multiplication and unit defines a new CDGA $A_\alpha = (A_\alpha, \mu, \eta)$. Note that in the case of a free CDGA $A = \mathrm{Sym}\, V$, the result $(\mathrm{Sym}\, V)_\alpha$ of such a deformation along a Maurer-Cartan derivation α provides a sfCDGA as in Definition 2.2.4.

Remark 2.3.6. We conclude this subsection by providing some comments about the Kähler differentials under the hypothesis that A is a fgsfCDGA, in which case the dg-module of derivations $\mathsf{T}_A \in {}_A\mathbf{dgMod}^{\mathrm{fgsf}}$ is dualizable. We define the A-dg-module of **Kähler differentials** by the internal hom

$$\Omega_A \;:=\; \underline{\hom}_A(\mathsf{T}_A, A) \;\in\; {}_A\mathbf{dgMod}^{\mathrm{fgsf}} \quad , \tag{2.3.8a}$$

which is fgsf as a consequence of Remark 2.2.8 and

$$(\Omega_A)^\sharp \;\cong\; A^\sharp \otimes (V^\sharp)^{**} \;\cong\; A^\sharp \otimes V^\sharp \quad . \tag{2.3.8b}$$

We denote the **duality pairings** between Ω_A and T_A by

$$\langle\,\cdot\,,\,\cdot\,\rangle \,:\, \Omega_A \otimes_A \mathsf{T}_A \;\to\; A \;,\;\; \omega \otimes_A D \;\mapsto\; \langle \omega, D\rangle \;:=\; \omega(D) \tag{2.3.9a}$$

and

$$\langle\,\cdot\,,\,\cdot\,\rangle \,:\, \mathsf{T}_A \otimes_A \Omega_A \;\to\; A \;,\;\; D \otimes_A \omega \;\mapsto\; \langle D, \omega\rangle \;:=\; (-1)^{|D||\omega|}\,\omega(D) \quad , \tag{2.3.9b}$$

for all homogeneous $\omega \in \Omega_A$ and $D \in \mathsf{T}_A$. The **de Rham differential** $\mathrm{d}_{\mathrm{dR}} : A \to \Omega_A$ is defined by

$$\big\langle D, \mathrm{d}_{\mathrm{dR}}(a)\big\rangle \;:=\; D(a) \quad , \tag{2.3.10}$$

for all $a \in A$ and $D \in \mathsf{T}_A$. From this definition one directly checks that $\mathrm{d}_{\mathrm{dR}}(a\,a') = \mathrm{d}_{\mathrm{dR}}(a) \cdot a' + a \cdot \mathrm{d}_{\mathrm{dR}}(a')$, for all $a, a' \in A$, and that $\mathrm{d}_{\Omega_A}\,\mathrm{d}_{\mathrm{dR}} = \mathrm{d}_{\mathrm{dR}}\,\mathrm{d}_A$. $\triangle$

2.3.2 The de Rham algebra and shifted symplectic forms

Remark 2.3.6 has introduced the Kähler differential forms via the dualisation (using the closed monoidal structure of ${}_A\mathbf{dgMod}$) of the derivations T_A. This subsection will construct the Kähler differential forms as objects in their own right (without referring to duals) by first constructing the de Rham algebra from a stated theorem. Recall that a symplectic form on a smooth manifold M is a 2-cocycle in the de Rham algebra (i.e., a 2-form $\omega \in \Omega^2(M)$ which is closed under the exterior derivative $\mathrm{d}_{\mathrm{dR}}\omega = 0$) whose pointwise restriction $\omega_p(\cdot,\cdot) : T_pM \wedge_{\mathbb{K}} T_pM \to \mathbb{K}$ is non-degenerate. The above mentioned stated theorem constructs the de Rham algebra as a *double* complex hence, in order to have only *one* notion of closedness, we take the product totalisation of this double complex. Assuming a non-negatively graded CDGA $(A^{\geq 0}, \delta)$, defining the notion of a shifted symplectic structure on A requires further introducing a quasi-isomorphism condition which generalises the non-degeneracy condition of a symplectic form. We finish this subsection by then showing that the *only* shifted symplectic structures on $\mathrm{CE}_{\mathfrak{g}}$ (for $\mathfrak{g}$ a fd Lie algebra) are simply the 2-shifted symplectic structures and these can be characterised as quadratic Lie algebra structures on $\mathfrak{g}$, i.e., non-degenerate symmetric ad-invariant pairings on $\mathfrak{g}$.

The de Rham algebra of a CDGA A

As mentioned above, the de Rham algebra of a CDGA A is a double complex hence let us recall the definition of such double complexes as in [MuRo19, Section 2].

Definition 2.3.7. A **double cochain complex** $V^{\bullet,\bullet} = (V^{\sharp,\sharp}, \mathrm{d}, \delta)$ consists of a $\mathbb{Z}\times\mathbb{Z}$-graded vector space, i.e., a family

$$V^{\sharp,\sharp} = \left\{V^{i,j}\right\}_{(i,j)\in\mathbb{Z}\times\mathbb{Z}} \tag{2.3.11a}$$

of vector spaces labelled by two integers, $i \in \mathbb{Z}$ (called **vertical degree**) and $j \in \mathbb{Z}$ (called **horizontal degree**), and two families

$$\mathrm{d} = \left\{\mathrm{d} : V^{i,j} \to V^{i+1,j}\right\}_{(i,j)\in\mathbb{Z}\times\mathbb{Z}} \quad , \quad \delta = \left\{\delta : V^{i,j} \to V^{i,j+1}\right\}_{(i,j)\in\mathbb{Z}\times\mathbb{Z}} \tag{2.3.11b}$$

of degree-increasing linear maps (called **vertical differential** and **horizontal differential**, respectively) which square to zero and commute with one another[7],

$$\mathrm{d}^2 = \delta^2 = \mathrm{d}\circ\delta - \delta\circ\mathrm{d} = 0 \quad . \tag{2.3.11c}$$

Remark 2.3.8. The fact that the differentials commute with one another means that: for fixed vertical degree $i \in \mathbb{Z}$, the vertical differential $\mathrm{d}_i := \{\mathrm{d} : V^{i,j} \to (V[1,0])^{i,j}\}_{j\in\mathbb{Z}}$ is a cochain map and, for fixed horizontal degree $j \in \mathbb{Z}$, the horizontal differential $\delta_j := \{\delta : V^{i,j} \to (V[0,1])^{i,j}\}_{i\in\mathbb{Z}}$ is a cochain map. △

A **double cochain map** $f : V^{\bullet,\bullet} \to W^{\bullet,\bullet}$ is a family $\{f : V^{i,j} \to W^{i,j}\}_{(i,j)\in\mathbb{Z}\times\mathbb{Z}}$ of linear maps which preserve both the vertical and horizontal degrees which are compatible with both differentials,

$$\mathrm{d}\circ f = f\circ\mathrm{d} \quad , \quad \delta\circ f = f\circ\delta \quad . \tag{2.3.12}$$

We denote by $\mathbf{Ch}^{\bullet,\bullet}$ the category of double cochain complexes and double cochain maps. The category $\mathbf{Ch}^{\bullet,\bullet}$ carries the following standard symmetric monoidal structure. The **tensor product**

[7] As opposed to **bicomplexes** where the only difference is that the differentials anti-commute.

$V^{\bullet,\bullet} \otimes W^{\bullet,\bullet} \in \mathbf{Ch}^{\bullet,\bullet}$ of two double cochain complexes $V^{\bullet,\bullet}, W^{\bullet,\bullet} \in \mathbf{Ch}^{\bullet,\bullet}$ is given by

$$(V^{\bullet,\bullet} \otimes W^{\bullet,\bullet})^{i,j} := \bigoplus_{(k,l)\in\mathbb{Z}\times\mathbb{Z}} V^{k,l} \otimes W^{i-k,j-l} \ , \tag{2.3.13a}$$

for all $(i,j) \in \mathbb{Z} \times \mathbb{Z}$, and with differentials on homogeneous elements

$$\begin{aligned} \mathrm{d}(v \otimes w) &:= (\mathrm{d}v) \otimes w + (-1)^{|v|_{\mathbf{v}}} v \otimes \mathrm{d}w \quad , \\ \delta(v \otimes w) &:= (\delta v) \otimes w + (-1)^{|v|_{\mathbf{h}}} v \otimes \delta w \quad , \end{aligned} \tag{2.3.13b}$$

where the subscript $\mathbf{v}$ refers to the vertical degree and the subscript $\mathbf{h}$ refers to the horizontal degree of the element. The unit for the monoidal product is once again the ground field $\mathbb{K}$ concentrated in degree (0,0) with both differentials annihilating every element.

The symmetric braiding is defined component-wise by the following Koszul sign rule,

$$\gamma_{V^{\bullet,\bullet}W^{\bullet,\bullet}} : V^{\bullet,\bullet} \otimes W^{\bullet,\bullet} \to W^{\bullet,\bullet} \otimes V^{\bullet,\bullet} \quad , \quad v \otimes w \mapsto (-1)^{|v|_{\mathbf{h}}|w|_{\mathbf{h}}+|v|_{\mathbf{v}}|w|_{\mathbf{v}}} w \otimes v \quad , \tag{2.3.14}$$

for all homogeneous $v \in V^{\bullet,\bullet}$ and $w \in W^{\bullet,\bullet}$.

A **commutative differential doubly-graded algebra** (in short, a **CDdGA**) is a commutative algebra object in the symmetric monoidal category $(\mathbf{Ch}^{\bullet,\bullet}, \otimes, \mathbb{K}, \gamma)$, i.e., a triple $(B^{\bullet,\bullet}, \mu, \eta)$, analogous to Subsection 2.2.2, whose double cochain multiplication $\mu : B^{\bullet,\bullet} \otimes B^{\bullet,\bullet} \to B^{\bullet,\bullet}$ is associative/commutative, analogous to (2.2.13c)/(2.2.16), and whose double cochain unit map $\eta : \mathbb{K} \to B^{\bullet,\bullet}$ is unital, analogous to (2.2.13d). We denote by $\mathbf{CDdGA} := \mathbf{CAlg}(\mathbf{Ch}^{\bullet,\bullet})$ the category of CDdGAs whose morphisms are double cochain algebra morphisms, i.e., double cochain maps which preserve the multiplication and unit analogous to (2.2.17).

We say that a double cochain complex is **vertically positive** if $V^{i,j} = 0$ for $i < 0$ and we denote the symmetric monoidal full subcategory of such objects as $\mathbf{Ch}^{\geq 0,\bullet}$. Furthermore, we denote by $\mathbf{CAlg}(\mathbf{Ch}^{\geq 0,\bullet})$ the category of **vertically positive CDdGAs** and their double cochain algebra morphisms. We are now ready to make sense of the statement from [CPTVV17, Proposition 1.3.8] which we state without proving.

Theorem 2.3.9. *Define the forgetful functor*

$$\mathcal{U} : \mathbf{CAlg}(\mathbf{Ch}^{\geq 0,\bullet}) \to \mathbf{CAlg}(\mathbf{Ch}^{\bullet}) \tag{2.3.15a}$$

to act on objects as

$$\mathcal{U}\left(B^{\geq 0,\bullet}\right) = \mathcal{U}(B^{\geq 0,\sharp}, \mathrm{d}, \delta) = (B^{0,\sharp}, \delta) = B^{0,\bullet} \tag{2.3.15b}$$

and on morphisms[8] *as*

$$\mathcal{U}\{f : B^{i,j} \to C^{i,j}\}_{(i,j)\in\mathbb{N}\times\mathbb{Z}} = \{f : B^{0,j} \to C^{0,j}\}_{j\in\mathbb{Z}} \quad . \tag{2.3.15c}$$

This functor admits a left adjoint

$$\Omega : \mathbf{CAlg}(\mathbf{Ch}^{\bullet}) \to \mathbf{CAlg}(\mathbf{Ch}^{\geq 0,\bullet}) \tag{2.3.16}$$

which we call the ***de Rham functor****.*

Definition 2.3.10. Given a CDGA (A, δ, μ, η), we define the **de Rham algebra** of A as the vertically positive CDdGA

$$\Omega_A := \Omega(A) = (\Omega_A^{\geq 0,\sharp}, \mathrm{d}_{\mathrm{dR}}, \delta, \wedge_A, \eta) \quad . \tag{2.3.17}$$

[8] Notice that this then tells us that the CDdGA structure on $B^{\geq 0,\bullet}$ is mapped to a CDGA structure on $B^{0,\bullet}$.

Remark 2.3.11. Note that we have $(\Omega_A^{0,\sharp}, \delta, \wedge_A, \eta) \cong (A, \delta, \mu, \eta)$ as CDGAs. Restricting to $\Omega_A^{1,\sharp}$, the graded $\mathbb{K}$-linear map $\delta : \Omega_A^{1,\sharp} \to \Omega_A^{1,\sharp}[0,1]$ still squares to zero hence provides a well-defined differential, we denote this cochain complex as $\Omega_A^1 := (\Omega_A^{1,\sharp}, \delta)$. The associative unital double cochain multiplication $\wedge_A$ restricts to a well-defined left A-action

$$\wedge_A : A \otimes \Omega_A^1 \to \Omega_A^1 \tag{2.3.18}$$

which is associative and unital. We thus define the **Kähler differential forms** of A as the A-dg-module $(\Omega_A^1, \wedge_A)$. △

We could invert the order of this construction by following, for example, [Eis95, Chapter 16].

Definition 2.3.12. A module of **Kähler differential forms** of A over $\mathbb{K}$ is a 2-tuple $(\Omega^1_{A/\mathbb{K}}, \mathrm{d}_{\mathrm{dR}})$ where $\Omega^1_{A/\mathbb{K}}$ is an A-dg-module and $\mathrm{d}_{\mathrm{dR}} : A \to \Omega^1_{A/\mathbb{K}}$ is a cochain derivation (called the **de Rham differential**) whose image generates $\Omega^1_{A/\mathbb{K}}$ as an A-dg-module. Let us expand the meaning of these terms, the cochain property means that:

(i) A homogeneous element $a \in A$ has degree $|\mathrm{d}_{\mathrm{dR}}(a)| = |a|$.

(ii) The de Rham differential is $\mathbb{K}$-linear,

$$\mathrm{d}_{\mathrm{dR}}(\alpha a + \beta b) = \alpha \mathrm{d}_{\mathrm{dR}}(a) + \beta \mathrm{d}_{\mathrm{dR}}(b) \quad , \tag{2.3.19a}$$

where $a, b \in A$ and $\alpha, \beta \in \mathbb{K}$.

(iii) The de Rham differential commutes with the CDGA differential (which is actually the differential of $\Omega^1_{A/\mathbb{K}}$),

$$\partial(\mathrm{d}_{\mathrm{dR}}) = [\delta, \mathrm{d}_{\mathrm{dR}}] = \delta \circ \mathrm{d}_{\mathrm{dR}} - \mathrm{d}_{\mathrm{dR}} \circ \delta = 0 \quad . \tag{2.3.19b}$$

The derivation property means that the Leibniz law holds:

$$\mathrm{d}_{\mathrm{dR}}(ab) = (\mathrm{d}_{\mathrm{dR}}a)b + a\mathrm{d}_{\mathrm{dR}}b \quad , \tag{2.3.20}$$

where $a, b \in A$.

Proposition 2.3.13. *The cochain derivation $\mathrm{d}_{\mathrm{dR}} : A \to \Omega^1_{A/\mathbb{K}}$ has the following universal property*[9]*: given any A-dg-module M and cochain derivation $\mathcal{D} : A \to M$, there exists a unique A-linear cochain map $D : \Omega^1_{A/\mathbb{K}} \to M$ such that $\mathcal{D} = D \circ \mathrm{d}_{\mathrm{dR}}$.*

Proof. Any given A-linear cochain map is determined by its restriction to the generators hence $D : \Omega^1_{A/\mathbb{K}} \to M$, if it exists, is determined by its values $D(\mathrm{d}_{\mathrm{dR}}a)$ and these values must be equal to $\mathcal{D}(a)$. If there were another such A-linear cochain map $D' : \Omega^1_{A/\mathbb{K}} \to M$ then we would have $D'(\mathrm{d}_{\mathrm{dR}}a) = \mathcal{D}(a) = D(\mathrm{d}_{\mathrm{dR}}a)$ ergo $D' = D$. Now we must prove existence, define $D(\mathrm{d}_{\mathrm{dR}}a) := \mathcal{D}(a)$ and lift to the whole A-dg-module $\Omega^1_{A/\mathbb{K}}$ by forcing A-linearity, i.e., $D(a\,\mathrm{d}_{\mathrm{dR}}b) := a\mathcal{D}(b)$. It is clear that this provides a graded $\mathbb{K}$-linear map hence the only well-definition check is merely to verify the cochain property,

$$\mathrm{d}_M\left(D(a\,\mathrm{d}_{\mathrm{dR}}b)\right) = \mathrm{d}_M\left(a\mathcal{D}(b)\right) = (\delta a)\mathcal{D}(b) + (-1)^{|a|}a\mathcal{D}(\delta b) = D\left((\delta a)\mathrm{d}_{\mathrm{dR}}b\right) + (-1)^{|a|}D(a\,\mathrm{d}_{\mathrm{dR}}(\delta b)) \ , \tag{2.3.21}$$

where we have used the Leibniz law (2.2.28d) and the fact that $\mathcal{D}$ is cochain for the second equality. It is clear that this last equality is the same as $D\left(\delta\big(a\,\mathrm{d}_{\mathrm{dR}}(b)\big)\right)$. □

[9]Which, as always, determines the map and the codomain uniquely up to canonical isomorphism.

Remark 2.3.14. This universal property justifies referring to $\Omega^1_{A/\mathbb{K}} = (\Omega^1_{A/\mathbb{K}}, \mathrm{d}_{\mathrm{dR}})$ as *the* Kähler differential forms. We could rephrase the universal property as the following isomorphism

$$Z^0\big(\mathsf{T}(A,M)\big) \cong Z^0\big(\underline{\hom}_A(\Omega^1_{A/\mathbb{K}}, M)\big) \tag{2.3.22}$$

natural in $M \in {}_A\mathbf{dgMod}$. This sometimes goes by the slogan: the module of Kähler differential forms "A-linearises" the cochain derivations from A. △

We define the higher forms by $\Omega^i_{A/\mathbb{K}} := \bigwedge^i_A(\Omega^1_{A/\mathbb{K}})$ for $i \in \mathbb{N}$ where we set $\Omega^0_{A/\mathbb{K}} = \bigwedge^0_A(\Omega^1_{A/\mathbb{K}}) := A$.

Lemma 2.3.15. *For $p \in \mathbb{N}^*$, there exists a unique cochain map $\mathrm{d}_{\mathrm{dR}} : \Omega^p_{A/\mathbb{K}} \to \Omega^{p+1}_{A/\mathbb{K}}$ given on elements by*

$$\mathrm{d}_{\mathrm{dR}}\big(a_0\mathrm{d}_{\mathrm{dR}}(a_1) \wedge_A ... \wedge_A \mathrm{d}_{\mathrm{dR}}(a_p)\big) = \mathrm{d}_{\mathrm{dR}}(a_0) \wedge_A \mathrm{d}_{\mathrm{dR}}(a_1) \wedge_A ... \wedge_A \mathrm{d}_{\mathrm{dR}}(a_p) \quad . \tag{2.3.23}$$

Proof. Uniqueness is obvious hence let us show existence. It is degree-preserving and $\mathbb{K}$-linear because both the de Rham differential d_{dR} and the A-relative wedge product are. It commutes with δ because d_{dR} does and because $\wedge_A$ (and the action of A) satisfy the Leibniz law. □

We define the following $\mathbb{N} \times \mathbb{Z}$-graded vector space $\Omega_{A/\mathbb{K}} := \bigwedge_A(\Omega^1_{A/\mathbb{K}})$.

Corollary 2.3.16. *The following is called the **de Rham algebra** of A over $\mathbb{K}$,*

$$A \xrightarrow{\mathrm{d}_{\mathrm{dR}}} \Omega^1_{A/\mathbb{K}} \xrightarrow{\mathrm{d}_{\mathrm{dR}}} \Omega^2_{A/\mathbb{K}} \xrightarrow{\mathrm{d}_{\mathrm{dR}}} \Omega^3_{A/\mathbb{K}} \xrightarrow{\mathrm{d}_{\mathrm{dR}}} \cdots \quad . \tag{2.3.24}$$

It is a vertically positive CDdGA in the sense that $\big(\Omega_{A/\mathbb{K}}, \mathrm{d}_{\mathrm{dR}}, \delta, \wedge_A, \eta\big) \in \mathbf{CAlg}(\mathbf{Ch}^{\geq 0,\bullet})$. The vertical differential is the de Rham differential d_{dR}, the horizontal differential is the CDGA differential δ, the double cochain unit map is the same as that of A and the double cochain multiplication map is the wedge product relative to A.

Proof. We already know that $[\delta, \mathrm{d}_{\mathrm{dR}}] = \delta^2 = 0$ so we first check that d_{dR} squares to zero,

$$\begin{aligned}
\mathrm{d}^2_{\mathrm{dR}}\big(a_0\mathrm{d}_{\mathrm{dR}}(a_1) \wedge_A ... \wedge_A \mathrm{d}_{\mathrm{dR}}(a_p)\big) &= \mathrm{d}_{\mathrm{dR}}\big(\mathrm{d}_{\mathrm{dR}}(a_0) \wedge_A \mathrm{d}_{\mathrm{dR}}(a_1) \wedge_A ... \wedge_A \mathrm{d}_{\mathrm{dR}}(a_p)\big) \\
&= \mathrm{d}_{\mathrm{dR}}(\mathbb{1}) \wedge_A \mathrm{d}_{\mathrm{dR}}(a_0) \wedge_A \mathrm{d}_{\mathrm{dR}}(a_1) \wedge_A ... \wedge_A \mathrm{d}_{\mathrm{dR}}(a_p) \\
&= 0
\end{aligned}$$

where we have used (2.3.23) for the first and second equalities whereas we have used the well-known fact that derivations annihilate algebra units, as in (2.3.4f), for the last equality. By construction $\eta : \mathbb{K} \to A = \Omega^0_{A/\mathbb{K}}$ is double cochain and provides a unit for the commutative associative multiplication $\wedge_A$. The multiplication $\wedge_A$ is also, by construction, degree-preserving and $\mathbb{K}$-linear as well as compatible with δ hence let us check it is compatible with d_{dR}. Given

$$w = a_0\mathrm{d}_{\mathrm{dR}}(a_1) \wedge_A ... \wedge_A \mathrm{d}_{\mathrm{dR}}(a_p) \tag{2.3.25a}$$

and

$$w' = a'_0\mathrm{d}_{\mathrm{dR}}(a'_1) \wedge_A ... \wedge_A \mathrm{d}_{\mathrm{dR}}(a'_p) \quad , \tag{2.3.25b}$$

we have

$$w \wedge_A w' = (-1)^{|a'_0||a_1 \cdots a_p|} a_0 a'_0 \mathrm{d}_{\mathrm{dR}}(a_1) \wedge_A ... \wedge_A \mathrm{d}_{\mathrm{dR}}(a_p) \wedge_A \mathrm{d}_{\mathrm{dR}}(a'_1) \wedge_A ... \wedge_A \mathrm{d}_{\mathrm{dR}}(a'_p) \tag{2.3.25c}$$

hence

$$\begin{aligned}
\mathrm{d}_{\mathrm{dR}}(w \wedge_A w') =& (-1)^{|a_0'||a_1\cdots a_p|}\mathrm{d}_{\mathrm{dR}}(a_0)a_0' \wedge_A \mathrm{d}_{\mathrm{dR}}(a_1) \wedge_A ... \wedge_A \mathrm{d}_{\mathrm{dR}}(a_p) \wedge_A \mathrm{d}_{\mathrm{dR}}(a_1') \wedge_A ... \wedge_A \mathrm{d}_{\mathrm{dR}}(a_p') \\
&+ (-1)^{|a_0'||a_1\cdots a_p|}a_0\mathrm{d}_{\mathrm{dR}}(a_0') \wedge_A \mathrm{d}_{\mathrm{dR}}(a_1) \wedge_A ... \wedge_A \mathrm{d}_{\mathrm{dR}}(a_p) \wedge_A \mathrm{d}_{\mathrm{dR}}(a_1') \wedge_A ... \wedge_A \mathrm{d}_{\mathrm{dR}}(a_p') \\
=& \mathrm{d}_{\mathrm{dR}}(a_0) \wedge_A \mathrm{d}_{\mathrm{dR}}(a_1) \wedge_A ... \wedge_A \mathrm{d}_{\mathrm{dR}}(a_p) \wedge_A w' \\
&+ (-1)^p w \wedge_A \mathrm{d}_{\mathrm{dR}}(a_0') \wedge_A \mathrm{d}_{\mathrm{dR}}(a_1') \wedge_A ... \wedge_A \mathrm{d}_{\mathrm{dR}}(a_p') \\
=& (\mathrm{d}_{\mathrm{dR}}w) \wedge_A w' + (-1)^p w \wedge_A \mathrm{d}_{\mathrm{dR}}w'
\end{aligned}$$

□

Example 2.3.17. Following on from Example 2.2.6, we take the base CDGA as $A = \mathrm{CE}_{\mathfrak{g}}$ for $\mathfrak{g}$ a fd Lie algebra. As mentioned in Remark 2.3.3, a derivation of a sfCDGA is completely determined by its restriction to the generators hence the $\mathrm{CE}_{\mathfrak{g}}$-dg-module of Kähler differential forms $\Omega^1_{\mathrm{CE}_{\mathfrak{g}}/\mathbb{K}}$ (from Definition 2.3.12) is a *semi-free* $\mathrm{CE}_{\mathfrak{g}}$-dg-module generated by $\{\mathrm{d}_{\mathrm{dR}}(\vartheta^i)\}_{1\leq i\leq \mathrm{dim}\mathfrak{g}}$ where $|\mathrm{d}_{\mathrm{dR}}(\vartheta^i)| = 1$. Dual to Example 2.3.4, we wish to relate the result of taking the differential δ of a generator $\mathrm{d}_{\mathrm{dR}}(\vartheta^i)$ to some familiar representation of $\mathfrak{g}$. We use the fact that the de Rham differential commutes with the CDGA differential (2.3.19b) in the following,

$$\begin{aligned}
\delta\left(\mathrm{d}_{\mathrm{dR}}(\vartheta^i)\right) &= \mathrm{d}_{\mathrm{dR}}\left(\delta(\vartheta^i)\right) \\
&= \mathrm{d}_{\mathrm{dR}}\left(-\tfrac{1}{2}\mathfrak{g}^i_{jk}\vartheta^j\vartheta^k\right) \\
&= -\tfrac{1}{2}\mathfrak{g}^i_{jk}\mathrm{d}_{\mathrm{dR}}(\vartheta^j)\vartheta^k - \tfrac{1}{2}\mathfrak{g}^i_{jk}\vartheta^j\mathrm{d}_{\mathrm{dR}}(\vartheta^k) \\
&= -\mathfrak{g}^i_{jk}\vartheta^j\mathrm{d}_{\mathrm{dR}}(\vartheta^k) \qquad . \qquad (2.3.26)
\end{aligned}$$

Recall the definition of the **coadjoint representation** $\mathrm{ad}^* : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}^*)$ where

$$\left(x_j.\theta^i\right)(x_k) := -\theta^i([x_j, x_k]) = -\mathfrak{g}^i_{jk} \qquad \Longrightarrow \qquad \left(x_j.\theta^i\right) = -\mathfrak{g}^i_{jk}\theta^k \quad , \qquad (2.3.27)$$

and notice that, as graded vector spaces

$$\mathrm{d}_{\mathrm{dR}}\mathfrak{g}^*[-1] \cong \mathfrak{g}^*[-1] \qquad , \qquad \mathrm{d}_{\mathrm{dR}}(\vartheta^i) \mapsto \vartheta^i \quad , \qquad (2.3.28)$$

hence we rewrite (2.3.26) as

$$\mathrm{d}_{\mathfrak{g}^*}(\vartheta^i) = \vartheta^j\left(x_j.(\vartheta^i)\right) = -\mathfrak{g}^i_{jk}\vartheta^j(\vartheta^k) \qquad . \qquad (2.3.29)$$

In sum, we have the following isomorphism between sf $\mathrm{CE}_{\mathfrak{g}}$-dg-modules fg in degree 1,

$$\Omega^1_{\mathrm{CE}_{\mathfrak{g}}/\mathbb{K}} \cong \mathrm{CE}_{\mathfrak{g}}(\mathfrak{g}^*_{\mathrm{ad}^*}[-1]) \quad . \qquad (2.3.30)$$

Now we can see that the de Rham algebra of $\mathrm{CE}_{\mathfrak{g}}$ over $\mathbb{K}$ from Corollary 2.3.16 is a sf CDdGA,

$$\Omega^{\sharp,\sharp}_{\mathrm{CE}_{\mathfrak{g}}/\mathbb{K}} \cong \mathbb{K}[\vartheta^1, \ldots, \vartheta^{\mathrm{dim}\mathfrak{g}}, \mathrm{d}_{\mathrm{dR}}(\vartheta^1), \ldots, \mathrm{d}_{\mathrm{dR}}(\vartheta^{\mathrm{dim}\mathfrak{g}})] \quad , \qquad (2.3.31)$$

where $\vartheta^i \in \Omega^{0,1}_{\mathrm{CE}_{\mathfrak{g}}/\mathbb{K}}$ and $\mathrm{d}_{\mathrm{dR}}(\vartheta^i) \in \Omega^{1,1}_{\mathrm{CE}_{\mathfrak{g}}/\mathbb{K}}$ thus

$$\vartheta^i\vartheta^j = -\vartheta^j\vartheta^i \quad , \qquad (2.3.32a)$$

$$\vartheta^i\mathrm{d}_{\mathrm{dR}}(\vartheta^j) = -\mathrm{d}_{\mathrm{dR}}(\vartheta^j)\vartheta^i \quad , \qquad (2.3.32b)$$

$$\mathrm{d}_{\mathrm{dR}}(\vartheta^i) \wedge_{\mathrm{CE}_{\mathfrak{g}}} \mathrm{d}_{\mathrm{dR}}(\vartheta^j) = \mathrm{d}_{\mathrm{dR}}(\vartheta^j) \wedge_{\mathrm{CE}_{\mathfrak{g}}} \mathrm{d}_{\mathrm{dR}}(\vartheta^i) \quad . \qquad (2.3.32c)$$

The two differentials are determined[10] by:

$$\mathrm{d}_{\mathrm{dR}}[\vartheta^i] := \mathrm{d}_{\mathrm{dR}}(\vartheta^i) \tag{2.3.33a}$$

$$\mathrm{d}_{\mathrm{dR}}\left(\mathrm{d}_{\mathrm{dR}}(\vartheta^i)\right) = 0 \tag{2.3.33b}$$

$$\delta(\vartheta^i) = -\tfrac{1}{2}\mathfrak{g}^i_{jk}\vartheta^j\vartheta^k \tag{2.3.33c}$$

$$\delta\left(\mathrm{d}_{\mathrm{dR}}(\vartheta^i)\right) = -\tfrac{1}{2}\mathfrak{g}^i_{jk}\mathrm{d}_{\mathrm{dR}}(\vartheta^j)\vartheta^k - \tfrac{1}{2}\mathfrak{g}^i_{jk}\vartheta^j\mathrm{d}_{\mathrm{dR}}(\vartheta^k) \quad . \tag{2.3.33d}$$

The fact that the generator 1-forms commute, (2.3.32c), means that there is no upper bound on the vertical form degree of the underlying double complex. Furthermore, the fact that the CDGA generators anti-commute, (2.3.32a), together with the fact that the bidegree of the generators is given by $|\vartheta^i| = (0,1)$ and $|\mathrm{d}_{\mathrm{dR}}(\vartheta^i)| = (1,1)$ means that the underlying double complex has a staircase shape, i.e., $\Omega^{\sharp,\sharp}_{\mathrm{CE}_\mathfrak{g}/\mathbb{K}} = (\Omega^{i,j}_{\mathrm{CE}_\mathfrak{g}/\mathbb{K}})_{0\leq i\leq j\leq i+\dim\mathfrak{g}}$. A visual depiction of part of this staircase shape is given below as a commutative diagram where $\mathrm{d}_{\mathrm{dR}}(\mathfrak{g}^*[-1]) := \mathbb{K}\langle \mathrm{d}_{\mathrm{dR}}(\vartheta^1), \dots, \mathrm{d}_{\mathrm{dR}}(\vartheta^{\dim\mathfrak{g}})\rangle$.

$$\begin{array}{ccccccccc}
\uparrow{\scriptstyle \mathrm{d}_{\mathrm{dR}}} & & \uparrow{\scriptstyle \mathrm{d}_{\mathrm{dR}}} & & \uparrow{\scriptstyle \mathrm{d}_{\mathrm{dR}}} & & \uparrow{\scriptstyle \mathrm{d}_{\mathrm{dR}}} & & \\
\{0\} & \xrightarrow{\delta} & \{0\} & \xrightarrow{\delta} & \{0\} & \xrightarrow{\delta} & \bigwedge^3_{\mathbb{K}}(\mathrm{d}_{\mathrm{dR}}(\mathfrak{g}^*[-1])) & \longrightarrow & \\
\uparrow{\scriptstyle \mathrm{d}_{\mathrm{dR}}} & & \uparrow{\scriptstyle \mathrm{d}_{\mathrm{dR}}} & & \uparrow{\scriptstyle \mathrm{d}_{\mathrm{dR}}} & & \uparrow{\scriptstyle \mathrm{d}_{\mathrm{dR}}} & & \\
\{0\} & \xrightarrow{\delta} & \{0\} & \xrightarrow{\delta} & \bigwedge^2_{\mathbb{K}}(\mathrm{d}_{\mathrm{dR}}(\mathfrak{g}^*[-1])) & \xrightarrow{\delta} & \mathfrak{g}^*[-1]\otimes\bigwedge^2_{\mathbb{K}}(\mathrm{d}_{\mathrm{dR}}(\mathfrak{g}^*[-1])) & \longrightarrow & \\
\uparrow{\scriptstyle \mathrm{d}_{\mathrm{dR}}} & & \uparrow{\scriptstyle \mathrm{d}_{\mathrm{dR}}} & & \uparrow{\scriptstyle \mathrm{d}_{\mathrm{dR}}} & & \uparrow{\scriptstyle \mathrm{d}_{\mathrm{dR}}} & & \\
\{0\} & \xrightarrow{\delta} & \mathrm{d}_{\mathrm{dR}}(\mathfrak{g}^*[-1]) & \xrightarrow{\delta} & \mathfrak{g}^*[-1]\otimes\mathrm{d}_{\mathrm{dR}}(\mathfrak{g}^*[-1]) & \xrightarrow{\delta} & \mathrm{Sym}^2_{\mathbb{K}}(\mathfrak{g}^*[-1])\otimes\mathrm{d}_{\mathrm{dR}}(\mathfrak{g}^*[-1]) & \longrightarrow & \\
\uparrow{\scriptstyle \mathrm{d}_{\mathrm{dR}}} & & \uparrow{\scriptstyle \mathrm{d}_{\mathrm{dR}}} & & \uparrow{\scriptstyle \mathrm{d}_{\mathrm{dR}}} & & \uparrow{\scriptstyle \mathrm{d}_{\mathrm{dR}}} & & \\
\mathbb{K} & \xrightarrow{\delta} & \mathfrak{g}^*[-1] & \xrightarrow{\delta} & \mathrm{Sym}^2_{\mathbb{K}}(\mathfrak{g}^*[-1]) & \xrightarrow{\delta} & \mathrm{Sym}^3_{\mathbb{K}}(\mathfrak{g}^*[-1]) & \longrightarrow &
\end{array}$$

A generic pure element of $\Omega^{i,i+k}_{\mathrm{CE}_\mathfrak{g}/\mathbb{K}}$, for $0 \leq k \leq \dim\mathfrak{g}$, has the following form

$$\omega_{a_1a_2\dots a_kb_1b_2\dots b_i}\vartheta^{a_1}\vartheta^{a_2}\cdots\vartheta^{a_k}\mathrm{d}_{\mathrm{dR}}(\vartheta^{b_1})\wedge_{\mathrm{CE}_\mathfrak{g}}\mathrm{d}_{\mathrm{dR}}(\vartheta^{b_2})\wedge_{\mathrm{CE}_\mathfrak{g}}\cdots\wedge_{\mathrm{CE}_\mathfrak{g}}\mathrm{d}_{\mathrm{dR}}(\vartheta^{b_i}) \tag{2.3.34}$$

with tensor of coefficients taking value in the ground field $\omega_{a_1a_2\dots a_kb_1b_2\dots b_i} \in \mathbb{K}$ with the relevant symmetries, e.g.,

$$\omega_{a_1a_2\dots a_kb_1b_2\dots b_i} = -\omega_{a_2a_1\dots a_kb_1b_2\dots b_i} = \omega_{a_1a_2\dots a_kb_2b_1\dots b_i} \quad . \tag{2.3.35}$$

▽

Shifted Symplectic Structures

As mentioned at the start of this subsection, the de Rham algebra $\Omega_{A/\mathbb{K}}$ is a double complex yet we wish to capture a single notion of closedness in order to unambiguously talk about cocycles. Following the construction laid out in [Pri18, Definition 1.6], we apply a product totalisation to the underlying double complex $(\Omega^{\sharp,\sharp}_{A/\mathbb{K}}, \mathrm{d}_{\mathrm{dR}}, \delta)$ as in [CW94, **Total Complexes** 1.2.6]. We call this product totalisation the **de Rham complex** and denote it as

$$DR := \mathrm{Tot}^{\Pi}(\Omega_{A/\mathbb{K}}) \quad . \tag{2.3.36a}$$

[10] The first equality is actually *not* tautological but rather tells us that the result of applying the vertical differential on a CDGA generator is a generator for the forms.

Explicitly, the m^{th}-degree of the resulting cochain complex is given by

$$DR^m = \prod_{i\in\mathbb{N}} \Omega^{i,m-i}_{A/\mathbb{K}} \quad , \tag{2.3.36b}$$

and a generic element $w \in DR^m$ is simply written as

$$w = (w_{(i)} \in \Omega^{i,m-i}_{A/\mathbb{K}})_{i\in\mathbb{N}} \quad . \tag{2.3.36c}$$

As with any product totalisation, the **total differential** is given by $\mathrm{d}^{\mathrm{Tot}} := \mathrm{d}_{\mathrm{dR}} \pm \delta$ where the sign flips along the indexing of the tuple and our convention is given by

$$\mathrm{d}^{\mathrm{Tot}} w := (\mathrm{d}^{\mathrm{Tot}} w_{(i)})_{i\in\mathbb{N}} \tag{2.3.37a}$$

where

$$\mathrm{d}^{\mathrm{Tot}} w_{(j)} := \begin{cases} \delta w_{(0)} \ , & j = 0 \\ \mathrm{d}_{\mathrm{dR}} w_{(j-1)} + (-1)^{j-1} \delta w_{(j)} \ , & j \geq 1 \end{cases} \tag{2.3.37b}$$

thus

$$\mathrm{d}^{\mathrm{Tot}} w = (\delta w_{(0)} \, , \mathrm{d}_{\mathrm{dR}} w_{(0)} + \delta w_{(1)} \, , \mathrm{d}_{\mathrm{dR}} w_{(1)} - \delta w_{(2)} \, , \ldots , \mathrm{d}_{\mathrm{dR}} w_{(j-1)} + (-1)^{j-1} \delta w_{(j)} \, , \ldots) \quad . \tag{2.3.37c}$$

A generic element $w \in DR^m$ is a cocycle (i.e., $\mathrm{d}^{\mathrm{Tot}} w = 0$) if and only if the following infinite tower of conditions holds:

$$\begin{aligned} \delta w_{(0)} &= 0 \quad , \\ \mathrm{d}_{\mathrm{dR}} w_{(0)} + \delta w_{(1)} &= 0 \quad , \\ \mathrm{d}_{\mathrm{dR}} w_{(1)} - \delta w_{(2)} &= 0 \quad , \\ \mathrm{d}_{\mathrm{dR}} w_{(2)} + \delta w_{(3)} &= 0 \quad , \\ &\vdots \end{aligned}$$

Such an infinite tower of conditions fails to highlight the 2-forms $w_{(2)} \in \Omega^{2,m-2}_{A/\mathbb{K}}$ as the forms of interest, instead singling out 0-forms $w_{(0)} \in \Omega^{0,m}_{A/\mathbb{K}}$. The simplest solution is to apply a filtration on the de Rham complex as in [Pri18, Definition 1.6].

Definition 2.3.18. For $p \in \mathbb{N}^*$, the **p-Filtered de Rham complex** $\mathrm{Fil}^p DR$ is the product totalisation of the following double complex,

$$\Omega^{\geq p}_{A/\mathbb{K}} := \begin{cases} 0 \ , & i < p \\ \Omega^i_{A/\mathbb{K}} \ , & i \geq p \end{cases} \tag{2.3.38}$$

Simply put, the m^{th}-degree,

$$(\mathrm{Fil}^p DR)^m = \prod_{i\geq p} (\Omega^{i,m-i}_{A/\mathbb{K}}) \quad , \tag{2.3.39}$$

is given just as before but now by nullifying all the forms lower than the p-forms.

Definition 2.3.19. An **n-shifted pre-symplectic structure** ω is an $(n+2)$-cocycle in $\mathrm{Fil}^2 DR$, i.e.,

$$\omega = (\omega_{(i)} \in \Omega^{i,n+2-i}_{A/\mathbb{K}})_{i\geq 2} \in \mathrm{Fil}^2 DR \tag{2.3.40}$$

such that

$$\mathrm{d}^{\mathrm{Tot}} \omega = (\mathrm{d}^{\mathrm{Tot}} \omega_{(i)})_{i\geq 2} = 0 \tag{2.3.41a}$$

where

$$\mathrm{d}^{\mathrm{Tot}}\omega_{(j)} = \begin{cases} \delta\omega_{(2)} \ , & j = 2 \\ \mathrm{d}_{\mathrm{dR}}\omega_{(j-1)} + (-1)^{j-1}\delta\omega_{(j)} \ , & j \geq 3 \end{cases} \quad . \tag{2.3.41b}$$

Remark 2.3.20. Expanding the requirement $\mathrm{d}^{\mathrm{Tot}}\omega = 0$ into a tuple of conditions:

$$\delta\omega_{(2)} = 0 \quad , \tag{2.3.42a}$$
$$\mathrm{d}_{\mathrm{dR}}\omega_{(2)} + \delta\omega_{(3)} = 0 \quad , \tag{2.3.42b}$$
$$\mathrm{d}_{\mathrm{dR}}\omega_{(3)} - \delta\omega_{(4)} = 0 \quad , \tag{2.3.42c}$$
$$\vdots$$

reveals that the tower of conditions now singles out 2-forms as the special grading; they are closed under the CDGA differential as well as being closed under the form differential up to a CDGA coboundary 3-form which itself is closed under the form differential up to a CDGA coboundary 4-form and so on. △

We say '*pre*-symplectic structure' because we are still missing the notion of invertibility. As mentioned before, working in the context of CDGAs requires that we work with something more relaxed than an isomorphism and this is fulfilled by a *quasi*-isomorphism as in Definition 2.2.2. For the rest of this subsection, we restrict to positively graded CDGAs $A = (A^{\geq 0}, \delta)$ in which case we have a canonical map called the **co-augmentation map**,

$$\epsilon : (A^{\geq 0}, \delta) \to (A^0, 0) \qquad , \qquad a \mapsto \delta_0^{|a|} a \quad . \tag{2.3.43}$$

It is obvious that the co-augmentation map (2.3.43) is a CDGA morphism and hence makes A^0 into an A-dg-module and we think of A^0 with this A-action from now on.

Definition 2.3.21. Given a positively graded CDGA $A = (A^{\geq 0}, \delta)$, an n-shifted pre-symplectic structure $\omega \in Z^{n+2}\left(\mathrm{Fil}^2 DR\right)$ is an **n-shifted symplectic structure** if the **contraction** with the 2-form component $\omega_{(2)} \in \Omega^{2,n}_{A/\mathbb{K}}$,

$$\omega_{(2)} \otimes_A \mathrm{id}_{A^0} : \mathsf{T}_A \otimes_A A^0 \to \Omega^1_{A/\mathbb{K}}[n] \otimes_A A^0 \quad , \tag{2.3.44}$$

induces a quasi-isomorphism.

The following example demonstrates that we do indeed have a genuine generalisation of the familiar notion of a symplectic form.

Example 2.3.22. If we were working with an ordinary affine algebraic variety and its $\mathbb{K}$-algebra of regular functions then we would have $(A, \delta) = (A^0, 0)$. In this case, Definition 2.3.12 of the Kähler differential forms of A^0 over $\mathbb{K}$ reduces simply to the familiar one, i.e., $\Omega^{1,0}_{A^0/\mathbb{K}}$ is an A^0-module generated by the image of the derivation $\mathrm{d}_{\mathrm{dR}} : A^0 \to \Omega^{1,0}_{A^0/\mathbb{K}}$. The de Rham algebra of Corollary 2.3.16 simplifies to the usual one, i.e., $\Omega_{A^0/\mathbb{K}} = \left(\bigwedge_{A^0}(\Omega^{1,0}_{A^0/\mathbb{K}}), \mathrm{d}_{\mathrm{dR}}, 0, \wedge_{A^0}, \mathbb{1}\right)$ and the underlying double complex is given by $\left(\bigoplus_{i\in\mathbb{N}} \bigwedge^i_{A^0}(\Omega^{1,0}_{A^0/\mathbb{K}}), \mathrm{d}_{\mathrm{dR}}, 0\right)$ thus the product totalisation of the double complex, as in (2.3.36b), acts by identity and the total differential, as in (2.3.37b), remains as merely the form differential d_{dR}. An n-shifted pre-symplectic structure $\omega \in Z^{n+2}\left(\mathrm{Fil}^2 DR\right)$ is then simply given by $\omega = \omega_{(n+2)} \in \Omega^{n+2,0}_{A/\mathbb{K}}$ such that $\mathrm{d}^{\mathrm{Tot}}\omega = \mathrm{d}_{\mathrm{dR}}\omega_{(n+2)} = 0$, i.e, it is closed in the traditional sense of algebraic geometry. For n-shifted symplectic structures, we see that the component $\omega_{(2)}$ is only nonzero (and thus capable of inducing a quasi-isomorphism) for $n = 0$. Furthermore, the derivations T_{A^0} reduce to the usual one and we have the following unitor

isomorphisms $\mathsf{T}_{A^0} \otimes_{A^0} A^0 \cong \mathsf{T}_{A^0}$ and $\Omega^{1,0}_{A^0/\mathbb{K}} \otimes_{A^0} A^0 \cong \Omega^{1,0}_{A^0/\mathbb{K}}$ in the category of A^0-modules, these modules obviously have trivial cohomology hence a quasi-isomorphism between them is simply an isomorphism. Of course, we also have $\omega_{(2)} \otimes_{A^0} \mathrm{id}_{A^0} = \omega_{(2)}$ hence the quasi-isomorphism condition (2.3.44) boils down to the requirement that

$$\omega_{(2)} : \mathsf{T}_{A^0} \to \Omega^{1,0}_{A^0/\mathbb{K}} \tag{2.3.45}$$

is an isomorphism. In sum for an ordinary affine algebraic variety, the only n-shifted symplectic structures are the 0-shifted symplectic structures and these are simply the non-degenerate 2-cocycles in the ordinary de Rham algebra. ▽

Example 2.3.23. This example starts from where Example 2.3.17 left off. Noting that $\mathrm{CE}^0_{\mathfrak{g}} = \mathbb{K}$, the CDGA base change in the contraction map (2.3.44) has the following effect on the module of Kähler differentials. $\Omega^1_{\mathrm{CE}_{\mathfrak{g}}/\mathbb{K}} \otimes_{\mathrm{CE}_{\mathfrak{g}}} \mathbb{K}$ is spanned by $\{\mathrm{d}_{\mathrm{dR}}(\vartheta^i) \otimes_{\mathrm{CE}_{\mathfrak{g}}} 1\}_{1 \leq i \leq \dim \mathfrak{g}}$ and the differential

$$\delta \otimes_{\mathrm{CE}_{\mathfrak{g}}} \mathrm{id}_{\mathbb{K}} + \mathrm{id}_{\Omega^1_{\mathrm{CE}_{\mathfrak{g}}/\mathbb{K}}} \otimes_{\mathrm{CE}_{\mathfrak{g}}} 0 = \delta \otimes_{\mathrm{CE}_{\mathfrak{g}}} \mathrm{id}_{\mathbb{K}} \tag{2.3.46}$$

annihilates every generator as follows,

$$\begin{aligned}
\delta\left(\mathrm{d}_{\mathrm{dR}}(\vartheta^i)\right) \otimes_{\mathrm{CE}_{\mathfrak{g}}} 1 =& \mathrm{d}_{\mathrm{dR}}\left(\delta(\vartheta^i)\right) \otimes_{\mathrm{CE}_{\mathfrak{g}}} 1 \\
=& \mathrm{d}_{\mathrm{dR}}\left(-\tfrac{1}{2}\mathfrak{g}^i_{jk}\vartheta^j\vartheta^k\right) \otimes_{\mathrm{CE}_{\mathfrak{g}}} 1 \\
=& -\tfrac{1}{2}\mathfrak{g}^i_{jk}\left[\mathrm{d}_{\mathrm{dR}}(\vartheta^j)\vartheta^k + \vartheta^j \mathrm{d}_{\mathrm{dR}}(\vartheta^k)\right] \otimes_{\mathrm{CE}_{\mathfrak{g}}} 1 \\
=& -\tfrac{1}{2}\mathfrak{g}^i_{jk}\left[\mathrm{d}_{\mathrm{dR}}(\vartheta^j) \otimes_{\mathrm{CE}_{\mathfrak{g}}} \epsilon(\vartheta^k) - \mathrm{d}_{\mathrm{dR}}(\vartheta^k) \otimes_{\mathrm{CE}_{\mathfrak{g}}} \epsilon(\vartheta^j)\right] \\
=& 0 \quad .
\end{aligned}$$

Thus the cochain complex $\Omega^1_{\mathrm{CE}_{\mathfrak{g}}/\mathbb{K}} \otimes_{\mathrm{CE}_{\mathfrak{g}}} \mathbb{K}$ indeed has trivial differential and is concentrated in degree 1 hence it is identical with its cohomology. Likewise, $\mathsf{T}_{\mathrm{CE}_{\mathfrak{g}}} \otimes_{\mathrm{CE}_{\mathfrak{g}}} \mathbb{K}$ is concentrated in degree -1 with basis given by $\{\frac{\partial}{\partial(\vartheta^i)} \otimes_{\mathrm{CE}_{\mathfrak{g}}} 1\}$. Recall the isomorphism (2.3.5a) from Example 2.3.4 which now gives us the following isomorphism between graded vector spaces,

$$\mathfrak{g}^*[-1] \cong \Omega^1_{\mathrm{CE}_{\mathfrak{g}}/\mathbb{K}} \otimes_{\mathrm{CE}_{\mathfrak{g}}} \mathbb{K} \qquad , \qquad \vartheta^i \mapsto \mathrm{d}_{\mathrm{dR}}(\vartheta^i) \otimes_{\mathrm{CE}_{\mathfrak{g}}} 1 \quad . \tag{2.3.47}$$

Recall as well the isomorphism (2.3.30) from Example 2.3.17 which gives us the isomorphism,

$$\mathfrak{g}[1] \cong \mathsf{T}_{\mathrm{CE}_{\mathfrak{g}}} \otimes_{\mathrm{CE}_{\mathfrak{g}}} \mathbb{K} \qquad , \qquad s^{-1}x_i \mapsto \frac{\partial}{\partial(\vartheta^i)} \otimes_{\mathrm{CE}_{\mathfrak{g}}} 1 \quad . \tag{2.3.48}$$

Now we can translate the quasi-isomorphism condition of Definition 2.3.44 to the requirement that the following graded $\mathbb{K}$-linear map

$$\omega_{(2)} : \mathfrak{g}[1] \to \mathfrak{g}^*[-1][n] \cong \mathfrak{g}^*[n-1] \tag{2.3.49}$$

is simply an isomorphism. Further still, through the composition

$$\mathfrak{g}[1] \otimes_{\mathbb{K}} \mathfrak{g}[1] \xrightarrow{\omega_{(2)} \otimes_{\mathbb{K}} \mathrm{id}} \mathfrak{g}^*[n-1] \otimes_{\mathbb{K}} \mathfrak{g}[1] \xrightarrow{\mathrm{ev}} \mathbb{K}[n] \quad , \tag{2.3.50}$$

we can rephrase the quasi-isomorphism condition to the requirement that the following $\mathbb{K}$-linear map $\mathrm{ev} \circ (\omega_{(2)} \otimes_{\mathbb{K}} \mathrm{id}) = \omega_{(2)} : \mathfrak{g}[1] \otimes_{\mathbb{K}} \mathfrak{g}[1] \to \mathbb{K}[n]$ be non-degenerate. ▽

Theorem 2.3.24. *The set of n-shifted symplectic structures on $\mathrm{CE}_{\mathfrak{g}}$ is:*

(i) For $n \neq 2$, empty.

(ii) For $n = 2$, bijective to the set $\mathbf{Rep}_{\mathbb{K}}\mathfrak{g}(\mathrm{Sym}^2_{\mathbb{K}}\mathfrak{g}, \mathbb{K})^{\times,\mathfrak{g}}$ of non-degenerate symmetric ad-invariant pairings on the Lie algebra $\mathfrak{g}$.

Proof. Throughout this proof we will refer to the following general tuple

$$\omega = (\omega_{(2)} \in \Omega^{2,n}_{\mathrm{CE}_\mathfrak{g}/\mathbb{K}}, \omega_{(3)} \in \Omega^{3,n-1}_{\mathrm{CE}_\mathfrak{g}/\mathbb{K}}, \ldots, \omega_{(p)} \in \Omega^{p,n+2-p}_{\mathrm{CE}_\mathfrak{g}/\mathbb{K}}, \ldots) \tag{2.3.51}$$

which is simply the expansion of (2.3.40). Let us first verify item (i) of Theorem 2.3.24 by dividing the claim into two separate cases ($n \leq 1$) and ($n \geq 3$):

($n \leq 1$) Recalling the staircase shape of the double complex, i.e., $\Omega^{\sharp,\sharp}_{\mathrm{CE}_\mathfrak{g}/\mathbb{K}} = (\Omega^{i,j}_{\mathrm{CE}_\mathfrak{g}/\mathbb{K}})_{0\leq i\leq j\leq i+\dim\mathfrak{g}}$ from Example 2.3.17, we can see that (2.3.51) is simply a trivial tuple and, in particular, $\omega_{(2)} = 0$ certainly fails to satisfy the non-degeneracy condition as rephrased through (2.3.50) above. In sum, the set of n-shifted symplectic structures is empty for $n \leq 1$.

($n \geq 3$) Looking at the above general tuple (2.3.51) and recalling the expression (2.3.34) for the generic element of $\Omega^{i,i+k}_{\mathrm{CE}_\mathfrak{g}/\mathbb{K}}$, we can see that

$$\omega_{(2)} = \omega_{a_1a_2\ldots a_{n-2}b_1b_2}\vartheta^{a_1}\vartheta^{a_2}\cdots\vartheta^{a_{n-2}}\mathrm{d}_{\mathrm{dR}}(\vartheta^{b_1}) \wedge_{\mathrm{CE}_\mathfrak{g}} \mathrm{d}_{\mathrm{dR}}(\vartheta^{b_2}) \in \Omega^{2,n}_{\mathrm{CE}_\mathfrak{g}/\mathbb{K}} \quad . \tag{2.3.52}$$

Consider the result of contracting (2.3.52) on a basis element $\frac{\partial}{\partial(\vartheta^i)} \otimes_{\mathrm{CE}_\mathfrak{g}} 1 \in \mathsf{T}_{\mathrm{CE}_\mathfrak{g}} \otimes_{\mathrm{CE}_\mathfrak{g}} \mathbb{K}$,

$$\begin{aligned}
\frac{\partial}{\partial(\vartheta^i)} \otimes_{\mathrm{CE}_\mathfrak{g}} 1 \mapsto & \omega_{(2)}\left(\frac{\partial}{\partial(\vartheta^i)}\right) \otimes_{\mathrm{CE}_\mathfrak{g}} 1 \\
&= -2\omega_{a_1a_2\ldots a_{n-2}b_1i}\vartheta^{a_1}\vartheta^{a_2}\cdots\vartheta^{a_{n-2}}\mathrm{d}_{\mathrm{dR}}(\vartheta^{b_1}) \otimes_{\mathrm{CE}_\mathfrak{g}} 1 \\
&= 2(-1)^{n-1}\omega_{a_1a_2\ldots a_{n-2}b_1i}\mathrm{d}_{\mathrm{dR}}(\vartheta^{b_1}) \otimes_{\mathrm{CE}_\mathfrak{g}} \epsilon(\vartheta^{a_1}\vartheta^{a_2}\cdots\vartheta^{a_{n-2}}) \\
&= 0
\end{aligned}$$

where the first equality arises because of the symmetry in the latter two indices in the coefficient tensor (2.3.35) together with the $\mathrm{CE}_\mathfrak{g}$-linear pairing map (2.3.10), the second equality comes from the action of $\mathrm{CE}_\mathfrak{g}$ on $\mathrm{CE}^0_\mathfrak{g} = \mathbb{K}$ through the coaugmentation map (2.3.43) and the third equality comes from the fact that $n \geq 3$ and thus

$$|\vartheta^{a_1}\vartheta^{a_2}\cdots\vartheta^{a_{n-2}}| = n - 2 \geq 1 \quad . \tag{2.3.53}$$

Again, both $\mathsf{T}_{\mathrm{CE}_\mathfrak{g}} \otimes_{\mathrm{CE}_\mathfrak{g}} \mathbb{K}$ and $\Omega^1_{\mathrm{CE}_\mathfrak{g}/\mathbb{K}}[n] \otimes_{\mathrm{CE}_\mathfrak{g}} \mathbb{K}$ have cohomologies identical to themselves, thus the contraction map $\omega_{(2)} \otimes_{\mathrm{CE}_\mathfrak{g}} \mathrm{id}_\mathbb{K} = 0$ fails to satisfy the quasi-isomorphism condition (2.3.44). In sum, the set of n-shifted symplectic structures is empty for $n \geq 3$.

Now we verify item (ii) of Theorem 2.3.24. For $n = 2$, the general tuple (2.3.51) becomes

$$\omega = (\omega_{(2)} \in \Omega^{2,2}_{\mathrm{CE}_\mathfrak{g}/\mathbb{K}}, \omega_{(3)} \in \Omega^{3,1}_{\mathrm{CE}_\mathfrak{g}/\mathbb{K}}, \ldots, \omega_{(p)} \in \Omega^{p,4-p}_{\mathrm{CE}_\mathfrak{g}/\mathbb{K}}, \ldots) \tag{2.3.54a}$$

but, again, the staircase shape of the double complex tells us that there are no nontrivial forms whose vertical degree is greater than that of their horizontal degree hence we use (2.3.34) to write the tuple as

$$\omega = (\omega_{(2)} = \omega_{b_1b_2}\mathrm{d}_{\mathrm{dR}}(\vartheta^{b_1}) \wedge_{\mathrm{CE}_\mathfrak{g}} \mathrm{d}_{\mathrm{dR}}(\vartheta^{b_2}), 0, 0, \ldots) \quad . \tag{2.3.54b}$$

As mentioned in (2.3.35), the matrix with components $\omega_{b_1b_2} \in \mathbb{K}$ is a symmetric matrix.

The tower of conditions (2.3.42a) for a 2-shifted pre-symplectic structure reduces to the following two conditions

$$\delta\omega_{(2)} = \mathrm{d}_{\mathrm{dR}}\omega_{(2)} = 0 \quad . \tag{2.3.55a}$$

The condition $\mathrm{d}_{\mathrm{dR}}\omega_{(2)} = 0$ is automatically satisfied by the coexactness of the 2-form; in fact the 2-form is actually a 2-coboundary, i.e., $\omega_{(2)} = \mathrm{d}_{\mathrm{dR}}\left(\omega_{b_1b_2}\vartheta^{b_1}\mathrm{d}_{\mathrm{dR}}(\vartheta^{b_2})\right)$.

The condition $\delta\omega_{(2)} = 0$ implies

$$\begin{aligned}\omega_{b_1b_2}\delta\left(\mathrm{d}_{\mathrm{dR}}(\vartheta^{b_1})\wedge_{\mathrm{CE}_\mathfrak{g}}\mathrm{d}_{\mathrm{dR}}(\vartheta^{b_2})\right) =&\omega_{b_1b_2}\mathrm{d}_{\mathrm{dR}}\left(\delta(\vartheta^{b_1})\right)\wedge_{\mathrm{CE}_\mathfrak{g}}\mathrm{d}_{\mathrm{dR}}(\vartheta^{b_2}) - \omega_{b_1b_2}\mathrm{d}_{\mathrm{dR}}(\vartheta^{b_1})\wedge_{\mathrm{CE}_\mathfrak{g}}\mathrm{d}_{\mathrm{dR}}\left(\delta(\vartheta^{b_2})\right)\\ =&2\omega_{b_1b_2}\mathrm{d}_{\mathrm{dR}}\left(\delta(\vartheta^{b_1})\right)\wedge_{\mathrm{CE}_\mathfrak{g}}\mathrm{d}_{\mathrm{dR}}(\vartheta^{b_2})\\ =&2\omega_{b_1b_2}\mathrm{d}_{\mathrm{dR}}\left(-\tfrac{1}{2}\mathfrak{g}^{b_1}_{jk}\vartheta^j\vartheta^k\right)\wedge_{\mathrm{CE}_\mathfrak{g}}\mathrm{d}_{\mathrm{dR}}(\vartheta^{b_2})\\ =&-2\omega_{b_1b_2}\mathfrak{g}^{b_1}_{jk}\vartheta^j\mathrm{d}_{\mathrm{dR}}\left(\vartheta^k\right)\wedge_{\mathrm{CE}_\mathfrak{g}}\mathrm{d}_{\mathrm{dR}}(\vartheta^{b_2})\\ =&0\qquad,\end{aligned}\tag{2.3.55b}$$

where the first equality uses the fact that each differential is a derivation of the algebra multiplication $\wedge_{\mathrm{CE}_\mathfrak{g}}$ and, in this case, the signs come from the horizontal CDGA degree. The second equality uses the symmetry $\omega_{b_1b_2} = \omega_{b_2b_1}$ together with the double Koszul sign rule (2.3.14) of a CDdGA. The fourth equality uses the skew-symmetry of the structure constants $\mathfrak{g}^{b_1}_{jk} = -\mathfrak{g}^{b_1}_{kj}$ together with the anti-commutativity (2.3.32b). Stripping (2.3.55b) down to an index equation leaves us with

$$\omega_{db}\mathfrak{g}^d_{ca} + \omega_{ad}\mathfrak{g}^d_{cb} = 0\quad.\tag{2.3.56a}$$

The symmetry of the tensor $\omega_{b_1b_2} = \omega_{b_2b_1}$ relates to the pairings on the Lie algebra through (2.3.50) which now becomes

$$\omega_{(2)} : \mathfrak{g}\otimes\mathfrak{g}\to\mathbb{K}\quad,\quad x_{i_1}\otimes x_{i_2}\mapsto\omega_{b_1b_2}\theta^{b_1}(x_{i_1})\theta^{b_2}(x_{i_2}) = \omega_{i_1i_2}\quad.\tag{2.3.56b}$$

In other words, we have a linear map $\omega_{(2)} : \mathrm{Sym}^2_{\mathbb{K}}\mathfrak{g}\to\mathbb{K}$. Reinterpreting (2.3.56a) as a further property of this map gives us

$$\begin{aligned}&\implies \omega_{(2)}(x_d\otimes x_b)\mathfrak{g}^d_{ca} + \omega_{(2)}(x_a\otimes x_d)\mathfrak{g}^d_{cb} = 0\quad,\\ &\implies \omega_{(2)}(\mathfrak{g}^d_{ca}x_d\otimes x_b) + \omega_{(2)}(x_a\otimes\mathfrak{g}^d_{cb}x_d) = 0\quad,\\ &\implies \omega_{(2)}([x_c,x_a]\otimes x_b + x_a\otimes[x_c,x_b]) = 0\quad.\end{aligned}$$

Explicitly, we now see that such pairings on the Lie algebra are not merely symmetric but also ad-invariant hence we actually have a linear map $\omega_{(2)} : (\mathrm{Sym}^2_{\mathbb{K}}\mathfrak{g})^\mathfrak{g}\to\mathbb{K}$.

The quasi-isomorphism condition (2.3.44) translates to the requirement that (2.3.56b) be non-degenerate.

In sum, a 2-shifted symplectic structure on $\mathrm{CE}_\mathfrak{g}$ is characterised as a non-degenerate linear map $\omega_{(2)} : (\mathrm{Sym}^2_{\mathbb{K}}\mathfrak{g})^\mathfrak{g}\to\mathbb{K}$. □

2.3.3 Shifted polyvectors and shifted Poisson structures

We now define n-shifted polyvectors. In order to compatibly extend the DGLA (and A-dg-module) structure on T_A together with the CDGA structure of A to the complex of n-shifted polyvectors in the most straightforward way possible, we construct the n-shifted polyvectors as the symmetric algebra over A. By this way of construction, the n-shifted polyvectors naturally inherit a CDGA structure over A (given simply by symmetrised tensor product relative to A).

Definition 2.3.25. The CDGA of **n-shifted polyvectors** is defined as the symmetric algebra

of the shifted (by $-n-1$) derivations where the symmetric powers are relative to A, i.e.,

$$\mathrm{Pol}(A,n) := \mathrm{Sym}_A\big(\mathsf{T}_A[-n-1]\big) = \bigoplus_{m\geq 0}\mathrm{Sym}_A^m\big(\mathsf{T}_A[-n-1]\big) \quad . \tag{2.3.57}$$

$\mathrm{Pol}(A,n)$ is a free CDGA over A. The differential structure is the unique extension of ∂ to the whole complex as a derivation of $\mathrm{Pol}(A,n)$ of degree 1 such that $\partial(a)=\delta(a)$ for $a\in A$.

Remark 2.3.26. The shifted derivations $\mathsf{T}_A[-n-1]$ inherit from the commutator bracket $[\,\cdot\,,\,\cdot\,]$ on T_A a degree $-n-1$ bracket $[\,\cdot\,,\,\cdot\,]' : \mathsf{T}_A[-n-1]\otimes\mathsf{T}_A[-n-1]\to\mathsf{T}_A[-n-1]$ which is defined by the following commutative diagram of cochain maps

$$\begin{array}{ccc} \mathsf{T}_A[-n-1]\otimes\mathsf{T}_A[-n-1] & \xrightarrow{\quad[\,\cdot\,,\,\cdot\,]'\quad} & \mathsf{T}_A[-n-1][-n-1] \\ \Big\downarrow\cong & & \Big\uparrow\cong \\ (\mathsf{T}_A\otimes\mathsf{T}_A)[-2n-2] & \xrightarrow[[\,\cdot\,,\,\cdot\,][-2n-2]]{} & \mathsf{T}_A[-2n-2] \end{array} \tag{2.3.58a}$$

or, as defined on elements,

$$[s^{n+1}X, s^{n+1}Y]' := (-1)^{|X|(n+1)}s^{n+1}[X,Y] \quad . \tag{2.3.58b}$$

With such a definition, it can easily be checked that the properties inherited by this bracket are as follows:

(i) Antisymmetry: For all homogeneous $s^{n+1}X, s^{n+1}Y\in\mathsf{T}_A[-n-1]$,

$$[s^{n+1}X, s^{n+1}Y]' = -(-1)^{n+1}(-1)^{(|X|+n+1)(|Y|+n+1)}[s^{n+1}Y, s^{n+1}X]' \quad . \tag{2.3.59a}$$

(ii) Jacobi identity: For all homogeneous $s^{n+1}X, s^{n+1}Y, s^{n+1}\in\mathsf{T}_A[-n-1]$,

$$\begin{aligned}[s^{n+1}X,[s^{n+1}Y,s^{n+1}Z]']' =& (-1)^{|X|(n+1)}[[s^{n+1}X,s^{n+1}Y]',s^{n+1}Z]' \\ &+(-1)^{|X||Y|}[s^{n+1}Y,[s^{n+1}X,s^{n+1}Z]']' \quad .\end{aligned} \tag{2.3.59b}$$

(iii) Derivation property: For all homogeneous $a\in A$ and $s^{n+1}X, s^{n+1}Y\in\mathsf{T}_A[-n-1]$,

$$[s^{n+1}X, as^{n+1}Y]' = X(a)s^{n+1}Y + (-1)^{|X||a|}a[s^{n+1}X, s^{n+1}Y]' \tag{2.3.59c}$$

△

Definition 2.3.27. The **Schouten-Nijenhuis bracket**

$$[\,\cdot\,,\,\cdot\,]_{\mathrm{S}} : \mathrm{Pol}(A,n)\otimes\mathrm{Pol}(A,n)\to\mathrm{Pol}(A,n) \tag{2.3.60}$$

is defined as the unique extension of the shifted bracket $[\,\cdot\,,\,\cdot\,]'$ on $\mathsf{T}_A[-n-1]$ to a $\mathbb{P}_{n+2}$**-algebra** structure on $\mathrm{Pol}(A,n)$, i.e., a degree $-n-1$ cochain bracket that satisfies the following properties:

(i) Antisymmetry: For all homogeneous $P,Q\in\mathrm{Pol}(A,n)$,

$$[P,Q]_{\mathrm{S}} = -(-1)^{n+1}(-1)^{|P||Q|}[Q,P]_{\mathrm{S}} \quad . \tag{2.3.61a}$$

(ii) Jacobi identity: For all homogeneous $P, Q, R \in \mathrm{Pol}(A, n)$,

$$[P,[Q,R]_{\mathrm{S}}]_{\mathrm{S}} = (-1)^{(|P|-n-1)(n+1)}[[P,Q]_{\mathrm{S}},R]_{\mathrm{S}} + (-1)^{(|P|-n-1)(|Q|-n-1)}[Q,[P,R]_{\mathrm{S}}]_{\mathrm{S}} \quad . \tag{2.3.61b}$$

(iii) Derivation property: For all homogeneous $a \in A$ and $P, Q \in \mathrm{Pol}(A, n)$,

$$[P,QR]_{\mathrm{S}} = [P,Q]_{\mathrm{S}}R + (-1)^{(|P|-n-1)|Q|}Q[P,R]_{\mathrm{S}} \tag{2.3.61c}$$

with the following values on shifted derivations $s^{n+1}X, s^{n+1}Y \in \mathsf{T}_A[-n-1]$ and CDGA elements $a, b \in A$:

$$[s^{n+1}X, s^{n+1}Y]_{\mathrm{S}} := [s^{n+1}X, s^{n+1}Y]' = (-1)^{|X|(n+1)}s^{n+1}[X,Y], \tag{2.3.62a}$$

$$[s^{n+1}X, a]_{\mathrm{S}} := X(a), \tag{2.3.62b}$$

$$[a,b]_{\mathrm{S}} := 0. \tag{2.3.62c}$$

The following left derivation property will be useful in computations later,

$$[PQ,R]_{\mathrm{S}} = (-1)^{|P|(n+1)}P[Q,R]_{\mathrm{S}} + (-1)^{|Q||R|}[P,R]_{\mathrm{S}}Q \ , \tag{2.3.63}$$

it can be proved using (2.3.61a) and (2.3.61c). Note that the Schouten–Nijenhuis bracket $[\,\cdot\,,\,\cdot\,]_{\mathrm{S}}$ decreases the weight by 1, i.e. for $P \in \mathrm{Pol}(A, n)$ of weight m_P and $Q \in \mathrm{Pol}(A, n)$ of weight m_Q, the weight of $[P, Q]_{\mathrm{S}}$ is $m_P + m_Q - 1$.

Proposition 2.3.28. *The Schouten-Nijenhuis bracket* $[\,\cdot\,,\,\cdot\,]_{\mathrm{S}} : \mathrm{Pol}(A, n) \otimes \mathrm{Pol}(A, n) \to \mathrm{Pol}(A, n)$ *is well-defined.*

Proof. Though (2.3.61c) and (2.3.61a) are used to extend the bracket to higher weights, it needs to be checked that such an extension is well-defined. By induction on the sum of the weights of the inputs, assuming that (2.3.61a) and (2.3.61c) holds for $(P = P_1, Q, R)$, we prove the corresponding claim for $(P = P_1P_2,\ Q,\ R)$.

$$\begin{aligned}
[P_1P_2,Q]_{\mathrm{S}} =& (-1)^{|P_1|(n+1)}P_1[P_2,Q]_{\mathrm{S}} + (-1)^{|P_2||Q|}[P_1,Q]_{\mathrm{S}}P_2 \\
=& -(-1)^{|P_1|(n+1)}(-1)^{n+1}(-1)^{|P_2||Q|}P_1[Q,P_2]_{\mathrm{S}} - (-1)^{n+1}(-1)^{|P_1||Q|}(-1)^{|P_2||Q|}[Q,P_1]_{\mathrm{S}}P_2 \\
=& -(-1)^{n+1}(-1)^{|P_1P_2||Q|}[Q,P_1P_2]_{\mathrm{S}} \ ,
\end{aligned} \tag{2.3.64}$$

the first line follows from the induction hypothesis that $[\,\cdot\,,\,\cdot\,]_{\mathrm{S}}$ is a biderivation on lower weights, the second line follows from the induction hypothesis that $[\,\cdot\,,\,\cdot\,]_{\mathrm{S}}$ is skew-commutative on lower weights. The third line once again follows from the induction hypothesis but now in the right slot of the bracket.

$$\begin{aligned}
[P_1P_2,QR]_{\mathrm{S}} =& (-1)^{|P_1|(n+1)}P_1[P_2,QR]_{\mathrm{S}} + (-1)^{|P_2||QR|}[P_1,QR]_{\mathrm{S}}P_2 \\
=& (-1)^{|P_1|(n+1)}P_1([P_2,Q]_{\mathrm{S}}R + (-1)^{|Q|(|P_2|-n-1)}Q[P_2,R]_{\mathrm{S}}) \\
&+ (-1)^{|P_2||QR|}([P_1,Q]_{\mathrm{S}}R + (-1)^{|Q|(|P_1|-n-1)}Q[P_1,R]_{\mathrm{S}})P_2 \\
=& (-1)^{|P_1|(n+1)}P_1[P_2,Q]_{\mathrm{S}}R + (-1)^{|P_1|(n+1)}(-1)^{|Q|(|P_2|-n-1)}P_1Q[P_2,R]_{\mathrm{S}} \\
&+ (-1)^{|P_2||QR|}[P_1,Q]_{\mathrm{S}}RP_2 + (-1)^{|P_2||QR|}(-1)^{|Q|(|P_1|-n-1)}Q[P_1,R]_{\mathrm{S}}P_2 \\
=& (-1)^{|P_1|(n+1)}P_1[P_2,Q]_{\mathrm{S}}R + (-1)^{|P_2||Q|}[P_1,Q]_{\mathrm{S}}P_2R \\
&+ (-1)^{|P_1|(n+1+|Q|)}(-1)^{|Q|(|P_2|-n-1)}QP_1[P_2,R]_{\mathrm{S}} + (-1)^{|P_2||QR|}(-1)^{|Q|(|P_1|-n-1)}Q[P_1,R]_{\mathrm{S}}P_2 \\
=& [P_1P_2,Q]_{\mathrm{S}}R + (-1)^{|Q|(|P_1P_2|-n-1)}Q[P_1P_2,R]_{\mathrm{S}} \quad ,
\end{aligned} \tag{2.3.65}$$

Finally, (2.3.61b) follows from the fact that $\mathrm{Pol}(A,n)$ is generated, as a CDGA, by $\mathsf{T}_A[-n-1]$ and A hence the Jacobi identity on $A \oplus \mathsf{T}_A[-n-1]$ is guaranteed by (2.3.62a), (2.3.62b), (2.3.62c) and (2.3.59b). □

Lemma 2.3.29. *We can use the Schouten-Nijenhuis bracket to write the differential on* $\mathrm{Pol}(A,n)$ *as* $\partial = [s^{n+1}\delta, \cdot\,]_{\mathrm{S}}$.

Proof. As mentioned in the proof of Proposition 2.3.28, (2.3.61c) allows us to reduce this claim to $\mathsf{T}_A[-n-1]$ and A. On A, (2.3.62b) shows this to be true whereas on $\mathsf{T}_A[-n-1]$, (2.3.62a) shows this to be true in light of the comment in Remark 2.3.2. □

This Lemma, together with the Jacobi identity (2.3.61b), shows the Schouten-Nijenhuis bracket $[\,\cdot\,,\,\cdot\,]_{\mathrm{S}}$ to indeed be a cochain.

The definition of nSPSs on a CDGA A uses a completion of the $\mathbb{P}_{n+2}$-algebra of n-shifted polyvectors $\mathrm{Pol}(A,n)$ in order to avoid bounds on the weights, see e.g. [Pri17, Pri18]. This completion[11] is obtained by replacing the direct sum of cochain complexes in Definition 2.3.25 by a product.

Definition 2.3.30. The **completed n-shifted polyvectors** on a CDGA A are defined as

$$\widehat{\mathrm{Pol}}(A,n) := \prod_{m\geq 0} \mathrm{Sym}_A^m\big(\mathsf{T}_A[-n-1]\big) \quad . \tag{2.3.66}$$

A completed n-shifted polyvector is thus a formal sum $P = \sum_{m\geq 0} P^{(m)} \in \widehat{\mathrm{Pol}}(A,n)$ of homogeneous weight components $P^{(m)} \in \mathrm{Sym}_A^m\big(\mathsf{T}_A[-n-1]\big)$. The $\mathbb{P}_{n+2}$-algebra structure on $\mathrm{Pol}(A,n)$ extends to the completion $\widehat{\mathrm{Pol}}(A,n)$ by setting

$$P\,Q := \sum_{m\geq 0}\Bigg(\sum_{k+l=m} P^{(k)}\,Q^{(l)}\Bigg) \quad , \tag{2.3.67a}$$

$$[P,Q]_{\mathrm{S}} := \sum_{m\geq 0}\Bigg(\sum_{k+l-1=m} [P^{(k)},Q^{(l)}]_{\mathrm{S}}\Bigg) \quad , \tag{2.3.67b}$$

for all $P,Q \in \widehat{\mathrm{Pol}}(A,n)$. Note that these are well-defined because the weight is bounded from below by 0 and hence the sums in the parentheses are finite.

Proposition 2.3.31. *The $\mathbb{P}_{n+2}$-algebra structure $[\,\cdot\,,\,\cdot\,]_{\mathrm{S}}$ on $\widehat{\mathrm{Pol}}(A,n)$ induces a DGLA structure $\{\cdot,\cdot\}$ on the shifted complex $\widehat{\mathrm{Pol}}(A,n)[n+1]$ which is a derivation of the algebra structure of $\widehat{\mathrm{Pol}}(A,n)$. The bracket*

$$\{\cdot,\cdot\} : \bigwedge{}^2 \widehat{\mathrm{Pol}}(A,n)[n+1] \to \widehat{\mathrm{Pol}}(A,n)[n+1] \tag{2.3.68}$$

operates on inputs as

$$\{s^{-n-1}P, s^{-n-1}Q\} := (-1)^{(|P|-1)(n+1)} s^{-n-1}[P,Q]_{\mathrm{S}} \ . \tag{2.3.69}$$

Proof. We first check the skew-symmetry property

$$\begin{aligned}\{s^{-n-1}P, s^{-n-1}Q\} &= -\,(-1)^{n+1}(-1)^{|P||Q|}(-1)^{(|P|-1)(n+1)} s^{-n-1}[Q,P]_{\mathrm{S}} \\ &= -\,(-1)^{|P||Q|}(-1)^{|P|(n+1)} s^{-n-1}[Q,P]_{\mathrm{S}} \\ &= -\,(-1)^{(|P|-n-1)(|Q|-n-1)}\{s^{-n-1}Q, s^{-n-1}P\},\end{aligned} \tag{2.3.70}$$

[11] We need this *completion* so that we may talk of n-shifted polyvectors with an infinitude of nontrivial terms as, for instance, in the case of the 1SPS on the Chevalley-Eilenberg algebra of a Lie 2-algebra.

then the Jacobi property

$$\begin{aligned}\{s^{-n-1}P,\{s^{-n-1}Q,s^{-n-1}R\}\} =&(-1)^{n+1}(-1)^{|Q|(n+1)}\{s^{-n-1}P,s^{-n-1}[Q,R]_{\mathrm{S}}\}\\ =&(-1)^{|Q|(n+1)}(-1)^{|P|(n+1)}s^{-n-1}[P,[Q,R]_{\mathrm{S}}]_{\mathrm{S}}\\ =&(-1)^{|Q|(n+1)}(-1)^{|P|(n+1)}s^{-n-1}((-1)^{(|P|-n-1)(n+1))}[[P,Q]_{\mathrm{S}},R]_{\mathrm{S}}\\ &+(-1)^{(|P|-n-1)(|Q|-n-1)}[Q,[P,R]_{\mathrm{S}}]_{\mathrm{S}})\\ =&(-1)^{(|Q|-n-1)(n+1)}s^{-n-1}[[P,Q]_{\mathrm{S}},R]_{\mathrm{S}}\\ &+(-1)^{|s^{-n-1}P||s^{-n-1}Q|}(-1)^{|PQ|(n+1)}s^{-n-1}[Q,[P,R]_{\mathrm{S}}]_{\mathrm{S}}\\ =&(-1)^{n+1}(-1)^{|P|(n+1)}\{s^{-n-1}[P,Q]_{\mathrm{S}},s^{-n-1}R\}\\ &+(-1)^{|s^{-n-1}P||s^{-n-1}Q|}(-1)^{n+1}(-1)^{|P|(n+1)}\{s^{-n-1}Q,s^{-n-1}[P,R]_{\mathrm{S}}\}\\ =&\{\{s^{-n-1}P,s^{-n-1}Q\},s^{-n-1}R\}\\ &+(-1)^{|s^{-n-1}P||s^{-n-1}Q|}\{s^{-n-1}Q,\{s^{-n-1}P,s^{-n-1}R\}\} \end{aligned} \tag{2.3.71}$$

and, finally, the Leibniz property,

$$\begin{aligned}\{s^{-n-1}P,s^{-n-1}(QR)\} =&(-1)^{n+1}(-1)^{|P|(n+1)}s^{-n-1}[P,QR]_{\mathrm{S}}\\ =&(-1)^{n+1}(-1)^{|P|(n+1)}s^{-n-1}([P,Q]_{\mathrm{S}}R+(-1)^{(|P|-n-1)|Q|}Q[P,R]_{\mathrm{S}})\\ =&\{s^{-n-1}P,s^{-n-1}Q\}R+(-1)^{|P||Q|}Q\{s^{-n-1}P,s^{-n-1}R\}. \end{aligned} \tag{2.3.72}$$

□

The DGLA structure $\{\cdot,\cdot\}$ on $\widehat{\mathrm{Pol}}(A,n)[n+1]$ takes the following values for low weights:

$$\{s^{-n-1}s^{n+1}X,s^{-n-1}s^{n+1}Y\} = [X,Y]\ , \tag{2.3.73a}$$

$$\{s^{-n-1}s^{n+1}X,s^{-n-1}a\} = (-1)^{|X|(n+1)}s^{-n-1}X(a)\ , \tag{2.3.73b}$$

$$\{s^{-n-1}a,s^{-n-1}b\} = 0\ . \tag{2.3.73c}$$

We call this complex $\widehat{\mathrm{Pol}}(A,n)[n+1]$ the complex of unshifted completed n-shifted polyvectors or, simply, the complex of **polyvectors**.

Example 2.3.32. Following on from Example 2.3.4, we wish to describe the complex of polyvectors when $A=\mathrm{CE}_{\mathfrak{g}}$ hence $\mathsf{T}_{\mathrm{CE}_{\mathfrak{g}}}\cong\mathrm{CE}_{\mathfrak{g}}(\mathfrak{g}_{\mathrm{ad}}[1])$. We describe the $(-n-1)$-shifted derivations simply as

$$\mathsf{T}_{\mathrm{CE}_{\mathfrak{g}}}[-n-1]\cong\mathrm{CE}_{\mathfrak{g}}(\mathfrak{g}_{\mathrm{ad}}[-n]) \tag{2.3.74}$$

and write the generators as

$$s^{n+1}\frac{\partial}{\partial(\vartheta^i)}=s^nx_i\quad . \tag{2.3.75}$$

Definition 2.3.25 constructs the n-shifted polyvectors as a free CDGA over $\mathrm{CE}_{\mathfrak{g}}$, i.e.,

$$\mathrm{Pol}(\mathrm{CE}_{\mathfrak{g}},n):=\mathrm{Sym}_{\mathrm{CE}_{\mathfrak{g}}}\big(\mathrm{CE}_{\mathfrak{g}}(\mathfrak{g}_{\mathrm{ad}}[-n])\big)\cong\mathrm{CE}_{\mathfrak{g}}\otimes\mathrm{Sym}(\mathfrak{g}_{\mathrm{ad}}[-n])\quad . \tag{2.3.76}$$

The polyvectors are thus given by

$$\widehat{\mathrm{Pol}}(\mathrm{CE}_{\mathfrak{g}},n)[n+1]\cong\Big(\mathrm{CE}_{\mathfrak{g}}\otimes\widehat{\mathrm{Sym}}(\mathfrak{g}_{\mathrm{ad}}[-n])\Big)[n+1]\quad , \tag{2.3.77}$$

and the cohomological degree of a weight m term of **arity** l is given by

$$|s^{-n-1}\vartheta^{i_1}\vartheta^{i_2}\dots\vartheta^{i_l}s^nx_{j_1}s^nx_{j_2}\dots s^nx_{j_m}|=(m-1)n+l-1 \tag{2.3.78}$$

For low weights, by (2.3.73) and (2.3.5f), the DGLA structure has the following simple form:

$$\{s^{-n-1}s^{n+1}s^{-1}x_j, s^{-n-1}s^{n+1}s^{-1}x_k\} = [s^{-1}x_j, s^{-1}x_k] = 0 \quad , \tag{2.3.79a}$$

$$\{s^{-n-1}s^{n+1}s^{-1}x_j, s^{-n-1}\vartheta^k\} = (-1)^{n+1}s^{-n-1}s^{-1}x_j(\vartheta^k) = (-1)^{n+1}s^{-n-1}\delta^k_j \quad , \tag{2.3.79b}$$

$$\{s^{-n-1}\vartheta^j, s^{-n-1}\vartheta^k\} = 0 \quad . \tag{2.3.79c}$$

For the concrete computations in this chapter, instead of using the DGLA structure $\{\cdot,\cdot\}$, we prefer to use the Schouten-Nijenhuis bracket $[\cdot,\cdot]_{\mathrm{S}}$ on the n-shifted polyvectors as in Definition 2.3.27. Recall the low weight values of this Schouten-Nijenhuis bracket (2.3.62) which now simplify further:

$$\left[s^{n+1}\frac{\partial}{\partial(\vartheta^i)}, s^{n+1}\frac{\partial}{\partial(\vartheta^j)}\right]_{\mathrm{S}} = 0 \quad , \tag{2.3.80a}$$

$$\left[s^{n+1}\frac{\partial}{\partial(\vartheta^i)}, \vartheta^j\right]_{\mathrm{S}} = \delta^j_i \quad , \tag{2.3.80b}$$

$$[\vartheta^i, \vartheta^j]_{\mathrm{S}} = 0 \quad . \tag{2.3.80c}$$

We will also need the relations:

$$s^{n+1}\frac{\partial}{\partial(\vartheta^i)}s^{n+1}\frac{\partial}{\partial(\vartheta^j)} = (-1)^n s^{n+1}\frac{\partial}{\partial(\vartheta^j)}s^{n+1}\frac{\partial}{\partial(\vartheta^i)} \quad , \tag{2.3.81a}$$

$$\vartheta^i s^{n+1}\frac{\partial}{\partial(\vartheta^j)} = (-1)^n s^{n+1}\frac{\partial}{\partial(\vartheta^j)}\vartheta^i \quad . \tag{2.3.81b}$$

$\triangledown$

The DGLA structure of Proposition 2.3.31 on polyvectors allows us to generalise the notion of Maurer-Cartan derivation (as in Definition 2.3.5) to the notion of Maurer-Cartan polyvector.

Definition 2.3.33. A **Maurer-Cartan polyvector** is a polyvector $s^{-n-1}\pi \in \widehat{\mathrm{Pol}}(A,n)[n+1]^1$ of cohomological degree 1 that satisfies the Maurer-Cartan equation:

$$\partial(s^{-n-1}\pi) + \frac{1}{2}\{s^{-n-1}\pi, s^{-n-1}\pi\} = 0 \quad . \tag{2.3.82a}$$

Equivalently, it is a completed n-shifted polyvector $\pi \in \widehat{\mathrm{Pol}}(A,n)^{n+2}$ of cohomological degree $n+2$ that satisfies

$$\partial\pi + \tfrac{1}{2}[\pi,\pi]_{\mathrm{S}} = 0 \quad . \tag{2.3.82b}$$

We now extend the comment mentioned below Definition 2.3.5 and show that, given a Maurer-Cartan polyvector, we can deform the differential of $\widehat{\mathrm{Pol}}(A,n)$ as the following lemma shows.

Lemma 2.3.34. *If $s^{-n-1}\pi \in \widehat{\mathrm{Pol}}(A,n)[n+1]^1$ is a Maurer-Cartan polyvector then*

$$\mathcal{D}_\pi(P) := (-1)^{n+1}s^{n+1}\{s^{-n-1}\pi, s^{-n-1}P\} = [\pi, P]_{\mathrm{S}} \tag{2.3.83}$$

defines a degree 1 derivation of the CDGA $\widehat{\mathrm{Pol}}(A,n)$ such that $\partial_\pi^2 = 0$ where $\partial_\pi := \partial + \mathcal{D}_\pi$.

Proof. First we prove the degree 1 derivation property,

$$\begin{aligned}\mathcal{D}_\pi(PQ) =&(-1)^{n+1}s^{n+1}\{s^{-n-1}\pi, s^{-n-1}PQ\}\\ =&(-1)^{n+1}s^{n+1}\{s^{-n-1}\pi, s^{-n-1}P\}Q + (-1)^{n+1}(-1)^{|P||\pi|}s^{n+1}P\{s^{-n-1}\pi, s^{-n-1}Q\}\\ =&\mathcal{D}_\pi(P)Q + (-1)^{n+1}(-1)^{|P|(n+2)}(-1)^{|P|(n+1)}Ps^{n+1}\{s^{-n-1}\pi, s^{-n-1}Q\}\\ =&\mathcal{D}_\pi(P)Q + (-1)^{|P|}P\mathcal{D}_\pi(Q)\ .\end{aligned}$$

The second line follows from (2.3.72), the third line follows from the fact that $|\pi| = n+2$. Now we prove the square-zero condition,

$$\begin{aligned}(\partial + \mathcal{D}_\pi)^2(P) =&\partial(\partial(P) + \mathcal{D}_\pi(P)) + \mathcal{D}_\pi(\partial(P) + \mathcal{D}_\pi(P))\\ =&\partial(\mathcal{D}_\pi(P)) + \mathcal{D}_\pi(\partial(P)) + \mathcal{D}_\pi(\mathcal{D}_\pi(P))\\ =&(-1)^{n+1}\partial(s^{n+1}\{s^{-n-1}\pi, s^{-n-1}P\}) + (-1)^{n+1}s^{n+1}\{s^{-n-1}\pi, s^{-n-1}\partial P\}\\ &+ s^{n+1}\{s^{-n-1}\pi, \{s^{-n-1}\pi, s^{-n-1}P\}\}\\ =&s^{n+1}\{\partial(s^{-n-1}\pi), s^{-n-1}P\} - s^{n+1}\{s^{-n-1}\pi, \partial(s^{-n-1}P)\} + (-1)^{n+1}s^{n+1}\{s^{-n-1}\pi, s^{-n-1}\partial P\}\\ &+ s^{n+1}\left\{\frac{1}{2}\{s^{-n-1}\pi, s^{-n-1}\pi\}, s^{-n-1}P\right\}\\ =&s^{n+1}\left\{\partial(s^{-n-1}\pi) + \frac{1}{2}\{s^{-n-1}\pi, s^{-n-1}\pi\}, s^{-n-1}P\right\} = 0 \quad .\end{aligned}$$

The second line follows from the fact that ∂ is a differential on $\widehat{\mathrm{Pol}}(A,n)$ hence $\partial^2 = 0$. The fourth line follows from the Jacobi identity (2.3.71) and the last line follows from the Maurer-Cartan equation. □

Remark 2.3.35. It is trivial to check that a weight 1 Maurer-Cartan polyvector $s^{-n-1}s^{n+1}\pi^{(1)} \in \widehat{\mathrm{Pol}}^1(A,n)[n+1]^1$ is indeed a Maurer-Cartan derivation $\pi^{(1)} \in \mathsf{T}^1_A$, i.e.,

$$\partial(s^{-n-1}s^{n+1}\pi^{(1)}) + \frac{1}{2}\{s^{-n-1}s^{n+1}\pi^{(1)}, s^{-n-1}s^{n+1}\pi^{(1)}\} = 0 \quad \Longrightarrow \quad \partial(\pi^{(1)}) + \frac{1}{2}[\pi^{(1)}, \pi^{(1)}] = 0 \quad . \tag{2.3.84}$$

And it is also clear that we recover the deformed differential $\delta_{\pi^{(1)}} := \delta + \pi^{(1)}$ of (2.3.7) when evaluating $\partial_{s^{n+1}\pi^{(1)}} := \partial + \mathcal{D}_{s^{n+1}\pi^{(1)}}$ on the base CDGA A,

$$\begin{aligned}\partial(a) + \mathcal{D}_{s^{n+1}\pi^{(1)}}(a) =&\delta(a) + (-1)^{n+1}s^{n+1}\{s^{-n-1}s^{n+1}\pi^{(1)}, s^{-n-1}a\}\\ =&\delta(a) + (-1)^{n+1}(-1)^{|\pi^{(1)}|(n+1)}s^{n+1}s^{-n-1}\pi^{(1)}(a)\\ =&\delta(a) + \pi^{(1)}(a) \quad ,\end{aligned} \tag{2.3.85}$$

where the second equality comes from (2.3.73b) and the third equality comes from the facts that $|\pi^{(1)}| = n+2$ and $s^{n+1}s^{-n-1} = (-1)^{n+1}$. △

We now define the concept of an n-shifted Poisson structure following [Pri17, Definition 1.5], see also [Pri18, Definition 2.5].

Definition 2.3.36. An **n-shifted Poisson structure** (nSPS) on a CDGA A is a Maurer-Cartan polyvector (as in Definition 2.3.33) whose weight terms are all at least 2.

Remark 2.3.37. Decomposing $\pi = \sum_{m\geq 2}\pi_m \in \widehat{\mathrm{Pol}}(A,n)^{n+2}$ into its weight components $\pi_m \in \widehat{\mathrm{Pol}}^m(A,n)^{n+2}$ and using (2.3.67), one observes that the Maurer-Cartan equation (2.3.82a) is

equivalent to the following tower of conditions

$$\partial\pi_2 = 0 \quad , \tag{2.3.86a}$$
$$\partial\pi_3 + \tfrac{1}{2}[\pi_2,\pi_2]_{\mathrm{S}} = 0 \quad , \tag{2.3.86b}$$
$$\vdots$$
$$\partial\pi_m + \tfrac{1}{2}\sum_{k+l=m+1}[\pi_k,\pi_l]_{\mathrm{S}} = 0 \quad , \tag{2.3.86c}$$

for all $m \geq 3$. The first condition states that the shifted bivector π_2 is closed with respect to the differential ∂ which is induced by the one of the CDGA A. Note that this condition is automatically satisfied in the case of an ordinary (i.e. non-dg) commutative algebra, so one usually does not encounter it when studying Poisson structures on algebraic varieties. The second condition is a shifted generalization and homotopical relaxation of the usual condition that the Schouten–Nijenhuis bracket of the bivector π_2 with itself vanishes. It implies in particular that the binary shifted Poisson bracket on A associated with π_2 satisfies the Jacobi identity only up to a homotopy determined by the shifted trivector π_3. The higher weight components π_m, for $m \geq 4$, provide a coherent tower of higher homotopies for the Jacobi identity. △

Example 2.3.38. Analogous to Example 2.3.22, we wish to show that the only nSPSs on an ordinary commutative $\mathbb{K}$-algebra $A = (A^0, 0)$ are simply the 0SPSs and these are precisely the ordinary Poisson structures, i.e., biderivations $\pi : \bigwedge^2_{\mathbb{K}} A \to A$ which satisfy the Jacobi identity.

First note that, in this case, the degree of an n-shifted polyvector $P \in \prod_{m\geq 0}\mathrm{Sym}^m_A\big(\mathsf{T}_A[-n-1]\big)$ is simply determined by its weight m, i.e., $|P| = m(n+1)$. The degree equation $n+2 = m(n+1)$ admits no solutions for $n = -1$ hence we rearrange as $m = 1 + \frac{1}{n+1}$ but $m \in \mathbb{N}$ hence this only makes sense if $n = 0$ and $m = 2$, i.e., for a skew-symmetric biderivation $P \in \mathrm{Sym}^2_A\big(\mathsf{T}_A[-1]\big)$. The tower of Remark 2.3.37 then boils down to $[P,P]_{\mathrm{S}} = 0$ which, for $n = 0$ and an ordinary algebra, is the Jacobi identity. ▽

Example 2.3.39. We are now in a position to give a simple characterisation theorem of the nSPSs, for $n \in \mathbb{N}^*$, on a generic sfCDGA fg in degree 1, $A \cong \mathrm{CE}_{\mathfrak{g}}$, in terms of data compatibly adjoined to the underlying fd Lie algebra $\mathfrak{g}$. This example will extend and generalise the result of Theorem 2.3.24. First let us figure out the admissible values for the shift n by setting (2.3.78) equal to 1,

$$(m-1)n + l = 2 \quad . \tag{2.3.87}$$

Given that the weight requirement is $m \geq 2$ and the arity must always be $l \geq 0$, we can see that this equation admits no solutions for $n \geq 3$. For $n = 2$, the only solution is $(m = 2, l = 0)$. For $n = 1$, there are two solutions given by $(m = 2, l = 1)$ and $(m = 3, l = 0)$.

Theorem 2.3.40. *The 2SPSs on* $\mathrm{CE}_{\mathfrak{g}}$ *can be characterised as symmetric ad-invariant 2-tensors, i.e.,* $\pi_{n=2} \in \big(\mathrm{Sym}^2\mathfrak{g}\big)^{\mathfrak{g}}$.

Proof. We write the $(m = 2, l = 0)$ 2-shifted polyvector as

$$\pi_2 = \tfrac{1}{2}\pi^{ij}s^3\frac{\partial}{\partial(\vartheta^i)}s^3\frac{\partial}{\partial(\vartheta^j)} \tag{2.3.88}$$

where the symmetry condition $\pi^{ij} = \pi^{ji} \in \mathbb{K}$ comes from (2.3.81a). Noting the (bi)derivation property (2.3.61c) of the Schouten bracket $[\cdot,\cdot]_{\mathrm{S}}$ together with the triviality of the result between the $s^{n+1}\frac{\partial}{\partial(\vartheta^i)}$ terms (2.3.80a), the third weight component (2.3.86b) of the Maurer-Cartan equation,

$$\tfrac{1}{2}[\pi_2,\pi_2]_{\mathrm{S}} = 0 \quad , \tag{2.3.89}$$

in the tower of Remark 2.3.37 is automatically upheld and imposes no nontrivial conditions on the 2-tensor π^{ij}. To handle the second weight component (2.3.86a) of the Maurer-Cartan equation,

$$\partial(\pi_2) = 0 \quad , \tag{2.3.90}$$

we must make use of (2.3.5b) and (2.3.81b),

$$\begin{aligned}\partial\left(\tfrac{1}{2}\pi^{ij}s^3\frac{\partial}{\partial(\vartheta^i)}s^3\frac{\partial}{\partial(\vartheta^j)}\right) &= \tfrac{1}{2}\pi^{ij}\left(\mathfrak{g}^l_{ki}\vartheta^k s^3\frac{\partial}{\partial(\vartheta^l)}s^3\frac{\partial}{\partial(\vartheta^j)} + \mathfrak{g}^l_{kj}\vartheta^k s^3\frac{\partial}{\partial(\vartheta^i)}s^3\frac{\partial}{\partial(\vartheta^l)}\right)\\ &= \pi^{ij}\mathfrak{g}^l_{ki}\vartheta^k s^3\frac{\partial}{\partial(\vartheta^l)}s^3\frac{\partial}{\partial(\vartheta^j)}\end{aligned} \tag{2.3.91}$$

where the last equality comes from a reindexing on the second term together with the symmetry condition $\pi^{ij} = \pi^{ji}$ and using (2.3.81a). Setting (2.3.91) to zero then stripping away the algebra terms gives us

$$\pi^{ij}\mathfrak{g}^l_{ki} + \pi^{il}\mathfrak{g}^j_{ki} = 0 \quad . \tag{2.3.92}$$

Setting

$$\pi_{n=2} = \pi^{ij}x_i \otimes x_j \in \mathrm{Sym}^2\mathfrak{g} \quad , \tag{2.3.93}$$

and taking the Lie bracket of this symmetric 2-tensor with some basis element x_k gives us,

$$[x_k, \pi^{ij}x_i \otimes x_j] = \pi^{ij}[x_k, x_i] \otimes x_j + \pi^{ij}x_i \otimes [x_k, x_j] = \pi^{ij}\mathfrak{g}^l_{ki}x_l \otimes x_j + \pi^{ij}\mathfrak{g}^l_{kj}x_i \otimes x_l \quad . \tag{2.3.94}$$

Reindexing the second term ($i \leftrightarrow j$) and setting the above expression to zero means that we can strip away the x terms to leave us with (2.3.92) again hence that relation is indeed the ad-invariance condition. ▽

Before we characterise the 1SPS, which we know to consist of the terms ($m = 2, l = 1$) and ($m = 3, l = 0$), we must first recall the definition of a quasi-Lie bialgebra.

Remark 2.3.41. Following the convention in [EnHa08, Subsection 2.3], the notation Alt_n means to sum over the cyclic permutations of n letters, $\sigma \in \mathrm{C}_n$, weighted by $\mathrm{sgn}\,\sigma$. Note that this notation allows us to compactly say that a Lie algebra structure on $\mathfrak{g} \in \mathbf{Vec}_{\mathbb{K}}$ is given by the datum of a linear map $[\cdot,\cdot] : \bigwedge^2_{\mathbb{K}}\mathfrak{g} \to \mathfrak{g}$ (called the Lie bracket) subject to a single condition

$$[\cdot,\cdot] \circ ([\cdot,\cdot] \otimes \mathrm{id}_{\mathfrak{g}}) \circ \mathrm{Alt}_3 = 0 : \mathfrak{g}^{\otimes 3} \to \mathfrak{g} \tag{2.3.95a}$$

(called the Jacobi identity). We arrive at the notion of a **Lie coalgebra** $(\mathfrak{g}, \Delta)$ by dualising[12] the datum and condition, i.e., a linear map $\Delta : \mathfrak{g} \to \bigwedge^2_{\mathbb{K}}\mathfrak{g}$ (called the **Lie cobracket**) subject to the **coJacobi identity**,

$$\mathrm{Alt}_3 \circ (\Delta \otimes \mathrm{id}_{\mathfrak{g}}) \circ \Delta = 0 : \mathfrak{g} \to \mathfrak{g}^{\otimes 3} \quad . \tag{2.3.95b}$$

The notion of a **Lie bialgebra** $(\mathfrak{g}, [\cdot,\cdot], \Delta)$ is a compatibilisation of these two notions; the Lie cobracket must be a derivation of the Lie bracket,

$$\Delta \circ [\cdot,\cdot] = [\![\Delta, \cdot]\!] + [\![\cdot, \Delta]\!] : \mathfrak{g}^{\otimes 2} \to \mathfrak{g}^{\otimes 2} \quad . \tag{2.3.95c}$$

We can rephrase (2.3.95c) in the language of Lie algebra cohomology as in [Kas95, Section XVIII.1], [CW94, Section 7.7] or [LGPV13, Subsection 4.1.1]. Given a Lie algebra representation (V, ρ) as in Example 2.2.10, we define the space of V**-valued** l**-cochains** as $C^l(\mathfrak{g}; V) :=$

[12]Note that in the case of a fd Lie algebra $\mathfrak{g}$, this dualisation should be read as literal in the sense that $\mathfrak{g}^*$ is then a Lie coalgebra with Lie cobracket $[\cdot,\cdot]^* : \mathfrak{g}^* \to (\bigwedge^2_{\mathbb{K}}\mathfrak{g})^*$.

$\mathrm{Hom}(\bigwedge^l_{\mathbb{K}}\mathfrak{g}, V)$ and, given $\Theta^{(l)} : \bigwedge^l_{\mathbb{K}}\mathfrak{g} \to V$, we define an $(l+1)$-linear map by

$$\begin{aligned}\big(\delta\Theta^{(l)}\big)(x_{i_1},\dots,x_{i_{l+1}}) := &\sum_{1\le j<k\le l+1}(-1)^{j+k}\Theta^{(l)}([\![x_{i_j},x_{i_k}]\!],x_{i_1},\dots,\hat{x}_{i_j},\dots,\hat{x}_{i_k},\dots,x_{i_{l+1}})\\ &+\sum_{1\le j\le l+1}(-1)^{j+1}x_{i_j}.\Theta^{(l)}(x_{i_1},\dots,\hat{x}_{i_j},\dots,x_{i_{l+1}})\end{aligned} \tag{2.3.96}$$

where the hat $\hat{}$ indicates that the input is omitted. The proof that the above is indeed skew-symmetric and that $\delta : \mathrm{Hom}(\bigwedge^l_{\mathbb{K}}\mathfrak{g}, V) \to \mathrm{Hom}(\bigwedge^{l+1}_{\mathbb{K}}\mathfrak{g}, V)$ satisfies $\delta^2 = 0$ is a standard exercise in making use of the Jacobi identity and the fact that $\rho : \mathfrak{g} \to \mathrm{End}(V)$ is a Lie algebra homomorphism. In fact, if $\mathfrak{g}$ is fd and $(\mathbb{K}, \rho_{\mathrm{Triv}} = 0)$ is the trivial representation then we recover the Chevalley-Eilenberg algebra from Example 2.2.6 whereas for a generic representation (V, ρ) we recover the Chevalley-Eilenberg module from Example 2.2.10. Now we can see that (2.3.95c) is equivalent to the condition that $\Delta \in Z^1(\mathfrak{g}; \bigwedge^2_{\mathbb{K}}\mathfrak{g}_{\mathrm{ad}})$. △

Definition 2.3.42. A **quasi-Lie bialgebra** is a triple $(\mathfrak{g}, \Delta, \varphi)$ where $\mathfrak{g}$ is a Lie algebra, the Lie cobracket $\Delta \in Z^1(\mathfrak{g}; \bigwedge^2_{\mathbb{K}}\mathfrak{g}_{\mathrm{ad}})$ satisfies the coJacobi identity up to coboundary in the $\bigwedge^3_{\mathbb{K}}\mathfrak{g}_{\mathrm{ad}}$-valued 0-cochain $\varphi \in \bigwedge^3_{\mathbb{K}}\mathfrak{g}_{\mathrm{ad}}$,

$$\mathrm{Alt}_3 \circ (\Delta \otimes \mathrm{id}_{\mathfrak{g}}) \circ \Delta = [\![\cdot, \varphi]\!] \quad , \tag{2.3.97a}$$

and this 0-cochain is coherent with the Lie cobracket,

$$\mathrm{Alt}_4 \circ (\Delta \otimes \mathrm{id}^{\otimes 2}_{\mathfrak{g}}) \circ \varphi = 0 \quad . \tag{2.3.97b}$$

Remark 2.3.43. We can write $\varphi \in \bigwedge^3_{\mathbb{K}}\mathfrak{g}$ in terms of the generators as

$$\varphi = \varphi^{mjk} x_m \wedge x_j \wedge x_k \tag{2.3.98a}$$

where $\varphi^{mjk} \in \mathbb{K}$ is totally anti-symmetric. Likewise

$$\Delta(x_l) = \Delta_l^{mk} x_m \wedge x_k \tag{2.3.98b}$$

is such that $\Delta_l^{mk} = -\Delta_l^{km} \in \mathbb{K}$. Upon application to x_l, the quasi-coJacobi identity (2.3.97a) now reads

$$\Delta_m^{ij}\Delta_l^{mk} x_i \wedge x_j \wedge x_k = \mathfrak{g}_{lm}^i \varphi^{mjk} x_i \wedge x_j \wedge x_k \tag{2.3.98c}$$

which yields

$$\Delta_m^{ij}\Delta_l^{mk} + \Delta_m^{jk}\Delta_l^{mi} + \Delta_m^{ki}\Delta_l^{mj} = \mathfrak{g}_{lm}^i\varphi^{mjk} + \mathfrak{g}_{lm}^j\varphi^{mki} + \mathfrak{g}_{lm}^k\varphi^{mij} \quad . \tag{2.3.98d}$$

The coherency condition (2.3.97b) reads

$$\Delta_m^{ij}\varphi^{mkl} x_i \wedge x_j \wedge x_k \wedge x_l = 0 \tag{2.3.98e}$$

which yields

$$\Delta_m^{ij}\varphi^{mkl} - \Delta_m^{jk}\varphi^{mli} + \Delta_m^{kl}\varphi^{mij} - \Delta_m^{li}\varphi^{mjk} - \Delta_m^{ik}\varphi^{mjl} + \Delta_m^{lj}\varphi^{mik} = 0 \quad . \tag{2.3.98f}$$

The 1-cocycle condition (2.3.95c) on generators $x_l \otimes x_m$ reads

$$\Delta_k^{ij}\mathfrak{g}_{lm}^k x_i \wedge x_j = 2(\mathfrak{g}_{km}^j\Delta_l^{ik} + \mathfrak{g}_{lk}^i\Delta_m^{kj}) x_i \wedge x_j \tag{2.3.98g}$$

which yields

$$\Delta_k^{ij}\mathfrak{g}_{lm}^k = \mathfrak{g}_{km}^j\Delta_l^{ik} + \mathfrak{g}_{lk}^i\Delta_m^{kj} - \mathfrak{g}_{km}^i\Delta_l^{jk} - \mathfrak{g}_{lk}^j\Delta_m^{ki} \quad . \tag{2.3.98h}$$

△

Proposition 2.3.44. *The 1SPSs on* $\mathrm{CE}_{\mathfrak{g}}$ *can be characterised as quasi-Lie bialgebra structures on* $\mathfrak{g}$.

Proof. We write the candidate 1-shifted polyvector as

$$\pi_2 + \pi_3 = \tfrac{1}{2!}\pi_k^{ij}\vartheta^k s^2\frac{\partial}{\partial(\vartheta^i)}s^2\frac{\partial}{\partial(\vartheta^j)} + \tfrac{1}{3!}\pi^{lmn}s^2\frac{\partial}{\partial(\vartheta^l)}s^2\frac{\partial}{\partial(\vartheta^m)}s^2\frac{\partial}{\partial(\vartheta^n)} \quad , \tag{2.3.99}$$

where $\pi_k^{ij} = -\pi_k^{ji} \in \mathbb{K}$ and $\pi^{lmn} \in \mathbb{K}$ is totally anti-symmetric. The tower of conditions in Remark 2.3.37 now boils down to the first four weight components:

$$\partial\pi_2 = 0 \quad , \tag{2.3.100a}$$

$$\partial\pi_3 + \frac{1}{2}[\pi_2,\pi_2]_{\mathrm{S}} = 0 \quad , \tag{2.3.100b}$$

$$[\pi_2,\pi_3]_{\mathrm{S}} = 0 \quad , \tag{2.3.100c}$$

$$[\pi_3,\pi_3]_{\mathrm{S}} = 0 \quad . \tag{2.3.100d}$$

The first equation yields

$$\begin{aligned}\partial\pi_2 &= -\tfrac{1}{4}\pi_k^{ij}\,\mathfrak{g}_{lm}^k\vartheta^l\vartheta^m s^2\frac{\partial}{\partial(\vartheta^i)}s^2\frac{\partial}{\partial(\vartheta^j)} - \pi_k^{ij}\,\mathfrak{g}_{mi}^p\vartheta^k\vartheta^m s^2\frac{\partial}{\partial(\vartheta^p)}s^2\frac{\partial}{\partial(\vartheta^j)}\\ &= \left(-\tfrac{1}{4}\pi_k^{ij}\,\mathfrak{g}_{lm}^k - \pi_l^{pj}\,\mathfrak{g}_{mp}^i\right)\vartheta^l\vartheta^m s^2\frac{\partial}{\partial(\vartheta^i)}s^2\frac{\partial}{\partial(\vartheta^j)}\\ &= \left(-\tfrac{1}{4}\pi_k^{ij}\,\mathfrak{g}_{lm}^k - \pi_l^{kj}\,\mathfrak{g}_{mk}^i\right)\vartheta^l\vartheta^m s^2\frac{\partial}{\partial(\vartheta^i)}s^2\frac{\partial}{\partial(\vartheta^j)}\end{aligned} \tag{2.3.101a}$$

where the second equality comes from a reindexing on the second term as $k \leftrightarrow l$ and $p \leftrightarrow i$. Setting (2.3.101a) to zero and then stripping away all algebra terms yields (2.3.98h) with $\Delta_k^{ij} = \pi_k^{ij}$. The second equation (2.3.100b) yields, upon using (2.3.80b),

$$\pi_k^{nm}\pi_n^{ij} + \pi_k^{ni}\pi_n^{jm} + \pi_k^{nj}\pi_n^{mi} = \mathfrak{g}_{kl}^m\pi^{lij} + \mathfrak{g}_{kl}^j\pi^{lmi} + \mathfrak{g}_{kl}^i\pi^{ljm} \tag{2.3.101b}$$

which is clearly equivalent to (2.3.98d) upon setting $\varphi^{lmn} := \pi^{lmn}$. The third equation (2.3.100c) yields the equation

$$\pi^{lmn}\pi_l^{jk} - \pi^{lkn}\pi_l^{jm} - \pi^{lmk}\pi_l^{jn} - \pi^{lmj}\pi_l^{nk} - \pi^{ljn}\pi_l^{mk} + \pi^{ljk}\pi_l^{mn} = 0, \tag{2.3.101c}$$

which is equivalent to (2.3.98f). The fourth equation (2.3.100d) is true by construction given that the Schouten bracket is a biderivation and has trivial result between the shifted derivations as in (2.3.80a). ▽

▽

Chapter 3

Shifted Poisson structures on Chevalley-Eilenberg algebras

The aim of this chapter is to specialize the concepts of shifted polyvectors and SPSs to the case where A is a fgsfCDGA and develop a graphical calculus which is practically useful to analyse the algebraic properties of nSPSs. The main results about our graphical calculus are stated in Corollaries 3.1.5 and 3.1.6. We observe in Remark 3.1.7 that the derived geometric concept of an nSPS in this fgsf context is equivalent to the $L_\infty[0, n-1]$-quasi-bialgebra structures from [BSZ13, Definition 2.5] and we thus provide a geometric interpretation for such objects.

In Section 3.2, we specialize our description of nSPSs further to the case where $A = \mathrm{CE}_{\mathfrak{g}}$ is the Chevalley-Eilenberg algebra of a Lie N-algebra $\mathfrak{g}$. We show in Lemma 3.2.6 that nSPSs on $A = \mathrm{CE}_{\mathfrak{g}}$ are necessarily trivial for $n > 2N$, which provides us with a useful guiding principle for our exploration. For ordinary Lie algebras (i.e. $N = 1$), we recover in Subsection 3.2.1 from our graphical calculus the characterisation in [Saf21] of $(n = 1)$- and $(n = 2)$- SPSs in terms of quasi-Lie bialgebras and, respectively, invariant symmetric tensors, see Propositions 3.2.11 and 3.2.13. The focus of Subsection 3.2.2 is on the case of Lie 2-algebras, which by Lemma 3.2.6 admit a priori non-trivial positively nSPSs for all $n \in \{1, 2, 3, 4\}$. In the case of $n \in \{2, 3, 4\}$, we characterize these shifted Poisson structures explicitly by using our graphical calculus and find that they consist of finitely many data, see Propositions 3.2.15, 3.2.17 and 3.2.18. The case of $n = 1$ is exceptional and more difficult because 1SPSs consist of a priori infinitely many data, see Remark 3.2.20. In the final Section 3.3 we provide a collection of explicit examples for SPSs on the Chevalley-Eilenberg algebras associated with Abelian Lie 2-algebras, shifted cotangent Lie 2-algebras, string Lie 2-algebras and derivation strict Lie 2-algebras. Our reason for this particular choice of examples it that these Lie 2-algebras appear naturally in mathematical physics, e.g., in Abelian gerbes [Bry08], in higher-dimensional Chern-Simons theory [Zuc21, SV24], in string theory [B+24].

3.1 Shifted polyvectors of free CDGAs as multilinear maps

To match our present notation and conventions with the ones we use when studying higher Lie algebras in Section 3.2 below, we consider a free CDGA of the form

$$A \;=\; \mathrm{Sym}\big(\mathfrak{g}^*[-1]\big) \quad , \tag{3.1.1}$$

where $\mathfrak{g}^* := \underline{\hom}(\mathfrak{g}, \mathbb{K}) \in \mathbf{Ch}$ is the dual of a fd cochain complex $\mathfrak{g} \in \mathbf{Ch}$. We denote the duality pairing by $\langle \cdot\,, \cdot \rangle : \mathfrak{g}^* \otimes \mathfrak{g} \to \mathbb{K}\,,\ \theta \otimes x \mapsto \langle \theta, x \rangle$. Later in Section 3.2, the cochain complex $\mathfrak{g}$ will

be equipped with the structure of an L_∞-algebra.

Recalling (2.3.4a), we obtain that the dg-module of derivations on the CDGA (3.1.1) is given by

$$\mathsf{T}_{\mathrm{Sym}(\mathfrak{g}^*[-1])} \,\cong\, \underline{\hom}\Big(\mathfrak{g}^*[-1], \mathrm{Sym}\big(\mathfrak{g}^*[-1]\big)\Big) \,\cong\, \mathrm{Sym}\big(\mathfrak{g}^*[-1]\big) \otimes \mathfrak{g}[1] \quad . \tag{3.1.2}$$

The evaluation of derivations on the CDGA $\mathrm{Sym}\big(\mathfrak{g}^*[-1]\big)$ is determined by shifting and permuting the duality pairing according to

$$\mathfrak{g}[1] \otimes \mathfrak{g}^*[-1] \xrightarrow{\cong} (\mathfrak{g} \otimes \mathfrak{g}^*)[1-1] = \mathfrak{g} \otimes \mathfrak{g}^* \xrightarrow{\gamma} \mathfrak{g}^* \otimes \mathfrak{g} \xrightarrow{\langle \cdot, \cdot \rangle} \mathbb{K} \quad , \tag{3.1.3a}$$

where the first cochain isomorphism is given in (2.2.9) and it moves the shifts to the left. With a slight abuse of notation, we denote the resulting pairing by the same symbol $\langle \cdot, \cdot \rangle$ as the duality pairing. For homogeneous $s^{-1}\, x \in \mathfrak{g}[1]$ and $s\, \theta \in \mathfrak{g}^*[-1]$, this pairing reads explicitly as

$$\langle s^{-1}x, s\,\theta \rangle \,:=\, (-1)^{|x|}\, (-1)^{|x|\,|\theta|}\, s^{-1}s\, \langle \theta, x \rangle \,=\, \langle \theta, x \rangle \quad , \tag{3.1.3b}$$

where in the last step we used that $|x| = -|\theta|$ whenever $\langle \theta, x \rangle \neq 0$.

Using (3.1.2), it follows that the n-shifted polyvectors from Definition 2.3.25 on the free CDGA $\mathrm{Sym}\big(\mathfrak{g}^*[-1]\big)$ are given by

$$\mathrm{Pol}\big(\mathrm{Sym}\big(\mathfrak{g}^*[-1]\big), n\big) \,\cong\, \mathrm{Sym}\big(\mathfrak{g}[-n]\big) \otimes \mathrm{Sym}\big(\mathfrak{g}^*[-1]\big) \,\cong\, \mathrm{Sym}\Big(\mathfrak{g}[-n] \oplus \mathfrak{g}^*[-1]\Big) \quad . \tag{3.1.4a}$$

Observe that, in addition to the cohomological degree of the underlying cochain complexes, there are two additional gradings

$$\mathrm{Pol}\big(\mathrm{Sym}\big(\mathfrak{g}^*[-1]\big), n\big) \,=\, \bigoplus_{m,l\geq 0} \mathrm{Pol}^{m,l}\big(\mathrm{Sym}\big(\mathfrak{g}^*[-1]\big), n\big) \,\cong\, \bigoplus_{m,l\geq 0} \Big(\mathrm{Sym}^m\big(\mathfrak{g}[-n]\big) \otimes \mathrm{Sym}^l\big(\mathfrak{g}^*[-1]\big)\Big) \tag{3.1.4b}$$

given by the **weight** m of shifted polyvectors and the **symmetric power** l in the underlying free algebra $\mathrm{Sym}\big(\mathfrak{g}^*[-1]\big)$. The completion from Definition 2.3.30 is then given by

$$\widehat{\mathrm{Pol}}\big(\mathrm{Sym}\big(\mathfrak{g}^*[-1]\big), n\big) \,=\, \prod_{m\geq 0} \bigoplus_{l\geq 0} \mathrm{Pol}^{m,l}\big(\mathrm{Sym}\big(\mathfrak{g}^*[-1]\big), n\big) \quad , \tag{3.1.5}$$

i.e., only the weight gets completed. As a side remark, let us note that completing also the symmetric power l in (3.1.5) defines the CDGA of completed n-shifted polyvectors on the completed symmetric algebra $\widehat{\mathrm{Sym}}\big(\mathfrak{g}^*[-1]\big) := \prod_{l\geq 0} \mathrm{Sym}^l\big(\mathfrak{g}^*[-1]\big)$.

In the present case of a free CDGA, the Schouten–Nijenhuis bracket (2.3.68) admits a simplified description in terms of the commutator of a **bullet composition** operation,

$$\bullet : \mathrm{Pol}\big(\mathrm{Sym}\big(\mathfrak{g}^*[-1]\big), n\big)[n+1] \otimes \mathrm{Pol}\big(\mathrm{Sym}\big(\mathfrak{g}^*[-1]\big), n\big)[n+1] \to \mathrm{Pol}\big(\mathrm{Sym}\big(\mathfrak{g}^*[-1]\big), n\big)[n+1]\,, \tag{3.1.6}$$

which we shall now describe. We define the cochain map $\bullet$ on the homogeneous generators

$s\,\theta, s\,\theta' \in \mathfrak{g}^*[-1]$ and $s^n x, s^n x' \in \mathfrak{g}[-n]$ of (3.1.4) by

$$
\begin{aligned}
(s^{-n-1} s\,\theta) \bullet (s^{-n-1} s\,\theta') &= 0 \quad , && \text{(3.1.7a)}\\
(s^{-n-1} s^n x) \bullet (s^{-n-1} s\,\theta) &= 0 \quad , && \text{(3.1.7b)}\\
(s^{-n-1} s\,\theta) \bullet (s^{-n-1} s^n x) &= (-1)^{|\theta|}\, s^{-n-1}\, \langle \theta, x \rangle \quad , && \text{(3.1.7c)}\\
(s^{-n-1} s^n x) \bullet (s^{-n-1} s^n x') &= 0 \quad , && \text{(3.1.7d)}
\end{aligned}
$$

and extend to $\mathrm{Pol}\big(\mathrm{Sym}\big(\mathfrak{g}^*[-1]\big), n\big)$ as a biderivation, i.e. by demanding the properties:

$$
(s^{-n-1} P) \bullet (s^{-n-1} Q\, R) = \big((s^{-n-1} P) \bullet (s^{-n-1} Q)\big)\, R + (-1)^{|P|\,|Q|}\, Q\, \big((s^{-n-1} P) \bullet (s^{-n-1} R)\big) \quad \text{(3.1.8a)}
$$

and

$$
\begin{aligned}
(s^{-n-1} P\, Q) \bullet (s^{-n-1} R) &= (-1)^{|P|\,(n+1)}\, P\, \big((s^{-n-1} Q) \bullet (s^{-n-1} R)\big)\\
&\quad + (-1)^{|Q|\,(|R|-n-1)}\, \big((s^{-n-1} P) \bullet (s^{-n-1} R)\big)\, Q \quad , \qquad \text{(3.1.8b)}
\end{aligned}
$$

for all homogeneous $P, Q, R \in \mathrm{Pol}\big(\mathrm{Sym}\big(\mathfrak{g}^*[-1]\big), n\big)$.

Proposition 3.1.1. *The Schouten–Nijenhuis bracket* (2.3.68) *on the n-shifted polyvectors of a free CDGA* $\mathrm{Sym}\big(\mathfrak{g}^*[-1]\big)$ *agrees with the commutator of the composition operation* (3.1.6)*, i.e.*

$$
\{s^{-n-1} P, s^{-n-1} Q\} = (s^{-n-1} P) \bullet (s^{-n-1} Q) - (-1)^{(|P|-n-1)\,(|Q|-n-1)}\, (s^{-n-1} Q) \bullet (s^{-n-1} P) \quad , \qquad \text{(3.1.9)}
$$

for all homogeneous $P, Q \in \mathrm{Pol}\big(\mathrm{Sym}\big(\mathfrak{g}^*[-1]\big), n\big)$.

Proof. The commutator of $\bullet$ is manifestly antisymmetric, as is the Schouten–Nijenhuis bracket (2.3.70), and it also satisfies the derivation property (2.3.72). The latter statement follows from (3.1.8a) and the following identity

$$
\begin{aligned}
(s^{-n-1} Q\, R) \bullet (s^{-n-1} P) &= (-1)^{|R|\,|Q|}\, (s^{-n-1} R\, Q) \bullet (s^{-n-1} P)\\
&= (-1)^{|R|\,(|Q|-n-1)}\, R\, \big((s^{-n-1} Q) \bullet (s^{-n-1} P)\big)\\
&\quad + (-1)^{|Q|\,(|R|+|P|-n-1)}\, \big((s^{-n-1} R) \bullet (s^{-n-1} P)\big)\, Q\\
&= (-1)^{|R|\,(|P|-n-1)}\, \big((s^{-n-1} Q) \bullet (s^{-n-1} P)\big)\, R\\
&\quad + (-1)^{|Q|\,(n+1)}\, Q\, \big((s^{-n-1} R) \bullet (s^{-n-1} P)\big) \quad , \qquad \text{(3.1.10)}
\end{aligned}
$$

where in the first and third step we used commutativity of the CDGA $\mathrm{Pol}\big(\mathrm{Sym}\big(\mathfrak{g}^*[-1]\big), n\big)$ and in the second step we used (3.1.8b). As a consequence of antisymmetry and the derivation property, it suffices to verify the identity (3.1.9) on the generators of (3.1.4). For the left-hand side given by the Schouten–Nijenhuis bracket, one computes using (2.3.73) and (3.1.3) that

$$
\begin{aligned}
\{s^{-n-1} s\,\theta, s^{-n-1} s\,\theta'\} &= 0 = \{s^{-n-1} s^n x, s^{-n-1} s^n x'\} \quad , && \text{(3.1.11a)}\\
\{s^{-n-1} s^n x, s^{-n-1} s\,\theta\} &= \{s^{-n-1} s^{n+1} s^{-1} x, s^{-n-1} s\,\theta\} = (-1)^{(|x|-1)\,(n+1)}\, s^{-n-1}\, \langle s^{-1} x, s\theta \rangle\\
&= (-1)^{(|x|-1)\,(n+1)}\, s^{-n-1}\, \langle \theta, x \rangle \quad , && \text{(3.1.11b)}
\end{aligned}
$$

for all homogeneous $s\,\theta, s\,\theta' \in \mathfrak{g}^*[-1]$ and $s^n x, s^n x' \in \mathfrak{g}[-n]$. Comparing with (3.1.7) one immediately observes that this coincides with the commutator of $\bullet$, which implies that (3.1.9) holds true for generators and hence for all shifted polyvectors. □

Corollary 3.1.2. *A completed n-shifted polyvector $\pi = \sum_{m\geq 2} \pi^{(m)} \in \widehat{\mathrm{Pol}}\big(\mathrm{Sym}\big(\mathfrak{g}^*[-1]\big)\big)^{n+2}$ of degree $n+2$ and weight ≥ 2 is an nSPS in the sense of Definition 2.3.36 if and only if*

$$\partial\big(s^{-n-1}\pi\big) + (s^{-n-1}\pi) \bullet (s^{-n-1}\pi) \;=\; 0 \quad . \tag{3.1.12}$$

Proof. From Proposition 3.1.1, it follows that

$$\begin{aligned} \{s^{-n-1}\pi, s^{-n-1}\pi\} \;&=\; (s^{-n-1}\pi) \bullet (s^{-n-1}\pi) - (-1)^{(n+2-n-1)^2}\; (s^{-n-1}\pi) \bullet (s^{-n-1}\pi) \\ &=\; 2\; (s^{-n-1}\pi) \bullet (s^{-n-1}\pi) \quad . \end{aligned} \tag{3.1.13}$$

This implies that (3.1.12) is equivalent to the Maurer-Cartan equation (2.3.82a). □

In Section 3.2 below, it will be convenient to identify n-shifted polyvectors and Poisson structures on the free CDGA $\mathrm{Sym}\big(\mathfrak{g}^*[-1]\big)$ in terms of maps $\underline{\hom}(\mathfrak{g}^{\otimes l}, \mathfrak{g}^{\otimes m})$ between tensor powers of the *unshifted* cochain complex $\mathfrak{g}$. This identification involves Koszul signs, e.g., those arising from the shifting isomorphisms (2.2.9), which contribute to the explicit form of the transferred composition operation (3.1.6) and hence the transferred Schouten–Nijenhuis bracket. The aim of this subsection is to work out these identifications in detail.

First, let us observe that, as a consequence of (2.2.9) and (2.2.10), we have cochain isomorphisms

$$\begin{aligned} \mathrm{Sym}\big(\mathfrak{g}^*[-1]\big) \;&\xrightarrow{\cong}\; \bigoplus_{l\geq 0} \big(\textstyle\bigwedge^l \mathfrak{g}^*\big)[-l] \quad , \\ s\theta_1 \cdots s\theta_l \;&\longmapsto\; (-1)^{\sum_{j=1}^l |\theta_j|\,(l-j)}\; s^l\theta_1 \cdots \theta_l \end{aligned} \tag{3.1.14a}$$

and

$$\begin{aligned} \mathrm{Sym}\big(\mathfrak{g}[-n]\big) \;&\xrightarrow{\cong}\; \bigoplus_{m\geq 0} \big(\mathrm{Sym}^m_{\pm}\mathfrak{g}\big)[-mn] \quad , \\ s^n x_1 \cdots s^n x_m \;&\longmapsto\; (-1)^{\sum_{k=1}^m |x_k|\,(m-k)n}\; s^{mn} x_1 \cdots x_m \quad , \end{aligned} \tag{3.1.14b}$$

where

$$\mathrm{Sym}^m_{\pm} V \;:=\; \begin{cases} \mathrm{Sym}^m V & , \text{ for } n \text{ even} \quad , \\ \bigwedge^m V & , \text{ for } n \text{ odd} \quad , \end{cases} \tag{3.1.14c}$$

denotes the symmetric/exterior powers of a cochain complex $V \in \mathbf{Ch}$. Using (2.2.9) once more for the n-shifted polyvectors (3.1.4), we obtain the cochain isomorphism

$$\begin{aligned} \mathrm{Pol}\big(\mathrm{Sym}\big(\mathfrak{g}^*[-1]\big), n\big) \;&\xrightarrow{\cong}\; \bigoplus_{m,l\geq 0} \Big(\mathrm{Sym}^m_{\pm}\,\mathfrak{g} \otimes \textstyle\bigwedge^l \mathfrak{g}^*\Big)[-mn-l] \quad , \\ s^n x_1 \cdots s^n x_m\, s\theta_1 \cdots s\theta_l \;&\longmapsto\; (-1)^{\sum_{k=1}^m |x_k|\,((m-k)n+l)}\; (-1)^{\sum_{j=1}^l |\theta_j|\,(l-j)}\; s^{mn+l} x_1 \cdots x_m\, \theta_1 \cdots \theta_l \quad . \end{aligned} \tag{3.1.15}$$

Since this isomorphism is defined weight-wise, it extends to the completed n-shifted polyvectors (3.1.5) replacing $\bigoplus_{m\geq 0}$ by $\prod_{m\geq 0}$ on both sides.

Next, we define a cochain map

$$\mathrm{Sym}^m_{\pm}\,\mathfrak{g} \otimes \textstyle\bigwedge^l \mathfrak{g}^* \;\longrightarrow\; \underline{\hom}\big(\mathfrak{g}^{\otimes l}, \mathfrak{g}^{\otimes m}\big) \quad , \tag{3.1.16a}$$

for all $m, l \geq 0$, by making use of the duality pairing $\langle\,\cdot\,,\,\cdot\,\rangle : \mathfrak{g}^* \otimes \mathfrak{g} \to \mathbb{K}$. To an element

$B = x_1 \cdots x_m \, \theta_1 \cdots \theta_l \in \mathrm{Sym}^m_\pm \, \mathfrak{g} \otimes \wedge^l \mathfrak{g}^*$, with all x_k and θ_j homogeneous, this cochain map assigns the element $L_B \in \underline{\hom}\big(\mathfrak{g}^{\otimes l}, \mathfrak{g}^{\otimes m}\big)$ which is defined by the evaluation formula

$$L_B(y_1, \dots, y_l) \;:=\; \sum_{\sigma \in S_l} \sum_{\rho \in S_m} (-1)^{|x_1 \cdots x_m|^\rho_\pm} \; (-1)^{|\theta_1 \cdots \theta_l|^\sigma_-} \; (-1)^{|y_1 \cdots y_l|^{\mathrm{rev}}_+} \bigotimes_{k=1}^{m} x_{\rho(k)} \; \prod_{j=1}^{l} \langle \theta_{\sigma(j)}, y_j \rangle \quad , \tag{3.1.16b}$$

for all homogeneous $y_1, \dots, y_l \in \mathfrak{g}$, where the Koszul signs are determined from permutations $\rho \in S_m$ in $\mathrm{Sym}^m_\pm V$, i.e. $v_1 \cdots v_m = (-1)^{|v_1 \cdots v_m|^\rho_\pm} \, v_{\rho(1)} \cdots v_{\rho(m)}$ for all homogeneous $v_1, \dots v_m \in V$ in a cochain complex $V \in \mathbf{Ch}$. The superscript $^{\mathrm{rev}}$ refers to the reversal permutation $(1, 2, \dots, l) \to (l, \dots, 2, 1)$. The sum over all permutations $\sigma \in S_l, \rho \in S_m$, together with the associated Koszul signs, encodes the (anti-)symmetry properties of $\mathrm{Sym}^m_\pm \, \mathfrak{g} \otimes \wedge^l \mathfrak{g}^*$ at the level of maps $\underline{\hom}\big(\mathfrak{g}^{\otimes l}, \mathfrak{g}^{\otimes m}\big)$. Indeed, the cochain map (3.1.16) defines an isomorphism when corestricting

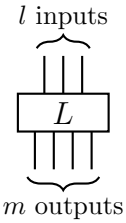
$$\mathrm{Sym}^m_\pm \, \mathfrak{g} \otimes \bigwedge^l \mathfrak{g}^* \;\; \xrightarrow{\cong} \;\; \underline{\hom}^-_\pm\big(\mathfrak{g}^{\otimes l}, \mathfrak{g}^{\otimes m}\big) \;\subseteq\; \underline{\hom}\big(\mathfrak{g}^{\otimes l}, \mathfrak{g}^{\otimes m}\big) \tag{3.1.16c}$$

to the subcomplex of all $L \in \underline{\hom}\big(\mathfrak{g}^{\otimes l}, \mathfrak{g}^{\otimes m}\big)$ whose input is totally antisymmetric and whose output is totally symmetric for n even or antisymmetric for n odd. The additional Koszul sign $(-1)^{|y_1 \cdots y_l|^{\mathrm{rev}}_+}$ is a convenient convention which simplifies some of the signs appearing below. Combining the above identifications, we obtain a cochain isomorphism

$$\mathrm{Pol}\big(\mathrm{Sym}\big(\mathfrak{g}^*[-1]\big), n\big) \;\cong\; \bigoplus_{m,l \geq 0} \underline{\hom}^-_\pm\big(\mathfrak{g}^{\otimes l}, \mathfrak{g}^{\otimes m}\big)[-mn - l] \quad , \tag{3.1.17}$$

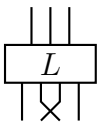
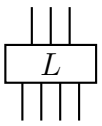
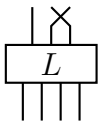
which extends to the completed n-shifted polyvectors (3.1.5) replacing $\bigoplus_{m \geq 0}$ by $\prod_{m \geq 0}$ on both sides.

It will be convenient to represent elements $L \in \underline{\hom}^-_\pm\big(\mathfrak{g}^{\otimes l}, \mathfrak{g}^{\otimes m}\big)$ graphically by diagrams of the form

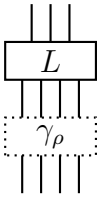
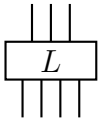
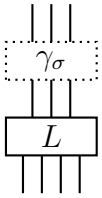


$$\in \;\; \underline{\hom}^-_\pm\big(\mathfrak{g}^{\otimes l}, \mathfrak{g}^{\otimes m}\big) \tag{3.1.18}$$

which should be read from top to bottom. Total (anti-)symmetry of the inputs and outputs under the symmetric braiding γ on $\mathbf{Ch}$ is graphically visualized by e.g.

$(-1)^n$  $=$  $= (-1)$  . (3.1.19a)

More generally, given any permutations $\sigma \in S_l$ and $\rho \in S_m$, we have that

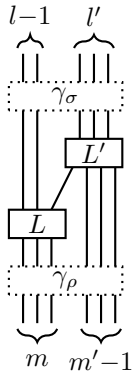
$(-1)^{|\rho| \, n}$  $=$  $= (-1)^{|\sigma|}$  , (3.1.19b)

where the dotted boxes represent the action of permutations via the symmetric braiding γ and

$|\sigma|, |\rho| \in \mathbb{Z}_2$ denotes the signatures of the permutations.

Proposition 3.1.3. *The transfer $\tilde{\bullet}$ of the composition operation $\bullet$ from* (3.1.6) *along the cochain isomorphism* (3.1.17) *reads as follows: For all homogeneous $L \in \underline{\hom}^-_{\pm}\big(\mathfrak{g}^{\otimes l}, \mathfrak{g}^{\otimes m}\big)$ and $L' \in \underline{\hom}^-_{\pm}\big(\mathfrak{g}^{\otimes l'}, \mathfrak{g}^{\otimes m'}\big)$,*

$$
\begin{aligned}
(s^{(m-1)n+l-1}L)\tilde{\bullet}(s^{(m'-1)n+l'-1}L') \;=\; & (-1)^{|L|\,((m'-1)n+l'-1)}\,(-1)^{n(m'-1)(l-1)} \\
& \times\; s^{(m+m'-2)n+l+l'-2} \sum_{\substack{\sigma\in\mathrm{Sh}(l-1,l')\\ \rho\in\mathrm{Sh}(m,m'-1)}} (-1)^{|\sigma|}\,(-1)^{|\rho|\,n}
\end{aligned}
\quad , \tag{3.1.20}
$$



where the sums are over all shuffle permutations.

Remark 3.1.4. When evaluated on homogeneous elements $y_1, \dots, y_{l+l'-1} \in \mathfrak{g}$, the composed map (3.1.20) reads as follows: Using a Sweedler-like notation

$$
L'(y_1, \dots, y_{l'}) \;=\; L'_{(0)}(y_1, \dots, y_{l'}) \otimes L'_{(1)}(y_1, \dots, y_{l'}) \;\in\; \mathfrak{g} \otimes \mathfrak{g}^{m'-1} \quad , \tag{3.1.21}
$$

with summations understood, in order to split off the first tensor factor $\mathfrak{g}$, we have that

$$
\begin{aligned}
\big((s^{(m-1)n+l-1}L)\tilde{\bullet}(s^{(m'-1)n+l'-1}L')\big)(y_1, \dots, y_{l+l'-1}) \;=\; & (-1)^{|L|\,((m'-1)n+l'-1)}\,(-1)^{n(m'-1)(l-1)} \\
& \times\; s^{(m+m'-2)n+l+l'-2} \sum_{\substack{\sigma\in\mathrm{Sh}(l-1,l')\\ \rho\in\mathrm{Sh}(m,m'-1)}} (-1)^{|y_1\cdots y_{l+l'-1}|^{\sigma}_{-}}\,(-1)^{|\rho|\,n}\,(-1)^{\sum_{i=1}^{l-1}|y_{\sigma(i)}|\,|L'|} \\
& \times\; \gamma_\rho\Big(L\Big(y_{\sigma(1)}, \dots, y_{\sigma(l-1)}, L'_{(0)}(y_{\sigma(l)}, \dots, y_{\sigma(l+l'-1)})\Big) \otimes L'_{(1)}(y_{\sigma(l)}, \dots, y_{\sigma(l+l'-1)})\Big) \quad .
\end{aligned}
\tag{3.1.22}
$$

The last Koszul sign in the second line arises from permuting L' with $y_{\sigma(1)} \otimes \cdots \otimes y_{\sigma(l-1)}$. $\triangle$

Proof. The proof is a straightforward but lengthy calculation using the explicit descriptions of the cochain isomorphisms (3.1.15) and (3.1.16). We will not spell out this calculation in full detail, but only give some hints. Using these isomorphisms, homogeneous elements $s^{mn+l}L \in \underline{\hom}^-_{\pm}(\mathfrak{g}^{\otimes l}, \mathfrak{g}^{\otimes m})[-mn-l]$ and $s^{m'n+l'}L' \in \underline{\hom}^-_{\pm}(\mathfrak{g}^{\otimes l'}, \mathfrak{g}^{\otimes m'})[-m'n-l']$ can be equivalently presented as n-shifted polyvectors, which we expand in a choice of basis $\{x_a \in \mathfrak{g}\}$ and its dual basis $\{\theta^a \in \mathfrak{g}^*\}$ as

$$
P \;=\; P^{a_1\cdots a_m}_{b_1\cdots b_l}\; s^n x_{a_1} \cdots s^n x_{a_m}\, s\theta^{b_1} \cdots s\theta^{b_l} \in \mathrm{Pol}^{m,l}\big(\mathrm{Sym}\big(\mathfrak{g}^*[-1]\big), n\big) \quad , \tag{3.1.23a}
$$

$$
P' \;=\; P'^{c_1\cdots c_{m'}}_{d_1\cdots d_{l'}}\; s^n x_{c_1} \cdots s^n x_{c_{m'}}\, s\theta^{d_1} \cdots s\theta^{d_{l'}} \in \mathrm{Pol}^{m',l'}\big(\mathrm{Sym}\big(\mathfrak{g}^*[-1]\big), n\big) \quad , \tag{3.1.23b}
$$

where we use Einstein's sum convention to suppress summation symbols. Using the derivation properties (3.1.8) and definition on generators (3.1.7) of the composition operation $\bullet$, one finds

$$
\begin{aligned}
(s^{-n-1}P) \bullet (s^{n-1}P') \;=\; & l\,m'\,(-1)^{|\theta^{b_l}|}\;(-1)^{\sum_{k=2}^{m'}(|x_{c_k}|+n)\,\sum_{j=1}^{l-1}(|\theta^{b_j}|+1)}\; P^{a_1\cdots a_m}_{b_1\cdots b_l}\; P'^{c_1\cdots c_{m'}}_{d_1\cdots d_{l'}}\; \langle \theta^{b_l}, x_{c_1}\rangle \\
& \times\; s^{-n-1} s^n x_{a_1} \cdots s^n x_{a_m}\, s^n x_{c_2} \cdots s^n x_{c_{m'}}\, s\theta^{b_1} \cdots s\theta^{b_{l-1}}\, s\theta^{d_1} \cdots s\theta^{d_{l'}} \quad .
\end{aligned}
\tag{3.1.24}
$$

Transferring this result through the cochain isomorphisms (3.1.15) and (3.1.16) yields the element-wise formula (3.1.22), expressed in the chosen of basis, which is equivalent to the graphical expression (3.1.20). □

Combining this result with Corollary 3.1.2, we obtain a graphical characterisation of nSPSs on a free CDGA.

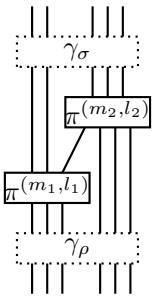
Corollary 3.1.5. *The datum of an nSPS on the free CDGA* $\mathrm{Sym}\big(\mathfrak{g}^*[-1]\big)$ *is equivalent to a family of maps*

$$\left\{ \boxed{\pi^{(m,l)}} \in \underline{\hom}^-_{\pm}\big(\mathfrak{g}^{\otimes l}, \mathfrak{g}^{\otimes m}\big)^{(1-m)n+2-l} \quad : \quad m \geq 2\ , \quad l \geq 0 \right\} \quad , \tag{3.1.25}$$

which, for every fixed $m \geq 2$*, is bounded in* l *and satisfies the following identities*

$$(-1)^{(m-1)n+l-1}\ \partial \boxed{\pi^{(m,l)}} + \sum_{m_1+m_2-1=m}\ \sum_{l_1+l_2-1=l} (-1)^{((1-m_1)n+2-l_1)\,((m_2-1)n+l_2-1)}$$

$$\times\ (-1)^{n(m_2-1)(l_1-1)} \sum_{\substack{\sigma\in \mathrm{Sh}(l_1-1,l_2)\\ \rho\in \mathrm{Sh}(m_1,m_2-1)}} (-1)^{|\sigma|}\,(-1)^{|\rho|\,n}$$



$$= \quad 0 \qquad , \tag{3.1.26}$$

for all $m \geq 2$ *and* $l \geq 0$.

Recall that Remark 2.3.35 expresses the identification between Maurer-Cartan derivations $\alpha \in \mathsf{T}^1_A$ and weight 1 Maurer-Cartan polyvectors $s^{-n-1}s^{n+1}\pi^{(1)} \in \widehat{\mathrm{Pol}^1}(A,n)[n+1]^1$. Using the deformed differential $\partial_{s^{n+1}\pi^{(1)}} = \partial + (-1)^{n+1}s^{n+1}\{s^{-n-1}s^{n+1}\pi^{(1)}, s^{-n-1}(\cdot)\} = \partial + [s^{n+1}\pi^{(1)}, \cdot]_{\mathrm{S}}$ of Lemma 2.3.34, the Maurer-Cartan equation for nSPSs $\pi = \sum_{m\geq 2}\pi^{(m)} \in \widehat{\mathrm{Pol}}(A_{\pi^{(1)}}, n)^{n+2}$ on $A_{\pi^{(1)}}$ from Definition 2.3.36 can then be rewritten as

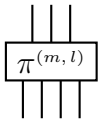
$$\partial_{s^{n+1}\pi^{(1)}}\pi + \tfrac{1}{2}[\pi,\pi]_{\mathrm{S}} = \partial(s^{n+1}\pi^{(1)} + \pi) + \tfrac{1}{2}[s^{n+1}\pi^{(1)} + \pi, s^{n+1}\pi^{(1)} + \pi]_{\mathrm{S}} = 0 \quad , \tag{3.1.27}$$

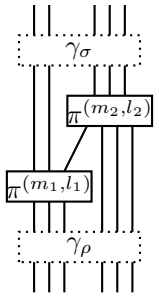
where the last step used the Maurer-Cartan equation for $s^{n+1}\pi^{(1)}$. In sum, an nSPS π on the deformed CDGA $A_{\pi^{(1)}}$ is equivalent to a Maurer-Cartan polyvector $s^{n+1}\pi^{(1)}+\pi$ in the undeformed n-shifted polyvectors $\widehat{\mathrm{Pol}}(A,n)$. Specializing to sfCDGAs, we obtain the following generalization of Corollary 3.1.5.

Corollary 3.1.6. *The combined datum of a sf deformation of the free CDGA* $\mathrm{Sym}\big(\mathfrak{g}^*[-1]\big)$ *and an nSPS on the associated sfCDGA is equivalent to a family of maps (including weight* $m = 1$ *components)*

$$\left\{ \boxed{\pi^{(m,l)}} \in \underline{\hom}^-_{\pm}\big(\mathfrak{g}^{\otimes l}, \mathfrak{g}^{\otimes m}\big)^{(1-m)n+2-l} \quad : \quad m \geq 1\ , \quad l \geq 0 \right\} \quad , \tag{3.1.28}$$

which, for every fixed $m \geq 1$, is bounded in l and satisfies the following identities

$$(-1)^{(m-1)n+l-1}\ \partial$$  $$+ \sum_{m_1+m_2-1=m}\ \sum_{l_1+l_2-1=l} (-1)^{((1-m_1)n+2-l_1)\,((m_2-1)n+l_2-1)}$$

$$\times\ (-1)^{n(m_2-1)(l_1-1)} \sum_{\substack{\sigma\in \mathrm{Sh}(l_1-1,l_2)\\ \rho\in\mathrm{Sh}(m_1,m_2-1)}} (-1)^{|\sigma|}\,(-1)^{|\rho|\,n}$$  $$= \quad 0 \qquad , \qquad (3.1.29)$$

for all $m \geq 1$ and $l \geq 0$.

Remark 3.1.7. We would like to observe that, in the case where $\mathfrak{g} \in \mathbf{Ch}$ is a non-positively graded N-term cochain complex as in (3.2.2) below, our combined datum from Corollary 3.1.6 of a sf deformation of the free CDGA $\mathrm{Sym}\big(\mathfrak{g}^*[-1]\big)$ and an nSPS on the associated sfCDGA is equivalent to that of an $L_\infty[0, n-1]$**-quasi-bialgebra** structure in the sense of [BSZ13, Definition 2.5]. In our notation, these authors introduce, for any non-positively graded vector space $\mathfrak{g}$ and any integers $c, d \in \mathbb{Z}$, the completed symmetric algebra

$$\prod_{l,m\geq 0} \mathrm{Sym}^m\big(\mathfrak{g}[-1-d]\big) \otimes \mathrm{Sym}^l\big(\mathfrak{g}^*[-1-c]\big) \tag{3.1.30}$$

and endow it with the so-called 'big bracket' $\langle\!\langle\, \cdot\,, \cdot\, \rangle\!\rangle$ which is given by extending the duality pairing $\langle\, \cdot\,, \cdot\, \rangle : \mathfrak{g}^* \otimes \mathfrak{g} \to \mathbb{K}$ to a biderivation. They then define an $L_\infty[c, d]$**-quasi-bialgebra** structure on $\mathfrak{g}$ in terms of a degree $3 + c + d$ element $t = \sum_{m\geq 1} \sum_{l\geq 0} t^{(m,l)}$ satisfying[1] $\langle\!\langle t, t \rangle\!\rangle = 0$. Comparing (3.1.30) with (3.1.4) and (3.1.5), we observe that, up to a completion in l which is inessential for N-term cochain complexes as in (3.2.2), the derived geometric concept of n-shifted polyvectors is given by choosing the integers $(c, d) = (0, n-1)$ in (3.1.30). Furthermore, from (3.1.11) we see that the Schouten–Nijenhuis bracket $\{\, \cdot\,, \cdot\, \}$ agrees with the 'big bracket' $\langle\!\langle\, \cdot\,, \cdot\, \rangle\!\rangle$ because both are given on the generators by the duality pairing. To relate the identity $\langle\!\langle t, t \rangle\!\rangle = 0$ with the Maurer-Cartan equation in Definition 2.3.36, note that [BSZ13, Definition 2.5] encodes the differential d on $\mathfrak{g}$ in terms of the $t^{(1,1)}$-component. Splitting off the differential, one can write $t = \mathrm{d} + t'$ and one observes that $\langle\!\langle t, t \rangle\!\rangle = 0$ is equivalent to the Maurer-Cartan equation for the remainder t'. △

3.2 Application to higher Chevalley-Eilenberg algebras

Let us first recall the definition of an L_∞-algebra in the sign conventions of, for instance, [KS22, Remark 3.6].

Definition 3.2.1. Given a graded vector space $\mathfrak{g}$ and a family of graded-linear maps

$$\Big\{\ell_i : \textstyle\bigwedge^i \mathfrak{g} \to \mathfrak{g}[2-i]\Big\}_{i\in\mathbb{N}^*} \qquad , \tag{3.2.1a}$$

we say $\mathfrak{g}$ is an L_∞**-algebra** if the **generalised Jacobi identity** is satisfied: given $x_1, \ldots, x_i \in \mathfrak{g}$,

[1] Let us also mention that if $t^{(m,0)} = 0$ then Bai, Sheng and Zhu call this an $L_\infty[c, d]$**-bialgebra** structure on $\mathfrak{g}$.

we have

$$\sum_{j+k-1=i} \sum_{\sigma\in \mathrm{Sh}(k,j-1)} (-1)^{|\sigma|}\chi_\sigma (-1)^{j-1}\ell_j\big(\ell_k(x_{\sigma(1)},\dots,x_{\sigma(k)}),x_{\sigma(k+1)},\dots,x_{\sigma(i)}\big)=0 \quad . \tag{3.2.1b}$$

where χ_σ is the Koszul sign coming from the permutation of *graded* vector space elements $x_1,\dots,x_i\in\mathfrak{g}$.

Notice that (3.2.1b) tells us that $\ell_1:\mathfrak{g}\to\mathfrak{g}[1]$ is a differential structure on $\mathfrak{g}$ and thus we usually denote $\mathrm{d}:=\ell_1$ and refer to $\mathfrak{g}$ as an underling cochain complex. This section focuses on the case where

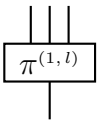
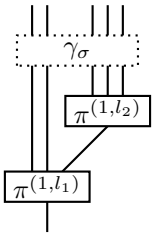
$$\mathfrak{g} := \Big(\, \mathfrak{g}^{-N+1} \xrightarrow{\mathrm{d}} \mathfrak{g}^{-N+2} \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \mathfrak{g}^{-1} \xrightarrow{\mathrm{d}} \mathfrak{g}^{0} \,\Big) \in \mathbf{Ch} \tag{3.2.2}$$

is an N-term cochain complex of fd vector spaces which is concentrated in non-positive degrees, where $N\in\mathbb{N}^*$ is a positive integer. Let us recall that a **Lie N-algebra** is a pair $(\mathfrak{g},\ell)$ consisting of an N-term cochain complex as in (3.2.2) and a family $\ell=\big\{\ell_l\in\underline{\hom}^-(\mathfrak{g}^{\otimes l},\mathfrak{g})^{2-l}\big\}_{l\geq 2}$ of totally antisymmetric linear maps of degree $2-l$ which satisfy the generalised Jacobi identities (3.2.1b). In other words, a Lie N-algebra is precisely an L_∞-algebra whose underlying cochain complex is of the N-term form (3.2.2). By a simple degree counting argument, one observes that $\ell_l=0$ must necessarily vanish for all $l>1+N$, hence the family of L_∞-brackets $\ell=\{\ell_l\}_{l\geq 2}$ is bounded for every Lie N-algebra.

It is well-known that L_∞-algebra structures on $\mathfrak{g}$ correspond to (a subclass of) the sf deformations of the free CDGA $\mathrm{Sym}\big(\mathfrak{g}^*[-1]\big)$, see, e.g., [KS22]. We recover this result as a special case of our graphical calculus from Corollary 3.1.6.

Corollary 3.2.2. *Let $\mathfrak{g}\in\mathbf{Ch}$ be an N-term cochain complex as in* (3.2.2) *and consider a family of maps $\{\pi^{(m,l)}\}_{m\geq 1,l\geq 0}$ as in Corollary 3.1.6 with $\pi^{(1,0)}=0$,[2] $\pi^{(1,1)}=0$ and $\pi^{(m,l)}=0$, for all $m\geq 2$ and $l\geq 0$. Then the identities* (3.1.29) *are equivalent to the generalised Jacobi identities* (3.2.1b).

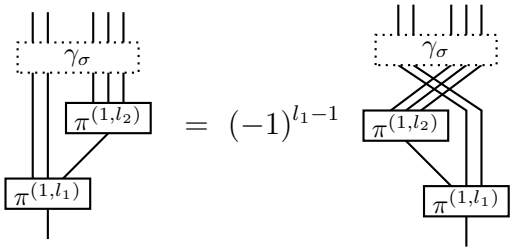
Proof. Let us first observe that the non-vanishing maps $\pi^{(1,l)}\in\underline{\hom}^-_{\pm}(\mathfrak{g}^{\otimes l},\mathfrak{g})^{2-l}$, for $l\geq 2$, have the correct degrees and antisymmetry properties required for an L_∞-algebra structure. Using that $\pi^{(m,l)}=0$, for all $m\geq 2$ and $l\geq 0$, the identities in (3.1.29) simplify to

$$(-1)^{l-1}\,\partial$$  $$+\sum_{l_1+l_2-1=l}(-1)^{(2-l_1)\,(l_2-1)}\sum_{\sigma\in\mathrm{Sh}(l_1-1,l_2)}(-1)^{|\sigma|}$$  $$= \quad 0 \quad , \tag{3.2.3}$$

for all $l\geq 2$. Using the antisymmetry properties (3.1.19), we can permute the inputs of $\pi^{(1,l_1)}$

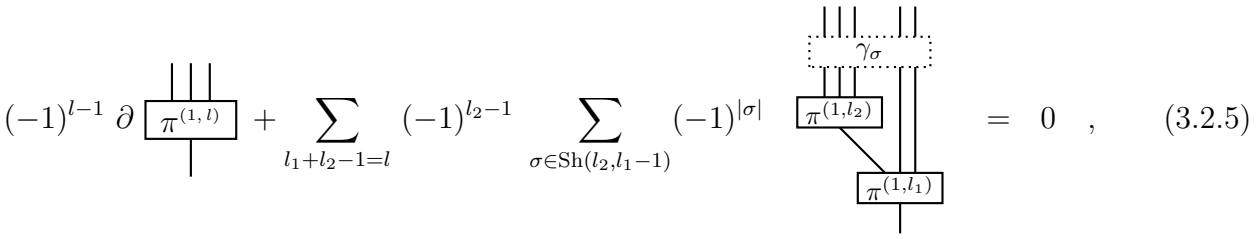
[2] Actually, this is automatic given that $\pi^{(1,0)}:\mathbb{K}\to\mathfrak{g}^2$.

and find



. (3.2.4)

Inserting this back into (3.2.3) and using also that the sum over shuffles yields an antisymmetric map, we obtain



(3.2.5)

for all $l \geq 2$, where the sum in this expression is over $(l_2, l_1{-}1)$-shuffles, in contrast to the $(l_1{-}1, l_2)$-shuffles in (3.2.3). Under the identification $\ell_l := (-1)^{l-1}\,\pi^{(1,l)}$, for all $l \geq 2$, this gives the L_∞-algebra structure identities in Definition 3.2.1. □

Remark 3.2.3. Considering cochain complexes with $\mathfrak{g}^2 \neq 0$ and dropping in Corollary 3.2.2 the requirement that $\pi^{(1,0)} = 0$ and $\pi^{(1,1)} = 0$ leads to more general sf deformations of the CDGA $\mathrm{Sym}\big(\mathfrak{g}^*[-1]\big)$. The component $\pi^{(1,1)}$ induces a deformation of the given differential d of $\mathfrak{g} \in \mathbf{Ch}$ and the component $\pi^{(1,0)}$ introduces a violation of the square-zero condition of the differential, which is called curvature and leads to the concept of **curved L_∞-algebras**. We do not consider nSPSs on these more general deformations in this chapter and restrict our attention to L_∞-algebras. That being said, we will briefly talk about the Chevalley-Eilenberg differential coming from a curved L_∞-algebra in Construction 3.2.4. △

Given any (bounded) L_∞-algebra structure $\ell = \{\ell_l\}_{l\geq 2}$ on the cochain complex $\mathfrak{g} \in \mathbf{Ch}$, the associated sf deformation

$$\mathrm{CE}_{\mathfrak{g}} \;:=\; \big(\mathrm{Sym}\big(\mathfrak{g}^*[-1]\big)\big)_\ell \tag{3.2.6}$$

of the free CDGA $\mathrm{Sym}\big(\mathfrak{g}^*[-1]\big)$ yields the **Chevalley-Eilenberg algebra** of the L_∞-algebra $\mathfrak{g} = (\mathfrak{g}, \ell)$.

Construction 3.2.4. We can explicate the one-to-one correspondence between fd curved L_∞-algebras $(\mathfrak{g}, \ell)$ and fgsfCDGAs

$$\mathrm{CE}_{\mathfrak{g}} := \big(\mathrm{Sym}\big(\mathfrak{g}^*[-1]\big), \delta\big)$$

by *constructing* the differential δ on generators $s\theta^i$ induced by ℓ. Explicitly, with chosen basis $\{x_i \in \mathfrak{g}\}_{i\in I}$, if

$$\pi^{(1,l)} := (-1)^{l-1}\,\ell_l : \textstyle\bigwedge^l \mathfrak{g} \to \mathfrak{g}[2-l]$$

for $l \in \mathbb{N}$, then we denote

$$\ell_l(x_{i_1}, \ldots, x_{i_l}) = \sum_{|x_j| = |x_{i_1}\cdots x_{i_l}|+2-l} \ell^j_{i_1,\ldots,i_l} x_j \quad .$$

Recall that the Maurer-Cartan derivation $\pi^{(1)} \in \mathsf{T}^1_{\mathrm{Sym}(\mathfrak{g}^*[-1])}$ deforms the zero differential of

$\mathrm{Sym}(\mathfrak{g}^*[-1])$ as in (2.3.83). Explicitly, this works out to

$$\delta(s\theta^m) = -\sum_{l\in\mathbb{N}} \sum_{|s\theta^{i_1}\cdots s\theta^{i_l}|=|s\theta^m|+1} \frac{1}{l!}(-1)^{|x_{i_1}\cdots x_{i_l}|_{\mathrm{rev}}}(-1)^{\sum_{r=1}^{l}|x_{i_r}|r}\ell^m_{i_1,\ldots,i_l}s\theta^{i_1}\cdots s\theta^{i_l}\,, \tag{3.2.7}$$

and it can be shown straightforwardly that the square-zero condition of the above differential follows from the generalised Jacobi identities that ℓ satisfies (equivalently, the Maurer-Cartan equation that $\pi^{(1)}$ satisfies). ▷

Remark 3.2.5. The previous construction can be specialised to the case of demonstrating the one-to-one correspondence between fd L_∞-algebras (simply restrict to $l \in \mathbb{N}^*$) and fgsfCDGAs which are **augmented over** $\mathbb{K}$, i.e. for which the canonical projection $A \to \mathbb{K}$ is a homomorphism of CDGAs. Finally, it can be restricted even further to the case of the one-to-one correspondence between $\mathbb{Z}^{\leq 0}$-graded fd L_∞-algebras and positively fgsfCDGAs. △

One can interpret the sfCDGA $\mathrm{CE}_\mathfrak{g}$ as an algebraic model for the classifying stack $B\mathfrak{g} = [*/\mathfrak{g}]$ of the L_∞-algebra, see, e.g., [Pri17, Example 3.6] and [Pri18, Examples 1.10]. Corollary 3.1.6 then provides a characterisation of the nSPSs on $\mathrm{CE}_\mathfrak{g}$ by fixing the weight $m = 1$ components of the family of maps $\{\pi^{(m,l)}\}_{m\geq 1, l\geq 0}$ in terms of the L_∞-algebra structure according to $\pi^{(1,0)} = 0$, $\pi^{(1,1)} = 0$ and $\pi^{(1,l)} = (-1)^{l-1}\,\ell_l$, for all $l \geq 2$. The three following Lemmas allow us to systematically eliminate those $\pi^{(m,l)}$'s which are necessarily trivial based purely on degree counting arguments. The aim of the following two subsections is then to work out these characterisations explicitly in the simplest two cases where $\mathfrak{g}$ is an ordinary Lie algebra and where $\mathfrak{g}$ is a Lie 2-algebra.

Lemma 3.2.6. *Let $\mathfrak{g} = (\mathfrak{g}, \ell)$ be any Lie N-algebra. Then every n-shifted Poisson structure on $\mathrm{CE}_\mathfrak{g}$ is necessarily trivial for $n > 2N$.*

Proof. We argue by degree counting that every component $\pi^{(m,l)} \in \underline{\hom}^-_{\pm}\big(\mathfrak{g}^{\otimes l}, \mathfrak{g}^{\otimes m}\big)^{(1-m)n+2-l}$, for $m \geq 2$ and $l \geq 0$, of an nSPS π on $\mathrm{CE}_\mathfrak{g}$ is necessarily trivial in the case where $n > 2N$. Since $\mathfrak{g}$ is, by hypothesis (3.2.2), concentrated in degrees $\{-N+1, \ldots, 0\}$, it follows that the internal hom complex $\underline{\hom}\big(\mathfrak{g}^{\otimes l}, \mathfrak{g}^{\otimes m}\big) \cong \mathfrak{g}^{\otimes m} \otimes \mathfrak{g}^{*\otimes l}$ is concentrated in degrees $\{m(-N+1), \ldots, l(N-1)\}$. Hence, for the existence of a non-trivial element $\pi^{(m,l)}$ of degree $(1-m)n+2-l$, the bounds

$$m(-N+1) \;\leq\; (1-m)n+2-l \;\leq\; l(N-1) \tag{3.2.8}$$

must be satisfied. Since $m \geq 2$, we have that $1-m < 0$ and hence the hypothesis $n > 2N$ of this lemma yields the inequality

$$(1-m)n+2-l \;<\; (1-m)2N+2-l \;\leq\; (1-m)2N+2 \quad , \tag{3.2.9}$$

where in the last step we also used that $l \geq 0$. Combining this inequality with the first inequality in (3.2.8) we find

$$m\,(-N+1) \;<\; (1-m)2N+2 \quad\Longleftrightarrow\quad m(N+1) \;<\; 2(N+1) \quad\Longleftrightarrow\quad m \;<\; 2 \quad . \tag{3.2.10}$$

This is inconsistent with the fact that every SPS has only components with $m \geq 2$, which completes our proof. □

Now that we have specified an upper bound on the shift n in terms of the **height** N as in Lemma 3.2.6, let us specify two bounds[3] on the weight m in terms of both the shift n and the height N.

[3] A lower bound and then an upper bound for the cases of $n \leq N-1$ and $N \leq n \leq 2N$, respectively.

Lemma 3.2.7. *In the context of Lemma 3.2.6: for $n \leq N-1$, an nSPS is such that none of its weights $m \geq 2$ are necessarily trivial. If $N \leq n \leq 2N$, an nSPS is such that its weights are necessarily trivial for*

$$m \leq \frac{n+2}{n-N+1} \quad . \tag{3.2.11}$$

Proof. A nontrivial nSPS on $\mathrm{CE}_{\mathfrak{g}}$ must have a nontrivial weight $m \geq 2$ component. Such a weight component can have an input component of *at most*

$$(1-m)n + m(N-1) + 2 \quad .$$

Schematically, we spend the cohomological degree $n+2$ on the lowest degree generators of both $\mathrm{CE}_{\mathfrak{g}}$ and $\mathsf{T}_{\mathrm{CE}_{\mathfrak{g}}}[-n-1]$ as follows,

$$\pi^{(m)} \sim \ (s\theta_0)^{(1-m)n+m(N-1)+2}(s^n x^{-N+1})^m \quad , \tag{3.2.12}$$

thus

$$(1-m)n + m(N-1) + 2 \geq 0 \quad . \tag{3.2.13}$$

For $n < N-1$, we can rearrange the inequality (3.2.13) to get the lower bound

$$m \geq \frac{-n-2}{N-1-n} \quad . \tag{3.2.14}$$

In other words, for $n < N-1$, only those weight components m satisfying (3.2.14) will be not necessarily trivial. For $n < N-1$, we already have $\frac{-n-2}{N-1-n} < 1$ hence, for $n < N-1$, (3.2.14) tells us that all weights $m \geq 2$ will be not necessarily trivial. If $n = N-1$ then (3.2.13) reads as $(1-m)n + m(N-1) + 2 = N+1 > 0$ hence the case $n = N-1$ is also such that all weights $m \geq 2$ will be not necessarily trivial.

For $N \leq n \leq 2N$, we can rearrange the inequality (3.2.13) to get the upper bound (3.2.11). In fact, we can limit this upper bound in terms of the height N alone; notice that

$$N \leq n \leq 2N \quad \Longrightarrow \quad N+2 \leq n+2 \leq 2N+2 \quad , \quad 1 \leq n-N+1 \leq N+1 \quad ,$$

which in turn gives us

$$2 \leq \frac{n+2}{n-N+1} \leq N+2 \quad .$$

□

Finally, we wish to specify a constraint on the triviality of a 0-input component as a lower bound on the shift n in terms of the weight m.

Lemma 3.2.8. *In the context of Lemma 3.2.7, a 0-input component of a weight m component of an nSPS is not necessarily trivial if*

$$n \geq \frac{2}{m-1} \quad . \tag{3.2.15}$$

Proof. Looking back to (3.2.12), we can reach the 0-input component if we can exchange *all* $s\theta_0$'s in promoting the $s^n x^{-N+1}$'s up to, at most, an $s^n x^0$ and for this we clearly need

$$(1-m)n + m(N-1) + 2 \leq m(N-1) \quad . \tag{3.2.16}$$

We rearrange (3.2.16) to get the desired lower bound (3.2.15). □

Remark 3.2.9. Putting Lemmas 3.2.6, 3.2.7 and 3.2.8 together:

1. The 1SPS has the not necessarily vanishing components

$$\pi_{n=1} = \bigoplus_{l=1}^{2N-1} \Big(\pi^{(2,l)} : \wedge^l\mathfrak{g} \to \wedge^2\mathfrak{g}[1-l]\Big) \oplus \prod_{m=3}^{\infty} \bigoplus_{l=0}^{3+m(N-2)} \Big(\pi^{(m,l)} : \wedge^l\mathfrak{g} \to \wedge^m\mathfrak{g}[3-m-l]\Big) \,. \tag{3.2.17}$$

The upper limit of the second sum over inputs in (3.2.17) must be $3+m(N-2) \geq 0$ hence, if $N = 1$, then (3.2.17) actually reads

$$\pi_{n=1} = \big(\pi^{(2,1)} : \mathfrak{g} \to \wedge^2\mathfrak{g}\big) \oplus \big(\pi^{(3,0)} : \mathbb{K} \to \wedge^3\mathfrak{g}\big) \quad .$$

2. For $2 \leq n \leq N-1$, we have

$$\pi = \prod_{m=2}^{\infty} \bigoplus_{l=0}^{(1-m)n+m(N-1)+2} \Big(\pi^{(m,l)} : \wedge^l\mathfrak{g} \to \mathrm{Sym}^m_{\pm}\mathfrak{g}[(1-m)n+2-l]\Big) \quad . \tag{3.2.18}$$

Note that these nSPSs are all necessarily trivial for $N \leq 2$.

3. For $N \leq n \leq 2N$, we have

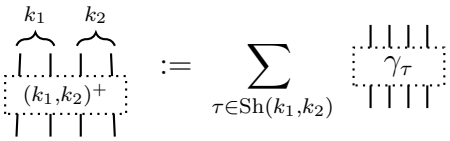
$$\pi = \bigoplus_{m=2}^{\lfloor \frac{n+2}{n-N+1} \rfloor} \bigoplus_{l=0}^{(1-m)n+m(N-1)+2} \Big(\pi^{(m,l)} : \wedge^l\mathfrak{g} \to \mathrm{Sym}^m_{\pm}\mathfrak{g}[(1-m)n+2-l]\Big) \quad . \tag{3.2.19}$$

If $N = 1$ then item 1 already spells out the 1SPS and (3.2.15) tells us to ignore the lower bound of $1 \leq n \leq 2$ hence (3.2.19) only spells out the 2SPS. In this case, the 2SPS has its not necessarily trivial components given by

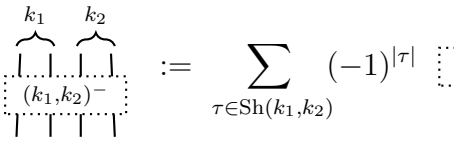
$$\pi_{n=2} = \pi^{(2,0)} : \mathbb{K} \to \mathrm{Sym}^2\mathfrak{g} \quad .$$

△

Convention 3.2.10. For better readability of the graphical identities in the following subsections, we will adopt the compact notation



, (3.2.20a)



, (3.2.20b)

which suppresses the summation over (k_1, k_2)-shuffle permutations and their signs. Note that in the identities for nSPSs in Corollary 3.1.6, the shuffles attached to the inputs of maps $\underline{\hom}^-_{\pm}\big(\mathfrak{g}^{\otimes l}, \mathfrak{g}^{\otimes m}\big)$ always carry odd signs as in (3.2.20b) while the shuffles attached to the outputs carry either odd signs (3.2.20b) or even signs (3.2.20a) depending on the parity of n. ▷

3.2.1 Ordinary Lie algebras

Suppose that the cochain complex $\mathfrak{g}$ is concentrated in degree 0, which necessarily forces the differential $\mathrm{d} = 0$ to be trivial. It follows that the internal homs $\underline{\hom}(\mathfrak{g}^{\otimes m}, \mathfrak{g}^{\otimes l})$ are concentrated

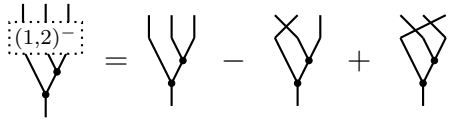
in degree 0 too and carry the trivial differential $\partial = 0$. This implies that the weight 1 components $\pi^{(1,l)} \in \underline{\hom}^-_\pm(\mathfrak{g}^{\otimes l}, \mathfrak{g})^{2-l}$ of the family of maps in Corollary 3.1.6 are necessarily trivial for $l \neq 2$. We shall visualize the only non-trivial component by

$$\pi^{(1,2)} \;=\; \in\; \underline{\hom}^-_\pm\big(\mathfrak{g}^{\otimes 2}, \mathfrak{g}\big)^0 \quad . \tag{3.2.21}$$



As a consequence of the identities (3.1.29) for $m = 1$ (see also Corollary 3.2.2), it follows that this defines a Lie algebra structure on $\mathfrak{g}$, i.e. the Jacobi identity

$$0 \;=\; \tag{3.2.22}$$



holds true. (The usual form of the Jacobi identity with cyclic permutations, in contrast to shuffle permutations, is obtained by using antisymmetry (3.1.19) at the top vertex of the second term.)

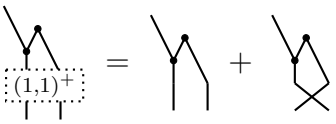
Our aim is to characterize the nSPSs on the associated Chevalley-Eilenberg algebra $\mathrm{CE}_\mathfrak{g}$ of this Lie algebra, for $n \in \{1, 2\}$ by using Remark 3.2.9. In particular, we recover from Corollary 3.1.6 the results of Safronov [Saf21, Proposition 2.6 and Theorem 2.8], which we shall restate in our graphical calculus in the following two propositions.

Proposition 3.2.11. *A 2SPS on the Chevalley-Eilenberg algebra* $\mathrm{CE}_\mathfrak{g}$ *of an ordinary Lie algebra* $\mathfrak{g}$ *with bracket* (3.2.21) *is given by the datum of a degree* 0 *map*

$$\pi^{(2,0)} \;=\; \in\; \underline{\hom}^-_+\big(\mathbb{K}, \mathfrak{g}^{\otimes 2}\big)^0 \quad , \tag{3.2.23}$$

which satisfies the identity

$$0 \;=\; \;=\; \;+\; \;=\; 0 \quad . \tag{3.2.24}$$



This datum is equivalent to an adjoint action invariant symmetric tensor in $\mathfrak{g}^{\otimes 2}$*, i.e., an element in* $(\mathrm{Sym}^2\,\mathfrak{g})^\mathfrak{g}$.

Proof. This result follows directly from (3.2.19) and Corollary 3.1.6. The adjoint action invariance condition follows then from the $(m, l) = (2, 1)$ component of (3.1.29). □

Example 3.2.12. Given a semisimple Lie algebra $\mathfrak{g}$, the space of invariant bilinear forms on $\mathfrak{g}$ has the same dimension as the number of simple components. In particular, a simple Lie algebra $\mathfrak{g}$ has a line of invariant bilinear forms and this is given by scalar multiples of the Killing form. Dually, a simple Lie algebra has a unique (up to scaling prefactor) symmetric ad-invariant 2-tensor and this is given by the quadratic Casimir. The classification of 2SPSs on the Chevalley-Eilenberg algebra of a semisimple Lie algebra is thus given by the space of quadratic Casimirs and such a space has the same dimension as the number of simple components. ▽

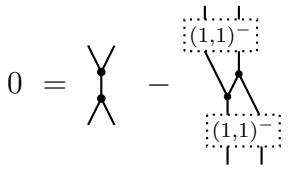
Proposition 3.2.13. *A 1SPS on the Chevalley-Eilenberg algebra* $\mathrm{CE}_\mathfrak{g}$ *of an ordinary Lie algebra* $\mathfrak{g}$ *with bracket* (3.2.21) *is given by the datum of two degree* 0 *maps*

$$\pi^{(2,1)} \;=\; \in\; \underline{\hom}^-_-\big(\mathfrak{g}, \mathfrak{g}^{\otimes 2}\big)^0 \quad , \qquad \pi^{(3,0)} \;=\; \in\; \underline{\hom}^-_-\big(\mathbb{K}, \mathfrak{g}^{\otimes 3}\big)^0 \quad , \tag{3.2.25}$$

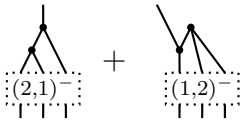




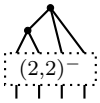
which satisfy the identities



(3.2.26a)



(3.2.26b)



(3.2.26c)

These data are equivalent to a quasi-Lie bialgebra structure on the Lie algebra $\mathfrak{g}$*, see e.g. [ES02, Section 16.2].*

Proof. The proof is a direct consequence of (3.2.17) and Corollary 3.1.6. □

3.2.2 Lie 2-algebras

Consider a 2-term cochain complex

$$\mathfrak{g} := \left(\mathfrak{g}^{-1} \xrightarrow{\mathrm{d}} \mathfrak{g}^{0} \right) \in \mathbf{Ch} \quad . \tag{3.2.27}$$

Applying Corollary 3.2.2 and a simple degree counting argument, one observes that an L_∞-algebra structure on $\mathfrak{g}$ consists of two maps

$$\pi^{(1,2)} = \; \in \; \underline{\hom}^-_{\pm}\big(\mathfrak{g}^{\otimes 2}, \mathfrak{g}\big)^0 \quad , \qquad \pi^{(1,3)} = \; \in \; \underline{\hom}^-_{\pm}\big(\mathfrak{g}^{\otimes 3}, \mathfrak{g}\big)^{-1} \tag{3.2.28}$$

of degree 0 and, respectively, degree -1. The L_∞-algebra structure identities in (3.1.29) (see also Corollary 3.2.2) specialize in the present case to the identities

$$\partial \; = 0 \quad , \tag{3.2.29a}$$

$$-\partial \; = \quad , \tag{3.2.29b}$$

$$0 = - \quad + \quad . \tag{3.2.29c}$$

The first identity expresses that the binary bracket is a cochain map and the second identity states that the ternary bracket is a homotopy witnessing the Jacobi identity. The third identity is an algebraic relation between the binary and ternary brackets. We would like to note that the quadratic identity for the ternary bracket

$$0 = \quad , \tag{3.2.30}$$

which arises from setting $(m,l)=(1,5)$ in (3.1.29), holds true automatically. This is due to the fact that the only non-trivial component of a degree -1 map $\pi^{(1,3)}$ on a 2-term complex (3.2.27) is given by $\pi^{(1,3)}:\mathfrak{g}^0\otimes\mathfrak{g}^0\otimes\mathfrak{g}^0\to\mathfrak{g}^{-1}$, hence this map composes trivially with itself.

Definition 3.2.14. A **Lie 2-algebra** is a quadruple $\mathfrak{g}=(\mathfrak{g},\mathrm{d},[\cdot,\cdot],[\cdot,\cdot,\cdot])$ that consists of: a graded vector space $\mathfrak{g}=\mathfrak{g}^{-1}\oplus\mathfrak{g}^0$ concentrated in degrees $\{-1,0\}$, a graded-linear map $\mathrm{d}:\mathfrak{g}\to\mathfrak{g}[1]$ (which automatically squares to zero because of the length of the complex), a cochain map $[\cdot,\cdot]:\bigwedge^2\mathfrak{g}\to\mathfrak{g}$ which satisfies (3.2.29b), i.e. the Jacobi identity up to adjoint differential of the Jacobiator $[\cdot,\cdot,\cdot]:\bigwedge^3\mathfrak{g}\to\mathfrak{g}[-1]$, and (3.2.29c).

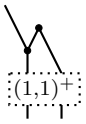
Our aim is to characterize the nSPSs on the associated Chevalley-Eilenberg algebra $\mathrm{CE}_{\mathfrak{g}}$ of this Lie 2-algebra for $n\in\{1,2,3,4\}$ using Remark 3.2.9 and Corollary 3.1.6.

Proposition 3.2.15. *A 4SPS on the Chevalley-Eilenberg algebra* $\mathrm{CE}_{\mathfrak{g}}$ *of a Lie 2-algebra* $\mathfrak{g}$ *with brackets* (3.2.28) *is given by the datum of a degree* -2 *map*

$$\pi^{(2,0)} \;=\; \wedge \;\in\; \underline{\hom}^-_+\big(\mathbb{K},\mathfrak{g}^{\otimes 2}\big)^{-2} \quad , \tag{3.2.31}$$

which satisfies the identities

$$\partial \wedge \;=\; 0 \quad , \qquad 0 \;=\; \tag{3.2.32}$$



Proof. Simply substitute $N=2$ and $n=4$ into (3.2.19). The identities (3.2.32) follow directly from the $(m,l)=(2,0)$ and $(m,l)=(2,1)$ components of (3.1.29). □

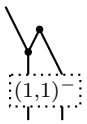
Remark 3.2.16. Note that the first condition of (3.2.32) tells us that the 4SPS is trivial if the differential $\mathrm{d}:\mathfrak{g}^{-1}\to\mathfrak{g}^0$ is injective. △

Proposition 3.2.17. *A 3SPS on the Chevalley-Eilenberg algebra* $\mathrm{CE}_{\mathfrak{g}}$ *of a Lie 2-algebra* $\mathfrak{g}$ *with brackets* (3.2.28) *is given by the datum of two maps*

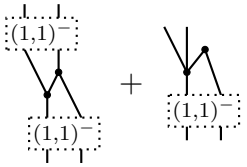
$$\pi^{(2,0)} \;=\; \wedge \;\in\; \underline{\hom}^-_-\big(\mathbb{K},\mathfrak{g}^{\otimes 2}\big)^{-1} \quad , \qquad \pi^{(2,1)} \;=\; \;\in\; \underline{\hom}^-_-\big(\mathfrak{g},\mathfrak{g}^{\otimes 2}\big)^{-2} \tag{3.2.33}$$

of degree -1 *and, respectively, degree* -2*, which satisfy the identities*

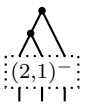
$$\partial \wedge \;=\; 0 \quad , \qquad -\partial \;=\; \tag{3.2.34a}$$



$$0 \;=\; \;-\; \;+\; \tag{3.2.34b}$$

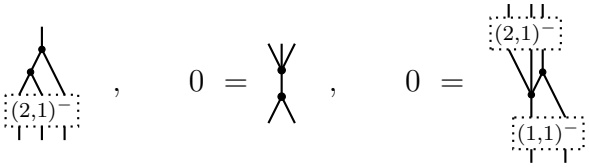


$$0 \;=\; \tag{3.2.34c}$$



Proof. Observing that $\pi^{(1,3)}:\mathfrak{g}^0\otimes\mathfrak{g}^0\otimes\mathfrak{g}^0\to\mathfrak{g}^{-1}$ and $\pi^{(2,1)}:\mathfrak{g}^0\to\mathfrak{g}^{-1}\otimes\mathfrak{g}^{-1}$, one already has:

$$0 \;=\; \quad , \qquad 0 \;=\; \quad , \qquad 0 \;=\; \tag{3.2.35}$$



The identities (3.2.34) follow directly from the $(m,l) = (2,0)$, $(2,1)$, $(2,2)$ and $(3,0)$ components of (3.1.29). □

Proposition 3.2.18. *A 2SPS on the Chevalley-Eilenberg algebra* $\mathrm{CE}_{\mathfrak{g}}$ *of a Lie 2-algebra* $\mathfrak{g}$ *with brackets* (3.2.28) *is given by the datum of one degree* 0 *map*

$$\pi^{(2,0)} \;=\; \in\; \underline{\hom}_+^-\big(\mathbb{K},\mathfrak{g}^{\otimes 2}\big)^0 \quad , \tag{3.2.36a}$$

one degree -1 *map*

$$\pi^{(2,1)} \;=\; \in\; \underline{\hom}_+^-\big(\mathfrak{g},\mathfrak{g}^{\otimes 2}\big)^{-1} \quad , \tag{3.2.36b}$$

two degree -2 *maps*

$$\pi^{(2,2)} \;=\; \in\; \underline{\hom}_+^-\big(\mathfrak{g}^{\otimes 2},\mathfrak{g}^{\otimes 2}\big)^{-2} \quad , \qquad \pi^{(3,0)} \;=\; \in\; \underline{\hom}_+^-\big(\mathbb{K},\mathfrak{g}^{\otimes 3}\big)^{-2} \quad , \tag{3.2.36c}$$

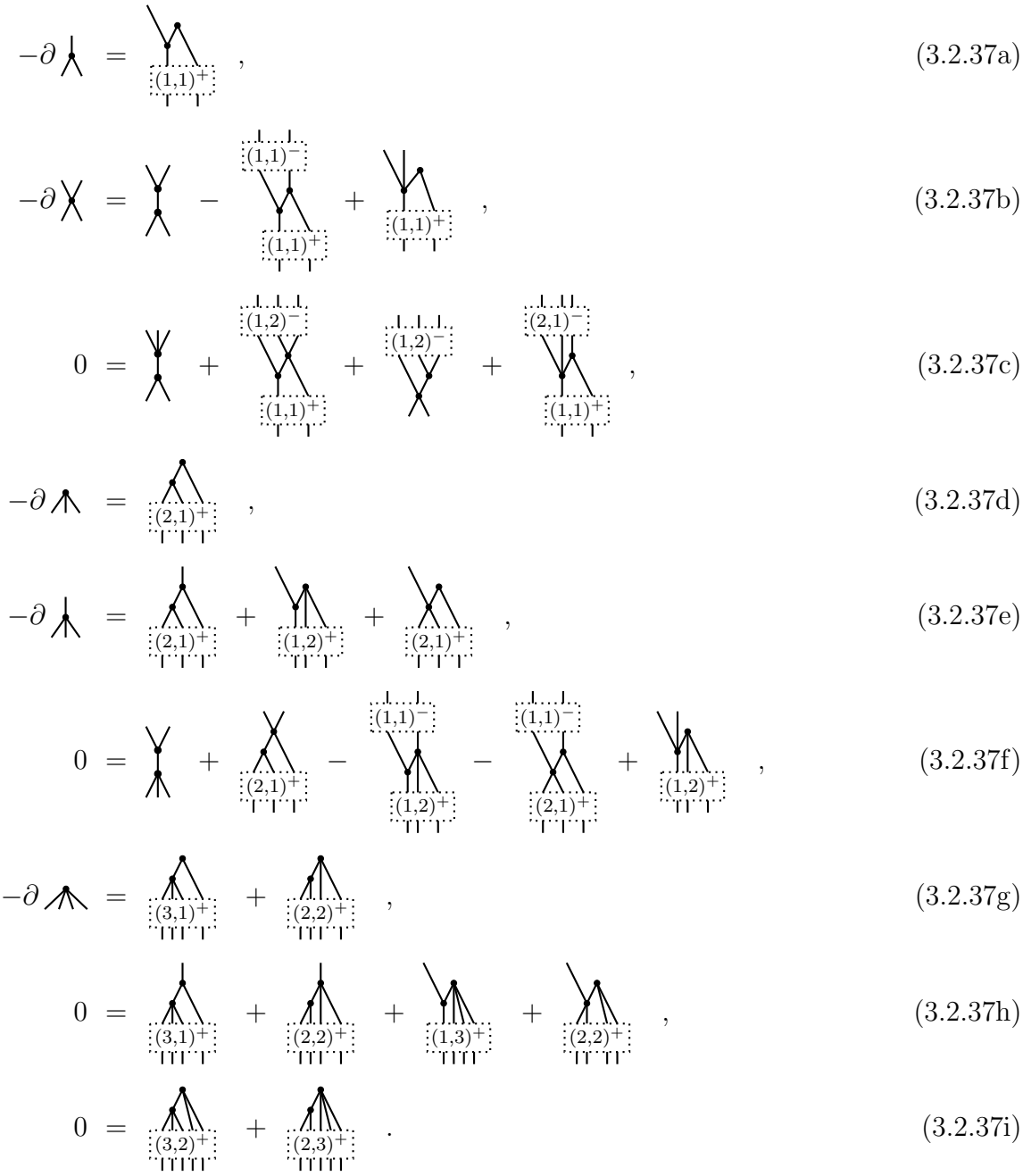
one degree -3 *map*

$$\pi^{(3,1)} \;=\; \in\; \underline{\hom}_+^-\big(\mathfrak{g},\mathfrak{g}^{\otimes 3}\big)^{-3} \quad , \tag{3.2.36d}$$

and one degree -4 *map*

$$\pi^{(4,0)} \;=\; \in\; \underline{\hom}_+^-\big(\mathbb{K},\mathfrak{g}^{\otimes 4}\big)^{-4} \quad . \tag{3.2.36e}$$

These data have to satisfy the following nine identities:



(3.2.37a)

(3.2.37b)

(3.2.37c)

(3.2.37d)

(3.2.37e)

(3.2.37f)

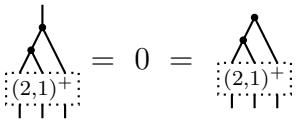
(3.2.37g)

(3.2.37h)

(3.2.37i)

Proof. In order to reduce the family of identities (3.1.29) to (3.2.37), one uses that various terms vanish as a consequence of the specific form of the maps $\pi^{(1,3)} : \mathfrak{g}^0 \otimes \mathfrak{g}^0 \otimes \mathfrak{g}^0 \to \mathfrak{g}^{-1}$, $\pi^{(2,0)} : \mathbb{K} \to \mathfrak{g}^0 \otimes \mathfrak{g}^0$, $\pi^{(2,2)} : \mathfrak{g}^0 \otimes \mathfrak{g}^0 \to \mathfrak{g}^{-1} \otimes \mathfrak{g}^{-1}$, $\pi^{(3,1)} : \mathfrak{g}^0 \to \mathfrak{g}^{-1} \otimes \mathfrak{g}^{-1} \otimes \mathfrak{g}^{-1}$ and $\pi^{(4,0)} : \mathbb{K} \to \mathfrak{g}^{-1} \otimes \mathfrak{g}^{-1} \otimes \mathfrak{g}^{-1} \otimes \mathfrak{g}^{-1}$. □

Example 3.2.19. Consider a 2-term DGLA $\mathfrak{g} = \left(\mathfrak{g}^{-1} \xrightarrow{\mathrm{d}} \mathfrak{g}^0, [\cdot,\cdot]\right)$ and a 2SPS on the Chevalley-Eilenberg algebra $\mathrm{CE}_\mathfrak{g}$, as in Proposition 3.2.18, such that only $\pi^{(2,0)}$ and $\pi^{(2,1)}$ are nontrivial (and $\pi^{(2,1)}$ annihilates $\mathfrak{g}^{-1}$). Such a 2SPS is a special case of CM's notion of a "symmetric quasi-invariant tensor in a differential crossed module" from [CFM15, Definition 30] whereby we assume a trivial $\mathfrak{g}^0$-invariant cocycle $c \in \ker(\mathrm{d}) \subseteq \mathfrak{g}^{-1}$ and we append the further two conditions:



(3.2.38)

▽

Remark 3.2.20. In contrast to the finite characterisations of $n = 2, 3, 4$-shifted Poisson structures given above, a 1SPS on the Chevalley-Eilenberg algebra $\mathrm{CE}_{\mathfrak{g}}$ of a Lie 2-algebra $\mathfrak{g}$ with brackets (3.2.28) involves in general an infinite number of data, i.e. (3.2.17) with $N = 2$ and $n = 1$. We consider the maps

$$\pi^{(2,l)} = \quad \in \ \underline{\hom}^-_-\big(\mathfrak{g}^{\otimes l}, \mathfrak{g}^{\otimes 2}\big)^{1-l} \quad , \quad \text{for all } l \in \{1,2,3\} \quad , \tag{3.2.39a}$$



and

$$\pi^{(m,l)} = \quad \in \ \underline{\hom}^-_-\big(\mathfrak{g}^{\otimes l}, \mathfrak{g}^{\otimes m}\big)^{3-m-l} \quad , \quad \text{for all } m \geq 3 \text{ and } l \in \{0,1,2,3\} \quad . \tag{3.2.39b}$$



These data must satisfy the infinite tower of identities which is obtained by specializing (3.1.29) to the present case. △

Remark 3.2.21. Given an ordinary Lie algebra $\mathfrak{g}$; Proposition 3.2.13 showed that a 1SPS on $\mathrm{CE}_{\mathfrak{g}}$ is equivalent to a quasi-Lie bialgebra structure on $\mathfrak{g}$. Continuing on from Remark 3.1.7, a **Lie 2-(quasi-)bialgebra** is defined by Bai, Sheng and Zhu [BSZ13, Definition 2.5] as a $L_\infty[0,1]$-(quasi-)bialgebra structure on $\mathfrak{g} = \mathfrak{g}^{-1} \oplus \mathfrak{g}^0$. In other words, it is *actually* the 2SPS of Proposition 3.2.18 that is the Lie 2-quasi-bialgebra structure and *not* the 1SPS of Remark 3.2.20 (which, instead, is a 2-term $L_\infty[0,0]$-quasi-bialgebra). △

3.3 Explicit examples for Lie 2-algebras

In this section we apply the general characterisation results for SPSs from Subsection 3.2.2 to specific classes of examples of Lie 2-algebras.

3.3.1 Abelian Lie 2-algebras

Let us recall that, for the ordinary Abelian Lie algebra $\mathbb{K}$, there exists a 1-parameter family of 2SPSs on $\mathrm{CE}_{\mathbb{K}}$ which is given by $\pi^{(2,0)} \in \big(\mathrm{Sym}^2 \mathbb{K}\big)^0 \cong \mathbb{K}$, see also Proposition 3.2.11. In stark contrast to this, the Abelian Lie 2-algebra $\mathbb{K}[1] = (\mathbb{K} \xrightarrow{0} 0)$, which arises for example in the theory of Abelian gerbes [Bry08], does not admit any non-trivial nSPSs for all $n \geq 1$. This claim can be verified from our characterisation results in Propositions 3.2.15, 3.2.17, 3.2.18 and Remark 3.2.20, together with the observation that, for every $m \geq 2$, we have $\underline{\hom}^-_+(\mathbb{K}[1]^{\otimes l}, \mathbb{K}[1]^{\otimes m}) \cong 0$ as a consequence of $\mathrm{Sym}^m\big(\mathbb{K}[1]\big) \cong \big(\bigwedge^m \mathbb{K}\big)[m] \cong 0$ for $m \geq 2$.

A Lie 2-algebraic analogue of the 1-parameter family $\pi^{(2,0)} \in \big(\mathrm{Sym}^2 \mathbb{K}\big)^0 \cong \mathbb{K}$ of 2SPSs on $\mathrm{CE}_{\mathbb{K}}$ can be found by considering the 2-dimensional Abelian Lie 2-algebra $\mathbb{K}^2[1] = (\mathbb{K}^2 \xrightarrow{0} 0)$. Then, by Proposition 3.2.15, there exists a 1-parameter family of 4SPSs on $\mathrm{CE}_{\mathbb{K}^2[1]}$ which is given by $\pi^{(2,0)} \in \mathrm{Sym}^2\big(\mathbb{K}^2[1]\big)^{-2} \cong \big(\bigwedge^2 \mathbb{K}^2\big)^0 \cong \mathbb{K}$. The required doubling of dimensions is thus a feature which arises from the odd parity of the non-trivial elements of $\mathbb{K}^2[1]$, which turns symmetric

powers into antisymmetric ones. Let us further emphasize that the shift of such SPSs gets doubled from $n = 2$ to $n = 4$ when passing from ordinary Lie algebras to Lie 2-algebras.

3.3.2 Shifted cotangent Lie 2-algebras

Let $\mathfrak{h}$ be a Lie algebra with Lie bracket $[\,\cdot\,,\,\cdot\,] : \mathfrak{h} \otimes \mathfrak{h} \to \mathfrak{h}$. Let us endow its dual $\mathfrak{h}^* := \underline{\hom}(\mathfrak{h}, \mathbb{K})$ with the coadjoint action $\mathrm{ad}^* : \mathfrak{h} \otimes \mathfrak{h}^* \to \mathfrak{h}^*$. The **shifted cotangent Lie 2-algebra** is defined by the 2-term cochain complex

$$T^*[1]\mathfrak{h} \;:=\; \mathfrak{h}^*[1] \oplus \mathfrak{h} \;=\; \Big(\mathfrak{h}^* \;\xrightarrow{\;0\;}\; \mathfrak{h} \Big) \;\in\; \mathbf{Ch} \tag{3.3.1a}$$

together with the Lie 2-algebra structure (see (3.2.28) and (3.2.29)) which is defined by the trivial higher bracket $\pi^{(1,3)} = 0$ and

$$\begin{aligned} \pi^{(1,2)} \;:\; T^*[1]\mathfrak{h} \otimes T^*[1]\mathfrak{h} \;&\longrightarrow\; T^*[1]\mathfrak{h} \quad , \\ (x, y) \;&\longmapsto\; [x, y] \quad , \\ (x, \theta) \;&\longmapsto\; \mathrm{ad}^*_x(\theta) \quad , \\ (\omega, \theta) \;&\longmapsto\; 0 \quad , \end{aligned} \tag{3.3.1b}$$

for all $x, y \in \mathfrak{h}$ and $\theta, \omega \in \mathfrak{h}^*[1]$. Such Lie 2-algebras are used for example in the context of higher-dimensional Chern-Simons theories which are based on 2-connections, see e.g. [Zuc21] and [SV24]. Applying our results from Propositions 3.2.15 and 3.2.17, we obtain the following characterisations of the $(n = 3)$- and $(n = 4)$SPSs on $\mathrm{CE}\big(T^*[1]\mathfrak{h}\big)$:

Example 3.3.1. A 4SPS on $\mathrm{CE}\big(T^*[1]\mathfrak{h}\big)$ is given by an element

$$\pi^{(2,0)} \;\in\; \underline{\hom}^-_+\big(\mathbb{K}, (T^*[1]\mathfrak{h})^{\otimes 2}\big)^{-2} \;\cong\; \bigwedge\nolimits^2 \mathfrak{h}^* \tag{3.3.2}$$

which is invariant under the tensor product coadjoint action. ▽

Example 3.3.2. A 3SPS on $\mathrm{CE}\big(T^*[1]\mathfrak{h}\big)$ is given by two elements

$$\pi^{(2,0)} \;\in\; \underline{\hom}^-\big(\mathbb{K}, (T^*[1]\mathfrak{h})^{\otimes 2}\big)^{-1} \;\cong\; \mathfrak{h}^* \otimes \mathfrak{h} \quad , \tag{3.3.3a}$$

$$\pi^{(2,1)} \;\in\; \underline{\hom}^-\big(T^*[1]\mathfrak{h}, (T^*[1]\mathfrak{h})^{\otimes 2}\big)^{-2} \;\cong\; \underline{\hom}\big(\mathfrak{h}, \mathrm{Sym}^2\,\mathfrak{h}^*\big) \quad . \tag{3.3.3b}$$

Using that the differential $\partial = 0$ is trivial in this example, one checks that the identities in the first line of (3.2.34) are equivalent to the invariance condition

$$\big(\mathrm{ad}^*_x \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}_x\big)\pi^{(2,0)} \;=\; 0 \quad , \tag{3.3.4a}$$

for all $x \in \mathfrak{h}$, where $\mathrm{ad} := [\,\cdot\,,\,\cdot\,] : \mathfrak{h} \otimes \mathfrak{h} \to \mathfrak{h}$ denotes the adjoint action. The identity in the second line of (3.2.34) is equivalent to the condition

$$\pi^{(2,1)}\big([x, y]\big) \;=\; \big(\mathrm{ad}^*_x \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}^*_x\big)\pi^{(2,1)}(y) - \big(\mathrm{ad}^*_y \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}^*_y\big)\pi^{(2,1)}(x) \quad , \tag{3.3.4b}$$

for all $x, y \in \mathfrak{h}$. Note that this is similar to the properties of a Lie bialgebra structure, with the difference that $\pi^{(2,1)} : \mathfrak{h} \to \mathrm{Sym}^2\,\mathfrak{h}^*$ takes values in the second symmetric power of the dual $\mathfrak{h}^*$, in contrast to the second exterior power of $\mathfrak{h}$ in the case of Lie bialgebras. The identity in the third line of (3.2.34) yields the condition

$$\sum_{\rho \in \mathrm{Sh}(2,1)} \gamma_\rho^{\mathbf{Vec}}\Big(\big(\pi^{(2,1)} \otimes \mathrm{id}\big)\pi^{(2,0)}\Big) \;=\; 0 \quad , \tag{3.3.4c}$$

where $\gamma^{\mathbf{Vec}}$ denotes the symmetric braiding on the category of vector spaces, i.e. it does *not* involve Koszul signs. (Note that the odd signs $(-1)^{|\rho|}$ in the sum over shuffles in (3.2.34) get compensated by the Koszul signs in the symmetric braiding (2.2.4) for $\mathfrak{h}^*[1] \otimes \mathfrak{h}^*[1] \otimes \mathfrak{h}^*[1]$.) ▽

Remark 3.3.3. In higher-dimensional Chern-Simons theory [Zuc21, SV24] the relevant 3-shifted Poisson structures are given by the coevaluation map $\pi^{(2,0)} = \mathrm{coev}(1) \in \mathfrak{h}^* \otimes \mathfrak{h}$ and $\pi^{(2,1)} = 0$. Such 3SPSs are non-degenerate, hence they have a corresponding 3-shifted symplectic structure which enters the construction of the action functional. Our characterisation in (3.3.3) and (3.3.4) shows that this is only a subclass of the 3SPSs on $\mathrm{CE}\big(T^*[1]\mathfrak{h}\big)$ and in particular there exists more flexibility which is given by the datum of the linear map $\pi^{(2,1)} : \mathfrak{h} \to \mathrm{Sym}^2\,\mathfrak{h}^*$. △

3.3.3 String Lie 2-algebras

String Lie 2-algebras are simple examples of L_∞-algebras, they appeared first in [BC04, Example 6.10] as shifted central extensions. Explicitly, let $\mathfrak{h}$ be a Lie algebra with Lie bracket $[\,\cdot\,,\,\cdot\,] : \mathfrak{h} \otimes \mathfrak{h} \to \mathfrak{h}$ and choose any 3-cocycle $\kappa : \mathfrak{h} \otimes \mathfrak{h} \otimes \mathfrak{h} \to \mathbb{K}$. Consider the 2-term cochain complex

$$\mathfrak{h}_\kappa \;:=\; \mathbb{K}[1] \oplus \mathfrak{h} \;=\; \Big(\mathbb{K} \xrightarrow{\;0\;} \mathfrak{h} \Big) \;\in\; \mathbf{Ch} \tag{3.3.5a}$$

and endow it with the Lie 2-algebra structure (see (3.2.28) and (3.2.29)) which is defined by:

$$\begin{aligned} \pi^{(1,2)} \,:\, \mathfrak{h}_\kappa \otimes \mathfrak{h}_\kappa \;&\longrightarrow\; \mathfrak{h}_\kappa \quad , \\ (x,y) \;&\longmapsto\; [x,y] \quad , \\ (a,y) \;&\longmapsto\; 0 \quad , \\ (a,b) \;&\longmapsto\; 0 \quad , \end{aligned} \qquad\qquad \begin{aligned} \pi^{(1,3)} \,:\, \mathfrak{h}_\kappa \otimes \mathfrak{h}_\kappa \otimes \mathfrak{h}_\kappa \;&\longrightarrow\; \mathfrak{h}_\kappa \quad , \\ (x,y,z) \;&\longmapsto\; \kappa(x,y,z) \quad , \\ (a,y,z) \;&\longmapsto\; 0 \quad , \\ (a,b,z) \;&\longmapsto\; 0 \quad , \\ (a,b,c) \;&\longmapsto\; 0 \quad , \end{aligned} \tag{3.3.5b}$$

where $x, y, z \in \mathfrak{h}$ and $a, b, c \in \mathbb{K}[1]$.

4SPSs

These are trivial by the same argument $\big(\mathrm{Sym}^2\,\mathfrak{h}_\kappa\big)^{-2} \cong \big(\bigwedge^2 \mathbb{K}\big)^0 \cong 0$ as in Subsection 3.3.1.

3SPSs

Proposition 3.2.17 shows that a 3SPS on $\mathrm{CE}_{\mathfrak{h}_\kappa}$ is given by two elements:

$$\pi^{(2,0)} \,\in\, \underline{\hom}^-\big(\mathbb{K}, \mathfrak{h}_\kappa^{\otimes 2}\big)^{-1} \,\cong\, \mathfrak{h} \quad , \qquad \pi^{(2,1)} \,\in\, \underline{\hom}^-\big(\mathfrak{h}_\kappa, \mathfrak{h}_\kappa^{\otimes 2}\big)^{-2} \,\cong\, \mathfrak{h}^* \quad . \tag{3.3.6}$$

Denoting these elements by

$$\mathbb{1} \,\in\, \mathfrak{h} \quad , \qquad \big\langle\,\cdot\,\big\rangle \,:\, \mathfrak{h} \longrightarrow \mathbb{K} \quad , \tag{3.3.7a}$$

the identities (3.2.34) for 3SPSs reduce to the properties

$$[x, \mathbb{1}] \,=\, 0 \quad , \qquad \big\langle \mathbb{1} \big\rangle \,=\, 0 \quad , \qquad \big\langle [x,y] \big\rangle \,=\, -\kappa(x,y,\mathbb{1}) \quad , \tag{3.3.7b}$$

for all $x, y \in \mathfrak{h}$. The first property demands that the distinguished element $\mathbb{1} \in \mathfrak{h}$ is central in the Lie algebra $\mathfrak{h}$ and the second property demands that it is annihilated by the linear form $\langle\,\cdot\,\rangle : \mathfrak{h} \to \mathbb{K}$. The third property demands that the linear form $\langle\,\cdot\,\rangle$ annihilates the Lie bracket

$[\,\cdot\,,\,\cdot\,]$ on $\mathfrak{h}$, up to a violation which is determined by the restricted the 3-cocycle $\kappa(\,\cdot\,,\,\cdot\,,\mathbb{1})$ on the distinguished element.

Remark 3.3.4. If the 3-cocycle comes from a symmetric invariant bilinear form on $\mathfrak{h}$, i.e. $\kappa(x,y,z) := \langle x,[y,z]\rangle$ then $\kappa(x,y,\mathbb{1}) = \langle x,[y,\mathbb{1}]\rangle = 0$. In this case, the form $\langle\,\cdot\,\rangle : \mathfrak{h} \to \mathbb{K}$ is a genuine Lie algebra morphism. △

Example 3.3.5. Assuming that the underlying Lie algebra $\mathfrak{h}$ is semisimple, it follows that the central element $\mathbb{1} \in \mathfrak{h}$ must necessarily be zero $\mathbb{1} = 0$. The properties in (3.3.7b) then simplify to the requirement that $\big\langle [x,y]\big\rangle = 0$, for all $x, y \in \mathfrak{h}$. Since $[\mathfrak{h},\mathfrak{h}] = \mathfrak{h}$ for semisimple Lie algebras, it follows that the linear form $\langle\,\cdot\,\rangle = 0$ must necessarily vanish. Hence, any 3SPS on $\mathrm{CE}_{\mathfrak{h}_\kappa}$ is trivial in the case where the underlying Lie algebra $\mathfrak{h}$ is semisimple. ▽

For a non-semisimple Lie algebra $\mathfrak{h}$, there can exist non-trivial 3SPSs. Let us consider the following two extremal cases:

Example 3.3.6. Suppose that $\mathbb{1} = 0$ is trivial. Then the properties in (3.3.7b) reduce to the 1-cocycle condition $\langle[\,\cdot\,,\,\cdot\,]\rangle = 0$. Hence, every non-trivial 1-cocycle on $\mathfrak{h}$ defines a 3SPS with $\mathbb{1} = 0$. As a concrete example, consider a matrix Lie algebra $\mathfrak{h} \subseteq \mathfrak{gl}_N(\mathbb{K})$ and observe that choosing $\mathbb{1} = 0$ and the matrix trace $\langle\,\cdot\,\rangle = \mathrm{Tr}$ defines a 3SPS.

Subexample 3.3.7. For the three-dimensional Abelian Lie algebra $\mathfrak{h} = \mathbb{K}^3$, every element is central hence choose $\mathbb{1} = aX + bY + cZ$ in which case the last relation of (3.3.7b) boils down to $\kappa(x,y,aX+bY+cZ) = 0$. Consecutively plugging in the pairs $(x = X, y = Y), (x = X, y = Z), (x = Y, y = Z)$ forces $c = b = a = 0$, respectively, thus $\mathbb{1} = 0$ and (3.3.7b) is automatically satisfied. In sum, the space of 3SPSs on $\mathrm{CE}_{\mathbb{K}^3_\kappa}$ is isomorphic to the space of forms $(\mathbb{K}^3)^*$. ▽

Subexample 3.3.8. Let $\mathfrak{h} = \mathfrak{H}$ be the **3-dimensional Heisenberg algebra** with basis $\{X,Y,Z\}$ and Lie bracket defined by:

$$[X,Y] = Z \qquad , \qquad [X,Z] = [Y,Z] = 0 \quad . \tag{3.3.8}$$

$\mathbb{1} = aZ$ is the only solution to $[\cdot,\mathbb{1}] = 0$ hence the second relation of (3.3.7b) reads as $a\langle Z\rangle = 0$. Feed $(x = X, y = Y)$ into the third relation of (3.3.7b),

$$\langle Z\rangle = -[X,Y,aZ] = -a \quad . \tag{3.3.9}$$

Hence, $a = \langle Z\rangle = 0$ and a 3SPS is given by a form which annihilates Z. ▽

▽

Example 3.3.9. Suppose that $\langle\,\cdot\,\rangle = 0$ is trivial. For a non-trivial central element $0 \neq \mathbb{1} \in \mathfrak{h}$ to exist, the Lie algebra $\mathfrak{h}$ must be a central extension $0 \to \mathbb{K} \to \mathfrak{h} \to \mathfrak{k} \to 0$. The third property in (3.3.7b) then requires that the 3-cocycle κ entering the string Lie 2-algebra $\mathfrak{h}_\kappa$ is induced by the map $\mathfrak{h} \to \mathfrak{k}$ from a 3-cocycle on $\mathfrak{k}$. If κ is chosen of this form, then there exists a 3SPS defined by the central element $\mathbb{1} \in \mathfrak{h}$ picked out by the map $\mathbb{K} \to \mathfrak{h}$ and the trivial linear form $\langle\,\cdot\,\rangle = 0$. ▽

2SPSs

Remark 3.3.10. Proposition 3.2.18 tells us that the set of 2SPSs on $\mathrm{CE}_{\mathfrak{h}_\kappa}$ is bijective to the set of symmetric pairs $\pi^{(2,0)} \in \mathrm{Sym}^2\mathfrak{h}$ and linear endomorphisms $\pi^{(2,1)} : \mathfrak{h} \to \mathfrak{h}$ such that:

(i) The symmetric pair is ad-invariant, $\pi^{(2,0)} \in (\mathrm{Sym}^2\mathfrak{h})^{\mathfrak{h}}$.

(ii) The linear endomorphism is a derivation up to Jacobiator of the symmetric pair,

$$[\,\cdot\,,\pi^{(2,1)}(\cdot)] + [\pi^{(2,1)}(\cdot)\,,\cdot\,] - \pi^{(2,1)}([\,\cdot\,,\cdot\,]) = \kappa(\cdot,\cdot,\pi^{(2,0)}) \quad . \tag{3.3.10}$$

(iii) The symmetric pair is in the kernel of the linear endomorphism,

$$(\pi^{(2,1)} \otimes_{\mathbb{K}} \mathrm{id}_{\mathfrak{h}} + \mathrm{id}_{\mathfrak{h}} \otimes_{\mathbb{K}} \pi^{(2,1)})(\pi^{(2,0)}) = 0 \quad . \tag{3.3.11}$$

△

Remark 3.3.11. Whereas Example 3.2.19 showed that certain 2SPSs (in the case of a 2-term DGLA) coincided with a special case of CM's notion of a "symmetric quasi-invariant tensor in a differential crossed module" [CFM15, Definition 30], the present case of a 2SPS and its data clearly diverges from this notion. Even if we strictified our string Lie 2-algebra $\mathfrak{h}_\kappa$ to a 2-term DGLA, the "quasi-derivation" $\pi^{(2,1)} : \mathfrak{h} \to \mathfrak{h}$ is a datum piece without an obvious analogue on the side of "symmetric quasi-invariant tensors". In fact, an honest comparison between the above 2SPS and CM's "symmetric quasi-invariant tensor in the String differential crossed module" [CFM15, Subsubsection 4.4.3] is difficult because our preferred model for the string Lie 2-algebra $\mathfrak{h}_\kappa$ is given by a fd L_∞-algebra with a Jacobiator $\pi^{(1,3)}$ (3.3.5), while [CFM15] use a different model which is given by an infinite-dimensional differential crossed module that encodes the cocycle κ less directly. Our suspicion that these two concepts are slightly different will be strengthened by the following example. △

Example 3.3.12. Take $\mathfrak{h} = \mathfrak{sl}_2$, i.e. the special linear Lie algebra of order 2. We work with the usual basis $\{x_+, x_-, x_3 \in \mathfrak{sl}_2\}$ whose Lie bracket relations are

$$[x_+, x_-] \;=\; x_3 \quad , \qquad [x_3, x_\pm] \;=\; \pm 2\, x_\pm \quad . \tag{3.3.12}$$

Using the (rescaled) Killing form $\langle\,\cdot\,,\,\cdot\,\rangle : \mathfrak{sl}_2 \otimes \mathfrak{sl}_2 \to \mathbb{K}$ given by $\langle x_3, x_3\rangle = 1$ and $\langle x_+, x_-\rangle = \langle x_-, x_+\rangle = \frac{1}{2}$, we define the 3-cocycle $\kappa(x,y,z) := \langle x, [y,z]\rangle$. In terms of our basis, this 3-cocycle is given by

$$\kappa(x_+, x_-, x_3) \;=\; 1 \tag{3.3.13}$$

and total antisymmetry in its arguments. This defines the string Lie 2-algebra $(\mathfrak{sl}_2)_\kappa$.

Since $\mathfrak{sl}_2$ is a simple Lie algebra, there exists a unique (up to scale) invariant symmetric tensor, which is concretely given by

$$\pi^{(2,0)} \;=\; \lambda \left(x_+\, x_- + \tfrac{1}{4}\, x_3^2 \right) \;\in\; (\mathrm{Sym}^2 \mathfrak{sl}_2)^{\mathfrak{sl}_2} \quad , \tag{3.3.14}$$

where $\lambda \in \mathbb{K}$ is any parameter. Assuming $\lambda \neq 0$ for the moment, the general solution for the linear map $\pi^{(2,1)} : \mathfrak{sl}_2 \to \mathfrak{sl}_2$ to the equation (3.3.11) is given by

$$\tilde{\pi}(x_+) \;=\; a\, x_+ + b\, x_3 \quad , \quad \tilde{\pi}(x_-) \;=\; -a\, x_- + c\, x_3 \quad , \quad \tilde{\pi}(x_3) \;=\; -2\, c\, x_+ - 2\, b\, x_- \quad , \tag{3.3.15}$$

where $a, b, c \in \mathbb{K}$ are arbitrary parameters. Evaluating (3.3.10) on $(x = x_+, y = x_-)$, one finds that it does not admit any solutions for $\lambda \neq 0$. In other words, every 2SPS on $\mathrm{CE}_{(\mathfrak{sl}_2)_\kappa}$ is characterised by a derivation $\tilde{\pi} : \mathfrak{sl}_2 \to \mathfrak{sl}_2$. This behaviour is drastically different to both: the case of ordinary Lie algebras $\mathfrak{g}$ from Proposition 3.2.11 and the "symmetric quasi-invariant tensors in the String differential crossed module" from [CFM15, Theorem 36] and [RW15, Section 5.2]. Furthermore, taking into account the natural concept of homotopies between SPSs (see, e.g., [Saf21]), it follows that every 2SPS on $\mathrm{CE}_{(\mathfrak{sl}_2)_\kappa}$ is homotopic to the trivial one because every derivation of $\mathfrak{sl}_2$ is inner. ▽

Example 3.3.13. The 'homotopically null' result of Example 3.3.12 holds for any generic simple Lie algebra $\mathfrak{h}$. For a simple Lie algebra: every symmetric invariant bilinear form is a multiple of the Killing form[4] $B : \mathrm{Sym}^2\mathfrak{h} \to \mathbb{K}$, every 3-cocyle $\kappa \in \mathsf{H}^3(\mathfrak{h};\mathbb{K})$ is a multiple of $B(\cdot,[\cdot,\cdot]) : \wedge^3\mathfrak{h} \to \mathbb{K}$ and there is a unique (up to scaling prefactor) symmetric invariant pair given by the quadratic Casimir, i.e. $\pi^{(2,0)} = \lambda B^{-1}$. Choosing $\kappa = B(\cdot,[\cdot,\cdot])$, we see that $\pi^{(2,1)} = \lambda\mathrm{id}_\mathfrak{h}$ produces a solution to (3.3.10). The general solution to such an equation would be $\pi^{(2,1)} = \lambda\mathrm{id}_\mathfrak{h} + D$ where $D : \mathfrak{h} \to \mathfrak{h}$ is a derivation. However, $\mathfrak{h}$ being a simple Lie algebra means that every derivation is an inner derivation, i.e. $D = [x,\cdot]$ for some $x \in \mathfrak{h}$. Plugging $\pi^{(2,1)} = \lambda\mathrm{id}_\mathfrak{h} + [x,\cdot]$ into (3.3.11) then forces $\lambda = 0$. In sum, for a simple Lie algebra $\mathfrak{h}$, every 2SPS on $\mathrm{CE}_{\mathfrak{h}_\kappa}$ is characterised by an inner derivation of $\mathfrak{h}$ and hence every 2SPS is homotopic to the trivial one. ▽

This vanishing of the symmetric invariant pair $\pi^{(2,0)}$ holds for some non-simple Lie algebras as well.

Example 3.3.14. For the three-dimensional Abelian Lie algebra $\mathfrak{h} = \mathbb{K}^3 = \mathbb{K}\langle x_1,x_2,x_3\rangle$ of Subexample 3.3.7, the 3-cocycle is $\kappa(x_j,x_k,x_l) = \varepsilon_{jkl}$ and we denote $\pi^{(2,0)} = \pi^{ij}x_ix_j$ hence (3.3.10) reads as $\pi^{ij}\varepsilon_{jkl} = 0$. Choosing $k = 2$ and $l = 3$ forces $\pi^{i1} = 0$ and the rest of the choices for k and m then forces $\pi^{(2,0)} = 0$ and a 2SPS is merely a linear endomorphism $\pi^{(1,2)} : \mathbb{K}^3 \to \mathbb{K}^3$. ▽

Of course, there are examples of Lie algebras for which the associated 2SPS can have nonzero $\pi^{(2,0)}$.

Example 3.3.15. Let $\mathfrak{h} = \mathfrak{H}$ be the 3-dimensional Heisenberg Lie algebra from Subexample 3.3.8. It can be easily checked that there are only two distinct classes of 2SPSs on $\mathrm{CE}_{\mathfrak{H}_\kappa}$ given by:

1. $(\pi_X^X,\pi_X^Y,\pi_X^Z,\pi_Y^X,\pi_Y^Y,\pi_Y^Z) \in \mathbb{K}^6$:

$$\begin{aligned}\pi^{(2,1)}(X) =& \pi_X^X X + \pi_X^Y Y + \pi_X^Z Z &,\\ \pi^{(2,1)}(Y) =& \pi_Y^X X + \pi_Y^Y Y + \pi_Y^Z Z &,\\ \pi^{(2,1)}(Z) =& 0 &,\\ \pi^{(2,0)} =& -\tfrac{1}{2}\left(\pi_X^X + \pi_Y^Y\right) Z\otimes Z &.\end{aligned}$$

2. $(\pi_X^X,\pi_X^Y,\pi_X^Z,\pi_Y^X,\pi_Y^Y,\pi_Y^Z) \in \mathbb{K}^6$:

$$\begin{aligned}\pi^{(2,1)}(X) =& \pi_X^X X + \pi_X^Y Y + \pi_X^Z Z &,\\ \pi^{(2,1)}(Y) =& \pi_Y^X X + \pi_Y^Y Y + \pi_Y^Z Z &,\\ \pi^{(2,1)}(Z) =& (\pi_X^X + \pi_Y^Y) Z &,\\ \pi^{(2,0)} =& 0 &.\end{aligned}$$

▽

3.3.4 Derivation strict Lie 2-algebras

Given a Lie algebra $\mathfrak{h}$, we denote the Lie algebra of Lie derivations by $\mathsf{T}_\mathfrak{h}$.

Definition 3.3.16. The **derivation Lie 2-algebra** is defined by

$$\mathsf{T}_\mathfrak{h}^\mathfrak{h} := \left(\mathfrak{h} \xrightarrow{\mathrm{ad}} \mathsf{T}_\mathfrak{h}, [\cdot,\cdot]\right)$$

[4]Which is non-degenerate for a simple Lie algebra.

where the bracket is defined for $X, Y \in \mathsf{T}_{\mathfrak{h}}$ and $x, y \in \mathfrak{h}$ as:

$$\begin{aligned}[X,Y] :=& X \circ Y - Y \circ X \quad , \\ [X,y] :=& X(y) \qquad\qquad\quad .\end{aligned}$$

Example 3.3.17. For a semisimple Lie algebra $\mathfrak{h}$, the adjoint representation $\mathfrak{h} \xrightarrow{\text{ad}} \mathsf{T}_{\mathfrak{h}}$ is faithful hence Remark 3.2.16 tells us that every 4SPS on $\text{CE}_{\mathsf{T}^{\mathfrak{h}}_{\mathfrak{h}}}$ is trivial. ▽

Remark 3.3.18. A few comparisons (with string Lie 2-algebras) are in order:

1. The construction of the string Lie 2-algebra (3.3.5) required a 3-cocycle $\kappa : \wedge^3 \mathfrak{h} \to \mathbb{K}$ whereas the construction of the derivation Lie 2-algebra has no requirement.
2. The derivation Lie 2-algebra is strict whereas the string Lie 2-algebra is a Lie 2-algebra with trivial differential, i.e. a "minimal model". In other words, they are on opposite poles of simplicity.
3. The 'homotopical part' $\mathfrak{g}^{-1}$ of the string Lie 2-algebra is always the ground field $\mathbb{K}$ whereas that of the derivation Lie 2-algebra is simply whatever Lie algebra $\mathfrak{h}$ we initially start with.

This last point plays a pivotal rôle; as we have already seen, the dimensionality of $\mathfrak{g}^{-1}$ controls, e.g., which terms of the 2SPS automatically vanish. In particular, only the weight 2 terms of the 2SPS may be nontrivial. We will find a very different story with the below example. △

Example 3.3.19. The **two-dimensional non-Abelian Lie algebra** $\mathfrak{2}$ has basis $\{x, y\}$ and Lie bracket given by

$$[x, y]_{\mathfrak{2}} = x \quad .$$

It is then simple to check that[5] $\text{ad} : \mathfrak{2} \cong \mathsf{T}_{\mathfrak{2}}$. Denoting $\{X = \text{ad}_x\,, Y = \text{ad}_y\}$, we have the following relations:

$$[X, Y] = X \qquad , \qquad [X, x] = 0 = [Y, y] \qquad , \qquad [X, y] = x = [x, Y] \quad .$$

The 2SPS

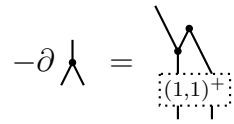
Proposition 3.2.18, together with the above dimension constraints and triviality of the Jacobiator, tells us that a 2SPS on $\text{CE}_{\mathsf{T}^{\mathfrak{2}}_{\mathfrak{2}}}$ is characterised as:

$\pi^{(2,0)} = \wedge \in \underline{\text{hom}}^-_+\big(\mathbb{K}, \mathfrak{g}^{\otimes 2}\big)^0$, $\pi^{(2,1)} =$  $\in \underline{\text{hom}}^-_+\big(\mathfrak{g}, \mathfrak{g}^{\otimes 2}\big)^{-1}$, $\pi^{(2,2)} =$  $\in \underline{\text{hom}}^-_+\big(\mathfrak{g}^{\otimes 2}, \mathfrak{g}^{\otimes 2}\big)^{-2}$,

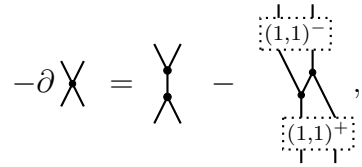
$$\pi^{(3,0)} = \wedge \in \underline{\text{hom}}^-_+\big(\mathbb{K}, \mathfrak{g}^{\otimes 3}\big)^{-2} \quad ,$$

[5] Again, Remark 3.2.16 then tells us that every 4SPSs on $\text{CE}_{\mathsf{T}^{\mathfrak{2}}_{\mathfrak{2}}}$ is trivial.

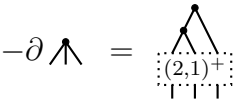
subject to the following four relations:



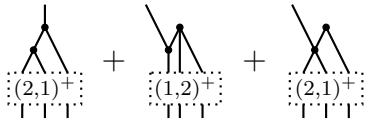
(3.3.16a)



(3.3.16b)



(3.3.16c)



(3.3.16d)

Let us denote:

$$\begin{aligned}
\pi^{(2,0)} &= \pi^{XX}X^2 + 2\pi^{XY}XY + \pi^{YY}Y^2 && && ,\\
\pi^{(2,1)}(X) &= \pi^{xX}_X xX + \pi^{xY}_X xY + \pi^{yX}_X yX + \pi^{yY}_X yY && , \quad \pi^{(2,1)}(x) = \pi_x xy && ,\\
\pi^{(2,1)}(Y) &= \pi^{xX}_Y xX + \pi^{xY}_Y xY + \pi^{yX}_Y yX + \pi^{yY}_Y yY && , \quad \pi^{(2,1)}(y) = \pi_y xy && ,\\
\pi^{(2,2)}(X \wedge Y) &= \pi^{xy}xy && && ,\\
\pi^{(3,0)} &= \pi^{X}Xxy + \pi^{Y}Yxy && && ,
\end{aligned}$$

and take $\pi^{(1,2)} = -[\cdot,\cdot]$ then the first relation

$$[\cdot, \pi^{(2,0)}] = \mathrm{ad} \circ \pi^{(2,1)} + \pi^{(2,1)} \circ \mathrm{ad}$$

imposes the following conditions on the following inputs:

$$\begin{aligned}
X &\implies 2\pi^{XY} = \pi^{xX}_X && , \quad 2\pi^{YY} = \pi^{xY}_X + \pi^{yX}_X && , \quad \pi^{yY}_X = 0\\
Y &\implies -2\pi^{XX} = \pi^{xX}_Y && , \quad -2\pi^{XY} = \pi^{xY}_Y + \pi^{yX}_Y && , \quad \pi^{yY}_Y = 0\\
x &\implies 2\pi^{YY} = \pi^{xY}_X - \pi_x && , \quad \pi_x + \pi^{yX}_X = 0\\
y &\implies -2\pi^{XY} = \pi^{xY}_Y - \pi_y && , \quad \pi_y + \pi^{yX}_Y = 0
\end{aligned}$$

which then reshapes the structures as

$$\begin{aligned}
\pi^{(2,0)} &= -\frac{1}{2}\pi^{xX}_Y X^2 - \left(\pi^{xY}_Y + \pi^{yX}_Y\right)XY + \frac{1}{2}\left(\pi^{xY}_X + \pi^{yX}_X\right)Y^2\\
\pi^{(2,1)}(X) &= -\left(\pi^{xY}_Y + \pi^{yX}_Y\right)xX + \pi^{xY}_X xY + \pi^{yX}_X yX \quad , \quad \pi^{(2,1)}(Y) = \pi^{xX}_Y xX + \pi^{xY}_Y xY + \pi^{yX}_Y yX\\
&\pi^{(2,1)}(x) = -\pi^{yX}_X xy \quad , \quad \pi^{(2,1)}(y) = -\pi^{yX}_Y xy \quad .
\end{aligned}$$

The second relation

$$\pi^{(2,1)} \circ [\cdot,\cdot] - [\pi^{(2,1)}, \cdot] - [\cdot, \pi^{(2,1)}] = \mathrm{ad} \circ \pi^{(2,2)} - \pi^{(2,2)} \circ \mathrm{ad}$$

imposes the following condition on the following input

$$X \otimes_{\mathbb{K}} Y \quad \Longrightarrow \quad \pi^{xy} = 0 \quad ,$$

and all other inputs and conditions are redundant given the previous impositions thus $\pi^{(2,2)} = 0$ is the only takeaway from the second relation. The third relation

$$\mathrm{S}^{+}_{2,1} \circ \left(\pi^{(2,1)} \otimes_{\mathbb{K}} \mathrm{id}_{\mathfrak{g}}\right)(\pi^{(2,0)}) = -\mathrm{ad}(\pi^{(3,0)})$$

only imposes the following two conditions:

$$\begin{aligned}\pi^{X} =& \frac{3}{2}\left[\pi^{xX}_{Y}\pi^{yX}_{X} + \pi^{yX}_{Y}\left(\pi^{xY}_{Y} + \pi^{yX}_{Y}\right)\right] \\ \pi^{Y} =& \frac{3}{2}\left[\pi^{xY}_{Y}\pi^{yX}_{X} - \pi^{yX}_{Y}\pi^{xY}_{X}\right]\end{aligned}$$

The fourth relation

$$[\cdot, \pi^{(3,0)}] = S^{+}_{2,1} \circ (\pi^{(2,1)} \otimes \mathrm{id}) \circ \pi^{(2,1)}$$

imposes no new conditions.

In sum, a 2SPS is given by:

$$\begin{gathered}\pi^{(2,0)} = -\frac{1}{2}\pi^{xX}_{Y}X^2 - \left(\pi^{xY}_{Y} + \pi^{yX}_{Y}\right)XY + \frac{1}{2}\left(\pi^{xY}_{X} + \pi^{yX}_{X}\right)Y^2 \quad , \\ \pi^{(2,1)}(X) = -\left(\pi^{xY}_{Y} + \pi^{yX}_{Y}\right)xX + \pi^{xY}_{X}xY + \pi^{yX}_{X}yX \quad , \quad \pi^{(2,1)}(Y) = \pi^{xX}_{Y}xX + \pi^{xY}_{Y}xY + \pi^{yX}_{Y}yX \quad , \\ \pi^{(2,1)}(x) = -\pi^{yX}_{X}xy \quad , \quad \pi^{(2,1)}(y) = -\pi^{yX}_{Y}xy \\ \pi^{(3,0)} = \frac{3}{2}\left[\pi^{xX}_{Y}\pi^{yX}_{X} + \pi^{yX}_{Y}\left(\pi^{xY}_{Y} + \pi^{yX}_{Y}\right)\right]Xxy + \frac{3}{2}\left[\pi^{xY}_{Y}\pi^{yX}_{X} - \pi^{yX}_{Y}\pi^{xY}_{X}\right]Yxy \quad ,\end{gathered}$$

where $\pi^{xX}_{Y}, \pi^{xY}_{Y}, \pi^{yX}_{Y}, \pi^{xY}_{X}, \pi^{yX}_{X} \in \mathbb{K}$ are free parameters. In sum, the space of 2SPSs on $\mathrm{CE}_{\mathbf{T}^2_{\mathfrak{z}}}$ is given by $\mathbb{K}^5$ where a point in that space specifies π as above. ▽

Chapter 4

Infinitesimal 2-braidings from 2-shifted Poisson structures

This chapter studies infinitesimal braidings in the 2-categorical context with the main result being that 2SPSs induce infinitesimal 2-braidings as in [KLS25, Theorem 3.10]. The presentation and ordering of ideas differs from that work in some significant ways: the preliminaries surrounding Derived Algebraic Geometry appearing in [KLS25, Section 2] have already been covered in Sections 2.2 and 2.3. We consider the tricategorical[1] universe $\mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}$ in more detail and do not defer its structure to an appendix as in [KLS25, Appendix A]; we do this because the rest of this thesis will frequently refer to subtle details surrounding this tricategory in order to introduce definitions and constructions. The exploration of higher-order deformations in [KLS25, Subsection 3.3] has been promoted to a chapter in its own right, i.e., Chapter 5.

Section 4.1 introduces the relevant tricategorical framework in which the deformation constructions of Section 4.2 will take place; the basic objects of our interest are as follows.

Definition 4.0.1. A **cochain 2-category** is a $\mathbf{Ch}_R^{[-1,0]}$-category, i.e. a category enriched over the symmetric monoidal category $\mathbf{Ch}_R^{[-1,0]}$ of 2-term cochain complexes (concentrated in degrees -1 and 0) of modules over a commutative $\mathbb{K}$-algebra R.

Remark 4.0.2. Note that a $\mathbf{Ch}_R^{[-1,0]}$-category is indeed a 2-category; the 1-morphisms are the degree 0 morphisms $f : U \to V$ whereas the 2-morphisms $\phi : f \Rightarrow f' : U \to V$ are the degree -1 morphisms such that $\partial\phi = f - f'$. The vertical composition of $f \overset{\phi}{\Rightarrow} f' \overset{\phi'}{\Rightarrow} f'' : U \to V$ is given by $\phi' \circ \phi = \phi' + \phi$ whereas the horizontal composition of $\phi : f \Rightarrow f' : U \to V$ and $\psi : g \Rightarrow g' : V \to W$ is given by $\psi * \phi = \psi f' + g\phi$. These compositions are evidently associative and admit units, furthermore the interchange law holds because of the truncation. △

Because $\mathbf{Ch}_R^{[-1,0]}$-categories are 2-categories, they assemble into a tricategory $\mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}$ where: 1-morphisms are $\mathbf{Ch}_R^{[-1,0]}$-functors, 2-morphisms are pseudonatural transformations and the 3-morphisms are modifications, see [JY21, Chapters 4, 11 and 12] for the more general context of a tricategory of bicategories, lax functors, lax natural transformations and modifications. We will not provide well-definition checks for said compositions in this chapter but, instead, we provide such checks in the technical Appendix A. We finish Subsection 4.1.1 by describing the symmetric monoidal structure on the tricategory $\mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}$.

Section 4.2 begins by making use of said symmetric monoidal structure to introduce the concept of a pseudonaturally-braided (strict) monoidal $\mathbf{Ch}_R^{[-1,0]}$-category, see Definition 4.2.1. We then follow [CFM15, Definition 15] in fixing the notion of a hexagonal infinitesimal 2-braiding in Definition

[1] See [GPS95] or [JY21, Section 11.2] for the relevant definition of a tricategory.

4.2.3. Theorem 4.2.7 shows that an infinitesimal 2-braiding provides a first-order deformation of a symmetric monoidal $\mathbf{Ch}_{\mathbb{K}}^{[-1,0]}$-category to a pseudonaturally-braided monoidal $\mathbf{Ch}_{\mathbb{K}[h]/(h^2)}^{[-1,0]}$-category if and only if the infinitesimal 2-braiding is hexagonal. This increased flexibility arising from pseudonaturality is essential for our construction of infinitesimal 2-braidings in Subsection 4.2.2 but it also comes at the cost of deforming the four-term relations as mentioned in Remark 4.2.9.

Subsection 4.2.1 introduces the notion of the homotopy 2-category of a dg-category and this is used in Subsection 4.2.2 to study the first-order deformation of the symmetric monoidal cochain 2-category

$$ {}_A\mathbf{C} := \mathsf{Ho}_2\big({}_A\mathbf{dgMod}^{\mathrm{fgsf}}\big) \quad . $$

The homotopy 2-category ${}_A\mathbf{C}$ has the same underlying objects, i.e. fgsf A-dg-modules, and its morphisms are given by the good truncations to cohomological degrees $\{-1, 0\}$ of the morphism complexes in ${}_A\mathbf{dgMod}^{\mathrm{fgsf}}$. Our main result is presented in Theorem 4.2.12 where we show that every 2SPS on a fgsfCDGA A defines, by a very explicit construction, a γ-equivariant[2] hexagonal infinitesimal 2-braiding on ${}_A\mathbf{C}$. This means that every 2SPS on A defines an explicit first-order braided monoidal deformation of the homotopy 2-category of the symmetric monoidal dg-category ${}_A\mathbf{dgMod}^{\mathrm{fgsf}}$, which realizes concretely the abstract deformation results from [CPTVV17] to first order in $\hbar$.

Section 4.3 studies our deformation construction from Subsection 4.2.2 using explicit examples associated with L_∞-algebras. Recall from Construction 3.2.4 that, for a fd L_∞-algebra $\mathfrak{g}$, the Chevalley-Eilenberg algebra $\mathrm{CE}_{\mathfrak{g}}$ is a fgsfCDGA hence Theorem 4.2.12 applies. Subsection 4.3.1 specialises this result to the case where $\mathfrak{g} = \mathfrak{h}$ is an ordinary Lie algebra, which leads to a homotopical generalization of the traditional constructions in [Car93] and [Kas95, Chapter XX]. Subsection 4.3.2 studies the case where $\mathfrak{g} = \mathfrak{h}_\kappa$ is a string Lie 2-algebra as in Subsection 3.3.3.

4.1 Tricategories with pseudofunctorial composition

We assume that the reader has some familiarity with enriched category theory as in Subsection 2.1.2; see, e.g. [Kel82] for a comprehensive treatment, or [JY21, Chapter 1.3] and [Rie14, Chapter 3] for more concise introductions.

Let us recall the following definition a 3-category.

Definition 4.1.1. A **3-category** is a category enriched over the category **2Cat** of 2-categories and 2-functors. A **3-functor** is a **2Cat**-functor.

We require a slightly weakened variant of this concept where the enrichment category is generalized to the category $\mathbf{2Cat}^{\mathrm{ps}}$ of 2-categories and pseudofunctors. See, e.g., [JY21, Theorem 4.1.30] for a detailed description of this category and, in particular, of its associative and unital composition. The category $\mathbf{2Cat}^{\mathrm{ps}}$ admits finite products and a terminal object, given by the 2-category $\mathbf{T} \in \mathbf{2Cat}^{\mathrm{ps}}$ consisting of a single 0-cell and its identity 1-cell and 2-cell, hence it is symmetric monoidal with respect to the Cartesian structure.

Definition 4.1.2. We denote by

$$ \mathbf{3Cat} := \mathbf{2Cat}^{\mathrm{ps}}\mathbf{Cat} \tag{4.1.1} $$

the 2-category whose:

[2] See Definition 4.2.5.

- objects are $\mathbf{2Cat}^{\mathrm{ps}}$-categories,
- 1-morphisms are $\mathbf{2Cat}^{\mathrm{ps}}$-functors,
- 2-morphisms are $\mathbf{2Cat}^{\mathrm{ps}}$-natural transformations.

To illustrate how an object $\mathcal{C} \in \mathbf{3Cat}$ admits an interpretation in terms of a weakened kind of 3-category, let us spell out, in some detail, the data defining $\mathcal{C}$. By definition of an enriched category, $\mathcal{C}$ has a class of objects and, for each pair of objects $\mathbf{C}, \mathbf{D} \in \mathcal{C}$, a 2-category $\mathcal{C}[\mathbf{C}, \mathbf{D}] \in \mathbf{2Cat}^{\mathrm{ps}}$ of morphisms. Since the 2-category $\mathcal{C}[\mathbf{C}, \mathbf{D}]$ consists of 0-cells, 1-cells and 2-cells, there are three layers of morphisms in $\mathcal{C} \in \mathbf{3Cat}$, as required for a 3-category. The enriched category $\mathcal{C}$ comes further with identity morphisms $\mathrm{id}_{\mathbf{C}} : \mathbf{T} \to \mathcal{C}[\mathbf{C}, \mathbf{C}]$, or equivalently a 0-cell $\mathrm{id}_{\mathbf{C}}$, 1-cell $\mathrm{Id}_{\mathrm{id}_{\mathbf{C}}}$ and 2-cell $\mathrm{ID}_{\mathrm{Id}_{\mathrm{id}_{\mathbf{C}}}}$, and composition morphisms

$$* \,:\, \mathcal{C}[\mathbf{D}, \mathbf{E}] \times \mathcal{C}[\mathbf{C}, \mathbf{D}] \;\longrightarrow\; \mathcal{C}[\mathbf{C}, \mathbf{E}] \tag{4.1.2}$$

in $\mathbf{2Cat}^{\mathrm{ps}}$, which in our case are pseudofunctors between 2-categories. The compositions $*$ must be associative and unital with respect to the identities $\mathrm{id}_{\mathbf{C}}$. Note that the relaxation to pseudofunctors makes our variant of weak 3-categories from Definition 4.1.2 more general than strict 3-categories, for which the analogous composition morphisms $*$ must be 2-functors[3].

The 2-category $\mathbf{3Cat}$ carries a symmetric monoidal structure which is determined by the Cartesian structure of the enrichment category $\mathbf{2Cat}^{\mathrm{ps}}$, see, e.g., [Kel82, Section 1.4] for the details. The monoidal product $\mathcal{C} \times \mathcal{C}' \in \mathbf{3Cat}$ of $\mathcal{C}, \mathcal{C}' \in \mathbf{3Cat}$ is the $\mathbf{2Cat}^{\mathrm{ps}}$-category whose objects are pairs $(\mathbf{C}, \mathbf{C}') \in \mathcal{C} \times \mathcal{C}'$ of objects $\mathbf{C} \in \mathcal{C}$ and $\mathbf{C}' \in \mathcal{C}'$ and whose morphisms

$$(\mathcal{C} \times \mathcal{C}')[(\mathbf{C}, \mathbf{C}'), (\mathbf{D}, \mathbf{D}')] := \mathcal{C}[\mathbf{C}, \mathbf{D}] \times \mathcal{C}'[\mathbf{C}', \mathbf{D}'] \in \mathbf{2Cat}^{\mathrm{ps}}$$

are given by the Cartesian structure on $\mathbf{2Cat}^{\mathrm{ps}}$. The monoidal unit $\mathcal{T} \in \mathbf{3Cat}$ is the $\mathbf{2Cat}^{\mathrm{ps}}$-category with a single object, say $\star \in \mathcal{T}$, and morphisms $\mathcal{T}[\star, \star] := \mathbf{T} \in \mathbf{2Cat}^{\mathrm{ps}}$ the terminal object. The symmetric braiding

$$\mathrm{flip}_{\mathcal{C},\mathcal{C}'} \,:\, \mathcal{C} \times \mathcal{C}' \;\longrightarrow\; \mathcal{C}' \times \mathcal{C} \tag{4.1.3a}$$

is the $\mathbf{2Cat}^{\mathrm{ps}}$-functor which acts on objects by flipping $(\mathbf{C}, \mathbf{C}') \mapsto (\mathbf{C}', \mathbf{C})$ and on morphisms through the symmetric braiding on $\mathbf{2Cat}^{\mathrm{ps}}$,

$$(\mathcal{C} \times \mathcal{C}')[(\mathbf{C}, \mathbf{C}'), (\mathbf{D}, \mathbf{D}')] = \mathcal{C}[\mathbf{C}, \mathbf{D}] \times \mathcal{C}'[\mathbf{C}', \mathbf{D}'] \xrightarrow{\cong} \mathcal{C}'[\mathbf{C}', \mathbf{D}'] \times \mathcal{C}[\mathbf{C}, \mathbf{D}] = (\mathcal{C}' \times \mathcal{C})[(\mathbf{C}', \mathbf{C}), (\mathbf{D}', \mathbf{D})]\,. \tag{4.1.3b}$$

Making use of this symmetric monoidal structure on $\mathbf{3Cat}$, one can define a concept of a symmetric monoidal category object in $\mathbf{3Cat}$.

Definition 4.1.3. A **symmetric monoidal category object in 3Cat** is a tuple $(\mathcal{C}, \boxtimes, u, a, l, r, \tau)$ consisting of the following data:

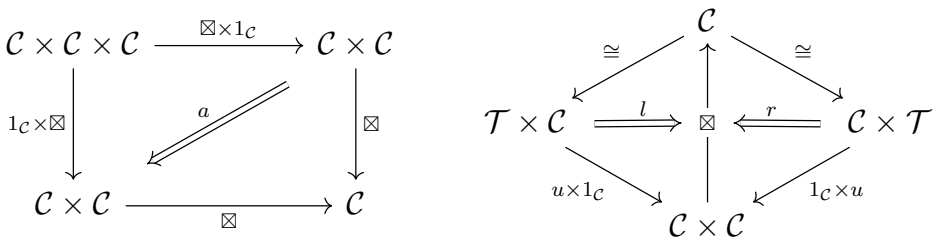
(i) an object $\mathcal{C} \in \mathbf{3Cat}$,

(ii) a 1-morphism $\boxtimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ in $\mathbf{3Cat}$ (called **monoidal composition**),

(iii) a 1-morphism $u : \mathcal{T} \to \mathcal{C}$ in $\mathbf{3Cat}$ (called **monoidal identity**),

[3]Note that our identity morphisms $\mathrm{id}_{\mathbf{C}} : \mathbf{T} \to \mathcal{C}[\mathbf{C}, \mathbf{C}]$ are already 2-functors.

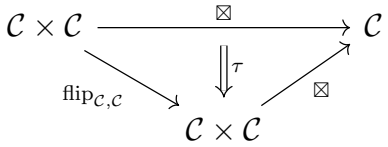
(iv) invertible 2-morphisms



(4.1.4)

in **3Cat** (called associator, left unitor and right unitor),

(v) an invertible 2-morphism



(4.1.5)

in **3Cat** (called symmetric braiding).

These data have to satisfy the usual coherence conditions for symmetric monoidal categories internal to the 2-category **3Cat**. When the associator a and left/right unitors l/r are all identities then we say $(\mathcal{C},\boxtimes,u,\tau)$ is a **symmetric strict monoidal category object in 3Cat**.

4.1.1 A closed symmetric monoidal tricategory of cochain 2-categories

Let R be a commutative $\mathbb{K}$-algebra and consider the symmetric monoidal category $\mathbf{Ch}_R$ of cochain complexes of R-modules from Subsection 2.2.1. We denote by

$$\mathbf{Ch}_R^{[-1,0]} \subseteq \mathbf{Ch}_R \tag{4.1.6}$$

the full subcategory whose objects are all cochain complexes $V \in \mathbf{Ch}_R$ concentrated in degrees $\{-1,0\}$, i.e., $V^i = 0$ for all $i \in \mathbb{Z}\setminus\{-1,0\}$. We shall often denote such objects by

$$V \;=\; \Big(V^{-1} \xrightarrow{\mathrm{d}_V} V^0 \Big) \in \mathbf{Ch}_R^{[-1,0]} \quad . \tag{4.1.7}$$

Note that the tensor product $\otimes_R$ on $\mathbf{Ch}_R$ does not directly restrict to the full subcategory $\mathbf{Ch}_R^{[-1,0]} \subseteq \mathbf{Ch}_R$ because $V\otimes_R W \in \mathbf{Ch}_R$ is, in general, concentrated in degrees $\{-2,-1,0\}$, for $V,W \in \mathbf{Ch}_R^{[-1,0]}$. This issue can be resolved by composing $\otimes_R$ with the good truncation functor $\tau^{[-1,0]} : \mathbf{Ch}_R \to \mathbf{Ch}_R^{[-1,0]}$ to define the truncated tensor product

$$\widetilde{\otimes}_R \;:\; \mathbf{Ch}_R^{[-1,0]} \times \mathbf{Ch}_R^{[-1,0]} \xrightarrow{\subseteq} \mathbf{Ch}_R \times \mathbf{Ch}_R \xrightarrow{\otimes_R} \mathbf{Ch}_R \xrightarrow{\tau^{[-1,0]}} \mathbf{Ch}_R^{[-1,0]} \quad . \tag{4.1.8a}$$

Explicitly, given two objects $V,W \in \mathbf{Ch}_R^{[-1,0]}$, the truncated tensor product reads as

$$V\widetilde{\otimes}_R W \;=\; \left(\frac{\big(V^{-1}\otimes_R W^0\big)\oplus\big(V^0\otimes_R W^{-1}\big)}{\mathrm{d}_{V\otimes_R W}\big(V^{-1}\otimes_R W^{-1}\big)} \xrightarrow{\mathrm{d}_{V\otimes_R W}} V^0\otimes_R W^0 \right) \quad . \tag{4.1.8b}$$

One directly checks that $\mathbf{Ch}_R^{[-1,0]}$ forms a symmetric monoidal category with respect to the truncated tensor product $\widetilde{\otimes}_R$, the monoidal unit $R \in \mathbf{Ch}_R^{[-1,0]}$ and the symmetric braiding

$$\widetilde{\gamma}_{V,W} : V \widetilde{\otimes}_R W \longrightarrow W \widetilde{\otimes}_R V \quad , \quad v \otimes_R w \longmapsto w \otimes_R v \quad , \tag{4.1.9}$$

where all Koszul signs are trivial because there are no odd-odd degree combinations in the truncated tensor product (4.1.8). For later use, we observe that the symmetric monoidal categories $(\mathbf{Ch}_R, \otimes_R, R, \gamma)$ and $(\mathbf{Ch}_R^{[-1,0]}, \widetilde{\otimes}_R, R, \widetilde{\gamma})$ are related by a lax symmetric monoidal functor.

Lemma 4.1.4. *The truncation functor $\tau^{[-1,0]} : \mathbf{Ch}_R \to \mathbf{Ch}_R^{[-1,0]}$ is canonically lax symmetric monoidal.*

Proof. The functor $\tau^{[-1,0]}$ preserves the monoidal units strictly, i.e., $\tau^{[-1,0]}(R) = R$, and the lax structure $\tau^{[-1,0]}(V) \,\widetilde{\otimes}_R\, \tau^{[-1,0]}(W) \to \tau^{[-1,0]}\big(V \otimes_R W\big)$ for the tensor product of $V, W \in \mathbf{Ch}_R$ is given by the obvious cochain map (drawn vertically)

$$\begin{array}{ccc} \frac{\left(\frac{V^{-1}}{\mathrm{d}_V V^{-2}} \otimes_R \mathsf{Z}^0(W)\right) \oplus \left(\mathsf{Z}^0(V) \otimes_R \frac{W^{-1}}{\mathrm{d}_W W^{-2}}\right)}{\mathrm{d}_{V\otimes_R W}\left(V^{-1} \otimes_R W^{-1}\right)} & \xrightarrow{\mathrm{d}_{V\otimes_R W}} & \mathsf{Z}^0(V) \otimes_R \mathsf{Z}^0(W) \\ \downarrow & & \downarrow \\ \frac{(V\otimes_R W)^{-1}}{\mathrm{d}_{V\otimes_R W}\left(V \otimes_R W\right)^{-2}} & \xrightarrow[\mathrm{d}_{V\otimes_R W}]{} & \mathsf{Z}^0(V \otimes_R W) \end{array} \tag{4.1.10}$$

□

The goal of the remaining part of this subsection is to provide details for the following result.

Proposition 4.1.5. *There exists a tricategory*

$$\mathbf{dgCat}_R^{[-1,0],\mathrm{ps}} \in \mathbf{3Cat} \tag{4.1.11}$$

in the sense of Definition 4.1.2 whose

- *objects are $\mathbf{Ch}_R^{[-1,0]}$-categories,*
- *1-morphisms are $\mathbf{Ch}_R^{[-1,0]}$-functors,*
- *2-morphisms are pseudonatural transformations, and*
- 3*-morphisms are modifications.*

This tricategory admits a symmetric strict monoidal structure in the sense of Definition 4.1.3 which agrees on $\mathbf{Ch}_R^{[-1,0]}$-categories, $\mathbf{Ch}_R^{[-1,0]}$-functors and $\mathbf{Ch}_R^{[-1,0]}$-natural transformations with the usual one from enriched category theory, see, e.g., [Kel82, Section 1.4].

The objects and 1-morphisms in $\mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}$ are defined according to the standard definitions of enriched category theory. The precise definitions of the 2-morphisms and 3-morphisms are as follows.

Definition 4.1.6. Given $\mathbf{Ch}_R^{[-1,0]}$-functors $F, G : \mathbf{C} \to \mathbf{D}$, a **pseudonatural transformation** $\xi : F \Rightarrow G$ consists of the following two pieces of data:

(i) For each 0-cell $U \in \mathbf{C}$, a 1-cell $\xi_U \in \mathbf{D}[F(U), G(U)]^0$ in the hom-object $\mathbf{D}[F(U), G(U)] \in \mathbf{Ch}_R^{[-1,0]}$ which we call a **cochain component**.

(ii) For each pair of 0-cells $U, U' \in \mathbf{C}$, a homotopy $\xi_{(-)} : \mathbf{C}[U, V] \to \mathbf{D}[F(U), G(V)][-1]$ whose output we call a **homotopy component**.

These two pieces of data have to satisfy the following two axioms:

(1) For all $f \in \mathbf{C}[U, V]$,

$$G(f)\,\xi_U - \xi_V F(f) \;=\; \partial\xi_f + \xi_{\partial f} \quad . \tag{4.1.12a}$$

(2) For all $f \in \mathbf{C}[U, V]$ and $g \in \mathbf{C}[V, W]$,

$$\xi_{gf} \;=\; \xi_g F(f) + G(g)\,\xi_f \quad . \tag{4.1.12b}$$

A **pseudonatural endomorphism** is one of the form $\upsilon : F \Rightarrow F$.

Remark 4.1.7. The present definition is noticeably more compact than [KLS25, Definition A.5]. Both F and G are $\mathbf{Ch}_R^{[-1,0]}$-functors hence they preserve units; plugging $g = f = 1_U$ into (4.1.12b) immediately yields

$$\xi_{1_U} = 2\xi_{1_U} \qquad \Longrightarrow \qquad \xi_{1_U} = 0 \quad . \tag{4.1.13}$$

In other words, the lax unity axiom [KLS25, (A.14)] is redundant given the lax associativity axiom [KLS25, (A.15)]. Furthermore, a homotopy $\xi_{(-)} : \mathbf{C}[U, V] \to \mathbf{D}\,[F(U), G(V)]\,[-1]$ between truncated homs automatically gives $\xi_k = 0$ for $k \in \mathbf{C}[U, V]^{-1}$ hence plugging $f = k$ into (4.1.12a) yields

$$G(k)\,\xi_U - \xi_V F(k) \;=\; \xi_{\partial k} \quad , \tag{4.1.14a}$$

whereas plugging $f \in \mathbf{C}[U, V]^0$ into (4.1.12a) yields

$$G(f)\,\xi_U - \xi_V F(f) \;= \partial\xi_f \quad . \tag{4.1.14b}$$

△

Remark 4.1.8. Let us note that a pseudonatural transformation with trivial homotopy components, $\xi_f = 0$, is simply a $\mathbf{Ch}_R^{[-1,0]}$-natural transformation. The **zero pseudonatural transformation** $0 : F \Rightarrow G$ is such an example. △

Example 4.1.9. The **identity pseudonatural transformation** $\mathrm{Id}_F : F \Rightarrow F : \mathbf{C} \to \mathbf{D}$, defined by $(\mathrm{Id}_F)_U = 1_{F(U)}$ and $(\mathrm{Id}_F)_f = 0$, is both a pseudonatural endomorphism and $\mathbf{Ch}_R^{[-1,0]}$-natural transformation, i.e., it is a $\mathbf{Ch}_R^{[-1,0]}$-natural endomorphism. We sometimes denote such identities as $1 := \mathrm{Id}_F$ when the context makes it clear what the $\mathbf{Ch}_R^{[-1,0]}$-functor F is. ▽

Definition 4.1.10. Given two pseudonatural transformations $\xi, \xi' \in \mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}[\mathbf{C}, \mathbf{D}](F, G)$, a **modification** $\Xi : \xi \Rrightarrow \xi' : F \Rightarrow G : \mathbf{C} \to \mathbf{D}$ consists of the following datum:

(i) For each 0-cell $U \in \mathbf{C}$, a 2-cell $\Xi_U \in \mathbf{D}[F(U), G(U)]^{-1}$ such that

$$\xi_U - \xi'_U = \partial\Xi_U \quad . \tag{4.1.15}$$

This datum is subject to the single axiom:

1. For every $f \in \mathbf{C}[U, V]$,

$$\Xi_V F(f) + \xi_f = \xi'_f + G(f)\Xi_U \quad . \tag{4.1.16}$$

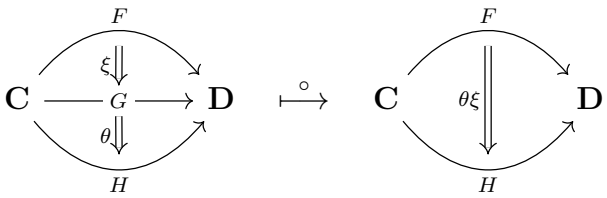
An **endomodification** is one of the form $\Xi : \xi \Rrightarrow \xi' : F \Rightarrow F$, i.e., it is a modification between pseudonatural endomorphisms and *not* an endomorphic modification $\Theta : \theta \Rrightarrow \theta : F \Rightarrow G$. We denote by $\mathrm{ID}_\xi := \mathbf{0} : \xi \Rrightarrow \xi$ the **zero modification** which is an instance of the latter.

In the following subsubsections we describe explicitly the symmetric strict monoidal $\mathbf{2Cat}^{\mathrm{ps}}$-category structure of $\mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}$ in the sense of Definitions 4.1.2 and 4.1.3.

The hom cochain 2-categories

Given two $\mathbf{Ch}_R^{[-1,0]}$-categories $\mathbf{C}$ and $\mathbf{D}$, we denote by $\mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}[\mathbf{C},\mathbf{D}] \in \mathbf{2Cat}^{\mathrm{ps}}$ the cochain 2-category whose objects are $\mathbf{Ch}_R^{[-1,0]}$-functors, 1-morphisms are pseudonatural transformations and 2-morphisms are modifications.

Construction 4.1.11. The **vertical composite pseudonatural transformation**



(4.1.17a)

in the 2-category $\mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}[\mathbf{C},\mathbf{D}]$ is defined by:

$$(\theta\xi)_U := \theta_U \xi_U \quad , \tag{4.1.17b}$$

$$(\theta\xi)_f := \theta_f \xi_U + \theta_V \xi_f \quad , \tag{4.1.17c}$$

see Subsection A.1.1 for the well-definition check. This composition is associative and unital with respect to the identity 1. ▷

Definition 4.1.12. A pseudonatural transformation $\xi : F \Rightarrow G : \mathbf{C} \to \mathbf{D}$ is a **pseudonatural isomorphism** if there exists another pseudonatural transformation $\xi^{-1} : G \Rightarrow F : \mathbf{C} \to \mathbf{D}$ such that $\xi^{-1}\xi = \mathrm{Id}_F$ and $\xi\,\xi^{-1} = \mathrm{Id}_G$. A **pseudonatural automorphism** $\xi \in \mathrm{PsNat}_F^\times$ is a pseudonatural endomorphism $\xi \in \mathrm{PsNat}_F$ which is also a pseudonatural isomorphism.

It is obvious that pseudonatural isomorphisms are closed under vertical composition.

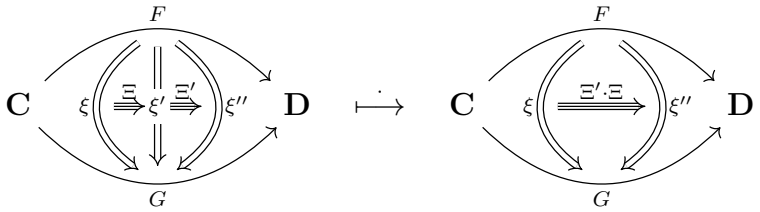
Remark 4.1.13. The formulae for the vertical composition/unit in Construction 4.1.11 tells us that a pseudonatural transformation $\xi : F \Rightarrow G : \mathbf{C} \to \mathbf{D}$ is a pseudonatural isomorphism if and only if the cochain components $\xi_U \in \mathbf{D}[F(U), G(U)]^0$ are isomorphisms $\xi_U : F(U) \cong G(U)$. In other words, it can be easily checked that setting:

$$\xi_U^{-1} := (\xi_U)^{-1} : G(U) \cong F(U) \quad , \tag{4.1.18a}$$

$$\xi_f^{-1} := -\,\xi_V^{-1}\xi_f\xi_U^{-1} \in \mathbf{D}\,[G(U), F(V)]^{-1} \quad , \tag{4.1.18b}$$

produces a well-defined **inverse pseudonatural transformation** $\xi^{-1} : G \Rightarrow F : \mathbf{C} \to \mathbf{D}$. △

Construction 4.1.14. The **lateral composite modification**



(4.1.19a)

in the category $\mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}[\mathbf{C},\mathbf{D}](F,G)$ is defined by

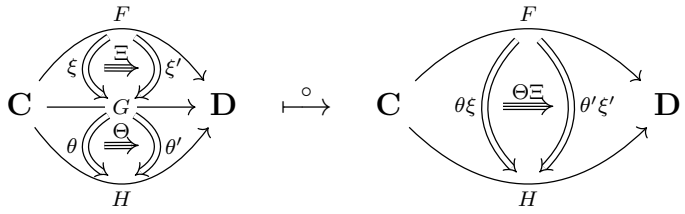
$$(\Xi' \cdot \Xi)_U := \Xi'_U + \Xi_U \quad , \tag{4.1.19b}$$

see Subsection A.2.2 for the well-definition check. This composition is associative, unital with respect to $\mathbf{0}$ and invertible[4] with respect to the **reverse** modifications $\overleftarrow{\Xi} : \xi' \Rrightarrow \xi : F \Rightarrow G : \mathbf{C} \to \mathbf{D}$ defined by

$$\overleftarrow{\Xi}_U := -\Xi_U \quad . \tag{4.1.20}$$

▷

Construction 4.1.15. The **vertical composite modification**



(4.1.21a)

in the 2-category $\mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}[\mathbf{C},\mathbf{D}]$ is defined by

$$(\Theta\Xi)_U := \Theta_U \xi'_U + \theta_U \Xi_U \quad , \tag{4.1.21b}$$

see Subsection A.2.3 for the well-definition check. This composition is associative, unital with respect to the particular identities $\mathrm{ID}_{\mathrm{Id}_F} : \mathrm{Id}_F \Rrightarrow \mathrm{Id}_F$ and functorial, i.e., the exchange law between vertical and lateral composition of modifications is upheld[5],

$$(\Theta'\Xi') \cdot (\Theta\Xi) = (\Theta' \cdot \Theta)(\Xi' \cdot \Xi) \quad , \tag{4.1.22}$$

and the vertical composition of two lateral unit modifications is again a lateral unit,

$$\mathrm{ID}_\theta \mathrm{ID}_\xi = \mathrm{ID}_{\theta\xi} \quad . \tag{4.1.23}$$

Using that $\partial\Xi_U = \xi_U - \xi'_U$ and $\partial\Theta_U = \theta_U - \theta'_U$ together with the truncation of the hom-complexes allows us to equally write the vertical composition of modifications (4.1.21b) as

$$(\Theta\Xi)_U = \Theta_U (\xi_U - \partial\Xi_U) + (\theta'_U + \partial\Theta_U)\, \Xi_U = \Theta_U \xi_U + \theta'_U \Xi_U \quad . \tag{4.1.24}$$

We denote the **pre/post-whiskering**, respectively, by

$$\Theta\xi' := \Theta\, \mathrm{ID}_{\xi'} \qquad , \qquad \theta\Xi := \mathrm{ID}_\theta\, \Xi \quad . \tag{4.1.25}$$

Using this notation, we can rewrite (4.1.21b) and (4.1.24), respectively, as

$$\Theta\Xi = \Theta\xi' \cdot \theta\Xi = \theta'\Xi \cdot \Theta\xi \quad . \tag{4.1.26}$$

▷

Remark 4.1.16. If $\xi, \xi' : F \Rightarrow G$ are pseudonatural isomorphisms then a modification between them $\Xi : \xi \Rrightarrow \xi' : F \Rightarrow G$ is called an **isomodification**, in which case

$$\Xi^{-1} := \xi^{-1} \overleftarrow{\Xi}\, \xi'^{-1} : \xi^{-1} \Rrightarrow \xi'^{-1} : G \Rightarrow F \tag{4.1.27}$$

[4]Every modification is thus an isomorphic modification and $*$ is *pseudo*functorial. Furthermore, $\mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}$ is actually a weak $(3,2)$-category.

[5]See Subsection A.2.6 for such proof.

clearly defines the **inverse modification**. An **automodification** $\Xi \in \mathbf{Mod}_F^\times$ is a modification between pseudonatural automorphisms. It is obvious from the definition that isomodifications are closed under vertical composition and whiskering by a pseudonatural isomorphism. △

To state various definitions we will need the notion of R-linear addition $+$ of pseudonatural transformations and modifications. Note that in the latter case R-linear addition of modifications is actually distinct from lateral composition $\cdot$, e.g., scaling a modification by[6] $\frac{1}{25}$ has no correlate as a lateral composite.

Construction 4.1.17. Suppose we have modifications $\Xi : \xi \Rrightarrow \xi' : F \Rightarrow G$ and $\Upsilon : \upsilon \Rrightarrow \upsilon' : F \Rightarrow G$ then, for $r, r' \in R$, we define $r\Xi + r'\Upsilon : r\xi + r'\upsilon \Rrightarrow z\xi' + w\upsilon' : F \Rightarrow G$ by

$$(r\xi+r'\upsilon)_U := r\xi_U+r'\upsilon_U \ , \ (r\xi+r'\upsilon)_f := r\xi_f+r'\upsilon_f \qquad , \qquad (r\Xi+r'\Upsilon)_U := r\Xi_U+r'\Upsilon_U \, . \tag{4.1.28}$$

These addition/scaling operations are evidently abelian and vertical composition $\circ$ of pseudonatural transformations (4.1.17)/modifications (4.1.21b) is R-bilinear. We define the **coboundary** of $\Xi : \xi \Rrightarrow \xi'$ as

$$\partial\Xi :\equiv \xi - \xi' \qquad . \tag{4.1.29}$$

▷

Finally, for every $\mathbf{Ch}_R^{[-1,0]}$-functor F, the map

$$\mathrm{Id}_F : \mathbf{1} \to \mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}[\mathbf{C},\mathbf{D}](F,F) \qquad , \qquad \star \mapsto \mathrm{Id}_F \quad , \quad 1_\star \mapsto \mathrm{ID}_{\mathrm{Id}_F} \, ,$$

is evidently functorial.

The horizontal composition/identity pseudofunctors

Given three $\mathbf{Ch}_R^{[-1,0]}$-categories $\mathbf{C}$, $\mathbf{D}$ and $\mathbf{E}$, we denote by

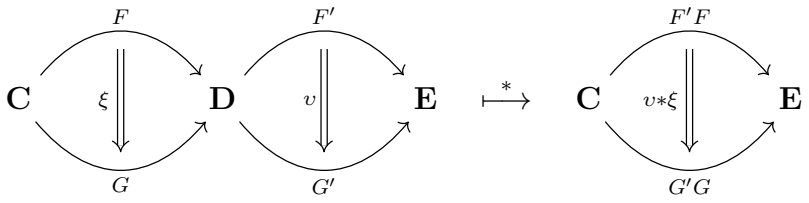
$$* \, : \, \mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}[\mathbf{D},\mathbf{E}] \times \mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}[\mathbf{C},\mathbf{D}] \ \longrightarrow \ \mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}[\mathbf{C},\mathbf{E}] \tag{4.1.30}$$

the horizontal composition pseudofunctor between the hom 2-categories from the previous subsubsection. On objects in the hom 2-categories, this pseudofunctor acts in terms of the usual (associative and unital) composition of enriched functors which we will denote by

$$\mathbf{C} \xrightarrow{F} \mathbf{D} \xrightarrow{F'} \mathbf{E} \quad \overset{*}{\longmapsto} \quad \mathbf{C} \xrightarrow{F'F} \mathbf{E} \quad . \tag{4.1.31}$$

On 1-morphisms in the hom 2-categories, the pseudofunctor $*$ acts as follows.

Construction 4.1.18. The **horizontal composite pseudonatural transformation**



(4.1.32a)

[6] That is, $\frac{1}{25}$ of the unit of R.

in the tricategory $\mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}$ is defined by

$$(\upsilon * \xi)_U := \upsilon_{G(U)}\, F'(\xi_U) \quad , \tag{4.1.32b}$$
$$(\upsilon * \xi)_f := \upsilon_{G(f)}\, F'(\xi_U) + \upsilon_{G(V)}\, F'(\xi_f) \quad , \tag{4.1.32c}$$

see Subsection A.1.2 for the well-definition check. This composition is associative and unital with respect to the particular identity pseudonatural transformations $\mathrm{Id}_{\mathrm{id}_\mathbf{C}} : \mathrm{id}_\mathbf{C} \Rightarrow \mathrm{id}_\mathbf{C} : \mathbf{C} \to \mathbf{C}$. ▷

Remark 4.1.19. The horizontal composition of two identity pseudonatural transformations is another identity pseudonatural transformation, i.e.,

$$\mathrm{Id}_{F'} * \mathrm{Id}_F = \mathrm{Id}_{F'F} \quad . \tag{4.1.33}$$

This fact that $*$ is *strictly unitary* will be implicit in many constructions throughout. △

Remark 4.1.20. Whiskering pseudonatural transformations by $\mathbf{Ch}_R^{[-1,0]}$-functors works as usual:

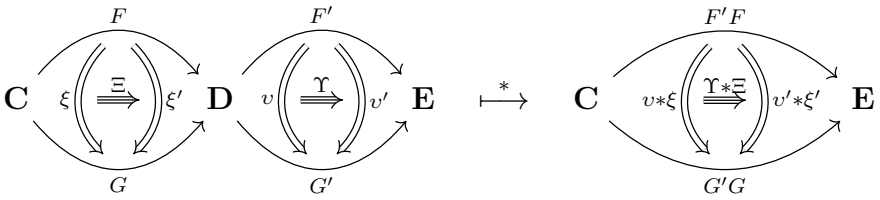
$$(\upsilon * \mathrm{Id}_G)_U = \upsilon_{G(U)} \ , \ (\upsilon * \mathrm{Id}_G)_f = \upsilon_{G(f)} \quad , \quad (\mathrm{Id}_{F'} * \xi)_U = F'(\xi_U) \ , \ (\mathrm{Id}_{F'} * \xi)_f = F'(\xi_f) \, , \tag{4.1.34}$$

hence pseudonatural isomorphisms are closed under whiskering. We will denote (4.1.34) by

$$\upsilon_G := \upsilon * \mathrm{Id}_G \qquad , \qquad F'(\xi) := \mathrm{Id}_{F'} * \xi \quad . \tag{4.1.35}$$

Associativity of horizontal composition gives us $H(\upsilon_F) = H(\upsilon)_F$. △

On 2-morphisms in the hom 2-categories, the pseudofunctor $*$ acts as



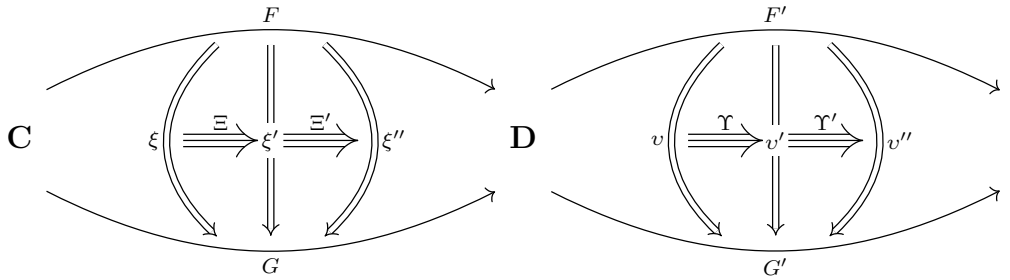
(4.1.36a)

with

$$(\Upsilon * \Xi)_U := \Upsilon_{G(U)}\, F'(\xi_U) + \upsilon'_{G(U)}\, F'(\Xi_U) \quad , \tag{4.1.36b}$$

see Subsection A.2.4 for the well-definition check. This composition is associative and unital with respect to the particular identity modifications $\mathrm{ID}_{\mathrm{Id}_{\mathrm{id}_\mathbf{C}}} : \mathrm{Id}_{\mathrm{id}_\mathbf{C}} \Rrightarrow \mathrm{Id}_{\mathrm{id}_\mathbf{C}} : \mathrm{id}_\mathbf{C} \Rightarrow \mathrm{id}_\mathbf{C} : \mathbf{C} \to \mathbf{C}$.

Local functoriality means that, given



, (4.1.37)

we have $(\cdot, *)$-exchange[7],

$$(\Upsilon' \cdot \Upsilon) * (\Xi' \cdot \Xi) = (\Upsilon' * \Xi') \cdot (\Upsilon * \Xi) \quad ,$$

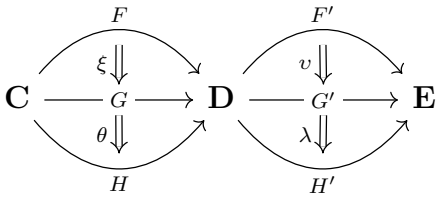
[7]See Subsection A.2.6 for such proof.

and that

$$\mathrm{ID}_{\upsilon} * \mathrm{ID}_{\xi} = \mathrm{ID}_{\upsilon * \xi} \quad .$$

As is well known, the vertical and horizontal composition of pseudonatural transformations does *not* satisfy the middle-four exchange law but rather this is relaxed by the exchanger. Many modifications appearing throughout will be instances of the exchanger.

Construction 4.1.21. Given any composable diagram of the form



, (4.1.38a)

the **exchanger** is the modification[8]

$$*^{2}_{(\lambda,\theta),(\upsilon,\xi)} \,:\, (\lambda * \theta)(\upsilon * \xi) \;\Rrightarrow\; \lambda\upsilon * \theta\xi \tag{4.1.38b}$$

whose components are defined by

$$\left(*^{2}_{(\lambda,\theta),(\upsilon,\xi)} \right)_{U} \;:=\; \lambda_{H(U)}\, \upsilon_{\theta_U} F'(\xi_U) \quad . \tag{4.1.38c}$$

▷

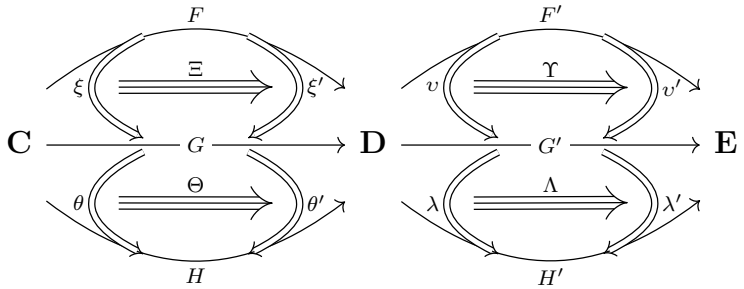
Remark 4.1.22. Notice that the exchanger (4.1.38) is trivial if υ is a $\mathbf{Ch}_R^{[-1,0]}$-natural transformation or $\theta = 1$ hence whiskering is functorial, i.e.:

$$(\lambda\upsilon)_F = \lambda_F \upsilon_F \qquad , \qquad F'(\theta\xi) = F'(\theta)F'(\xi) \quad , \tag{4.1.39a}$$

$$1_F = 1 \qquad , \qquad F'(1) = 1 \quad . \tag{4.1.39b}$$

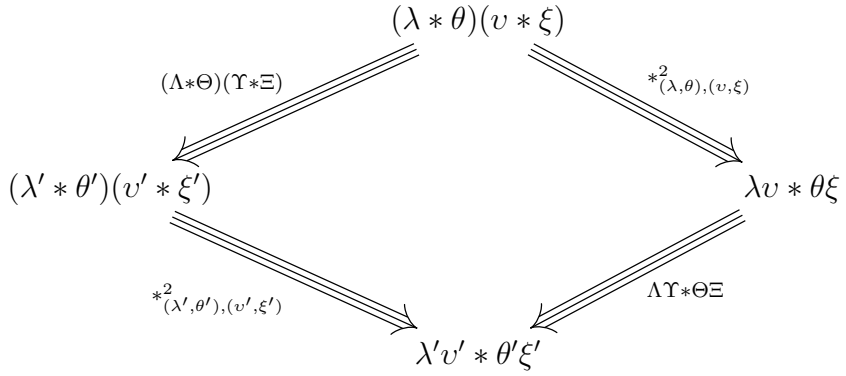
△

Naturality of the pseudo composition constraint means that, for modifications

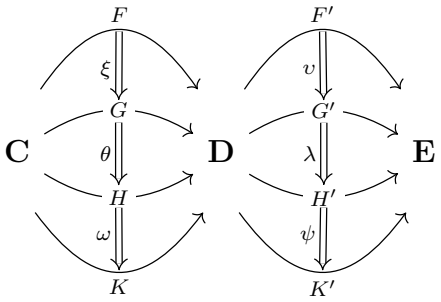


[8] See Subsection A.2.1 for the well-definition check.

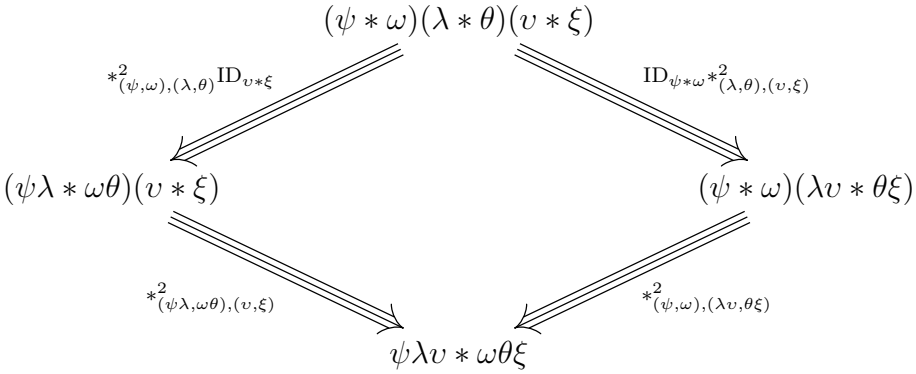
the following diagram commutes (composition of modifications is lateral in this diagram)



see Subsection A.2.5 for the proof. The associativity axiom of the pseudo composition constraint demands that, for pseudonatural transformations



the following diagram commutes

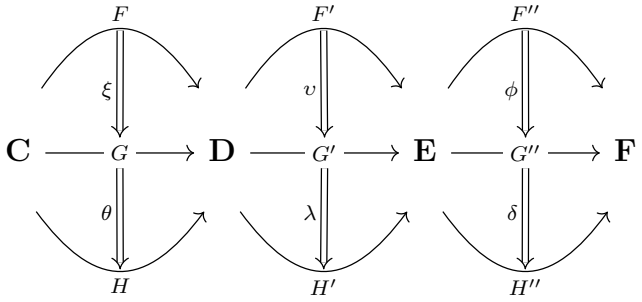


For each $\mathbf{C} \in \mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}$, the map $\mathrm{id}_{\mathbf{C}} : \mathbf{T} \to \mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}[\mathbf{C}, \mathbf{C}]$ mentioned in the discussion right after Definition 4.1.2 is already known to be a 2-functor.

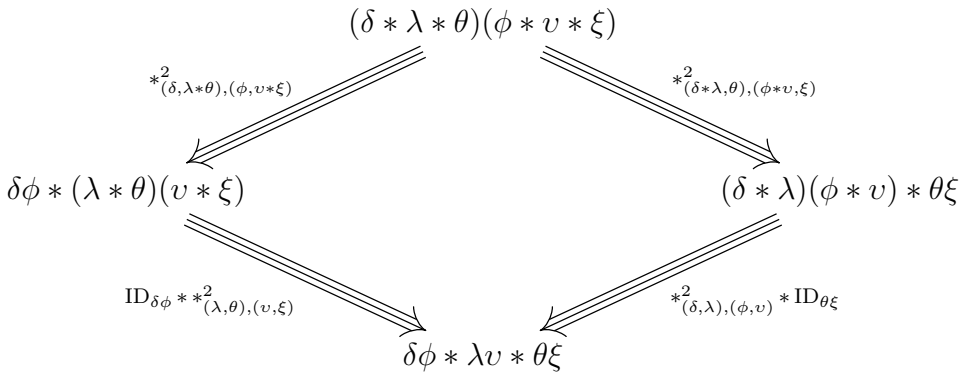
Associativity/unitality of horizontal composition

As already mentioned, the horizontal composition of $\mathbf{Ch}_R^{[-1,0]}$-functors, pseudonatural transformations and modifications is associative. The associativity of horizontal composition also means that the two *a priori distinct* ways to extract an exchanger modification from the following ambiguous

pasting diagram



are, in fact, the same modification, i.e., the following diagram commutes



The identity: $\mathbf{Ch}_R^{[-1,0]}$-functors $\mathrm{id}_{\mathbf{C}}$, pseudonatural transformations $\mathrm{Id}_{\mathrm{id}_{\mathbf{C}}}$ and modifications $\mathrm{ID}_{\mathrm{Id}_{\mathrm{id}_{\mathbf{C}}}}$ are unital with respect to the horizontal composition. The unitality of horizontal composition also requires left and right triviality of the exchanger, i.e., for

$$F \overset{\xi}{\Rightarrow} G \overset{\theta}{\Rightarrow} H : \mathbf{C} \to \mathbf{D} \quad ,$$

we need

$$*^2_{(\mathrm{Id}_{\mathrm{id}_{\mathbf{D}}},\theta),(\mathrm{Id}_{\mathrm{id}_{\mathbf{D}}},\xi)} = \mathrm{ID}_{\theta\xi} = *^2_{(\theta,\mathrm{Id}_{\mathrm{id}_{\mathbf{C}}}),(\xi,\mathrm{Id}_{\mathrm{id}_{\mathbf{C}}})} : \theta\xi \Rrightarrow \theta\xi \quad .$$

Recalling the comment at the end of Construction 4.1.21, we can see that the above requirement indeed holds.

The symmetric strict monoidal structure

The monoidal composition

$$\boxtimes \; : \; \mathbf{dgCat}_R^{[-1,0],\mathrm{ps}} \times \mathbf{dgCat}_R^{[-1,0],\mathrm{ps}} \;\longrightarrow\; \mathbf{dgCat}_R^{[-1,0],\mathrm{ps}} \tag{4.1.40}$$

is the $\mathbf{2Cat}^{\mathrm{ps}}$-functor which assigns to a pair of objects $\mathbf{C}, \mathbf{C}' \in \mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}$ the usual $\mathbf{Ch}_R^{[-1,0]}$-category $\mathbf{C} \boxtimes \mathbf{C}' \in \mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}$ and to a pair of 1-morphisms $F : \mathbf{C} \to \mathbf{D}$ and $J : \mathbf{C}' \to \mathbf{D}'$ in $\mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}$ the usual $\mathbf{Ch}_R^{[-1,0]}$-functor $F \boxtimes J : \mathbf{C} \boxtimes \mathbf{C}' \to \mathbf{D} \boxtimes \mathbf{D}'$ from [Kel82, Section 1.4].

Construction 4.1.23. The $\mathbf{Ch}_R^{[-1,0]}$-category $\mathbf{C} \boxtimes \mathbf{C}'$ is defined by:

(i) Objects are simply 2-tuples which we denote by juxtaposition, i.e., $U\,U' \in \mathbf{C} \boxtimes \mathbf{C}'$ where $U \in \mathbf{C}$ and $U' \in \mathbf{C}'$.

(ii) The hom 2-term complexes are given by the truncated tensor product, i.e.,

$$f \boxtimes f' := f \mathbin{\widetilde{\otimes}_R} f' := \begin{cases} 0\,, & |f| = |f'| = -1 \\ f \otimes_R f'\,, & \text{otherwise} \end{cases} : U\,U' \to V\,V' \quad . \tag{4.1.41}$$

(iii) The cochain composition is given by the middle-four exchange (which law is upheld because of the truncation), i.e.,

$$(g \boxtimes g')(f \boxtimes f') = (gf) \boxtimes (g'f') \tag{4.1.42}$$

has no sign for passing f through g' because if they both were of degree -1 then the truncation would kill this composite.

(iv) The cochain units are simply given by mapping $\mathbb{1} \in \mathbb{K} \subseteq R$ to $1_U \boxtimes 1_{U'} = 1_U \otimes_{\mathbb{R}} 1_{U'} =: 1_{U\,U'}$.

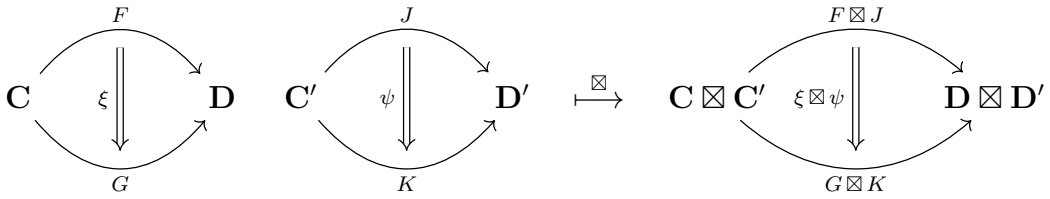
This data clearly satisfies the axioms of associativity and unitality required of a $\mathbf{Ch}_R^{[-1,0]}$-category. ▷

The $\mathbf{Ch}_R^{[-1,0]}$-functor $F \boxtimes J : \mathbf{C} \boxtimes \mathbf{C}' \to \mathbf{D} \boxtimes \mathbf{D}'$ acts simply as:

$$\begin{aligned} U\,U' &\mapsto F(U)J(U') \quad , \\ f \boxtimes f' &\mapsto F(f) \boxtimes J(f') \quad . \end{aligned}$$

The latter assignments are clearly functorial because of the above defined composition/units of the monoidal composite $\mathbf{Ch}_R^{[-1,0]}$-category. The latter assignments are also clearly R-linear cochain because both of F and J are.

Construction 4.1.24. The **monoidal composite pseudonatural transformation**



(4.1.43a)

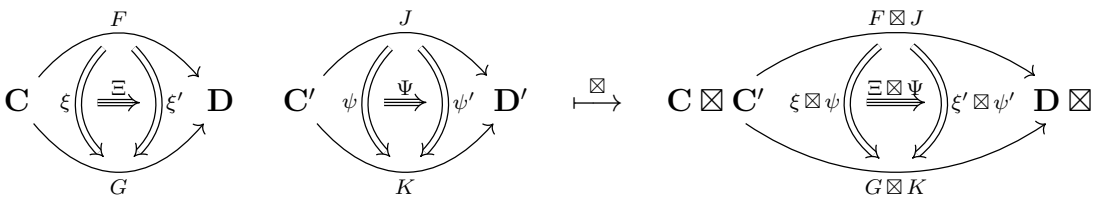
in the tricategory $\mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}$ is defined by:

$$(\xi \boxtimes \psi)_{U\,U'} := \xi_U \boxtimes \psi_{U'} \quad , \tag{4.1.43b}$$

$$(\xi \boxtimes \psi)_{f \boxtimes f'} := \xi_f \boxtimes K(f')\psi_{U'} + \xi_V F(f) \boxtimes \psi_{f'} \quad . \tag{4.1.43c}$$

Note that for two $\mathbf{Ch}_R^{[-1,0]}$-natural transformations, ξ and ψ, the monoidal composite, $\xi \boxtimes \psi$, agrees with the usual one from enriched category theory. ▷

Construction 4.1.25. The **monoidal composite modification**

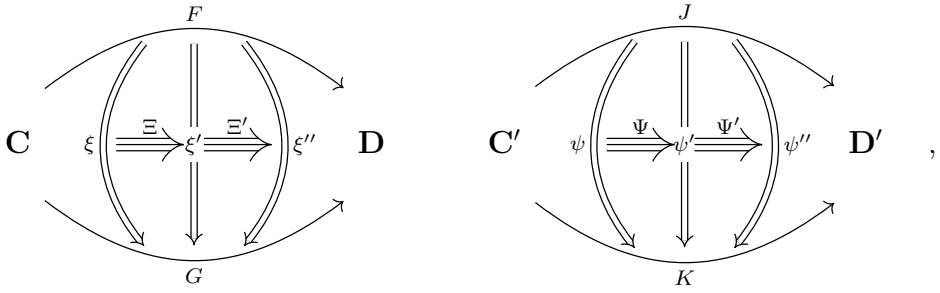


(4.1.44a)

in the tricategory $\mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}$ is defined by

$$(\Xi \boxtimes \Psi)_{U\,U'} := \Xi_U \;\boxtimes\; \psi_{U'} + \xi'_U \;\boxtimes\; \Psi_{U'} \quad . \tag{4.1.44b}$$

Local bifunctoriality requires that lateral identity/composition of modifications is preserved by monoidal composition, i.e., given



,

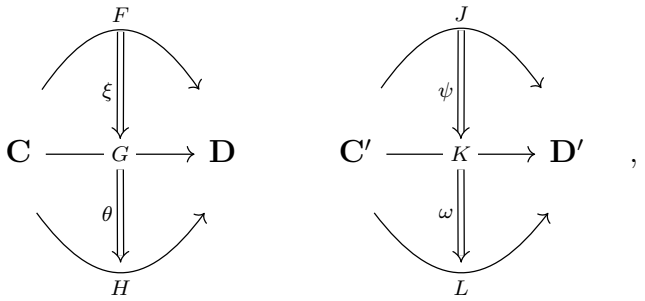
we have $(\cdot, \boxtimes)$-exchange,

$$(\Xi' \cdot \Xi) \boxtimes (\Psi' \cdot \Psi) = (\Xi' \boxtimes \Psi') \cdot (\Xi \boxtimes \Psi) \quad ,$$

and

$$\mathrm{ID}_\xi \boxtimes \mathrm{ID}_\psi = \mathrm{ID}_{\xi \boxtimes \psi} \quad .$$

As with the horizontal composition $*$, the monoidal composition $\boxtimes$ is strictly unitary, i.e., we have $\mathrm{Id}_F \boxtimes \mathrm{Id}_J = \mathrm{Id}_{F \boxtimes J}$. In fact, given pseudonatural transformations

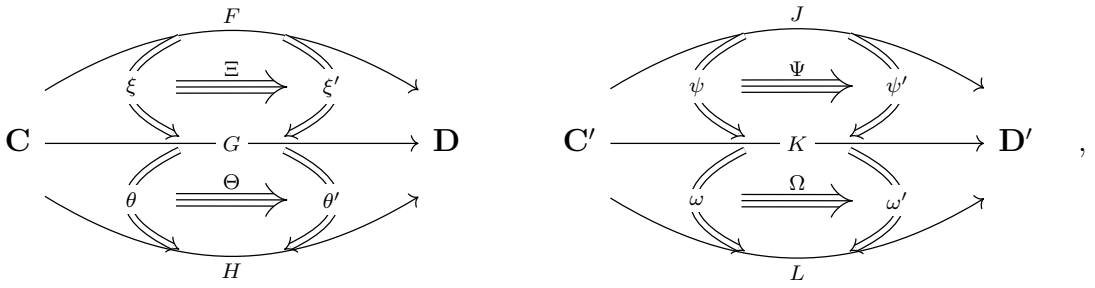


,

we have

$$(\theta\xi) \boxtimes (\omega\psi) = (\theta \boxtimes \omega)(\xi \boxtimes \psi) \quad .$$

In other words, the pseudo composition constraint is trivial[9] and the naturality condition translates into yet again another exchange law, namely, given



,

we have

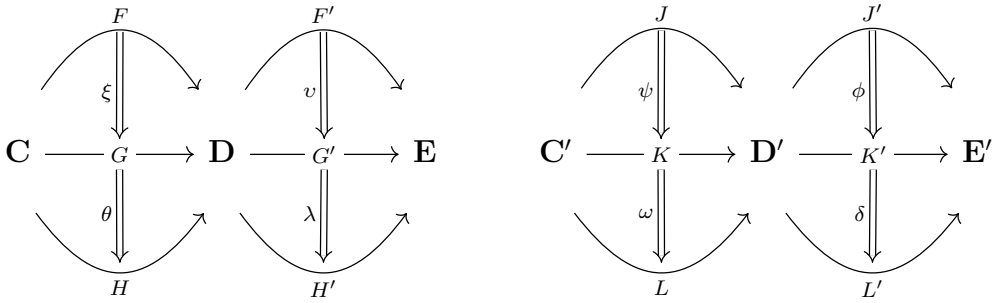
$$\Theta\Xi \boxtimes \Omega\Psi = (\Theta \boxtimes \Omega)(\Xi \boxtimes \Psi) \quad .$$

In summary, the $(\circ, \boxtimes)$-exchange law is upheld.

The preservation of horizontal composition means that the $(*, \boxtimes)$-exchange law is upheld (for all three different levels of morphisms) as well as there being equality between the relevant pseudo

[9]Meaning that we need not prove that the associativity axiom of the pseudo composition constraint is upheld.

composition constraints, i.e., given pseudonatural transformations



the following diagram commutes

$$\begin{array}{ccc}
\big[(\lambda * \theta) \boxtimes (\delta * \omega)\big]\big[(\upsilon * \xi) \boxtimes (\phi * \psi)\big] & = & \big[(\lambda \boxtimes \delta) * (\theta \boxtimes \omega)\big]\big[(\upsilon \boxtimes \phi) * (\xi \boxtimes \psi)\big] \\
\| & & \Downarrow *^2_{(\lambda\boxtimes\delta,\theta\boxtimes\omega),(\upsilon\boxtimes\phi,\xi\boxtimes\psi)} \\
(\lambda * \theta)(\upsilon * \xi) \boxtimes (\delta * \omega)(\phi * \psi) & & (\lambda \boxtimes \delta)(\upsilon \boxtimes \phi) * (\theta \boxtimes \omega)(\xi \boxtimes \psi) \\
\Downarrow *^2_{(\lambda,\theta),(\upsilon,\xi)} \boxtimes *^2_{(\delta,\omega),(\phi,\psi)} & & \| \\
(\lambda\upsilon * \theta\xi) \boxtimes (\delta\phi * \omega\psi) & = & (\lambda\upsilon \boxtimes \delta\phi) * (\theta\xi \boxtimes \omega\psi)
\end{array}$$

Preservation of horizontal identity 2-functor morphisms $\mathrm{id}_{\mathbf{C}} : \mathbf{T} \to \mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}[\mathbf{C},\mathbf{C}]$ means:

$$\mathrm{id}_{\mathbf{C}} \boxtimes \mathrm{id}_{\mathbf{C}'} = \mathrm{id}_{\mathbf{C}\boxtimes\mathbf{C}'} \quad , \tag{4.1.45a}$$

$$\mathrm{Id}_{\mathrm{id}_{\mathbf{C}}} \boxtimes \mathrm{Id}_{\mathrm{id}_{\mathbf{C}'}} = \mathrm{Id}_{\mathrm{id}_{\mathbf{C}\boxtimes\mathbf{C}'}} \quad , \tag{4.1.45b}$$

$$\mathrm{ID}_{\mathrm{Id}_{\mathrm{id}_{\mathbf{C}}}} \boxtimes \mathrm{ID}_{\mathrm{Id}_{\mathrm{id}_{\mathbf{C}'}}} = \mathrm{ID}_{\mathrm{Id}_{\mathrm{id}_{\mathbf{C}\boxtimes\mathbf{C}'}}} \quad , \tag{4.1.45c}$$

but the second relation follows from the first together with the fact of strict unitarity of $\boxtimes$ whereas the third relation follows from the second together with local bifunctoriality of $\boxtimes$.

The monoidal identity $u : \mathcal{T} \to \mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}$ is the $\mathbf{Ch}_R^{[-1,0]}$-category

$$\mathbf{R} \in \mathbf{dgCat}_R^{[-1,0],\mathrm{ps}} \tag{4.1.46}$$

consisting of a single object, say $\star$, with morphisms $\mathbf{R}(\star,\star) := R \in \mathbf{Ch}_R^{[-1,0]}$ given by the monoidal unit of the enrichment category and the cochain composition given by multiplication of scalars (which is associative and unital with respect to $\mathbb{1} \in R$). The pseudofunctor

$$u_\star : \mathbf{T} \to \mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}(\mathbf{R},\mathbf{R})$$

is actually a specific instance of the 2-functors $\mathrm{id}_{\mathbf{C}} : \mathbf{T} \to \mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}[\mathbf{C},\mathbf{C}]$ for the particular case $\mathbf{C} = \mathbf{R}$, i.e., $\mathrm{id}_{\mathbf{R}} = u_\star$. Likewise, preservation of horizontal composition

$$* \diamond (u_\star \times u_\star) = u_\star \tag{4.1.47}$$

is just a special case of $* \diamond (\mathrm{id}_{\mathbf{C}} \times \mathrm{id}_{\mathbf{C}}) = \mathrm{id}_{\mathbf{C}}$ which we already know to hold from the unitality of horizontal composition $*$ in the previous subsubsection. The preservation of horizontal units reduces to the tautology $u_\star = u_\star$.

Remark 4.1.26. We have seen that the monoidal composition $\boxtimes$ is a *3-functor* meaning that

the exchange law and preservation of identities is upheld for all three different levels of morphisms (thus pseudonatural isomorphisms and isomodifications are preserved) in the tricategory $\mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}$, this fact will become relevant, e.g., when discussing the pentagon axiom holding at order h^2 in Lemma 5.2.13. Note also that this monoidal composition is strictly associative (and unital with respect to u) in large part because of the truncation but also because we regard $\mathbf{Ch}_R^{[-1,0]}$ as a strict monoidal category. The symmetric braiding τ has components $\tau_{\mathbf{C},\mathbf{C}'} : \mathbf{C} \boxtimes \mathbf{C}' \cong \mathbf{C}' \boxtimes \mathbf{C}$ as the obvious $\mathbf{Ch}_R^{[-1,0]}$-involution. This family of components is clearly natural and satisfies the hexagon axiom

$$\tau_{\mathbf{C},(\mathbf{C}' \boxtimes \mathbf{C}'')} = (\mathrm{id}_{\mathbf{C}'} \boxtimes \tau_{\mathbf{C},\mathbf{C}''})(\tau_{\mathbf{C},\mathbf{C}'} \boxtimes \mathrm{id}_{\mathbf{C}''}) \quad . \tag{4.1.48}$$

△

4.2 Infinitesimal 2-braidings

This section presents material in the opposite direction to that of [KLS25, Subsection 3.1]. That is, rather than starting with a first-order deformation of the symmetric braiding [KLS25, (3.4)] and then proceeding to derive the imposed relations on the pseudonatural endomorphism $t : \otimes \Rightarrow \otimes$, we start with the definition of a hexagonal infinitesimal 2-braiding and then proceed to show, in the proof to Theorem 4.2.7, that such a condition is necessary and sufficient for the deformed braiding to satisfy the strict hexagon axioms (4.2.4).

Definition 4.2.1. A **pseudonaturally-braided strict monoidal $\mathbf{Ch}_R^{[-1,0]}$-category** $(\mathbf{C}, \otimes, I, \sigma)$ consists of the following four pieces of data:

(i) $\mathbf{C} \in \mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}$.

(ii) A $\mathbf{Ch}_R^{[-1,0]}$-functor $\otimes : \mathbf{C} \boxtimes \mathbf{C} \to \mathbf{C}$ called the **monoidal product** whose object map is denoted by bracketing the juxtaposition, i.e.,

$$U \times V \overset{\boxtimes}{\longmapsto} UV \overset{\otimes}{\longmapsto} (UV) \quad . \tag{4.2.1a}$$

Again, the truncation of the homs (and the fact that $\otimes$ preserves degrees) means that the exchange never incurs a sign, i.e.,

$$(f' \otimes g')(f \otimes g) = (f'f) \otimes (g'g) \quad . \tag{4.2.1b}$$

(iii) A $\mathbf{Ch}_R^{[-1,0]}$-functor $\eta : \mathbf{R} \to \mathbf{C}$ called the **monoidal unit** given by $\eta(\star) = I \in \mathbf{C}$.

(iv) A pseudonatural isomorphism $\sigma : \otimes \Rightarrow \otimes \tau_{\mathbf{C},\mathbf{C}}$ called the **braiding**.

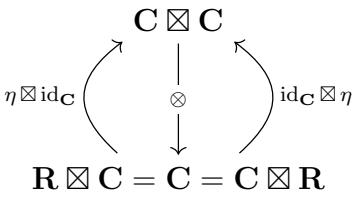
This data must satisfy the following three axioms:

1. **Associativity** $\otimes(\otimes \boxtimes \mathrm{id}_{\mathbf{C}}) = \otimes(\mathrm{id}_{\mathbf{C}} \boxtimes \otimes)$. In terms of objects and morphisms, this reads:

$$((UV)W) =: (UVW) := (U(VW)) \qquad , \qquad (f \otimes g) \otimes k =: f \otimes g \otimes k := f \otimes (g \otimes k)\,. \tag{4.2.2}$$

2. **Unitality**, i.e., the monoidal product $\otimes$ is a retraction of both the left unit $\eta \boxtimes \mathrm{id}_{\mathbf{C}}$ and the

right unit $\mathrm{id}_{\mathbf{C}} \boxtimes \eta$,



$$\otimes(\eta \boxtimes \mathrm{id}_{\mathbf{C}}) = \mathrm{id}_{\mathbf{C}} = \otimes(\mathrm{id}_{\mathbf{C}} \boxtimes \eta) \quad . \tag{4.2.3a}$$

In terms of objects and morphisms, this demands:

$$(IU) = U = (UI) \qquad , \qquad 1_I \otimes f = f = f \otimes 1_I \quad . \tag{4.2.3b}$$

3. **Strict left hexagon axiom**,

$$\sigma_{\mathrm{id}_{\mathbf{C}} \boxtimes \otimes} \equiv \otimes\left([1 \boxtimes \sigma]_{\tau_{\mathbf{C},\mathbf{C}} \boxtimes \mathrm{id}_{\mathbf{C}}}[\sigma \boxtimes 1]\right) \quad . \tag{4.2.4a}$$

4. **Strict right hexagon axiom**,

$$\sigma_{\otimes \boxtimes \mathrm{id}_{\mathbf{C}}} \equiv \otimes\left([\sigma \boxtimes 1]_{\mathrm{id}_{\mathbf{C}} \boxtimes \tau_{\mathbf{C},\mathbf{C}}}[1 \boxtimes \sigma]\right) \quad . \tag{4.2.4b}$$

Remark 4.2.2. If $\gamma : \otimes \Rightarrow \otimes\tau_{\mathbf{C},\mathbf{C}}$ is a $\mathbf{Ch}_R^{[-1,0]}$-natural **involution**[10], then the involutive property of γ together with the functoriality of $\otimes$ means that γ satisfies the strict left hexagon axiom (4.2.4a),

$$\gamma_{U(VW)} = (1_V \otimes \gamma_{UW})(\gamma_{UV} \otimes 1_W) \quad , \tag{4.2.6a}$$

if and only if it satisfies the strict right hexagon axiom (4.2.4b),

$$\gamma_{(UV)W} = (\gamma_{UW} \otimes 1_V)(1_U \otimes \gamma_{VW}) \quad , \tag{4.2.6b}$$

in which case we call $(\mathbf{C}, \otimes, I, \gamma)$ a **symmetric strict monoidal $\mathbf{Ch}_R^{[-1,0]}$-category**. △

Definition 4.2.3. An **infinitesimal 2-braiding** on a symmetric monoidal $\mathbf{Ch}_R^{[-1,0]}$-category $(\mathbf{C}, \otimes, I, \gamma)$ is a pseudonatural endomorphism $t : \otimes \Rightarrow \otimes : \mathbf{C} \boxtimes \mathbf{C} \to \mathbf{C}$. We say it is **hexagonal** if it satisfies both the **left infinitesimal hexagon relation**,

$$t_{\mathrm{id}_{\mathbf{C}} \boxtimes \otimes} \equiv \otimes(t \boxtimes 1) + \otimes\left([\gamma^{-1} \boxtimes 1](1 \boxtimes t)_{\tau_{\mathbf{C},\mathbf{C}} \boxtimes \mathrm{id}_{\mathbf{C}}}[\gamma \boxtimes 1]\right) \quad , \tag{4.2.7a}$$

and the **right infinitesimal hexagon relation**,

$$t_{\otimes \boxtimes \mathrm{id}_{\mathbf{C}}} \equiv \otimes(1 \boxtimes t) + \otimes\left([1 \boxtimes \gamma^{-1}](t \boxtimes 1)_{\mathrm{id}_{\mathbf{C}} \boxtimes \tau_{\mathbf{C},\mathbf{C}}}[1 \boxtimes \gamma]\right) \quad . \tag{4.2.7b}$$

As in [CFM15, Remark 16], we say a *hexagonal* infinitesimal 2-braiding to contrast with the more general scenario of (4.2.7a) and (4.2.7b) being replaced with modifications which satisfy higher coherence conditions[11]. We wish to keep the adjective "hexagonal" rather than absorb it into the name of "infinitesimal 2-braiding" because it will be interesting throughout to note which lemmas and propositions depend on this property and which do not.

[10] That is, a $\mathbf{Ch}_R^{[-1,0]}$-natural isomorphism which further satisfies

$$\gamma_{\tau_{\mathbf{C},\mathbf{C}}}\gamma = \mathrm{Id}_{\otimes} \qquad \Longrightarrow \qquad \gamma_{VU}\gamma_{UV} = 1_{(UV)} \quad . \tag{4.2.5}$$

[11] We will see discussion of hexagonators become relevant at *second* order in the deformation parameter h.

Remark 4.2.4. Reading (4.2.7a) as an equality between the cochain components we reproduce the familiar left infinitesimal hexagon relation, namely for $U, V, W \in \mathbf{C}$,

$$t_{U(VW)} = t_{UV} \otimes 1_W + (\gamma_{VU} \otimes 1_W)(1_V \otimes t_{UW})(\gamma_{UV} \otimes 1_W) \tag{4.2.8a}$$

and as an equality between the homotopy components, for $f \in \mathbf{C}[U, U']^0$, $g \in \mathbf{C}[V, V']^0$ and $k \in \mathbf{C}[W, W']^0$, we find[12] something analogous

$$t_{f \boxtimes (g \otimes k)} = t_{f \boxtimes g} \otimes k + (\gamma_{V'U'} \otimes 1_{W'})(g \otimes t_{f \boxtimes k})(\gamma_{UV} \otimes 1_W) \quad . \tag{4.2.8b}$$

Likewise, reading (4.2.7b) as an equality between the cochain components we reproduce the familiar right infinitesimal hexagon relation,

$$t_{(UV)W} = 1_U \otimes t_{VW} + (1_U \otimes \gamma_{WV})(t_{UW} \otimes 1_V)(1_U \otimes \gamma_{VW}) \tag{4.2.9a}$$

and as an equality between the homotopy components, we find

$$t_{(f \otimes g) \boxtimes k} = f \otimes t_{g \boxtimes k} + (1_{U'} \otimes \gamma_{W'V'})(t_{f \boxtimes k} \otimes g)(1_U \otimes \gamma_{VW}) \quad . \tag{4.2.9b}$$

△

Definition 4.2.5. An infinitesimal 2-braiding $\otimes \overset{t}{\Rightarrow} \otimes$ is said to be γ**-equivariant** if it intertwines with the symmetric braiding γ, i.e.,

$$\gamma\, t \;\equiv\; t_{\tau_{\mathbf{C},\mathbf{C}}\gamma} \quad . \tag{4.2.10}$$

Plainly:

$$\gamma_{UV} t_{UV} = t_{VU} \gamma_{UV} \quad , \tag{4.2.11a}$$

$$\gamma_{U'V'} t_{f \boxtimes g} = t_{g \boxtimes f} \gamma_{UV} \quad . \tag{4.2.11b}$$

Lemma 4.2.6. *If an infinitesimal 2-braiding $\otimes \overset{t}{\Rightarrow} \otimes$ is γ-equivariant then* (4.2.7a) $\Longleftrightarrow$ (4.2.7b).

Proof. We simply show that (4.2.8a) $\Longrightarrow$ (4.2.9a),

$$\begin{aligned} t_{(UV)W} =& \gamma_{W(UV)} t_{W(UV)} \gamma_{(UV)W} \\ =& (1_U \otimes \gamma_{WV})(\gamma_{WU} \otimes 1_V)\left[t_{WU} \otimes 1_V + (\gamma_{UW} \otimes 1_V)(1_U \otimes t_{WV})(\gamma_{WU} \otimes 1_V)\right](\gamma_{UW} \otimes 1_V)(1_U \otimes \gamma_{VW}) \\ =& 1_U \otimes t_{VW} + (1_U \otimes \gamma_{WV})(t_{UW} \otimes 1_V)(1_U \otimes \gamma_{VW}) \quad , \end{aligned}$$

where: the first equality comes from the involutivity of γ and the γ-equivariance (4.2.11a), the second equality comes from the hexagon axiom that γ satisfies (as in Remark 4.2.2) together with the cochain components of the left infinitesimal hexagon relation (4.2.8a) and the last equality uses the functoriality of $\otimes$ together with, again, the involutivity of γ and the γ-equivariance (4.2.11a). Analogous arguments show (4.2.8b) $\Longrightarrow$ (4.2.9b) upon using the homotopy components of the γ-equivariance (4.2.11b). Showing (4.2.9) $\Longrightarrow$ (4.2.8) works the same way. □

Theorem 4.2.7. *An infinitesimal 2-braiding t on a symmetric monoidal $\mathbf{Ch}_{\mathbb{K}}^{[-1,0]}$-category $(\mathbf{C}, \otimes, I, \gamma)$ integrates to a pseudonaturally-braided strict monoidal $\mathbf{Ch}_{\mathbb{K}[h]/(h^2)}^{[-1,0]}$-category as in Definition 4.2.1,*

$$\left(\mathbf{C}_{\mathbb{K}[h]/(h^2)}, \otimes, I, \sigma :\equiv \gamma[1 + \tfrac{h}{2}t]\right) \quad , \tag{4.2.12}$$

if and only if t is a hexagonal infinitesimal 2-braiding as in Definition 4.2.3.

[12] Recall that γ is a $\mathbf{Ch}_R^{[-1,0]}$-natural transformation hence its homotopy components are trivial, i.e., $\gamma_{f \boxtimes g} = 0$.

Proof. Let us show that the order h relation of the strict left hexagon axiom (4.2.4a) is equivalent to the left infinitesimal hexagon relation (4.2.7a). The order h relation of the strict left hexagon axiom (4.2.4a) reads as

$$(\gamma t)_{\mathrm{id}_{\mathbf{C}}\boxtimes\otimes} \equiv \otimes\big([1\boxtimes\gamma]_{\tau_{\mathbf{C},\mathbf{C}}\boxtimes\mathrm{id}_{\mathbf{C}}}(\gamma t\boxtimes 1)\big)+\otimes\big([1\boxtimes\gamma t]_{\tau_{\mathbf{C},\mathbf{C}}\boxtimes\mathrm{id}_{\mathbf{C}}}(\gamma\boxtimes 1)\big) \quad . \tag{4.2.13a}$$

We use Remark 4.1.22 and the fact that $\boxtimes$ is a 3-functor (as in Remark 4.1.26) to rewrite this as

$$\gamma_{\mathrm{id}_{\mathbf{C}}\boxtimes\otimes}\, t_{\mathrm{id}_{\mathbf{C}}\boxtimes\otimes} \equiv \otimes\big([1\boxtimes\gamma]_{\tau_{\mathbf{C},\mathbf{C}}\boxtimes\mathrm{id}_{\mathbf{C}}}(\gamma\boxtimes 1)(t\boxtimes 1)\big)+\otimes\big([1\boxtimes\gamma]_{\tau_{\mathbf{C},\mathbf{C}}\boxtimes\mathrm{id}_{\mathbf{C}}}[1\boxtimes t]_{\tau_{\mathbf{C},\mathbf{C}}\boxtimes\mathrm{id}_{\mathbf{C}}}(\gamma\boxtimes 1)\big) \quad . \tag{4.2.13b}$$

Using Remark 4.1.22 again, we rewrite the RHS as

$$\otimes\big([1\boxtimes\gamma]_{\tau_{\mathbf{C},\mathbf{C}}\boxtimes\mathrm{id}_{\mathbf{C}}}(\gamma\boxtimes 1)\big)\otimes(t\boxtimes 1)+\otimes[1\boxtimes\gamma]_{\tau_{\mathbf{C},\mathbf{C}}\boxtimes\mathrm{id}_{\mathbf{C}}}\otimes\big([1\boxtimes t]_{\tau_{\mathbf{C},\mathbf{C}}\boxtimes\mathrm{id}_{\mathbf{C}}}(\gamma\boxtimes 1)\big) \quad . \tag{4.2.13c}$$

Finally, we use the fact that γ satisfies the strict left hexagon axiom

$$\gamma_{\mathrm{id}_{\mathbf{C}}\boxtimes\otimes} \equiv \otimes\big([1\boxtimes\gamma]_{\tau_{\mathbf{C},\mathbf{C}}\boxtimes\mathrm{id}_{\mathbf{C}}}(\gamma\boxtimes 1)\big) \equiv \otimes[1\boxtimes\gamma]_{\tau_{\mathbf{C},\mathbf{C}}\boxtimes\mathrm{id}_{\mathbf{C}}}\otimes(\gamma\boxtimes 1) \tag{4.2.13d}$$

together with functoriality of whiskering (Remark 4.1.22). Analogously, one shows that the strict right hexagon axiom (4.2.4b) is equivalent to the right infinitesimal hexagon relation (4.2.7b). □

Remark 4.2.8. The braiding $\sigma :\equiv \gamma[1+\frac{h}{2}t]$ is not necessarily involutive because, modulo h^2,

$$\sigma_{VU}\sigma_{UV}-1_{(UV)} = (\gamma_{VU}+\tfrac{h}{2}\gamma_{VU}t_{VU})(\gamma_{UV}+\tfrac{h}{2}\gamma_{UV}t_{UV})-1_{(UV)} = \tfrac{h}{2}(t_{UV}+\gamma_{VU}t_{VU}\gamma_{UV}) \tag{4.2.14a}$$

is, in general, non-zero unless t is **γ-antivariant**, i.e.,

$$t+(\gamma\, t)_{\tau_{\mathbf{C},\mathbf{C}}}\gamma = 0 \quad . \tag{4.2.14b}$$

In particular, if t is a nontrivial γ-equivariant hexagonal infinitesimal 2-braiding then the deformed braiding is necessarily *not* involutive. △

Remark 4.2.9. Pseudonaturality of t, specifically (4.1.14b), deforms the left 4-term relations as

$$(t_{UV}\otimes 1_W)t_{(UV)W}-t_{(UV)W}(t_{UV}\otimes 1_W) = \partial t_{t_{UV}\boxtimes 1_W} \quad . \tag{4.2.15}$$

The 4-term relations, $[t_{UV}\otimes 1_W, t_{(UV)W}]=0=[1_U\otimes t_{VW}, t_{U(VW)}]$, are essential for a deformation quantisation to all orders in the ordinary problem of 1-categorical Cartier integration, see [Car93], [Kas95, Theorem XX.6.1] and [BRW23, Section 2]. △

4.2.1 Homotopy 2-categories of symmetric monoidal dg-categories

Let R be a commutative $\mathbb{K}$-algebra and denote by $\mathbf{dgCat}_R$ the 2-category of $\mathbf{Ch}_R$-categories, $\mathbf{Ch}_R$-functors and $\mathbf{Ch}_R$-natural transformations. We shall regard $\mathbf{dgCat}_R$ as a 3-category (with trivial 3-morphisms) in the sense of Definition 4.1.2. The lax symmetric monoidal truncation functor $\tau^{[-1,0]}:\mathbf{Ch}_R\to\mathbf{Ch}_R^{[-1,0]}$ from Lemma 4.1.4 induces a change of base 3-functor

$$\mathsf{Ho}_2 : \mathbf{dgCat}_R \longrightarrow \mathbf{dgCat}_R^{[-1,0]} \subseteq \mathbf{dgCat}_R^{[-1,0],\mathrm{ps}} \quad , \tag{4.2.16}$$

which sends $\mathbf{Ch}_R$-natural transformations to $\mathbf{Ch}_R^{[-1,0]}$-natural transformations. Observe that, at the level of objects, this 3-functor assigns to a $\mathbf{Ch}_R$-category $\mathbf{C}\in\mathbf{dgCat}_R$, with objects $\mathbf{C}_0$ and morphism cochain complexes $\mathbf{C}[U,U']\in\mathbf{Ch}_R$, its homotopy 2-category $\mathsf{Ho}_2(\mathbf{C})\in\mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}$, i.e., the $\mathbf{Ch}_R^{[-1,0]}$-category with the same objects $\mathbf{C}_0$ and morphisms given by the truncated cochain

complexes $\tau^{[-1,0]}\big(\mathbf{C}[U,U']\big) \in \mathbf{Ch}_R^{[-1,0]}$.

Since the enrichment category $\mathbf{Ch}_R$ of cochain complexes of R-modules is symmetric monoidal with respect to the relative tensor product $\otimes_R$, there exists an induced symmetric monoidal structure on $\mathbf{dgCat}_R$, see, e.g., [Kel82, Section 1.4]. The 3-functor (4.2.16) assigning homotopy 2-categories is lax symmetric monoidal with the lax structure $\mathsf{Ho}_2(\mathbf{C}) \boxtimes \mathsf{Ho}_2(\mathbf{D}) \to \mathsf{Ho}_2(\mathbf{C} \boxtimes \mathbf{D})$ induced from the lax structure of the good truncation functor in Lemma 4.1.4, for all $\mathbf{C}, \mathbf{D} \in \mathbf{dgCat}_R$. For later use, we record the following direct consequence of this result.

Corollary 4.2.10. *Let $(\mathbf{C}, \otimes, I, \gamma)$ be a symmetric monoidal $\mathbf{Ch}_R$-category. Then the homotopy 2-category $\mathsf{Ho}_2(\mathbf{C}) \in \mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}$ is canonically a symmetric monoidal $\mathbf{Ch}_R^{[-1,0]}$-category.*

Recall the symmetric monoidal $\mathbf{Ch}_{\mathbb{K}}$-category ${}_A\mathbf{dgMod}^{\mathrm{fgsf}}$ of fgsf dg-modules over a fg sfCDGA A over the field $\mathbb{K}$. Our starting point is thus given by the homotopy 2-category

$$ {}_A\mathbf{C} \;:=\; \mathsf{Ho}_2\big({}_A\mathbf{dgMod}^{\mathrm{fgsf}}\big) \;\in\; \mathbf{dgCat}_{\mathbb{K}}^{[-1,0],\mathrm{ps}} \tag{4.2.17} $$

which we endow with the symmetric monoidal structure that is induced from the standard one on ${}_A\mathbf{dgMod}^{\mathrm{fgsf}}$ given in (2.2.30) and (2.2.31). We then simply strictify this symmetric monoidal structure so as to work with a symmetric strict monoidal $\mathbf{Ch}_{\mathbb{K}}^{[-1,0]}$-category.

The effect of forming the homotopy 2-category is that the cochain complex $\underline{\hom}_A(U,V) \in \mathbf{Ch}_{\mathbb{K}}$ of morphisms between any two dg-modules $U, V \in {}_A\mathbf{dgMod}^{\mathrm{fgsf}}$ from (2.2.29) gets replaced by its good truncation ${}_A\mathbf{C}(U,V) := \tau^{[-1,0]}\big(\underline{\hom}_A(U,V)\big) \in \mathbf{Ch}_{\mathbb{K}}^{[-1,0]}$ to degrees -1 and 0. Concretely, this means that the homotopy 2-category ${}_A\mathbf{C} = \mathsf{Ho}_2\big({}_A\mathbf{dgMod}^{\mathrm{fgsf}}\big)$ describes only degree 0 morphisms between A-dg-modules which commute with the differentials and 1-homotopies (modulo 2-homotopies) between such morphisms, but it does not include information about higher homotopies. Our motivation behind this truncation of the dg-category ${}_A\mathbf{dgMod}^{\mathrm{fgsf}}$ is that the homotopy 2-category ${}_A\mathbf{C} = \mathsf{Ho}_2\big({}_A\mathbf{dgMod}^{\mathrm{fgsf}}\big)$ is a 2-categorical object whose deformations can be described very explicitly in the context of the tricategory $\mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}$ from Proposition 4.1.5. Most importantly, every tower of homotopy coherence data will automatically truncate beyond the level of 1-homotopies.

4.2.2 Construction from 2-shifted Poisson structures

Suppose that $\pi = \sum_{m\geq 2} \pi^{(m)} \in \widehat{\mathrm{Pol}}(A,2)^4$ is a 2SPS (recall Definition 2.3.36) on the fgsfCDGA A. In this subsection, we will present an explicit construction which assigns to this datum a γ-equivariant hexagonal infinitesimal 2-braiding $t : \otimes_A \Rightarrow \otimes_A$ on the homotopy 2-category ${}_A\mathbf{C} = \mathsf{Ho}_2\big({}_A\mathbf{dgMod}^{\mathrm{fgsf}}\big) \in \mathbf{dgCat}_{\mathbb{K}}^{[-1,0],\mathrm{ps}}$. We will illustrate later in Section 4.3 that our construction is a non-trivial homotopical generalisation of the standard construction of an infinitesimal braiding on the representation category $\mathbf{Rep}_{\mathbb{K}}\mathfrak{g}$ of a Lie algebra $\mathfrak{g}$ from an invariant symmetric tensor in $(\mathrm{Sym}^2\mathfrak{g})^{\mathfrak{g}}$, see, e.g., [Car93] and [Kas95, Chapter XX].

Our infinitesimal 2-braiding t is built as a (vertical) composite of more primitive pseudonatural transformations which we are going to define now. Let us denote by

$$ L_{\Omega_A[1]} := \Omega_A[1] \otimes_A (\,\cdot\,) \;:\; {}_A\mathbf{C} \;\longrightarrow\; {}_A\mathbf{C} \tag{4.2.18} $$

the $\mathbf{Ch}_{\mathbb{K}}^{[-1,0]}$-functor of taking the relative tensor product with the 1-shift $\Omega_A[1] \in {}_A\mathbf{C}$ of the dg-module of Kähler differentials from (2.3.8). We define a pseudonatural transformation

$$ \xi \;:\; \mathrm{id}_{{}_A\mathbf{C}} \;\Longrightarrow\; L_{\Omega_A[1]} \;:\; {}_A\mathbf{C} \;\longrightarrow\; {}_A\mathbf{C} \tag{4.2.19a} $$

by specifying its components as in Definition 4.1.6: For each object $U \in {}_A\mathbf{C}$, the component

$$\big(\xi_U : U \to \Omega_A[1] \otimes_A U\big) \;\in\; {}_A\mathbf{C}\big[U, \Omega_A[1] \otimes_A U\big]^0 \tag{4.2.19b}$$

is the A-linear cochain map which is defined on generators $\mathbb{1} \otimes_{\mathbb{K}} u \in U^\sharp \cong A^\sharp \otimes_{\mathbb{K}} U_0^\sharp$ of U by

$$\xi_U\big(\mathbb{1} \otimes_{\mathbb{K}} u\big) \;:=\; s^{-1}\,(\mathrm{d}_{\mathrm{dR}} \otimes_{\mathbb{K}} 1_{U_0})\big(\mathrm{d}_U(\mathbb{1} \otimes_{\mathbb{K}} u)\big) \quad , \tag{4.2.19c}$$

where we recall our notation from Subsection 2.2.1 that $s^{-1} \in \mathbb{K}[1]$ denotes the degree -1 element represented by the unit of $\mathbb{K}$. For each A-linear map $f \in {}_A\mathbf{C}[U, U']$, the component

$$\big(\xi_f : U \to \Omega_A[1] \otimes_A U'\big) \;\in\; {}_A\mathbf{C}\big[U, \Omega_A[1] \otimes_A U'\big]^{|f|-1} \tag{4.2.19d}$$

is defined on the generators by

$$\xi_f\big(\mathbb{1} \otimes_{\mathbb{K}} u\big) \;:=\; -s^{-1}\,(\mathrm{d}_{\mathrm{dR}} \otimes_{\mathbb{K}} 1_{U_0'})\big(f(\mathbb{1} \otimes_{\mathbb{K}} u)\big) \quad . \tag{4.2.19e}$$

Lemma 4.2.11. *The components* (4.2.19) *define a pseudonatural transformation*

$$\xi : \mathrm{id}_{{}_A\mathbf{C}} \Rightarrow L_{\Omega_A[1]} \quad .$$

Proof. Before verifying properties (4.1.12a) and (4.1.12b) of Definition 4.1.6, let us first comment on the well-definedness of the components of ξ. Concerning (i) from Definition 4.1.6, we have to show that $\partial\xi_U := \mathrm{d}_{\Omega_A[1]\otimes_A U}\,\xi_U - \xi_U\,\mathrm{d}_U = 0$, i.e., that ξ_U commutes with the differentials, which can be verified by picking a basis $\{u_i \in U_0^\sharp\}$ and evaluating on the associated generators $\mathbb{1} \otimes_{\mathbb{K}} u_i \in U$. This allows us to write $\mathrm{d}_U(\mathbb{1} \otimes_{\mathbb{K}} u_i) =: \underline{U_i}^j \otimes_{\mathbb{K}} u_j$ and hence $\xi_U(\mathbb{1} \otimes_{\mathbb{K}} u_i) = s^{-1}\,\mathrm{d}_{\mathrm{dR}}(\underline{U_i}^j) \otimes_{\mathbb{K}} u_j$, with $\underline{U_i}^j \in A$ and summations over repeated indices understood. We then compute

$$(\partial\xi_U)\big(\mathbb{1} \otimes_{\mathbb{K}} u_i\big) = -s^{-1}\,\big[\mathrm{d}_{\mathrm{dR}}\big(\delta(\underline{U_i}^j)\big) \otimes_{\mathbb{K}} u_j + (-1)^{|\underline{U_i}^j|}\,\mathrm{d}_{\mathrm{dR}}\big(\underline{U_i}^j\,\underline{U_j}^k\big) \otimes_{\mathbb{K}} u_k\big] = 0\,, \tag{4.2.20}$$

where the last equality follows from expressing the square-zero condition $\mathrm{d}_U\mathrm{d}_U(\mathbb{1} \otimes_{\mathbb{K}} u_i) = 0$ in our index notation. Concerning (ii) from Definition 4.1.6, we note that ξ_f as defined in (4.2.19) is clearly $\mathbb{K}$-linear in the argument f.

To verify (4.1.12a), we use also the index notations $f(\mathbb{1} \otimes_{\mathbb{K}} u_i) =: \underline{f_i}^j \otimes_{\mathbb{K}} u'_j$ and $\mathrm{d}_{U'}(\mathbb{1} \otimes_{\mathbb{K}} u'_i) =: \underline{U'_i}^{\,j} \otimes_{\mathbb{K}} u'_j$, where $\{u'_i \in U_0'^\sharp\}$ is a choice of basis for the generators of U', and compute

$$\begin{aligned}
&\big(1_{\Omega_A[1]} \otimes_A f\big)\big(\xi_U(\mathbb{1} \otimes_{\mathbb{K}} u_i)\big) - \xi_{U'}\big(f(\mathbb{1} \otimes_{\mathbb{K}} u_i)\big) - \xi_{\partial f}(\mathbb{1} \otimes_{\mathbb{K}} u_i) \\
&\quad= s^{-1}\Big(\mathrm{d}_{\mathrm{dR}}\big(\delta(\underline{f_i}^j)\big) \otimes_{\mathbb{K}} u'_j + (-1)^{|\underline{f_i}^j|}\,\mathrm{d}_{\mathrm{dR}}(\underline{f_i}^j)\,\underline{U'_j}^{\,k} \otimes_{\mathbb{K}} u'_k - (-1)^{|f||s\underline{U_i}^j|}\,\underline{U_i}^j\,\mathrm{d}_{\mathrm{dR}}(\underline{f_j}^k) \otimes_{\mathbb{K}} u'_k\Big) \\
&\quad= (\partial\xi_f)(\mathbb{1} \otimes_{\mathbb{K}} u_i) \quad ,
\end{aligned} \tag{4.2.21}$$

It remains to verify (4.1.12b). Given $f \in {}_A\mathbf{C}[U, U']^0$ and $f' \in {}_A\mathbf{C}[U', U'']^0$, we introduce one more index notation $f'(\mathbb{1} \otimes_{\mathbb{K}} u'_i) =: \underline{f'_i}^{\,j} \otimes_{\mathbb{K}} u''_j$, where $\{u''_i \in U_0''^\sharp\}$ is a choice of basis for the generators of U'', and compute

$$\begin{aligned}
\xi_{f'f}\big(\mathbb{1} \otimes_{\mathbb{K}} u_i\big) \;&=\; -s^{-1}\,\mathrm{d}_{\mathrm{dR}}\big(\underline{f_i}^j\,\underline{f'_j}^{\,k}\big) \otimes_{\mathbb{K}} u''_k \\
&=\; -s^{-1}\Big(\mathrm{d}_{\mathrm{dR}}(\underline{f_i}^j)\,\underline{f'_j}^{\,k} \otimes_{\mathbb{K}} u''_k + \underline{f_i}^j\,\mathrm{d}_{\mathrm{dR}}(\underline{f'_j}^{\,k}) \otimes_{\mathbb{K}} u''_k\Big) \\
&=\; \xi_{f'}\big(f(\mathbb{1} \otimes_{\mathbb{K}} u_i)\big) + (1_{\Omega_A[1]} \otimes_A f')\big(\xi_f(\mathbb{1} \otimes_{\mathbb{K}} u_i)\big) \quad .
\end{aligned} \tag{4.2.22}$$

□

In addition to the pseudonatural transformation ξ from (4.2.19), we require two $\mathbf{Ch}_{\mathbb{K}}^{[-1,0]}$-natural transformations to build our infinitesimal 2-braiding. First, using the original symmetric braiding γ, we obtain a family of isomorphisms

$$1_{\Omega_A[1]} \otimes_A \gamma_{U,\Omega_A[1]} \otimes_A 1_V : \Omega_A[1] \otimes_A U \otimes_A \Omega_A[1] \otimes_A V \longrightarrow \Omega_A[1] \otimes_A \Omega_A[1] \otimes_A U \otimes_A V \quad (4.2.23a)$$

in ${}_A\mathbf{C}$, for all $U, V \in {}_A\mathbf{C}$, which defines a $\mathbf{Ch}_{\mathbb{K}}^{[-1,0]}$-natural isomorphism

$$\otimes_A\big(L_{\Omega_A[1]} \boxtimes L_{\Omega_A[1]}\big) \xRightarrow{\cong} L^2_{\Omega_A[1]} \otimes_A \quad . \quad (4.2.23b)$$

Second, consider the extension $\langle \cdot, \cdot \rangle : \mathrm{Sym}^2_A\big(\mathsf{T}_A[-1]\big) \otimes_A \Omega_A[1] \otimes_A \Omega_A[1] \to A$ of the duality pairing (2.3.9) defined by

$$\big\langle sDsD', s^{-1}\omega \otimes_A s^{-1}\omega' \big\rangle := (-1)^{|\omega|+|D|}\big(\langle D, \langle D', \omega\rangle\omega'\rangle - (-1)^{|D||D'|}\langle D', \langle D, \omega\rangle\omega'\rangle\big), \quad (4.2.24)$$

for all homogeneous $D, D' \in \mathsf{T}_A$ and $\omega, \omega' \in \Omega_A$. Considering, in particular, the weight 2 component $\pi^{(2)} \in \mathrm{Sym}^2_A\big(\mathsf{T}_A[-3]\big)^4 \cong \mathrm{Sym}^2_A\big(\mathsf{T}_A[-1]\big)^0$ of the 2SPS (recall that this is a cocycle, see Remark 2.3.37), we obtain a family of morphisms

$$\langle \pi^{(2)}, \cdot \rangle \otimes_A 1_{U\otimes_A V} : \Omega_A[1] \otimes_A \Omega_A[1] \otimes_A U \otimes_A V \longrightarrow A \otimes_A U \otimes_A V = U \otimes_A V \quad (4.2.25a)$$

in ${}_A\mathbf{C}$, for all $U, V \in {}_A\mathbf{C}$, which defines a $\mathbf{Ch}_{\mathbb{K}}^{[-1,0]}$-natural transformation

$$\iota_{\pi^{(2)}} : L^2_{\Omega_A[1]}\otimes_A \Longrightarrow \otimes_A \quad . \quad (4.2.25b)$$

Using the $\circ$-composition of pseudonatural transformations from Construction 4.1.11, we define the pseudonatural endomorphism

$$\begin{array}{ccccc} \otimes_A & & \xRightarrow{\quad t \quad} & & \otimes_A \\ \| & & & & \uparrow{\scriptstyle \iota_{\pi^{(2)}}} \\ \otimes_A(\mathrm{id}_{{}_A\mathbf{C}} \boxtimes \mathrm{id}_{{}_A\mathbf{C}}) & \xRightarrow[\otimes_A(\xi\boxtimes\xi)]{} & \otimes_A(L_{\Omega_A[1]} \boxtimes L_{\Omega_A[1]}) & \xRightarrow[\cong]{} & L^2_{\Omega_A[1]}\otimes_A \end{array} \quad (4.2.26)$$

as our infinitesimal 2-braiding on ${}_A\mathbf{C}$. Note that this pseudonatural endomorphism of $\otimes_A$ depends only on the bivector component $\pi^{(2)}$ of the 2SPS $\pi = \sum_{m\geq 2}\pi^{(m)} \in \widehat{\mathrm{Pol}}(A,2)^4$.

Theorem 4.2.12. *Given any 2SPS $\pi \in \widehat{\mathrm{Pol}}(A,2)^4$ on a fg sfCDGA A over the field $\mathbb{K}$, the pseudonatural endomorphism $t : \otimes_A \Rightarrow \otimes_A$ from (4.2.26) defines a γ-equivariant hexagonal infinitesimal 2-braiding in the sense of Definitions 4.2.5 and 4.2.3 on the homotopy 2-category ${}_A\mathbf{C} = \mathsf{Ho}_2\big({}_A\mathbf{dgMod}^{\mathrm{fgsf}}\big) \in \mathbf{dgCat}_{\mathbb{K}}^{[-1,0],\mathrm{ps}}$ of the dg-category of fgsf dg-modules over A.*

Proof. The most straightforward proof of this result is to compute the components of the pseudonatural transformation t from (4.2.26), and then to verify directly that these components are γ-equivariant (4.2.10) and satisfy the infinitesimal hexagons from Remark 4.2.4. Using the composition operations from Subsection 4.1.1, one finds by a straightforward computation that

$$\begin{array}{ccc} U \otimes_A V & \xrightarrow{\quad t_{UV} \quad} & U \otimes_A V \\ {\scriptstyle \xi_U\otimes_A\xi_V}\downarrow & & \uparrow{\scriptstyle \langle\pi^{(2)},\cdot\rangle\otimes_A 1_{U\otimes_A V}} \\ \Omega_A[1] \otimes_A U \otimes_A \Omega_A[1] \otimes_A V & \xrightarrow[1_{\Omega_A[1]}\otimes_A\gamma_{U,\Omega_A[1]}\otimes_A 1_V]{} & \Omega_A[1] \otimes_A \Omega_A[1] \otimes_A U \otimes_A V \end{array} \quad (4.2.27a)$$

for all $U, V \in {}_A\mathbf{C}$, and

$$\begin{array}{ccc} U \otimes_A V & \xrightarrow{\quad t_{f\boxtimes g} \quad} & U' \otimes_A V' \\ {\scriptstyle \xi_f \otimes_A (1_{\Omega_A[1]} \otimes_A g)\xi_V + \xi_{U'} f \otimes_A \xi_g} \big\downarrow & & \big\uparrow {\scriptstyle \langle \pi^{(2)}, \cdot \rangle \otimes_A 1_{U' \otimes_A V'}} \\ \Omega_A[1] \otimes_A U' \otimes_A \Omega_A[1] \otimes_A V' & \xrightarrow[1_{\Omega_A[1]} \otimes_A \gamma_{U', \Omega_A[1]} \otimes_A 1_{V'}]{} & \Omega_A[1] \otimes_A \Omega_A[1] \otimes_A U' \otimes_A V' \end{array} \tag{4.2.27b}$$

for all $f \in {}_A\mathbf{C}[U, U']^0$ and $g \in {}_A\mathbf{C}[V, V']^0$.

The γ-equivariance (4.2.11a) for the cochain components t_{UV} follows immediately by using symmetry of the extended pairing (4.2.24), i.e., $\langle \pi^{(2)}, \cdot \rangle \, \gamma = \langle \pi^{(2)}, \cdot \rangle$, and the hexagon identity for the original symmetric braiding γ. To prove the γ-equivariance (4.2.11b) for the homotopy components $t_{f\boxtimes g}$, one has to use additionally that

$$\begin{aligned} \gamma_{\Omega_A[1]\otimes_A U', \Omega_A[1]\otimes_A V'} \big(\xi_f \otimes_A (1_{\Omega_A[1]} \otimes_A g)\xi_V + \xi_{U'} f \otimes_A \xi_g\big) \gamma_{V,U} & \\ = (1_{\Omega_A[1]} \otimes_A g)\xi_V \otimes_A \xi_f + \xi_g \otimes_A \xi_{U'} f & \\ = \xi_{V'} g \otimes_A \xi_f + \xi_g \otimes_A (1_{\Omega_A[1]} \otimes_A f)\xi_U + \partial\big(\xi_g \otimes_A \xi_f\big) & \\ = \xi_{V'} g \otimes_A \xi_f + \xi_g \otimes_A (1_{\Omega_A[1]} \otimes_A f)\xi_U \,, & \end{aligned} \tag{4.2.28}$$

where in the second step we used (4.1.14b) of Definition 4.1.6 of pseudonatural transformations and in the last step we used that the morphism complexes of ${}_A\mathbf{C}$ are, by definition, truncated to degrees $\{-1, 0\}$, hence the 2-homotopy in the last term gets identified with 0.

It remains to verify the left infinitesimal hexagon relation (4.2.7a) since the right infinitesimal hexagon relation (4.2.7b) then follows from Lemma 4.2.6 and γ-equivariance of t. For this we recall the definition of the components of ξ from (4.2.19) and observe that

$$\xi_{V\otimes_A W} \;=\; \xi_V \otimes_A 1_W + (\gamma_{V,\Omega_A[1]} \otimes_A 1_W)\,(1_V \otimes_A \xi_W) \quad , \tag{4.2.29a}$$

for all $V, W \in {}_A\mathbf{C}$, and

$$\xi_{g\otimes_A k} \;=\; \xi_g \otimes_A k + (\gamma_{V',\Omega_A[1]} \otimes_A 1_{W'})\,(g \otimes_A \xi_k) \quad , \tag{4.2.29b}$$

for all $g \in {}_A\mathbf{C}[V, V']^0$ and $k \in {}_A\mathbf{C}[W, W']^0$. Using also the hexagon axiom for the original symmetric braiding γ, one verifies the cochain/homotopy components of the left infinitesimal hexagon relation (4.2.8a)/(4.2.8b). □

Remark 4.2.13. For later use in our discussion of examples in Section 4.3, let us expand the components t_{UV} and $t_{f\boxtimes g}$ of the infinitesimal 2-braiding (4.2.26) in terms of a choice of bases for the fg sf A-dg-modules $U, V, U', V' \in {}_A\mathbf{C}$. Using similar notations as in the proof of Lemma 4.2.11, we denote by $\{u_i \in U_0^\sharp\}$ a basis for $U^\sharp \cong A^\sharp \otimes_{\mathbb{K}} U_0^\sharp$, by $\{v_q \in V_0^\sharp\}$ a basis for $V^\sharp \cong A^\sharp \otimes_{\mathbb{K}} V_0^\sharp$, by $\{u'_i \in U_0'^\sharp\}$ a basis for $U'^\sharp \cong A^\sharp \otimes_{\mathbb{K}} U_0'^\sharp$, and by $\{v'_q \in V_0'^\sharp\}$ a basis for $V'^\sharp \cong A^\sharp \otimes_{\mathbb{K}} V_0'^\sharp$. Making use of these bases, we write

$$\begin{aligned} \mathrm{d}_U(\mathbb{1} \otimes_{\mathbb{K}} u_i) &= \underline{U}_i{}^j \otimes_{\mathbb{K}} u_j \quad , & \mathrm{d}_V(\mathbb{1} \otimes_{\mathbb{K}} v_q) &= \underline{V}_q{}^r \otimes_{\mathbb{K}} v_r \quad , \\ \mathrm{d}_{U'}(\mathbb{1} \otimes_{\mathbb{K}} u'_i) &= \underline{U}'_i{}^j \otimes_{\mathbb{K}} u'_j \quad , & \mathrm{d}_{V'}(\mathbb{1} \otimes_{\mathbb{K}} v'_q) &= \underline{V}'_q{}^r \otimes_{\mathbb{K}} v'_r \quad , \end{aligned}$$

for the differentials, and for the A-linear cochain maps $f \in {}_A\mathbf{C}[U, U']^0$ and $g \in {}_A\mathbf{C}[V, V']^0$ we write

$$f(\mathbb{1} \otimes_{\mathbb{K}} u_i) = \underline{f}_i{}^j \otimes_{\mathbb{K}} u'_j \qquad \text{and} \qquad g(\mathbb{1} \otimes_{\mathbb{K}} v_q) = \underline{g}_q{}^r \otimes_{\mathbb{K}} v'_r \quad , \tag{4.2.30}$$

where again all summations over repeated indices are suppressed. Evaluating the diagram

(4.2.27a), we then find the explicit basis expression

$$\begin{aligned} t_{UV}\big(\mathbb{1}\otimes_{\mathbb{K}} u_i\otimes_{\mathbb{K}} v_q\big) &= (t_{UV})_{iq}{}^{jr}\otimes_{\mathbb{K}} u_j\otimes_{\mathbb{K}} v_r \\ &= (-1)^{|u_j|\,(|\underline{V}_q{}^r|-1)}\,\big\langle \pi^{(2)}, s^{-1}\mathrm{d}_{\mathrm{dR}}(\underline{U}_i{}^j)\otimes_A s^{-1}\mathrm{d}_{\mathrm{dR}}(\underline{V}_q{}^r)\big\rangle\otimes_{\mathbb{K}} u_j\otimes_{\mathbb{K}} v_r \quad . \end{aligned} \tag{4.2.31a}$$

Furthermore, evaluating the diagram (4.2.27b), we find

$$\begin{aligned} t_{f\boxtimes g}(\mathbb{1}\otimes_{\mathbb{K}} u_i\otimes_{\mathbb{K}} v_q) &= \big(t_{f\boxtimes g}\big)_{iq}{}^{jr}\otimes u'_j\otimes_{\mathbb{K}} v'_r \\ &= -(-1)^{|u'_j|\,(|\underline{V}_q{}^s|+|\underline{g}_s{}^r|-1)}\,\big\langle \pi^{(2)}, s^{-1}\mathrm{d}_{\mathrm{dR}}(\underline{f}_i{}^j)\otimes_A s^{-1}\mathrm{d}_{\mathrm{dR}}(\underline{V}_q{}^s)\,\underline{g}_s{}^r\big\rangle\otimes_{\mathbb{K}} u'_j\otimes_{\mathbb{K}} v'_r \\ &\quad -(-1)^{|\underline{f}_i{}^k|+|u'_j|\,(|\underline{g}_q{}^r|-1)}\,\big\langle \pi^{(2)}, s^{-1}\underline{f}_i{}^k\,\mathrm{d}_{\mathrm{dR}}(\underline{U}'_k{}^j)\otimes_A s^{-1}\mathrm{d}_{\mathrm{dR}}(\underline{g}_q{}^r)\big\rangle\otimes u'_j\otimes_{\mathbb{K}} v'_r \quad . \end{aligned} \tag{4.2.31b}$$

Given a concrete example for the bivector $\pi^{(2)}\in \mathrm{Sym}^2_A\big(\mathsf{T}_A[-3]\big)^4\cong \mathrm{Sym}^2_A\big(\mathsf{T}_A[-1]\big)^0$, one can evaluate these expressions further by using the definition (4.2.24) of the duality pairing. △

Example 4.2.14. We can use the formulae from Remark 4.2.13 in order to provide a more explicit basis expression for the obstruction $t_{t_{UV}\boxtimes 1_W}$ to the left 4-term relations in (4.2.15),

$$\begin{aligned} &t_{t_{UV}\boxtimes 1_W}(\mathbb{1}\otimes_{\mathbb{K}} u_i\otimes_{\mathbb{K}} v_q\otimes_{\mathbb{K}} w_s) \\ &= -(-1)^{|u_j|\,(|\underline{W}_s{}^t|-1)}\,\big\langle \pi^{(2)}, s^{-1}\mathrm{d}_{\mathrm{dR}}\big((t_{UV})_{iq}{}^{jr}\big)\otimes_A s^{-1}\mathrm{d}_{\mathrm{dR}}(\underline{W}_s{}^t)\big\rangle\otimes_{\mathbb{K}} u_j\otimes_{\mathbb{K}} v_r\otimes_{\mathbb{K}} w_t \\ &= -(-1)^{|u_j||\underline{V}_q{}^r\underline{W}_s{}^t|}\big\langle \pi^{(2)}, s^{-1}\mathrm{d}_{\mathrm{dR}}\langle \pi^{(2)}, s^{-1}\mathrm{d}_{\mathrm{dR}}\underline{U}_i{}^j\otimes_A s^{-1}\mathrm{d}_{\mathrm{dR}}\underline{V}_q{}^r\rangle\otimes_A s^{-1}\mathrm{d}_{\mathrm{dR}}\underline{W}_s{}^t\big\rangle\otimes_{\mathbb{K}} u_j\otimes_{\mathbb{K}} v_r\otimes_{\mathbb{K}} w_t \end{aligned} \tag{4.2.32}$$

▽

4.3 Examples

This section studies concrete examples of our infinitesimal 2-braidings from Theorem 4.2.12. The examples we consider are the L_∞-algebras from Sections 3.2 and 3.3. More specifically, recall that Corollary 3.2.2 shows that to every such L_∞-algebra $\mathfrak{g}$ is associated its Chevalley-Eilenberg sfCDGA $\mathrm{CE}_\mathfrak{g}$ and the underlying CGA (3.2.6) makes it obvious that $\mathrm{CE}_\mathfrak{g}$ is fg if $\mathfrak{g}$ is fd hence it satisfies our hypotheses from Subsection 4.2.2.

The symmetric monoidal dg-category ${}_{\mathrm{CE}_\mathfrak{g}}\mathbf{dgMod}^{\mathrm{fgsf}}$ of fgsf $\mathrm{CE}_\mathfrak{g}$-dg-modules describes weak representations of the L_∞-algebra $\mathfrak{g}$ (see [All14, Definition 2.2]) and their equivariant maps (see [All14, Definition 3.3]). Our result in Theorem 4.2.12 then implies that, given any 2SPS on the CDGA $\mathrm{CE}_\mathfrak{g}$, one obtains a first-order deformation of the symmetric strict monoidal cochain 2-category $\mathsf{Ho}_2\big({}_{\mathrm{CE}_\mathfrak{g}}\mathbf{dgMod}^{\mathrm{fgsf}}\big)$ of the representation category of $\mathfrak{g}$ into a pseudonaturally-braided strict monoidal cochain 2-category. Drawing analogies with the standard deformation theory of the representation category $\mathbf{Rep}(\mathfrak{g})$ of an ordinary Lie algebra $\mathfrak{g}$ and its relationship to the quantum group $U_q\mathfrak{g}$ (see, e.g., [Car93] and [Kas95, Theorem XX.6.1]), one may interpret our deformation constructions as a first step towards an L_∞-algebraic generalisation of quantum groups.

4.3.1 Ordinary Lie algebras

Let us start with the simplest example which is given by the case where the L_∞-algebra $\mathfrak{g}$ is an ordinary fd Lie algebra $\mathfrak{g}=(\mathfrak{g},[\cdot,\cdot])$. Recall from Example 2.2.6, if we choose a basis $\{x_a\in\mathfrak{g} : a=1,\dots,\dim(\mathfrak{g})\}$ and denote the structure constants by $[x_a,x_b]=f^c_{ab}\,x_c$ then the

Chevalley-Eilenberg CDGA $\mathrm{CE}_{\mathfrak{g}}$ is given by the $\mathbb{Z}$-graded commutative algebra

$$\mathrm{CE}_{\mathfrak{g}}^{\sharp} \,=\, \mathbb{K}[\{s\theta^a\}] \tag{4.3.1a}$$

which is freely generated by $\dim(\mathfrak{g})$-many generators $s\theta^a$ of degree 1, for $a = 1, \dots, \dim(\mathfrak{g})$, together with the differential

$$\delta(s\theta^a) \,=\, -\tfrac{1}{2}\, f^a_{bc}\, s\theta^b s\theta^c \tag{4.3.1b}$$

which is determined by the structure constants.

Consider now the homotopy 2-category ${}_{\mathrm{CE}_{\mathfrak{g}}}\mathbf{C} = \mathsf{Ho}_2\big({}_{\mathrm{CE}_{\mathfrak{g}}}\mathbf{dgMod}^{\mathrm{fgsf}}\big)$ of the symmetric monoidal dg-category of fgsf $\mathrm{CE}_{\mathfrak{g}}$-dg-modules, which is the object of focus in our deformation construction from Theorem 4.2.12. Let us first illustrate the sense in which this category describes weak representations of the Lie algebra $\mathfrak{g}$. By definition, given any object $U \in {}_{\mathrm{CE}_{\mathfrak{g}}}\mathbf{C}$, one has a $\mathbb{Z}$-graded vector space of generators $U_0^\sharp$ such that $U^\sharp \cong \mathrm{CE}_{\mathfrak{g}}^\sharp \otimes U_0^\sharp$. Choosing any basis $\{u_i \in U_0^\sharp\}$ for the generators, the differential on U is completely characterized by specifying $\mathrm{d}_U(\mathbb{1} \otimes u_i) \in U$, for all i, which when expanded in terms of the chosen bases reads as

$$\mathrm{d}_U(\mathbb{1} \otimes u_i) \,=\, \underline{U}_i{}^j \otimes u_j \,=\, \sum_{n=0}^{\infty} \tfrac{1}{n!}\, U_{a_1 \dots a_n i}{}^j\, s\theta^{a_1} \cdots s\theta^{a_n} \otimes u_j \quad , \tag{4.3.2}$$

where $U_{a_1 \dots a_n i}{}^j \in \mathbb{K}$ are coefficients. The square-zero condition $\mathrm{d}_U \mathrm{d}_U(\mathbb{1} \otimes u_i) = 0$ gives a tower of compatibility conditions between the coefficients $U_{a_1 \dots a_n i}{}^j$ and the structure constants f^a_{bc} of the Lie algebra $\mathfrak{g}$, which express a concept of Lie algebra representation up to coherent homotopies. Strict representations of $\mathfrak{g}$ on a cochain complex U_0 are recovered in the special case where $\mathrm{d}_U(\mathbb{1} \otimes u_i) = U_i{}^j \otimes u_j + U_{ai}{}^j\, s\theta^a \otimes u_j$ is of polynomial degree ≤ 1 in $s\theta^a$. The first term in this expression defines a differential d_{U_0} on $U_0^\sharp$ and the second term defines a strict Lie algebra action map $\mathfrak{g} \otimes U_0 \to U_0$ which is a cochain map for the differential of $U_0 = (U_0^\sharp, \mathrm{d}_{U_0})$.

In addition to weak representations of the Lie algebra $\mathfrak{g}$, the homotopy 2-category ${}_{\mathrm{CE}_{\mathfrak{g}}}\mathbf{C} = \mathsf{Ho}_2\big({}_{\mathrm{CE}_{\mathfrak{g}}}\mathbf{dgMod}^{\mathrm{fgsf}}\big)$ also describes weakly $\mathfrak{g}$-equivariant maps. (The 2-morphisms in ${}_{\mathrm{CE}_{\mathfrak{g}}}\mathbf{C}$ are homotopies between such weakly $\mathfrak{g}$-equivariant maps.) Indeed, a 1-morphism $h : U \to U'$ in ${}_{\mathrm{CE}_{\mathfrak{g}}}\mathbf{C}$ is by definition a cochain map which commutes with the left $\mathrm{CE}_{\mathfrak{g}}$-actions. Hence, by choosing a basis $\{u_i \in U_0^\sharp\}$ for $U^\sharp \cong \mathrm{CE}_{\mathfrak{g}}^\sharp \otimes U_0^\sharp$ and a basis $\{u'_i \in U_0'^\sharp\}$ for $U'^\sharp \cong \mathrm{CE}_{\mathfrak{g}}^\sharp \otimes U_0'^\sharp$, we can expand

$$h(\mathbb{1} \otimes u_i) \,=\, \underline{h}_i{}^j \otimes u'_j \,=\, \sum_{n=0}^{\infty} \tfrac{1}{n!}\, h_{a_1 \dots a_n i}{}^j\, s\theta^{a_1} \cdots s\theta^{a_n} \otimes u'_j \quad , \tag{4.3.3}$$

where $h_{a_1 \dots a_n i}{}^j \in \mathbb{K}$ are coefficients. The cochain map condition $\mathrm{d}_{U'}\big(h(\mathbb{1} \otimes u_i)\big) = h\big(\mathrm{d}_U(\mathbb{1} \otimes u_i)\big)$ gives a tower of compatibility conditions between the coefficients $h_{a_1 \dots a_n i}{}^j$ and the structure constants f^a_{bc} of the Lie algebra $\mathfrak{g}$, which express a concept of $\mathfrak{g}$-equivariance up to coherent homotopies. Strict morphisms $h : U \to U'$ between strict $\mathfrak{g}$-representations on cochain complexes U_0 and U'_0 are recovered in the special case where $h(\mathbb{1} \otimes u_i) = h_i{}^j \otimes u'_j$ is constant in $s\theta^a$.

The input datum for Theorem 4.2.12 is a 2SPS on $\mathrm{CE}_{\mathfrak{g}}$ and Theorem 2.3.40 already characterises such 2SPSs in terms of invariant symmetric tensors hence we have the following immediate corollary.

Corollary 4.3.1. *Every invariant symmetric tensor, i.e., $\pi = \frac{1}{2}\, \pi^{ab} s^3 \frac{\partial}{\partial(s\theta^a)} s^3 \frac{\partial}{\partial(s\theta^b)}$ such that $\pi^{ab} = \pi^{ba}$ and $f^a_{cd}\pi^{db} + f^b_{cd}\pi^{ad} = 0$, defines via Theorem 4.2.12 a γ-equivariant hexagonal infinitesimal 2-braiding t on the homotopy 2-category ${}_{\mathrm{CE}_{\mathfrak{g}}}\mathbf{C}$ of the symmetric monoidal dg-category of fgsf $\mathrm{CE}_{\mathfrak{g}}$-dg-modules.*

Remark 4.3.2. Let us recall that explicit formulae for the components of the γ-equivariant hexagonal infinitesimal 2-braiding t are given in (4.2.31). Note that the morphism components t_{UV} in (4.2.31a) involve the de Rham differential

$$(\mathrm{d}_{\mathrm{dR}}\otimes\mathrm{id})\big(\mathrm{d}_U(\mathbb{1}\otimes u_i)\big) \;=\; \sum_{n=1}^{\infty}\tfrac{1}{(n-1)!}\,U_{a_1\dots a_n i}{}^{j}\;s\theta^{a_1}\cdots s\theta^{a_{n-1}}\mathrm{d}_{\mathrm{dR}}(s\theta^{a_n})\otimes u_j \tag{4.3.4}$$

of (4.3.2), as well as $(\mathrm{d}_{\mathrm{dR}}\otimes\mathrm{id})\big(\mathrm{d}_V(\mathbb{1}\otimes v_q)\big)$. Hence, the components t_{UV} in (4.2.31a) are in general non-constant polynomials in $s\theta^a$, unless both U and V are strict $\mathfrak{g}$-representations. Similarly, the homotopy components $t_{f\boxtimes g}$ in (4.2.31b) involve the de Rham differential

$$(\mathrm{d}_{\mathrm{dR}}\otimes\mathrm{id})\big(f(\mathbb{1}\otimes u_i)\big) \;=\; \sum_{n=1}^{\infty}\tfrac{1}{(n-1)!}\,f_{a_1\dots a_n i}{}^{j}\;s\theta^{a_1}\cdots s\theta^{a_{n-1}}\mathrm{d}_{\mathrm{dR}}(s\theta^{a_n})\otimes u'_j \tag{4.3.5}$$

of (4.3.3), as well as $(\mathrm{d}_{\mathrm{dR}}\otimes\mathrm{id})\big(g(\mathbb{1}\otimes v_q)\big)$. Hence, the homotopies $t_{f\boxtimes g}$ in (4.2.31b) are in general non-trivial, unless both f and g are strict morphisms. △

As a direct consequence of Remark 4.2.9, we obtain the following result.

Corollary 4.3.3. *Every invariant symmetric tensor $\pi\in(\mathrm{Sym}^2\mathfrak{g})^{\mathfrak{g}}$ defines via Corollary 4.3.1 a pseudonaturally-braided monoidal deformation to all orders in h of the full subcategory of ${}_{\mathrm{CE}_{\mathfrak{g}}}\mathbf{C}\in\mathbf{dgCat}^{[-1,0],\mathrm{ps}}_{\mathbb{K}[[\hbar]]}$ which consists of all $\mathrm{CE}_{\mathfrak{g}}$-dg-modules corresponding to strict $\mathfrak{g}$-representations.*

Proof. Combining Example 4.2.14 and Remark 4.3.2, we obtain that the obstructions $t_{t_{UV}\boxtimes 1_W}$ in Remark 4.2.9 are all trivial for strict $\mathfrak{g}$-representations because in this case $(t_{UV})_{iq}{}^{jr}$ are constant polynomials in $s\theta^a$, hence $\mathrm{d}_{\mathrm{dR}}\big((t_{UV})_{iq}{}^{jr}\big)=0$. This implies that the obstructions in Remark 4.2.9 are void thus yielding the desired braided monoidal deformation to all orders in h. □

Remark 4.3.4. Our result in Corollary 4.3.3 is a homotopical generalisation of the deformation constructions in [Car93] and [Kas95, Chapter XX]. We would like to stress that our generalisation is non-trivial because, even though we restrict ourselves to strict $\mathfrak{g}$-representations, the morphisms in the *full* subcategory from Corollary 4.3.3 are weakly $\mathfrak{g}$-equivariant maps (and their homotopies) between strict $\mathfrak{g}$-representations. Recalling our observations from Remark 4.3.2, this implies in particular that the infinitesimal 2-braiding t is a genuine pseudo-natural transformation, i.e., it has non-trivial homotopy components $t_{f\boxtimes g}$, and so are the associator and braiding of the deformed category from Corollary 4.3.3. △

4.3.2 String Lie 2-algebras

Recall from Subsection 3.3.3 that a string Lie 2-algebra $\mathfrak{h}_\kappa$ is constructed from an ordinary Lie algebra $\mathfrak{h}=(\mathfrak{h},[\cdot,\cdot])$ and a 3-cocycle $\kappa:\bigwedge^3\mathfrak{h}\to\mathbb{K}$. The resulting Lie 2-algebra $\mathfrak{h}_\kappa$ is given by the $[-1,0]$-concentrated graded vector space defined by $\mathfrak{h}_\kappa^0:=\mathfrak{h}$ and $\mathfrak{h}_\kappa^{-1}:=\mathbb{K}$ together with the non-trivial brackets:

$$\ell_2(x,y)\;:=\;[x,y]\quad,\qquad \ell_3(x,y,z)\;:=\;\kappa(x,y,z)\quad, \tag{4.3.6}$$

for all $x,y,z\in\mathfrak{h}$ and zero otherwise. Let us choose again a basis $\{x_a\in\mathfrak{h}\}$ and denote the structure constants by $[x_a,x_b]=f_{ab}^c\,x_c$ and $\kappa(x_a,x_b,x_c)=\kappa_{abc}$.

The Chevalley-Eilenberg CDGA $\mathrm{CE}_{\mathfrak{h}_\kappa}$ of the string Lie 2-algebra $\mathfrak{h}_\kappa$ is then given by the $\mathbb{Z}$-graded

commutative algebra

$$\mathrm{CE}^{\sharp}_{\mathfrak{h}_\kappa} \;=\; \mathbb{K}[\{s\theta^a\}, s\nu] \tag{4.3.7a}$$

which is freely generated by $\dim(\mathfrak{h})$-many generators $s\theta^a$ of degree 1, for $a = 1, \dots, \dim(\mathfrak{h})$, and a single generator $s\nu$ of degree 2, together with the differential

$$\delta(s\theta^a) \;=\; -\tfrac{1}{2}\, f^a_{bc} s\theta^b s\theta^c \qquad , \qquad \delta(s\nu) \;=\; -\tfrac{1}{3!}\, \kappa_{abc} s\theta^a s\theta^b s\theta^c \quad . \tag{4.3.7b}$$

Note that the square-zero condition $\delta^2(s\theta^a) = 0$ follows from the Jacobi identity of the Lie bracket and $\delta^2(s\nu) = 0$ follows from the 3-cocycle condition of κ.

The following proposition is an immediate corollary of Remark 3.3.10.

Proposition 4.3.5. *Any 2SPS $\pi = \sum_{m\geq 2} \pi^{(m)} \in \widehat{\mathrm{Pol}}\big(\mathrm{CE}_{\mathfrak{h}_\kappa}, 2\big)^4$ on $\mathrm{CE}_{\mathfrak{h}_\kappa}$ is of the form*

$$\pi \;=\; \pi^{(2)} \;=\; \tfrac{1}{2}\pi^{ab} s^3 \frac{\partial}{\partial(s\theta^a)} s^3 \frac{\partial}{\partial(s\theta^b)} + \tilde{\pi}^a_b s\theta^b\, s^3 \frac{\partial}{\partial(s\theta^a)} s^3 \frac{\partial}{\partial(s\nu)} \quad , \tag{4.3.8a}$$

where $\pi^{ab} \in \mathbb{K}$ and $\tilde{\pi}^a_b \in \mathbb{K}$ are coefficients which must satisfy the properties

$$f^a_{cd}\, \pi^{db} + f^b_{cd}\, \pi^{ad} \;=\; 0 \quad , \tag{4.3.8b}$$

$$f^d_{bc}\, \tilde{\pi}^a_d - f^a_{bd}\, \tilde{\pi}^d_c + f^a_{cd}\, \tilde{\pi}^d_b - \kappa_{bcd}\, \pi^{da} \;=\; 0 \quad , \tag{4.3.8c}$$

$$\pi^{ac}\, \tilde{\pi}^b_c + \pi^{bc}\, \tilde{\pi}^a_c \;=0 \quad . \tag{4.3.8d}$$

The following result is a direct consequence of Theorem 4.2.12 and Proposition 4.3.5.

Corollary 4.3.6. *Every pair $(\pi, \tilde{\pi})$ consisting of an invariant symmetric tensor $\pi \in (\mathrm{Sym}^2\mathfrak{h})^{\mathfrak{h}}$ and a quasi-derivation $\tilde{\pi} : \mathfrak{h} \to \mathfrak{h}$ defines via Proposition 4.3.5 and Theorem 4.2.12 a γ-equivariant hexagonal infinitesimal 2-braiding t on the homotopy 2-category ${}_{\mathrm{CE}_{\mathfrak{h}_\kappa}}\mathbf{C}$ of the symmetric monoidal dg-category of fgsf $\mathrm{CE}_{\mathfrak{h}_\kappa}$-dg-modules.*

Remark 4.3.7. Recalling the explicit formulae for the components of the γ-equivariant hexagonal infinitesimal 2-braiding t from (4.2.31), we immediately observe by using also (4.3.8a) that the morphism components t_{UV} are in general non-constant polynomials in $s\theta^a$ in the case where $\tilde{\pi}^a_b \neq 0$ is non-trivial. (As illustrated in Examples 3.3.13, 3.3.14 and 3.3.15, this seems to be the most relevant case.) Hence, the obstructions $t_{t_{UV} \boxtimes 1_W}$ from Example 4.2.14 are non-trivial if the quasi-derivation $\tilde{\pi}$ is nontrivial. This even holds true for strict representations of the string Lie 2-algebra $\mathfrak{h}_\kappa$, i.e., the expansions (4.3.2) of the differential d_U are polynomials in $s\theta^a$ and $s\nu$ of degree ≤ 1. As a consequence, for these trivial invariant tensor 2SPSs, we currently do not have a meaningful generalisation of the all-order deformation result from Corollary 4.3.3 to the case of $\mathrm{CE}_{\mathfrak{h}_\kappa}$. However, if we use the nontrivial invariant tensor 2SPS of Item 1 of Example 3.3.15 then we *do* have a meaningful generalisation of the all-order deformation result from Corollary 4.3.3 to the case of $\mathrm{CE}_{\mathfrak{H}_\kappa}$ if the expansions (4.3.2) of the differential d_U are polynomials in $s\theta^a$ of degree ≤ 1 and constant in $s\nu$. △

Chapter 5

Second-order integration of infinitesimal 2-braidings

This chapter is based on the solo preprint [Kemp25]. Specifically, we are now going to explore the problem of second-order integration of infinitesimal 2-braidings. Recall from Remark 4.2.9 that an infinitesimal 2-braiding does not necessarily satisfy the 4-term relations thus we are obstructed from integrating to all orders via the usual methods in 1-category theory. The central aim of this chapter is to show that such obstructions are actually part of the structure of a braided monoidal cochain 2-category as in Definition 5.1.3.

Subsection 5.2.1 recalls the subscript index notation $t_{ij} : \otimes^n \Rightarrow \otimes^n$ from [KLS25, (3.32)] and then uses this notation to introduce our notions of "left/right pre-hexagonators", upon choosing Drinfeld's Knizhnik-Zamolodchikov series (see [BRW23, Theorem 20]) $\Phi_{\mathrm{KZ}}(t_{12}, t_{23})$ as our ansatz associator and $\gamma e^{\frac{h}{2}t}$ as our ansatz braiding.

Section 5.2 is about showing that the (pre-)hexagonators admit nontrivial second-order terms and satisfy the requisite axioms of a braided strictly-unital monoidal $\mathbf{Ch}^{[-1,0]}_{\mathbb{K}[h]/(h^3)}$-category. Subsection 5.2.3 recalls CM's [CFM15] important notions of coherency (see Definition 5.2.30) and total γ-equivariance (see Definition 5.2.20) of an infinitesimal 2-braiding. We demonstrate in Propositions 5.2.18, 5.2.29 and 5.2.33 that a coherent totally γ-equivariant hexagonal infinitesimal 2-braiding provides a second-order deformation quantisation of a symmetric strict monoidal $\mathbf{Ch}^{[-1,0]}_{\mathbb{K}}$-category to a braided strictly-unital monoidal $\mathbf{Ch}^{[-1,0]}_{\mathbb{K}[h]/(h^3)}$-category.

We include Section 5.3 for two reasons: the first is that we want to show that the pentagonator Π is nontrivial and this nontriviality begins at order h^3. The second reason is that we wish to determine the order h^3 term of the right pre-hexagonator (5.2.4c) for later computations in Subsection 6.2.3.

5.1 Braided strictly-unital monoidal cochain 2-categories

This section specifies the definition of a braided monoidal bicategory in [CSP14, Appendix C] to our context where: we work with $\mathbf{Ch}^{[-1,0]}_R$-categories instead of bicategories, the monoidal product $\otimes$ is a $\mathbf{Ch}^{[-1,0]}_R$-functor rather than a pseudofunctor, the associator and braiding are pseudonatural isomorphisms (as in Definition 4.1.12) instead of "pseudonatural equivalences", the unitors and 2-unitors are all identities. We already know that pseudonatural isomorphisms are closed under vertical and monoidal composition (see Remark 4.1.26) as well as whiskering (see Remark 4.1.20) hence the pentagonator (5.1.1a) and left/right hexagonators (5.1.8a)/(5.1.9a) will automatically be isomodifications and admit inverses as in Remark 4.1.16.

The following definition is our specification of the definition of a monoidal bicategory as found in [CSP14, Definition C.1].

Definition 5.1.1. A **strictly-unital monoidal $\mathbf{Ch}_R^{[-1,0]}$-category** $(\mathbf{C}, \otimes, I, \alpha, \Pi)$ consists of a quintuple of data:

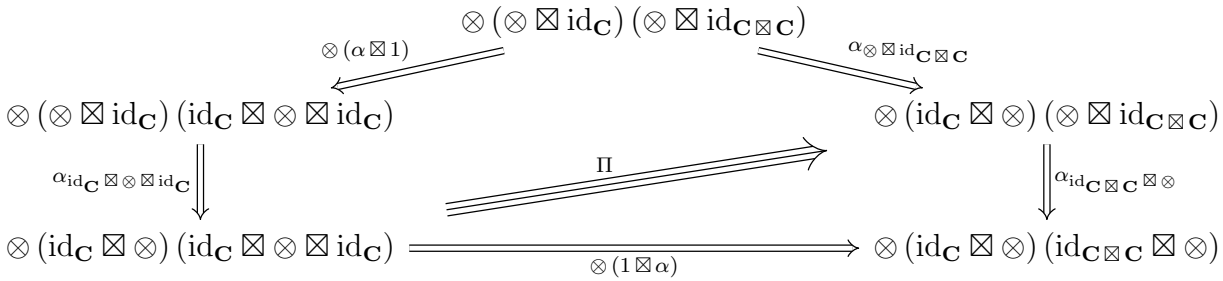
(i) $\mathbf{C} \in \mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}$.

(ii) A $\mathbf{Ch}_R^{[-1,0]}$-functor $\otimes : \mathbf{C} \boxtimes \mathbf{C} \to \mathbf{C}$.

(iii) A $\mathbf{Ch}_R^{[-1,0]}$-functor $\eta : \mathbf{R} \to \mathbf{C}$.

(iv) A pseudonatural isomorphism $\alpha : \otimes(\otimes \boxtimes \mathrm{id}_{\mathbf{C}}) \Rightarrow \otimes(\mathrm{id}_{\mathbf{C}} \boxtimes \otimes) : \mathbf{C} \boxtimes \mathbf{C} \boxtimes \mathbf{C} \to \mathbf{C}$ called the **associator**.

(v) An isomodification[1],



called the **pentagonator**,

$$\Pi : [\otimes(1 \boxtimes \alpha)]\, \alpha_{\mathrm{id}_{\mathbf{C}} \boxtimes \otimes \boxtimes \mathrm{id}_{\mathbf{C}}}\, [\otimes(\alpha \boxtimes 1)] \Rrightarrow \alpha_{\mathrm{id}_{\mathbf{C}\boxtimes\mathbf{C}} \boxtimes \otimes}\, \alpha_{\otimes \boxtimes \mathrm{id}_{\mathbf{C}\boxtimes\mathbf{C}}}\,. \tag{5.1.1a}$$

For every $UVWX \in \mathbf{C}^{\boxtimes 4}$, we must have

$$(1_U \otimes \alpha_{VWX})\alpha_{U(VW)X}(\alpha_{UVW} \otimes 1_X) - \alpha_{UV(WX)}\alpha_{(UV)WX} = \partial\Pi_{UVWX} \quad . \tag{5.1.1b}$$

For every $f \boxtimes g \boxtimes k \boxtimes x \in \mathbf{C}^{\boxtimes 4}[UVWX, U'V'W'X']^0$, we must have

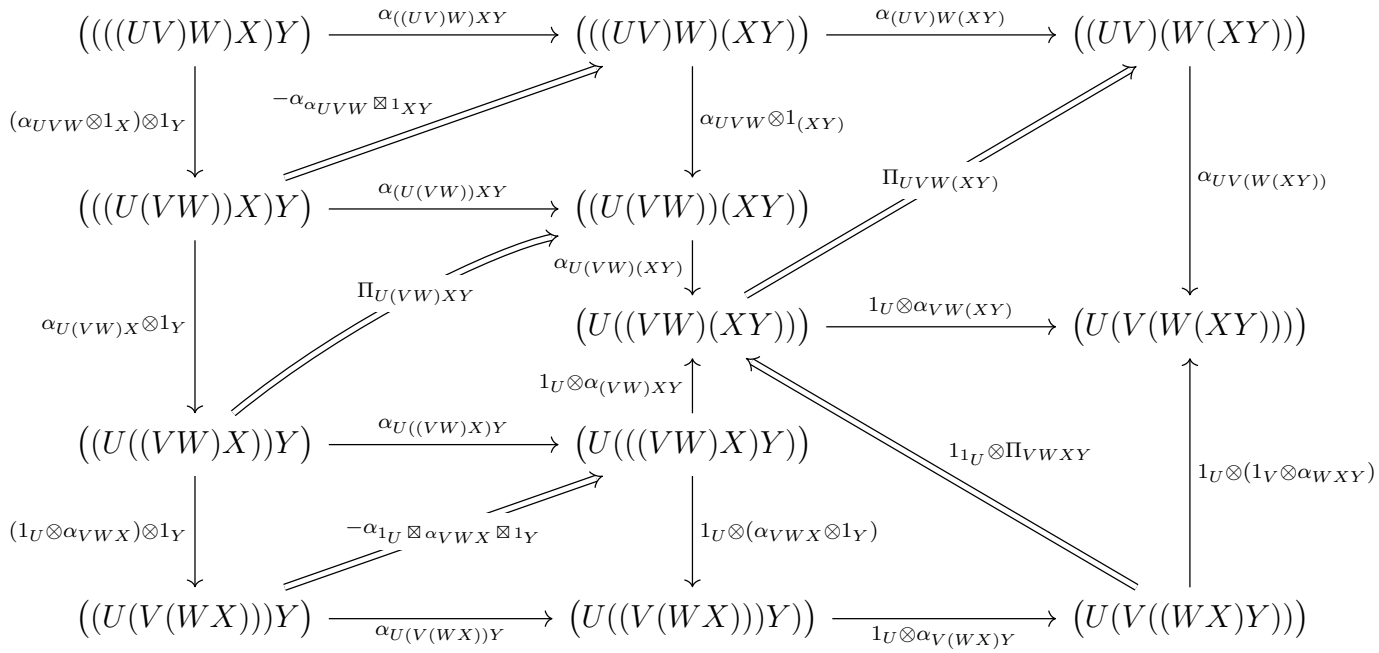
$$\begin{aligned} &\Pi_{U'V'W'X'}\left[\big((f \otimes g) \otimes k\big) \otimes x\right] + (f \otimes \alpha_{g\boxtimes k\boxtimes x})\alpha_{U(VW)X}(\alpha_{UVW} \otimes 1_X) \\ &+ (1_{U'} \otimes \alpha_{V'W'X'})\alpha_{f\boxtimes(g\otimes k)\boxtimes x}(\alpha_{UVW} \otimes 1_X) + (1_{U'} \otimes \alpha_{V'W'X'})\alpha_{U'(V'W')X'}(\alpha_{f\boxtimes g\boxtimes k} \otimes x) \\ &= \\ &\alpha_{f\boxtimes g\boxtimes(k\otimes x)}\alpha_{(UV)WX} + \alpha_{U'V'(W'X')}\alpha_{(f\otimes g)\boxtimes k\boxtimes x} + \left[f \otimes \big(g \otimes (k \otimes x)\big)\right]\Pi_{UVWX} \end{aligned} \tag{5.1.1c}$$

This data is required to satisfy the following four axioms:

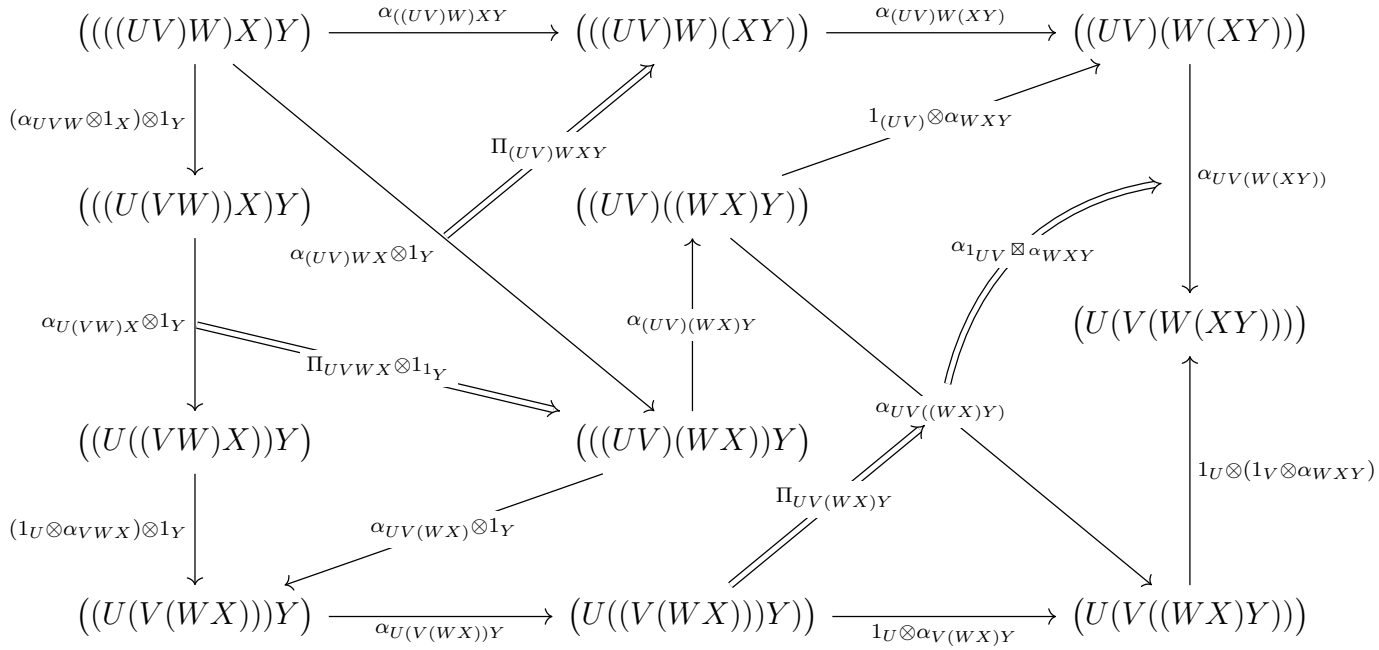
[1] Note that we implicitly use the fact that $\boxtimes$ is a 3-functor to write things such as

$$\otimes(\mathrm{id}_{\mathbf{C}} \boxtimes \otimes)(\otimes \boxtimes \mathrm{id}_{\mathbf{C}\boxtimes\mathbf{C}}) = \otimes(\otimes \boxtimes \otimes) = \otimes(\otimes \boxtimes \mathrm{id}_{\mathbf{C}})(\mathrm{id}_{\mathbf{C}\boxtimes\mathbf{C}} \boxtimes \otimes) \quad .$$

1. **Associahedron**, i.e., the following two pasting diagrams in $\mathbf{C}$ are equivalent:



and



As an equality between homotopies, this reads,

$$
\begin{aligned}
&(1_U\otimes\alpha_{VW(XY)})\Pi_{U(VW)XY}\left([\alpha_{UVW}\otimes 1_X]\otimes 1_Y\right)-\left(1_U\otimes\alpha_{VW(XY)}\right)\alpha_{U(VW)(XY)}\alpha_{\alpha_{UVW}\boxtimes 1_{XY}}\\
&+(1_U\otimes\Pi_{VWXY})\alpha_{U((VW)X)Y}\left(\alpha_{U(VW)X}[\alpha_{UVW}\otimes 1_X]\otimes 1_Y\right)+\Pi_{UVW(XY)}\alpha_{((UV)W)XY}\\
&-\left(1_U\otimes[1_V\otimes\alpha_{WXY}]\alpha_{V(WX)Y}\right)\alpha_{1_U\boxtimes\alpha_{VWX}\boxtimes 1_Y}\left(\alpha_{U(VW)X}[\alpha_{UVW}\otimes 1_X]\otimes 1_Y\right)\\
&=\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad(5.1.2)\\
&\left(1_U\otimes[1_V\otimes\alpha_{WXY}]\alpha_{V(WX)Y}\right)\alpha_{U(V(WX))Y}\left(\Pi_{UVWX}\otimes 1_Y\right)+\alpha_{UV(W(XY))}\Pi_{(UV)WXY}\\
&+(1_U\otimes[1_V\otimes\alpha_{WXY}])\,\Pi_{UV(WX)Y}\left(\alpha_{(UV)WX}\otimes 1_Y\right)+\alpha_{1_{UV}\boxtimes\alpha_{WXY}}\alpha_{(UV)(WX)Y}(\alpha_{(UV)WX}\otimes 1_Y)
\end{aligned}
$$

2. Unitality as in (4.2.3a).

3. **Triangle**; the associator α is a retraction of the middle insertion of the unit $\mathrm{Id}_{\mathrm{id}_{\mathbf{C}} \boxtimes \eta \boxtimes \mathrm{id}_{\mathbf{C}}}$,

$$\alpha_{\mathrm{id}_{\mathbf{C}} \boxtimes \eta \boxtimes \mathrm{id}_{\mathbf{C}}} = \mathrm{Id}_{\otimes} : \otimes (\otimes \boxtimes \mathrm{id}_{\mathbf{C}}) (\mathrm{id}_{\mathbf{C}} \boxtimes \eta \boxtimes \mathrm{id}_{\mathbf{C}}) = \otimes : \mathbf{C} \boxtimes \mathbf{C} \to \mathbf{C} \quad . \tag{5.1.3a}$$

In terms of objects and morphisms, this demands:

$$\alpha_{UIV} = 1_{(UV)} \qquad , \qquad \alpha_{f \boxtimes 1_I \boxtimes g} = 0 \quad . \tag{5.1.3b}$$

4. **Prism**; the pentagonator Π is a retraction of both middle insertions of the unit,

$$\Pi_{UIWX} = 0 = \Pi_{UVIX} \quad . \tag{5.1.4}$$

If the pentagon axiom is upheld and the pentagonator Π is trivial then we say $(\mathbf{C}, \otimes, I, \alpha)$ is a **pentagonal strictly-unital monoidal $\mathbf{Ch}_R^{[-1,0]}$-category**.

Remark 5.1.2. In the familiar context of an ordinary monoidal 1-category $(\mathbf{C}, \otimes, I, \alpha, \lambda, \rho)$, the triangle axiom

$$\begin{array}{ccc} (U \otimes I) \otimes W & \xrightarrow{\alpha_{UIW}} & U \otimes (I \otimes W) \\ & {\scriptstyle \rho_U \otimes 1_W} \searrow \quad \swarrow {\scriptstyle 1_U \otimes \lambda_W} & \\ & U \otimes W & \end{array} \tag{5.1.5a}$$

implies both the "left" triangle identity

$$\begin{array}{ccc} (I \otimes V) \otimes W & \xrightarrow{\alpha_{IVW}} & I \otimes (V \otimes W) \\ & {\scriptstyle \lambda_V \otimes 1_W} \searrow \quad \swarrow {\scriptstyle \lambda_{V \otimes W}} & \\ & V \otimes W & \end{array} \tag{5.1.5b}$$

and the "right" triangle identity

$$\begin{array}{ccc} (U \otimes V) \otimes I & \xrightarrow{\alpha_{UVI}} & U \otimes (V \otimes I) \\ & {\scriptstyle \rho_{U \otimes V}} \searrow \quad \swarrow {\scriptstyle 1_U \otimes \rho_V} & \\ & U \otimes V & \end{array} \tag{5.1.5c}$$

The proof of this usually relies on the naturality of the associator; for example, see the diagrammatic proof of [Kas95, Lemma XI.2.2]. In the case of Definition 5.1.1, the triangle axiom (5.1.3b) still provides these left and right triangle identities despite the fact that the associator α is a *pseudo*natural isomorphism. This is in large part because of strict unitality (4.2.3b) and the fact that the pentagon axiom holds if one of the middle tensorands is the monoidal unit (5.1.4). Let us show this for the left triangle identity by plugging in $U = V = I$ into (5.1.1b),

$$\alpha_{II(WX)} \alpha_{(II)WX} = (1_I \otimes \alpha_{IWX}) \alpha_{I(IW)X} (\alpha_{IIW} \otimes 1_X) \quad \Longrightarrow \quad \alpha_{IWX} = 1_{(WX)} \quad , \tag{5.1.6a}$$

and $f = g = 1_I$ into (5.1.1c),

$$\alpha_{(1_I \otimes 1_I) \boxtimes k \boxtimes x} = 1_I \otimes \alpha_{1_I \boxtimes k \boxtimes x} + \alpha_{1_I \boxtimes (1_I \otimes k) \boxtimes x} \quad \Longrightarrow \quad \alpha_{1_I \boxtimes k \boxtimes x} = 0 \quad . \tag{5.1.6b}$$

By plugging $W = X = I$ into (5.1.1b) and $k = x = 1_I$ into (5.1.1c) one shows the right triangle identity also holds hence the three 2-unitors in [CSP14, Definition C.1] are all trivial.

Furthermore, we can plug $U = V = I$ into (5.1.2). We use the fact that all three aforementioned

triangle identities hold together with (5.1.4) to get

$$2\Pi_{IWXY} = \Pi_{IWXY} \qquad \Longrightarrow \qquad \Pi_{IWXY} = 0 \quad . \tag{5.1.7}$$

Plugging $X = Y = I$ into (5.1.2) would likewise give use that $\Pi_{UVWI} = 0$.

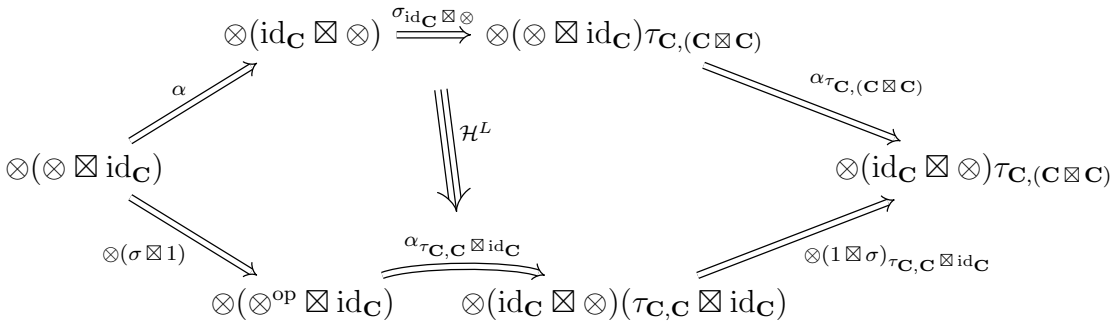
In summary, the prismatic axioms [CSP14, Axiom SM2 of Definition C.1] indeed hold because: both the left and right unitor are identities, all three 2-unitors are trivial, the pentagonator is trivial if one of its subscripts is I, the associator homotopy components of the unitors (which are unit 1-cells) vanishes as in (4.1.13). △

The following definition is our specification of the definition of a braiding on a monoidal bicategory as found in [CSP14, Definition C.2].

Definition 5.1.3. A **braided strictly-unital monoidal $\mathbf{Ch}_R^{[-1,0]}$-category** consists of a strictly-unital monoidal $\mathbf{Ch}_R^{[-1,0]}$-category $(\mathbf{C}, \otimes, I, \alpha, \Pi)$ together with a triple $(\sigma, \mathcal{H}^L, \mathcal{H}^R)$ of data:

(i) A pseudonatural isomorphism $\sigma : \otimes \Rightarrow \otimes\, \tau_{\mathbf{C},\mathbf{C}} : \mathbf{C} \boxtimes \mathbf{C} \to \mathbf{C}$.

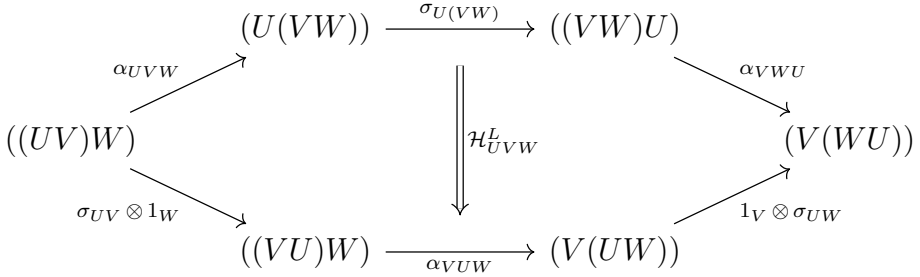
(ii) The **left hexagonator**[2],



is an isomodification,

$$\mathcal{H}^L : \alpha_{\tau_{\mathbf{C},(\mathbf{C}\boxtimes\mathbf{C})}}\, \sigma_{\mathrm{id}_{\mathbf{C}}\boxtimes\otimes}\, \alpha \Rrightarrow \big(\big[\otimes\,(1\boxtimes\sigma)\big]\alpha\big)_{\tau_{\mathbf{C},\mathbf{C}}\boxtimes\mathrm{id}_{\mathbf{C}}}\,[\otimes(\sigma\boxtimes 1)]\ . \tag{5.1.8a}$$

The following diagrams in $\mathbf{C}$ will appear throughout the enumeration of the axioms:

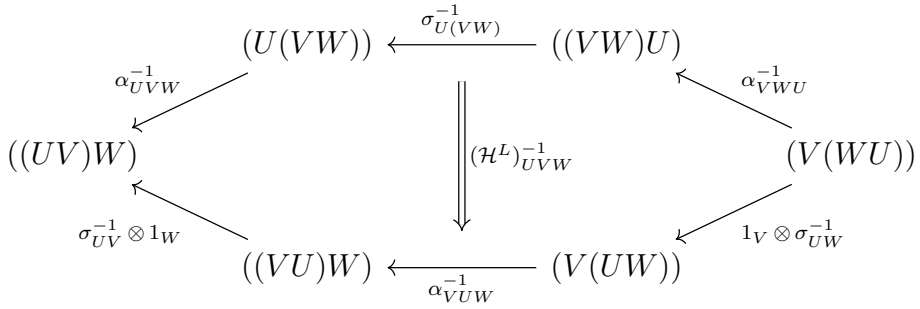


(5.1.8b)

[2]Note that we implicitly use the fact that τ is a natural transformation to write things such as

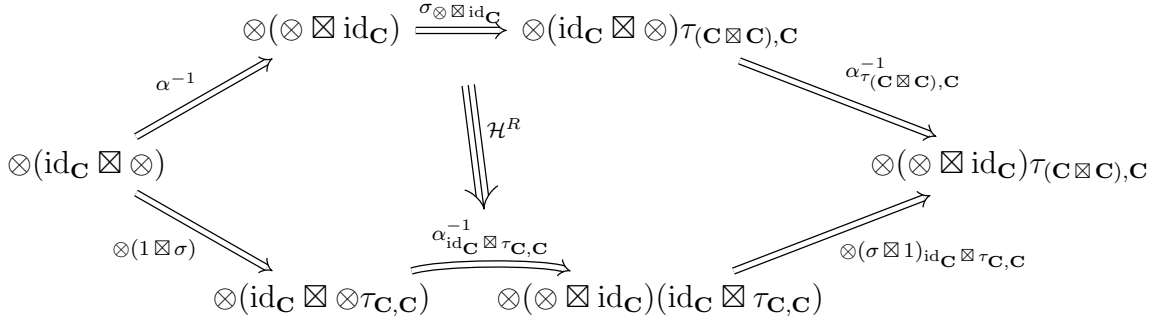
$$\tau_{\mathbf{C},\mathbf{C}}\,(\mathrm{id}_{\mathbf{C}}\boxtimes\otimes) = (\otimes\boxtimes\mathrm{id}_{\mathbf{C}})\tau_{\mathbf{C},(\mathbf{C}\boxtimes\mathbf{C})}$$

and we also use the fact that τ is a symmetric braiding hence satisfies the hexagon axiom (4.1.48).



(5.1.8c)

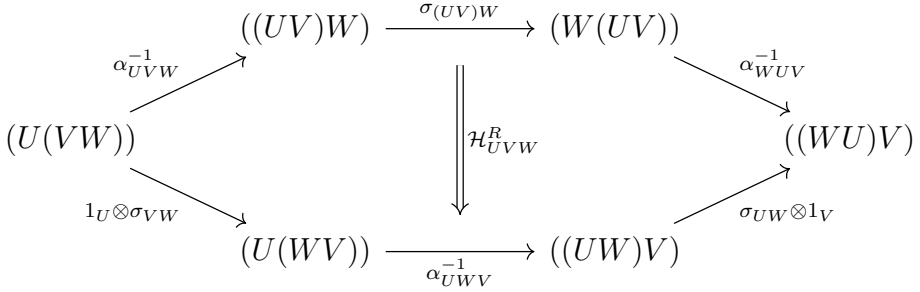
(iii) The **right hexagonator**,



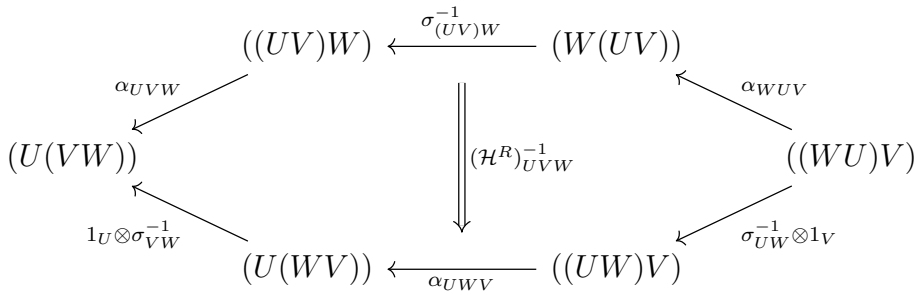
is an isomodification,

$$\mathcal{H}^R : \alpha^{-1}_{\tau_{(\mathbf{C}\boxtimes\mathbf{C}),\mathbf{C}}}\,\sigma_{\otimes\boxtimes \mathrm{id}_{\mathbf{C}}}\,\alpha^{-1} \Rrightarrow \big(\big[\otimes(\sigma\boxtimes 1)\big]\alpha^{-1}\big)_{\mathrm{id}_{\mathbf{C}}\boxtimes\tau_{\mathbf{C},\mathbf{C}}}\,[\otimes(1\boxtimes\sigma)]\ . \tag{5.1.9a}$$

The following diagrams in $\mathbf{C}$ will also appear throughout the enumeration of the axioms:



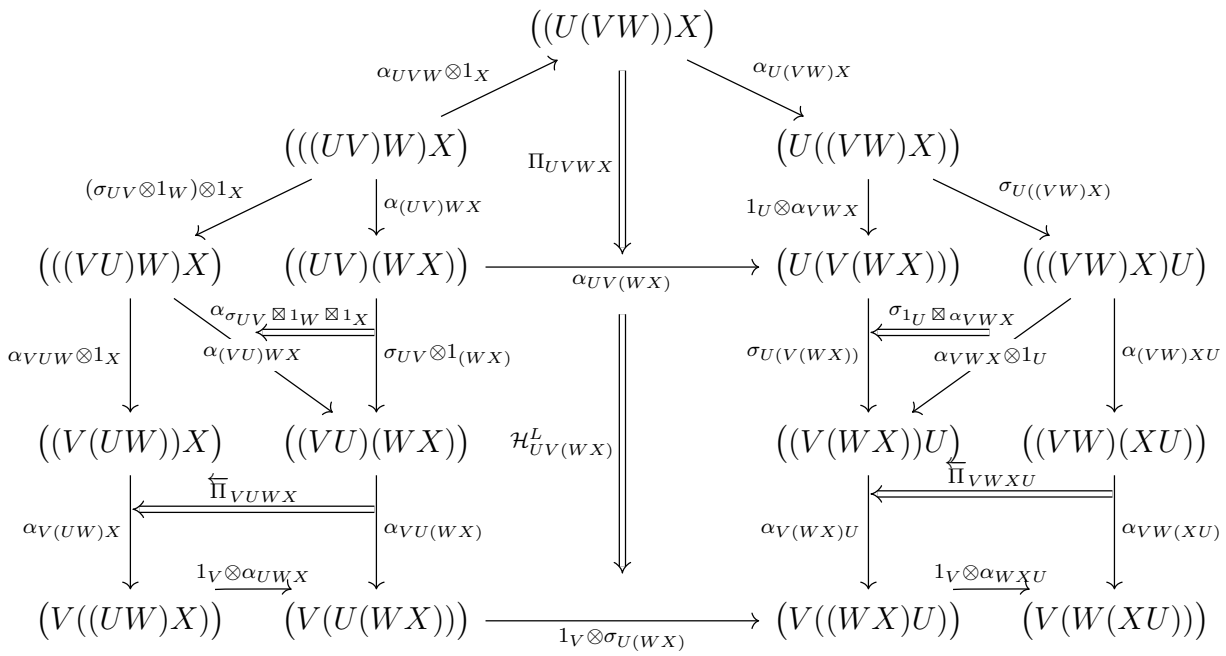
(5.1.9b)



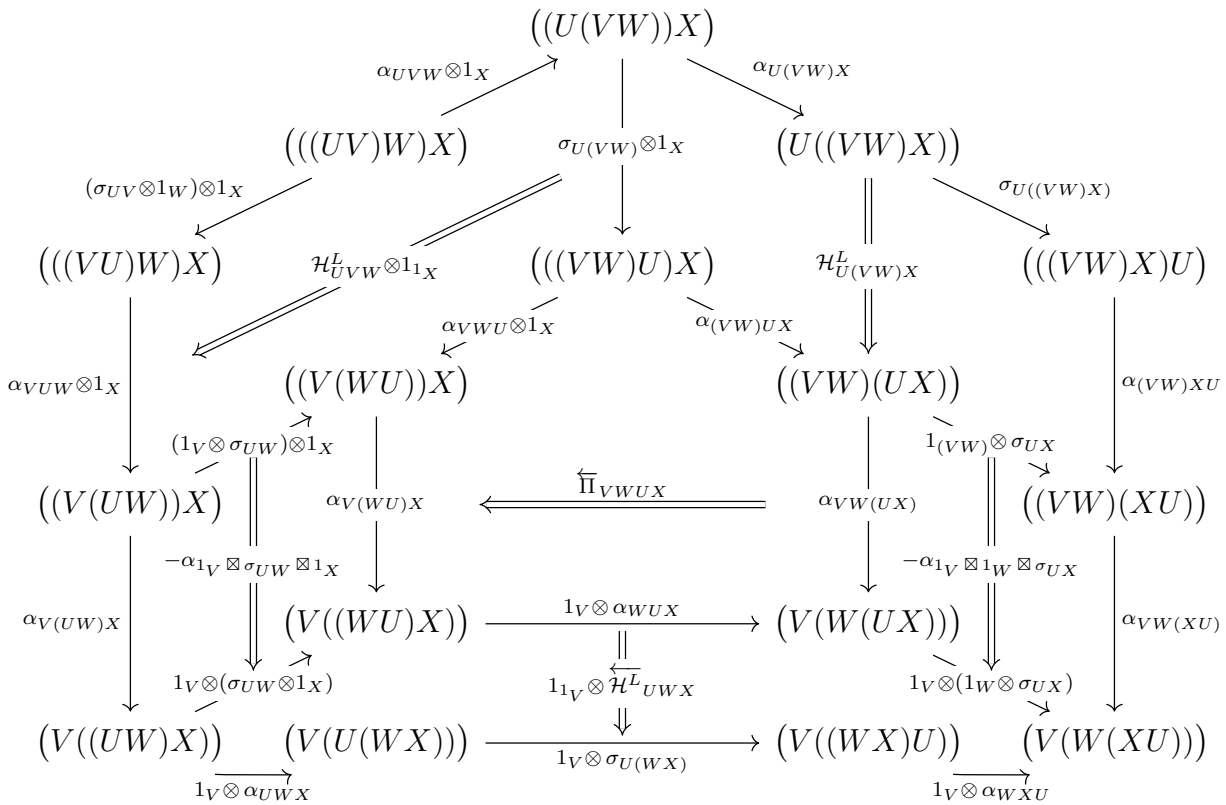
(5.1.9c)

This data is required to satisfy the following four axioms:

1. **Left tetrahedron**, i.e., the following two pasting diagrams in $\mathbf{C}$ are equivalent:



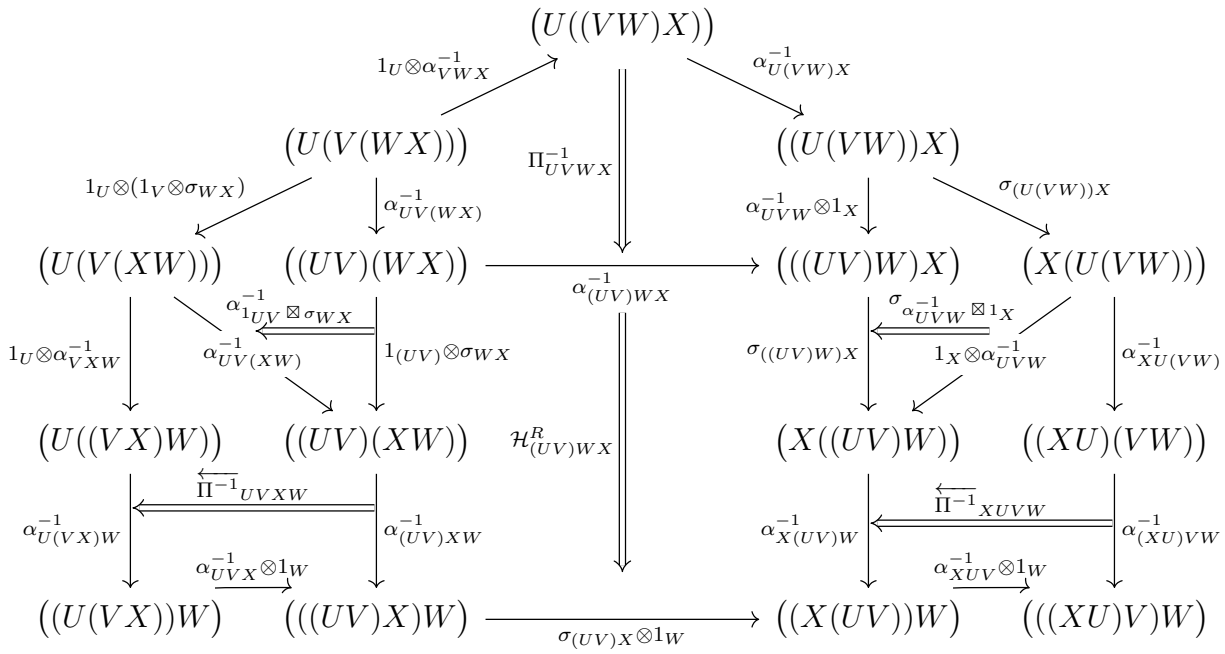
and



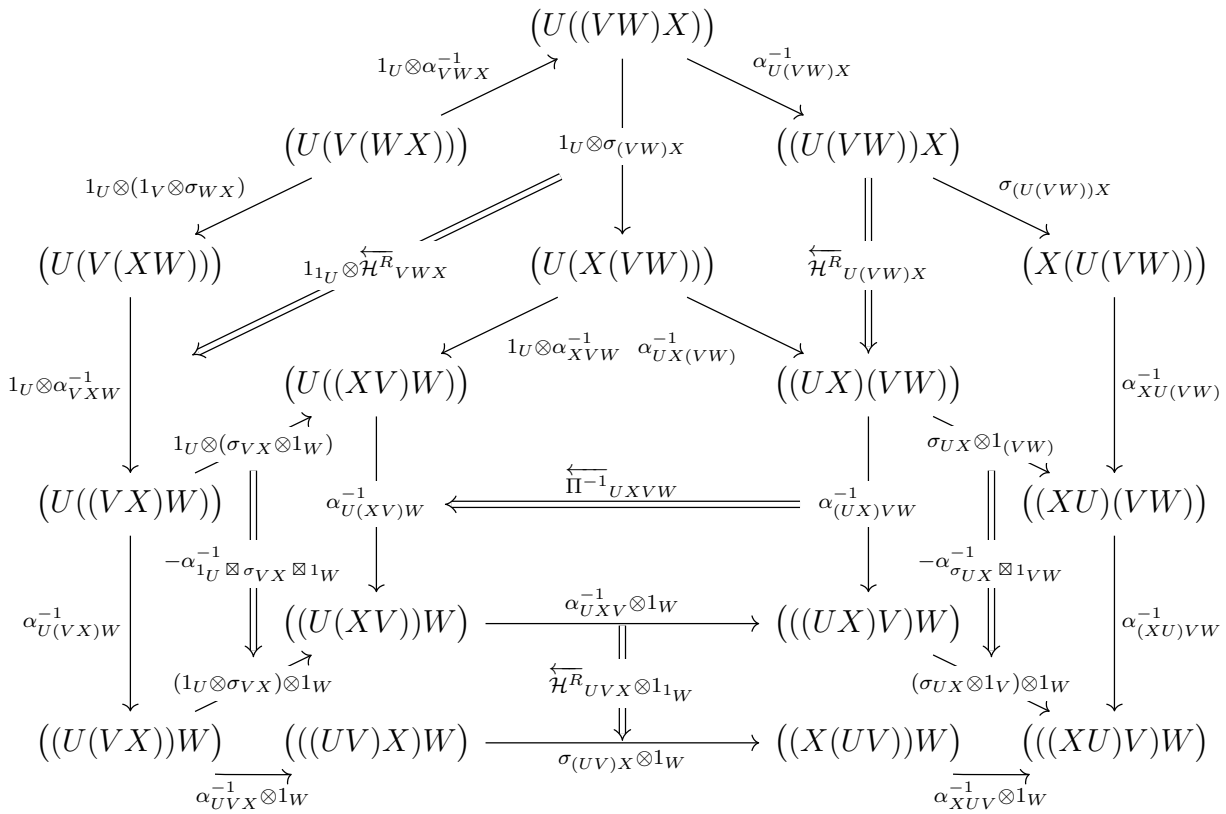
which reads explicitly as,

$$
\begin{aligned}
&(1_V \otimes \alpha_{WXU})\alpha_{V(WX)U}\sigma_{1_U \boxtimes \alpha_{VWX}}\alpha_{U(VW)X}(\alpha_{UVW} \otimes 1_X) - \Pi_{VWXU}\sigma_{U((VW)X)}\alpha_{U(VW)X}(\alpha_{UVW} \otimes 1_X) \\
&+ (1_V \otimes \alpha_{WXU})\mathcal{H}^L_{UV(WX)}\alpha_{(UV)WX} + (1_V \otimes \alpha_{WXU}\sigma_{U(WX)})\alpha_{VU(WX)}\alpha_{\sigma_{UV} \boxtimes 1_W \boxtimes 1_X} \\
&+ (1_V \otimes \alpha_{WXU})\alpha_{V(WX)U}\sigma_{U(V(WX))}\Pi_{UVWX} - (1_V \otimes \alpha_{WXU}\sigma_{U(WX)})\Pi_{VUWX}\big((\sigma_{UV} \otimes 1_W) \otimes 1_X\big) \\
&= \qquad\qquad (5.1.10) \\
&\alpha_{VW(XU)}\mathcal{H}^L_{U(VW)X}(\alpha_{UVW} \otimes 1_X) - (1_V \otimes (1_W \otimes \sigma_{UX})\alpha_{WUX})\,\alpha_{1_V \boxtimes \sigma_{UW} \boxtimes 1_X}\,(\alpha_{VUW}(\sigma_{UV} \otimes 1_W) \otimes 1_X) \\
&- \big(1_V \otimes (1_W \otimes \sigma_{UX})\big)\Pi_{VWUX}\big(\sigma_{U(VW)}\alpha_{UVW} \otimes 1_X\big) - \alpha_{1_V \boxtimes 1_W \boxtimes \sigma_{UX}}\alpha_{(VW)UX}(\sigma_{U(VW)}\alpha_{UVW} \otimes 1_X) \\
&+ (1_V \otimes (1_W \otimes \sigma_{UX})\alpha_{WUX})\,\alpha_{V(WU)X}(\mathcal{H}^L_{UVW} \otimes 1_X) - \big(1_V \otimes \mathcal{H}^L_{UWX}\big)\,\alpha_{V(UW)X}\,(\alpha_{VUW}(\sigma_{UV} \otimes 1_W) \otimes 1_X)
\end{aligned}
$$

2. **Right tetrahedron**, i.e., the following two pasting diagrams in $\mathbf{C}$ are equivalent:



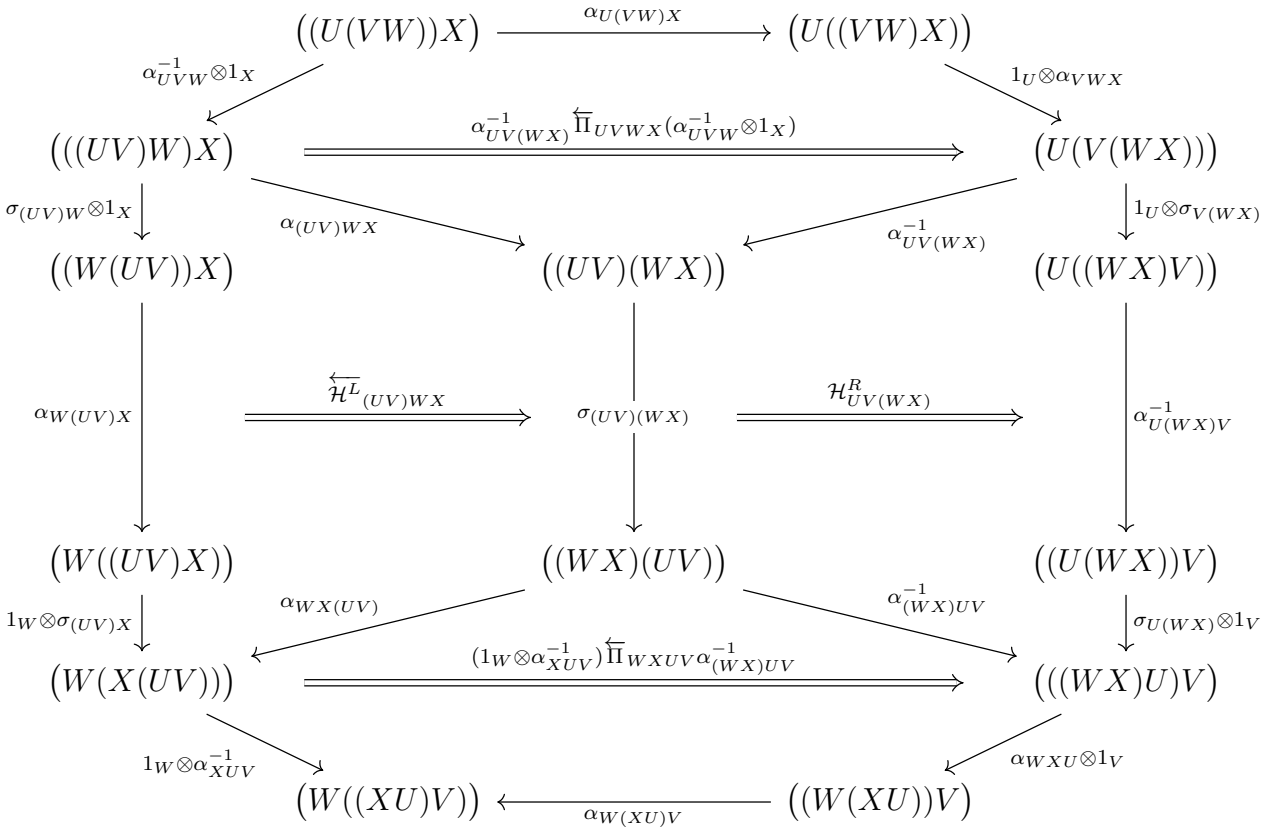
and



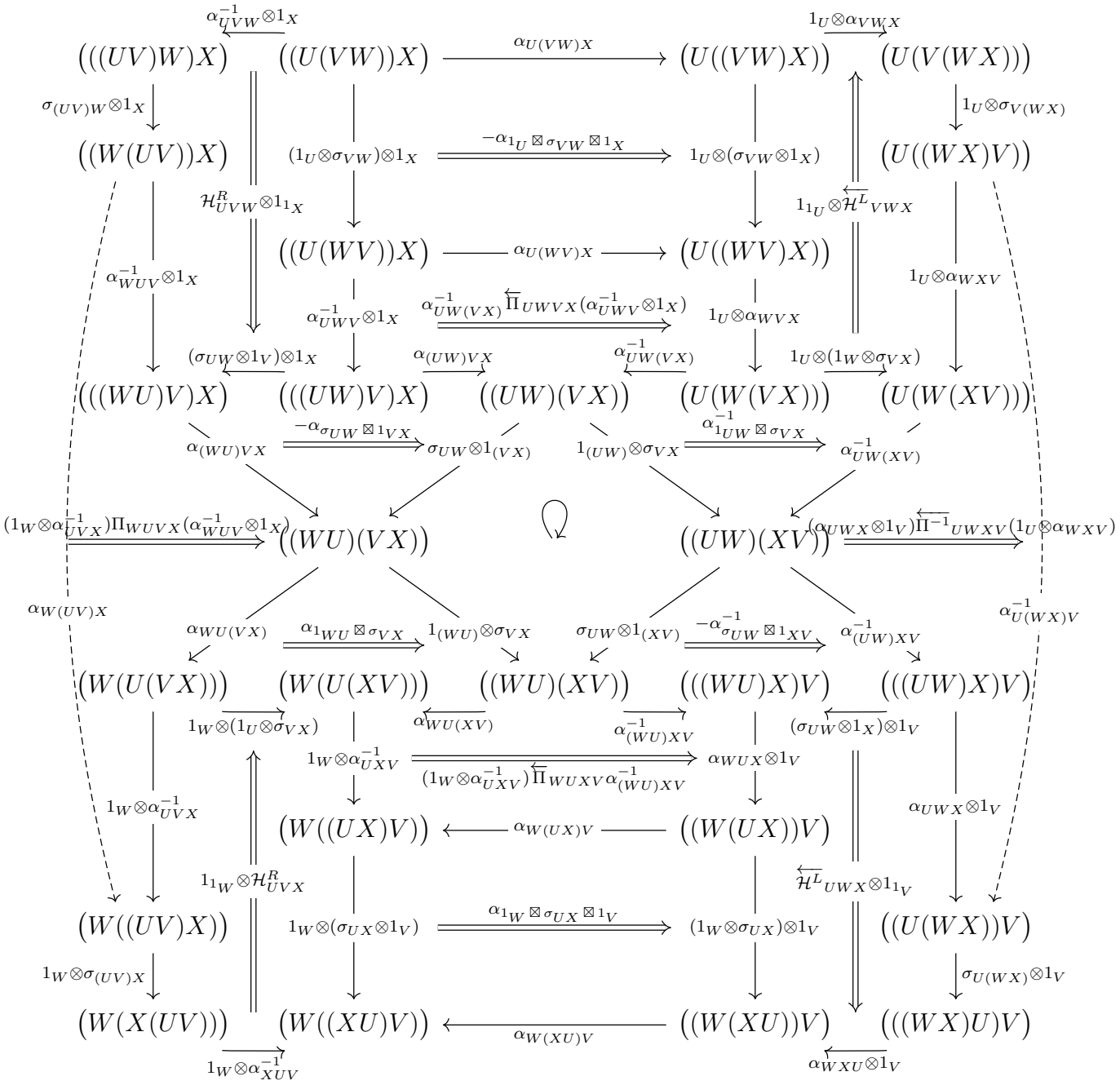
which reads explicitly as,

$$
\begin{aligned}
&(\alpha^{-1}_{XUV}\otimes 1_W)\alpha^{-1}_{X(UV)W}\sigma_{\alpha^{-1}_{UVW}\boxtimes 1_X}\alpha^{-1}_{U(VW)X}(1_U\otimes\alpha^{-1}_{VWX})-\Pi^{-1}_{XUVW}\sigma_{(U(VW))X)}\alpha^{-1}_{U(VW)X}(1_U\otimes\alpha^{-1}_{VWX})\\
&+(\alpha^{-1}_{XUV}\otimes 1_W)\mathcal{H}^R_{(UV)WX}\alpha^{-1}_{UV(WX)}+(\alpha^{-1}_{XUV}\sigma_{(UV)X}\otimes 1_W)\alpha^{-1}_{(UV)XW}\alpha^{-1}_{1_{UV}\boxtimes\sigma_{WX}}\\
&+(\alpha^{-1}_{XUV}\otimes 1_W)\alpha^{-1}_{X(UV)W}\sigma_{((UV)W)X}\Pi^{-1}_{UVWX}-(\alpha^{-1}_{XUV}\sigma_{(UV)X}\otimes 1_W)\Pi^{-1}_{UVXW}\big(1_U\otimes(1_V\otimes\sigma_{WX})\big)\\
&=\qquad\qquad(5.1.11)\\
&-\alpha^{-1}_{(XU)VW}\mathcal{H}^R_{U(VW)X}(1_U\otimes\alpha^{-1}_{VWX})-\big((\sigma_{UX}\otimes 1_V)\alpha^{-1}_{UXV}\otimes 1_W\big)\,\alpha^{-1}_{1_U\boxtimes\sigma_{VX}\boxtimes 1_W}\,\big(1_U\otimes\alpha^{-1}_{VXW}(1_V\otimes\sigma_{WX})\big)\\
&-\big((\sigma_{UX}\otimes 1_V)\otimes 1_W\big)\Pi^{-1}_{UXVW}\big(1_U\otimes\sigma_{(VW)X}\alpha^{-1}_{VWX}\big)-\alpha^{-1}_{\sigma_{UX}\boxtimes 1_{VW}}\alpha^{-1}_{U(X(VW))}\big(1_U\otimes\sigma_{(VW)X}\alpha^{-1}_{VWX}\big)\\
&-\big((\sigma_{UX}\otimes 1_V)\alpha^{-1}_{UXV}\otimes 1_W\big)\,\alpha^{-1}_{U(XV)W}(1_U\otimes\mathcal{H}^R_{VWX})-\big(\mathcal{H}^R_{UVX}\otimes 1_W\big)\,\alpha^{-1}_{U(VX)W}\,\big(1_U\otimes\alpha^{-1}_{VXW}(1_V\otimes\sigma_{WX})\big)
\end{aligned}
$$

3. **Hexahedron**, i.e., the following two pasting diagrams in $\mathbf{C}$ are equivalent:



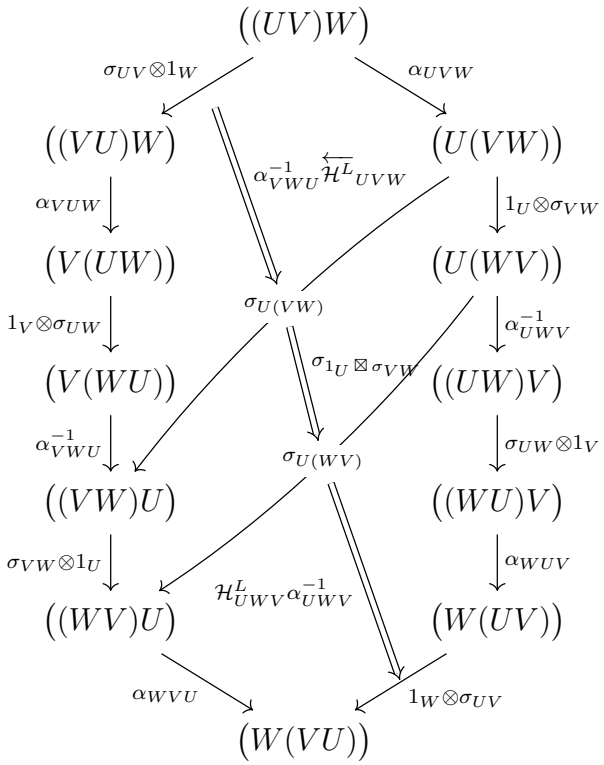
and

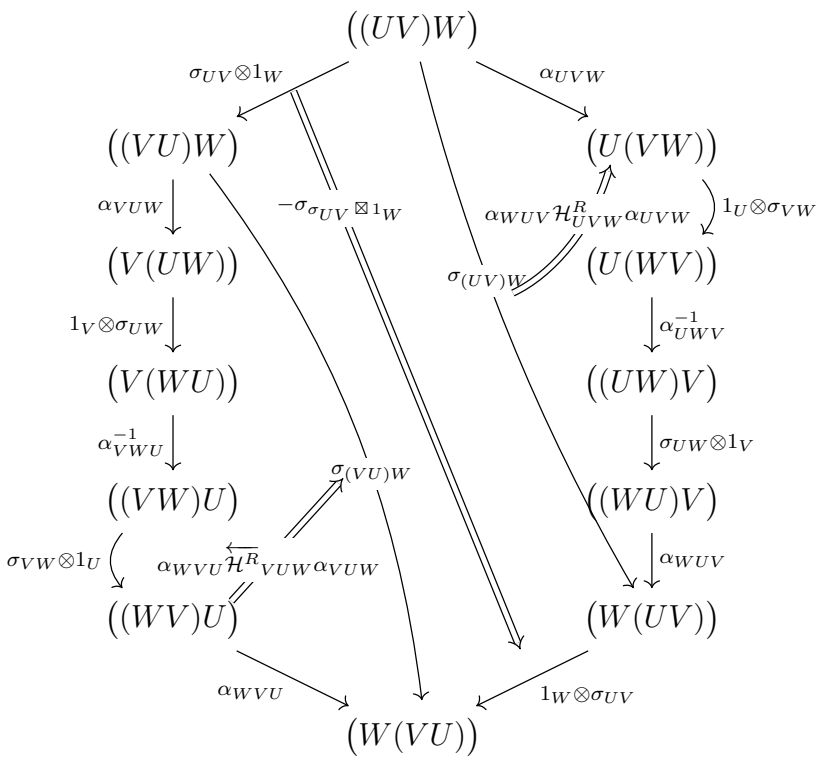

$\alpha^{-1}_{UVW}\otimes 1_X$
$\alpha_{U(VW)X}$
$1_U\otimes\alpha_{VWX}$
$\sigma_{(UV)W}\otimes 1_X$
$1_U\otimes\sigma_{V(WX)}$
$-\alpha_{1_U\boxtimes\sigma_{VW}\boxtimes 1_X}$
$\mathcal{H}^R_{UVW}\otimes 1_{1_X}$
$1_{1_U}\otimes\overleftarrow{\mathcal{H}^L}_{VWX}$
$\alpha^{-1}_{WUV}\otimes 1_X$
$1_U\otimes\alpha_{WVX}$
$\alpha^{-1}_{UWV}\otimes 1_X$
$\alpha^{-1}_{UW(VX)}\overleftarrow{\Pi}_{UWVX}(\alpha^{-1}_{UWV}\otimes 1_X)$
$(\sigma_{UW}\otimes 1_V)\otimes 1_X$
$\alpha_{(UW)VX}$
$\alpha^{-1}_{UW(VX)}$
$1_U\otimes(1_W\otimes\sigma_{VX})$
$\alpha_{(WU)VX}$
$-\alpha_{\sigma_{UW}\boxtimes 1_{VX}}$
$\sigma_{UW}\otimes 1_{(VX)}$
$1_{(UW)}\otimes\sigma_{VX}$
$\alpha^{-1}_{1_{UW}\boxtimes\sigma_{VX}}$
$\alpha^{-1}_{UW(XV)}$
$(1_W\otimes\alpha^{-1}_{UVX})\Pi_{WUVX}(\alpha^{-1}_{WUV}\otimes 1_X)$
$(\alpha_{UWX}\otimes 1_V)\overleftarrow{\Pi^{-1}}_{UWXV}(1_U\otimes\alpha_{WXV})$
$\alpha_{W(UV)X}$
$\alpha^{-1}_{U(WX)V}$
$\alpha_{WU(VX)}$
$\alpha_{1_{WU}\boxtimes\sigma_{VX}}$
$1_{(WU)}\otimes\sigma_{VX}$
$\sigma_{UW}\otimes 1_{(XV)}$
$-\alpha^{-1}_{\sigma_{UW}\boxtimes 1_{XV}}$
$\alpha^{-1}_{(UW)XV}$
$1_W\otimes(1_U\otimes\sigma_{VX})$
$\alpha^{-1}_{WU(XV)}$
$\alpha^{-1}_{(WU)XV}$
$(\sigma_{UW}\otimes 1_X)\otimes 1_V$
$1_W\otimes\alpha^{-1}_{UXV}$
$(1_W\otimes\alpha^{-1}_{UXV})\overleftarrow{\Pi}_{WUXV}\alpha^{-1}_{(WU)XV}$
$\alpha_{WUX}\otimes 1_V$
$1_W\otimes\alpha^{-1}_{UVX}$
$\alpha_{W(UX)V}$
$\alpha_{UWX}\otimes 1_V$
$1_{1_W}\otimes\mathcal{H}^R_{UVX}$
$\overleftarrow{\mathcal{H}^L}_{UWX}\otimes 1_{1_V}$
$1_W\otimes(\sigma_{UX}\otimes 1_V)$
$\alpha_{1_W\boxtimes\sigma_{UX}\boxtimes 1_V}$
$(1_W\otimes\sigma_{UX})\otimes 1_V$
$1_W\otimes\sigma_{(UV)X}$
$\sigma_{U(WX)}\otimes 1_V$
$\alpha_{W(XU)V}$
$1_W\otimes\alpha^{-1}_{XUV}$
$\alpha^{-1}_{WXU}\otimes 1_V$

which reads explicitly as,

$$
\begin{aligned}
&\alpha_{W(XU)V}(\alpha_{WXU}\otimes 1_V)\left[\mathcal{H}^R_{UV(WX)}(1_U\otimes\alpha_{VWX})\alpha_{U(VW)X}-\alpha^{-1}_{(WX)UV}\sigma_{(UV)(WX)}\alpha^{-1}_{UV(WX)}\Pi_{UVWX}(\alpha^{-1}_{UVW}\otimes 1_X)\right]\\
&-(1_W\otimes\alpha^{-1}_{XUV})\left[\Pi_{WXUV}\alpha^{-1}_{(WX)UV}\sigma_{(UV)(WX)}\alpha_{(UV)WX}+\mathcal{H}^L_{(UV)WX}\right](\alpha^{-1}_{UVW}\otimes 1_X)\\
&= \qquad (5.1.12)\\
&\left[(1_W\otimes\alpha^{-1}_{XUV}\sigma_{(UV)X}\alpha^{-1}_{UVX})\Pi_{WUVX}+(1_W\otimes\mathcal{H}^R_{UVX})\alpha_{WU(VX)}\alpha_{(WU)VX}\right](\alpha^{-1}_{WUV}\sigma_{(UV)W}\alpha^{-1}_{UVW}\otimes 1_X)\\
&+[1_W\otimes(\sigma_{UX}\otimes 1_V)\alpha^{-1}_{UXV}(1_U\otimes\sigma_{VX})]\alpha_{WU(VX)}\left[\alpha_{(WU)VX}(\mathcal{H}^R_{UVW}\otimes 1_X)-\alpha_{\sigma_{UW}\boxtimes 1_{VX}}\left(\alpha^{-1}_{UWV}(1_U\otimes\sigma_{VW})\otimes 1_X\right)\right]\\
&+(1_W\otimes[\sigma_{UX}\otimes 1_V]\alpha^{-1}_{UXV})\alpha_{1_{WU}\boxtimes\sigma_{VX}}(\sigma_{UW}\otimes 1_{(VX)})\alpha_{(UW)VX}\left(\alpha^{-1}_{UWV}[1_U\otimes\sigma_{VW}]\otimes 1_X\right)\\
&+\alpha_{1_W\boxtimes\sigma_{UX}\boxtimes 1_V}(\alpha_{WUX}\otimes 1_V)\alpha^{-1}_{(WU)XV}[\sigma_{UW}\otimes\sigma_{VX}]\alpha_{(UW)VX}\left[\alpha^{-1}_{UWV}[1_U\otimes\sigma_{VW}]\otimes 1_X\right]\\
&-\left[1_W\otimes[\sigma_{UX}\otimes 1_V]\alpha^{-1}_{UXV}\right]\Pi_{WUXV}\alpha^{-1}_{(WU)XV}[\sigma_{UW}\otimes\sigma_{VX}]\alpha_{(UW)VX}\left[\alpha^{-1}_{UWV}[1_U\otimes\sigma_{VW}]\otimes 1_X\right]\\
&-\alpha_{W(XU)V}\left[(1_W\otimes\sigma_{UX})\alpha_{WUX}\otimes 1_V\right]\alpha^{-1}_{(WU)XV}(\sigma_{UW}\otimes\sigma_{VX})\alpha^{-1}_{UW(VX)}\Pi_{UWVX}[\alpha^{-1}_{UWV}(1_U\otimes\sigma_{VW})\otimes 1_X]\\
&-\alpha_{W(XU)V}\left[(1_W\otimes\sigma_{UX})\alpha_{WUX}\otimes 1_V\right]\alpha^{-1}_{(WU)XV}(\sigma_{UW}\otimes\sigma_{VX})\alpha^{-1}_{UW(VX)}(1_U\otimes\alpha_{WVX})\alpha_{1_U\boxtimes\sigma_{VW}\boxtimes 1_X}\\
&-\alpha_{W(XU)V}\left[(1_W\otimes\sigma_{UX})\alpha_{WUX}\otimes 1_V\right]\alpha^{-1}_{\sigma_{UW}\boxtimes 1_{XV}}(1_{(UW)}\otimes\sigma_{VX})\alpha^{-1}_{UW(VX)}[1_U\otimes\alpha_{WVX}(\sigma_{VW}\otimes 1_X)]\alpha_{U(VW)X}\\
&+\alpha_{W(XU)V}\left[(1_W\otimes\sigma_{UX})\alpha_{WUX}(\sigma_{UW}\otimes 1_X)\otimes 1_V\right]\alpha^{-1}_{(UW)XV}\alpha^{-1}_{1_{UW}\boxtimes\sigma_{VX}}[1_U\otimes\alpha_{WVX}(\sigma_{VW}\otimes 1_X)]\alpha_{U(VW)X}\\
&-\alpha_{W(XU)V}(\mathcal{H}^L_{UWX}\otimes 1_V)\alpha^{-1}_{(UW)XV}\alpha^{-1}_{UW(XV)}[1_U\otimes(1_W\otimes\sigma_{VX})\alpha_{WVX}(\sigma_{VW}\otimes 1_X)]\alpha_{U(VW)X}\\
&-\alpha_{W(XU)V}\left(\alpha_{WXU}\sigma_{U(WX)}\alpha_{UWX}\otimes 1_V\right)\alpha^{-1}_{(UW)XV}\alpha^{-1}_{UW(XV)}(1_U\otimes\mathcal{H}^L_{VWX})\alpha_{U(VW)X}\\
&-\alpha_{W(XU)V}\left(\alpha_{WXU}\sigma_{U(WX)}\alpha_{UWX}\otimes 1_V\right)\Pi^{-1}_{UWXV}(1_U\otimes\alpha_{WXV}\sigma_{V(WX)}\alpha_{VWX})\alpha_{U(VW)X}
\end{aligned}
$$

4. **Breen polytope**, i.e., the following two pasting diagrams in $\mathbf{C}$ are equivalent:





which reads explicitly as,

$$
\begin{aligned}
&-\alpha_{WVU}(\sigma_{VW}\otimes 1_U)\alpha^{-1}_{VWU}\mathcal{H}^L_{UVW}+\alpha_{WVU}\sigma_{1_U\boxtimes\sigma_{VW}}\alpha_{UVW}+\mathcal{H}^L_{UWV}\alpha^{-1}_{(UW)V}(1_U\otimes\sigma_{VW})\alpha_{UVW}\\
&= \qquad (5.1.13)\\
&-\alpha_{WVU}\mathcal{H}^R_{VUW}\alpha_{VUW}(\sigma_{UV}\otimes 1_W)-\sigma_{\sigma_{UV}\boxtimes 1_W}+(1_W\otimes\sigma_{UV})\alpha_{WUV}\mathcal{H}^R_{UVW}\alpha_{UVW} \quad .
\end{aligned}
$$

A **hexagonally-braided** strict monoidal $\mathbf{Ch}^{[-1,0]}_R$-category $(\mathbf{C},\otimes,I,\sigma)$ is a pseudonaturally-

braided strict monoidal $\mathbf{Ch}_R^{[-1,0]}$-category whose braiding further satisfies the **strict hexagonal Breen polytope**,

$$\sigma_{1_U \boxtimes \sigma_{VW}} + \sigma_{\sigma_{UV} \boxtimes 1_W} = 0 \quad . \tag{5.1.14}$$

Definition 5.1.4. An infinitesimal 2-braiding $\otimes \overset{t}{\Rightarrow} \otimes$ is **Breen** if it satisfies

$$\gamma_{U(WV)} t_{1_U \boxtimes \gamma_{VW}} + \gamma_{(VU)W} t_{\gamma_{UV} \boxtimes 1_W} = 0 \quad . \tag{5.1.15}$$

Example 5.1.5. It is clear from (4.2.30) and (4.2.31b) that those γ-equivariant hexagonal infinitesimal 2-braidings induced by 2SPSs are Breen. ▽

Theorem 4.2.7 showed that an infinitesimal 2-braiding t provides a first-order deformation of a symmetric strict monoidal cochain 2-category to a pseudonaturally-braided strict monoidal cochain 2-category if and only if t is hexagonal.

Theorem 5.1.6. *An infinitesimal 2-braiding t provides a first-order deformation quantisation of the symmetric strict monoidal $\mathbf{Ch}_{\mathbb{K}}^{[-1,0]}$-category $(\mathbf{C}, \otimes, I, \gamma)$ to a hexagonally-braided strict monoidal $\mathbf{Ch}_{\mathbb{K}[h]/(h^2)}^{[-1,0]}$-category*

$$\big(\mathbf{C}_{\mathbb{K}[h]/(h^2)}, \otimes, I, \sigma :\equiv \gamma[1 + \tfrac{h}{2}t]\big) \tag{5.1.16}$$

if and only if t is a hexagonal Breen infinitesimal 2-braiding.

Proof. The Breen property (5.1.15) of the infinitesimal 2-braiding is obviously equivalent to the order h relation coming from the strict hexagonal Breen axiom (5.1.14). □

5.2 Second-order deformation quantisation

This section recalls CM's important notions of coherency (see Definition 5.2.30) and total γ-equivariance (see Definition 5.2.20) of an infinitesimal 2-braiding. We demonstrate in Propositions 5.2.18, 5.2.29 and 5.2.33 that a coherent totally γ-equivariant hexagonal infinitesimal 2-braiding provides a second-order deformation quantisation of a symmetric strict monoidal $\mathbf{Ch}_{\mathbb{K}}^{[-1,0]}$-category to a braided pentagonal strictly-unital monoidal $\mathbf{Ch}_{\mathbb{K}[h]/(h^3)}^{[-1,0]}$-category. We then show, in Remark 6.2.5, that the coboundary σ-equivariant syllepsis of Remark 6.2.1 is also a σ-equivariant syllepsis on the aforementioned braided pentagonal strictly-unital monoidal $\mathbf{Ch}_{\mathbb{K}[h]/(h^3)}^{[-1,0]}$-category.

5.2.1 Associativity of the monoidal product and subscript index notation

The following index notation follows that of [KLS25, (3.32)].

Remark 5.2.1. The following point can not be stressed enough, the monoidal product $\otimes$ will *always* be associative $\otimes(\otimes \boxtimes \mathrm{id}_{\mathbf{C}}) = \otimes(\mathrm{id}_{\mathbf{C}} \boxtimes \otimes)$; we are deforming the associator to a nontrivial pseudonatural *auto*morphism. Thus, $\otimes^n : \mathbf{C}^{\boxtimes(n+1)} \to \mathbf{C}$ will always be well-defined $\forall n \in \mathbb{N}$ and this permits the following index notation:

1. We write things such as:

$$t_{12} :\equiv \otimes(t \boxtimes \mathrm{Id}_{\mathrm{id}_{\mathbf{C}}}) \quad , \quad \alpha_{1(23)4} :\equiv \alpha_{\mathrm{id}_{\mathbf{C}} \boxtimes \otimes \boxtimes \mathrm{id}_{\mathbf{C}}} \quad , \quad \sigma_{13} :\equiv \otimes(\mathrm{Id}_{\mathrm{id}_{\mathbf{C}}} \boxtimes \sigma)_{\tau_{\mathbf{C},\mathbf{C}} \boxtimes \mathrm{id}_{\mathbf{C}}} . \tag{5.2.1a}$$

2. The fact that γ is natural, involutive and satisfies the hexagon axiom means that we have, for example,

$$t_{13} :\equiv \otimes\big([\gamma^{-1}\boxtimes 1](1\boxtimes t)_{\tau_{\mathbf{C},\mathbf{C}}\boxtimes \mathrm{id}_{\mathbf{C}}}[\gamma\boxtimes 1]\big) \equiv \otimes\big([1\boxtimes\gamma^{-1}](t\boxtimes 1)_{\mathrm{id}_{\mathbf{C}}\boxtimes\tau_{\mathbf{C},\mathbf{C}}}[1\boxtimes\gamma]\big)\ . \quad (5.2.1b)$$

3. If the infinitesimal 2-braiding is hexagonal then this means, for example,

$$t_{1(23)} :\equiv t_{\mathrm{id}_{\mathbf{C}}\boxtimes\otimes} \equiv t_{12}+t_{13} \quad . \quad (5.2.1c)$$

In general, for $|\{i,j,k\}|=3$, we would have:

$$t_{k(ij)} \equiv t_{ki}+t_{kj} \qquad , \qquad t_{(ij)k}\equiv t_{ik}+t_{jk} \quad . \quad (5.2.1d)$$

4. If t is γ-equivariant then this means, for example,

$$t_{12} :\equiv t \equiv \gamma^{-1}t_{\tau_{\mathbf{C},\mathbf{C}}}\gamma \equiv: t_{21} \quad . \quad (5.2.1e)$$

In general, for $k\neq i\neq j\neq k$, we would have:

$$t_{ij}\equiv t_{ji} \qquad , \qquad t_{(ij)k}\equiv t_{k(ij)} \quad . \quad (5.2.1f)$$

△

This subscript index notation allows us to rewrite: the pentagonator (5.1.1a) as

$$\Pi : \alpha_{234}\alpha_{1(23)4}\alpha_{123} \Rrightarrow \alpha_{12(34)}\alpha_{(12)34} \quad , \quad (5.2.2a)$$

the left hexagonator (5.1.8a) as

$$\mathcal{H}^L : \alpha_{231}\sigma_{1(23)}\alpha \Rrightarrow \sigma_{13}\alpha_{213}\sigma_{12} \quad , \quad (5.2.2b)$$

and the right hexagonator (5.1.9a) as

$$\mathcal{H}^R : \alpha^{-1}_{312}\sigma_{(12)3}\alpha^{-1} \Rrightarrow \sigma_{13}\alpha^{-1}_{132}\sigma_{23} \quad . \quad (5.2.2c)$$

Construction 5.2.2. Choosing Drinfeld's Knizhnik-Zamolodchikov series as our ansatz associator $\alpha\equiv\Phi_{\mathrm{KZ}}(t_{12},t_{23})$, the braiding as $\sigma\equiv\gamma e^{\frac{h}{2}t}$, the left hexagonator as $\mathcal{H}^L=\gamma_{1(23)}\mathbf{L}$ and the right hexagonator as $\mathcal{H}^R=\gamma_{(12)3}\mathbf{R}$, we rewrite: the pentagonator (5.2.2a) as

$$\Pi : \Phi(t_{23},t_{34})\Phi(t_{1(23)},t_{(23)4})\Phi(t_{12},t_{23}) \Rrightarrow \Phi(t_{12},t_{2(34)})\Phi(t_{(12)3},t_{34}) \quad , \quad (5.2.3a)$$

the left hexagonator (5.2.2b) as the **left pre-hexagonator**,

$$\mathbf{L} : \Phi(t_{23},t_{31})e^{\frac{h}{2}t_{1(23)}}\Phi(t_{12},t_{23}) \Rrightarrow e^{\frac{h}{2}t_{13}}\Phi(t_{21},t_{13})e^{\frac{h}{2}t_{12}} \quad , \quad (5.2.3b)$$

and the right hexagonator (5.2.2c) as[3] the **right pre-hexagonator**,

$$\mathbf{R} : \Phi(t_{12}\,,t_{31})e^{\frac{h}{2}t_{(12)3}}\Phi(t_{23}\,,t_{12}) \Rrightarrow e^{\frac{h}{2}t_{13}}\Phi(t_{32}\,,t_{13})e^{\frac{h}{2}t_{23}} \quad . \quad (5.2.3c)$$

▷

Remark 5.2.3. For a γ-equivariant hexagonal t, we use (5.2.1d) and (5.2.1f) to rewrite (5.2.3)

[3] Note that $\Phi_{\mathrm{KZ}}(A,B)=\Phi_{\mathrm{KZ}}(B,A)^{-1}$, see [BRW23, (2.24)].

as:

$$\Pi : \Phi(t_{23}, t_{34})\Phi(t_{12} + t_{13}, t_{24} + t_{34})\Phi(t_{12}, t_{23}) \Rrightarrow \Phi(t_{12}, t_{23} + t_{24})\Phi(t_{13} + t_{23}, t_{34}) \quad , \tag{5.2.4a}$$

$$\mathbf{L} : \Phi(t_{23}, t_{13})e^{\frac{h}{2}(t_{12}+t_{13})}\Phi(t_{12}, t_{23}) \Rrightarrow e^{\frac{h}{2}t_{13}}\Phi(t_{12}, t_{13})e^{\frac{h}{2}t_{12}} \quad , \tag{5.2.4b}$$

$$\mathbf{R} : \Phi(t_{12}\,, t_{13})e^{\frac{h}{2}(t_{13}+t_{23})}\Phi(t_{23}\,, t_{12}) \Rrightarrow e^{\frac{h}{2}t_{13}}\Phi(t_{23}\,, t_{13})e^{\frac{h}{2}t_{23}} \quad . \tag{5.2.4c}$$

△

Remark 5.2.4. For a γ-equivariant hexagonal t, it might look as if we can define $\mathbf{L} := \mathbf{R}_{321}$ however we will show in Remark 5.2.12 that this fails already at order h^2. We show in Lemma 5.2.25 that this succeeds if t is totally γ-equivariant stict. △

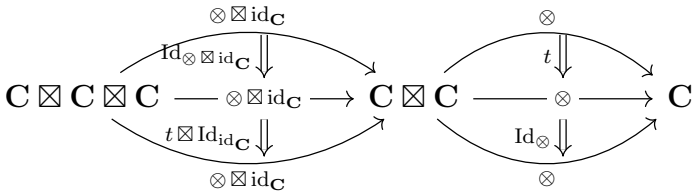
5.2.2 Four-term relationators and infinitesimal hexagonators

Definition 5.2.5. A **left 4-term relationator** is an endomodification of the following form,

$$\mathcal{L} : [t_{12}\,, t_{(12)3}] \Rrightarrow 0 : \otimes^2 \Rightarrow \otimes^2 : \mathbf{C}^{\boxtimes 3} \to \mathbf{C} \quad . \tag{5.2.5}$$

Remark 5.2.6. Given some modification $\Xi : \xi \Rrightarrow \xi'$, we refer to Construction 4.1.17 to define a new modification $\widehat{\Xi} := \Xi - \mathrm{ID}_{\xi'} : \xi - \xi' \Rrightarrow 0$ which obviously has the same components. △

Proposition 5.2.7. *The specific instance of the exchanger*



$$, \tag{5.2.6a}$$

i.e.,

$$*^2_{(\mathrm{Id}_\otimes, t \boxtimes \mathrm{Id}_{\mathrm{id}_\mathbf{C}}), (t, \mathrm{Id}_{\otimes \boxtimes \mathrm{id}_\mathbf{C}})} : \otimes(t \boxtimes 1)\, t_{\otimes \boxtimes \mathrm{id}_\mathbf{C}} \Rrightarrow t * (t \boxtimes 1) : \otimes^2 \Rightarrow \otimes^2 : \mathbf{C}^{\boxtimes 3} \to \mathbf{C} \quad , \tag{5.2.6b}$$

encodes a left 4-term relationator (5.2.5).

Proof. Observe that

$$[t_{12}\,, t_{(12)3}] :\equiv \otimes(t \boxtimes 1)\, t_{\otimes \boxtimes \mathrm{id}_\mathbf{C}} - t_{\otimes \boxtimes \mathrm{id}_\mathbf{C}} \otimes (t \boxtimes 1) \equiv \otimes(t \boxtimes 1)\, t_{\otimes \boxtimes \mathrm{id}_\mathbf{C}} - t * (t \boxtimes 1)\,, \tag{5.2.7}$$

where the last equality comes from the condition for triviality of the exchanger in Remark 4.1.22. We now use the overhat notation from Remark 5.2.6 to define

$$\mathcal{L} := \hat{*}^2_{(\mathrm{Id}_\otimes, t \boxtimes \mathrm{Id}_{\mathrm{id}_\mathbf{C}}), (t, \mathrm{Id}_{\otimes \boxtimes \mathrm{id}_\mathbf{C}})} \quad . \tag{5.2.8}$$

□

Remark 5.2.8. We can express the components of (5.2.8) by using the formula (4.1.38),

$$\mathcal{L}_{UVW} = \left(\hat{*}^2_{(\mathrm{Id}_\otimes, t \boxtimes \mathrm{Id}_{\mathrm{id}_\mathbf{C}}), (t, \mathrm{Id}_{\otimes \boxtimes \mathrm{id}_\mathbf{C}})}\right)_{UVW} = t_{t_{UV} \boxtimes 1_W} \quad . \tag{5.2.9}$$

Notice that the benefit of regarding the left 4-term relationator as a specific instance of the exchanger is that, in order to show that the components (5.2.9) constitute a modification, we do

not need to show that the condition (4.1.16) holds (cf. [KLS25, Proposition 3.13]) given that we *already* know the exchanger to be a modification in general. △

Because of (5.2.9), we can compactly write

$$\mathcal{L} = t_{t_{12}} : [t_{12}\,, t_{(12)3}] \Rrightarrow 0 \quad . \tag{5.2.10}$$

Of course, we also have a **right 4-term relationator**,

$$\mathcal{R} := t_{t_{23}} = \hat{*}^2_{(\mathrm{Id}_\otimes, \mathrm{Id}_{\mathrm{id}_{\mathbf{C}}} \boxtimes t), (t, \mathrm{Id}_{\mathrm{id}_{\mathbf{C}}} \boxtimes \otimes)} : [t_{23}\,, t_{1(23)}] \Rrightarrow 0 \quad . \tag{5.2.11}$$

Lemma 5.2.9. *If the infinitesimal 2-braiding t is γ-equivariant then:*

$$\mathcal{L} = \mathcal{R}_{312} \;\;,\;\; \mathcal{L}_{213} = \mathcal{R}_{321} \;\;,\;\; \mathcal{R} = \mathcal{L}_{231} \;\;,\;\; \mathcal{R}_{132} = \mathcal{L}_{321} \;\;,\;\; \mathcal{L}_{132} = \mathcal{R}_{213} \;\;,\;\; \mathcal{L}_{312} = \mathcal{R}_{231}\,. \tag{5.2.12}$$

Proof. Let us demonstrate the fifth relation $\mathcal{L}_{132} = \mathcal{R}_{213}$, i.e.,

$$(1_U \otimes \gamma_{WV})\mathcal{L}_{UWV}(1_U \otimes \gamma_{VW}) = (\gamma_{VU} \otimes 1_W)\mathcal{R}_{VUW}(\gamma_{UV} \otimes 1_W) \quad . \tag{5.2.13a}$$

Using the fact that γ is involutive and satisfies the hexagon axiom, we rewrite this as

$$\mathcal{L}_{UWV} = \gamma_{V(UW)}\mathcal{R}_{VUW}\gamma_{(UW)V} \qquad \Longrightarrow \qquad t_{t_{UW} \boxtimes 1_V} = \gamma_{V(UW)} t_{1_V \boxtimes t_{UW}} \gamma_{(UW)V} \quad , \tag{5.2.13b}$$

but the latter equality is a special case of (4.2.11b). The other relations are proved similarly. □

Lemma 5.2.10. *For a hexagonal infinitesimal 2-braiding, a third 4-term relationator is given by*

$$\mathcal{L} + \mathcal{R} = t_{t_{12}+t_{23}} : [t_{12} + t_{23}\,, t_{13}] \Rrightarrow 0 \quad . \tag{5.2.14a}$$

Proof. Applying Construction 4.1.17 and Item 3 of Remark 5.2.1 tells us that

$$\mathcal{L}+\mathcal{R} : [t_{12}\,, t_{(12)3}] + [t_{23}\,, t_{1(23)}] \equiv [t_{12}\,, t_{13}+t_{23}] + [t_{23}\,, t_{12}+t_{13}] \equiv [t_{12}+t_{23}\,, t_{13}] \Rrightarrow 0 \quad . \tag{5.2.14b}$$

□

Infinitesimal hexagonators

Construction 5.2.11. We expand the **pre-braiding** $\gamma^{-1}\sigma :\equiv e^{\frac{h}{2}t}$ to order h^2 and consult [Kas95, Corollary XIX.6.5] for the order h^2 expansion of the Drinfeld associator series $\Phi(t_{12}\,, t_{23})$:

$$\Phi(t_{12}\,, t_{23}) \equiv 1 + \frac{h^2}{24}[t_{12}\,, t_{23}] \quad , \tag{5.2.15a}$$

$$e^{\frac{h}{2}t} \equiv 1 + \frac{h}{2}t + \frac{h^2}{8}t^2 \quad . \tag{5.2.15b}$$

Assuming a γ-equivariant hexagonal infinitesimal 2-braiding, substituting the above ansatz into (5.2.4b) and taking into account only order h^2 terms, we have

$$\frac{1}{24}[t_{23}\,, t_{13}] + \frac{1}{8}(t_{12}+t_{13})^2 + \frac{1}{24}[t_{12}\,, t_{23}] \Rrightarrow \frac{1}{8}t_{13}^2 + \frac{1}{4}t_{13}t_{12} + \frac{1}{8}t_{12}^2 + \frac{1}{24}[t_{12}\,, t_{13}] \quad . \tag{5.2.16a}$$

Balancing terms as in Remark 5.2.6, we have the following constraint

$$\frac{1}{24}(2\mathcal{L} + \mathcal{R}) : \frac{1}{24}([t_{12}+t_{23}\,, t_{13}] + [t_{12}\,, t_{(12)3}]) \Rrightarrow 0 : \otimes^2 \Rightarrow \otimes^2 : \mathbf{C}^{\boxtimes 3} \to \mathbf{C} \quad . \tag{5.2.16b}$$

In other words, we have a candidate left hexagonator

$$\mathcal{H}^L := \frac{h^2}{24}\gamma_{1(23)}(2\mathcal{L}+\mathcal{R}) + \mathcal{O}(h^3) \tag{5.2.17a}$$

and, of course, we also have a candidate right hexagonator given by

$$\mathcal{H}^R := \frac{h^2}{24}\gamma_{(12)3}(\mathcal{L}+2\mathcal{R}) + \mathcal{O}(h^3) \quad . \tag{5.2.17b}$$

We denote the **left/right infinitesimal hexagonator**, respectively, by:

$$\mathrm{h}^L := 2\mathcal{L}+\mathcal{R} \qquad , \qquad \mathrm{h}^R := \mathcal{L}+2\mathcal{R} \quad . \tag{5.2.18}$$

▷

Remark 5.2.12. Following on from Remark 5.2.4, Lemma 5.2.9 gives, for a γ-equivariant hexagonal t,

$$\mathrm{h}^R_{321} = \mathcal{L}_{321} + 2\mathcal{R}_{321} = \mathcal{R}_{132} + 2\mathcal{L}_{213} \neq \mathrm{h}^L \quad . \tag{5.2.19}$$

△

5.2.3 Verifying the braided monoidal 2-category axioms are satisfied

Lemma 5.2.13. *If the infinitesimal 2-braiding t is hexagonal then the associator* (5.2.15a) *satisfies the pentagon axiom modulo h^2.*

Proof. Referring to (5.2.4a),

$$[t_{12}\,, t_{23}+t_{24}] + [t_{13}+t_{23}\,, t_{34}] - [t_{23}\,, t_{34}] - [t_{12}+t_{13}\,, t_{24}+t_{34}] - [t_{12}\,, t_{23}] \Rrightarrow 0 \tag{5.2.20a}$$

simply boils down to

$$-[t_{12}\,, t_{34}] - [t_{13}\,, t_{24}] \Rrightarrow 0 \tag{5.2.20b}$$

but the LHS is 0 for the usual reason. More explicitly,

$$\begin{aligned}
[t_{12}\,, t_{34}] &\equiv \otimes(t \boxtimes \mathrm{Id}_\otimes) \otimes (\mathrm{Id}_\otimes \boxtimes t) - \otimes(\mathrm{Id}_\otimes \boxtimes t) \otimes (t \boxtimes \mathrm{Id}_\otimes) \\
&\equiv \otimes\left([t \boxtimes \mathrm{Id}_\otimes][\mathrm{Id}_\otimes \boxtimes t] - [\mathrm{Id}_\otimes \boxtimes t][t \boxtimes \mathrm{Id}_\otimes]\right) \\
&\equiv \otimes\left([t \boxtimes t] - [t \boxtimes t]\right) \\
&\equiv 0
\end{aligned} \tag{5.2.20c}$$

where the second equality comes from the functoriality of whiskering (as in Remark 4.1.22) and the third equality comes from the fact that the monoidal composition $\boxtimes$ is a 3-functor (as mentioned in Remark 4.1.26). Also, we have the expression for $t_{24}t_{13}$,

$$\otimes\left(\left[\otimes(1 \boxtimes \gamma^{-1}) \boxtimes 1\right]\left[([\mathrm{Id}_\otimes \boxtimes t][t \boxtimes \mathrm{Id}_\otimes])_{\mathrm{id}_{\mathbf{C}} \boxtimes \tau_{\mathbf{C},\mathbf{C}} \boxtimes \mathrm{id}_{\mathbf{C}}}\right][\otimes(1 \boxtimes \gamma) \boxtimes 1]\right) \tag{5.2.20d}$$

but, as before, we have $(\mathrm{Id}_\otimes \boxtimes t)(t \boxtimes \mathrm{Id}_\otimes) \equiv (t \boxtimes \mathrm{Id}_\otimes)(\mathrm{Id}_\otimes \boxtimes t)$ hence $t_{24}t_{13} \equiv t_{13}t_{24}$. □

Lemma 5.2.13 allows us to choose $\Pi = \mathbf{0}$ thus the associahedron axiom (5.1.2) simplifies as follows.

Lemma 5.2.14. *Given a hexagonal infinitesimal 2-braiding t,*

$$\left(\mathbf{C}_{\mathbb{K}[h]/(h^3)}\,, \otimes\,, I\,, \alpha :\equiv 1 + \frac{h^2}{24}[t_{12}\,, t_{23}]\right) \tag{5.2.21}$$

satisfies

$$\begin{aligned}\alpha_{1_{UV}\boxtimes\alpha_{WXY}}\alpha_{(UV)(WX)Y}(\alpha_{(UV)WX}\otimes 1_Y)+\big(1_U\otimes\alpha_{VW(XY)}\big)\,\alpha_{U(VW)(XY)}\alpha_{\alpha_{UVW}\boxtimes 1_{XY}} \qquad (5.2.22)\\ +\big(1_U\otimes[1_V\otimes\alpha_{WXY}]\alpha_{V(WX)Y}\big)\,\alpha_{1_U\boxtimes\alpha_{VWX}\boxtimes 1_Y}\,\big(\alpha_{U(VW)X}[\alpha_{UVW}\otimes 1_X]\otimes 1_Y\big)=0\,.\end{aligned}$$

Proof. Identity pseudonatural transformations 1 have trivial homotopy components (see Example 4.1.9) and the homotopy components annihilate units (4.1.13). □

Definition 5.2.15. An infinitesimal 2-braiding t is **unital** if: for every object $U\in\mathbf{C}$, we have

$$t_{UI}=0=t_{IU}\quad , \qquad (5.2.23a)$$

and, for every 1-cell $f\in\mathbf{C}[U,U']^0$, we have

$$t_{f\boxtimes 1_I}=0=t_{1_I\boxtimes f}\quad . \qquad (5.2.23b)$$

Lemma 5.2.16. *If the infinitesimal 2-braiding is unital then so too are the 4-term relationators,*

$$\mathcal{L}_{UVI}=\mathcal{L}_{UIW}=\mathcal{L}_{IVW}=\mathcal{R}_{UVI}=\mathcal{R}_{UIW}=\mathcal{R}_{IVW}=0\quad . \qquad (5.2.24)$$

Proof. Obvious from the components (5.2.9). □

Lemma 5.2.17. *A hexagonal infinitesimal 2-braiding is unital.*

Proof. The monoidal product $\otimes$ is strictly unital (4.2.3b) thus $V=(VI)$ and $g=g\otimes 1_I$. We first use the cochain components of the left infinitesimal hexagon relation (4.2.8a),

$$t_{UV}=t_{U(VI)}=t_{UV}\otimes 1_I+(\gamma_{VU}\otimes 1_I)(1_V\otimes t_{UI})(\gamma_{UV}\otimes 1_I)=t_{UV}+\gamma_{VU}(1_V\otimes t_{UI})\gamma_{UV}\,, \qquad (5.2.25)$$

thus $1_V\otimes t_{UI}=0$ hence setting $V=I$ forces $t_{UI}=0$. Now let us use the homotopy components of the left infinitesimal hexagon relation (4.2.8b),

$$t_{f\boxtimes g}=t_{f\boxtimes(g\otimes 1_I)}=t_{f\boxtimes g}\otimes 1_I+(\gamma_{V'U'}\otimes 1_I)(g\otimes t_{f\boxtimes 1_I})(\gamma_{UV}\otimes 1_I)=t_{f\boxtimes g}+\gamma_{V'U'}(g\otimes t_{f\boxtimes 1_I})\gamma_{UV}\,. \qquad (5.2.26)$$

This gives us $g\otimes t_{f\boxtimes 1_I}=0$ hence setting $g=1_I$ forces $t_{f\boxtimes 1_I}=0$. Analogous arguments using the right infinitesimal hexagon relations (4.2.9) gives us $t_{IU}=0$ and $t_{1_I\boxtimes g}=0$. □

Proposition 5.2.18. *Given a hexagonal infinitesimal 2-braiding t, the second-order deformation* (5.2.21) *is a pentagonal strictly-unital monoidal* $\mathbf{Ch}^{[-1,0]}_{\mathbb{K}[h]/(h^3)}$*-category.*

Proof. The only remaining axiom to check is the triangle axiom (5.1.3b). We have

$$\alpha_{UIV}=\Phi(t_{UI},t_{IV})=\Phi(0,0)=1_{((UI)V)}=1_{(UV)} \qquad (5.2.27a)$$

and

$$\begin{aligned}\alpha_{f\boxtimes 1_I\boxtimes g}&=1_{f\boxtimes 1_I\boxtimes g}+\frac{h^2}{24}[t_{12}t_{23}-t_{23}t_{12}]_{f\boxtimes 1_I\boxtimes g}\\ &=\frac{h^2}{24}\big[(t_{f\boxtimes 1_I}\otimes g)(1_U\otimes t_{I\boxtimes V})+(t_{U'\boxtimes I}\otimes 1_{V'})(f\otimes t_{1_I\boxtimes g})-(f\otimes t_{1_I\boxtimes g})(t_{U\boxtimes I}\otimes 1_V)\\ &\qquad -(1_{U'}\otimes t_{I\boxtimes V'})(t_{f\boxtimes 1_I}\otimes g)\big]\\ &=0\quad . \end{aligned} \qquad (5.2.27b)$$

□

Remark 5.2.19. The above proof makes it clear that the triangle axiom (5.1.3b) is held to all orders (beyond h^2) because of: the unitality of t, the definition of the homotopy components of the vertical composition (4.1.17c) and the fact that the associator is chosen to be Drinfeld's KZ series $\alpha := \Phi_{\mathrm{KZ}}(t_{12}, t_{23})$. Furthermore, Lemma 5.2.16 makes it obvious that the prism axiom (5.1.4) will also hold to all orders because the pentagonator series will be constructed out of the infinitesimal 2-braiding and the 4-term relationators (see (5.3.10) for the third order term). △

Totally symmetric infinitesimal 2-braidings

Before we tackle the tetrahedra, hexahedron and Breen polytope axioms in Definition 5.1.3, we recall the important notion of total γ-equivariance as in CM[4] [CFM15, Definition 18].

Definition 5.2.20. A **totally γ-equivariant** t is a γ-equivariant t which satisfies

$$*^2_{(\mathrm{Id}_\otimes, \gamma \boxtimes \mathrm{Id}_{\mathrm{id}_{\mathbf{C}}}), (t, \mathrm{Id}_\otimes \boxtimes \mathrm{id}_{\mathbf{C}})} = \mathrm{ID}_{\gamma_{12} t_{(12)3}} = \mathbf{0} \tag{5.2.28a}$$

or, in short,

$$t_{\gamma_{UV} \boxtimes 1_W} = 0 \quad . \tag{5.2.28b}$$

Example 5.2.21. As in Example 5.1.5, it is clear from (4.2.30) and (4.2.31b) that those infinitesimal 2-braidings induced by 2SPSs are totally γ-equivariant. ▽

Remark 5.2.22. Under the condition of γ-equivariance, (5.2.28b) is equivalent to

$$t_{1_U \boxtimes \gamma_{VW}} = 0 \quad . \tag{5.2.29}$$

Furthermore, in [CFM15, Definition 18] conditions (27) and (29) are also redundant given that we can categorify these conditions as:

$$\gamma_{23} t_{1(23)} \equiv t * (1 \boxtimes \gamma) \quad , \tag{5.2.30a}$$
$$\gamma_{12} t_{(12)3} \equiv t * (\gamma \boxtimes 1) \quad , \tag{5.2.30b}$$

respectively, yet these are clear instances of the exchanger which Definition 5.2.20 *already* tells us are identity modifications. △

Lemma 5.2.23. *A totally γ-equivariant infinitesimal 2-braiding is a Breen infinitesimal 2-braiding.*

Proof. Obvious from (5.2.28b) and (5.2.29). □

Remark 5.2.24. Rearranging (5.2.30b), we have

$$t_{(12)3} \equiv \otimes(\gamma^{-1} \boxtimes 1)[t * (\gamma \boxtimes 1)] \equiv \otimes(\gamma^{-1} \boxtimes 1)[t * \mathrm{Id}_{\otimes \tau_{\mathbf{C},\mathbf{C}} \boxtimes \mathrm{id}_{\mathbf{C}}}] \otimes (\gamma \boxtimes 1) \equiv: t_{(21)3} \quad , \tag{5.2.31a}$$

where the second equality comes from the condition for triviality of the exchanger in Remark 4.1.22. In general, for $k \neq i \neq j \neq k$,

$$t_{(ij)k} \equiv t_{(ji)k} \qquad , \qquad t_{i(jk)} \equiv t_{i(kj)} \quad . \tag{5.2.31b}$$

△

Lemma 5.2.25. *Following Lemma 5.2.9, if t is totally γ-equivariant then:*

$$\mathcal{L} = \mathcal{R}_{312} = \mathcal{L}_{213} = \mathcal{R}_{321} \ , \ \mathcal{R} = \mathcal{L}_{231} = \mathcal{R}_{132} = \mathcal{L}_{321} \ , \ \mathcal{L}_{132} = \mathcal{R}_{213} = \mathcal{L}_{312} = \mathcal{R}_{231} \, . \tag{5.2.32}$$

[4]Note that, throughout their joint publications, compositions of morphisms are written from left to right.

Proof. Let us show that $\mathcal{L} = \mathcal{L}_{213}$, i.e.,

$$\mathcal{L}_{UVW} = (\gamma_{VU} \otimes 1_W)\mathcal{L}_{VUW}(\gamma_{UV} \otimes 1_W) \quad \Longrightarrow \quad t_{t_{UV} \boxtimes 1_W} = (\gamma_{VU} \otimes 1_W) t_{t_{VU} \boxtimes 1_W} (\gamma_{UV} \otimes 1_W)\,. \tag{5.2.33}$$

We use total γ-equivariance (5.2.28b) and the fact that the homotopy components of a pseudonatural transformation splits products (4.1.12b) to write the latter expression as $t_{\gamma_{VU} t_{VU} \gamma_{UV} \boxtimes 1_W}$ which equals $t_{t_{UV} \boxtimes 1_W}$ because of (4.2.11a). The proofs for the other relations work in the same way. □

Remark 5.2.26. Following Remark 5.2.12, Lemma 5.2.25 gives us, for a totally γ-equivariant t,

$$\mathrm{h}^R_{321} = \mathcal{R}_{132} + 2\mathcal{L}_{213} = 2\mathcal{L} + \mathcal{R} = \mathrm{h}^L \quad . \tag{5.2.34a}$$

In fact, under this condition of total γ-equivariance,

$$\mathbf{R}_{321} : \Phi(t_{32}, t_{13}) e^{\frac{h}{2} t_{(32)1}} \Phi(t_{21}, t_{32}) \Rrightarrow e^{\frac{h}{2} t_{31}} \Phi(t_{12}, t_{31}) e^{\frac{h}{2} t_{21}} \tag{5.2.34b}$$

but the (co)domain pseudonatural automorphism is equal to that of (5.2.3b) upon using (5.2.1f) and Remark 5.2.24. In sum, a totally γ-equivariant t means that we can choose $\mathbf{L} := \mathbf{R}_{321}$. △

Lemma 5.2.27. *Given a totally γ-equivariant hexagonal t, the left 4-term relationator is* ***primitive****:*

$$\mathcal{L}_{12(34)} = \mathcal{L}_{123} + \mathcal{L}_{124} \quad , \quad \mathcal{L}_{1(23)4} = \mathcal{L}_{124} + \mathcal{L}_{134} \quad , \quad \mathcal{L}_{(12)34} = \mathcal{L}_{134} + \mathcal{L}_{234}\,. \tag{5.2.35}$$

Proof.

$$\begin{aligned} \mathcal{L}_{UV(WX)} =& t_{t_{UV} \boxtimes (1_W \otimes 1_X)} \\ =& t_{t_{UV} \boxtimes 1_W} \otimes 1_X + (\gamma_{W(UV)} \otimes 1_X)(1_W \otimes t_{t_{UV} \boxtimes 1_X})(\gamma_{(UV)W} \otimes 1_X) \\ =& \mathcal{L}_{UVW} \otimes 1_X + (\gamma_{W(UV)} \otimes 1_X)(1_W \otimes \mathcal{L}_{UVX})(\gamma_{(UV)W} \otimes 1_X) \quad , \end{aligned} \tag{5.2.36a}$$

where the 2nd step used the homotopy components of the left infinitesimal hexagon (4.2.8b).

$$\begin{aligned} \mathcal{L}_{U(VW)X} =& t_{t_{U(VW)} \boxtimes 1_X} \\ =& t_{[t_{UV} \otimes 1_W + (\gamma_{VU} \otimes 1_W)(1_V \otimes t_{UW})(\gamma_{UV} \otimes 1_W)] \boxtimes 1_X} \\ =& t_{UV} \otimes t_{1_W \boxtimes 1_X} + (1_{(UV)} \otimes \gamma_{XW})(t_{t_{UV} \boxtimes 1_X} \otimes 1_W)(1_{(UV)} \otimes \gamma_{WX}) \\ & + (\gamma_{VU} \otimes 1_{(WX)}) t_{(1_V \otimes t_{UW}) \boxtimes 1_X} (\gamma_{UV} \otimes 1_{(WX)}) \\ =& (1_{(UV)} \otimes \gamma_{XW})(\mathcal{L}_{UVX} \otimes 1_W)(1_{(UV)} \otimes \gamma_{WX}) + (\gamma_{VU} \otimes 1_{(WX)})(1_V \otimes \mathcal{L}_{UWX})(\gamma_{UV} \otimes 1_{(WX)}), \end{aligned} \tag{5.2.36b}$$

where: the 2nd step used the left infinitesimal hexagon's cochain components (4.2.8a), the first two terms of the 3rd step used the right infinitesimal hexagon's homotopy components (4.2.9b) and the third term used total γ-equivariance (5.2.28b) and the fact that the homotopy components split products (4.1.12b), the 4th step used the fact that the homotopy components kill units (4.1.13).

$$\begin{aligned} \mathcal{L}_{(UV)WX} =& t_{t_{(UV)W} \boxtimes 1_X} \\ =& t_{[1_U \otimes t_{VW} + (1_U \otimes \gamma_{WV})(t_{UW} \otimes 1_V)(1_U \otimes \gamma_{VW})] \boxtimes 1_X} \\ =& 1_U \otimes \mathcal{L}_{VWX} + (1_U \otimes \gamma_{(WX)V})(\mathcal{L}_{UWX} \otimes 1_V)(1_U \otimes \gamma_{V(WX)}) \quad , \end{aligned} \tag{5.2.36c}$$

where the last step used the right infinitesimal hexagon multiple times together with the total γ-equivariance and the facts that the homotopy components split products and annihilate units. □

Corollary 5.2.28. *$\mathcal{R}$ is also primitive and so too are the infinitesimal hexagonators* (5.2.18).

This corollary is quite powerful as exemplified by the following proposition.

Proposition 5.2.29. *Given a totally γ-equivariant hexagonal infinitesimal 2-braiding t,*

$$\left(\mathbf{C}_{\mathbb{K}[h]/(h^3)}, \otimes, I, \alpha :\equiv 1 + \frac{h^2}{24}[t_{12}, t_{23}], \sigma :\equiv \gamma e^{\frac{h}{2}t}, \mathcal{H}^L := \frac{h^2}{24}\gamma_{1(23)}\mathrm{h}^L, \mathcal{H}^R := \frac{h^2}{24}\gamma_{(12)3}\mathrm{h}^R\right) \quad (5.2.37)$$

satisfies the left/right tetrahedron axiom (5.1.10)/(5.1.11) *and the hexahedron axiom* (5.1.12).

Proof. With trivial pentagonator, the left tetrahedron axiom (5.1.10) reads as

$$\begin{aligned}
&(1_V \otimes \alpha_{WXU})\alpha_{V(WX)U}\sigma_{1_U \boxtimes \alpha_{VWX}}\alpha_{U(VW)X}(\alpha_{UVW} \otimes 1_X) \\
&+ (1_V \otimes \alpha_{WXU})\mathcal{H}^L_{UV(WX)}\alpha_{(UV)WX} + (1_V \otimes \alpha_{WXU}\sigma_{U(WX)})\alpha_{VU(WX)}\alpha_{\sigma_{UV} \boxtimes 1_W \boxtimes 1_X} \\
&= \qquad (5.2.38a) \\
&\alpha_{VW(XU)}\mathcal{H}^L_{U(VW)X}(\alpha_{UVW} \otimes 1_X) - \alpha_{1_V \boxtimes 1_W \boxtimes \sigma_{UX}}\alpha_{(VW)UX}(\sigma_{U(VW)}\alpha_{UVW} \otimes 1_X) \\
&+ (1_V \otimes (1_W \otimes \sigma_{UX})\alpha_{WUX})\,\alpha_{V(WU)X}(\mathcal{H}^L_{UVW} \otimes 1_X) - \left(1_V \otimes \mathcal{H}^L_{UWX}\right)\alpha_{V(UW)X}\left(\alpha_{VUW}(\sigma_{UV} \otimes 1_W) \otimes 1_X\right) \\
&- (1_V \otimes (1_W \otimes \sigma_{UX})\alpha_{WUX})\,\alpha_{1_V \boxtimes \sigma_{UW} \boxtimes 1_X}\left(\alpha_{VUW}(\sigma_{UV} \otimes 1_W) \otimes 1_X\right) \quad .
\end{aligned}$$

The order h^2 relation is given by

$$\begin{aligned}
&\left(\gamma_{1(23)}\mathrm{h}^L\right)_{UV(WX)} + (1_V \otimes \gamma_{U(WX)})[t_{12}\,, t_{23}]_{\gamma_{UV} \boxtimes 1_W \boxtimes 1_X} \\
&= \qquad (5.2.38b) \\
&\left(\gamma_{1(23)}\mathrm{h}^L\right)_{U(VW)X} - [t_{12}\,, t_{23}]_{1_V \boxtimes 1_W \boxtimes \gamma_{UX}}(\gamma_{U(VW)} \otimes 1_X) \\
&+ \left(1_{(VW)} \otimes \gamma_{UX}\right)\left(\left(\gamma_{1(23)}\mathrm{h}^L\right)_{UVW} \otimes 1_X\right) - \left(1_V \otimes \left(\gamma_{1(23)}\mathrm{h}^L\right)_{UWX}\right)\left(\gamma_{UV} \otimes 1_{(WX)}\right) \\
&- \left(1_{(VW)} \otimes \gamma_{UX}\right)[t_{12}\,, t_{23}]_{1_V \boxtimes \gamma_{UW} \boxtimes 1_X}\left(\gamma_{UV} \otimes 1_{(WX)}\right) \quad .
\end{aligned}$$

Terms such as

$$[t_{12}, t_{23}]_{\gamma_{UV} \boxtimes 1_W \boxtimes 1_X} = (t_{\gamma_{UV} \boxtimes 1_W} \otimes 1_X)\left(1_{(UV)} \otimes t_{WX}\right) - \left(1_{(VU)} \otimes t_{WX}\right)(t_{\gamma_{UV} \boxtimes 1_W} \otimes 1_X) \quad (5.2.38c)$$

are trivial by the condition of total γ-equivariance (5.2.28b) thus (5.2.38b) simplifies to

$$\begin{aligned}
\left(\gamma_{1(23)}\mathrm{h}^L\right)_{UV(WX)} = \left(\gamma_{1(23)}\mathrm{h}^L\right)_{U(VW)X} &+ \left(1_{(VW)} \otimes \gamma_{UX}\right)\left(\left(\gamma_{1(23)}\mathrm{h}^L\right)_{UVW} \otimes 1_X\right) \\
&- \left(1_V \otimes \left(\gamma_{1(23)}\mathrm{h}^L\right)_{UWX}\right)\left(\gamma_{UV} \otimes 1_{(WX)}\right) \quad . \qquad (5.2.38d)
\end{aligned}$$

This is simply

$$\begin{aligned}
\gamma_{U(VWX)}\mathrm{h}^L_{UV(WX)} = \gamma_{U(VWX)}\mathrm{h}^L_{U(VW)X} &+ \left(1_{(VW)} \otimes \gamma_{UX}\right)\left(\gamma_{U(VW)}\mathrm{h}^L_{UVW} \otimes 1_X\right) \\
&- \left(1_V \otimes \gamma_{U(WX)}\mathrm{h}^L_{UWX}\right)\left(\gamma_{UV} \otimes 1_{(WX)}\right) \quad . \qquad (5.2.38e)
\end{aligned}$$

The symmetric braiding γ satisfies the hexagon axiom hence we can simplify even further,

$$\mathrm{h}^L_{UV(WX)} = \mathrm{h}^L_{U(VW)X} + \mathrm{h}^L_{UVW} \otimes 1_X - \left(\gamma_{VU} \otimes 1_{(WX)}\right)\left(1_V \otimes \mathrm{h}^L_{UWX}\right)\left(\gamma_{UV} \otimes 1_{(WX)}\right) . \quad (5.2.38f)$$

We finally rewrite this as

$$\mathrm{h}^L_{12(34)} = \mathrm{h}^L_{1(23)4} + \mathrm{h}^L_{123} - \mathrm{h}^L_{134} \quad (5.2.38g)$$

and apply Corollay 5.2.28.

Comparing (5.1.10) with (5.1.11) reveals that the right tetrahedron axiom would eventually reduce down to a similar equation as the above but now with h^L replaced with h^R.

With trivial pentagonator, the hexahedron axiom (5.1.12) reads as

$$
\begin{aligned}
&\alpha_{W(XU)V}(\alpha_{WXU}\otimes 1_V)\mathcal{H}^R_{UV(WX)}(1_U\otimes\alpha_{VWX})\alpha_{U(VW)X}-(1_W\otimes\alpha^{-1}_{XUV})\mathcal{H}^L_{(UV)WX}(\alpha^{-1}_{UVW}\otimes 1_X)\\
&= \qquad (5.2.38h)\\
&(1_W\otimes\mathcal{H}^R_{UVX})\alpha_{WU(VX)}\alpha_{(WU)VX}(\alpha^{-1}_{WUV}\sigma_{(UV)W}\alpha^{-1}_{UVW}\otimes 1_X)\\
&+[1_W\otimes(\sigma_{UX}\otimes 1_V)\alpha^{-1}_{UXV}(1_U\otimes\sigma_{VX})]\alpha_{WU(VX)}\left[\alpha_{(WU)VX}(\mathcal{H}^R_{UVW}\otimes 1_X)-\alpha_{\sigma_{UW}\boxtimes 1_{VX}}\left(\alpha^{-1}_{UWV}(1_U\otimes\sigma_{VW})\otimes 1_X\right)\right]\\
&+(1_W\otimes[\sigma_{UX}\otimes 1_V]\alpha^{-1}_{UXV})\alpha_{1_{WU}\boxtimes\sigma_{VX}}(\sigma_{UW}\otimes 1_{(VX)})\alpha_{(UW)VX}\left(\alpha^{-1}_{UWV}[1_U\otimes\sigma_{VW}]\otimes 1_X\right)\\
&+\alpha_{1_W\boxtimes\sigma_{UX}\boxtimes 1_V}(\alpha_{WUX}\otimes 1_V)\alpha^{-1}_{(WU)XV}[\sigma_{UW}\otimes\sigma_{VX}]\alpha_{(UW)VX}\left[\alpha^{-1}_{UWV}[1_U\otimes\sigma_{VW}]\otimes 1_X\right]\\
&-\alpha_{W(XU)V}\left[(1_W\otimes\sigma_{UX})\alpha_{WUX}\otimes 1_V\right]\alpha^{-1}_{(WU)XV}(\sigma_{UW}\otimes\sigma_{VX})\alpha^{-1}_{UW(VX)}(1_U\otimes\alpha_{WVX})\alpha_{1_U\boxtimes\sigma_{VW}\boxtimes 1_X}\\
&-\alpha_{W(XU)V}\left[(1_W\otimes\sigma_{UX})\alpha_{WUX}\otimes 1_V\right]\alpha^{-1}_{\sigma_{UW}\boxtimes 1_{XV}}(1_{(UW)}\otimes\sigma_{VX})\alpha^{-1}_{UW(VX)}[1_U\otimes\alpha_{WVX}(\sigma_{VW}\otimes 1_X)]\alpha_{U(VW)X}\\
&+\alpha_{W(XU)V}\left[(1_W\otimes\sigma_{UX})\alpha_{WUX}(\sigma_{UW}\otimes 1_X)\otimes 1_V\right]\alpha^{-1}_{(UW)XV}\alpha^{-1}_{1_{UW}\boxtimes\sigma_{VX}}[1_U\otimes\alpha_{WVX}(\sigma_{VW}\otimes 1_X)]\alpha_{U(VW)X}\\
&-\alpha_{W(XU)V}(\mathcal{H}^L_{UWX}\otimes 1_V)\alpha^{-1}_{(UW)XV}\alpha^{-1}_{UW(XV)}[1_U\otimes(1_W\otimes\sigma_{VX})\alpha_{WVX}(\sigma_{VW}\otimes 1_X)]\alpha_{U(VW)X}\\
&-\alpha_{W(XU)V}\left(\alpha_{WXU}\sigma_{U(WX)}\alpha_{UWX}\otimes 1_V\right)\alpha^{-1}_{(UW)XV}\alpha^{-1}_{UW(XV)}(1_U\otimes\mathcal{H}^L_{VWX})\alpha_{U(VW)X}\qquad .
\end{aligned}
$$

We use the comment from above that total γ-equivariance annihilates the associator homotopy components of the symmetric braiding (e.g., $\alpha_{\gamma_{UW}\boxtimes 1_{VX}}=0$) to derive the order h^2 relation,

$$
\begin{aligned}
&[\gamma_{(12)3}\mathrm{h}^R]_{UV(WX)}-[\gamma_{1(23)}\mathrm{h}^L]_{(UV)WX}\\
&= \qquad (5.2.38i)\\
&(1_W\otimes[\gamma_{(12)3}\mathrm{h}^R]_{UVX})(\gamma_{(UV)W}\otimes 1_X)+[1_W\otimes(\gamma_{UX}\otimes 1_V)(1_U\otimes\gamma_{VX})]([\gamma_{(12)3}\mathrm{h}^R]_{UVW}\otimes 1_X)\\
&-([\gamma_{1(23)}\mathrm{h}^L]_{UWX}\otimes 1_V)[1_U\otimes(1_W\otimes\gamma_{VX})(\gamma_{VW}\otimes 1_X)]-\left(\gamma_{U(WX)}\otimes 1_V\right)(1_U\otimes[\gamma_{1(23)}\mathrm{h}^L]_{VWX})\,.
\end{aligned}
$$

This is simply

$$
\begin{aligned}
&\gamma_{(UV)(WX)}\mathrm{h}^R_{UV(WX)}-\gamma_{(UV)(WX)}\mathrm{h}^L_{(UV)WX}\\
&= \qquad (5.2.38j)\\
&\left[1_W\otimes\gamma_{(UV)X}\mathrm{h}^R_{UVX}\right]\left[\gamma_{(UV)W}\otimes 1_X\right]+\left[1_W\otimes\gamma_{(UV)X}\right]\left[\gamma_{(UV)W}\mathrm{h}^R_{UVW}\otimes 1_X\right]\\
&-\left[\gamma_{U(WX)}\mathrm{h}^L_{UWX}\otimes 1_V\right]\left[1_U\otimes\gamma_{V(WX)}\right]-\left[\gamma_{U(WX)}\otimes 1_V\right]\left[1_U\otimes\gamma_{V(WX)}\mathrm{h}^L_{VWX}\right]\qquad .
\end{aligned}
$$

Again, the hexagon axiom that γ satisfies as a symmetric braiding means we can write

$$
\gamma_{(UV)(WX)}=\left[1_W\otimes\gamma_{(UV)X}\right]\left[\gamma_{(UV)W}\otimes 1_X\right]=\left[\gamma_{U(WX)}\otimes 1_V\right]\left[1_U\otimes\gamma_{V(WX)}\right] \qquad (5.2.38k)
$$

which finally leaves us with

$$
\begin{aligned}
\mathrm{h}^R_{UV(WX)}-\mathrm{h}^L_{(UV)WX}=&\left(\gamma_{W(UV)}\otimes 1_X\right)\left(1_W\otimes\mathrm{h}^R_{UVX}\right)\left(\gamma_{(UV)W}\otimes 1_X\right)+\mathrm{h}^R_{UVW}\otimes 1_X\\
&-\left(1_U\otimes\gamma_{(WX)V}\right)\left(\mathrm{h}^L_{UWX}\otimes 1_V\right)\left(1_U\otimes\gamma_{V(WX)}\right)-1_U\otimes\mathrm{h}^L_{VWX} \qquad (5.2.38l)
\end{aligned}
$$

but, once again, we know the infinitesimal hexagonators to be primitive as in Corollay 5.2.28. □

Coherent infinitesimal 2-braidings and the Breen polytope

Before we discuss the Breen polytope axiom (5.1.13), we quickly recall another important notion in CM; this time the notion of coherency [CFM15, Defintion 17].

Definition 5.2.30. An infinitesimal 2-braiding is **coherent** if it satisfies

$$
-\mathcal{R}_{213}=\mathcal{L}+\mathcal{R}=-\mathcal{L}_{132}, \qquad (5.2.39a)
$$

that is,

$$-(\gamma_{VU}\otimes 1_W)t_{1_V\boxtimes t_{UW}}(\gamma_{UV}\otimes 1_W)=t_{t_{UV}\boxtimes 1_W}+t_{1_U\boxtimes t_{VW}}=-(1_U\otimes\gamma_{WV})t_{t_{UW}\boxtimes 1_V}(1_U\otimes\gamma_{VW})\,. \tag{5.2.39b}$$

Remark 5.2.31. As in Lemma 5.2.9, a γ-equivariant t gives $\mathcal{R}_{213}=\mathcal{L}_{132}$ thus, in contrast to the remarks after [CFM15, Defintion 17], including γ-equivariance as a condition makes the definition natural from the categorical perspective; the Breen polytope can only enforce *one* equality at order h^2. Given a γ-equivariant t, coherency demands

$$\mathcal{L}+\mathcal{L}_{231}+\mathcal{L}_{132}=\mathbf{0}\quad . \tag{5.2.40}$$

△

Remark 5.2.32. Using the basis expression for $\mathcal{L}=t_{t_{12}}$ in (4.2.32), (5.2.40) becomes

$$\begin{aligned}&\langle\pi^{(2)},s^{-1}\mathrm{d}_{\mathrm{dR}}\langle\pi^{(2)},s^{-1}\mathrm{d}_{\mathrm{dR}}\underline{U}_i{}^j\otimes_A s^{-1}\mathrm{d}_{\mathrm{dR}}\underline{V}_q{}^r\rangle\otimes_A s^{-1}\mathrm{d}_{\mathrm{dR}}\underline{W}_s{}^t\\&+(-1)^{|v_q||s\underline{W}_s{}^t|+|w_t||s\underline{V}_q{}^r|}s^{-1}\mathrm{d}_{\mathrm{dR}}\langle\pi^{(2)},s^{-1}\mathrm{d}_{\mathrm{dR}}\underline{U}_i{}^j\otimes_A s^{-1}\mathrm{d}_{\mathrm{dR}}\underline{W}_s{}^t\rangle\otimes_A s^{-1}\mathrm{d}_{\mathrm{dR}}\underline{V}_q{}^r\\&+(-1)^{|v_r||\underline{U}_i{}^j\underline{W}_s{}^t|+|s\underline{U}_i{}^j|(|v_q|+|w_s|)}s^{-1}\mathrm{d}_{\mathrm{dR}}\langle\pi^{(2)},s^{-1}\mathrm{d}_{\mathrm{dR}}\underline{V}_q{}^r\otimes_A s^{-1}\mathrm{d}_{\mathrm{dR}}\underline{W}_s{}^t\rangle\otimes_A s^{-1}\mathrm{d}_{\mathrm{dR}}\underline{U}_i{}^j\rangle u_j\otimes_A v_r\otimes_A w_t=0\end{aligned}$$

To understand this requirement let us recall the following operations (3.29) and (3.25) of [LGPV13, Chapter 3], remembering that in our case A is a CDGA. In fact, looking at Cartan's formula as in [LGPV13, Proposition 3.6] suggests that the above LHS relates to the interior product $\iota_{[\pi^{(2)},\pi^{(2)}]_{\mathrm{S}}}$ but the third weight component of the Maurer-Cartan equation $\partial\pi^{(3)}+\frac{1}{2}[\pi^{(2)},\pi^{(2)}]_{\mathrm{S}}=0$ then implies that this is proportional to $\iota_{\partial\pi^{(3)}}$. But this then means that the above LHS is the output of the inner hom differential ∂ of the degree -2 morphism which is constructed analogously to (4.2.27a) but with three ξ's and the dual pairing $\langle\pi^{(3)},\cdot\rangle$. The truncation would then guarantee that the above requirement is satisfied. Now we can see that coherency is *also* natural from the perspective of Derived Algebraic Geometry. △

Proposition 5.2.33. *Given a totally γ-equivariant hexagonal infinitesimal 2-braiding t, the second-order deformation* (5.2.37) *satisfies the Breen polytope axiom* (5.1.13) *if and only if t is coherent.*

Proof. The order h^2 relation of (5.1.13) is given as

$$\begin{aligned}&-(\gamma_{VW}\otimes 1_U)\left(\frac{1}{24}\gamma_{1(23)}\mathrm{h}^L\right)_{UVW}+\frac{1}{2}(\gamma t)_{1_U\boxtimes\frac{1}{2}\gamma_{VW}t_{VW}}+\left(\frac{1}{24}\gamma_{1(23)}\mathrm{h}^L\right)_{UWV}(1_U\otimes\gamma_{VW})\\&=\\&-\left(\frac{1}{24}\gamma_{(12)3}\mathrm{h}^R\right)_{VUW}(\gamma_{UV}\otimes 1_W)-\frac{1}{2}(\gamma t)_{\frac{1}{2}\gamma_{UV}t_{UV}\boxtimes 1_W}+(1_W\otimes\gamma_{UV})\left(\frac{1}{24}\gamma_{(12)3}\mathrm{h}^R\right)_{UVW}\end{aligned} \tag{5.2.41a}$$

which we rewrite,

$$\begin{aligned}&(\gamma_{VW}\otimes 1_U)\gamma_{U(VW)}\mathrm{h}^L_{UVW}-6\gamma_{U(WV)}t_{1_U\boxtimes\gamma_{VW}t_{VW}}-\gamma_{U(WV)}\mathrm{h}^L_{UWV}(1_U\otimes\gamma_{VW})\\&=\\&\gamma_{(VU)W}\mathrm{h}^R_{VUW}(\gamma_{UV}\otimes 1_W)+6\gamma_{(VU)W}t_{\gamma_{UV}t_{UV}\boxtimes 1_W}-(1_W\otimes\gamma_{UV})\gamma_{(UV)W}\mathrm{h}^R_{UVW}\quad .\end{aligned} \tag{5.2.41b}$$

Now we simultaneously use: the fact that the homotopy components of a pseudonatural transformation splits products (4.1.12b), the total γ-equivariance of t and the hexagon axiom that γ

satisfies,

$$\mathrm{h}^L_{UVW} - 6\mathcal{R}_{UVW} - (1_U \otimes \gamma_{WV})\mathrm{h}^L_{UWV}(1_U \otimes \gamma_{VW}) = (\gamma_{VU} \otimes 1_W)\mathrm{h}^R_{VUW}(\gamma_{UV} \otimes 1_W) + 6\mathcal{L}_{UVW} - \mathrm{h}^R_{UVW}\,. \tag{5.2.41c}$$

We use the definitions of the infinitesimal hexagonators (5.2.18) and the subscript index notation,

$$2\mathcal{L} + \mathcal{R} - 6\mathcal{R} - 2\mathcal{L}_{132} - \mathcal{R}_{132} = \mathcal{L}_{213} + 2\mathcal{R}_{213} + 6\mathcal{L} - \mathcal{L} - 2\mathcal{R} \quad . \tag{5.2.41d}$$

As in Lemma 5.2.25, total γ-equivariance of t gives us: $\mathcal{R}_{213} = \mathcal{L}_{132}$, $\mathcal{R}_{132} = \mathcal{R}$ and $\mathcal{L}_{213} = \mathcal{L}$. We thus rewrite the above as

$$4\mathcal{L} + 4\mathcal{R} + 4\mathcal{L}_{132} = 0 \tag{5.2.41e}$$

and this is equivalent to the condition of coherency for a γ-equivariant t. □

5.3 Third-order deformation data

We take $\mathbb{K} = \mathbb{C}$ because the coefficient of the third-order term of Drinfeld's KZ series [Kas95, Corollary XIX.6.5],

$$\alpha :\equiv \Phi_{\mathrm{KZ}}(t_{12}, t_{23}) \equiv 1 + \frac{h^2}{24}[t_{12}, t_{23}] + \frac{h^3\zeta(3)}{(2\pi i)^3}\big[[t_{12}, t_{23}], t_{12} + t_{23}\big] + \dots \quad , \tag{5.3.1}$$

is imaginary and irrational. We substitute $h := 2\pi i\hbar$ hence (5.3.1) becomes

$$\Phi(t_{12}, t_{23}) \equiv 1 - \frac{\hbar^2\pi^2}{6}[t_{12}, t_{23}] + \hbar^3\zeta(3)\big[[t_{12}, t_{23}], t_{12} + t_{23}\big] + \dots \tag{5.3.2}$$

5.3.1 The pentagonator

We substitute (5.3.2) into (5.2.2a) and read off the third-order terms,

$$\begin{aligned}\frac{1}{\zeta(3)}(\Pi)_3 : \big[[t_{23}, t_{34}], t_{23} + t_{34}\big] + \big[[t_{1(23)}, t_{(23)4}], t_{1(23)} + t_{(23)4}\big] + \big[[t_{12}, t_{23}], t_{12} + t_{23}\big] \\ \Rrightarrow \big[[t_{12}, t_{2(34)}], t_{12} + t_{2(34)}\big] + \big[[t_{(12)3}, t_{34}], t_{(12)3} + t_{34}\big]\,.\end{aligned} \tag{5.3.3}$$

Let us assume hexagonality of t and write[5]:

$$\begin{aligned}\big[[t_{12}, t_{2(34)}], t_{12} + t_{2(34)}\big] &= \big[[t_{12}, t_{23}], t_{12} + t_{23}\big] + \big[[t_{12}, t_{23}], t_{24}\big] + \big[[t_{12}, t_{24}], t_{12} + t_{23} + t_{24}\big]\,, \\ \big[[t_{(12)3}, t_{34}], t_{(12)3} + t_{34}\big] &= \big[[t_{23}, t_{34}], t_{23} + t_{34}\big] + \big[[t_{23}, t_{34}], t_{13}\big] + \big[[t_{13}, t_{34}], t_{13} + t_{23} + t_{34}\big]\,, \\ \big[[t_{1(23)}, t_{(23)4}], t_{1(23)} + t_{(23)4}\big] &= \big[[t_{12}, t_{24}] + [t_{13}, t_{34}], t_{12} + t_{13} + t_{24} + t_{34}\big]\,.\end{aligned} \tag{5.3.4}$$

Now we balance terms as usual,

$$\frac{(\widehat{\Pi})_3}{\zeta(3)} : \big[t_{13}, [t_{23}, t_{34}]\big] - \big[t_{24}, [t_{23}, t_{12}]\big] + \big[[t_{13}, t_{34}], t_{12} + t_{24} - t_{23}\big] + \big[[t_{12}, t_{24}], t_{13} + t_{34} - t_{23}\big] \Rrightarrow 0\,. \tag{5.3.5}$$

[5]Throughout this subsection, we will make use of the fact that

$$[t_{ij}, t_{kl}] = 0$$

for $\{i, j, k, l\} = \{1, 2, 3, 4\}$. These "disjoint-commuting" relations were demonstrated in the proof of Lemma 5.2.13.

With a view to 4-term relationators, we use the disjoint-commuting relations to rewrite (5.3.5) as

$$\big[t_{13},[t_{23},t_{34}]\big]-\big[t_{24},[t_{23},t_{12}]\big]+\big[t_{23},[t_{1(23)},t_{(23)4}]\big]+\big[[t_{13},t_{34}],t_{12}+t_{24}\big]+\big[[t_{12},t_{24}],t_{13}+t_{34}\big]\Rrightarrow 0\,.$$

(i) The first two terms give us

$$\begin{aligned}\big[t_{13},[t_{23},t_{34}]\big]-\big[t_{24},[t_{23},t_{12}]\big]&=\big[t_{13},[t_{23},t_{(23)4}-t_{24}]\big]-\big[t_{24},[t_{23},t_{1(23)}-t_{13}]\big]\\&=\big[t_{13},\partial\mathcal{L}_{234}\big]-\big[t_{24},\partial\mathcal{R}_{123}\big]-\big[t_{13},[t_{23},t_{24}]\big]+\big[t_{24},[t_{23},t_{13}]\big]\\&=\partial\left(\big[t_{13},\mathcal{L}_{234}\big]-\big[t_{24},\mathcal{R}_{123}\big]\right)\quad,\end{aligned}\tag{5.3.6}$$

where we used the Jacobi identity to eliminate the last two terms of the second equality.

(ii) The third term straightforwardly gives us

$$\big[[t_{23},t_{1(23)}],t_{(23)4}\big]+\big[t_{1(23)},[t_{23},t_{(23)4}]\big]=\partial\left([t_{12}+t_{13},\mathcal{L}_{234}]-[t_{24}+t_{34},\mathcal{R}_{123}]\right)\tag{5.3.7}$$

(iii) The last two terms $\big[[t_{13},t_{34}],t_{12}+t_{24}\big]+\big[[t_{12},t_{24}],t_{13}+t_{34}\big]$ can be rewritten as

$$\begin{aligned}&\big[t_{34},[t_{12},t_{13}]\big]+\big[t_{13},[t_{34},t_{24}]\big]-\big[t_{24},[t_{12},t_{13}]\big]-\big[t_{12},[t_{34},t_{24}]\big]\\&=\big[t_{34},[t_{12},t_{(12)3}-t_{23}]\big]+\big[t_{13},[t_{34},t_{2(34)}-t_{23}]\big]-\big[t_{24},[t_{12},t_{(12)3}-t_{23}]\big]-\big[t_{12},[t_{34},t_{2(34)}-t_{23}]\big]\\&=\partial\left([t_{34}-t_{24},\mathcal{L}_{123}]+[t_{13}-t_{12},\mathcal{R}_{234}]\right)+\big[t_{24}-t_{34},[t_{12},t_{23}]\big]+\big[t_{12}-t_{13},[t_{34},t_{23}]\big]\end{aligned}\tag{5.3.8}$$

and the term $\big[t_{24},[t_{12},t_{23}]\big]-\big[t_{13},[t_{34},t_{23}]\big]-\big[t_{34},[t_{12},t_{23}]\big]+\big[t_{12},[t_{34},t_{23}]\big]$ gives

$$\begin{aligned}\big[t_{13},[t_{23},t_{34}]\big]-\big[t_{24},[t_{23},t_{12}]\big]&=\partial\left([t_{13},\mathcal{L}_{234}]-[t_{24},\mathcal{R}_{123}]\right)-\big[t_{13},[t_{23},t_{24}]\big]+\big[t_{24},[t_{23},t_{13}]\big]\\&=\partial\left([t_{13},\mathcal{L}_{234}]-[t_{24},\mathcal{R}_{123}]\right)\quad.\end{aligned}\tag{5.3.9}$$

Putting it all together, the pentagonator series (modulo $\hbar^4$) is given by

$$\Pi=\hbar^3\zeta(3)\left([t_{34}-t_{24},\mathcal{L}_{123}]-[3t_{24}+t_{34},\mathcal{R}_{123}]+[t_{12}+3t_{13},\mathcal{L}_{234}]+[t_{13}-t_{12},\mathcal{R}_{234}]\right)\tag{5.3.10}$$

5.3.2 The right pre-hexagonator

We compute the third-order term of the right pre-hexagonator (5.2.4c),

$$\begin{aligned}(\mathbf{R})_3:&\zeta(3)\left(\big[[t_{12},t_{13}],t_{1(23)}\big]+\big[[t_{23},t_{12}],t_{12}+t_{23}\big]\right)-\frac{\pi^3 i}{6}\left([t_{12},t_{13}]t_{(12)3}+t_{(12)3}[t_{23},t_{12}]+t_{(12)3}^3\right)\\&\Rrightarrow\zeta(3)\big[[t_{23},t_{13}],t_{(12)3}\big]-\frac{\pi^3 i}{6}\left([t_{23},t_{13}]t_{23}+t_{13}[t_{23},t_{13}]+t_{13}^3+t_{23}^3+3t_{13}^2t_{23}+3t_{13}t_{23}^2\right)\,.\end{aligned}\tag{5.3.11}$$

Balancing terms, we get

$$\begin{aligned}&\zeta(3)\left(\big[[t_{12},t_{13}],t_{1(23)}\big]-\big[[t_{23},t_{13}],t_{(12)3}\big]+\big[[t_{23},t_{12}],t_{12}+t_{23}\big]\right)\\&-\frac{\pi^3 i}{6}\left([t_{23},t_{13}^2]+[t_{23}^2,t_{13}]+[t_{12},t_{13}]t_{(12)3}+t_{(12)3}[t_{23},t_{12}]\right)\Rrightarrow 0\quad.\end{aligned}\tag{5.3.12}$$

We rewrite this as

$$\begin{aligned}&\zeta(3)\left(\big[t_{(12)3},[t_{23},t_{13}]\big]-\big[t_{1(23)},[t_{12},t_{13}]\big]+\big[t_{23},[t_{12},t_{23}]\big]-\big[t_{12},[t_{23},t_{12}]\big]\right)\\&-\frac{\pi^3 i}{6}\left([t_{12}+t_{23},t_{13}]t_{(12)3}+t_{(12)3}[t_{23},t_{1(23)}]\right)\Rrightarrow 0\quad .\end{aligned}\tag{5.3.13}$$

The terms with coefficient $-\frac{\pi^3 i}{6}$ can be simply written as,

$$\left([t_{12},t_{(12)3}]+[t_{23},t_{1(23)}]\right)t_{(12)3}+t_{(12)3}[t_{23},t_{1(23)}]=(\partial\mathcal{L}+\partial\mathcal{R})\,t_{(12)3}+t_{(12)3}\partial\mathcal{R}\quad .\tag{5.3.14}$$

The coefficient of $\zeta(3)$ takes a bit more work,

$$\begin{aligned}&\big[t_{(12)3},[t_{23},t_{1(23)}-t_{12}]\big]-\big[t_{1(23)},[t_{12},t_{(12)3}-t_{23}]\big]+\big[t_{23},[t_{12},t_{23}]\big]-\big[t_{12},[t_{23},t_{12}]\big]\\&=[t_{(12)3},\partial\mathcal{R}]-\big[t_{(12)3},[t_{23},t_{12}]\big]-[t_{1(23)},\partial\mathcal{L}]+\big[t_{1(23)},[t_{12},t_{23}]\big]+\big[t_{23},[t_{12},t_{23}]\big]-\big[t_{12},[t_{23},t_{12}]\big]\\&=[t_{(12)3},\partial\mathcal{R}]-[t_{1(23)},\partial\mathcal{L}]-\big[[t_{(12)3},t_{23}],t_{12}\big]-\big[t_{23},[t_{(12)3},t_{12}]\big]+\big[[t_{1(23)},t_{12}],t_{23}\big]+\big[t_{12},[t_{1(23)},t_{23}]\big]\\&\qquad+\big[t_{23},[t_{12},t_{23}]\big]-\big[t_{12},[t_{23},t_{12}]\big]\\&=[t_{(12)3},\partial\mathcal{R}]-[t_{1(23)},\partial\mathcal{L}]-\big[2t_{12},[t_{23},t_{1(23)}]\big]+\big[2t_{23},[t_{12},t_{(12)3}]\big]\\&=[t_{(12)3}-2t_{12},\partial\mathcal{R}]+[2t_{23}-t_{1(23)},\partial\mathcal{L}]\end{aligned}\tag{5.3.15}$$

In sum, the third-order term of the the right pre-hexagonator series $\mathbf{R}$ is given by

$$(\mathbf{R})_3=\zeta(3)\big([t_{(12)3}-2t_{12},\mathcal{R}]+[2t_{23}-t_{1(23)},\mathcal{L}]\big)-\frac{\pi^3 i}{6}\big(\,(\mathcal{L}+\mathcal{R})\,t_{(12)3}+t_{(12)3}\mathcal{R}\big)\,.\tag{5.3.16}$$

Chapter 6

Sylleptic deformations

This chapter is also based on the solo preprint [Kemp25]. Definition 6.1.1 specifies the general notion of a σ-equivariant syllepsis on a braided monoidal bicategory [CSP14, Definition C.5] to our context of braided strictly-unital monoidal $\mathbf{Ch}_R^{[-1,0]}$-categories. In Subsection 6.1.1, Remark 6.1.2 analyses the scenario of γ-equivariant syllepses on symmetric strict monoidal $\mathbf{Ch}_R^{[-1,0]}$-categories and Definition 6.1.3 introduces our notion of an "infinitesimal syllepsis". Theorem 6.1.5 shows that 3SPSs induce infinitesimal syllepses in much the same way that 2SPSs induce infinitesimal 2-braidings and Example 6.1.6 studies these infinitesimal syllepses in the concrete cases of the shifted cotangent Lie 2-algebras (as in Remark 3.3.3) and matrix string Lie 2-algebras (as in Example 3.3.6).

We introduce in Subsection 6.2.1, our notion of a "coboundary infinitesimal syllepsis" and show that these are induced by 2SPSs whose weight 2 term is coboundary and Example 6.2.4 shows that the homotopically null 2SPSs (for the case of a simple string Lie 2-algebra) of Example 3.3.13 induce nontrivial coboundary syllepses. Remark 6.2.5 shows that the coboundary σ-equivariant syllepsis of Remark 6.2.1 is also a σ-equivariant syllepsis on the braided pentagonal strictly-unital monoidal $\mathbf{Ch}^{[-1,0]}_{\mathbb{K}[h]/(h^3)}$-category (5.2.37).

Even though we did not go through the exercise of showing that the third-order expansions of the pentagonator and hexagonators (see Section 5.3) satisfy their requisite axioms, Subsection 6.2.3 shows that the third-order expansion of the coboundary syllepsis satisfies its requisite axioms.

6.1 Syllepses on braided strictly-unital monoidal cochain 2-categories

Following on from Definition 5.1.3, we specify the general notion of a σ-equivariant syllepsis on a braided monoidal bicategory as in [CSP14, Definition C.5] to our context of braided strictly-unital monoidal $\mathbf{Ch}_R^{[-1,0]}$-categories.

Definition 6.1.1. If $(\mathbf{C}, \otimes, I, \alpha, \Pi, \sigma, \mathcal{H}^L, \mathcal{H}^R)$ is a braided strictly-unital monoidal $\mathbf{Ch}_R^{[-1,0]}$-category then a **syllepsis** Σ consists of the following datum:

(i) A modification $\Sigma : \sigma \Rrightarrow \sigma^{-1}_{\tau_{\mathbf{C},\mathbf{C}}} : \otimes \Rightarrow \otimes \tau_{\mathbf{C},\mathbf{C}} : \mathbf{C} \boxtimes \mathbf{C} \to \mathbf{C}$. Specifying Definition 4.1.10, this requires:

- $\forall U, V \in \mathbf{C}$, we have the following homotopy $\Sigma_{UV} : \sigma_{UV} \Rightarrow \sigma^{-1}_{VU} : (UV) \to (VU)$, i.e.,

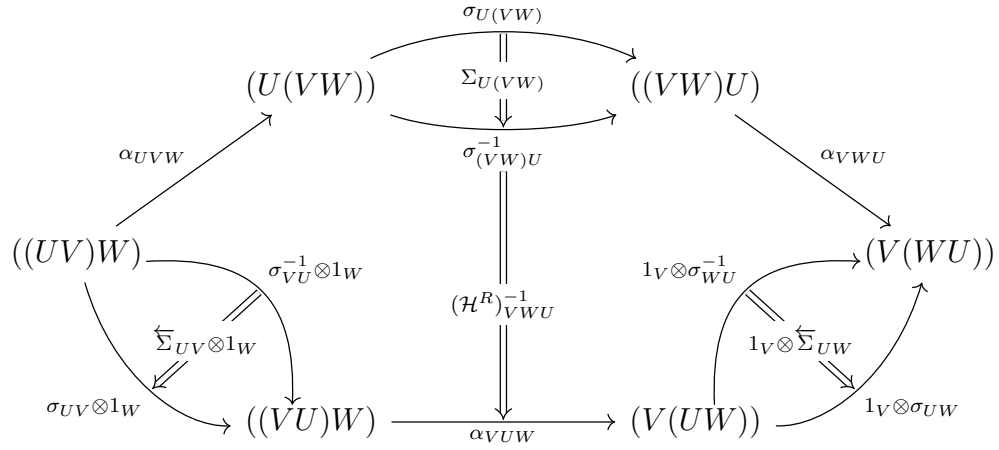
$$\partial \Sigma_{UV} = \sigma_{UV} - \sigma^{-1}_{VU} \quad . \tag{6.1.1a}$$

- For all $f \in \mathbf{C}[U,U']^0$ and $g \in \mathbf{C}[V,V']^0$,

$$\Sigma_{U'V'}(f \otimes g) + \sigma_{f \boxtimes g} = \sigma^{-1}_{g \boxtimes f} + (g \otimes f)\Sigma_{UV} \quad . \tag{6.1.1b}$$

This datum is required to satisfy the following two axioms:

1. **Left factorisation**, i.e., for all $UVW \in \mathbf{C}^{\boxtimes 3}$, the component of the left hexagonator (5.1.8b) is equal to the following pasting diagram in $\mathbf{C}$,

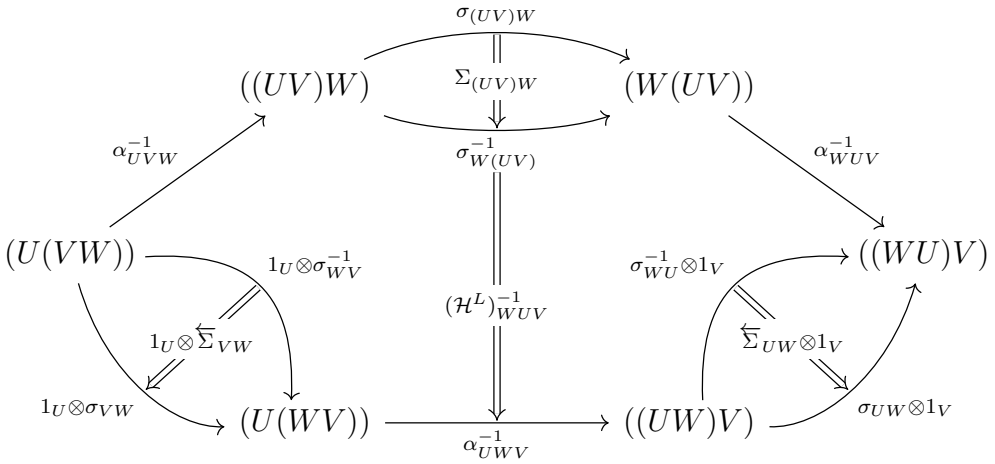


(6.1.2a)

In other words,

$$\mathcal{H}^L_{UVW} = \alpha_{VWU}\Sigma_{U(VW)}\alpha_{UVW} + (\mathcal{H}^R)^{-1}_{VWU} - (1_V \otimes \sigma^{-1}_{WU})\alpha_{VUW}(\Sigma_{UV}\otimes 1_W) - (1_V \otimes \Sigma_{UW})\alpha_{VUW}(\sigma_{UV}\otimes 1_W)\,. \tag{6.1.2b}$$

2. **Right factorisation**, i.e., for all $UVW \in \mathbf{C}^{\boxtimes 3}$, the component of the right hexagonator (5.1.9b) is equal to the following pasting diagram in $\mathbf{C}$,



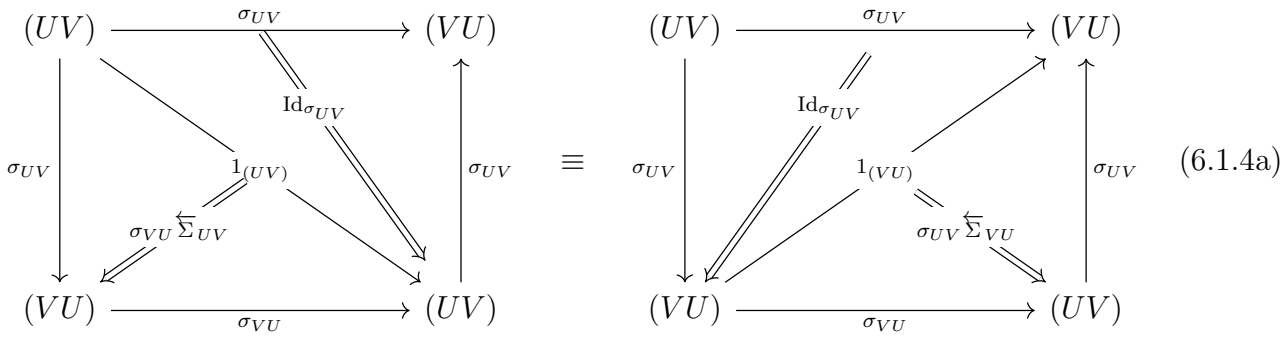
(6.1.3a)

In other words,

$$\mathcal{H}^R_{UVW} = \alpha^{-1}_{WUV}\Sigma_{(UV)W}\alpha^{-1}_{UVW} + (\mathcal{H}^L)^{-1}_{WUV} - (\sigma^{-1}_{WU}\otimes 1_V)\alpha^{-1}_{UWV}(1_U\otimes\Sigma_{VW}) - (\Sigma_{UW}\otimes 1_V)\alpha^{-1}_{UWV}(1_U\otimes\sigma_{VW})\,. \tag{6.1.3b}$$

A syllepsis is said to be **σ-equivariant** if, for all $UV \in \mathbf{C}\boxtimes\mathbf{C}$, we have the following equality

between pasting diagrams in $\mathbf{C}$,



(6.1.4a)

In other words,

$$\Sigma_{UV} = \sigma_{VU}^{-1}\Sigma_{VU}\sigma_{UV} \quad . \tag{6.1.4b}$$

6.1.1 Syllepses from 3-shifted Poisson structures

We now introduce "infinitesimal syllepses" on symmetric strict monoidal $\mathbf{Ch}_R^{[-1,0]}$-categories and show that 3SPSs give rise to them analogously to (4.2.27a). Remark 6.1.4 discusses the curious fact that infinitesimal syllepses can be 'integrated' to all orders simply by post-whiskering with the symmetric braiding γ.

Remark 6.1.2. Given a symmetric strict monoidal $\mathbf{Ch}_R^{[-1,0]}$-category $(\mathbf{C}, \otimes, I, \gamma)$, a γ-equivariant syllepsis $\Sigma : \gamma \Rrightarrow \gamma$ is strictly factorisable and intertwines with the symmetric braiding γ. For clarity's sake, let us specialise Definition 6.1.1 to the present context:

1. We have a family of endomorphic homotopies $\Sigma_{UV} \in \mathbf{C}[(UV),(VU)]^{-1}$, i.e.,

$$\partial\Sigma_{UV} = 0 \quad . \tag{6.1.5a}$$

2. The endomorphic homotopies are natural, i.e., for all $f \in \mathbf{C}[U,U']^0$ and $g \in \mathbf{C}[V,V']^0$,

$$\Sigma_{U'V'}(f \otimes g) = (g \otimes f)\Sigma_{UV} \quad . \tag{6.1.5b}$$

3. The endomorphic homotopies are **strictly left factorisable**, i.e., for all $UVW \in \mathbf{C}^{\boxtimes 3}$,

$$\Sigma_{U(VW)} = (1_V \otimes \gamma_{UW})(\Sigma_{UV} \otimes 1_W) + (1_V \otimes \Sigma_{UW})(\gamma_{UV} \otimes 1_W) \quad . \tag{6.1.5c}$$

4. The endomorphic homotopies are **strictly right factorisable**, i.e., for all $UVW \in \mathbf{C}^{\boxtimes 3}$,

$$\Sigma_{(UV)W} = (\gamma_{UW} \otimes 1_V)(1_U \otimes \Sigma_{VW}) + (\Sigma_{UW} \otimes 1_V)(1_U \otimes \gamma_{VW}) \quad . \tag{6.1.5d}$$

5. The endomorphic homotopies are γ-equivariant, i.e., for all $UV \in \mathbf{C} \boxtimes \mathbf{C}$,

$$\Sigma_{UV} = \gamma_{UV}\Sigma_{VU}\gamma_{UV} \quad . \tag{6.1.5e}$$

△

Definition 6.1.3. An **infinitesimal syllepsis** T on a symmetric strict monoidal $\mathbf{Ch}_R^{[-1,0]}$-category $(\mathbf{C}, \otimes, I, \gamma)$ consists of the following datum:

(i) An endomodification $T : \mathrm{Id}_\otimes \Rrightarrow \mathrm{Id}_\otimes : \otimes \Rightarrow \otimes : \mathbf{C} \boxtimes \mathbf{C} \to \mathbf{C}$. Again, this requires:

- A family of endomorphic homotopies $T_{UV} \in \mathbf{C}[(UV),(UV)]^{-1}$, i.e.,

$$\partial T_{UV} = 0 \quad . \tag{6.1.6a}$$

- The family of endomorphic homotopies is natural, i.e.,

$$T_{U'V'}(f \otimes g) = (f \otimes g) T_{UV} \quad . \tag{6.1.6b}$$

This datum is required to satisfy the following two axioms:

1. **Left infinitesimal factorisation**, i.e., for all $UVW \in \mathbf{C}^{\boxtimes 3}$,

$$T_{U(VW)} = T_{UV} \otimes 1_W + (\gamma_{VU} \otimes 1_W)(1_V \otimes T_{UW})(\gamma_{UV} \otimes 1_W) \quad . \tag{6.1.7a}$$

2. **Right infinitesimal factorisation**, i.e., for all $UVW \in \mathbf{C}^{\boxtimes 3}$,

$$T_{(UV)W} = 1_U \otimes T_{VW} + (1_U \otimes \gamma_{WV})(T_{UW} \otimes 1_V)(1_U \otimes \gamma_{VW}) \quad . \tag{6.1.7b}$$

An infinitesimal syllepsis is said to be γ**-equivariant** if, for all $UV \in \mathbf{C} \boxtimes \mathbf{C}$,

$$T_{UV} = \gamma_{VU} T_{VU} \gamma_{UV} \quad . \tag{6.1.8}$$

Remark 6.1.4. Recall that an infinitesimal braiding $t : \otimes \Rightarrow \otimes$ on a symmetric strict monoidal $\mathbf{Vec}_{\mathbb{K}}$-category $(\mathbf{C}, \otimes, I, \gamma)$ does not admit the simple integration as the braided *strict* monoidal $\mathbf{Vec}_{\mathbb{K}[[h]]}$-category $(\mathbf{C}[[h]], \otimes, I, \sigma := \gamma\, e^{\frac{h}{2}t})$ because, in general,

$$e^{\frac{h}{2}(t_{13}+t_{23})} \neq e^{\frac{h}{2}t_{13}} e^{\frac{h}{2}t_{23}} \quad , \quad e^{\frac{h}{2}(t_{13}+t_{12})} \neq e^{\frac{h}{2}t_{13}} e^{\frac{h}{2}t_{12}} \quad .$$

On the other hand, an infinitesimal syllepsis $T : \mathrm{Id}_{\otimes} \Rrightarrow \mathrm{Id}_{\otimes}$ on a symmetric strict monoidal $\mathbf{Ch}_R^{[-1,0]}$-category $(\mathbf{C}, \otimes, I, \gamma)$ admits the even simpler integration as the sylleptic symmetric strict monoidal $\mathbf{Ch}_R^{[-1,0]}$-category $(\mathbf{C}, \otimes, I, \gamma, \Sigma := \gamma T)$. This is because the strict factorisability relations (6.1.5c)/(6.1.5d) are already "linear" whereas the strict hexagon axioms (4.2.4) are "grouplike" and the infinitesimal hexagon relations (4.2.7) are linearisations thereof. △

Theorem 6.1.5. *Given any 3SPS $\pi_{n=3} \in \widehat{\mathrm{Pol}}(A,3)^5$ on a fg sf CDGA A over the field $\mathbb{K}$, the family*

$$\begin{array}{ccc} U \otimes_A V & \xrightarrow{\quad T_{UV} \quad} & (U \otimes_A V)[-1] \\ {\scriptstyle \xi_{U} \otimes_A \xi_V} \downarrow & & \uparrow {\scriptstyle \langle \pi^{(2)}_{n=3}, \cdot \rangle \otimes_A 1_{U \otimes_A V}} \\ \Omega_A[1] \otimes_A U \otimes_A \Omega_A[1] \otimes_A V & \xrightarrow[1_{\Omega_A[1]} \otimes_A \gamma_{U,\Omega_A[1]} \otimes_A 1_V]{} & \Omega_A[1] \otimes_A \Omega_A[1] \otimes_A U \otimes_A V \end{array} \tag{6.1.9}$$

defines a γ-equivariant infinitesimal syllepsis in the sense of Definition 6.1.3 on the homotopy 2-category ${}_A\mathbf{C} = \mathsf{Ho}_2\big({}_A\mathbf{dgMod}^{\mathrm{fgsf}}\big) \in \mathbf{dgCat}_{\mathbb{K}}^{[-1,0],\mathrm{ps}}$ of the dg-category of fgsf dg-modules over A.

Proof. Remark 2.3.37 tells us that the weight 2 term of a 3SPS $\pi^{(2)}_{n=3} \in \mathrm{Sym}^2_A\left(\mathsf{T}_A[-3-1]\right)^{3+2} \cong \mathrm{Sym}^2_A\left(\mathsf{T}_A[-1]\right)^{-1}$ satisfies $\partial \pi^{(2)}_{n=3} = 0$ thus the A-linear cochain map $\langle \cdot, \cdot \rangle : \mathrm{Sym}^2_A\big(\mathsf{T}_A[-1]\big) \otimes_A \Omega_A[1] \otimes_A \Omega_A[1] \to A$, explicitly given by (4.2.24), induces the coexact A-linear homotopy $\langle \pi^{(2)}_{n=3}, \cdot \rangle \in {}_A\mathbf{C}\left[\Omega_A[1] \otimes_A \Omega_A[1], A\right]^{-1}$. Though Lemma 4.2.11 told us that the family of maps $\big(\xi_U \in {}_A\mathbf{C}\big[U, \Omega_A[1] \otimes_A U\big]^0\big)_{U \in {}_A\mathbf{C}}$ is not natural but *pseudo*natural, the family (6.1.9) is actually natural because of the

coexactness of $\langle \pi^{(2)}_{n=3}, \cdot \rangle$ and the truncation on the homs. It can also be shown to satisfy the infinitesimal factorisation relations (6.1.7a)/(6.1.7b) and the γ-equivariance relation (6.1.8) by analogy with the proof of Theorem 4.2.12. □

Example 6.1.6. Looking back to Proposition 3.2.17, if we are considering the 3SPS on the Chevalley-Eilenberg algebra of a Lie 2-algebra $\mathfrak{g} = \left(\mathfrak{g}^{-1} \xrightarrow{\mathrm{d}} \mathfrak{g}^0, [\cdot,\cdot], [\cdot,\cdot,\cdot]\right)$ then the only possible nontrivial terms are $\pi^{(2,0)} : \mathbb{K} \to (\wedge^2 \mathfrak{g})[-1]$ and $\pi^{(2,1)} : \mathfrak{g} \to (\wedge^2 \mathfrak{g})[-2]$. Using the explicit basis expression (4.2.31a), these bivectors enter as

$$\begin{aligned} T_{UV}\big(u_i \otimes v_q\big) &= (T_{UV})_{iq}{}^{jr} u_j \otimes v_r \\ &= (-1)^{|u_j|\,(|\underline{V}_q{}^r|-1)} \left\langle \pi^{a\mu} s\partial_a s\partial_\mu + \pi_b^{\mu\nu} s\theta^b s\partial_\mu s\partial_\nu, s^{-1}\mathrm{d}_{\mathrm{dR}}(\underline{U}_i{}^j) \otimes_A s^{-1}\mathrm{d}_{\mathrm{dR}}(\underline{V}_q{}^r) \right\rangle u_j \otimes v_r \end{aligned} \tag{6.1.10}$$

where $\{\theta^a\}_{1\leq a\leq \dim(\mathfrak{g}^0)}$ denotes the basis for $(\mathfrak{g}^0)^*$ and $\partial_a := \frac{\partial}{\partial(s\theta^a)}$, likewise $\{\zeta^\mu\}_{1\leq \mu\leq \dim(\mathfrak{g}^{-1})}$ denotes the basis for $(\mathfrak{g}^{-1})^*$ and $\partial_\mu := \frac{\partial}{\partial(s\zeta^\mu)}$.

Subexample 6.1.7. If $(\pi^{(2,0)} \neq 0, \pi^{(2,1)} = 0)$ then, in order for the syllepsis to be nontrivial, we need to study those $\mathrm{CE}_{\mathfrak{g}}$-dg-modules which nontrivially represent both parts of the Lie 2-algebra $\mathfrak{g} = \mathfrak{g}^{-1} \oplus \mathfrak{g}^0$. An example of this case is provided by the 3-shifted *symplectic* structure on the Chevalley-Eilenberg algebra of the shifted cotangent Lie 2-algebra $T^*[1]\mathfrak{h}$ from Remark 3.3.3. Given that the shifted cotangent Lie 2-algebra $T^*[1]\mathfrak{h}$ of an ordinary Lie algebra $\mathfrak{h}$ is constructed by placing the dual $\mathfrak{h}^*$ in degree -1 as in (3.3.1), nontrivially representing $(T^*[1]\mathfrak{h})^0 := \mathfrak{h}$ automatically provides nontrivial action of $(T^*[1]\mathfrak{h})^{-1} := \mathfrak{h}^*$ by simply dualising. As said Remark mentions, such 3-shifted symplectic structures enter the construction of the action functional for higher-dimensional Chern-Simons theory hence (6.1.10) can be roughly thought of as a 'sylleptic deformation quantisation' of the action functional. Another example of this case is provided by those string Lie 2-algebras $\mathfrak{h}_\kappa$ where the underlying Lie algebra $\mathfrak{h}$ is a central extension as in Example 3.3.9. A nontrivial action of $\mathfrak{h}_\kappa^{-1} := \mathbb{K}$ is simply given by scalar multiplication. ▽

Subexample 6.1.8. If $(\pi^{(2,0)} = 0, \pi^{(2,1)} \neq 0)$ then we only need study those $\mathrm{CE}_{\mathfrak{g}}$-dg-modules which nontrivially represent $\mathfrak{g}^{-1}$. Examples of this case are provided by: the Heisenberg string Lie 2-algebra from Subexample 3.3.8, the Abelian string Lie 2-algebra from Subexample 3.3.7 and the matrix trace of Example 3.3.6. ▽

Finally, recall that Example 3.3.5 told us that all 3SPSs on the Chevalley-Eilenberg algebra of a string Lie 2-algebra of a semisimple Lie algebra are trivial hence we can say "semisimple string Lie 2-groups only admit trivial sylleptic deformations". ▽

6.2 Coboundary syllepses

6.2.1 Construction from coboundary 2-shifted Poisson structures

The following remark is a specification of Definition 6.1.1 and makes use of the formulae for the cochain/homotopy components of an inverse pseudonatural isomorphism (4.1.18).

Remark 6.2.1. Given a γ-equivariant hexagonal Breen t, consider the first-order deformation (5.1.16). A σ-equivariant syllepsis $\Sigma := h\gamma T$ is such that T must satisfy the five relations:

1. For all $UV \in \mathbf{C} \boxtimes \mathbf{C}$,

$$\partial T_{UV} = t_{UV} \quad . \tag{6.2.1a}$$

2. **Quasinaturality**, i.e., for all $f \in \mathbf{C}[U,U']^0$ and $g \in \mathbf{C}[V,V']^0$,

$$(f \otimes g)T_{UV} - T_{U'V'}(f \otimes g) = t_{f \boxtimes g} \quad . \tag{6.2.1b}$$

3. For all $UVW \in \mathbf{C}^{\boxtimes 3}$,

$$T_{U(VW)} = T_{UV} \otimes 1_W + (\gamma_{VU} \otimes 1_W)(1_V \otimes T_{UW})(\gamma_{UV} \otimes 1_W) \quad . \tag{6.2.1c}$$

4. For all $UVW \in \mathbf{C}^{\boxtimes 3}$,

$$T_{(UV)W} = 1_U \otimes T_{VW} + (1_U \otimes \gamma_{WV})(T_{UW} \otimes 1_V)(1_U \otimes \gamma_{VW}) \quad . \tag{6.2.1d}$$

5. For all $UV \in \mathbf{C} \boxtimes \mathbf{C}$,

$$T_{UV} = \gamma_{VU} T_{VU} \gamma_{UV} \quad . \tag{6.2.1e}$$

Because of (6.2.1a), we refer to $T : t \Rrightarrow 0$ as a **coboundary infinitesimal syllepsis**. △

Definition 6.2.2. A **coboundary 2SPS** $\pi_{n=2} \in \widehat{\mathrm{Pol}}(A,2)^4$ is one whose weight 2 term $\pi^{(2)}_{n=2} \in \mathrm{Sym}^2_A(\mathsf{T}_A[-1])^0$ is itself coboundary, i.e., $\exists\, \tilde{\pi}^{(2)}_{n=2} \in \mathrm{Sym}^2_A(\mathsf{T}_A[-1])^{-1}$ such that $\partial(\tilde{\pi}^{(2)}_{n=2}) = \pi^{(2)}_{n=2}$.

Theorem 6.2.3. *Given any coboundary 2SPS $\pi_{n=2}$ on a fg sf CDGA A over the field $\mathbb{K}$, the family*

$$\begin{array}{ccc} U \otimes_A V & \xrightarrow{\quad T_{UV} \quad} & (U \otimes_A V)[-1] \\ {\scriptstyle \xi_U \otimes_A \xi_V}\downarrow & & \uparrow{\scriptstyle \langle \tilde{\pi}^{(2)}_{n=2}, \cdot \rangle \otimes_A 1_{U \otimes_A V}} \\ \Omega_A[1] \otimes_A U \otimes_A \Omega_A[1] \otimes_A V & \xrightarrow[1_{\Omega_A[1]} \otimes_A \gamma_{U,\Omega_A[1]} \otimes_A 1_V]{} & \Omega_A[1] \otimes_A \Omega_A[1] \otimes_A U \otimes_A V \end{array} \tag{6.2.2}$$

defines a γ-equivariant coboundary infinitesimal syllepsis in the sense of Remark 6.2.1 on the homotopy 2-category ${}_A\mathbf{C} = \mathsf{Ho}_2\big({}_A\mathbf{dgMod}^{\mathrm{fgsf}}\big) \in \mathbf{dgCat}^{[-1,0],\mathrm{ps}}_{\mathbb{K}}$.

Proof. Example 5.1.5 told us $\pi_{n=2}$ induces a γ-equivariant hexagonal *Breen* infinitesimal 2-braiding t and it is obvious that, by construction, the first condition (6.2.1a) of Remark 6.2.1 is satisfied. The only relation in Remark 6.2.1 which remains to be checked is the quasinaturality (6.2.1b),

$$\begin{aligned} T_{U'V'}(f \otimes_A g) =& \iota_{\tilde{\pi}}(1 \otimes_A \gamma \otimes_A 1)(\xi_{U'} f \otimes_A \xi_{V'} g) \\ =& \iota_{\tilde{\pi}}(1 \otimes_A \gamma \otimes_A 1)\left(\left[(1 \otimes_A f)\xi_U - \partial\xi_f\right] \otimes_A \left[(1 \otimes_A g)\xi_V - \partial\xi_g\right]\right) \\ =& (f \otimes_A g)T_{UV} - \iota_\pi(1 \otimes_A \gamma \otimes_A 1)\left(\xi_f \otimes_A [1 \otimes_A g]\xi_V + [(1 \otimes_A f)\xi_U - \partial\xi_f] \otimes_A \xi_g\right) \\ =& (f \otimes_A g)T_{UV} - t_{f \boxtimes g} \quad , \end{aligned} \tag{6.2.3}$$

where the second equality uses the pseudonaturality (4.2.21) of ξ, the third equality uses the hom-truncation and the last equality uses the pseudonaturality (4.2.21) of ξ, again, on the last term together with the definition (4.2.27b) of the homotopy components of t. □

Example 6.2.4. Consider the case of a 2SPS on the Chevalley-Eilenberg algebra of a string Lie 2-algebra $\mathfrak{h}_\kappa$ as in Proposition 4.3.5. The string Lie 2-algebra has a trivial differential and $\mathfrak{h}^{-1}_\kappa := \mathbb{K}$ acts trivially on $\mathfrak{h}^0_\kappa := \mathfrak{h}$ thus $\delta(s\theta^a)$ and $\delta(s\nu)$ have no $s\nu$ term as in (4.3.7b) hence $\partial\left(\frac{\partial}{\partial(s\nu)}\right) = 0$ thus $\pi^{(2,0)}_{n=2} = \frac{1}{2}\pi^{ab} s^3 \frac{\partial}{\partial(s\theta^a)} s^3 \frac{\partial}{\partial(s\theta^b)}$ can only be coboundary if $\pi^{ab} = 0$. On the other

hand,

$$\partial\left(\frac{\partial}{\partial(s\theta^i)}\right) \sim f^a_{ib} s\theta^b \frac{\partial}{\partial(s\theta^a)} + \kappa_{ibc} s\theta^b s\theta^c \frac{\partial}{\partial(s\nu)} \tag{6.2.4}$$

yet $s^3\frac{\partial}{\partial(s\nu)} s^3 \frac{\partial}{\partial(s\nu)} = 0$ in $\mathrm{Pol}(\mathrm{CE}_{\mathfrak{h}_\kappa}, 2)^4$ therefore $\pi^{(2,1)}_{n=2} = \pi^a_b s\theta^b\, s^3\frac{\partial}{\partial(s\theta^a)} s^3 \frac{\partial}{\partial(s\nu)}$ admits the coboundary $\tilde{\pi}^{(2,1)}_{n=2} = \tilde{\pi}^i s^3 \frac{\partial}{\partial(s\theta^i)} s^3 \frac{\partial}{\partial(s\nu)}$ if $\pi^a_b = \tilde{\pi}^i f^a_{ib}$. Now recall that Example 3.3.13 told us that every 2SPS is merely given by an inner derivation $\pi^{(2,1)}_{n=2} = [y, \cdot] : \mathfrak{h} \to \mathfrak{h}$ if the underlying Lie algebra $\mathfrak{h}$ is simple. In this case, we expand the Lie algebra element y in terms of a basis $\{x_i\}_{1\leq i\leq \dim(\mathfrak{h})}$ as $y = \beta^i x_i$ so that the inner derivation acts on basis elements as $[y, x_b] = \beta^i f^a_{ib} x_a$. In other words,

$$\pi^{(2,1)}_{n=2} = [y, \cdot] = \beta^i f^a_{ib} s\theta^b\, s^3 \frac{\partial}{\partial(s\theta^a)} s^3 \frac{\partial}{\partial(s\nu)} \qquad \Longrightarrow \qquad \tilde{\pi}^i = \beta^i \quad . \tag{6.2.5}$$

In sum, though the 2SPSs on the Chevalley-Eilenberg algebra of a string Lie 2-algebra $\mathfrak{h}_\kappa$ for a simple Lie algebra $\mathfrak{h}$ are merely given by inner derivations of $\mathfrak{h}$ thus homotopically null and inducing homotopically trivial deformations of the symmetric braiding, these 2SPSs are nontrivial in the coboundary sense hence they induce nontrivial deformations of the identity syllepsis. ▽

6.2.2 Second-order coboundary syllepses

For a coherent totally γ-equivariant hexagonal t, consider the σ-equivariant syllepsis $\Sigma := h\gamma T$ from Remark 6.2.1 and the braided pentagonal strictly-unital monoidal $\mathbf{Ch}^{[-1,0]}_{\mathbb{K}[h]/(h^3)}$-category (5.2.37).

Remark 6.2.5. First we mention that (6.1.1a) and (6.1.1b) hold to order h^2 because of the γ-equivariance of t and because Σ has no order h^2 term. The order h^2 relation of (6.1.2b) reads as

$$2\mathcal{L} + \mathcal{R} = -(\mathcal{L}_{231} + 2\mathcal{R}_{231}) + 12 t_{13} T_{12} - 12 T_{13} t_{12} \quad , \tag{6.2.6a}$$

where we used Remark 4.1.16 for the inverse $(\mathcal{H}^R_{231})^{-1}$. The order h^2 relation of (6.1.3b) reads

$$\mathcal{L} + 2\mathcal{R} = -(2\mathcal{L}_{312} + \mathcal{R}_{312}) + 12 t_{13} T_{23} - 12 T_{13} t_{23} \quad . \tag{6.2.6b}$$

Lemma 5.2.25 gives us: $\mathcal{L}_{231} = \mathcal{R}$, $\mathcal{R}_{312} = \mathcal{L}$ and $\mathcal{R}_{231} = \mathcal{L}_{312} = \mathcal{L}_{132}$, and coherency gives us $\mathcal{L}_{132} = -(\mathcal{L} + \mathcal{R})$ hence (6.2.6a) and (6.2.6b) become, respectively:

$$t_{13} T_{12} = T_{13} t_{12} \quad , \quad t_{13} T_{23} = T_{13} t_{23} \quad . \tag{6.2.6c}$$

Likewise, the order h^2 relation of (6.1.4b) reads as

$$[t, T] = 0 \quad . \tag{6.2.6d}$$

Truncation of the homs together with $\partial T = t$ gives us

$$t_{13} T_{12} = (\partial T_{13}) T_{12} = T_{13} \partial T_{12} = T_{13} t_{12} \quad . \tag{6.2.6e}$$

The other relation in (6.2.6c) and the relation (6.2.6d) are proved similarly. △

6.2.3 Third and higher-order coboundary syllepses

Definition 6.2.6. Given a coherent totally γ-equivariant hexagonal t and the relevant monoidal data,

$$\big(\mathbf{C}[[\hbar]]\,,\, \otimes\,,\, \alpha :\equiv \Phi(t_{12}, t_{23})\,,\, \sigma :\equiv \gamma e^{\pi i \hbar t}\,,\, \mathcal{H}^L := \gamma_{1(23)}\mathbf{L}\,,\, \mathcal{H}^R := \gamma_{(12)3}\mathbf{R}\big) \quad , \tag{6.2.7}$$

if we have a syllepsis $\Sigma := \gamma\Gamma : \gamma e^{\pi i\hbar t} \Rrightarrow \gamma e^{-\pi i\hbar t}$, then we call $\Gamma : e^{\pi i\hbar t} \Rrightarrow e^{-\pi i\hbar t}$ a **pre-syllepsis**.

Consider the right factorisation axiom (6.1.3b),

$$\mathcal{H}^R = \alpha^{-1}_{312}\Sigma_{(12)3}\alpha^{-1} + (\mathcal{H}^L)^{-1}_{312} - \sigma^{-1}_{31}\alpha^{-1}_{132}\Sigma_{23} - \Sigma_{13}\alpha^{-1}_{132}\sigma_{23} \quad . \tag{6.2.8}$$

We use the formula in Remark 4.1.16 to write the inverse of the left hexagonator (5.2.2b) as

$$(\mathcal{H}^L)^{-1} = -\sigma^{-1}_{12}\alpha^{-1}_{213}\sigma^{-1}_{12}\mathcal{H}^L\alpha^{-1}\sigma^{-1}_{1(23)}\alpha^{-1}_{231} \implies (\mathcal{H}^L)^{-1}_{312} = -\sigma^{-1}_{31}\alpha^{-1}_{132}\sigma^{-1}_{32}\mathcal{H}^L_{312}\alpha^{-1}_{312}\sigma^{-1}_{3(12)}\alpha^{-1}\,.$$

Hence, we arrive at the following index equation

$$\begin{aligned}\mathbf{R} = \Phi(t_{12},t_{13})\Gamma_{(12)3}\Phi(t_{23},t_{12}) - e^{-\pi i\hbar t_{13}}\Phi(t_{23},t_{13})e^{-\pi i\hbar t_{23}}\mathbf{R}_{213}\Phi(t_{12},t_{13})e^{-\pi i\hbar t_{(12)3}}\Phi(t_{23},t_{12})\\ - e^{-\pi i\hbar t_{13}}\Phi(t_{23},t_{13})\Gamma_{23} - \Gamma_{13}\Phi(t_{23},t_{13})e^{\pi i\hbar t_{23}} \quad .\end{aligned} \tag{6.2.9}$$

Analogous arguments show the left factorisation axiom (6.1.2b) to yield

$$\begin{aligned}\mathbf{R}_{321} = \Phi(t_{23},t_{13})\Gamma_{1(23)}\Phi(t_{12},t_{23}) - e^{-\pi i\hbar t_{13}}\Phi(t_{12},t_{13})e^{-\pi i\hbar t_{12}}\mathbf{R}_{231}\Phi(t_{23},t_{13})e^{-\pi i\hbar t_{1(23)}}\Phi(t_{12},t_{23})\\ - e^{-\pi i\hbar t_{13}}\Phi(t_{12},t_{13})\Gamma_{12} - \Gamma_{13}\Phi(t_{12},t_{13})e^{\pi i\hbar t_{12}} \quad .\end{aligned} \tag{6.2.10}$$

The σ-equivariance property (6.1.4b) simply translates as

$$\Gamma = e^{-\pi i\hbar t}\Gamma_{21}e^{\pi i\hbar t} = e^{-\pi i\hbar \mathrm{ad}_t}(\Gamma_{21}) \quad . \tag{6.2.11}$$

Remark 6.2.7. Recalling the coboundary infinitesimal syllepsis $T : t \Rrightarrow 0$ of Remark 6.2.1, we consider the candidate pre-syllepsis

$$\Gamma = 2\pi i\hbar\sum_{m=0}^{\infty}\frac{(\pi i\hbar t)^{2m}}{(2m+1)!}T \quad . \tag{6.2.12}$$

Remark 6.2.1 tells us that T satisfies:

$$T_{1(23)} = T_{12} + T_{13} \quad , \quad T_{(12)3} = T_{23} + T_{13} \quad , \quad T_{12} = T_{21} \quad . \tag{6.2.13}$$

In particular, $T_{12} = T_{21}$ and $t_{12} = t_{21}$ implies $\Gamma_{12} = \Gamma_{21}$ thus simplifying (6.2.11) to

$$\Gamma = e^{-\pi i\hbar \mathrm{ad}_t}(\Gamma) \quad . \tag{6.2.14}$$

Now (6.2.6d) reduces the all-orders σ-equivariance relation (6.2.14) to the tautology $\Gamma = \Gamma$ hence the only relevant relations going forward are the factorisation relations (6.2.9)/(6.2.10). △

We now wish to show that

$$\Gamma = 2\pi i\hbar T - \frac{\pi^3 i\hbar^3}{3}t^2T \tag{6.2.15}$$

satisfies the factorisation relations (6.2.9)/(6.2.10) required of a σ-equivariant pre-syllepsis on the braided strictly-unital monoidal $\mathbf{Ch}^{[-1,0]}_{\mathbb{C}[\hbar]/(\hbar^4)}$-category $\big(\mathbf{C}_{\mathbb{C}[\hbar]/(\hbar^4)}, \otimes, I, \alpha, \Pi, \sigma, \mathcal{H}^L, \mathcal{H}^R\big)$. At order

$\hbar^3$, the right factorisation relation (6.2.9) reads as

$$\begin{aligned}
&\zeta(3)\big([t_{(12)3}-2t_{12},\mathcal{R}]+[2t_{23}-t_{1(23)},\mathcal{L}]\big)-\frac{\pi^3 i}{6}\big(\,(\mathcal{L}+\mathcal{R})\,t_{(12)3}+t_{(12)3}\mathcal{R}\big) \qquad (6.2.16)\\
&=-\frac{\pi^3 i}{3}t_{(12)3}^2T_{(12)3}-\frac{\pi^3 i}{3}[t_{12},t_{13}]T_{(12)3}-\frac{\pi^3 i}{3}T_{(12)3}[t_{23},t_{12}]\\
&\quad-\zeta(3)\big([t_{(12)3}-2t_{12},\mathcal{R}_{213}]+[2t_{13}-t_{2(13)},\mathcal{L}_{213}]\big)+\frac{\pi^3 i}{6}\big(\,(\mathcal{L}_{213}+\mathcal{R}_{213})\,t_{(12)3}+t_{(12)3}\mathcal{R}_{213}\big)\\
&\quad-\frac{\pi^3 i}{6}\big(\,(\mathcal{L}_{213}+2\mathcal{R}_{213})\,t_{(12)3}+t_{(12)3}\,(\mathcal{L}_{213}+2\mathcal{R}_{213})\,\big)\\
&\quad+\frac{\pi^3 i}{3}t_{23}^2T_{23}+\frac{\pi^3 i}{3}[t_{23},t_{13}]T_{23}+\pi^3 i t_{13}^2T_{23}+\frac{\pi^3 i}{3}t_{13}^2T_{13}+\frac{\pi^3 i}{3}T_{13}[t_{23},t_{13}]+\pi^3 i T_{13}t_{23}^2\quad .
\end{aligned}$$

Coherency of t gives us $\mathcal{R}_{213}=-(\mathcal{L}+\mathcal{R})$ and total γ-equivariance gives us $\mathcal{L}_{213}=\mathcal{L}$ which leaves

$$\frac{\pi^3 i}{3}\big(\,(\mathcal{L}+\mathcal{R})\,t_{(12)3}+t_{(12)3}\mathcal{R}\big)=\frac{\pi^3 i}{3}\big([t_{12}+t_{23},t_{13}]T_{(12)3}+T_{(12)3}[t_{23},t_{1(23)}]\big) \qquad (6.2.17)$$

on the RHS. Using the hexagonality of t and T together with the 2^{nd}-order relations (6.2.6c)/(6.2.6d) guarantees the above 3^{rd}-order relation. The left factorisation relation (6.2.10) is proved likewise.

Chapter 7

Conclusion and Outlook

Let us quickly summarise what has been achieved in this thesis and then follow up such a conclusion with a motivated outlook. Chapter 3 showed that a shifted Poisson structure (semiclassical data) on a CDGA A with nice properties is given by a specific family of multilinear maps on the underlying higher Lie algebra $\mathfrak{g}$ where $A \cong \mathrm{CE}_{\mathfrak{g}}$. Section 3.3 classified the 2, 3 and 4-shifted Poisson structures for explicit examples of Lie 2-algebras. For instance, Example 3.3.13 demonstrated that, for the case of the string Lie 2-algebra of a simple Lie algebra $\mathfrak{h}$, a 2-shifted Poisson structure is given by a derivation of $\mathfrak{h}$. Chapter 4 showed exactly how 2-shifted Poisson structures induce an infinitesimal 2-braiding t on the cochain 2-category $\mathsf{Ho}_2\big({}_A\mathbf{dgMod}^{\mathrm{sf}}\big)$ of semi-free A-dg-modules, A-linear cochain maps and A-linear homotopies modulo 2-homotopies, i.e., we explicitly described how the semiclassical data infinitesimally deforms the symmetric braiding of the 2-category of representations of $\mathfrak{g}$. Chapter 5 explored the 2^{nd}-order integration of such infinitesimal 2-braidings upon choosing the usual ansatz of the braiding as the exponential $\sigma := \gamma\, e^{i\pi\hbar t}$ and the associator as Drinfeld's KZ series $\alpha := \Phi_{\mathrm{KZ}}(t_{12}, t_{23})$. It was demonstrated in Subsection 5.2.2 that the left/right hexagon axioms do not hold but are obstructed by combinations of four-term relationators $\mathcal{L} := t_{t_{12}}$ and $\mathcal{R} := t_{t_{23}}$. Subsection 5.2.3 verified that such obstructions are indeed left/right hexagonators $\mathcal{H}^{L/R}$ (as in Definition 5.1.3) provided that t satisfies some niceness properties that Cirio and Faria Martins introduced [CFM15]. Finally, Chapter 6 proved that 3-shifted Poisson structures induces syllepses on $\mathsf{Ho}_2\big({}_A\mathbf{dgMod}^{\mathrm{sf}}\big)$ and coboundary 2-shifted Poisson structures induce infinitesimal coboundary syllepses $T : t \Rrightarrow 0$, the latter were integrated alongside that of the infinitesimal 2-braiding.

7.1 Cartier integration of infinitesimal 2-braidings

As mentioned towards the end of Section 1.2, the Cartier integration of an infinitesimal braiding t is carried out by studying parallel transport over $Y_n := \{(z_1, \ldots, z_n) \in \mathbb{C}^n |$ if $i \neq j$ then $z_i \neq z_j\}$ with respect to the (flat) KZ connection,

$$\nabla_{\mathrm{KZ}} := \hbar \sum_{1 \leq i < j \leq n} \omega_{ij} t_{ij} \quad ,$$

where $\omega_{ij} := \frac{\mathrm{d}z_i - \mathrm{d}z_j}{z_i - z_j}$. The braiding $\sigma := \gamma\, e^{i\pi\hbar t}$ and associator $\alpha := \Phi_{\mathrm{KZ}}(t_{12}, t_{23})$ are shown to satisfy the hexagon/pentagon axioms by constructing contractible 1-loops in Y_3 [BRW23, Section 2.3]/Y_4 [BRW23, Section 2.4]. Given an infinitesimal 2-braiding t and defining

$$\Delta_{\mathrm{CM}} := 2\hbar \sum_{1 \leq i < j < k \leq n} \omega_{ij} \wedge \omega_{ki} \mathcal{L}_{ijk} + \omega_{jk} \wedge \omega_{ki} \mathcal{R}_{ijk} \quad , \tag{7.1.1}$$

CM proved [CFM15, Section 3] that $(\nabla_{\mathrm{KZ}}, \Delta_{\mathrm{CM}})$ is 2-flat if t is hexagonal and coherent.

Hexagonators and the Breen polytope [Kemp25'] showed that one can construct hexagonator series $\mathcal{H}^{L/R}_{\mathrm{CM}}$ as a 2-holonomy with respect to $(\nabla_{\mathrm{KZ}}, \Delta_{\mathrm{CM}})$ over Y_3. The 2^{nd}-order term of $\mathcal{H}^{L/R}_{\mathrm{CM}}$ was computed and found to agree with the infinitesimal hexagonator $\mathrm{h}^{L/R}$ from Construction 5.2.11. Assuming a coherent totally γ-equivariant hexagonal t, it was demonstrated that the Breen polytope axiom (5.1.13) is satisfied by translating it into a 2-loop in Y_3.

Integrating infinitesimal coboundary syllepses to all orders Looking back to Subsection 6.2.3, the right pre-hexagonator $\mathbf{R} := \gamma^{-1}_{(12)3}\mathcal{H}^R$ and pre-syllepsis $\Gamma := \gamma^{-1}\Sigma = 2\pi i\hbar \sum_{m=0}^{\infty} \frac{(\pi i\hbar t)^{2m}}{(2m+1)!} T$ must together satisfy, for example,

$$\mathbf{R} = \Phi_{213}\Gamma_{(12)3}\Phi_{321} - e^{-\pi i\hbar t_{13}}\Phi_{231}\left[e^{-\pi i\hbar t_{23}}\mathbf{R}_{213}\Phi_{213}e^{-\pi i\hbar t_{(12)3}}\Phi_{321} - \Gamma_{23}\right] - \Gamma_{13}\Phi_{231}e^{\pi i\hbar t_{23}}\,, \quad (7.1.2)$$

where $\Phi_{ijk} := \Phi_{\mathrm{KZ}}(t_{12}, t_{23})$. One would like to prove (7.1.2) by constructing a suitable 2-loop in Y_3 but there is an obvious problem; in order to construct, e.g., $\Gamma_{23} : e^{i\pi\hbar t_{23}} \Rrightarrow e^{-i\pi\hbar t_{23}}$ as a 2-holonomy of a 2-path, such a 2-path would have to have its edges either side of the singularity $z_2 = z_3$. This suggests that the only way to prove (7.1.2) would be by substituting the explicit series formulae for all the terms involved as in Remarks 6.2.5 and 6.2.7.

The pentagonator and the rest of the axioms To complete the Cartier integration of an infinitesimal 2-braiding (initiated in [Kemp25']), we still need to construct a pentagonator series Π_{CM} and show that it satisfies the associahedron (5.1.2), left/right tetrahedron (5.1.10)/(5.1.11) and hexahedron (5.1.12) axioms. We expect that Π_{CM} would be constructed as a 2-holonomy with respect to $(\nabla_{\mathrm{KZ}}, \Delta_{\mathrm{CM}})$ over Y_4. A sanity check would be to compare the 3^{rd}-order expansion of Π_{CM} and compare it with the known one (5.3.10). The associahedron axiom would be proved by constructing a 2-loop in Y_5 whereas the left/right tetrahedron and hexahedron axioms would require a 2-loop in Y_4.

Tannaka-Krein reconstruction of Drinfeld-Jimbo quantum 2-groups Given a Lie algebra $\mathfrak{g}$ and a symmetric invariant 2-tensor $t \in (\mathrm{Sym}^2\mathfrak{g})^{\mathfrak{g}}$, the infinitesimal braiding induced on $\mathbf{Rep}_{\mathbb{K}}\mathfrak{g} \cong {}_{U\mathfrak{g}}\mathbf{Mod}$ can be integrated as above and the reconstructed Drinfeld-Jimbo quantum group is the cocommutative quasitriangular Hopf algebra

$$\big((U\mathfrak{g})[[\hbar]], \Delta, \varepsilon, R := e^{i\pi\hbar t}, \Phi := \Phi_{\mathrm{KZ}}(t_{12}, t_{23})\big) \quad . \quad (7.1.3)$$

Given an L_∞-algebra $\mathfrak{g}$, there exists the notion of a weak representation [All14, Definition 2.2] and an equivariant map between them [All14, Definition 3.3]. To the best of our knowledge though, the notion of higher homotopies is not known nor is the composition of maps. In other words, the dg-category $\mathbf{wRep}_{\mathbb{K}}\mathfrak{g}$ has not been explicitly described in the literature nor has its compatible symmetric monoidal structure. Such an investigation has been undertaken by the author and it was recently discovered exactly how to induce an infinitesimal 2-braiding on $\mathsf{Ho}_2\big(\mathbf{wRep}_{\mathbb{K}}\mathfrak{g}\big)$ from the weight 2 term of a 2SPS π_2 as characterised in Corollary 3.1.6. It was also recently found that such an infinitesimal 2-braiding is indeed coherent (because of the third weight component of the Maurer-Cartan equation) whereas the dg-module perspective only allowed us to speculate this connection in Remark 5.2.32.

7.2 Towards infinitesimal 3-braidings and beyond

We can ask what would happen if we lifted the truncation in the slightest manner and considered $\mathbf{Ch}^{[-2,0]}$-categories, $\mathbf{Ch}^{[-2,0]}$-functors, pseudonatural transformations, pseudomodifications and "mutations". In this context, a pseudonatural transformation $\xi : F \Rightarrow G : \mathbf{C} \to \mathbf{D}$ additionally has **2-homotoper components** $\xi_{(\cdot,\cdot)} : \mathbf{C}[V,W] \otimes \mathbf{C}[U,V] \to \mathbf{D}[FU,GW][-2]$ and these must satisfy, for all $f \in \mathbf{C}[U,V]^0$, $g \in \mathbf{C}[V,W]^0$ and $h \in \mathbf{C}[W,X]^0$:

$$G(g)\xi_f + \xi_g F(f) \;=\; \xi_{gf} + \partial(\xi_{g,f}) \quad , \tag{7.2.1}$$

$$G(h)\xi_{g,f} - \xi_{h,g}F(f) \;=\; \xi_{hg,f} - \xi_{h,gf} \quad . \tag{7.2.2}$$

Looking back to the proof of Lemma 4.2.11, we can see that the de Rham pseudonatural transformation $\xi : \mathrm{id}_{{}_A\mathbf{C}} \Rightarrow L_{\Omega_A[1]}$ has trivial 2-homotoper components! However, the monoidal composite $\xi \boxtimes \psi : F \boxtimes G \Rightarrow F' \boxtimes G' : \mathbf{C} \boxtimes \mathbf{D} \to \mathbf{C}' \boxtimes \mathbf{D}'$ has 2-homotoper components

$$(\xi \boxtimes \psi)_{f'\boxtimes g', f\boxtimes g} := \xi_{f',f} \boxtimes G'(g')\psi_V + \xi_{f'}F(f) \boxtimes G'(g')\psi_g + \xi_{U''}F(f'f) \boxtimes \psi_{g',g} \quad , \tag{7.2.3a}$$

where $U \xrightarrow{f} U' \xrightarrow{f'} U''$ and $V \xrightarrow{g} V' \xrightarrow{g'} V''$ hence $\xi \boxtimes \xi$ has the 2-homotoper components

$$(\xi \boxtimes \xi)_{f'\boxtimes g', f\boxtimes g} := \xi_{f'f} \boxtimes (1_{\Omega_A[1]} \otimes_A g')\xi_g \quad , \tag{7.2.4}$$

and the infinitesimal 3-braiding (4.2.26) has nontrivial 2-homotoper components. That being said, it now becomes clear that completely lifting the truncation and considering the dg-category ${}_A\mathbf{dgMod}^{\mathrm{sf}}$ would mean that the infinitesimal dg-braiding t^{π_2} would only have cochain, homotoper and 2-homotoper components because the n-homotoper components of the monoidal composite are defined "affinely" as in (7.2.3). Furthermore, the infinitesimal dg-braiding t^{π_2} would still satisfy the infinitesimal hexagonal relations as in Definition 4.2.3 simply because ξ does. What is surprising though is that the γ-equivariance does not hold for (2-)homotoper components; for example, the infinitesimal 3-braiding t^{π_2} satisfies:

$$\gamma_{V'U'}t_{g\boxtimes f}\gamma_{UV} = t_{f\boxtimes g} + \partial(t_{f\boxtimes 1_{V'}, 1_U\boxtimes g}) - t_{\partial(f)\boxtimes 1_{V'}, 1_U\boxtimes g} - t_{f\boxtimes 1_{V'}, 1_U\boxtimes\partial(g)} \quad , \tag{7.2.5}$$

$$\gamma_{V''U''}t_{g'\boxtimes f', g\boxtimes f}\gamma_{UV} = -t_{f'\boxtimes g', f\boxtimes g} + t_{f'\boxtimes 1_{V''}, f\boxtimes g'g} - t_{f'f\boxtimes 1_{V''}, 1_U\boxtimes g'}(1_U \otimes_A g) \quad . \tag{7.2.6}$$

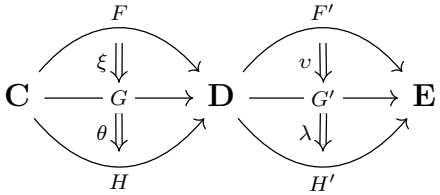
Using (7.2.4) and setting $g = 1_V$, we can simplify (7.2.5) to $t_{f\boxtimes 1_V} = \gamma_{VU'}t_{1_V\boxtimes f}\gamma_{UV}$ hence, for example, $t_{t_{UV}\boxtimes 1_V} = \gamma_{W(UV)}t_{1_W\boxtimes t_{UV}}\gamma_{(UV)W}$. Lemma 5.2.9 made use of such relations to reduce the independence between the four-term relationators but now they are pseudomodifications.

Four-term relationators A **pseudomodification** $\Xi : \xi \Rrightarrow \xi' : F \Rightarrow G : \mathbf{C} \to \mathbf{D}$ also has 2-homotoper components $\Xi_{(\cdot)} : \mathbf{C}[U,V] \to \mathbf{D}[FU,GV][-2]$ and these satisfy, for all $f \in \mathbf{C}[U,V]^0$ and $g \in \mathbf{C}[V,W]^0$:

$$G(f)\,\Xi_U - \Xi_V\,F(f) \;=\; \xi_f - \xi'_f - \partial(\Xi_f) \quad , \tag{7.2.7}$$

$$G(g)\,\Xi_f + \Xi_g\,F(f) \;=\; \xi_{g,f} - \xi'_{g,f} + \Xi_{gf} \quad . \tag{7.2.8}$$

In this context, given the diagram



, (7.2.9)

we have:

$$(\theta\xi)_{g,f} := \theta_{g,f}\xi_U + \theta_g\xi_f + \theta_W\xi_{g,f} \quad , \quad (\upsilon * \xi)_{g,f} = \upsilon_{Gg,Gf}F'\xi_U + \upsilon_{Gg}F'\xi_f + \upsilon_{GW}F'\xi_{g,f} \tag{7.2.10}$$

hence the exchanger pseudomodification $*^2_{\lambda,\theta|\upsilon,\xi} : (\lambda * \theta)(\upsilon * \xi) \Rrightarrow \lambda\upsilon * \theta\xi : F'F \Rightarrow H'H$ has 2-homotoper components given by

$$(*^2_{\lambda,\theta|\upsilon,\xi})_f := \lambda_{HV}\upsilon_{\theta_V}F'\xi_f + \lambda_{HV}(\upsilon_{\theta_f} + \upsilon_{\theta_V,Gf} - \upsilon_{Hf,\theta_U})F'\xi_U - \lambda_{Hf}\upsilon_{\theta_U}F'\xi_U \quad , \tag{7.2.11}$$

and the left four-term relationator $\mathcal{L} := *^2_{\mathrm{Id}_\otimes, t\boxtimes \mathrm{Id}_{\mathrm{id}_\mathbf{C}}|t,\mathrm{Id}_\otimes\boxtimes \mathrm{id}_\mathbf{C}}$ has 2-homotoper components

$$\mathcal{L}_{f\boxtimes g\boxtimes h} = t_{t_{f\boxtimes g}\boxtimes h} + t_{t_{U'V'}\boxtimes 1_{W'},(f\otimes g)\boxtimes h} - t_{(f\otimes g)\boxtimes h, t_{UV}\boxtimes 1_W} \tag{7.2.12}$$

where $f \in \mathbf{C}[U,U']^0$, $g \in \mathbf{C}[V,V']^0$ and $h \in \mathbf{C}[W,W']^0$. For our concrete t^{π_2}, the last term of (7.2.12) vanishes. Recall as well from Definition 5.2.20 that total γ-equivariance of t required $*^2_{\mathrm{Id}_\otimes,\gamma\boxtimes \mathrm{Id}_{\mathrm{id}_\mathbf{C}}|t,\mathrm{Id}_\otimes\boxtimes \mathrm{id}_\mathbf{C}} = 0$; this now imposes the additional constraint

$$(*^2_{\mathrm{Id}_\otimes,\gamma\boxtimes \mathrm{Id}_{\mathrm{id}_\mathbf{C}}|t,\mathrm{Id}_\otimes\boxtimes \mathrm{id}_\mathbf{C}})_{f\boxtimes g\boxtimes h} = t_{\gamma_{U'V'}\boxtimes 1_{W'},(f\otimes g)\boxtimes h} - t_{(g\otimes f)\boxtimes h,\gamma_{UV}\boxtimes 1_W} = \mathbf{0} \tag{7.2.13}$$

and such constraint is indeed satisfied by the infinitesimal 3-braiding t^{π_2}.

Higher weight components of SPSs This 3-categorical context also reveals how the weight 3 term of the 2SPS π_3 contributes to the deformation data; again, Remark 5.2.32 conjectured that the coherency of the infinitesimal 2-braiding is secured through the third weight component of the Maurer-Cartan equation together with the truncation to $\mathbf{Ch}^{[-1,0]}$-homs. Recall as well that Proposition 5.2.33 stated how coherency is related to the satisfaction of the Breen polytope axiom. Our infinitesimal 3-braiding t^{π_2} will not be coherent but, rather, quasi-coherent with π_3 inducing an "infinitesimal Breen polytoper". A **mutation** $\mathsf{M} : \Xi \rightsquigarrow \tilde{\Xi} : \xi \Rrightarrow \xi' : F \Rightarrow G : \mathbf{C} \to \mathbf{D}$ is a family of 2-homotopies $\mathsf{M}_U \in \mathbf{D}[FU, GU]^{-2}$, i.e., $\partial(\mathsf{M}_U) = \Xi_U - \tilde{\Xi}_U$ such that, for $f \in \mathbf{C}[U,V]^0$,

$$G(f)\,\mathsf{M}_U - \mathsf{M}_V F(f) \;=\; \tilde{\Xi}_f - \Xi_f \quad . \tag{7.2.14}$$

The infinitesimal Breen polytoper $\mathsf{B}^{\pi_3} : \mathcal{L} + \mathcal{L}_{231} + \mathcal{L}_{132} \rightsquigarrow \mathbf{0} : 0 \Rrightarrow 0 : \otimes^2 \Rightarrow \otimes : \mathbf{C}^{\boxtimes 3} \to \mathbf{C}$ would satisfy analogous infinitesimal hexagon and γ-equivariance relations.

Braided monoidal cochain 3-categories There is no consensus in the literature on a definition of a "Braided monoidal tricategory" in the same way as there is for that of braided monoidal bicategories [CSP14, Appendix C]. Nevertheless one can build a sensible definition as follows: first define a (strictly-unital) monoidal cochain 3-category as additionally having a "associahedronator" mutation and use the Stasheff polytopes [Sta63] to derive its higher coherence condition. Then define a braiding as now having "tetrahedronator/hexahedronator/Breen poltyoper" mutations and use Kapranov and Voevodsky's shuffle polytopes [KV94b] together with permutohedra [GR63] to derive all the higher coherence conditions. The 'niceness' properties of an infinitesimal 2-braiding became relevant at 2nd-order hence we would expect that of an infinitesimal 3-braiding to be revealed at 3rd-order. Such properties would motivate exactly how one would define a KZ 3-connection and its 3-holonomy [FMP11] would provide the means to Cartier integrate.

Appendix A

Tricategorical checks

This appendix provides the well-definition checks needed in Subsection 4.1.1 for the compositions of the tricategorical universe $\mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}$. Note that this appendix will not provide checks for associativity, unitality or preservation of units because these properties are relatively immediate and, importantly, *do not* require making use of the truncation.

A.1 Pseudonatural transformations

A.1.1 Vertical composition

Given $F \overset{\xi}{\Rightarrow} G \overset{\theta}{\Rightarrow} H : \mathbf{C} \to \mathbf{D}$, we define $F \overset{\theta\xi}{\Longrightarrow} H : \mathbf{C} \to \mathbf{D}$ by setting:

$$(\theta\xi)_U := \theta_U \xi_U \qquad , \qquad (\theta\xi)_f := \theta_f \xi_U + \theta_V \xi_f \qquad . \tag{A.1.1}$$

Checking (4.1.12a) is straightforward,

$$\partial(\theta\xi)_f + (\theta\xi)_{\partial f} = (\partial\theta_f + \theta_{\partial f})\xi_U + \theta_V(\partial\xi_f + \xi_{\partial f}) = Hf\,\theta_U\xi_U - \theta_V\xi_V Ff \tag{A.1.2}$$

where the second equality used the fact that θ and ξ satisfy (4.1.12a). Checking (4.1.12b) is less straightforward,

$$\begin{aligned}(\theta\xi)_{gf} &= (Hg\,\theta_f + \theta_g Gf)\xi_U + \theta_W(Gg\,\xi_f + \xi_g Ff) \\ &= Hg\,\theta_f\xi_U + \theta_g(\xi_V Ff + \partial\xi_f) + (Hg\,\theta_V - \partial\theta_g)\xi_f + \theta_W\xi_g Ff \\ &= Hg\,(\theta_f\xi_U + \theta_V\xi_f) + (\theta_g\xi_V + \theta_W\xi_g)Ff\end{aligned} \tag{A.1.3}$$

where the first equality used the fact that θ and ξ satisfy (4.1.12b), the second equality used the truncation to omit terms like $\theta_g\xi_{\partial f}$ whereas the last equality used the truncation to annihilate the term $\theta_g\partial\xi_f - (\partial\theta_g)\xi_f$.

A.1.2 Horizontal composition

Given $F \overset{\xi}{\Rightarrow} G : \mathbf{C} \to \mathbf{D}$ and $F' \overset{\upsilon}{\Rightarrow} G' : \mathbf{D} \to \mathbf{E}$, we define $F'F \overset{\upsilon*\xi}{\Longrightarrow} G'G : \mathbf{C} \to \mathbf{E}$ by setting:

$$(\upsilon * \xi)_U := \upsilon_{GU} F'\xi_U \qquad , \qquad (\upsilon * \xi)_f := \upsilon_{Gf} F'\xi_U + \upsilon_{GV} F'\xi_f \quad .$$

This time, checking (4.1.12a) requires using the fact that $\mathbf{Ch}_R^{[-1,0]}$-functors are locally cochain,

$$\partial(\upsilon * \xi)_f + (\upsilon * \xi)_{\partial f} = (\partial \upsilon_{Gf} + \upsilon_{\partial(Gf)})F'\xi_U + \upsilon_{GV}F'(\partial\xi_f + \xi_{\partial f}) = G'Gf\,\upsilon_{GU}F'\xi_U - \upsilon_{GV}F'\xi_V\,F'Ff\,. \tag{A.1.4}$$

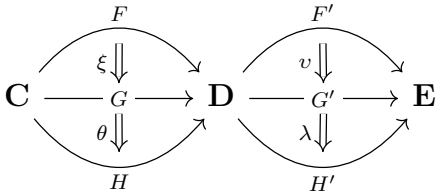
Again, checking (4.1.12b) requires using the truncation,

$$\begin{aligned}(\upsilon * \xi)_{gf} &= (G'Gg\,\upsilon_{Gf} + \upsilon_{Gg}F'Gf)F'\xi_U + \upsilon_{GW}F'(Gg\,\xi_f + \xi_g Ff)\\ &= G'Gg\,\upsilon_{Gf}F'\xi_U + \upsilon_{Gg}F'(\xi_V Ff + \partial\xi_f) + (G'Gg\,\upsilon_{GV} - \partial\upsilon_{Gg})F'\xi_f + \upsilon_{GW}F'\xi_g\,F'Ff\\ &= G'Gg\,(\upsilon_{Gf}F'\xi_U + \upsilon_{GV}F'\xi_f) + (\upsilon_{GW}F'\xi_g + \upsilon_{Gg}F'\xi_V)F'Ff \quad .\end{aligned} \tag{A.1.5}$$

A.2 Modifications

A.2.1 The exchanger

Given



, (A.2.1a)

the modification $*^2_{(\lambda,\theta),(\upsilon,\xi)} : (\lambda * \theta)(\upsilon * \xi) \Rrightarrow \lambda\upsilon * \theta\xi : F'F \Rightarrow H'H$ has components

$$(*^2_{(\lambda,\theta),(\upsilon,\xi)})_U := \lambda_{HU}\upsilon_{\theta_U}F'\xi_U \quad . \tag{A.2.1b}$$

Checking (4.1.15) is straightforward,

$$\partial(*^2_{(\lambda,\theta),(\upsilon,\xi)})_U = [\lambda_{HU}G'\theta_U][\upsilon_{GU}F'\xi_U] - (\lambda\upsilon)_{HU}F'(\theta\xi)_U = [(\lambda * \theta)(\upsilon * \xi)]_U - [\lambda\upsilon * \theta\xi]_U \quad ,$$

where the first equality used the fact that $\partial\theta_U = 0$ hence $\partial\upsilon_{\theta_U} = G'\theta_U\,\upsilon_{GU} - \upsilon_{HU}F'\theta_U$. Checking (4.1.16) is more involved,

$$\begin{aligned}&H'Hf\,\lambda_{HU}\upsilon_{\theta_U}F'\xi_U - \lambda_{HV}\upsilon_{\theta_V}F'(\xi_V Ff)\\ &= [\lambda_{HV}G'Hf + \partial\lambda_{Hf}]\upsilon_{\theta_U}F'\xi_U - \lambda_{HV}\upsilon_{\theta_V}F'(Gf\xi_U - \partial\xi_f)\\ &= \lambda_{HV}[\upsilon_{Hf\,\theta_U} - \upsilon_{Hf}F'\theta_U + G'\theta_V\upsilon_{Gf} - \upsilon_{\theta_V Gf}]F'\xi_U\\ &\quad + \partial[\lambda_{Hf}\upsilon_{\theta_U}F'\xi_U - \lambda_{HV}\upsilon_{\theta_V}F'\xi_f] + \lambda_{Hf}[\partial\upsilon_{\theta_U}]F'\xi_U + \lambda_{HV}[\partial\upsilon_{\theta_V}]F'\xi_f\\ &= \lambda_{HV}[\partial\upsilon_{\theta_f} + \upsilon_{\partial\theta_f} - \upsilon_{Hf}F'\theta_U + G'\theta_V\upsilon_{Gf}]F'\xi_U + \lambda_{Hf}[G'\theta_U\,\upsilon_{GU} - \upsilon_{HU}F'\theta_U]F'\xi_U\\ &\quad + \lambda_{HV}[G'\theta_V\,\upsilon_{GV} - \upsilon_{HV}F'\theta_V]F'\xi_f\\ &= [\lambda_{HV}G'\theta_f + \lambda_{Hf}G'\theta_U]\upsilon_{GU}F'\xi_U + \lambda_{HV}G'\theta_V[\upsilon_{Gf}F'\xi_U + \upsilon_{GV}F'\xi_f] - [\lambda_{Hf}\upsilon_{HU} + \lambda_{HV}\upsilon_{Hf}]F'(\theta_U\xi_U)\\ &\quad - \lambda_{HV}\upsilon_{HV}F'(\theta_V\xi_f + \theta_f\xi_U) \quad ,\end{aligned}$$

where the second equality uses (4.1.12b) on the first and third term whereas the last two equalities use the linearity of the homotopy components together with (4.1.12a).

A.2.2 Lateral composition

Given $\xi \overset{\Xi}{\Rrightarrow} \xi' \overset{\Xi'}{\Rrightarrow} \xi'' : F \Rightarrow G : \mathbf{C} \to \mathbf{D}$, we define $\Xi' \cdot \Xi : \xi \Rrightarrow \xi'' : F \Rightarrow G : \mathbf{C} \to \mathbf{D}$ by setting

$$(\Xi' \cdot \Xi)_U := \Xi'_U + \Xi_U \quad . \tag{A.2.2}$$

Checking well-definedness is incredibly easy:

$$\partial(\Xi'_U + \Xi_U) = \xi'_U - \xi''_U + \xi_U - \xi'_U = \xi_U - \xi''_U \quad , \tag{A.2.3a}$$

$$Gf\,\Xi'_U - \Xi'_V Ff + Gf\,\Xi_U - \Xi_V Ff = \xi'_f - \xi''_f + \xi_f - \xi'_f = \xi_f - \xi''_f \quad . \tag{A.2.3b}$$

A.2.3 Vertical composition

Given $\Xi : \xi \Rrightarrow \xi' : F \Rightarrow G$ and $\Theta : \theta \Rrightarrow \theta' : G \Rightarrow H$, we define $\Theta\Xi : \theta\xi \Rrightarrow \theta'\xi' : F \Rightarrow H$ as

$$(\Theta\Xi)_U := \Theta_U \xi_U + \theta'_U \Xi_U \quad . \tag{A.2.4}$$

Checking (4.1.15) is straightforward,

$$\partial(\Theta\Xi)_U = (\partial\Theta_U)\xi_U + \theta'_U \partial\Xi_U = (\theta_U - \theta'_U)\xi_U + \theta'_U(\xi_U - \xi'_U) = \theta_U\xi_U - \theta'_U\xi'_U \,. \tag{A.2.5}$$

Checking (4.1.16) requires that we use the properties of a pseudonatural transformation,

$$\begin{aligned}
&Hf\,[\Theta_U\xi_U + \theta'_U\Xi_U] - [\Theta_V\xi_V + \theta'_V\Xi_V]Ff \\
&= [\Theta_V Gf + \theta_f - \theta'_f]\xi_U + [\theta'_V Gf + \partial\theta'_f]\Xi_U - \Theta_V[Gf\,\xi_U - \partial\xi_f] - \theta'_V[Gf\,\Xi_U - \xi_f + \xi'_f] \\
&= \theta_f\xi_U - \theta'_f(\partial\Xi_U + \xi'_U) + (\partial\theta'_f)\Xi_U + \Theta_V\partial\xi_f + (\theta_V - \partial\Theta_V)\xi_f - \theta'_V\xi'_f \\
&= \theta_f\xi_U + \theta_V\xi_f - \theta'_f\xi'_U - \theta'_V\xi'_f \quad ,
\end{aligned} \tag{A.2.6}$$

where the first equality used (4.1.15) on the first and fourth term together with (4.1.12a) on the second and third term.

A.2.4 Horizontal composition

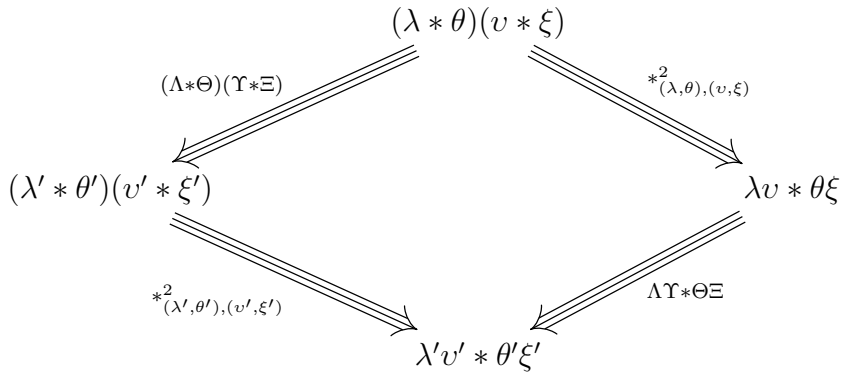
Given modifications $\Xi : \xi \Rrightarrow \xi' : F \Rightarrow G : \mathbf{C} \to \mathbf{D}$ and $\Upsilon : \upsilon \Rrightarrow \upsilon' : F' \Rightarrow G' : \mathbf{D} \to \mathbf{E}$, we define $\Upsilon * \Xi : \upsilon * \xi \Rrightarrow \upsilon' * \xi' : F'F \Rightarrow G'G : \mathbf{C} \to \mathbf{E}$ as

$$(\Upsilon * \Xi)_U := \Upsilon_{GU} F'\xi_U + \upsilon'_{GU} F'\Xi_U \quad . \tag{A.2.7}$$

Comparing the well-definition check of the vertical composition of pseudonatural transformations in Subsection A.1.1 with that of their horizontal composition in Subsection A.1.2, we can see that the well-definition check of the horizontal composition of modifications (A.2.7) will use the same steps as was used in the well-definition check of the vertical composition of modifications in Subsection A.2.3 but now with mention that the $\mathbf{Ch}_R^{[-1,0]}$-functors are locally cochain.

A.2.5 Naturality of pseudo composition constraint

The pseudo composition constraint $*^2$ is a natural isomorphism hence the following diagram must commute



We can determine the components of the mutation via the following computation,

$$
\begin{aligned}
&[(\Lambda * \Theta)(\Upsilon * \Xi)]_U - [\Lambda\Upsilon * \Theta\Xi]_U + [*^2_{(\lambda',\theta'),(\upsilon',\xi')}]_U - [*^2_{(\lambda,\theta),(\upsilon,\xi)}]_U \\
&= \Lambda_{HU}[G'\theta_U \upsilon_{GU} - \upsilon_{HU}F'\theta_U]F'\xi_U + \lambda'_{HU}[G'\Theta_U \upsilon_{GU} - \upsilon'_{HU}F'\Theta_U]F'\xi_U - \lambda_{HU}\upsilon_{\theta_U}F'\xi_U \\
&\quad + \lambda'_{HU}[G'\theta'_U \Upsilon_{GU} - \Upsilon_{HU}F'\theta_U]F'\xi_U + \lambda'_{HU}[G'\theta'_U \upsilon'_{GU} - \upsilon'_{HU}F'\theta'_U]F'\Xi_U + \lambda'_{HU}\upsilon'_{\theta'_U}F'\xi'_U \\
&= \Lambda_{HU}(\partial\upsilon_{\theta_U})F'\xi_U + \lambda'_{HU}[(\partial\Upsilon_{HU})F'\Theta_U + \upsilon_{\theta_U - \theta'_U}]F'\xi_U - \lambda_{HU}\upsilon_{\theta_U}F'\xi_U \\
&\quad + \lambda'_{HU}[\upsilon_{\theta'_U} - \upsilon'_{\theta'_U} - \Upsilon_{HU}\partial(F'\Theta_U)]F'\xi_U + \lambda'_{HU}(\partial\upsilon'_{\theta'_U})F'\Xi_U + \lambda'_{HU}\upsilon'_{\theta'_U}F'\xi'_U \\
&= 0
\end{aligned}
$$

where the second equality used, for example, $G'\Theta_U\, \upsilon_{GU} = \upsilon_{HU}F'\Theta_U + \upsilon_{\partial\Theta_U}$ together with $\partial\Upsilon_{HU} = \upsilon_{HU} - \upsilon'_{HU}$ and $\partial\Theta_U = \theta_U - \theta'_U$.

A.2.6 Exchange for lateral and vertical/horizontal composition

In this tricategorical framework, the vertical composition $\circ$ is a functor hence we must have

$$(\Theta'\Xi') \cdot (\Theta\Xi) = (\Theta' \cdot \Theta)(\Xi' \cdot \Xi) : \theta\xi \Rrightarrow \theta''\xi'' : F \Rightarrow H : \mathbf{C} \to \mathbf{D} \quad , \tag{A.2.8a}$$

which is easy to check,

$$
\begin{aligned}
(\Theta'\Xi')_U + (\Theta\Xi)_U &= \Theta'_U\xi'_U + \theta''_U\Xi'_U + \Theta_U\xi_U + \theta'_U\Xi_U \\
&= \Theta'_U(\xi_U - \partial\Xi_U) + \theta''_U\Xi'_U + \Theta_U\xi_U + (\theta''_U + \partial\Theta'_U)\Xi_U \\
&= (\Theta' \cdot \Theta)_U\xi_U + \theta''_U(\Xi' \cdot \Xi)_U \quad .
\end{aligned} \tag{A.2.9a}
$$

The horizontal composition is locally functorial hence we must have lateral and horizontal exchange but, as in Subsection A.2.4, this check will follow the same steps as above.

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